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DELAY INTEGRO–DIFFERENTIAL EQUATIONS OF MIXED TYPE IN BANACH SPACES

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Abstract. This paper contains sufficient conditions under which there exist extremal solutions of initial value problems for delay integro–differential equations of mixed type in Banach spaces. We use the monotone iterative technique for proving existence results. Some comparison results are also established.

1. Let B denote a real Banach space with a norm $\|\cdot\|$, and $\bar{B}$ be a cone in B which defines a partial ordering in the space B by relation $x \le y$ iff $y - x \in \bar{B}$. By θ we denote the zero element in B. The cone $\bar{B}$ is said to be regular if every nondecreasing and bounded in order sequence in B has a limit, i.e., $x_1 \le x_2 \le \cdots \le x_n \le \cdots \le y$ implies $\|x_n - x\| \to 0$ as $n \to \infty$ for some $x \in B$ (for details, see for example [2, 7, 8, 10]).

In this paper we consider the following initial value problem

$$x'(t) = f(t, x(t), x(\alpha(t)), Tx(t), Sx(t)),\ t \in J = [0, b],\ b > 0,\ x(0) = x_0 \tag{1.1}$$

where $f \in C(J \times B^4, B),\ \alpha \in C(J, J),\ 0 \le \alpha(t) \le t,\ t \in J$, and the operators T and S are defined by

$$Tx(t) = \int_0^{\beta(t)} k(t,s)x(s)ds, \quad Sx(t) = \int_0^b l(t,s)x(s)ds, \quad t \in J$$

with $\beta \in C(J, J),\ 0 \le \beta(t) \le t,\ t \in J,\ k \in C(D_1, R_+),\ l \in C(D_2, R_+),\ D_1 = \{(t,s) \in J \times J : t \ge s\}, D_2 = J \times J,\ R_+ = [0, \infty)$.

Indeed, $T, S : C(J, B) \to C(J, B)$. The method of lower and upper solutions is very useful for proving existence results to differential problems

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(for example, see for details [11]). In this paper we use this technique proving existence of extremal solutions to problem (1.1). In section 3, we study a delay linear integro–differential equation of Volterra type giving sufficient conditions under which such problem has a unique solution. To apply the monotone method we need some comparison results from section 4. The main result is given in section 5. We construct monotone sequences giving sufficient conditions under which they are convergent to extremal solutions of problem (1.1). Some existence and comparison results for corresponding linear delay problems are needed but the method of this paper is similar to that of [6]. If $f(t,u,v,w,z)$ does not depend on v and z, and $\beta(t)=t,\ t\in J$, then we have the problem from [6], and if f does not depend on the last three arguments, then we have problem considered in [3]; see also [1, 4, 9, 12]. This paper generalizes the results of [6]. Periodic boundary value problems for second order integro–differential equations are considered, for example, in [5, 13].

2. A function $u\in C^1(J,B)$ is said to be a lower solution of problem (1.1) if

$$u'(t)\le f(t,u(t),u(\alpha(t)),Tu(t),Su(t)),\quad t\in J,\quad u(0)\le x_0,$$

and an upper solution of (1.1) if the inequalities are reversed.

Let us introduce the following assumptions for later use:

(H_1) $f\in C(J\times B^4,B),\ l\in C(D_2,R_+)$,
(H_2) $\alpha,\beta\in C(J,J),\ 0\le\alpha(t)\le t,\ 0\le\beta(t)\le t,\ t\in J,\ k\in C(D_1,R_+)$,
(H_3) y_0,z_0 are lower and upper solutions of (1) and $y_0(t)\le z_0(t)$ on J,
(H_4) there exist nonnegative constants M,N,P such that

$$f(t,\bar u,\bar v,\bar w,\bar z)-f(t,u,v,w,\bar z)\ge -M(\bar u-u)-N(\bar v-v)-P(\bar w-w)$$

for $y_0(t)\le u\le\bar u\le z_0(t)$, $y_0(\alpha(t))\le v\le\bar v\le z_0(\alpha(t))$, $Ty_0(t)\le w\le\bar w\le Tz_0(t)$, $Sy_0(t)\le\bar z\le Sz_0(t),\ t\in J$,
(H_5) function f is nondecreasing in the last argument,
(H_6) $bNe^{Mb}+\dfrac{Pk_0b}{M}\left(e^{Mb}-1\right)\le 1$ if $M>0$, and $Nb+Pk_0b^2\le 1$ if $M=0$, where $k_0=\max\{k(t,s):(t,s)\in D_1\}$.

3. Now we consider a delay linear integro–differential problem.

Lemma 3.1. *Let Assumption H_2 hold. Let $M,N,P\ge 0,\ f_1\in C(J,B)$. Then the problem*

$$y'(t)=f_1(t)-My(t)-Ny(\alpha(t))-PTy(t),\ t\in J,\quad y(0)=x_0 \tag{3.1}$$

has a unique solution.

PROOF. Replace (3.1) by

$$y(t) = e^{-Mt}\left\{x_0 + \int_0^t e^{Ms}\left[f_1(s) - Ny(\alpha(s)) - PTy(s)\right]ds\right\} \equiv Ay(t), \quad t \in J.$$

Let $\|y\|_* = \max_{t\in J}[|y(t)|e^{-Kt}]$, where $K > N + k_0Pb$. Then

$$\begin{aligned}\|Ay - A\bar{y}\|_* &= \max_{t\in J} e^{-(K+M)t}\left|\int_0^t e^{Ms}\right. \\ &\quad \Big[-Ne^{-K\alpha(s)}e^{K\alpha(s)}[y(\alpha(s)) - \bar{y}(\alpha(s))] \\ &\quad \left.-P\int_0^{\beta(s)} k(s,r)[y(r) - \bar{y}(r)]e^{-Kr}e^{Kr}dr\Big]\,ds\right| \\ &\le \|y - \bar{y}\|_* Q,\end{aligned}$$

where

$$Q = \max_{t\in J} e^{-(K+M)t}\int_0^t e^{Ms}\left[Ne^{K\alpha(s)} + k_0P\int_0^{\beta(s)} e^{Kr}dr\right]ds,$$

$$k_0 = \max_{(t,s)\in D_1} k(t,s).$$

Note that

$$Q \le [N + k_0Pb]\max_{t\in J}\left\{e^{-(M+K)t}\int_0^t e^{(M+K)s}ds\right\} < 1 - e^{-(M+K)b} \equiv \bar{Q}.$$

By the Banach fixed point theorem, problem (3.1) has a unique solution because

$$\|Ay - A\bar{y}\|_* < \bar{Q}\|y - \bar{y}\|_* \quad \text{and} \quad \bar{Q} < 1.$$

It ends the proof. ∎

4. To apply the monotone iterative technique we need some comparison results.

LEMMA 4.1. *Let Assumptions H_2 and H_6 hold. Assume that $M, N, P \ge 0$ and*

$$p'(t) \le -Mp(t) - Np(\alpha(t)) - PTp(t), \quad p(0) \le \theta. \tag{4.1}$$

Then $p(t) \le \theta$ on J.

PROOF. Let $\bar{B}^*$ be the set of all continuous linear functionals g on B such that $g(x) \ge 0$ for all $x \in \bar{B}$. For any $g \in \bar{B}^*$, let $m(t) = g(p(t))$. Then $m \in C^1(J, \mathbb{R})$, and $m'(t) = g(p'(t))$, $g(Tp(t)) = Tm(t)$, $g(Sp(t)) = Sm(t)$. By (4.1), we have

$$m'(t) \le -Mm(t) - Nm(\alpha(t)) - PTm(t), \quad t \in J, \quad m(0) \le 0. \tag{4.2}$$

Let $v(t) = e^{Mt}m(t),\ t \in J$, so $v(0) \le 0$, and

$$Tm(t) = \int_0^{\beta(t)} k(t,s)m(s)ds = \int_0^{\beta(t)} k(t,s)e^{-Ms}v(s)ds.$$

Then, (4.2) yields

$$\begin{aligned} v'(t) &= Me^{Mt}m(t) + e^{Mt}m'(t) \le -e^{Mt}[Nm(\alpha(t)) + PTm(t)] \\ &= -Ne^{M[t-\alpha(t)]}v(\alpha(t)) - P\int_0^{\beta(t)} k^*(t,s)v(s)ds \end{aligned} \tag{4.3}$$

with $k^*(t,s) = e^{M(t-s)}k(t,s)$.

We need to show that $v(t) \le 0$ on J. Assume that it is not true, so there exists $t_0 \in (0,b]$ such that $v(t_0) > 0$. Let $\min\limits_{t\in[0,t_0]} = -A,\ A \ge 0$. If $A = 0$, then $v(t) \ge 0,\ t \in [0,t_0]$. Hence, $v'(t) \le 0,\ t \in [0,t_0]$, by (4.3). It shows that $v(t) \le 0,\ t \in [0,t_0]$, so $v(t_0) \le 0$. It is a contradiction. Let $A > 0$. Then there exists $t_1 \in [0,t_0)$ such that $v(t_1) = -A$. Moreover, there exists $t_2 \in (t_1,t_0)$ such that $v(t_2) = 0$. Now the mean value theorem gives

$$v(t_2) - v(t_1) = v'(t_3)(t_2 - t_1),\ \ t_3 \in (t_1,t_2),$$

so

$$v'(t_3) = \frac{A}{t_2 - t_1} > \frac{A}{b}.$$

On the other hand we obtain

$$\begin{aligned} v'(t_3) &\le -Ne^{M[t_3-\alpha(t_3)]}v(\alpha(t_3)) - P\int_0^{\beta(t_3)} k^*(t_3,s)v(s)ds \\ &\le NAe^{M[t_3-\alpha(t_3)]} + PA\int_0^{\beta(t_3)} k^*(t_3,s)ds \\ &\le \begin{cases} NAe^{Mb} + \frac{PAk_0}{M}\left(e^{Mb}-1\right) & \text{if } M > 0, \\ NA + PAk_0b & \text{if } M = 0 \end{cases} \end{aligned}$$

showing that $1 < Nbe^{Mb} + \frac{Pk_0b}{M}\left(e^{Mb}-1\right)$ if $M > 0$, and $1 < Nb + Pk_0b^2$ if $M = 0$. It is a contradiction because of Assumption H_6. Hence $v(t) \le 0$ on J and therefore $m(t) \le 0, t \in J$. Since $g \in \bar{B}^*$ is arbitrary, we get $p(t) \le \theta,\ t \in J$. It ends the proof. □

Remark 4.2. If $N = 0$ and $\beta(t) = t,\ t \in J$, then Lemma 4.1 becomes Lemma 3.1 of [6].

Lemma 4.3. *Let Assumptions H_1 to H_6 hold. Assume that u, v are lower and upper solutions of problem (1.1) and such that* $y_0(t) \le u(t) \le v(t) \le$

$z_0(t),\ t \in J$. *Let*

$$
(4.4)\qquad \begin{cases} y'(t) = f(t, u(t), u(\alpha(t)), Tu(t), Su(t)) + F(t, u(t), y(t)), \\ \quad t \in J, \quad y(0) = x_0, \\ z'(t) = f(t, v(t), v(\alpha(t)), Tv(t), Sv(t)) + F(t, v(t), z(t)), \\ \quad t \in J, \quad z(0) = x_0, \end{cases}
$$

where

$$
F(t, u(t), y(t)) = -M[y(t) - u(t)] - N[y(\alpha(t)) - u(\alpha(t))] - P[Ty(t) - Tu(t)].
$$

Then

(*i*) $u(t) \le y(t) \le z(t) \le v(t),\ \ t \in J$,
(*ii*) *y and z are lower and upper solutions of (1.1), respectively.*

Proof. Lemma 3.1 shows that system (4.4) has a unique solution (y, z). First, we show (i). Put $p = u - y$. Then $p(0) \le \theta$, and

$$
\begin{aligned} p'(t) &\le f(t, u(t), u(\alpha(t)), Tu(t), Su(t)) - f(t, u(t), u(\alpha(t)), Tu(t), Su(t)) \\ &\quad -F(t, u(t), y(t)) \\ &= -Mp(t) - Np(\alpha(t)) - PTp(t), \quad t \in J \end{aligned}
$$

since u is a lower solution of (1.1). This and Lemma 4.1 yield $p(t) \le \theta$ on J showing that $u(t) \le y(t)$ on J. In the same way, we can show that $z(t) \le v(t)$ on J. Now, we put $p = y - z$. Then, using Assumptions H_4 and H_5, we obtain

$$
\begin{aligned} p'(t) &= f(t, u(t), u(\alpha(t)), Tu(t), Su(t)) - f(t, v(t), v(\alpha(t)), Tv(t), Sv(t)) \\ &\quad +F(t, u(t), y(t)) - F(t, v(t), z(t)) \\ &\le M[v(t) - u(t)] + N[v(\alpha(t)) - u(\alpha(t))] + P[Tv(t) - Tu(t)] \\ &\quad +F(t, u(t), y(t)) - F(t, v(t), z(t)) \\ &= -Mp(t) - Np(\alpha(t)) - PTp(t), \quad t \in J, \quad p(0) = \theta. \end{aligned}
$$

Hence, by Lemma 4.1, $y(t) \le z(t)$ on J showing that property (i) holds. Now we need to show that y and z are lower and upper solutions of (1.1), respectively. Using Assumptions H_4 and H_5 we get

$$
\begin{aligned} y'(t) &= f(t, u(t), u(\alpha(t)), Tu(t), Su(t)) + F(t, u(t), y(t)) \\ &\quad -f(t, y(t), y(\alpha(t)), Ty(t), Sy(t)) + f(t, y(t), y(\alpha(t))), Ty(t), Sy(t)) \\ &\le f(t, y(t), y(\alpha(t)), Ty(t), Sy(t)) + M[y(t) - u(t)] \\ &\quad +N[y(\alpha(t)) - u(\alpha(t))] + P[Ty(t) - Tu(t)] + F(t, u(t), y(t)) \\ &= f(t, y(t), y(\alpha(t)), Ty(t), Sy(t)), \quad t \in J, \end{aligned}
$$

and

$$\begin{aligned} z'(t) &= f(t, v(t), v(\alpha(t)), Tv(t), Sv(t)) + F(t, v(t), z(t)) \\ &\quad - f(t, z(t), z(\alpha(t)), Tz(t), Sz(t)) + f(t, z(t), z(\alpha(t)), Tz(t), Sz(t)) \\ &\geq f(t, z(t), z(\alpha(t)), Tz(t), Sz(t)), \ \ t \in J \end{aligned}$$

showing that (ii) holds.

It ends the proof. □

5. The next section gives sufficient conditions on existence of extremal solutions for problems of type (1.1).

THEOREM 5.1. *Let cone $\bar{B}$ be regular. Assume that Assumptions H_1 to H_6 are satisfied. Then there exist monotone sequences $\{y_n\}$, $\{z_n\}$ such that $y_n \to y$, $z_n \to z$ as $n \to \infty$ uniformly and monotonically on J and y, z are minimal and maximal solutions of problem (1.1) on $[y_0, z_0]$, respectively.*

PROOF. Let $y_{n+1}(0) = z_{n+1}(0) = x_0$, and

$$\begin{cases} y'_{n+1}(t) = f(t, y_n(t), y_n(\alpha(t)), Ty_n(t), Sy_n(t)) + F(t, y_n(t), y_{n+1}(t)), \ t \in J, \\ z'_{n+1}(t) = f(t, z_n(t), z_n(\alpha(t)), Tz_n(t), Sz_n(t)) + F(t, z_n(t), z_{n+1}(t)), \ t \in J \end{cases}$$

for $n = 0, 1, \ldots$, where F is defined as in Lemma 4.3. Note that y_1, z_1 are well defined, by Lemma 3.1. Using Lemma 4.3, we obtain

$$y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t), \ \ t \in J,$$

and moreover y_1, z_1 are lower and upper solutions of (1.1).

Let us assume that

$$y_0(t) \leq y_1(t) \leq \cdots \leq y_{k-1}(t) \leq y_k(t) \leq z_k(t) \leq z_{k-1}(t) \leq \cdots \leq z_1(t) \leq z_0(t),$$

for $t \in J$ and let y_k, z_k be lower and upper solutions of problem (1.1) for some $k \geq 1$. Then, by Lemma 3.1, the elements y_{k+1}, z_{k+1} are well defined. Lemma 4.3 yields

$$y_k(t) \leq y_{k+1}(t) \leq z_{k+1}(t) \leq z_k(t), \ \ t \in J.$$

Hence, by induction, we have

$$y_0(t) \leq y_1(t) \leq \cdots \leq y_n(t) \leq z_n(t) \leq \cdots \leq z_1(t) \leq z_0(t), \ \ t \in J$$

for all n. The regularity of $\bar{B}$ and continuity of f imply that the sequences $\{y_n\}, \{z_n\}$ converge uniformly to the limit functions y, z, so $y_n \to y$, $z_n \to z$, and $y(t) \leq z(t)$ on J. Indeed, y, z are solutions of problem (1.1).

To prove that y, z are minimal and maximal solutions of (1.1) on the segment $[y_0, z_0]$, we need to show that if w is any solution of (1.1) such that $y_0(t) \leq w(t) \leq z_0(t)$ on J, then

$$y_0(t) \leq y(t) \leq w(t) \leq z(t) \leq z_0(t), \ \ t \in J.$$

To do this, suppose that for some k, $y_k(t) \le w(t) \le z_k(t)$ on J, and put $p = y_{k+1} - w$, $q = w - z_{k+1}$. Then, Assumptions H_4 and H_5 yield

$$\begin{aligned}
p'(t) &= f(t, y_k(t), y_k(\alpha(t)), Ty_k(t), Sy_k(t)) \\
&\quad -F(t, w(t), w(\alpha(t)), Tw(t), Sw(t)) + F(t, y_k(t), y_{k+1}(t)) \\
&\le M[w(t) - y_k(t)] + N[w(\alpha(t)) - y_k(\alpha(t))] + P[Tw(t) - Ty_k(t)] \\
&\quad +F(t, y_k(t), y_{k+1}(t)) \\
&\le -Mp(t) - Np(\alpha(t)) - PTp(t), \ \ t \in J, \ \ p(0) = \theta, \\
q'(t) &= F(t, w(t), w(\alpha(t)), Tw(t), Sw(t)) \\
&\quad -F(t, z_k(t), z_k(\alpha(t)), Tz_k(t), Sz_k(t)) - F(t, z_k(t), z_{k+1}(t)) \\
&\le -Mq(t) - Nq(\alpha(t)) - PTq(t), \ \ t \in J, \ \ q(0) = \theta.
\end{aligned}$$

By Lemma 4.1, we obtain $p(t) \le \theta$, $q(t) \le \theta$ on J showing that $y_{k+1}(t) \le w(t) \le z_{k+1}(t)$, $t \in J$. Since $y_0(t) \le w(t) \le z_0(t)$ it proves, by induction, that $y_n(t) \le w(t) \le z_n(t)$ on J for all n. Taking the limit as $n \to \infty$, we conclude that $y(t) \le w(t) \le z(t)$, $t \in J$.

The proof is complete. □

Remark 5.2. Note that Assumption H_4 holds if we assume that $f(t, u, v, w, z)$ is nondecreasing in u, v, w for fixed t and z. Indeed, in this case, we have

$$f(t, \bar{u}, \bar{v}, \bar{w}, z) - f(t, u, v, w, z) \ge 0 \ge -M(\bar{u} - u) - N(\bar{v} - v) - P(\bar{w} - w)$$

for some nonnegative M, N, P and $\bar{u} \ge u$, $\bar{v} \ge v$, $\bar{w} \ge w$.

Remark 5.3. If $N = 0$, $\beta(t) = t$, $t \in J$ and f does not depend on the last argument, then Theorem 5.1 becomes Theorem 2 of [6].

Example 5.4. Consider the initial value problem of an infinite system for scalar delay integro–differential equations of type

$$(5.1) \quad \begin{cases} x_n'(t) = \frac{1}{4n^2}\left[\frac{1}{2n^2} - x_n(t) - x_{n+1}\left(\frac{1}{2}t\right)\right] - \frac{1}{2(n+1)^2}\left[\int_0^{\frac{1}{3}t} x_{n+1}(s)ds\right]^2 \\ \qquad\qquad + \frac{1}{2(n+1)^3}\left[\int_0^1 x_{2n}(s)\cos^4(t-s)ds\right]^3, \ t \in J = [0,1], \\ x_n(0) = 0 \end{cases}$$

for $n = 1, 2, \ldots$. Let $B = \{u : (u_1, \ldots, u_n, \ldots) : u_n \in \mathbb{R}, \ \sum_{n=1}^{\infty} |u_n| < \infty\}$ with the norm $\|u\| = \sum_{n=1}^{\infty} |u_n|$ and $\bar{B} = \{u \in B : u_n \ge 0, \ n = 1, 2, \ldots\}$. Then $\bar{B}$ is a normal cone in B. Since B is weakly complete, we know from Remarks 4.3.1 and 1.2.4 of [8] that $\bar{B}$ is regular. In this case $f = (f_1, \ldots, f_n, \ldots)$ with

$$f_n(t, x, y, z, w) = \frac{1}{4n^2}\left[\frac{1}{2n^2} - x_n - y_{n+1}\right] - \frac{1}{2(n+1)^2} z_{n+1}^2 + \frac{1}{2(n+1)^3} w_{2n}^3.$$

Indeed, $f \in C(J \times B^4, B)$, $\alpha = \frac{1}{2}t$, $\alpha \in C(J, J)$, $0 \le \alpha(t) \le t$, $\beta(t) = \frac{1}{3}t$, $\beta \in C(J, J)$, $0 \le \beta(t) \le t$, $k(t, s) = 1$ for $(t, s) \in J \times J$, and $l(t, s) = \cos^4(t - s)$ for $(t, s) \in J \times J$. Let

$$y_0(t) = (0, \dots, 0, \dots), \quad z_0(t) = \left(1, \dots, \frac{1}{n^2}, \dots\right), \quad t \in J.$$

Indeed, $y_0(t) < z_0(t)$, $t \in J$. We see that

$$f_n(t, y_0(t), y_0(\alpha(t)), Ty_0(t), Sy_0(t)) = \frac{1}{8n^4} > 0, \quad t \in J, \quad n = 1, 2, \dots,$$

and

$$\begin{aligned} &f_n(t, z_0(t), z_0(\alpha(t)), Tz_0(t), Sz_0(t)) = \frac{1}{4n^2}\left[\frac{1}{2n^2} - \frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \\ &-\frac{1}{2(n+1)^2}\left[\int_0^{\frac{1}{3}t} \frac{1}{(n+1)^2} ds\right]^2 + \frac{1}{2(n+1)^3}\left[\int_0^1 \frac{1}{4n^2}\cos^4(t-s) ds\right]^3 \\ &\le -\frac{3n^2 + 2n}{8n^4(n+1)^2} - \frac{t^2}{18(n+1)^6} < 0, \quad t \in J, \quad n = 1, 2, \dots. \end{aligned}$$

It proves that y_0, z_0 are lower and upper solutions of problem (5.1) respectively, so assumption H_3 holds.

Let $y_0(t) \le u \le \bar{u} \le z_0(t)$, $y_0(\alpha(t)) \le v \le \bar{v} \le z_0(\alpha(t))$, $Ty_0(t) \le w \le \bar{w} \le Tz_0(t)$, $Sy_0(t) \le \bar{z} \le Sz_0(t)$ for all $t \in J$. Then

$$\begin{aligned} &f_n(t, \bar{u}, \bar{v}, \bar{w}, \bar{z}) - f_n(t, u, v, w, \bar{z}) = \\ &= \frac{1}{4n^2}\left[\frac{1}{2n^2} - \bar{u}_n - \bar{v}_{n+1}\right] - \frac{1}{2(n+1)^2}\bar{w}_{n+1}^2 \\ &\quad -\frac{1}{4n^2}\left[\frac{1}{2n^2} - u_n - v_{n+1}\right] + \frac{1}{2(n+1)^2} w_{n+1}^2 \\ &= -\frac{1}{4n^2}[\bar{u}_n - u_n] - \frac{1}{4n^2}[\bar{v}_{n+1} - v_{n+1}] \\ &\quad -\frac{1}{2(n+1)^2}[\bar{w}_{n+1} + w_{n+1}][\bar{w}_{n+1} - w_{n+1}] \\ &\ge -\frac{1}{4}[\bar{u}_n - u_n] - \frac{1}{4}[\bar{v}_{n+1} - v_{n+1}] - \frac{1}{3}[\bar{w}_n - w_n]. \end{aligned}$$

It yields $M = N = \frac{1}{4}$, $P = \frac{1}{3}$ and therefore

$$bNe^{Mb} + \frac{Pk_0 b}{M}\left(e^{Mb} - 1\right) \approx 0.6997 < 1.$$

It proves that assumption H_6 holds. Hence, problem (5.1) has extremal solutions in the segment $[y_0, z_0]$, by Theorem 5.1.

EXAMPLE 5.5. Consider the following infinite problem of scalar equations

$$(5.2)\quad \begin{cases} x_n'(t) &= \frac{1}{4}\left[\frac{t}{2n^2} - x_n(t)\right] + \frac{1}{10}x_{n+1}\left(\frac{1}{2}t\right) + \left[\int_0^{\frac{1}{4}t} x_{n+2}(s)ds\right]^2 \\ & \quad +t\left[\int_0^1 x_{2n}(s)\sin^2(t-s)ds\right]^4,\ t \in J = [0,1], \\ x_n(0) &= 0 \end{cases}$$

for $n = 1, 2, \ldots$. Indeed, $\alpha(t) = \frac{1}{2}t,\ \beta(t) = \frac{1}{4}t$. Let B and $\bar{B}$ be defined as in Example 1. Note that

$$f_n(t, x, y, z, w) = \frac{1}{4}\left[\frac{t}{2n^2} - x_n\right] + \frac{1}{10}y_{n+1} + z_{n+2}^2 + tw_{2n}^4.$$

Let

$$y_0(t) = (0, \ldots, 0, \ldots),\ \ z_0(t) = \left(t, \ldots, \frac{t}{n^2}, \ldots\right), \quad t \in J.$$

Then $y_0(t) \le z_0(t), \quad t \in J$, and

$$y_0'(t) = (0, \ldots, 0, \ldots), \quad z_0'(t) = \left(1, \ldots, \frac{1}{n^2}, \ldots\right).$$

It yields

$$f_n(t, y_0(t), y_0(\alpha(t)), Ty_0(t), Sy_0(t)) = \frac{t}{8n^2} \ge 0, \quad t \in J, \quad n = 1, 2, \ldots,$$

and

$$\begin{aligned} f_n(t, z_0(t), z_0(\alpha(t)), Tz_0(t), Sz_0(t)) &= \frac{1}{4}\left[\frac{t}{2n^2} - \frac{t}{n^2}\right] + \frac{t}{20(n+1)^2} \\ &+ \left[\int_0^{\frac{1}{4}t} \frac{s}{(n+2)^2}ds\right]^2 + t\left[\int_0^1 \frac{s}{4n^2}\sin^2(t-s)ds\right]^4 \\ &\le -\frac{t}{8n^2} + \frac{t}{20(n+1)^2} + \frac{1}{(n+2)^4}\left[\int_0^{\frac{1}{4}t} sds\right]^2 + \frac{t}{256n^8}\left[\int_0^1 sds\right]^4 \\ &= -\frac{t}{8n^2} + \frac{t}{20(n+1)^2} + \frac{t^4}{1024(n+2)^4} + \frac{t}{4096n^8} \\ &\le 0 < z_{0n}'(t),\ t \in J,\ n = 1, 2, \ldots \end{aligned}$$

It proves that y_0, z_0 are lower and upper solutions of problem (5.2) respectively, so assumption H_3 holds.

Let $y_0(t) \le u \le \bar{u} \le z_0(t),\ y_0(\alpha(t)) \le v \le \bar{v} \le z_0(\alpha(t)),\ Ty_0(t) \le w \le \bar{w} \le Tz_0(t),\ Sy_0(t) \le \bar{z} \le Sz_0(t)$ for all $t \in J$. Then

$$\begin{aligned} &f_n(t, \bar{u}, \bar{v}, \bar{w}, \bar{z}) - f_n(t, u, v, w, \bar{z}) = \\ &\quad = \frac{1}{4}\left[-\bar{u}_n + u_n\right] + \frac{1}{10}[\bar{v}_{n+1} - v_{n+1}] + \bar{w}_{n+2}^2 - w_{n+2}^2 \ge -\frac{1}{4}[\bar{u}_n - u_n], \end{aligned}$$

so $M = \frac{1}{4}$, $N = P = 0$. Assumption H_6 is satisfied and problem (5.2) has extremal solutions in the segment $[y_0, z_0]$, by Theorem 5.1.

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