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FREE STEINER LOOPS

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ABSTRACT. A Steiner loop, or a sloop, is a groupoid $(L; \cdot, 1)$, where $\cdot$ is a binary operation and 1 is a constant, satisfying the identities $1 \cdot x = x$, $x \cdot y = y \cdot x$, $x \cdot (x \cdot y) = y$. There is a one-to-one correspondence between Steiner triple systems and finite sloops.

Two constructions of free objects in the variety of sloops are presented in this paper. They both allow recursive construction of a free sloop with a free base X, provided that X is recursively defined set. The main results besides the constructions, are: Each subsloop of a free sloop is free too. A free sloop $\mathbf{S}$ with a free finite base X, $|X| \geq 3$, has a free subsloop with a free base of any finite cardinality and a free subsloop with a free base of cardinality ω as well; also $\mathbf{S}$ has a (non free) base of any finite cardinality $k \geq |X|$. We also show that the word problem for the variety of sloops is solvable, due to embedding property.

1. PRELIMINARIES

A Steiner loop, or a sloop, is an algebra $(L; \cdot, 1)$, where $\cdot$ is a binary operation and 1 is a constant, that satisfies the following identities

(S1) $$1 \cdot x = x$$
(S2) $$x \cdot y = y \cdot x$$
(S3) $$x \cdot (x \cdot y) = y$$

A Steiner triple system (STS) is a pair (L, M) where L is a finite set, M is a set containing three-element subsets of L with the property that for any $a, b \in L$ $(a \neq b)$ there is a unique $c \in L$ such that $\{a, b, c\} \in M$. It is evident that any STS on a set L enables a construction of a sloop on the set $L \cup \{1\}$ where $1 \notin L$, and vice versa. So, there is a one-to-one correspondence between Steiner triple systems and finite sloops (see [4], [7]).

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The class of involutory commutative loops is defined by the laws:
$1x = x,\ xx = 1,\ xy = yx,\ \forall x \forall y \exists! z \exists! u (xz = y \wedge ux = y)$.

Proposition 1. *The variety of sloops is a proper subvariety of the class of involutory commutative loops.*

Proof. If $(L; \cdot, 1)$ is a sloop then the equation $ax = b$ for any $a, b \in L$ has a unique solution $x = ab$. What follows is an example of an involutory commutative loop which is not a sloop:

$\cdot$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	a	1	c	d	e	b
b	b	c	1	e	a	d
c	c	d	e	1	b	a
d	d	e	a	b	1	c
e	e	b	d	a	c	1

$\square$

Further on we use the term *base* for a minimal generating set of an algebra, and *free base* for a base of an algebra in a given variety which has the universal mapping property. So, a set X is a free base of a sloop $\mathbf{S} = (S; \cdot, 1)$ iff X is its base and each mapping from X to L, where $\mathbf{L} = (L; \cdot, 1)$ is a sloop, can be extended to a homomorphism from $\mathbf{S}$ into $\mathbf{L}$.

2. Free sloops - construction 1

Let X be a given set. We define a chain of sets X_i and a set F_X by:

$$X_1 := X, \quad X_{i+1} := X_i \cup \{\{u, v\} \subseteq X_i |\ u \neq v,\ u \notin v,\ v \notin u\},$$
$$F_X := (\cup\, (X_i |\ i \geq 1)) \cup \{1\} \quad \textit{where } 1 \notin \cup\, (X_i |\ i \geq 1).$$

Proposition 2. *An element $x \in X_{i+1} \setminus X_i$ iff $x = \{u, v\}$ for some uniquely determined u and v such that $u \in X_i \setminus X_{i-1}$ or $v \in X_i \setminus X_{i-1}$.*

Define an operation $*$ on F_X as follows. If $u, v \in F_X \setminus \{1\}$ then

$$u * v := \begin{cases} \{u, v\} & u \neq v,\ u \notin v,\ v \notin u \\ 1 & u = v \\ t & v = \{u, t\} \textit{ or } u = \{v, t\} \end{cases}$$

and $1 * u := u, \quad u * 1 := u, \quad 1 * 1 := 1.$

Theorem 2.1. *$\mathbf{F}_X = (F_X; *, 1)$ is a free object in the variety of sloops with free base X.*

Proof. The commutativity is obvious. We check the identity $u*(u*v) = v$ in the following cases.

1) $u \neq v,\ u \notin v,\ v \notin u$: $u * (u * v) = u * \{u, v\} = v$,
2) $v = \{u, t\}$: $u * (u * v) = u * t = \{u, t\} = v$,
3) $u = \{v, t\}$: $u * (u * v) = u * t = v$.

In every other case, the statement is straightforward. So, $\mathbf{F}_X$ is a sloop.

It is clear that X is a base of $\mathbf{F}_X$ and it is a free one too. Namely, let $(L; \cdot, 1)$ be a sloop and $\phi : X \longrightarrow L$ a mapping. Define inductively a chain of mappings $(\phi_i : X_i \longrightarrow L \mid i \geq 1)$ as follows. $\phi_1 = \phi$ and if ϕ_i is defined, then for $x \in X_{i+1}$,

$$\phi_{i+1}(x) := \begin{cases} \phi_i(x) & x \in X_i \\ \phi_i(u) \cdot \phi_i(v) & x = \{u, v\} \in X_{i+1} \setminus X_i \end{cases}$$

By *Proposition 2*, ϕ_i is well defined for each $i \geq 1$.

Let $\phi^* := \cup(\phi_i \mid i \geq 1) \cup \{(1,1)\}$. In order to prove that ϕ^* is a homomorphism we consider the following cases.

1) $u \neq v,\ u \notin v,\ v \notin u\ (u, v \in X_i$ for some $i \geq 1)$: $\phi^*(u * v) = \phi^*(\{u, v\}) = \phi_{i+1}(\{u, v\}) = \phi_i(u) \cdot \phi_i(v) = \phi^*(u) \cdot \phi^*(v)$.

2) $u = \{v, t\} \in X_i$ for some $i > 1$: $\phi^*(u * v) = \phi^*(t) = \phi_{i-1}(t) = (\phi_{i-1}(v) \cdot \phi_{i-1}(t)) \cdot \phi_{i-1}(v) = \phi_i(\{v, t\}) \cdot \phi_{i-1}(v) = \phi^*(u) \cdot \phi^*(v)$, since $t = v * (t * v)$ by (S2) and (S3).

The remaining cases are trivial. ∎

Assuming that the set X is already well ordered (i.e. we work with sets of ZFC set theory (see [8])), we define an order on F_X extending the order of X, by induction on the number of pairs of braces, in the following way.

The element 1 is the smallest in F_X. If $\alpha, \beta \in F_X$ and α has smaller number of (pairs of) braces than β, then $\alpha < \beta$. If $\{\alpha, \beta\} \neq \{\gamma, \delta\} \in F_X$, $\{\alpha, \beta\}$, $\{\gamma, \delta\}$ have the same number of pairs of braces and $\alpha < \beta$, $\gamma < \delta$, then we set

$\{\alpha, \beta\} < \{\gamma, \delta\}$ if either $\alpha < \gamma$ or $\alpha = \gamma, \beta < \delta$ and
$\{\gamma, \delta\} < \{\alpha, \beta\}$ if either $\gamma < \alpha$ or $\alpha = \gamma, \delta < \beta$.

Proposition 3. *$(F_X, \leq)$ is a well ordered set.*

Proof. Let $A \subseteq F_X$. If A contains an element without braces, then the smallest element in $(X \cup \{1\}) \cap A$ is the smallest in A. Else, let $k > 0$ be the smallest number of braces of an element of A and $A' = \{a \in A \mid$ the number of braces in a is $k\}$. Consider the set $A'' = \{u \in F_X \mid \{u, v\} \in A', u < v\}$. By the inductive hypothesis A'' has the least element α and $A''' = \{v \in F_X \mid \{\alpha, v\} \in A'\}$ has the least element β. Then $\{\alpha, \beta\}$ is the least element in A' i.e. A. ∎

Note that if X is a recursive set, then F_X is recursive too.

3. Free sloops - construction 2

Here we will present another description of the free sloops by using the free term algebra $\mathbf{Term}_X = (Term; \cdot, 1)$ (i.e. the absolutely free algebra) over a set of free generators X, in the signature $\cdot, 1$. Any free sloop with free base

X can be obtained as a quotient algebra of $\mathbf{Term}_X$ ([7, 2]). Instead of that, our new construction will use a subset of $Term$ as a universe of a free sloop.

Define inductively a mapping $d : Term \longrightarrow \mathbb{N}$, where $\mathbb{N}$ is the set of non-negative integers, by:

$d(1) := 0, \quad d(x) := 0$ for $x \in X, \quad d(t_1 \cdot t_2) := d(t_1) + d(t_2) + 1.$

We shall refer to $d(t)$ as *weight* of the term $t \in Term$.

By induction on weight, define a mapping $C : Term \longrightarrow F_X$ in the following way:

$$C(t) := \begin{cases} 1 & t = 1 \textit{ or } t = t_1 \cdot t_2,\ C(t_1) = C(t_2) \\ t & t \in X \\ C(t_1) & t = t_1 \cdot t_2,\ C(t_2) = 1 \\ C(t_2) & t = t_1 \cdot t_2,\ C(t_1) = 1 \\ C(t_3) & t = t_1 \cdot t_2,\ C(t_2) = \{C(t_1), C(t_3)\} \textit{ or} \\ & t = t_1 \cdot t_2,\ C(t_1) = \{C(t_2), C(t_3)\} \\ \{C(t_1), C(t_2)\} & t = t_1 \cdot t_2 \textit{ and none of the previous holds} \end{cases}$$

Proposition 4. *The mapping C is an epimorphism from $\mathbf{Term}_X$ onto $\mathbf{F}_X$.*

Now, by the homomorphism theorem we have $\mathbf{Term}_X / kerC \cong \mathbf{F}_X$. Further on we will determine a canonical representative for each congruence class as follows.

Assuming that X is a well ordered set we define a mapping $T : F_X \longrightarrow Term$ using the well ordering of F_X, by:

$T(1) := 1, \quad T(x) := x$ for $x \in X, \quad T(\{u, v\}) := T(u) \cdot T(v)$ where $u < v$.

Proposition 5. *T is injective.*

Proposition 6. *$TCT = T, \ \ CTC = C$.*

Proof. Let $\alpha \in F_X$. If $\alpha = 1$ or $\alpha \in X$, the statement holds trivially. Let $\alpha = \{u, v\},\ u, v \in F_X,\ u < v$. Assume that the statement holds for any element of F_X smaller than α. So, $TCT(u) = T(u),\ TCT(v) = T(v)$. Then $CT(u) = u,\ CT(v) = v$ by *Proposition 5*, and since $\alpha \in F_X$ we have $CT(u) \neq CT(v),\ CT(u) \notin CT(v),\ CT(v) \notin CT(u)$. Hence, $TCT(\alpha) = TC(T(u) \cdot T(v)) = T(\{CT(u), CT(v)\}) = T(\{u, v\}) = T(\alpha)$.

Now, $CTC = C$ follows by *Proposition 5* and $TCT = T$. ☐

For an element $t \in Term$ we say that it is *reduced* if $TC(t) = t$. The mapping $R = TC$ will be called *reduction*. Note that $(R(t), t) \in kerC$ and in each congruence class there is only one reduced element which will be the canonical representative of the class.

The mapping R has the following properties.

Proposition 7. *$R^n = R$, for each $n \geq 2$, and for all $t, s \in Term$ we have:*

(i) $R(1 \cdot t) = R(t)$;

(ii) $R(t \cdot s) = R(s \cdot t)$;
(iii) $R(t \cdot (t \cdot s)) = R(s)$;
(iv) $R(t \cdot s) = t \cdot s \implies R(t) = t,\ R(s) = s$;
(v) $R(R(t) \cdot s) = R(t \cdot s)$;
(vi) $R(R(t) \cdot R(s)) = R(t \cdot s)$.

Proof. $R^n = R$ for $n \geq 2$ follows from *Proposition 6*. (i), (ii) and (iii) are straightforward since C is a homomorphism and $\mathbf{F}_X$ is a sloop, and (vi) is a consequence of (ii) and (v).

(iv) $R(t \cdot s) = TC(t \cdot s) = t \cdot s$ implies that $C(t \cdot s) = \{\alpha, \beta\}$ where $\alpha < \beta,\ T(\alpha) = t,\ T(\beta) = s$. Now, $TC(t) = TCT(\alpha) = T(\alpha) = t$ by *Proposition 6*, and in the same way $TC(s) = s$.

(v) Since $CR(t) = CTC(t) = C(t)$ we have

$$C(R(t) \cdot s) = \begin{cases} 1 & C(t) = C(s) \\ C(s) & C(t) = 1 \\ C(t) & C(s) = 1 \\ C(l) & C(t) = \{C(s), C(l)\} \text{ or } C(s) = \{C(t), C(l)\} \\ \{C(t), C(s)\} & \text{otherwise} \end{cases}$$

i.e. $C(R(t) \cdot s) = C(t \cdot s)$ and hence $TC(R(t) \cdot s) = TC(t \cdot s)$. ∎

Let G_X be the set of reduced terms i.e. $G_X = R(Term) = T(F_X)$. Define an operation $\circ$ on G_X by

$$t \circ s := R(t \cdot s) \text{ for all } t, s \in G_X.$$

Theorem 3.1. *$\mathbf{G}_X = (G_X; \circ, 1)$ is a free sloop with free base X.*

Proof. We will prove that the bijective mapping T is an isomorphism between $(F_X; *, 1)$ and $(G_X; \circ, 1)$. For each $t, s \in Term_X$ by *Proposition* 4 and 5 we have that $t/kerC \cdot s/kerC = (t \cdot s)/kerC = (R(t \cdot s))/kerC = R(R(t) \cdot R(s))/kerC = (R(t) \circ R(s))/kerC$. Since $\mathbf{Term}_X/kerC \cong \mathbf{F}_X$ we obtain $C(t) * C(s) = C(R(t) \circ R(s))$ and if $u = C(t),\ v = C(s)$, then $T(u * v) = T(C(t) * C(s)) = T(C(R(t) \circ R(s))) = R(R(t) \circ R(s)) = R(t) \circ R(s) = TC(t) \circ TC(s) = T(u) \circ T(v)$. ∎

Note that if X is a recursive set, then since C and T are recursively defined, we have that G_X is a recursive set too.

4. Some properties of free sloops

Proposition 8. *If X is a free base of a free sloop $\mathbf{S}$, then $\mathbf{S}$ is finite if and only if $|X| \leq 2$.*

Proof. $S = \{1\}$ for $X = \emptyset$, $S = \{1, a\}$ for $X = \{a\}$ and $S = \{1, a, b, ab\}$ for $X = \{a, b\}$. If $X = \{a, b, c, \dots\}$ where $|\{a, b, c\}| = 3$, consider the set $M = \{x_i \mid i \geq 1\}$ where $x_1 = a,\ x_2 = b,\ x_{2n+1} = ax_{2n},\ x_{2n+2} = cx_{2n+1}$ for $n \geq 1$. We have $M \subseteq S$, and M is infinite since $x_i \neq x_j$ for $i \neq j$. ∎

Theorem 4.1. *Every subsloop of a free sloop is free too.*

Proof. Let $\mathbf{G}_X = (G_X; \circ, 1)$ be a free sloop as in Construction 2, and let G' be a subsloop of $\mathbf{G}_X$. Recall that $R(t) = t$ for each $t \in G_X$.

If $x, y \in G' \setminus \{1\}$, then we say that x is a divisor of y if and only if there is a $t \in Term \setminus \{1\}$ such that $y = t \cdot x$ or $y = x \cdot t$. Then also $t \in G' \setminus \{1\}$, since by *Proposition 7*, (iv), the definition of $\circ$ and (S2), (S3) we have $t \in G_X$ and $t = x \circ y$. Note that if x is a divisor of y then $d(x) < d(y)$, which implies that any sequence $t_1, t_2, \ldots, t_n, \ldots$ such that t_{i+1} is a divisor of t_i, $i \geq 1$, is finite.

We shall prove that $B = \{t \in G' \setminus \{1\} | \ t \text{ has no divisors}\}$ is a free base for G'.

At first, by an induction on weight we show that B is a generating set of G'. Let $z \in G' \setminus \{1\}$. If $z \notin B$, then z has divisors, i.e. $z = x \cdot y$ for some $x, y \in G'$ and $z = R(z) = R(x \cdot y) = x \circ y$. By the inductive hypothesis x and y are generated by B and so is z.

Next we show that B is a base of G'. Namely, let $b \in B$ and let G'' be the subsloop of G' generated by $B \setminus \{b\}$. Then $G'' = \cup(G''_i \mid i \geq 1)$ where $G''_1 = B \setminus \{b\}$, $G''_{i+1} = \{t \circ s \mid t, s \in G''_i\}$. Now, $b \notin G''_2$ since if $b = t \circ s$ for some $t, s \in G''_1 = B \setminus \{b\}$, then $b = t \cdot s$. If $b \in G''_{i+1} \setminus G''_i$ for some $i \geq 2$, then $b = t \circ s$ for some $t, s \in G''_i$ such that $t \in G''_i \setminus G''_{i-1}$ (or $s \in G''_i \setminus G''_{i-1}$). We have to consider several cases. The case $t = s$ is not possible, since $t \circ t = 1$ and $b \neq 1$. If $t = s \circ u$ (or $s = t \circ u$) for some $u \in G''_{i-1}$, then $b = u \in G''_{i-1}$. The only case left is $b = t \cdot s$, contradicting $b \in B$.

Let $(L; *, 1)$ be an arbitrary sloop and $f : B \to L$ a mapping. We extend f to homomorphism $f' : G' \to L$ by an induction on weight in the following way: $f'(1) := 1$, $f'(b) := b$ for each $b \in B$, $f'(t) := f'(x) * f'(y)$ when $t = x \cdot y \in G' \setminus B$.

Then for any $t, s \in G'$ we have:

$$t \circ s = R(t \cdot s) = \begin{cases} 1 & t = s \\ s & t = 1 \\ t & s = 1 \\ l & s = t \cdot l \textit{ or } s = l \cdot t \textit{ or} \\ & t = s \cdot l \textit{ or } t = l \cdot s \\ t \cdot s & \textit{if none of the previous} \\ & \textit{holds and } C(t) < C(s) \\ s \cdot t & \textit{otherwise} \end{cases}$$

In all of the cases listed, from the definition of f' and the fact that $(L; *, 1)$ is a sloop, it follows that $f'(t \circ s) = f'(t) * f'(s)$. □

Corollary 1. *Every free sloop with at least 3 element free base has a free subsloop with infinite free base, and a free subsloop with free base of any finite cardinality.*

PROOF. Let $\mathbf{G}_X$ be the free sloop with free base X obtained by the construction 2, and let $a, b, c \in X$. Let $M = \{x_i \mid i \geq 1\} \subseteq Term$, where $x_1 = ab,\ x_2 = ac,\ x_{2n+1} = (x_{2n-1}c)(x_{2n}b),\ x_{2n+2} = (x_{2n}b)(x_{2n+1}c)$. Let G' be the subsloop of $\mathbf{G}_X$ generated by M. Since M is the set of elements of G' that have no divisors, by *Theorem 3* we have that G' is a free subsloop of $\mathbf{G}_X$ with infinite free base M. Out of the same reason, if $K = \{x_1, x_2, \ldots, x_k\} \subset M$, then the subsloop of $\mathbf{G}_X$ generated by K is a free one with k-element free base K. ∎

PROPOSITION 9. *A free sloop with free base X, $|X| \geq 3$, has infinitely many free bases.*

PROOF. Let $X = \{a, b, c\}$ be a free base of a free sloop S. Denote a sequence of elements of S by $b_0 = b,\ b_{2k+1} = ab_{2k},\ b_{2k+2} = cb_{2k+1},\ k \geq 0$. Then $X_i = \{a, b_i, c\}$ is a free base of S as well. ∎

The variety of sloops has nontrivial finite algebras, so there are no two isomorphic free sloops with finite free bases of different cardinality [7]. Nevertheless, we will show that any free sloop with finite base X, $|X| \geq 3$, has a base of any finite cardinality greater than $|X|$. Namely, it is a consequence of the following property, where $\mathbf{G}_X$ denotes the free sloop of construction 2.

PROPOSITION 10. *If $X = \{b_1, b_2, b_3, \ldots, b_k\},\ k \geq 3$, is a base of $\mathbf{G}_X$, then $\mathbf{G}_X$ has also a base $\{b_1, b_2, b_3', b_3'', b_4, \ldots, b_k\}$, where*

$$b_3' = (b_1 \cdot (b_2 \cdot b_3)) \cdot (b_2 \cdot (b_1 \cdot b_3)), \quad b_3'' = b_3' \cdot b_3.$$

PROOF. Let S be the subsloop of $\mathbf{G}_X$ generated by $\{b_1, b_2, b_3\}$, and let S' be the subsloop of $\mathbf{G}_X$ generated by $\{b_1, b_2, b_3', b_3''\}$. Since $b_3 = b_3' \cdot b_3''$, it is clear that $S' = S$. We shall prove that $\{b_1, b_2, b_3', b_3''\}$ is a base for S'.

Let S'' be the subsloop of $\mathbf{G}_X$ generated by $\{b_1, b_2, b_3'\}$. We shall prove that $b_3 \notin S''$.

For this purpose, first note that $S'' = \cup(S_i'' \mid i \geq 1)$ where $S_1'' = \{b_1, b_2, b_3'\}$ and $S_{i+1}'' = S_i'' \cup \{x \circ y \mid x, y \in S_i''\}$.

It is clear that $b_3,\ t_1 = b_1 \cdot (b_2 \cdot b_3),\ t_2 = b_2 \cdot (b_1 \cdot b_3) \notin S_2''$. Let $b_3, t_1, t_2 \notin S_i''$. Then $b_3 \notin S_{i+1}''$ since in order to extract b_3, b_3' must be multiplied by t_1 or t_2. Also, since $b_3 \notin S_i''$ we have $t_1, t_2 \notin S_{i+1}''$.

In a similar manner, it follows that the subsloops of S' generated by each of the sets $\{b_1, b_2, b_3''\}, \{b_2, b_3', b_3''\}, \{b_1, b_3', b_3''\}$ are proper subsets of S'. ∎

5. The word problem for sloops

We show that the word problem for the variety of sloops is solvable. Namely, we use the following T. Evans' result ([3]):

If V is a variety with the property that any incomplete V-algebra can be embedded in a V-algebra, then the word problem is solvable for V.

According to Evans' definition of incomplete algebras, an incomplete sloop with universe G is a quadruple $(G, \cdot, 1, D)$, where $D \subseteq G^2, 1 \in G, \cdot : D \to G$

is a mapping (called an incomplete operation on G), satisfying the following conditions:

(IS1) $(x,x)\in D \implies x\cdot x = 1$
(IS2) $(x,y)\in D \implies (y,x)\in D,\ x\cdot y = y\cdot x$
(IS3) $(x,1)\in D \implies x\cdot 1 = x$
(IS4) $(x,y)\in D \implies (x, x\cdot y)\in D,\ x\cdot(x\cdot y) = y$

Proposition 11. *Any incomplete sloop can be embedded into a sloop.*

Proof. Let $(G,\cdot,1,D)$ be an incomplete sloop. Denote $G_0 = G$, $D_0 = D\cup\{(x,x)|x\in G\}\cup\{(1,x),(x,1)|x\in G\}$ and let $\cdot_0 : D_0\to G$ be defined by $x\cdot_0 y := x\cdot y$, for $(x,y)\in D$, $x\cdot_0 x := 1, x\cdot_0 1 := x, 1\cdot_0 x := x$ for $x\in G$. Then $(G_0,\cdot_0,1,D_0)$ is an incomplete sloop such that $D\subseteq D_0\subseteq G_0^2$.

If $(G_i,\cdot_i,1,D_i)$ is defined incomplete sloop, we form a new one as follows.

Denote $C_i = \{\{x,y\}|x,y\in G_i,(x,y)\notin D_i\}$ and put $G_{i+1} = G_i\cup C_i$ (assuming that $C_i\cap G_i=\emptyset$). Define an incomplete operation $\cdot_{i+1}$ by:

$$\begin{array}{lcl}
(x,y)\in D_i & \implies & x\cdot_{i+1} y := x\cdot_i y,\\
(x,y)\in G_i^2\setminus D_i & \implies & x\cdot_{i+1} y := \{x,y\},\\
x\in G_{i+1} & \implies & x\cdot_{i+1} x := 1,\ x\cdot_{i+1} 1 := 1\cdot_{i+1} x := x,\\
x\in G_i,\ \{x,y\}\in C_i & \implies & x\cdot_{i+1}\{x,y\} := y,\ \{x,y\}\cdot_{i+1} x := y.
\end{array}$$

Let D_{i+1} be the set of all $(x,y)\in G_{i+1}$ for which $x\cdot_{i+1} y$ is defined.

It is clear that (IS1) - (IS3) hold for $(G_{i+1},\cdot_{i+1},1,D_{i+1})$. Several cases have to be considered in order to check (IS4) and the nontrivial ones are:

$(x,y)\in G_i^2\setminus D_i \implies x\cdot_{i+1} y = \{x,y\} \implies x\cdot_{i+1}(x\cdot_{i+1} y) = x\cdot_{i+1}\{x,y\} = y$;

$x\in G_i,\ y=\{x,z\}\in C_i \implies x\cdot_{i+1} y = x\cdot_{i+1}\{x,z\} = z \implies x\cdot_{i+1}(x\cdot_{i+1} y) = x\cdot_{i+1} z = \{x,z\} = y$.

That way we obtained chains of sets $(G_i|\ i\geq 0)$, $(D_i|\ i\geq 0)$, $(\cdot_i|\ i\geq 0)$, with the properties:

$$G_i\subseteq G_{i+1},\ D_i\subseteq G_i^2\subseteq D_{i+1},\ \cdot_i\subseteq\cdot_{i+1}.$$

Let

$$G^* = \bigcup_{i\geq 0} G_i,\ D^* = \bigcup_{i\geq 0} D_i,\ \cdot^* = \bigcup_{i\geq 0}\cdot_i.$$

Now for $x,y\in G^*$, there exists $i\geq 0$ such that $x,y\in G_i$, so $(x,y)\in D_{i+1}$, i.e. $(x,y)\in D^*$. Hence, $D^* = (G^*)^2$ i.e. $(G^*,\cdot^*,1)$ is a sloop in which $(G,\cdot,1,D)$ is embedded. □

As a corrolary of *Proposition 11* and [3] we get the following result.

Theorem 5.1. *The word problem for the variety of sloops is solvable.*

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