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A UNIFORMLY ACCURATE FINITE ELEMENTS METHOD FOR SINGULAR PERTURBATION PROBLEMS

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Abstract. We consider piecewise polynomial finite elements method for a singular perturbation problem. The finite elements method of [9] for a problem with non-constant coefficients was adapted by introducing piecewise polynomial approximation. We generate the tridiagonal difference schemes which are second order accurate in uniform norm.

1. Introduction

Ordinary differential equations with a small parameter ϵ multiplying the highest-order derivative terms are called singularly perturbed equation. We consider numerical methods which are ϵ-uniform of order p on the mesh Δ_n : $x_0 < x_1 < ... < x_n$, $h_i = x_i - x_{i-1}$, $i = 1, ..., n$ if there exists some h_0 independent of ϵ such that for all $h_j \leq h_0$, $\sup_{0<\epsilon\leq 1} \max_{\Delta_n} |u(x_i) - v_i| \leq M \max_{0\leq j\leq i} h_j^{-p}$ where u is the solution of the differential equation, v_i is the computed value to u, and M and p are independent of ϵ and n (cf. [11, 10]).

Petrov Galerkin method (PGM) is an old well established tool for solving linear and nonlinear ODEs and PDEs. Description of Galerkin method in Sobolev spaces, its qualitative analysis, energy estimates for different types of PDEs and their transformation into a system of ODEs, existence and uniqueness theorems for the solutions for some classes of PDEs can be found in, say, [8]. Recall it.

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We intend to find a weak solution to PDE, say of parabolic type

$$\begin{cases} u_t + Du = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times [0,T], \\ u = g & \text{on } \Omega \times \{t=0\}, \end{cases} \tag{1.1}$$

where D is an elliptic operator, first by constructing solutions of certain finite-dimensional approximation and then passing to the limits. Assume the functions $w_k = w_k(x)$, $(k = 1, ..)$ are smooth, $\{w_k\}_{k=1}^{\infty}$ is an orthogonal basis of $H_0^1(\Omega)$, Ω is an open set in $\mathbf{R}^n$ and $\{w_k\}_{k=1}^{\infty}$ is an orthonormal basis of $L^2(\Omega)$. Fix m as a positive integer. Find a function $u_m : [0,T] \to H_0^1(\Omega)$ of the form

$$u_m(t) := \sum_{k=1}^{m} d_m^k(t) w_k, \tag{1.2}$$

where we choose $d_m^k(t)$, $(0 \leq t \leq T)$, $(k = 1, ..., m)$ such to satisfy the initial condition

$$d_m^k(0) = (g, w_k),\ k = 1, ..., m, \tag{1.3}$$

and the weak form

$$(u_m', w_k) + B[u_m, w_k; t] = (f, w_k),\ \ 0 \leq t \leq T,\ k = 1, ..., m, \tag{1.4}$$

where B is the corresponding bilinear form. Thus, we look for a function u_m of the form (1.2) that satisfies the projection (1.4) of our problem into the finite dimensional subspace spanned by $\{w_k\}_{k=1}^{m}$. This transforms our problem to a system of ordinary differential equations

$$\begin{cases} d_m^k(0) = (g, w_k), & k = 1, ..., m, \\ (d_m^k(t))' + \sum_{l=1}^{m} e^{kl}(t) d_m^l(t) = f^k(t), & k = 1, ..., m, \end{cases} \tag{1.5}$$

where $f^k(t) := (f(t), w_k)$, $k = 1, ..., m$, $e^{kl}(t) := B[w_l, w_k; t]$. Standard existence theory of ODEs gives a unique absolutely continuous function $d_m(t) = (d_m^1(t), ..., d_m^m(t))$ satisfying (1.4) and (1.5) for a.e. $0 \leq t \leq T$. Then, (1.2) solves (1.4). So, we construct the approximate solutions. For each integer $m = 1, ...$ there exists a unique function u_m of the form (1.2) satisfying (1.3) and (1.4). Then we set $m \to \infty$ and show that the subsequence of solutions u_m of the approximate problems (1.3),(1.4) converges to a weak solution to (1.1). We pass to the limits as $m \to \infty$, to build a weak solution to PDE (1.1). Such solution exists.

The development of this method with applications in numerical analysis of singularly perturbed problems takes place 2D. In [12] is given stabilized Galerkin method with a high accuracy away from the boundary and interior layers where solution is smooth. Evaluation of this method goes towards finite elements method (FEM) to PDEs, especially to stabilized Galerkin method [12] for second order advection-diffusion-reaction model with the emphases on singularly perturbed problems appearing in the iterative solution of coupled incompressible Navier-Stokes problems (cf. [5, 15, 16]). It gives resolution of boundary layers on layer-adopted meshes using anisotropic interpolation estimates and sharp estimates of derivatives. Standard finite elements Galerkin method is applied to singularly perturbed convection-diffusion problems in 2D, using adaptive refinement procedure for meshes which overcomes many difficulties in getting higher order accuracy. Recently, three books appeared dealing with the numerical solution of singularly perturbed problems [17, 19, 24]. Also combination of different meshes equidistant away from the boundary layer and non-equidistant in boundary layer and combination with standard FEM are of great use (cf. [21, 22, 25]). Adaptive local mesh refinement Galerkin FEM procedure is applied for vector system of parabolic PDEs in 1D in [20]. The development of this area goes to different schemes for boundary layer and away from it with rectangular meshes and their combinations in 2D. Also, the fitted difference operators on piecewise uniform, say Shishkin meshes, are used in [18]. Then, the solutions with two sharp layers are considered: boundary layer and spike layer solution [6]. Families of rectangular 2D and 3D mixed FEM for the approximation of acoustic wave equations are given in [3].

Thus, the main directions in PG method continuous and discontinues are towards the solutions of Navier-Stokes equations the problem of new millennium because of its great importance for practice.

In this paper we use PGM to construct a difference scheme which approximates the solution to the problem (2.1) with appropriate accuracy. Classical methods are appropriately adapted to singularly perturbed problems in one dimension space.

We revise PGM in 1D for singular perturbation two-point boundary value problem in order to obtain $(O(h^2))$ uniformly accurate convergent scheme.

In [23] El-Mistikawy and Werle difference scheme was derived by using PGM of finite elements. For the test and trial space the exponential functions were used which are the solution of the "comparison" problem $\epsilon u'' + \bar{p}_i u' = \bar{f}_i$ and its adjoint on each subinterval $[x_{i-1}, x_i]$, where $\bar{p}_i$, $\bar{f}_i$ are piecewise constants. Using the properties of distributional derivatives they proved the second order of uniform convergence of this scheme. In [1], was given an analysis of PGM of finite elements as applied to the asymmetric two-point boundary value problem with constant coefficients, for the space of a hat trial functions and for a family of test spaces involving parameter α. We return once again to classical PGM in order to improve the result for non-constants coefficients. We try to find solution of the non-self adjoint problem using polynomial method of finite elements. For this purpose we use the same test and trial spaces as in [1], but for the approximation of the functions $p(x)$, $f(x)$ we use the piecewise polynomials:

1. the approximation by piecewise quadratics at the points x_{i-1}, $x_{i-1/2}$, x_i;
2. the approximation by piecewise cubic polynomial which is interpolated at both end points and two interior one of subinterval $[x_{i-1}, x_i]$.

We adopt this method to the exponential feature of the exact solution by introducing parameter α_i whose form is exponential and which follows the exponential form of the exact solution. We try to improve this result by introducing better approximation for the functions $p(x)$, $f(x)$ but the order of uniform convergence still remains two because we introduced the same fitting factor α_i obtained as a solution of $Lu = 0$ for $p = \text{const}$ in both cases. Special discretization of non-self adjoint perturbation equation by finite elements leads to a system of linear equations. The obtained difference schemes are second order accurate in uniform norm.

The advantage of this method is its simplicity and explicit form of the solution. We solve the tridiagonal system of linear equations and due to (2.3) the given solution has a global character.

The paper is organized as follows: In the first part we give a description of PGM of finite elements and its polynomial form. Then, we generate two difference schemes by using different approximation for driving terms. We give in the second section proof of the uniform convergence of these schemes. Finally, we give some numerical tests to confirm the theoretical predictions.

Throughout the paper M, δ, β denote different constants independent of the mesh size h and perturbation parameter ϵ; $p(x_i)$ and $f(x_i)$ are exact values at the point x_i; p_i and f_i are computed values; Q is the negligible part in the error estimate.

2. Scheme generation

Consider the non-self adjoint perturbation problem

$$Lu \equiv \epsilon u'' + p(x)u' = f(x), \quad u(0) = \gamma_0, \quad u(1) = \gamma_1, \tag{2.1}$$

the functions $p, f \in C^2([0,1])$, $p(x) \geq \bar{p} > 0$, $x \in (0,1)$, $0 < \epsilon << 1$, γ_0, γ_1 being given constants.

$u \in H^1((0,1))$ is a solution to (2.1) if and only if it is a solution of the weak form to (2.1)

$$u \in H^1((0,1)) \text{ and } B_\epsilon(u, w) = (u', -\epsilon w' + pw) = (f, w) \tag{2.2}$$

where $(,)$ is the usual inner product in $L^2([0,1])$.

Choosing in $H^1([0,1])$ two finite subspaces T^h and S^h of equal dimension referred to as test and trial space, we obtain the following PG discretization to (2.1). Let $\{\eta_i | i = 0(1)n\}$ and $\{\zeta_i | i = 1(1)n-1\}$ be the basis in the trial and the test space T^h and S^h respectively.

The unit interval is subdivided into n equal elements by the nodes $x_i = ih$ $(i = 0(1)n)$, and so $nh = 1$. For trial space we take the space of linear (hat) functions

$$\eta_i(x) = \eta(x/h - i), \;\; i = 0(1)n$$

where

$$\eta(s) = \begin{cases} 0, & |s| > 1, \\ 1+s, & -1 \le s \le 0, \\ 1-s, & 0 \le s \le 1. \end{cases}$$

For the test space we use a family spaces involving parameter α_i :

$$\zeta_i(x) = \eta_i(x) + \alpha_i \sigma(x/h - i), \;\; i = 1(1)n - 1$$

where

$$\sigma(s) = \begin{cases} 0, & |s| > 1, \\ -3s(s-1), & -1 \le s \le 0, \\ -\sigma(-s), & 0 \le s \le 1. \end{cases}$$

These test and trial spaces are taken from [9] and discussed in [4]. The functions in the test and trial spaces satisfy the following properties:

1. $\mathrm{supp}(\eta_i(x)) = [x_{i-1}, x_{i+1}]$;
2. $\eta_i(x_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol;
3. $\sum_{j=1}^{n-1} \eta_j(x) = 1$, for all $x \in [x_1, x_{n-1}]$.

We seek an approximation $\{u^h\}$, applying PGM of the form

$$u^h = \sum_{i=0}^{n} v_i \eta_i, \;\; v_0 = \gamma_0, \;\; v_n = \gamma_1, \tag{2.3}$$

where $\{v_i\}$, $i = 0(1)n$ satisfies the system of equations

$$\sum_{k=i-1}^{i+1} v_k B_\epsilon(\eta_k, \zeta_i) = (f, \zeta_i), \;\; i = 1(1)n - 1. \tag{2.4}$$

Case (a) : Replace p and f in (2.4) by $\bar{p}$ and $\bar{f}$ where $\bar{p} = \bar{p}_i$, $\bar{f} = \bar{f}_i$ on interval $[x_{i-1}, x_i]$, $i = 1(1)n$, where

$$\begin{aligned} \bar{f}_i = & f_{i-1}\Big(2/(h^2)(x - x_{i-1/2})(x - x_i)\Big) \\ & + f_i\Big(2/(h^2)(x - x_{i-1/2})(x - x_{i-1})\Big) \\ & + f_{i-1/2}\Big(- 4/(h^2)(x - x_{i-1})(x - x_i)\Big), \end{aligned}$$

$\bar{p}_i$ analogously. Hence, to (2.4) is then associated difference scheme

$$\begin{aligned} & r_i^- v_{i-1} + r_i^c v_i + r_i^+ v_{i+1} = \\ & \qquad h^2/\epsilon(q_i^- f_{i-1} + q_i^c f_i + q_i^+ f_{i+1} + f_{i-1/2} q_{i1/2}^- + f_{i+1/2} q_{i1/2}^+), \end{aligned} \tag{2.5}$$

or in shortened form $Rv_i = h^2/\epsilon QLv_i$, $i = 1(1)n-1$, $v_0 = \gamma_0$, $v_1 = \gamma_1$, where the corresponding coefficients are

$$r_i^- = 1 - \rho_{i-1}(\alpha_i/20) - \rho_i(1/6 + \alpha_i/20) - \rho_{i-1/2}(1/3 + 2/5\alpha_i)$$
$$r_i^+ = 1 + \rho_i(1/6 - \alpha_{i+1}/20) - \rho_{i+1}(\alpha_{i+1}/20) + \rho_{i+1/2}(1/3 - 2/5\alpha_{i+1})$$

$$r_i^c = -r_i^- - r_i^+, \quad q_i^- = \alpha_i/20, \quad q_i^c = (1/6 + \alpha_i/20) + (1/6 - \alpha_{i+1}/20),$$

$$q_i^+ = -\alpha_{i+1}/20, \quad q_{i1/2}^- = 1/3 + 2/5\alpha_i, \quad q_{i1/2}^+ = 1/3 - 2/5\alpha_{i+1}, \ \rho_i = p_i h/\epsilon,$$

where v_i denotes approximating solutions to (2.1) obtained by (2.4).

Case (b). Set in (2.4) instead of f and p the interpolation cubic polynomials $\bar{p}$ and $\bar{f}$ where $\bar{p} = \bar{p}_i$, $\bar{f} = \bar{f}_i$ on $[x_{i-1}, x_i]$, $i = 1(1)n$, such that $\bar{f}_i$ has the following form:

$$\begin{aligned}\bar{f}_i =& \Big(9/(2h^3)(x - x_{i-1})(x - x_{i-1/3})(x - x_{i-2/3})\Big) f_i \\ &- \Big(9/(2h^3)(x - x_{i-1/3})(x - x_{i-2/3})(x - x_i)\Big) f_{i-1} \\ &+ \Big(27/(2h^3)(x - x_{i-1})(x - x_{i-2/3})(x - x_i)\Big) f_{i-1/3} \\ &- \Big(27/(2h^3)(x - x_{i-1})(x - x_{i-1/3})(x - x_i)\Big) f_{i-2/3},\end{aligned}$$

$\bar{p}_i$ analogously. The difference scheme is now $Rv_i = h^2/\epsilon QLv_i$, $i = 1(1)n-1$, $v_0 = \gamma_0$, $v_1 = \gamma_1$, or

$$\begin{aligned}r_i^- v_{i-1} + r_i^c v_i + r_i^+ v_{i+1} =& h^2/\epsilon(q_i^- f_{i-1} + q_i^c f_i + q_i^+ f_{i+1} + f_{i-1/3} q_{i1/3}^- \\ &+ f_{i-2/3} q_{i2/3}^- + f_{i+1/3} q_{i1/3}^+ + f_{i+2/3} q_{i2/3}^+),\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}r_i^- =& 1 - \rho_i(13/120 + \alpha_i/40) - \rho_{i-1}(1/60 + \alpha_i/40) \\ &- \rho_{i-1/3}(3/40 + 9\alpha_i/40) - \rho_{i-2/3}(3/10 + 9\alpha_i/40) \\ r_i^+ =& 1 + \rho_{i+1}(1/60 - \alpha_{i+1}/40) + \rho_i(13/120 - \alpha_{i+1}/40) \\ &+ \rho_{i+1/3}(3/10 - 9\alpha_{i+1}/40) + \rho_{i+2/3}(3/40 - 9\alpha_{i+1}/40) \\ r_i^c =& - r_i^- - r_i^+\end{aligned}$$

$$q_i^c = (13/120 + \alpha_i/40) + (13/120 - \alpha_{i+1}/40), \quad q_i^- = 1/60 + \alpha_i/40$$

$$q_{i1/3}^- = 3/40 + 9\alpha_i/40, \quad q_{i2/3}^- = 3/10 + 9\alpha_i/40, \quad q_i^+ = 1/60 - \alpha_{i+1}/40$$

$$q_{i1/3}^+ = 3/10 - 9\alpha_{i+1}/40, \quad q_{i2/3}^+ = 3/40 - 9\alpha_{i+1}/40, \quad \rho_i = p_i h/\epsilon.$$

3. Determination of the parameter

We determine parameter α_i such that the truncated error to (2.5) (resp. (2.6)) would be equal to zero for $p = \text{const}$, i.e. $\tau_i(u) = Ru_i - QLu_i = 0$, on each subinterval $[x_{i-1}, x_i]$, $i = 1(1)n$. We set

1^0. $\quad \alpha_i = 2/\tilde{\rho}_i - \coth(\tilde{\rho}_i/2)$, $\ \tilde{\rho}_i = (p_i + p_{i-1})h/(2\epsilon)$, $\ p_i = p(x_i)$.

By requiring that $r_i^-/r_i^+ = \exp(-\rho_0)$, $\rho_0 = p_0 h/(2\epsilon)$, $p_0 = p(0)$, we obtain

2^0. $\quad \alpha_{1i} = (1 - \rho_i/2 - \exp(-\rho_0)(1 + \rho_i/2))/(1 - \exp(-\rho_i))$, $\ i = 1(1)n-1$.

The both give the same rate of uniform convergence, but the second one gives the better approximation to the exact solution.

4. Proof of the uniform convergence

Consider two comparison functions $\varphi_i = -2 + x_i$ and $\psi_i = -\exp(-\beta x_i/\epsilon)$ (cf. [13]).

Lemma 4.1 ([2]). *Let $p, f \in C^2([0,1])$. Then the solution to (2.1) has the form $u(x) = u_0(x) + w_0(x)$ where*

$$u_0(x) = -\epsilon u'(0)\exp(-p(0)x/\epsilon)/p(0), \quad \gamma = \gamma(\epsilon) = -\epsilon u'(0)/p(0),$$

$$|w_0^{(i)}(x)| \le M(1 + \epsilon^{-i+1}\exp(-2\delta x/\epsilon)), \quad i = 0(1)4, \quad |\gamma| \le M, \tag{4.1}$$

i.e.

$$|u(x)| \le \exp(-p(0)x/\epsilon) + |w_0(x)|.$$

Lemma 4.2 ([1], Maximum principle). *Let $\{v_i\}$ be a set of values at mesh points x_i, satisfying $v_0 \le 0$, $v_n \le 0$ and $Rv_i \ge 0$, $i = 1(1)n-1$. Then, $v_i \le 0$, $i = 0(1)n$.*

Let $z_i = u(x_i) - v_i$ where v_i is computed solution and $u(x_i)$ is the exact one at mesh point x_i and let τ_i be the truncated error to (2.5) (resp. (2.6)). Then, the truncated error of the difference scheme is $\tau_i = Ru_i - h^2/\epsilon QLu_i$, $i = 0(1)n$.

Corollary 4.3. *If $k_1(h,\epsilon) \ge 0$, $k_2(h,\epsilon) \ge 0$ are such functions that $R(k_1\varphi_i + k_2\psi_i) \ge R(\pm z_i) = \pm\tau_i$, then $|z_i| \le k_1|\varphi_i| + k_2|\psi_i|$.*

Lemma 4.4. *Let $p \in C^2([0,1])$. There are constants M and β such that the following inequalities hold:*

1. $R\varphi_i \ge Mh^2/\epsilon$, *when* $h < 1$, $\epsilon \in (0,1]$;
2. $R\psi_i \ge M\mu^i(\beta)(h/\epsilon)\min(h/\epsilon, 1)$,

where $\mu^i(\beta) = \exp(-\beta x_i/\epsilon)$, ($\beta$ is a constant which will be chosen).

Proof. (a) We have: $R\varphi_i = h(r_i^+ - r_i^-)$. When $h/\epsilon \le 1$ after Taylor expansion at the point $\rho_i = p_i h/\epsilon$, where $p_i = p(x_i)$ on each subinterval $[x_{i-1}, x_i]$, $i = 1(1)n$, we obtain $|R\varphi_i| \ge h\rho_i + O(h^4/\epsilon^3)$, i.e. $|R\varphi_i| \ge Mh^2/\epsilon$.

When $h/\epsilon \geq 1$ we have $|R\varphi_i| \geq Mh^2/\epsilon$. Thus, $|R\varphi_i| \geq Mh^2/\epsilon$ for $h < 1$, $\epsilon \in (0, 1]$.

$(b) R\psi_i = \mu^{i-1}(\beta) r_i^+ (1 - \mu(\beta))(\mu(\beta) - r_i^-/r_i^+)$.

Since $r_i^-/r_i^+ = \exp(-\rho_i) + O(h^2/\epsilon)$, $|\mu(\beta) - r_i^-/r_i^+| \geq Mh/\epsilon \exp(-\theta h/\epsilon)$, $0 < \theta < 1$. $r_i^+ = 1 + \rho_i/2 + \rho_i^2/12 + O(\rho_i^4)$, $|r_i^+| \geq M$, we obtain $|R\psi_i| \geq \mu(h^2/\epsilon^2)\mu^{i-1}(\beta)$, when $h/\epsilon \leq 1$. In the opposite case we have $|r_i^+| \geq Mh/\epsilon$, $|1 - \mu(\beta)| \geq M$, $|\mu(\beta) - r_i^-/r_i^+| \geq M$. Then, $|R\psi_i| \geq Mh/\epsilon$. ◻

THEOREM 4.5. *Let $p, f \in C^2([0, 1])$, and let $\{v_i\}$, $i = 1(1)n - 1$ be the set of computed values for the solution to (2.1) obtained by (2.5). Then, the following inequality holds*

$$|z_i| \leq Mh^2, \ \ i = 0(1)n.$$

PROOF. Truncated error to (2.5) is according to Lemma 4.1 sum of the truncated errors of dividing functions, $\tau_i(u) = \tau_i(u_0) + \tau_i(w_0)$. We shall estimate the first contribution to the nodal errors of the function w_0. Since τ_i is the linear operator then $\tau_i(w_0) = Rw_{0i} - h^2/\epsilon QLw_{0i}$. Set in $Rw_{0i} = h^2/\epsilon QLw_{0i}$, $f_{i\pm1} = \epsilon w''_{0i\pm1} + p_{i\pm1}w'_{0i\pm1}$, and $f_{i\pm1/2} = \epsilon w''_{0i\pm1/2} + p_{i\pm1/2}w'_{0i\pm1/2}$, expend into Taylor series each term w_0 at the point $w_0(x_i)$ and collect the terms followed the same derivatives. The appropriate expansion for $\tau_i(w_0)$ obtained by Taylor expansion of w_0 at the point x_i has the form

$$\tau_i(w_0) = \tau_i^{(0)} w_0(x_i) + \tau_i^{(1)} w'_0(x_i) + \tau_i^{(2)} w''_0(x_i) + \tau_i^{(3)} w'''_0(x_i) + ... + R$$

where $\tau_i^{(0)} = r_i^- + r_i^c + r_i^+$,

$$\begin{aligned}
\tau_i^{(1)} =& h(r_i^+ - r_i^- - \rho_{i-1}q_i^- - \rho_i q_i^c - \rho_{i+1}q_i^+ - \rho_{i+1/2}q_{i1/2}^+ - \rho_{i-1/2}q_{i1/2}^-),\\
\tau_i^{(2)} =& h^2\Big(1/2(r_i^+ + r_i^-) + \rho_{i-1}q_i^- - \rho_{i+1}q_i^+ + 1/2\rho_{i-1/2}q_{i1/2}^-\\
& - 1/2\rho_{i+1/2}q_{i1/2}^+ - q_i^- - q_i^+ - q_i^c - q_{i1/2}^- - q_{i1/2}^+\Big),\\
\tau_i^{(3)} =& h^3\Big(1/6(r_i^+ - r_i^-) - q_i^+ + q_i^- - 1/2(q_{i1/2}^+ - q_{i1/2}^-) - 1/2(\rho_{i-1}q_i^-\\
& + \rho_{i+1}q_i^+ + 1/4(\rho_{i+1/2}q_{i1/2}^+ + \rho_{i-1/2}q_{i1/2}^-))\Big),\\
R =& \tau_i^{(n+1)} w_0^{(n+1)}(\xi_i), \ x_{i-1/2} \leq \xi_i \leq x_{i+1/2}, \ i = 1(1)n - 1.
\end{aligned}$$

In our case $\tau_i^{(0)} = \tau_i^{(1)} = 0$. For $p = \text{const}$, $\tau_i^{(2)} = 0$. Setting the coefficients in $\tau_i^{(2)}$ we obtain $\tau_i^{(2)} = h^2\{\alpha_{i+1}/40(\rho_{i+1} - \rho_i) + \alpha_i/40(\rho_{i-1} - \rho_i) + 1/2(\alpha_{i+1} - \alpha_i)\}$. Since $|\rho_{i\pm1} - \rho_i| \leq Mh^2/\epsilon$, $|\alpha_{i+1} - \alpha_i| \leq Mh^2/\epsilon$ and $|\alpha_{i+1}| \leq M$, we obtain when $h/\epsilon \geq 1$ that $|\tau_i^{(2)}| \leq Mh^4/\epsilon$. When $h/\epsilon \leq 1$ we have

$$\begin{aligned}
\tau_i^{(2)} =& h^2\{\alpha_i/40(\rho_{i+1} - \rho_{i-1}) + 1/40(-1/6 + \rho_i/120)(\rho_{i+1} - \rho_i)^2\\
& + 1/2(\rho_{i+1} - \rho_i)(-1/6 + \rho_i^2/120)\} + Q.
\end{aligned}$$

Since $\alpha_{i+1} = \alpha_i + (\rho_{i+1} - \rho_i)\frac{\partial \alpha_{i_{i+1}}}{\partial \rho_{i+1}}(\rho_i) = -\rho_i/6 + 1/200\rho_i^3 + (\rho_{i+1} - \rho_i)(-1/6 + 1/120\rho_i^2) + Q$ and $\alpha_{i+1} - \alpha_i = (\rho_{i+1} - \rho_i)(-1/6 + 1/120\rho_i^2) + q$, $|\rho_{i\pm 1} - \rho_i| \leq Mh^2/\epsilon$, $|\rho_{i+1} - \rho_{i-1}| \leq Mh^2/\epsilon$, $\alpha_i = -\rho_i/6 + \rho_i/200 + Q$ we have $|\tau_i^{(2)}| \leq Mh^4/\epsilon$, when $h/\epsilon \leq 1$. Then, by (4.1)

$$|\tau_i^{(2)} w_0''| \leq Mh^4/\epsilon(1 + \epsilon^{-1} \exp -2\delta x_i/\epsilon),\ h < 1,\ \epsilon \in (0, 1].$$

Setting the coefficients of the scheme into $\tau_i^{(3)}$ we obtain that $\tau_i^{(3)} \equiv h^3(\rho_i/12 + \alpha_i/2)$, where $\rho_i = p_i h/\epsilon + Q$, $p_i = \text{const}$. When $h/\epsilon \geq 1$, $|\tau_i^{(3)}| \leq Mh^4/\epsilon$ and when $h/\epsilon \leq 1$, after Taylor expansion of $\coth \rho_i = 1/\rho_i + \rho_i/3 - \rho_i^3/45 + Q$ in α_i and after ordering the terms we obtain $\tau^{(3)} = h^3\rho_i^3/720 + O(h^8/\epsilon^5)$. Thus, $|\tau_i^{(3)}| \leq Mh^6/\epsilon^3$ for $h/\epsilon \leq 1$. Similarly we have that the remainders are of the lower order.Thus,

$$|\tau_i(w_0)| \leq Mh^4/\epsilon(1 + \epsilon^{-1} \exp(-2\delta x_i/\epsilon)),\ h < 1,\ \epsilon \in (0, 1].$$

The last inequality, Lemma 4.4 (1) and Corollary 4.3 give the contribution to the nodal errors due to the function w_0:

$$|z(w_0)| \leq Mh^2(1 + \epsilon^{-1} \exp(-2\delta x_i/\epsilon))\ h < 1,\ \epsilon \in (0, 1]. \tag{4.2}$$

Now, we shall consider the nodal errors due to the boundary layer function u_0. We determined α_i so that the truncated error for boundary layer function equals zero when $p(x) = \text{const}$. Denote the parts in the truncation error Ru_i and $h^2/\epsilon Q L u_i$ by τ_r and τ_q respectively. We have $\tau_i(u_0) = \tau_r - \tau_q$ and

$$\begin{aligned}
\tau_r =& u_{0i}\{r_i^-(\exp(\rho_0) - 1) + r_i^+(\exp(-\rho_0) - 1)\},\\
\tau_q =& u_{0i}\rho_0\{(\rho_0 - \rho_{i-1})q_i^- \exp(\rho_0) + (\rho_0 - \rho_i)q_i^c + (\rho_0 - \rho_{i+1})\exp(-\rho_0)q_i^+\\
&+ (\rho_0 - \rho_{i-1/2})q_{i1/2}^- \exp(\rho_0/2) + (\rho_0 - \rho_{i+1/2})q_i^+ \exp(-\rho_0/2)\}
\end{aligned}$$

where $u_{oi} = \exp(-p(0)x_i/\epsilon)$. When $h/\epsilon \geq 1$ since $|q_i^{\pm c}| \leq M$, $|r_{i1/2}^{\pm}| \leq M$, and since by mean value theorem $|p_0 - p_{i\pm 1}| \leq Mx_{i\pm 1}$, $|p_0 - p_{i\pm 1/2}| \leq Mx_{i\pm 1/2}$ and $(x/\epsilon)^k \exp(-cx/\epsilon) \leq M, C > 0, x \geq 0$ we obtain

$$|\tau_q| \leq Mh^2/\epsilon \exp(-\delta x_{i-1}/\epsilon),\ h/\epsilon \geq 1.$$

Recall, when $p_i = \text{const}$ we have $r_i^-/r_i^+ = \exp(-\rho_i)$, $\rho_i = p_i h/\epsilon$ and $\tau_i(u_0) = 0$. Since $\tau_r = u_{0i}\{(r_i^- - r_i^-(\rho_i))(\exp(\rho_0) - 1) + (r_i^+ - r_i^+(\rho_i))(\exp(-\rho_0) - 1)\}$ and by $|r_i^{\pm} - r_i^{\pm}(\rho_i)| \leq Mh^2/\epsilon$ we obtain

$$|\tau_r| \leq Mh^2/\epsilon \exp(-\delta x_{i-1}/\epsilon)),\ h/\epsilon \geq 1.$$

Thus,

$$|\tau_i(u_0)| \leq Mh^2/\epsilon \exp(-\delta x_{i-1}/\epsilon),\ h/\epsilon \geq 1. \tag{4.3}$$

By Lemma 4.4 (b), Corollary 4.3 and by the inequality

$$t^k \exp(-t) \leq C(\theta) \exp(-\theta t),\ t \in [0, \infty),\ c(\theta) = \text{const}(\theta),$$

we obtain the contribution to the nodal errors of the part $\tau_i(u_0)$ as

$$|z(u_0)| \le M\epsilon \exp(-\delta x_{i-1}/\epsilon),\ h/\epsilon \ge 1. \tag{4.4}$$

Let $h/\epsilon \le 1$. We shall expand each term in $\tau_i(u_0)$ into Taylor series and cancel the hardest parts of τ_r and τ_q. We have

$$\tau_r = u_{0i} r_i^+ \{r_i^-/r_i^+ (\exp(\rho_0) - 1) + (\exp(-\rho_0) - 1)\}.$$

Since for $p_i = \text{const}$, $r_i^-/r_i^+ = \exp(-\rho_i)$ we have

$$\tau_r = u_{0i} r_i^+ \{\exp(\rho_0 - \rho_i) - \exp(-\rho_i) + \exp(\rho_0) - 1)\} + Q,$$

where Q is negligible part. After expanding into Taylor series and after ordering the terms we obtain

$$\begin{aligned}\tau_r =& u_{0i} r_i^+ (\rho_0 - \rho_i)\rho_0(1 - \rho_i/3) + O(h^5/\epsilon^4)\\ =& u_{0i}(\rho_0 - \rho_i)\rho_0(1 + 1/6\rho_i) + O(h^5/\epsilon^4).\end{aligned}$$

In a case of τ_q we have

$$\begin{aligned}\tau_q =& u_{0i}\rho_0(\rho_0 - \rho_i)\{\alpha_i/20 \exp(\rho_0) + 1/3 + \alpha_i/20 - \alpha_{i+1}/20\\ &- \alpha_{i+1}/20 \exp(-\rho_0) + (1/3 + 2/5\alpha_i)\exp(\rho_0/2)\\ &+ (1/3 - 2/5\alpha_{i+1})\exp(-\rho_0/2))\}.\end{aligned}$$

Since $\alpha_{i+1} - \alpha_i = (\rho_{i+1} - \rho_i)(-1/6 + 1/120\rho_i^2) + Q$, we obtain after expanding into Taylor series and ordering the terms that

$$\tau_q = u_{0i}\rho_0(\rho_0 - \rho_i)[1 + \rho_0(\rho_0 - \rho_i)/12] + O(h^5/\epsilon^4).$$

Then, $\tau_i(u_0) = \tau_r - \tau_q = u_{0i}\rho_0(\rho_0 - \rho_i)[1 + 1/6\rho - 1 - \rho_0(\rho_0 - \rho_i)/12]$ $+O(h^5/\epsilon^4)$. After cancelling the hardest part we obtain $|\tau_i(u_0)| \le u_{0i}\rho_0(\rho_0 - \rho_i)\rho_i/6 + O(h^5/\epsilon^4)$. Thus,

$$|\tau_i(u_0)| \le Mh^4/\epsilon^3 \exp(-\delta x_i/\epsilon),\ h/\epsilon \le 1. \tag{4.5}$$

From (4.5), Lemma 4.4 (b) and Corollary 4.3 we obtain contribution to the nodal errors due to the function u_0 as

$$|z_i(u_0)| \le Mh^2/\epsilon \exp(-\delta x_i/\epsilon),\ h/\epsilon \le 1. \tag{4.6}$$

From (4.2), (4.4) and (4.6) we have the nodal errors for difference scheme (2.5) applied to the problem (2.1)

$$|z_i(u_0)| \le M(h^2 + \epsilon \min((h/\epsilon)^2, 1)\exp(-\delta x_{i-1}/\epsilon)),\ i = 0(1)n, \tag{4.7}$$

which leads to the conclusion of Theorem 4.5. ∎

Following precisely this approach we obtain the proof for the following Theorem.

THEOREM 4.6. *Let $p, f \in C^2([0,1])$ and let $\{v_i\}$ be the set of computed values for the solution to (2.1) obtained by (2.6). Then, the following estimate holds*

$$|z_i| \leq Mh^2, \ \ i = 0(1)n.$$

5. Numerical Evidence

We consider two different examples

$$-\epsilon u'' + u' = \exp(x), \ \ u(0) = u(1) = 0, \tag{5.1}$$

with the exact solution $u(x) = 1/(1-\epsilon)\ (\exp(x) - 1 + (\exp(x/\epsilon) - 1)(e - 1)(-\exp(1/\epsilon) + 1)$ taken from [14] and

$$\epsilon u'' + (1 + x^2)u' = -(\exp(x) + x^2), \ \ u(0) = -1, \ u(1) = 0 \tag{5.2}$$

taken from [7]. The mesh length $h = 1/I$ was halved starting with $I = 32$ and ending with $I = 1024$. The maximum nodal errors $E_\infty = \max_i |u(x_i) - v_i|$ is listed under E_∞. The rate of convergence is determined from E_∞ values for two consecutive values of I. We have *rate* $\equiv (\ln E^1_\infty - \ln E^2_\infty)/\ln 2$ where E^1_∞ and E^2_∞ correspond to $h = 1/I$ and $h = 1/(2I)$ respectively.

In Table 1 and 2 are given difference between exact and computed solution listed in E_∞ for the schemes (2.5) and (2.6) respectively. The example (5.2) is tested in Table 3 on the scheme (2.5) by using the double mesh principle ([7, 23]); described above. α_i is used.

TABLE 1

ϵ/I	32	64	128	256	512	1024
$\cdot$	E_∞	E_∞	E_∞	E_∞	E_∞	E_∞
2^{-4}	.43E-6	.27E-7	.17E-8	.11E-9	.66E-11	.64E-12
2^{-6}	.77E-5	.51E-6	.38E-7	.21E-8	.13E-9	.82E-11

TABLE 2

ϵ/I	32	64	128	256	512	1024
$\cdot$	E_∞	E_∞	E_∞	E_∞	E_∞	E_∞
2^{-3}	.86E-7	.54E-8	.33E-9	.21E-10	.14E-11	.44E-12
2^{-6}	.77E-5	.51E-6	.33E-7	.21E-8	.13E-9	.85E-11

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TABLE 3

ϵ/I	32		64		128		256		512	
$\cdot$	E_∞	Rate	E_∞	Rate	E_∞	Rate	E_∞	Rate	E_∞	Rate
1	.23E-4	2.00	.57E-5	2.00	.14E-5	2.00	.37E-6	2.00	.89E-7	2.00
2^{-1}	.67E-4	2.00	.17E-4	2.00	.46E-5	2.00	.11E-5	2.00	.26E-6	2.00
2^{-2}	.13E-3	2.00	.33E-4	2.00	.82E-5	2.00	.21E-5	2.00	.52E-6	2.00
2^{-3}	.17E-3	1.99	.43E-4	1.99	.11E-4	2.00	.27E-5	2.00	.68E-6	2.00
2^{-4}	.19E-3	1.90	.48E-4	1.99	.12E-4	2.00	.30E-5	2.00	.75E-6	2.00
2^{-5}	.19E-3	1.99	.50E-4	1.91	.13E-4	1.99	.31E-4	2.00	.78E-6	2.00
2^{-6}	.19E-3	1.81	.47E-4	1.99	.13E-4	1.92	.32E-5	1.99	.79E-6	2.00
2^{-7}	.17E-3	2.27	.47E-4	1.81	.12E-4	1.99	.32E-5	1.92	.79E-6	1.99
2^{-8}	.20E-3	2.24	.47E-4	2.28	.12E-4	1.81	.30E-5	1.99	.79E-6	1.91

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