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# ON GROUPS $E_{25} \cdot Z_4$ AS AUTOMORPHISM GROUPS OF $(100, 45, 20)$ SYMMETRIC DESIGNS

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Abstract. Nonabelian groups of order 100, that are extensions of the elementary abelian group $E_{25}$ by the group $Z_4$ are considered as possible automorphism groups of symmetric designs with parameters $(100, 45, 20)$. By the action of these groups new $(100, 45, 20)$ symmetric designs are constructed. Their full automorphism groups are given.

## 1. Introduction and preliminaries

Up to isomorphism there are 12 nonabelian groups of order 100. They are semidirect products of a group of order 25, either cyclic or elementary abelian, by a group of order 4. So far, eight $(100, 45, 20)$ symmetric designs having one of these groups as automorphism group are known. Seven of them are developments of difference sets in one nonabelian group of the type $E_{25} \cdot Z_4$ (see [8], [9]), and one has full automorphism group of the type $E_{25} \cdot E_4$, [5]. On all these designs the subgroup $E_{25}$ acts standardly and fixed point free, hence in orbits of length 25. This indicated the possibility that all nonabelian extensions of $E_{25}$ of order 100 would act on $(100, 45, 20)$ symmetric design under similar conditions. Here we check on that possibility for groups $E_{25} \cdot Z_4$ and construct forty-two nonisomorphic such designs using the method of tactical decomposition. This method ([4]) is based on the assumption that certain group acts on the design as its automorphism group, in which case orbit partition of points and blocks forms a tactical decomposition for the design, [1]. In this section we'll denote by $G$ any group under consideration.

The research is performed in several steps, the first one being orbit distributions determination, that is, determination of all possible $G$-orbit lengths

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on the sets of points and blocks. Calculating orbit structures (matrices), related to a particular distribution of orbit lengths, comes next. The entries of these matrices give the information on the number of points from $G$-point orbits that lie on each block of a specified $G$-block orbit.

Computationally most demanding stage is indexing, that is, specifying which points from certain point orbit lie on the representative block of each block orbit. A block orbit representative is chosen to be fixed by the subgroup of $G$ that stabilizes the orbit. It has to be determined so as to have exactly 20 points in common with each block of its own orbit (i.e. its $G$-images) and also with all other blocks of the design constructed up to that moment. To this accomplish, computer programs are made according to the algorithm given in [3], backtracking and lexicographic ordering included. Here one faces a combinatorial explosion of possibilities to investigate and, in order to put final result within reach, it is necessary to make admissible reductions which will not diminish the generality of the final result. One of the possibilities, which proves to be very efficient and helpful, is to refine orbit matrix regarding the action of a certain normal subgroup $H \lhd G$. This means to determine the subdivision of points from $G$-orbits into orbits of $H$. Additionally, reduction can be obtained using the elements of normalizer $N(G)$ (taken in the symmetric group over the set of points of the design), assured that one has corresponding permutation representations at disposal.

However, after carrying out various available reductions, it regularly shows that among constructed designs there retains a great number of mutually isomorphic ones, up to a couple of thousand. To distinguish nonisomorphic ones, as well as to define the full automorphism group of a constructed design, we make use of the available computer programs (packages), [6], [2].

## 2. Action of groups $E_{25} \cdot Z_4$

There are six semidirect products of the group $E_{25}$ by $Z_4$. In terms of generators and relations they can be described as

$$G_{\alpha,\beta} = \left\langle a, b, c \mid a^5 = b^5 = [a,b] = 1, c^4 = 1, c^{-1}ac = a^{\alpha}, c^{-1}bc = b^{\beta} \right\rangle ,$$

where $(\alpha, \beta) \in \{(1,3), (1,4), (2,2), (2,3), (2,4), (4,4)\}$.

We consider possible $G_{\alpha,\beta}$-action on $(100, 45, 20)$ symmetric design so that the subgroup $E_{25}$ acts semiregularly, excluding transitive action of $G_{\alpha,\beta}$. In that case possible block and point orbit lengths are 25 and 50, while orbit matrices can be of order 2, 3 and 4. Their entries are calculated ([4]) to be as follows:

$$\begin{array}{cc|c} 50 & 50 & \\ \hline 25 & 20 & 50 \\ 20 & 25 & 50 \end{array} \quad (O1), \qquad \begin{array}{ccc|c} 25 & 25 & 50 & \\ \hline 15 & 10 & 20 & 25 \\ 10 & 15 & 20 & 25 \\ 10 & 10 & 25 & 50 \end{array} \quad (O2),$$

$$\begin{array}{cccc|c} 25 & 25 & 25 & 25 & \\ \hline 15 & 10 & 10 & 10 & 25 \\ 10 & 15 & 10 & 10 & 25 \\ 10 & 10 & 15 & 10 & 25 \\ 10 & 10 & 10 & 15 & 25 \end{array} \quad (O3), \qquad \begin{array}{cccc|c} 25 & 25 & 25 & 25 & \\ \hline 14 & 11 & 12 & 8 & 25 \\ 11 & 14 & 8 & 12 & 25 \\ 12 & 8 & 11 & 14 & 25 \\ 8 & 12 & 14 & 11 & 25 \end{array} \quad (O4).$$

For indexing these matrices we need a permutation representation of the $G_{\alpha,\beta}$-generators of degree 25 and 50, provided by J. Hrabě de Angelis computer program.

Lemma 2.1. *Groups $E_{25} \cdot Z_4$ cannot act on $(100, 45, 20)$ symmetric design having an orbit of length 25.*

Proof. Action of groups $E_{25} \cdot Z_4$ which includes an orbit of length 25 corresponds to one of the orbit matrices $O2$, $O3$ or $O4$. The stabilizer of such an orbit is the subgroup $Z_4 = \langle c \rangle$ and its representative block, fixed by $Z_4$, consists of $Z_4$-point orbits as a whole.

Regarding the groups $G_{2,2}$ and $G_{2,3}$, subgroup $Z_4$ has one fixed point and six orbits of length 4 on 25-point orbits. This implies that corresponding entries of given orbit matrices should be congruent 0 or $1 (\mathrm{mod} 4)$, which is not satisfied.

As for the groups $G_{1,4}$ and $G_{4,4}$, the automorphism $c^2$ fixes all points on both 25 and 50-point orbits, so the number of its fixed points exceeds the upper limit for a nontrivial automorphism ([7], page 82).

Finally, the groups $G_{1,3}$ and $G_{2,4}$ are eliminated because indexing the first row of the orbit matrices $O2$, $O3$ and $O4$, performed by computer, in their case fails. ∎

Consequently, the only possible orbit matrix induced by the action of groups $E_{25} \cdot Z_4$ on $(100, 45, 20)$ symmetric design under given conditions is $O1$.

## 3. Construction of designs

Theorem 3.1. *Up to isomorphism and duality there exist forty-two $(100, 45, 20)$ symmetric designs having as automorphism group one of the groups $E_{25} \cdot Z_4$ so that it acts nontransitively on the design and its subgroup $E_{25}$ acts standardly and fixed point free.*

Proof. From Lemma 2.1 it follows that the construction of our designs is to be performed by indexing the single orbit structure $O1$. The $G_{\alpha,\beta}$-orbits of designs, $(\alpha, \beta) \in \{(1,3), (1,4), (2,2), (2,3), (2,4), (4,4)\}$, are stabilized by the subgroup $Z_2 = \langle c^2 \rangle$. $\{1, 2, \ldots, 50\}$ will denote the set of points of any point orbit.

*(i)* The group $G_{1,3}$ generators' permutation representation is:

generator a –> (no fixed point)
( 1 2 3 4 5 )( 6 10 11 12 13 )( 7 19 23 24 22 )( 8 20 35 37 44 )
( 9 21 36 47 41 )( 14 16 27 18 17 )( 15 28 42 49 38 )
( 25 32 45 50 43 )( 26 31 40 34 33 )( 29 48 39 46 30 )
generator b –> (no fixed point)
( 1 6 7 8 9 )( 2 10 19 20 21 )( 3 11 23 35 36 )( 4 12 24 37 47 )
( 5 13 22 44 41 )( 14 15 48 25 26 )( 16 28 39 32 31 )
( 17 38 29 43 33 )( 18 49 30 50 34 )( 27 42 46 45 40 )
generator c –> (no fixed point)
( 1 14 )( 2 16 )( 3 27 )( 4 18 )( 5 17 )( 6 25 9 48 )( 7 15 8 26 )
( 10 32 21 39 )( 11 45 36 46 )( 12 50 47 30 )( 13 43 41 29 )
( 19 28 20 31 )( 22 38 44 33 )( 23 42 35 40 )( 24 49 37 34 ).

It is obvious that $Z_2$ has 10 fixed points and 20 orbits of length 2 on $G_{1,3}$ point orbits, which gives enormous number of possibilities for composing a representative block of an orbit. Therefore, to accomplish the construction we will refine the orbit structure $O1$ regarding the action of the subgroup $E_{25}$. The obtained refined orbit matrices are $O3$ and $O4$. Indexing these matrices, after isomorphic structures elimination performed by mappings from $N(G_{1,3})$ and computer program by V. Krčadinac based on Nauty ([6]), finally leads to 44 pairs of base blocks of nonisomorphic, pairwise dual symmetric designs. Up to isomorphism and duality they are the following.

$D^1_{1,3}$ :
1 2 3 4 5 7 8 10 11 14 15 17 19 20 21 22 26 32 34 36 39 40 42 44 49
6 7 8 9 11 15 16 18 19 20 24 26 29 30 36 37 43 45 46 50
1 3 6 9 10 13 15 21 24 26 30 32 34 37 39 40 41 42 49 50
1 2 3 6 9 10 12 13 15 21 22 23 25 26 28 29 30 31 35 41 43 44 47 48 50

$D^2_{1,3}$ :
1 2 3 4 5 7 8 12 13 14 16 19 20 22 28 31 33 34 38 41 44 45 46 47 49
10 12 15 17 19 20 21 22 23 26 27 30 32 35 39 44 45 46 47 50
1 3 10 11 12 15 21 22 26 30 32 34 36 39 40 42 44 47 49 50
1 2 3 6 9 10 12 13 21 22 23 28 29 30 31 35 40 41 42 43 44 45 46 47 50

$D^3_{1,3}$ :
1 6 7 8 9 10 12 15 17 18 21 22 23 24 26 29 30 35 37 43 44 45 46 47 50
2 5 6 9 12 13 14 16 17 23 25 27 28 30 31 35 41 47 48 50
1 2 3 4 13 15 16 18 19 20 23 26 29 34 35 40 41 42 43 49
1 2 3 6 9 10 11 13 14 15 16 17 18 19 20 21 24 26 28 30 31 36 37 41 50

$D^4_{1,3}$ :
1 6 7 8 9 12 13 17 18 19 20 23 24 25 30 33 35 37 38 41 45 46 47 48 50
2 5 6 9 11 13 14 16 17 24 25 27 32 34 36 37 39 41 48 49
1 2 3 4 10 16 18 21 22 23 25 33 34 35 38 40 42 44 48 49
1 2 3 6 9 10 12 13 14 15 16 17 18 19 20 21 23 26 32 34 35 39 41 47 49

$D^5_{1,3}$ :
1 6 7 8 9 11 12 13 15 17 18 19 20 23 26 29 32 35 36 39 41 43 45 46 47
2 5 6 9 12 13 14 16 17 23 25 27 28 31 33 35 38 41 47 48
1 2 3 4 10 12 16 18 21 23 25 33 34 35 38 40 42 47 48 49
1 2 3 6 9 10 12 13 14 16 17 18 21 22 23 28 29 30 31 35 41 43 44 47 50
$D^6_{1,3}$ :
1 6 9 11 12 15 16 19 20 22 23 24 26 27 29 30 35 36 37 43 44 45 46 47 50
3 5 6 9 10 11 14 17 18 21 22 25 27 30 33 36 38 44 48 50
1 2 3 4 7 8 11 15 16 18 19 20 26 28 29 31 34 36 43 49
1 2 3 6 9 10 12 13 15 16 17 18 21 22 23 26 27 28 29 31 35 41 43 44 47
$D^7_{1,3}$ :
1 6 7 8 9 10 11 12 16 21 22 24 25 27 29 34 36 37 43 44 45 46 47 48 49
3 5 6 9 12 13 14 17 18 19 20 27 30 33 38 40 41 42 47 50
1 2 3 4 10 12 15 16 18 21 23 26 28 29 31 35 40 42 43 47
1 2 3 6 9 10 12 13 16 17 18 21 22 23 27 28 29 30 31 35 41 43 44 47 50
$D^8_{1,3}$ :
1 7 8 11 12 13 17 18 19 20 23 24 25 30 32 35 36 37 39 40 41 42 47 48 50
1 3 6 9 11 13 14 16 17 24 25 27 28 29 31 36 37 41 43 48
1 2 3 4 11 14 15 19 20 24 26 27 33 34 36 37 38 45 46 49
1 2 3 6 9 10 12 13 14 16 17 18 21 23 24 28 31 34 35 37 41 45 46 47 49
$D^9_{1,3}$ :
1 2 4 7 8 10 12 13 15 18 21 23 24 26 27 33 34 35 37 38 40 41 42 47 49
2 3 7 8 12 13 14 16 22 28 31 33 34 38 41 44 45 46 47 49
1 2 7 8 13 14 17 23 24 28 30 31 35 37 40 41 42 45 46 50
1 2 3 6 7 8 9 10 11 12 13 15 17 21 26 27 29 32 36 39 40 41 42 43 47
$D^{10}_{1,3}$ :
1 2 4 7 8 10 12 13 15 18 21 23 24 26 27 33 34 35 37 38 40 41 42 47 49
1 5 10 11 14 16 21 23 24 25 30 33 35 36 37 38 45 46 48 50
1 2 10 12 13 16 21 23 25 27 29 33 34 35 38 41 43 47 48 49
1 2 3 6 7 8 9 10 11 12 13 15 17 21 26 27 29 32 36 39 40 41 42 43 47
$D^{11}_{1,3}$ :
1 2 4 7 8 10 12 13 15 18 21 23 24 26 27 33 34 35 37 38 40 41 42 47 49
2 3 7 8 10 12 13 14 16 21 30 33 34 38 41 45 46 47 49 50
1 2 12 13 14 17 23 24 25 28 30 31 35 37 41 45 46 47 48 50
1 2 3 6 7 8 9 10 11 12 13 15 17 21 26 27 29 32 36 39 40 41 42 43 47
$D^{12}_{1,3}$ :
1 2 4 7 8 10 12 13 15 18 21 23 24 26 27 33 34 35 37 38 40 41 42 47 49
1 5 7 8 10 14 16 21 23 24 30 33 34 35 37 38 45 46 49 50
1 2 12 13 16 23 24 25 27 28 31 33 34 35 37 38 41 47 48 49
1 2 3 6 7 8 9 10 11 12 13 15 17 21 26 27 29 32 36 39 40 41 42 43 47

$D^{13}_{1,3}$ :
1 2 4 6 9 11 13 18 19 20 22 23 25 27 29 30 35 36 41 43 44 45 46 48 50
2 3 6 9 10 14 16 21 23 24 28 31 33 34 35 37 38 45 46 49
1 2 6 9 11 12 14 15 17 22 26 28 29 30 31 36 43 44 47 50
1 2 3 6 7 8 9 10 11 12 13 15 17 21 26 27 30 32 34 36 39 41 47 49 50
$D^{14}_{1,3}$ :
1 2 4 7 8 10 11 13 15 18 21 22 23 26 27 33 34 35 36 38 40 41 42 44 49
2 3 7 8 10 12 13 14 16 21 25 28 29 31 40 41 42 43 47 48
1 2 7 8 10 11 14 17 21 22 25 28 30 31 36 44 45 46 48 50
1 2 3 6 7 8 9 10 11 12 13 15 17 21 26 27 30 32 34 36 39 41 47 49 50
$D^{15}_{1,3}$ :
1 2 3 7 8 10 12 13 15 18 21 23 24 26 27 33 34 35 37 38 40 41 42 47 49
1 3 6 9 10 11 14 15 16 21 24 26 30 32 33 36 37 38 39 50
1 2 6 9 12 16 17 19 20 23 25 29 32 34 35 39 43 47 48 49
1 2 3 6 7 8 9 10 11 12 13 15 18 21 26 27 29 32 36 39 40 41 42 43 47
$D^{16}_{1,3}$ :
1 2 4 6 7 8 9 10 15 17 18 21 22 23 24 26 30 32 33 35 37 38 39 44 50
3 4 6 9 10 11 12 15 16 21 26 27 29 32 34 36 39 43 47 49
1 2 11 13 14 15 17 19 20 24 26 28 31 33 36 37 38 40 41 42
1 2 3 6 7 8 9 10 13 14 15 18 21 23 24 26 32 33 34 35 37 38 39 41 49
$D^{17}_{1,3}$ :
1 2 3 7 8 10 11 12 13 14 17 21 24 25 32 33 34 36 37 38 39 41 47 48 49
2 4 6 9 11 18 22 23 25 27 30 32 35 36 39 44 45 46 48 50
1 2 6 9 10 11 12 14 21 27 28 30 31 32 33 36 38 39 47 50
1 2 3 6 9 10 13 14 17 19 20 21 23 24 28 30 31 33 35 37 38 40 41 42 50
$D^{18}_{1,3}$ :
1 2 3 7 8 10 11 12 13 14 17 21 24 25 32 33 34 36 37 38 39 41 47 48 49
3 5 7 8 10 12 15 18 21 24 26 27 33 34 37 38 40 42 47 49
1 2 7 8 15 16 17 19 20 22 24 25 26 34 37 44 45 46 48 49
1 2 3 6 9 10 13 14 17 19 20 21 23 24 28 30 31 33 35 37 38 40 41 42 50
$D^{19}_{1,3}$ :
1 2 3 6 9 12 13 15 18 22 23 24 26 27 28 29 31 34 35 37 41 43 44 47 49
1 3 6 9 12 14 16 22 23 25 30 32 35 39 44 45 46 47 48 50
1 2 6 9 10 12 13 16 17 21 30 32 33 38 39 40 41 42 47 50
1 2 3 6 9 10 12 13 15 18 21 22 23 25 26 27 28 29 31 35 41 43 44 47 48
$D^{20}_{1,3}$ :
1 2 3 7 8 11 12 13 18 22 24 25 27 30 32 33 36 37 38 39 41 44 47 48 50
1 3 7 8 11 13 14 15 16 24 26 28 31 34 36 37 40 41 42 49
1 2 6 9 10 12 13 16 17 21 30 32 33 38 39 40 41 42 47 50
1 2 3 6 9 10 12 13 15 18 21 22 23 25 26 27 28 29 31 35 41 43 44 47 48

$D^{21}_{1,3}$ :
1 2 3 6 9 12 13 15 18 22 23 24 26 27 28 29 31 34 35 37 41 43 44 47 49
2 5 11 13 14 15 16 19 20 24 26 28 31 33 34 36 37 38 41 49
1 2 7 8 14 15 19 20 23 24 26 27 29 34 35 37 43 45 46 49
1 2 3 6 9 10 12 13 15 18 21 22 23 25 26 27 28 29 31 35 41 43 44 47 48
$D^{22}_{1,3}$ :
1 2 3 7 8 11 12 13 18 22 24 25 27 30 32 33 36 37 38 39 41 44 47 48 50
2 5 10 12 14 16 21 22 23 25 29 30 32 35 39 43 44 47 48 50
1 2 7 8 14 15 19 20 23 24 26 27 29 34 35 37 43 45 46 49
1 2 3 6 9 10 12 13 15 18 21 22 23 25 26 27 28 29 31 35 41 43 44 47 48

Here, each of the two base blocks is recorded in two rows; points of its first point-orbit are given in the first row, and the points of its second point-orbit in the second row. The other blocks of the design are produced from the pair of base blocks by action of the subgroup $\langle a, b, c^2 \rangle$.

*(ii)* For the group $G_{2,2}$, up to isomorphism, the following three designs are constructed:

$D^{1}_{2,2}$ :
1 3 4 6 9 10 13 17 18 19 21 23 27 29 30 31 33 35 36 37 38 41 43 45 46
2 5 6 9 10 11 16 17 18 23 24 25 27 29 38 40 43 44 49 50
3 4 11 13 17 20 23 28 29 31 34 35 37 38 40 42 44 47 48 49
1 3 4 6 7 8 9 10 11 12 13 15 18 22 26 27 31 32 35 37 39 40 43 44 49
$D^{2}_{2,2}$ :
1 2 3 4 5 7 8 10 11 16 18 20 21 24 25 26 28 32 33 41 42 44 46 47 50
3 4 10 12 13 18 21 22 23 24 26 31 32 33 34 36 38 45 48 50
7 8 11 12 17 21 22 23 24 27 29 33 35 37 38 41 43 44 46 50
1 3 4 6 7 8 9 10 11 12 15 18 19 22 23 26 30 32 38 39 40 41 44 46 49
$D^{3}_{2,2}$ :
1 6 9 10 13 17 18 19 20 21 23 26 27 29 30 31 32 33 35 37 38 41 42 43 46
2 3 4 5 6 9 16 17 19 23 24 25 26 28 29 30 32 38 47 50
3 4 10 12 13 16 17 18 22 23 25 29 31 34 35 36 37 38 45 48
1 3 4 6 7 8 9 10 11 13 15 16 18 19 25 26 30 31 32 35 37 39 40 44 49

They prove to be selfdual. The subgroup $\langle a, b, c^2 \rangle$ generates all the blocks of the designs. The group $G_{2,2}$ generators' permutation representation is:

generator a –> (no fixed point)
( 1 2 3 4 5 )( 6 10 11 12 13 )( 7 29 20 33 19 )( 8 30 21 42 17 )
( 9 31 22 44 18 )( 14 16 35 37 25 )( 15 43 36 38 46 )
( 23 45 27 39 41 )( 24 47 26 48 49 )( 28 50 40 34 32 )
generator b –> (no fixed point)
( 1 6 7 8 9 )( 2 10 29 30 31 )( 3 11 20 21 22 )( 4 12 33 42 44 )
( 5 13 19 17 18 )( 14 15 34 48 39 )( 16 43 32 49 41 )

( 23 35 36 28 24 )( 25 46 40 26 27 )( 37 38 50 47 45 )
generator c –> (no fixed point)
( 1 14 )( 2 35 5 37 )( 3 25 4 16 )( 6 34 9 48 )( 7 39 8 15 )
( 10 28 18 47 )( 11 40 44 49 )( 12 32 22 26 )( 13 50 31 24 )
( 17 38 29 23 )( 19 45 30 36 )( 20 27 42 43 )( 21 46 33 41 ).

*(iii)* Using the group $G_{2,3}$ 27 nonisomorphic $(100, 45, 20)$ symmetric designs are constructed, one of them being selfdual and the others pairwise dual. Up to duality we record 14 designs by their base blocks, starting with selfdual one.

$D^1_{2,3}$ :
1 6 9 10 13 15 18 19 20 21 22 23 24 25 26 28 29 31 36 41 42 43 45 47 49
2 3 4 5 6 9 15 16 17 19 20 22 25 29 33 36 38 43 46 48
3 4 6 7 8 9 10 13 16 17 18 21 25 26 27 32 36 44 45 50
1 3 4 10 11 12 13 15 16 17 18 19 20 21 22 26 29 30 34 35 37 39 40 43 45
$D^2_{2,3}$ :
1 2 3 4 5 6 9 11 12 18 19 23 26 28 29 30 32 33 34 41 42 44 46 47 49
6 9 10 12 16 17 23 24 28 30 31 33 35 37 38 42 45 46 47 48
3 4 10 11 13 21 23 25 28 34 36 38 39 40 41 42 45 47 48 49
1 3 4 6 7 8 9 10 11 12 15 20 22 27 30 33 34 38 41 43 45 46 48 49 50
$D^3_{2,3}$ :
1 6 9 10 11 13 18 19 20 21 22 24 26 29 31 32 33 34 41 42 44 45 46 47 49
2 3 4 5 6 9 13 16 17 20 21 22 33 35 37 38 42 46 47 48
3 4 10 11 13 16 17 18 21 23 25 26 27 28 33 34 36 45 46 50
1 3 4 6 7 8 9 10 11 12 15 16 17 18 20 22 26 30 34 39 40 42 43 45 47
$D^4_{2,3}$ :
1 6 9 10 12 13 18 19 20 21 22 23 26 27 28 29 30 38 41 42 45 47 48 49 50
2 3 4 5 6 9 13 16 17 20 21 22 33 35 37 38 42 46 47 48
3 4 10 12 13 16 17 18 21 24 25 26 30 31 32 36 38 44 45 48
1 3 4 6 7 8 9 10 11 12 15 16 17 18 20 22 26 30 34 39 40 42 43 45 47
$D^5_{2,3}$ :
1 2 3 4 5 6 9 10 12 18 23 24 26 28 30 31 32 33 41 42 44 45 46 47 49
6 9 11 12 16 17 19 23 28 29 30 33 34 35 37 38 42 46 47 48
3 4 10 11 13 21 23 25 28 32 34 35 36 37 39 40 42 44 45 47
1 3 4 6 7 8 9 10 11 12 15 20 22 27 30 32 33 34 35 37 43 44 45 46 50
$D^6_{2,3}$ :
1 2 3 4 5 6 9 10 18 19 20 22 24 26 27 29 31 39 40 41 42 45 47 49 50
6 9 13 16 17 19 20 21 22 23 27 28 29 33 38 39 40 46 48 50
3 4 10 12 13 20 21 22 25 27 30 32 33 36 42 44 45 46 47 50
1 3 4 6 7 8 9 10 11 13 15 19 21 29 33 34 35 37 39 40 42 43 45 46 47

$D^7_{2,3}$ :
1 6 9 10 11 13 18 19 20 21 22 24 26 27 29 31 32 34 39 40 41 44 45 49 50
2 3 4 5 6 9 13 16 17 20 21 22 27 35 37 38 39 40 48 50
3 4 10 11 12 16 17 18 20 22 25 26 27 30 33 34 36 45 46 50
1 3 4 6 7 8 9 10 11 13 15 16 17 18 21 23 26 28 34 39 40 42 43 45 47
$D^8_{2,3}$ :
1 6 9 10 12 13 18 19 20 21 22 23 26 27 28 29 30 32 35 37 42 44 45 47 50
2 3 4 5 6 9 10 16 17 19 29 32 33 35 37 39 40 44 45 46
3 4 10 11 12 16 17 18 20 22 25 26 27 30 33 34 36 45 46 50
1 3 4 6 7 8 9 10 11 13 15 16 17 18 21 23 26 28 34 39 40 42 43 45 47
$D^9_{2,3}$ :
1 3 4 6 9 11 12 13 20 21 22 24 27 30 31 32 34 35 37 38 41 44 48 49 50
3 4 6 9 10 12 23 27 28 30 33 35 37 39 40 42 45 46 47 50
10 11 12 16 17 19 24 25 29 30 31 33 34 35 36 37 41 45 46 49
1 2 3 4 5 6 7 8 9 10 11 15 18 24 26 27 31 33 34 41 43 45 46 49 50
$D^{10}_{2,3}$ :
1 3 4 6 9 10 11 12 19 23 27 28 29 30 32 33 34 39 40 42 44 45 46 47 50
3 4 6 9 10 12 23 27 28 30 33 35 37 39 40 42 45 46 47 50
11 12 13 16 17 20 21 22 23 25 28 30 34 36 38 39 40 42 47 48
1 2 3 4 5 6 7 8 9 10 11 15 18 24 26 27 31 33 34 41 43 45 46 49 50
$D^{11}_{2,3}$ :
1 2 3 4 5 10 11 12 15 18 19 24 26 29 30 31 32 34 38 42 43 44 45 47 48
6 7 8 9 12 13 16 17 21 23 25 28 30 33 35 36 37 41 46 49
3 4 6 9 10 12 15 23 25 28 30 35 36 37 38 41 43 45 48 49
1 3 4 6 9 10 11 12 15 19 23 25 28 29 30 32 34 36 38 41 43 44 45 48 49
$D^{12}_{2,3}$ :
1 6 7 8 9 10 11 12 15 18 19 24 26 29 30 31 32 34 38 42 43 44 45 47 48
2 3 4 5 12 13 16 17 21 23 25 28 30 33 35 36 37 41 46 49
3 4 6 9 10 12 16 17 18 23 26 28 30 35 37 38 41 45 48 49
1 3 4 6 9 10 11 12 16 17 18 19 23 26 28 29 30 32 34 38 41 44 45 48 49
$D^{13}_{2,3}$ :
1 2 3 4 5 10 11 12 15 18 19 24 26 29 30 31 32 33 34 41 43 44 45 46 49
6 7 8 9 12 13 16 17 21 23 25 28 30 35 36 37 38 42 47 48
3 4 6 9 10 11 15 23 25 28 33 34 36 39 40 42 43 45 46 47
1 3 4 6 9 10 11 12 15 20 22 23 25 27 28 30 33 34 36 42 43 45 46 47 50
$D^{14}_{2,3}$ :
1 6 7 8 9 10 11 12 15 18 19 24 26 29 30 31 32 33 34 41 43 44 45 46 49
2 3 4 5 12 13 16 17 21 23 25 28 30 35 36 37 38 42 47 48
3 4 6 9 10 11 16 17 18 23 26 28 33 34 39 40 42 45 46 47
1 3 4 6 9 10 11 12 16 17 18 20 22 23 26 27 28 30 33 34 42 45 46 47 50

The group $G_{2,3}$ generators' permutation representation is:

generator a –> (no fixed point)
( 1 2 3 4 5 )( 6 10 11 12 13 )( 7 19 23 24 22 )( 8 20 31 28 29 )
( 9 21 30 34 45 )( 14 16 26 18 17 )( 15 41 44 27 42 )
( 25 39 48 33 35 )( 32 49 43 47 50 )( 36 37 46 38 40 )
generator b –> (no fixed point)
( 1 6 7 8 9 )( 2 10 19 20 21 )( 3 11 23 31 30 )( 4 12 24 28 34 )
( 5 13 22 29 45 )( 14 15 36 25 43 )( 16 41 37 39 47 )
( 17 42 40 35 49 )( 18 27 38 33 32 )( 26 44 46 48 50 )
generator c –> (no fixed point)
( 1 14 )( 2 26 5 18 )( 3 17 4 16 )( 6 25 9 36 )( 7 15 8 43 )
( 10 48 45 38 )( 11 35 34 37 )( 12 39 30 40 )( 13 33 21 46 )
( 19 44 29 32 )( 20 50 22 27 )( 23 42 28 47 )( 24 41 31 49 ).

*(iv)* Eight nonisomorphic designs are constructed using the action of group $G_{2,4}$. They are pairwise dual and we note one representative of each dual pair.

$D^1_{2,4}$ :
1 2 3 4 5 6 7 11 12 17 18 20 29 30 33 34 37 38 39 41 42 43 44 45 47
2 5 11 12 16 17 18 19 20 22 25 28 31 33 35 36 43 45 46 49
1 3 4 7 11 12 17 18 21 27 30 31 32 33 35 39 40 45 48 50
1 3 4 6 7 8 9 11 12 14 15 16 19 20 21 22 28 32 34 42 43 44 47 48 50
$D^2_{2,4}$ :
1 2 3 4 5 6 7 11 12 17 18 20 27 31 33 35 37 38 40 43 45 46 48 49 50
3 4 10 13 19 21 22 25 27 29 31 32 35 36 37 38 40 41 42 47
1 7 11 12 16 20 23 24 27 28 34 37 38 40 42 43 44 46 47 49
1 3 4 6 7 8 9 11 12 14 15 16 19 20 21 22 28 32 34 42 43 44 47 48 50
$D^3_{2,4}$ :
1 3 4 6 8 10 11 12 13 17 18 21 23 24 29 30 32 33 34 39 41 42 44 45 47
2 5 11 12 15 19 21 22 30 31 32 35 36 37 38 39 42 46 47 49
1 3 4 6 16 20 28 29 30 31 33 35 37 38 39 41 43 45 48 50
1 3 4 6 7 8 9 11 12 14 16 19 20 21 22 26 28 30 32 39 43 46 48 49 50
$D^4_{2,4}$ :
1 3 4 6 8 10 11 12 13 16 21 23 24 28 29 30 31 32 33 35 39 41 42 45 47
2 5 11 12 15 17 18 19 20 22 29 30 33 36 39 41 43 45 46 49
1 6 11 12 16 20 23 24 28 29 34 37 38 41 43 44 46 48 49 50
1 3 4 6 7 8 9 11 12 14 16 19 20 21 22 26 28 30 32 39 43 46 48 49 50

The group $G_{2,4}$ generators' permutation representation is:

generator a –> (no fixed point)
( 1 2 3 4 5 )( 6 10 11 12 13 )( 7 19 23 24 22 )( 8 20 37 38 43 )
( 9 21 33 45 32 )( 14 16 18 17 28 )( 15 40 41 29 27 )
( 25 39 46 49 30 )( 26 34 31 35 44 )( 36 48 47 42 50 )
generator b –> (no fixed point)
( 1 6 7 8 9 )( 2 10 19 20 21 )( 3 11 23 37 33 )( 4 12 24 38 45 )

( 5 13 22 43 32 )( 14 15 36 26 25 )( 16 40 48 34 39 )
( 17 29 42 35 49 )( 18 41 47 31 46 )( 27 50 44 30 28 )
generator c −> (no fixed point)
( 1 14 )( 2 18 5 17 )( 3 28 4 16 )( 6 25 )( 7 26 )( 8 36 )( 9 15 )
( 10 46 13 49 )( 11 30 12 39 )( 19 31 22 35 )( 20 47 43 42 )
( 21 41 32 29 )( 23 44 24 34 )( 27 45 40 33 )( 37 50 38 48 ).

*(v)* From the groups $G_{1,4}$ and $G_{4,4}$ generators' permutation representation it is obvious that these groups cannot act on $(100, 45, 20)$ design in orbits of length 50 because the automorphism $c^2$ of order 2 would, in this case, fix all the points of design.

Testing all the quoted designs on mutual isomorphism gives that the design $D^2_{2,2}$ is isomorphic to the design $D^1_{2,3}$ and this completes the proof. □

THEOREM 3.2. $AutD^i_{1,3} \cong G_{1,3}, i = 1, \ldots, 22$; $AutD^i_{2,3} \cong G_{2,3}, i = 3, \ldots, 14$;
$AutD^i_{2,4} \cong G_{2,4}, i = 1, \ldots, 4$; $AutD^1_{2,2} \cong AutD^3_{2,2} \cong G_{2,2}$;
$\mid AutD^2_{2,2} \mid = \mid AutD^1_{2,3} \mid = \mid AutD^2_{2,3} \mid = 200$;

$$AutD^2_{2,2} \cong AutD^1_{2,3} \cong \langle a, b, c, d \mid a^5 = b^5 = 1,\ ab = ba,\ c^4 = 1,\ c^{-1}ac = a^2,\ c^{-1}bc = b^2, d^2 = 1,\ dad = a,\ dbd = b^4,\ cd = dc\rangle;$$

$$AutD^2_{2,3} \cong \langle a, b, c, d \mid a^5 = b^5 = 1,\ ab = ba,\ c^4 = 1,\ c^{-1}ac = a^2,\ c^{-1}bc = b^3\ d^2 = c^2,\ d^{-1}ad = b,\ d^{-1}bd = a^4,\ c^{-1}dc = d^3\rangle.$$

PROOF. The order of the cited full automorphism groups, as well as the permutation representation of their generators of degree 100, were obtained with a help of computer program by V. Tonchev. The groups $AutD^2_{2,2}$ and $AutD^2_{2,3}$, in terms of generators $a, b, c$ and $d$, were identified using GAP package, [2]. □

## 4. Final survey of the results

A survey of the results given in Theorems 3.1 and 3.2, regarding groups used for the construction, we give in table (4.1).

(4.1)

| Group | Number of nonisom. designs | Number of selfdual designs | Number of dual pairs | Full automorphism group |
|---|---|---|---|---|
| $G_{1,3}$ | 44 | 0 | 22 | $G_{1,3}$ for all |
| $G_{1,4}$ | 0 | 0 | 0 | - |
| $G_{2,2}$ | 3 | 3 | 0 | $G_{2,2}$ except for $D^2_{2,2}$ |
| $G_{2,3}$ | 27 | 1 | 13 | $G_{2,3}$ except for $D^1_{2,3}$ and $D^2_{2,3}$ |
| $G_{2,4}$ | 8 | 0 | 4 | $G_{2,4}$ for all |
| $G_{4,4}$ | 0 | 0 | 0 | - |

Up to isomorphism, two groups of order greater than 100 are obtained as full automorphism groups of the constructed designs: $H_1 \cong AutD^2_{2,2} \cong AutD^1_{2,3}$ and $H_2 \cong AutD^2_{2,3}$. From the generating relations it is obvious that their Sylow 2-subgroups are $Z_4 \times Z_2 \leq H_1$ and quaternion group $Q_8 \leq H_2$.

Note that, in addition to $G_{2,3} \leq AutD^2_{2,3}$, $G_{2,2}$ is a subgroup of $AutD^2_{2,3}$. This corresponds to the fact $D^2_{2,2} \cong D^1_{2,3}$.

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