On some cost allocation problems in communication networks*

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Abstract. *New technologies prompted an explosion in the development of communication networks. Modern network optimization techniques usually lead to a design of the most profitable, or the least cost network that will provide some service to customers. There are various costs and gains associated with building and using a communication network. Moreover, the involved multiple network users and/or owners possibly have conflicting objectives. However, they might cooperate in order to decrease their joint cost or increase their joint profit. Clearly, these individuals or organizations will support a globally 'attractive' solution(s) only if their expectations for a 'fair share' of the cost or profit are met. Consequently, providing network developers, users and owners with efficiently computable 'fair' cost allocation solution procedures is of great importance for strategic management. This work is an overview of some recent results (some already published as well as some new) in the development of cooperative game theory based mechanisms to efficiently compute 'attractive' cost allocation solutions for several important classes of communication networks.*

Key words: *communication networks, cost allocation, cooperative games, mathematical programming*

Sažetak.*Nove tehnologije izazvale su dramatičan razvoj komunikacijskih mreža. Moderne optimizacijske tehnike obično teže dizajnu što ekonomičnijih mreža, koje će zadovoljiti određene potrebe korisnika. Uz gradnju i korištenje komunikacijskih mreža vezani su razni troškovi i zarade, koji često uključuju korisnike i/ili vlasnike mreža potencijalno konfliktnih ciljeva i interesa. Unatoč razlikama, za očekivati je da bi oni htjeli surađivati ako međusobna suradnja umanjuje zajedničke troškove ili pak povećava zajedničke profite. Jasno je da će pojedinci ili organizacije podržavati zajednički atraktivna rješenja samo ako su ispunjena njihova individualna očekivanja glede "pravedne" raspodjele troškova ili profita. Posljedica toga je da je strateški važno za dizajnere, vlasnike i korisnike*

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komunikacijskih mreža imati na raspolaganju efikasne, izračunljive procedure za "pravednu" raspodjelu troškova. Ovaj rad je pregled novijih rezultata (nekih već publiciranih, kao i nekih sasvim novih) iz područja kooperativne teorije igara koji se bavi razvojem mehanizama za efikasno izračunavanje "atraktivnih" rješenja problema raspodjele troškova za nekoliko važnih klasa komunikacijskih mreža.

Ključne riječi: *komunikacijske mreže, raspodjela troškova, kooperativne igre, matematičko programiranje*

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1. Introduction

Recent technology advances prompted an explosion in the development of communication networks. It is a general opinion that efficient networking is necessary to preserve a competitive edge in today's society. There are various costs associated with building a communication network, which involves multiple network users and owners. The various network users and owners may wish to cooperate in order to decrease their joint cost or profit. Clearly, they will support a globally 'attractive' solution(s) if they are charged a 'fair' share of the total cost. The main objectives of network cost allocation studies is to define so called 'fair' cost allocation solution concepts and to develop mechanisms to efficiently compute 'fair' cost allocation solutions. In this paper we are concerned with the cooperative game theory approach to the area of cost allocation in communication networks. The paper presents an overview of some cooperative game theory-based formulations of the network cost allocation problems and some recently developed mechanisms to efficiently solve those problems for several important classes of networks.

There is a body of literature that studies the computation of cost allocation solution concepts for some related network cooperative games. The combinatorial issues are particularly complex, and the literature mostly deals with deriving properties of various cost allocation solutions restricted to special classes of networks and in the context of a particular problem. Nevertheless, many techniques used therein are not necessarily conceptually limited to such special networks and are expected to be helpful in further developments for more general and practical cases.

The first, and most studied is the class of Minimum Cost Spanning Trees (MCST) (see for example Bird [1976]) in which the set of users should be linked to a single special node (hub) at minimum cost. The weight of each link (edge) indicates the cost of that link and we must then allocate the cost of edges to the users, who are represented as nodes. The cost allocation problem associated with MCST problem is formulated as a cooperative game referred to as the MCST game. The main objective of cooperative game approach to this problem is to find stable cost allocations which will encourage everyone to stay with the grand coalition, and will give no monetary incentive to any subset of users to secede and build their own competing subnetwork. It turns out that stable cost allocations exist for the MCST problem. Moreover, we will be interested in further refining of such stable cost allocations in order to support network growth and dynamic cost changes (Kent and Skorin-Kapov [1996, 1997]). Namely, it is of great importance to set up a cost allocation scheme in such a way that will encourage new users to join the network, and will not give any of already present users a reason to block others from joining. In addition, we would like that the cost allocation scheme gives no incentive to anyone to inflate the cost of any link.

An important class of games that generalizes MCST games is the class of Steiner tree games (Skorin-Kapov [1995]). The Steiner tree game is once again concerned with the allocation of the cost of a certain service that is provided to users inhabited at network nodes. The main difference from the MCST game is that in the Steiner tree games some nodes may be switching points (there are no users residing at them). This assumption has profound practical implications in the modeling of communication networks. Unfortunately, it also increases the computational difficulty. By contrast to MCST games, the stable cost allocations for Steiner tree network game do not necessarily exist. (Tamir [1991]). We will comment on a heuristic algorithm (Skorin-Kapov [1995]) for finding core points when they exist.

Another class of even more practical design communication network problems is the class of Capacitated Network Design (CND) problems. Here the set of users is in need of a certain service that can be provided by connecting them (possibly through other users and/or switching points) to capacitated facilities, yet to be constructed. The objective is to build a network that will provide the above service at minimum cost. The associated cost allocation problem is formulated as a cooperative CND-game (Skorin-Kapov and Beltran [1994]). The CND game properly generalizes MCST games, Steiner tree games and several other classes of games studied in the literature. We outline the polynomial characterizations (Skorin-Kapov and Beltran [1994]) of the ϵ-core and the nucleolus of the CND game.

The plan of the paper follows. In *Section 2* we present some game theoretic definitions and preliminaries needed for understanding of the rest of the paper. In *Section 3* we formally define the MCST game and give an overview of some well known, as well as some new results. In *Section 4* we outline some findings from studies on Steiner tree games. In *Section 5* known results on CND-games are summarized. We finish with some concluding remarks in *Section 6*.

2. Definitions and preliminaries

In order to analyze the cost allocation problem associated with the optimization problems in communication networks, we need to introduce the following game theoretic definitions and notation. Let $\mathrm{P}=\{1,2,\cdots,\mathrm{n}\}$ be a finite set of *players* and let $\mathrm{c}: 2^{\mathrm{P}} \rightarrow \mathbb{R}$, with $\mathrm{c}(\emptyset)=0$, be a *characteristic function* defined over subsets of P referred to as *coalitions*. The characteristic function c is *submodular* if $\mathrm{c(S)+c(T)} \geq \mathrm{c(S\cup T)+c(S\cap T)}$ for all $\mathrm{S,T} \subseteq \mathrm{P}$. If c is submodular, (N;c) is said to be *concave*. If $\mathrm{c}(P)$ designates a cost that has to be shared by all the players, then the pair $(P;\mathrm{c})$ is called a (cost) *cooperative game*, or simply a *game*. For $\mathrm{x} \in \mathrm{R}^{|\mathrm{P}|}$ and $\mathrm{S} \subseteq P$, let $\mathrm{x(S)} \equiv \sum_{\mathrm{j}\in \mathrm{S}} \mathrm{x_j}$. We can interpret x(S) as the part of the total cost paid by the coalition S. A *cost allocation* vector x in a game $(P;\mathrm{c})$ satisfies $\mathrm{x}(P)=\mathrm{c}(P)$, and the solution theory of cooperative games is concerned with the selection of a reasonable subset of cost allocation vectors.

In cooperative game theory, several different solution concepts for fair cost allocation have been suggested (for a survey of these concepts see, for example, Young [1985] or Driessen [1988]). Central to the solution theory is the concept of solution referred to as the core of a game. The *core* of a game (P;c) consists of all vectors $x \in R^{|P|}$ such that $x(S) \leq c(S)$ for all $S \subseteq P$, and $x(P) = c(P)$. Observe that the core consists of all allocation vectors x which provide no incentive for any coalition to secede. In general, the core of a game may be empty. For a real number ε, the ε*-core* of a game (P;c) consists of all vectors $x \in R^n$ such that $x(S) \leq c(S) + \varepsilon$ for all $\emptyset \neq S \subset P$, and $x(P) = c(P)$. Clearly for ε big enough the ε-core of the game (P;c) is not empty. The *least* ε*-core* is the intersection of all nonempty ε-cores. Equivalently, let ε_0 be the smallest ε such that the ε-core is not empty. Then the least ε-core is the ε_0-core.

The *nucleolus*, introduced by Schmeidler [1969], is another well known solution concept. Intuitively, the nucleolus is an allocation that makes least-well-off coalition S as well-off as possible in a lexicographic sense. We say that a coalition S is better-off than T, relative to an allocation x, if $c(S)-x(S) > c(T)-x(T)$. Formally, the nucleolus can be presented as follows. For a game (P;c) and an associated cost allocation vector x, let quantity $e(x,S) = c(S)-x(S)$ be referred to as the *excess* of S relative to x, and let e(x) be a vector in $R^{(2^{|P|}-2)}$ whose entries are e(x,S), $\emptyset \neq S \subset P$, arranged in a non decreasing order. The nucleolus is the vector x that maximizes e(x) lexicographically. In contrast to the core, a nucleolus always exists. Moreover, it is unique and it is contained in the core if the core is not empty.

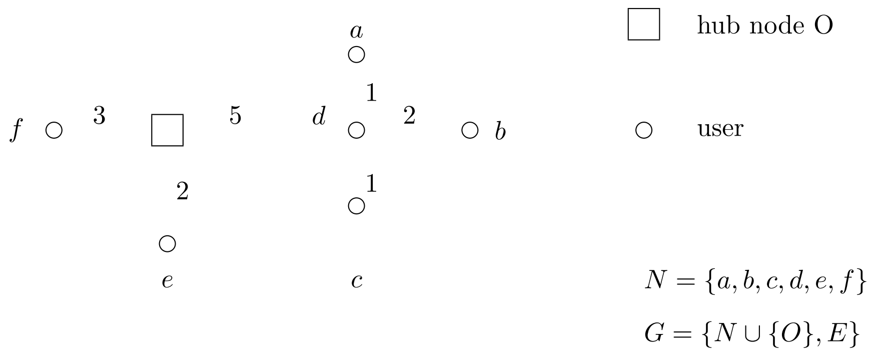
Another reasonable approach is to define the excess of a coalition on a per capita basis: $\tilde{e}(x,S) = (1/|S|)(c(S)-x(S)$. Let $\tilde{e}(x)$ be a vector in $R^{(2^n-2)}$, whose entries are $\tilde{e}(x,S)$, $\emptyset \neq S \subset P$, arranged in a nondecreasing order. The *per capita nucleolus* (Grotte [1970]) is the vector x that maximizes $\tilde{e}(x)$ lexicographically.

The *population monotonic cost allocation scheme* is a vector $x = (x_{iS})_{i \in S, S \subseteq P}$,of the game (P, c) if and only if:(i) for all $S \subseteq P$, $x_S(S) = c(S)$, and (ii) for all $S \subseteq T \subseteq P$ and $i \in S, x_{iS} \leq x_{iT}$ (no player's cost increases if additional user joins the network). A scheme that produces stable (core) cost allocation is said to be *distance monotonic* as long as no player's cost increases when any single edge weight is decreased (and symmetrically no player's cost decreases when any single weight is increased).

3. Minimum cost spanning tree (MCST) game

Consider a connected undirected network $G = (N \cup \{O\}, E)$ with a set of nodes $N \cup \{O\}$ and a set of arcs E. A common supplier O provides service which is required by users, and any node receiving the service can in turn deliver it to adjacent nodes. Each user in N is required to be connected, perhaps through other nodes, to a common supplier. There is a cost, $w((i,j)) = w_{ij} \geq 0$, $(i,j) \in E$, if arc (i,j) is used to deliver service. The objective is to provide service to the communities in N at a minimum cost. We will refer to the above optimization problem as the Minimum Cost Spanning Tree (MCST) problem.

The cost allocation is concerned with the fair distribution of the cost of providing the service among customers. In order to analyze the cost allocation problem associated with the MCST problem, we formulate this cost allocation problem as a cooperative game. Consider the MCST problem on a network $G = (N \cup \{O\}, E)$, with a set of users N. Denote by $\mathrm{MCST_Q}$, for Q⊆ N, the MCST problem obtained from the original problem by simply requesting that only nodes in Q have to be connected. Then, the pair (N;c), where $c: 2^{|N|} \to R$ is such that $c(\emptyset) = 0$ and for each Q⊆ N c(Q) is the minimum objective function value of $\mathrm{MCST_Q}$, is a game to be referred to as the MCST-game. For $x \in R^{|N|}$ and Q⊆ N, let $x(Q) \equiv \sum_{j \in Q} x_j$. We can interpret x(Q) as the part of the total cost paid by the coalition Q. A cost allocation vector x in a game (N;c) satisfies x(N) = c(N), and the solution theory of cooperative games is concerned with the selection of a reasonable subset of cost allocation vectors. Two versions were studied. The non-monotone version of the MCST game allows a coalition S⊆ N to use only nodes in S in order to construct a minimal cost network, while a monotone MCST game allows a coalition S to use nodes in N\S in constructing a minimal cost network.



Cost allocation $x = (x_a, x_b, x_c, x_d, x_e, x_f) = (1, 2, 1, 5, 2, 3)$ is in the core of the game (N, c).

Figure 1. *The core allocation of the MCST game*

Recall that the cost allocation vector x belongs to the core if for each $Q, Q \subset N, \mathrm{x}(Q) \leq c(Q)$. Observe that the core consists of all allocation vectors x which provide no incentive for any coalition to secede. The exponential number of core constraints, coupled with the fact that MCST_Q problem (in monotone version) is NP-complete whenever $2< |Q| < |N|$ (this is the Steiner tree problem, known to be NP-complete, see Garey and Johnson, [1979]), makes the core computation very difficult. Clearly, in case of monotone MCST-game determining whether a given cost allocation is in the core is NP-complete. However, Bird [1976] and D. Granot and Huberman [1981] showed that the MCST game has a non-empty core and found some points in the core. For example, consider the tree shown in *Figure 1*, and assume that it is a MCST in the complete graph G= $\{N \cup \{O\}, E\}$. It can be shown that the cost allocation in which each user j pays the cost of a unique edge (i, j) of the MCST is in the core of the associated MCST-game .

It is useful to consider the directed formulation of the MCST problem. The Minimum Cost Directed Spanning Tree (MCDST) problem is defined with respect to a directed weighted graph G= (N∪{O},E) with a weight (cost) function $w: E \to R^+$. Namely, find a directed spanning tree T= $(N_T \cup \{O\}, E_T)$ in G, rooted away from node O, such that the total edge-weight of T is minimum. It is clear that any MCST problem can be solved by considering an appropriate MCDST problem, obtained by replacing each edge of the given network by two arcs of opposite directions.

The MCDST-game is then based on a well known integer programming formulation of the minimum cost spanning tree problem. To describe the formulation, we need the following notation. Let G= (N∪{O}) be a directed graph and N the set of users. For a directed edge l= (i , j) we refer to i as the tail and j as a head of l , and for a subset of vertices S, S⊆ N, we denote by δ(S) the set of all directed edges having their heads, but not their tails, in S. A subset S, S⊆ N, is said to be a cut-set of G, if S∩N≠ ∅ and the subgraph G(S) of G induced by S is connected. We denote by $\mathcal{S}_{\mathrm{N}}$ the set of all cut-sets of G. The MCDST problem can be formulated as the following integer programming problem:

$$IP(N): \min\Big\{ wt\colon t(\delta(S)) \geq 1, S \in \mathcal{S}_{\mathrm{N}},\ t \in \{0,1\}\Big\}$$

where $t(i,j) \equiv 1$ if $(i,j) \in E$ is used in the MCDST, and zero otherwise.

For a subset Q⊆ N let IP(N) be an integer programming problem obtained from the original problem IP(Q) by simply replacing N by Q. Then, our MCST-game based on the above formulation of the MCDST problem is the pair (N;c), where $\mathrm{c}: 2^{|\mathrm{N}|} \rightarrow \mathrm{R}$ is such that c(∅) = 0 and for each Q⊆ N, c(Q) is the minimum objective function value of IP(Q).

The nonemptiness of the core of the MCST game is also alternatively shown in Granot [1986] based on results of Edmonds [1967]. Therein, the MCST game is formulated as the generalized linear production game (Granot [1986]), which is in turn a generalization of Owens's [1975] linear production game. So far it was not possible to characterize the entire core. On the other hand, some subsets of the core were characterized by several authors. It was shown (Granot [1986]) that in a linear production game feasible duals belong to the core, Aaarts and Driessen [1991] and Feltkamp et. al. [1994} further analyzed the structure of the core and the subset of the core called *irreducible core* of a MCST game.

The work of Kent and Skorin-Kapov [1996] leads to algorithmic finding of some core points based on the dual of the linear programming relaxation LP(N) of IP(N). The dual DP(N) of the LP(N) is:

$$DP(N)\colon \max\Big\{ \sum_{\mathrm{S}\in\mathcal{S}_{\mathrm{N}}} y_{\mathrm{S}}\colon \sum_{\mathrm{S}:\mathrm{e}\in\delta(\mathrm{S})} y_{\mathrm{S}} \leq w_{\mathrm{e}},\ \text{for all } e \in E; S \in \mathcal{S}_N\, , y_{\mathrm{S}} \geq 0 \text{ for all S}\Big\}.$$

Let y_{S} , S∈ $\mathcal{S}_N$ be a feasible solution to DP(N), and for each S∈ $\mathcal{S}_N$ allocate the amount y_{S} arbitrarily to users in S. Let $\mathrm{x}\in\mathrm{R}^N$ be the vector of costs allocated by this operation. It was shown by Skorin-Kapov [1995] that this partial cost allocation satisfies the core constraints. Since the objective function of LP(N) is a lower bound to IP(N), so is the value of the dual objective for any feasible dual solution. Thus, the vector $\mathrm{x}\in\mathrm{R}^N$ constructed above, gives us an allocation of some fraction of the total cost c(N).

Kent and Skorin-Kapov [1996] constructed an algorithm that finds an optimal solution to DP(N). Moreover, they proved that the optimal value of DP(N) coincides with the optimal value of IP(N). This further implies that their algorithm delivers core points. For completeness, we will outline below the slight modification of their algorithm. The algorithm starts with all dual variables at zero value. The k-th iteration of the algorithm increases y_S for each subset of users S of size k as much as possible without violating dual feasibility.

The Core Algorithm:

Input: A weighted complete network G= (N $\cup O$,E). The minimum cost directed panning tree T= ($N_T \cup \{O\}$, E_T).

Initialization: Set $x_p = 0$, for all $p \in$N, $w_{ij} = w_{ji} = W_{ij}$ for all node pairs (i, j), and $y_S = 0$ for all S $\in \mathcal{S}_N$. (x_p, $p \in$N is the cost allocated to a user p, and y_S, S $\in \mathcal{S}_N$ are the values of dual variables).

Main Algorithm

"Begin by finding a potential allocating set S".
    Do for k= 1,. . .,|N|,
        Do for each node $p \in N$
            Let S= {p}.
            Do until $|S| = k$ or no additional nodes can be added to S.
                If there exist $(i, j) \in \delta(S)$ such that $w(i, j) =$, $i \neq O$; let $S = S \cup \{i\}$.
            EndDo
"Next we allocate as much as possible to S."
            If for all $e \in \delta(S)$, $w(e) > 0$
                Let $y_S = min\{w(e), e \in \delta(S)\}$.
                Choose arbitrary a_p , p∈S such that $\sum_{p \in S} a_p = y_S$.
                Do for all p∈S
                    Let $x_p = x_p + a_p$
                EndDo
                Do for all $e \in \delta(S)$
                    Let $w(e) = w(e) - y_S$
                EndDo
            EndIf
        EndDo
    EndDo
End.

Theorem 1. *Given an instance of the MCST game on a graph G, the Core Algorithm allocates the entire cost of the MCST, and the resulting allocation is in the core of the MCST game on G.*

The cost allocations generated by the Core Algorithm are related to the irreducible core. For a given minimal cost spanning tree T, the *irreducible core* is the set of all allocations which are core allocations for every MCST game which has T as a minimal tree. It can be shown (Kent [1997]) that the set of core allocations generated by the core algorithm coincides with the irreducible core.

The above core algorithm was also presented in studies on population and distance monotonic cost allocation schemes for the MCST problem by Kent and Skorin-Kapov [1996,1997]). Therein they proposed a modification of the above Core Algorithm, thus providing a cost allocation scheme which is supporting dynamic changes in the network. In the proposed modification instead of allocating the dual value y_S arbitrarily to users in S we allocate $y_S/|S|$ to all users in S. We refer to a modified algorithm as to a Population Monotonic Scheme (PMS). It turns out that the PMS is generating stable cost allocations (core points) which are simultaneously population and distance monotonic. Namely, when a new player (user) joins the network and the new core allocation is found by the PMS, the cost will not increase to any of old players. Consequently, nobody will have the incentive to block the new user from joining the network. Moreover, if the cost of some link goes down (respectively up) no player's cost will go up (respectively down). Thus nobody will have incentive to inflate the cost of some link.

Theorem 2. *The PMS yields a population monotonic cost allocation scheme.*

Theorem 3. *The PMS yields a distance monotonic cost allocation scheme.*

4. Steiner tree (ST) game

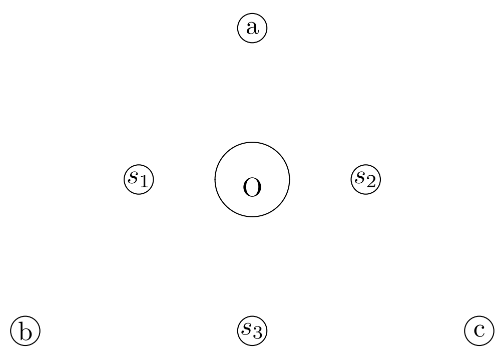
It is rare that theoretical terms find their usage in law statements. Therefore, it is particularly interesting and exciting to find out that the research on cost allocation in networks is directly relevant to policy decision making. Namely, Federal USA Law has landed added importance to work on cost allocation on minimum spanning trees and Steiner trees. Federal tariffs (see F.P. Preparata and M.F. Shamos, [1985]) require the following: "*When the Long Lines Department of the Telephone Company establishes a communication hookup for a customer, federal tariffs require that the billing rate be proportional to the length of a minimum spanning tree connecting the customers' termini ...*" The same source gives the following comment: "*This law is a Solomon-like compromise between what is desirable and what is practical to compute, for the minimum spanning tree is not the shortest possible interconnecting network if new vertices may be added to the original set. With this restriction lifted, the shortest tree is called a Steiner tree*". Therefore, from the Federal Communications Commission's (FCC) point of view, the Steiner tree approach was recognized as desirable, but due to its complexity, not practical to compute.

In this Section we will outline the results from recent work on the cost allocation for the minimum cost Steiner Tree networks by Skorin-Kapov [1995]. Therein it was shown for the first time (theoretically and computationally) that the fair cost allocation solutions, or their good approximations, could be often efficiently found on Steiner trees.

As already mentioned it is desirable that communication networks often have nodes that serve as switches. Those switches might be used by network users who do not necessarily reside at switch nodes. The associated optimization model is referred to as the *Minimum Cost Steiner Tree* problem. Formally, it is defined with respect to a weighted graph $G = (N \cup \{O\}, E)$ with a weight (cost) function $w: E \rightarrow R^+$, where N is the set of nodes and a (proper) subset of nodes D represents users. The objective is to find a directed tree $T = (N_T \cup \{O\}, E_T)$ in G, rooted away from node O and whose node set contains D, such that the total edge-weight of T is minimum.

Clearly the ST-game generalizes the class of minimum cost spanning tree games. It is well known that in general the core of the ST-game may be empty. Consider an example of the MCST game informally described in *Figure 2* (this example was suggested by Tamir [1991]). Clearly the entire cost of the minimum cost Steiner tree is c(N) = 5. On the other hand, for any core allocation each two member coalition would not pay more than 3, i.e. $x_a+x_b \leq 3$, $x_b+x_c \leq 3$, $x_a+x_c \leq 3$). The latter implies that the entire cost allocated could not exceed 4.5. Hence, the core is empty.

The ST-game is equivalent to the Fixed Cost Spanning Forest (FCSF) game, and was studied by D. Granot and F. Granot [1992a] for a special case when the underlying network G is a tree and by Skorin-Kapov [1992] when the underlying network has a series-parallel structure. It is also somewhat related to Sharkey's [1990] study of the shared facility game. Therein, he defines a simple facility game, and shows that the core of a simple facility game is nonempty if and only if the optimal values of the associated IP (Integer Program) and LP (Linear Program) are equal. In the ST-game the relationship between certain IP and LP associated with the ST-game plays an important role. Note however, that an ST-game is not a simple facility game. The work of Skorin-Kapov [1995] provided the first analysis of the ST-game for general networks.



$N = \{a, b, c\}$, $G = \{N \cup \{O\}, E\}$, $w(e) = 1$ for every $e \in E$.
For $S \subseteq N$, $c(S) =$ the total weight of the min. ST spanning $O \cup S$.

Figure 2. *Empty core of the ST-game* (N, c)

Therein the directed version of the optimization problem is considered. With the notation similar to that of *Section 3* we now denote by δ(S) the set of all directed edges having their heads, but not their tails, in S. A subset S, S$\subseteq$N, is said to be a Steiner cut-set of G, if S$\cap$D$\neq \emptyset$ (where D is the set of users) and the subgraph G(S) of G induced by S is connected. We denote by $\mathcal{S}_D$ the set of all Steiner cut-sets of G. The DST problem can be formulated (Prodon et. al. [1985]) as the following integer programming problem:

$$IP(D): \; min\left\{wx: x(\delta(S)) \geq 1, \; S \in \mathcal{S}_D, \; x \in \{0,1\}\right\}.$$

It is clear that any minimum cost ST problem can be solved by considering an appropriate minimum cost DST problem, obtained by replacing each edge of the given network by two arcs of opposite directions.

The associated cost allocation problem can be described with the following cooperative game. For a subset Q$\subseteq$D, let IP(Q) be an integer programming problem obtained from the original problem IP(D) by simply replacing D by Q. Then, our ST-game based on the above formulation of the DST problem is the pair (D;c), where $c: 2^{|D|} \rightarrow R$ is such that $c(\emptyset) = 0$ and for each Q$\subseteq$D, c(Q) is the minimum objective function value of IP(Q).

Consider the linear programming relaxation LP(D) of IP(D), defined as follows:

$$LP(D) : \min\Big\{wx : \ x(\delta(S)) \geq 1, S \in \mathcal{S}_{\mathrm{D}}\,, x \geq 0\Big\}.$$

It turns out that the sufficient condition for non-emptiness of the core of the ST-game is that the incidence vector of a minimum cost DST in G, rooted away from O and whose vertex set contains D, is an optimal solution to LP(D). This result follows for example, from Granot's [1986] generalized linear production model or equivalently from Wong's [1984] and Tamir's [1991] multicomodity flow formulation (for details see Skorin-Kapov [1995]. The work of Prodon et. al. [1985] implies that the above sufficient condition for the nonemptiness of the core is also necessary if the underlying network G is series-parallel (for details see also Skorin-Kapov [1992]).

Next we present a counterexample (from Skorin-Kapov [1995]) which demonstrates that, in general, the above sufficient condition is not necessary for the nonemptiness of the core. Indeed, consider the network G= (N$\cup${O}, E) in *Fig. 3.* Assume that all edge weights in G are 1 and let D= {5,6,7} be the set of users. Let (D;c) be the associated ST-game.

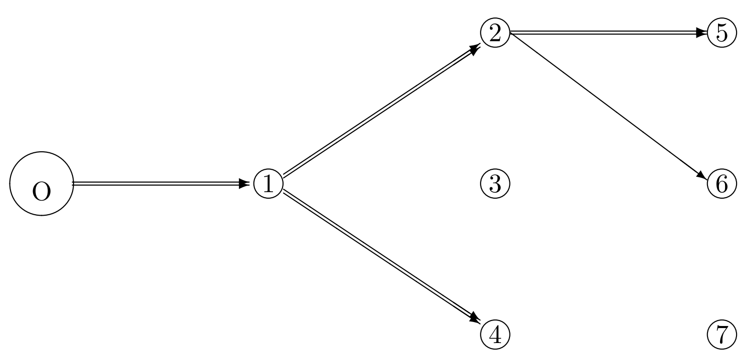


Figure 3. $G = (N \cup \{O\}, E)$

An optimal solution to IP(D) is indicated by bold arcs and has total weight c(D) = 6. It is easy to check that vector $(x_5,x_6,x_7) = (2,2,2)$ is in the core $\mathcal{C}$(D,c). On the other hand, one can verify that $x^* \in R^E_+$ defined as follows: $x^*_{(0,1)} = 1$, $x^*_{(1,2)} = \frac{1}{2}$, $x^*_{(1,3)} = \frac{1}{2}$, $x^*_{(1,4)} = \frac{1}{2}$, $x^*_{(2,5)} = \frac{1}{2}$, $x^*_{(2,6)} = \frac{1}{2}$, $x^*_{(3,6)} = \frac{1}{2}$, $x^*_{(3,7)} = \frac{1}{2}$, $x^*_{(4,5)} = \frac{1}{2}$, $x^*_{(4,7)} = \frac{1}{2}$, and $x^*_{(i,j)} = 0$ otherwise, is feasible to LP(D) associated with G and has the objective function value of 5.5.

Skorin-Kapov [1995] derived some technical sufficient conditions under which the incidence vector of a given ST and the optimal solution for LP(D) coincide. Moreover, he constructed a Core Heuristic whose first phase is similar to the Core

Algorithm of *Section 3*. Namely, the feasible solution to the dual of LP(D) is constructed, and then the cost of dual variables is arbitrarily distributed among users in dual sets. The heuristic is trying to construct feasible dual variables that satisfy complementary slackness conditions, and the construction of dual sets heavily depends on the optimal, or best known Steiner tree. This heuristic can be used to allocate a fraction of the total cost while satisfying the core constraints. If the total cost was not allocated by the first phase, then phase II of the core heuristic contains some other complex technical conditions under which sometimes one can allocate even more than the optimal dual value, and still preserve core conditions. It was shown theoretically and experimentally that the portion of the allocated cost that satisfies core constraints can be improved beyond the best value of dual objective. Computational analyses performed on 76 well known (Wong's [1984], Beasley's [1989] and Aneja's [1980]) Steiner tree problems confirmed that the Core Heuristic gives very tight bounds on the amount that can be allocated while satisfying the core constraints. Indeed, in 68 out of the above 76 cases the core heuristic produced core points, and in the remaining cases it provided "good" lower bounds.

5. Capacitated network design game

Although the results presented in *Sections 3* and *4* are encouraging, they are not sufficient to solve even more complex network problems regularly appearing in practice. Namely, networks often have facilities that have limited capacities. This is particularly true for switching points (line capacities often have unlimited capacities for practical needs, for example, fiber optic cables).

Consequently, there is a need to study capacitated network design problems. We now consider the following important class of (telecommunications) capacitated network design problems. The set of users is in need of a certain service that can be provided by connecting them (possibly through other users) to capacitated facilities, yet to be constructed. The objective is to build a network that will provide the above service at minimum cost. Formally, let G= (N,E) be an underlying complete undirected network with a set of nodes N and a set of links E. The set N represents potential users, as well as potential facility sites. For each $i \in N$, let $d_i \geq 0$ be the demand for service at node i. A facility with capacity u_i can be constructed ("opened") at node $i \in N$, at cost $c_i \geq 0$. Note that for some $i \in N$, we can have for example $d_i = 0$ or $c_i = \infty$ meaning, respectively, that user i does not require any service, and that a facility can not be opened at site i. Open facilities provide service required by the users, and any node receiving the service can pass an unused portion of it to adjacent nodes. Each node i should receive the service from a single facility and the facility opened at node i should serve at least i itself. There is a cost, $c((i,j)) = c_{ij} \geq 0$, $(i,j) \in E$, if arc (i,j) is used to deliver service. In the spirit of new technologies (satellites, optical fiber) we assume that links have virtually no capacity limits. Each customer with a positive demand for service should be connected, perhaps through other nodes, to an open facility. The objective is to satisfy the demand for service, by opening facilities and connecting the users to facilities, while satisfying the capacity constraints at minimum cost. We will refer to the above optimization problem as the Capacitated Network Design (CND) problem.

The CND problem generalizes many important problems that appear in the design of communication networks. Among them are: Capacitated Minimum Spanning Tree (CMST) problem, Capacitated Concentrator Location (CCL) problem, Capacitated Fixed Cost Spanning Forest (CFCSF) problem and Capacitated Steiner Tree (CST) problem. With all these problems, naturally arises the problem of allocating the corresponding total minimum cost among customers in a 'fair' manner. As in previous sections, we would like to allocate the cost in such a way that no subset of users would have incentive to secede and build their own network.

The cost allocation problem associated with the CND problem (first formulated in Skorin-Kapov and Beltran [1994]) is concerned with the allocation of the cost incurred by satisfying the users' demand for service. Consider a CND problem on the network G= (N,E). Denote by CND_S, for $S \subseteq N$, the CND problem obtained from the original problem by simply setting demands d_i, for all $i \in N \setminus S$, to zero. The pair (N; c), where $c: 2^N \to R$ is such that $c(\emptyset) = 0$ and for each $S \subseteq N$, c(S) is the minimum objective function value of CND_S, is a cooperative game in characteristic function form, to be referred to as the Capacitated Network Design (CND) game. In the definition of the characteristic function c we assume that the coalition S, while acting on its own, can establish a facility at a node i even if it is not owned by S ($i \notin S$), and can use a link (i,j) even if it is not owned by S (i or j $\notin S$). Similar approach in defining characteristic function was taken in some related uncapacitated network games (for example, FCSF game (Granot and Granot [1992a]) and ST game (Skorin-Kapov [1992,1995])). However, it might be reasonable to argue that nodes out of S will not necessarily allow (at least not free of charge) a coalition S to use nodes and links which are not controlled by S. Consequently, more restrictive model in which coalition S can not use nodes out of S may also be of interest. If such more restrictive approach is taken the CND game would be no longer monotone. Nevertheless, the entire analysis that follows would still hold.

The first work in this direction was presented in Skorin-Kapov [1993]. Therein he considered the cost allocation problem associated with the Capacitated Concentrator Covering (CCC) Problem. That problem is a special case of the CND problem in which the cost of links were ignored, and only the cost of capacitated switching facilities was allocated among users. Clearly, the cost allocation associated with the CCC problem is a very complex combinatorial problem, since the CCC optimization problem is NP-hard. In addition, it can easily be shown that the core of the CCC game (and the CND game) may be empty. Consider for example, the network G= (N,E) consisting of a three node ring with N= {1,2,3} and E= {(1,2),(2,3),(1,3)}. Assume that $c_i = C$, $d_i = 1$, and $U_i = 2$ for every $i \in N$. Now, one can easily verify that the core constraints induced by the two-member coalitions are: (i) $x_1+x_2 \leq C$, $x_1+x_3 \leq C$, $x_2+x_3 \leq C$, which implies that $x_1+x_2+x_3 \leq 1.5C$. On the other hand, the entire cost is $x_1+x_2+x_3 = 2C$. Thus, we conclude that the core of the CCC game associated with G is empty.

Nevertheless, Skorin-Kapov [1993] provided an interesting game theoretic analysis of the CCC game which enabled the computation of certain game theoretic solution concepts in polynomial time. That analysis was later extended (Skorin-Kapov and Beltran [1994]) to the general case of the CND game which considers both, the cost of links and the cost of capacitated facilities.

The main feature of these papers is that the introduction of capacities imposes a limit on the number of users that can be served by a single facility. Throughout the analysis this feature was used to show that several game theoretic solution concepts can be characterized with the same set of constraints (polynomial in size). This further suggested that the computation of the above cost allocation solutions may be feasible for a large class of quite realistic and practical capacitated network design problems. In particular, an efficient representation of the core, which often enables us to test whether the core of a CND game is empty, and generate core points (if they exist) in polynomial time was derived.

Theorem 4. *Let $\mathcal{S} = \{ S \,|\, S$ is a subset of N, such that an optimal solution to CND_S has a single opened facility$\}$. Then, the core of a CND game is given by all cost allocations $x \in R^{|N|}$, satisfying: $c(S) - x(S) \geq 0$, for all $S \in \mathcal{S}$, and $c(N) - x(N) = 0$.*

It is reasonable to assume that capacities u_i , $i \in N$ and demands d_i , $i \in N$ are such that the maximum number of users that can be served by a single facility is bounded with some fixed upper bound K. Then the size of each set $S \in \mathcal{S}$ is bounded by K. Moreover, the size of a family of sets $\mathcal{S}$ is in the worst case bounded by a K-degree polynomial and consequently sets in $\mathcal{S}$ can be generated in polynomial time. This further implies that if c(N) is obtained as an optimal solution to the CND problem, then the nonemptiness of the core of the CND game can be, at least in theory, efficiently tested.

If the core of a CND game is not empty it may consist of many cost allocations which are not equally “attractive“. Consider a simple example of a CND game in which the underlying network G= (N,E) consists of a three node chain, with N= {1,2,3} and E= {(1,2),(2,3)}. Assume that demands are $d_1 = d_2 = d_3 = 1$, potential facility costs are $c_1 = c_2 = c_3 = 2$, link costs are $c_{12} = 0$, $c_{23} = 2$, and capacities of potential facilities are $u_1 = u_2 = u_3 = 2$. It is easy to check that the cost allocations $x_1 = 2$, $x_2 = 0$, $x_3 = 2$ and $x'_1 = 1$, $x'_2 = 1$, $x'_3 = 2$ are both in the core of the game (N;c) associated with G. Clearly, the second solution is preferable to player 1, while the first solution is preferable to player 2.

For the case when the core of a CND problem is not empty we will try to determine an “attractive” point in the core, namely the nucleolus. A method for computing the nucleolus by solving a sequence of linear programming (LP) problems was implicitly suggested by Schmeidler [1969] and then further studied by numerous authors. The k^{th} LP problem, LP_k, solved by this method is:

$$\max \Big\{ \varepsilon : \varepsilon_j = c(S) - x(S), S \in P_j, j = 1, \ldots, k-1, \varepsilon \leq c(S) - x(S), \quad (LP_k)$$
$$S \notin \cup_{j=1}^{k-1} P_j,\ x_i \leq c(\{i\}), i = 1, \ldots, n,\ \ x(N) = c(N) \Big\},$$

where for $j \geq 1$, ε_j and P_j are, respectively, the optimal value and the set of subsets whose corresponding inequality constraints are satisfied as equalities at an optimal solution of LP_j . The nucleolus is obtained at problem LP_i if the optimal solution to LP_i is unique. We will refer to this method as the Linear Programming (LP) procedure for computing the nucleolus.

It was shown by Skorin-Kapov and Beltran [1994] that only a small portion (polynomial in size) of core constraints might be needed in the computation of the nucleolus. Namely, for the family of coalitions $\mathcal{S}$ used in *Theorem 4* the following holds:

Theorem 5. *If the core of a CND game is not empty then the nucleolus of a CND game is completely determined by constraints associated with the family of coalitions $\mathcal{S}$.*

Recall that the nucleolus is unique and it always exists. However, it appears to be difficult to efficiently characterize the nucleolus of a CND game when the core is empty. In the same paper Skorin-Kapov and Beltran [1994] proposed the least weighted ε-core of the CND game as a solution concept for a fair cost allocation associated with the CND problem.

For each coalition S, $S \subseteq N$, let w_S, be the weight associated with S. Then the weighted $\varepsilon-$core is a set of cost allocation vectors such that for all coalitions S, $\emptyset \neq S \subset N$: $c(S) - x(S) \geq w_S\,\varepsilon$ and $c(N) - x(N) = 0$.

The weighted ε-core can be interpreted as the set of efficient cost allocations that can not be improved upon by any coalition S if forming a coalition entails a cost $-w_S\,\varepsilon$. Clearly, the weighted ε-core is not empty if ε is sufficiently small. Let ε' be the largest ε for which the weighted ε-core is not empty. Then the weighted ε'-core is the least weighted ε-core. It appears that ε' is particularly interesting when it is negative, i.e. when the core of a game (N;c) is empty. For $S \subseteq N$, we can think of$-w_S\,\varepsilon'$ as the minimal weighted cost $-w_S\,\varepsilon$, or cross-subsidy, for which the weighted ε-core is not empty. Note that the above cost based subsidization does not use the information about the willingness of a coalition S to pay. Also note that even for the case of subsidy-free cost allocations, there is no generally preferable method of choosing one subsidy-free allocation rather than another (see for example Sharkey [1995]).

It was shown, that with a suitable choice of weights the least weighted ε-core of a CND game can be characterized by the collection of constraints associated with the family of coalitions $\mathcal{S}$. It seems reasonable to assume that the potential cross-subsidy absorbed by the coalition S should be proportional to the size of S or the amount of total demand of users in S. Under such assumption additivity of the weight function is a natural choice.

Let w be a real valued weight function, defined on the partition set of N, and for $S \subseteq N$, let $w(S) = w_S$. We say that the weight function w is *additive*, if for any two subsets $S_1, S_2 \subseteq N$, such that $S_1 \cap S_2 = \emptyset$, we have : $w_{S_1} + w_{S_2} = w_{S_1 \cup S_2}$.

The weights w_i, $i \in N$, should be decided upon on a case by case basis. One reasonable practical suggestion for the choice of a weight function is an additive function w, dependent on nodes demands, defined as follows: for $i \in N$, $w_i = d_i/D$, where $D = \sum_{i \in N} d_i$ is the total demand for service of the entire network.

Theorem 6. *Let w_S, $S \subseteq N$, be the weights generated by an additive weight function w. Then the least weighted ε-core of the CND game (N,c) is completely determined by constraints associated with the collection of coalitions $\mathcal{S}$.*

For a special case, when the above weights in *Theorem 6* are $w_S = |\, S \,|$, for all $S \subset N$, the least weighted ε-core is called the least per capita ε-core.

In order to further analyze the least weighted ε-core we introduce, for an arbitrary ε, the game $(N; c_\varepsilon)$ whose characteristic function c_ε is defined as follows:

$$c_\varepsilon(S) = \begin{cases} \mathrm{c(S)} - \mathrm{w_S}\,\varepsilon, & \text{if } \emptyset \neq \mathrm{S} \subset \mathrm{N} \\ \mathrm{c(S)}, & \text{if } \mathrm{S} = \mathrm{N} \end{cases}$$

Observe that the core of the game (N;c_ε) coincides with the weighted ε-core of the game (N;c). Let ε' be the largest ε for which the game (N;c_ε) has a nonempty core. Then the core of a game (N;$c_{\varepsilon'}$) coincides with the least weighted ε-core of a game (N;c). A sensible way of selecting a unique point from the least weighted ε-core is selecting the nucleolus of the game (N;$c_{\varepsilon'}$). For example, when the core is empty, under the assumption that every coalition has agreed to participate in a cross-subsidy with additional cost $-\mathrm{w_S}\,\varepsilon'$, such a cost allocation would lexicographically maximize minimal excess.

It appears that the nucleolus of a game (N;$c_{\varepsilon'}$) can also be characterized by the set of constraints associated with the collection $\mathcal{S}$.

Theorem 7. *Let* $\mathrm{w_S}$, $\mathrm{S} \subseteq \mathrm{N}$*, be weights generated by an additive weight function w. Assume that the core of a game (N;c) is empty, and let* ε' *be the largest* ε *for which the weighted* ε*-core of a game (N;c) is not empty. Then the nucleolus of a corresponding game (N;*$c_{\varepsilon'}$*) is completely determined by constraints induced by the collection of coalitions* $\mathcal{S}$*.*

Observe that if $\mathrm{w_S} = |\mathrm{S}|$, for all $\mathrm{S} \subseteq \mathrm{N}$, and if the nucleolus of (N;$c_{\varepsilon'}$) is the unique solution to the first linear program in the LP procedure for computing the nucleolus, then the nucleolus of the game (N;$c_{\varepsilon'}$) coincides with the nucleolus per capita of the game (N;c).

6. Some concluding remarks

Due to globalization and overall networking trends and today's technology developments the research in the area of cost allocation in communication network is becoming more important. However, this kind of research is hampered with the combinatorial difficulty of associated computational problems. Moreover, there are no definite answers to what could be considered a "fair" cost allocation (for more detailed discussion see Sharkey [1995]).

Please note that the overview of cost allocation problems and solutions presented in this paper is not exhaustive. It is mostly concentrated on classes of networks that posses tree structure. Moreover, it is mostly concerned with authors recent contributions to the field. In order to better inform the reader about the state of knowledge in the area of cost allocation in networks, we will mention in the sequel several other contributions and their relation to the work presented in this paper.

An important class of games that generalizes MCST games is the class of spanning network games (see. Granot and Maschler [1991], D. Granot and F. Granot [1992a], Feltkamp et. al. [1994], Skorin-Kapov [1995]). Nouweland et al. [1993] showed that the subclass of spanning network games coincides with monotonic games. The spanning network game is once again concerned with the allocation of the cost of a certain service that is provided to users inhabited at network nodes. The main difference from the MCST game is that in the spanning network games, some nodes may be switching points and in addition to the cost of links the cost of nodes might be considered.

A fruitful approach in the analysis of the cores of the above network games, just like in *Section 3* and *4* of this paper, is to seek a polynomial representation of such games in the form of linear production games. The core of a linear production game is a superset of the associated dual set. For example, this approach was also used to analyze the core of the Uncapacitated Plant Location game on trees (Kolen [1983], Kolen and Tamir [1990]), Location games (Granot [1987], Tamir [1992]), the Synthesis game (Tamir [1991]), and the MCST game (Granot [1986]). There are some special cases in which the core of a linear production game coincides with the associated dual set. Specifically, these cases are the assignment games (Shapley and Shubik [1972]), simple network flow games (Kalai and Zemel [1982]) and location games (Tamir [1992]). Note that if the core properly contains the dual set, then to verify whether a given vector is in the core is NP-complete (Chvatal [1978]). On the other hand, if the core coincides with the dual set then the core membership can be verified in polynomial time.

In principal, it seems that a 'good' polyhedral representation of the optimization problem paves the way for a 'good' cost allocation heuristic. It was shown in *Section 4* that a 'good' feasible dual solution of the optimization problem can be used to generate lower bounds for the part of the cost that can be allocated while satisfying the core constraints. Skorin-Kapov [1995] used this theoretical insight to construct an efficient heuristic for producing good feasible dual solutions for the Steiner tree problem on general networks, and then used those solutions to generate point(s) in the core or close to it. Samet and Zemel [1984] and Dubey and Shapley [1984] describe related results on the core and dual set of linear programming games on the more general convex programming game. Linear programming games (see, Owen [1982]) are well studied in the literature. However, most of the network optimization problems discussed herein require integrality of decision variables. That makes their linear programming relaxation only a starting tool. In a work on shared facility game, Sharkey [1990] defines a related game (referred therein as a simple game), and shows that the core of that game is nonempty if and only if the optimal values of the respective objective functions of the associated integer program and linear program coincide. Similar results are shown for the Fixed Cost Spanning Forest (FCSF) game on a tree (D. Granot and F. Granot [1992a]), and for the FCSF game on a series parallel graph (Skorin-Kapov [1992]).

Cost allocation solutions associated with capacitated network design problems were analyzed by Skorin-Kapov [1993], and Skorin-Kapov and Beltran [1994a,b] (and summarized in this paper *Section 5*). Therein they show that the characterization of the least ϵ-core and the computation of the nucleolus is feasible even for quite complex class of capacitated network problems.

In the cost allocation problems associated with network flow problems, the duality theory of LP is represented with the maximum flow minimum cut theorem of Ford and Fulkerson [1962]. For the basic network flow model see Shapley [1961] and for the analysis of the core of the associated cooperative flow games see Kalai and Zemel [1982a, 1982b], Granot and Hojati [1990], D. Granot and F. Granot

[1992b] and Tamir [1992]. Some other related cost allocation models were studied in communications and transportation networks. Granot and Hojati [1990] considered a simultaneous and non-simultaneous network synthesis game, and Sharkey [1992] defined a game closely related to a simultaneous network synthesis game. Tamir [1989] and Potters, Curiel and Tijs [1987] analyzed the traveling salesman cost allocation problem. Bittlingmayer [1990] and Woroch [1995] analyzed the triangular network as a model of an airline's hub and spoke network, Derks and Kuipers [1992] investigated the cost allocation associated with the routing problems. The allocation of value for jointly provided services in telecommunication network was studied by Linhart et. al. [1995]. The cost allocation in hub networks was studied by Skorin-Kapov [1998].

Most of the above studies are concerned with attempts to characterize the core. However, the core of the associated game may be empty, or the core may contain points considered 'unfair' by some users. In several network cost allocation papers, some other cost allocation solutions were analyzed. For example, the nucleolus (Granot [1984], Granot and Maschler [1991], Derks and Kuipers [1992]), the Shapley value (Myerson [1977], Owen [1986], Granot and Hojati [1990]) and the weighted least ε-core and nucleolus per capita (Skorin-Kapov [1993]). The sufficient conditions for the existence of the population monotonic refinement of the core allocation for cooperative games with transferable utility was studied by Sprumont [1990].

Clearly, the existing exact polynomial algorithms and characterizations of cost allocation solutions developed for the above network problems are not sufficient to handle the complexity of realistic cases of the cost allocation problem in general communication networks. Yet, they would be a basis for the development of heuristic algorithms. In addition, to overcome certain computational difficulties it is worth seeking approximations or useful modifications of cost allocation solutions we wish to compute.

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