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A DISCRETE BOUNDARY VALUE PROBLEM IN BANACH SPACES

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Abstract. In this paper we present existence theorems for the second order discrete boundary value problem in Banach spaces under weaker conditions we have known. We suppose the weak sequential continuity and some conditions expressed in terms of the measure of weak noncompactness.

1. Introduction

Throught this paper E denotes a Banach space with the norm $\|\cdot\|$, $N = \{1, 2, \dots, T\}$, $N^+ = \{0, 1, \dots, T+1\}$ where $T \in \{1, 2, 3, \dots\}$. We shall consider the abstract discrete boundary value problem

$$
\begin{cases} \Delta^2 y(i-1) + \quad f(i, y(i)) = 0, \quad i \in N, \\ y(0) = 0, \qquad\qquad y(T+1) = 0 \end{cases} \tag{1.1}
$$

where $y : N^+ \to E$ and

$$
f : N \times E \to E \tag{1.2}
$$

is weakly-weakly sequentially continuous i.e.

$$
\underset{x_n \in E}{\forall} \; \underset{i \in N}{\forall} (x_n \underset{weakly}{\longrightarrow} x) \Rightarrow (f(i, x_n) \underset{weakly}{\longrightarrow} f(i, x)).
$$

Let $B = \{x \in E : \|x\| \leq 1\}$ and let H be a bounded subset of E. The $\beta(H)$ measure of weak noncompactness of H is defined by

$$
\beta(H) = \inf\{t \geq 0 : H \subset K + t \cdot B \quad \text{for some weakly compact} \quad K \subset E\}.
$$

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Remark 1.1. Since N^+ is a discrete space, then any mapping of N^+ to a topological space E is continuous. We shall denote the set of all such mappings by $C(N^+, E)$.

Our result will be proved by the following fixed point theorem (see [7], Th. 2).

Theorem 1.2. *Let D be a nonempty, weakly closed, convex and bounded subset of Banach space E.*

Let $F : D \to D$ be a weakly sequentially continuous mapping, which is condensing with respect to the measure of weak noncompactness β i.e.

$$\beta(F(V)) < \beta(V)$$

for $\beta(V) > 0,\ V \subset D$ (i.e. F is a Darbo map), then F has fixed point.

We shall use the following properties of measure of weak noncompactness β.

Theorem 1.3. [3.6] *Let A, B are bounded subsets of E. Then*

I. $\beta(A) = 0 \Rightarrow A$ *is relatively weakly compact subset,*
II. $\beta(\overline{conv} A) = \beta(A)$,
III. $A \subset B \Rightarrow \beta(A) \leq \beta(B)$,
IV. $\beta(kA) = k\beta(A)$ *for* $k \in (0, \infty)$,
V. $\beta(\{x_0\} \cup A) = \beta(A)$.

Remark 1.4. The properties of measure of weak noncompactness are analogoues to the properties of measure of noncompactness (see [3], [4], [5], [8]).

Theorem 1.5. [8]. *Let $V \subset C(N^+, E)$ be a family of strongly equicontinuous functions. Then*

$$\beta(V) = \beta(V(N^+)) = \sup\{\beta(V(i)) : i \in N^+\}$$

where $\beta(V)$ denotes the measure of weak noncompactness in $C(N^+, E)$.

The semi-inner product on E is defined by

$$\langle x, y\rangle_+ = \|x\| \lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}.$$

The following properties are well known [1].

Theorem 1.6.

(a) $|\langle x, y\rangle_+| \leq \|x\| \cdot \|y\|$ *for* $x, y \in E$.

(b) $\langle y, x + \alpha y\rangle_+ = \langle y, x\rangle_+ + \alpha\|y\|^2$ *for all* $\alpha \in R,\ x, y \in E$.

(c) *If* $x : N^+ \to E$ *and* $i \in \{0, 1, \ldots, T-1\}$

then

$$\|x(i+1)\| \cdot \Delta^2\|x(i)\| \geq \langle x(i+1), \Delta^2 x(i)\rangle_+.$$

2. Existence theorems

Let G be a Green function for the discrete boundary value problem

$$\begin{cases} \Delta^2 y(i-1) = 0, & i \in N \\ y(0) = y(T+1) = 0 \end{cases}$$

i.e.

$$G(i,j) = \begin{cases} \frac{j(T+1-i)}{T+1}, & 0 \le j \le i-1 \\ \frac{i(T+1-j)}{T+1}, & i \le j \le T+1. \end{cases} \tag{2.3}$$

Solving (1.1) is equivalent to finding a $y \in C(N^+, E)$ which satisfies

$$y(i) = \sum_{j=1}^{T} G(i,j) f(j, y(j)), \quad i \in N^+$$

i.e. to finding a fixed point of the operator S defined by

$$(Sy)(i) = \sum_{j=1}^{T} G(i,j) f(j, y(j)) \quad \text{for} \quad y \in C(N^+, E),\ i \in N^+. \tag{2.4}$$

Theorem 2.1. *Assume that*

1° *$f : N \times E \to E$ is weakly-weakly sequentially continuous,*
2° *f is bounded,*
3° *$\beta(f(N \times A)) \le k\beta(A)$ for all bounded subset A of E where $k \ge 0$ is a constant,*
4° *$rTk < 1$, where*

$$r = \max\{r_i : i \in N^+\},$$
$$r_i = \max\{G(i,j) : j \in N\}.$$

Then (1.1) *has a solution $y \in C(N^+, E)$.*

Proof. Define the operator $S : C(N^+, E) \to C(N^+, E)$ by (2.4). Because G and f are bounded so S is bounded on $C(N^+, E)$. Of course S is weakly-weakly sequentially continuous. We will show that S is a Darbo map. To see this let V be a bounded subset of $C(N^+, E)$. Fix $i \in N^+$. Then

$$\begin{aligned}
\beta(SV(i)) &= \beta\Big(\Big\{\sum_{j=1}^{T} G(i,j) f(j, y(j)) : y \in V\Big\}\Big) = \\
&= \beta(T\,\overline{conv}\{G(i,j) f(j, y(j)) : y \in V,\ j \in N\}) = \\
&= T\beta(\{G(i,j) \cdot f(j, y(j)) : y \in V,\ j \in N\}) \le \\
&\le Tr_i \beta(\{f(j, y(j)) : y \in V,\ j \in N\}) \le \\
&\le Tr_i \beta(f(N \times V(N))) \le Tr_i k \beta(V(N)) = \\
&= Tr_i k \beta(V) \le Trk\beta(V).
\end{aligned}$$

By the Theorem 1.5

$$\beta(SV) \leq Trk\beta(V).$$

Since $Trk < 1$, then $S : C(N^+, E) \to C(N^+, E)$ is bounded Darbo map. So by Theorem 1.2 S has a fixed point. □

Remark 2.2. If E is a reflexive Banach space, then in Theorem 2.1 we assume only 1° and 2° (as the closed ball is weakly compact).

Remark 2.3. The condition 3° in Theorem 2.1 can be replaced by the condition with use axiomatic measure of weak-noncompactness μ (see [3], [4], [5]).

Analogously as Theorem 2.1 we can prove the following results.

Theorem 2.4. *Assume that*
1° $f : N \times E \to E$ *is weakly-weakly sequentially continuous,*
2° f *is bounded,*
3° $\mu(f(N \times A)) \leq k\mu(f(A))$ *for all bounded subset of* E *where* $k \geq 0$,
4° *the condition 4° in Theorem 2.1 be satisfied.*
The (1.1) has solution $y \in C(N^+, E)$.

Remark 2.5. Observe that the class of continuous functions is different to the class of weakly-weakly sequentially continuous functions and weakly-weakly continuous.

There exist many important examples of mappings, which are weakly sequentially continuous but not weakly continuous.

The relationship between strong weak and weak sequential continuity for mappings is studied in [2].

Theorem 2.6. *Assume that*
1° $f : N \times E \to E$ *is weakly-weakly sequentially continuous,*
2° *there exists* $v \in C(N^+, E)$ *and* $M \in C(N^+, (0, \infty))$ *with* $\langle y - v(i), -f(i, y) - \Delta^2 v(i-1)\rangle_+ \geq M(i)\Delta^2 M(i-1)$ *for all* $i \in N$ *and all* $y \in E$ *with* $\|y - v(i)\| = M(i)$,
3° $\|v(0)\| \leq M(0)$ *and* $\|v(T+1)\| \leq M(T+1)$,
4° $\beta(f(N \times A)) \leq k\beta(f(A))$ *for all bounded subset* A *of* E, *where* $k \geq 0$ *is a constant,*
5° *the condition 4° in Theorem 2.1 be satisfied,*
6° *for each* $b > 0$ *there exists a constant* $K_b \geq 0$ *such that*

$$\|f(j, u)\| \leq K_b \quad \text{for all} \quad j \in N \quad \text{and} \quad u \in E \quad \text{with} \quad \|u\| \leq b.$$

Then (1.1) *has a solution* $y \in C(N^+, E)$ *with* $\|y(i) - v(i)\| \leq M(i)$ *for* $i \in N$.

PROOF. Analogously as in [1] consider boundary value problem

$$\begin{cases} \Delta^2 y(i-1) + f(i,p(i,y(i))) = 0, \quad i \in N \\ y(0) = y(T+1) = 0 \end{cases} \tag{2.5}$$

where

$$p(i,y) = \lambda_i y + (1-\lambda_i)v(i)$$
$$\lambda_i = \min\Big\{1, \frac{M(i)}{\|y-v(i)\|}\Big\},\ i \in N$$

i.e. $\lambda_i \in (0,1\rangle$

$$p(i,y) = \begin{cases} y & if \|y-v(i)\| \le M(i) \\ M(i) \cdot \frac{y-v(i)}{\|y-v(i)\|} + v(i) & if \|y-v(i)\| > M(i) \end{cases}$$

is the redial retraction of E onto $\{y : \|y-v(i)\| \le M(i)\}$.

Define the operator $S : C(N^+,E) \to C(N^+,E)$ by setting

$$(Sy)(i) = \sum_{j=1}^{T} G(i,j) f(j,p(j,y(j)))$$

where $G(i,j)$ is defined by (2.3).

Now (2.5) is equivalent to the fixed point problem $y = Sy$. We claim $S : C(N^+,E) \to C(N^+,E)$ is a Darbo map. To see this let W be a bounded subset of $C(N^+,E)$. The condition $6°$ implies S is bounded map.

Fix $i \in N^+$. Then

$$\begin{aligned} \beta(SW(i)) &= \beta\Big(\Big\{\sum_{j=1}^{T} G(i,j) f(j,y(j)) : y \in V\Big\}\Big) \le \\ &\le \beta(T \cdot \overline{conv}\{G(i,j) f(j,p(j,y(j))) : y \in W,\ j \in N\}) = \\ &= T\beta(\{G(i,j) \cdot f(j,p(j,y(j))) : y \in W,\ j \in N\}) = \\ &= Tr_i\beta(\{f(j,p(j,y(j))) : y \in W,\ j \in N\}) \le \\ &\le Tr_i\beta(f(N \times \overline{conv}(W(N) \cup v(N)))) \end{aligned}$$

since if $y \in W$ and $j \in N$ we have

$$p(j,y(j)) = \lambda_j \cdot y(j) + (1-\lambda_j)v(j) \in \overline{conv}(W(N) \cup v(N))$$

because $\lambda_j \in (0,1\rangle$.

This together with the assumption $4°$ implies

$$\begin{aligned} \beta(SW(i)) &\le Tr_i k\beta(\text{overlineconv}(W(N) \cup v(N))) = \\ &= Tr_i k\beta(W(N) \cup v(N)) = Tr_i k\beta(W(N)) \le \\ &\le Trk\beta(W(N^+)) = Trk\beta(W). \end{aligned}$$

Now Theorem 1.5 implies

$$\beta(SW) = \sup_{i\in N^+} \beta(SW(i)) \leq Trk\beta(W).$$

Since $Trk < 1$ then S is a Darbo map.

By Theorem 1.2 the operator S has a fixed point. Consequently (2.5) has a solution $y \in C(N^+, E)$.

The proof of the inequality $\|y(i) - v(i)\| \leq M(i)$ for $i \in N$ is just the same as in [1]. Since $\|y(i) - v(i)\| \leq M(i)$ for $i \in N$ we have

$$p(i, y(i)) = y(i)$$

and consequently y is a solution of (1.1). □

It is possible to discuss the case, when in the assumption 2° of Theorem 2.6 the function M may take the value zero. Similar as Theorem 2.2 in [1] we can prove the following Theorem:

THEOREM 2.7. *Let conditions* 1°, 3°, 4°, 5°, 6° *of Theorem* 2.6 *hold. Furthermore suppose that the following conditions are satisfied:*

(2′) *there exists* $v \in C(N^+, E)$ *and* $M \in C(N^+, \langle 0, \infty))$ *with* $\langle y - v(i), -f(i, y) - \Delta^2 v(i-1)\rangle_+ \geq M(i)\Delta^2 M(i-1$ *for all* $i \in N$ *and* $y \in E$ *with* $\|y - v(i)\| = M(i)$ *and* $M(i) \neq 0$,

(2″) *there exists* v *and* M *as in* 2′ *with*

$$\frac{\langle y - v(i), -f(i, v(i)) - \Delta^2 v(i-1)\rangle_+}{\|y - v(i)\|} \geq \Delta^2 M(i-1)$$

for all $i \in N$ *and all* $y \in E$ *with* $\|y - v(i)\| > M(i)$ *and* $M(i) = 0$.

Then (1.1) *has a solution* $y \in C(N^+, E)$ *with* $\|y(i) - v(i)\| \leq M(i)$ *for* $i \in N$.

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