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MOVABLE CATEGORIES

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Abstract. The notion of movability for metrizable compacta was introduced by K.Borsuk [1]. In this paper we define the notion of a movable category and prove that the movability of a topological space X coincides with the movability of a suitable category, which is generated by the topological space X (i.e., the category $\mathcal{W}^X$, defined by S.Mardešić [9]).

1. Introduction

The notion of movability was introduced, for metrizable compacta, by K. Borsuk [1]. For the more general cases this notion was extended by S. Mardešić and J. Segal [10], J. Segal [12], P. Shostak [13]. In the equivariant theory of shape this notion was studied in the works of the author of present article [4, 5, 6, 7, 8] and of Z. Čerin [3].

It is necessary to note that in the works mentioned above the movability of topological spaces was defined by means of neighborhoods of the given space (embedded as closed set in a certain AR-space) or by means of inverse systems, depending on the approach to shape theory used. However, the categorical approach to shape theory of S. Mardešić [9] lacks a suitable categorical definition of movability.

In this article we define the notion of movable category and prove that the movability of a topological space is equivalent to the movability of a certain category.

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2. Basic notions and conventions concerning shape and movability

Let $\mathcal{HTOP}$ denote the homotopy category of topological spaces and homotopy classes of maps and $\mathcal{HCW}$ the full subcategory of $\mathcal{HTOP}$ whose objects are all topological spaces having the homotopy type of a CW-complex.

Definition 2.1 (K. Morita [11]). *An inverse system $\{X_\alpha, p_{\alpha\alpha'}, A\}$ in $\mathcal{HCW}$ is called* **associated** *with or an* **expansion** *of a topological space X if there are homotopy classes $p_\alpha : X \to X_\alpha$ for $\alpha \in A$ such that the following conditions are satisfied.*

(1) $p_{\alpha\alpha'} p_{\alpha'} = p_\alpha$, *if* $\alpha < \alpha'$.
(2) *For any homotopy class $f : X \to Q$ with $Q \in Ob(\mathcal{HCW})$, there exists $\alpha \in A$ and a homotopy class $f_\alpha : X_\alpha \to Q$ such that $f = f_\alpha p_\alpha$.*
(3) *For $\alpha \in A$ and for homotopy classes $f_\alpha, g_\alpha : X_\alpha \to Q$ with $Q \in Ob(\mathcal{HCW})$ such that $f_\alpha p_\alpha = g_\alpha p_\alpha$, there exist $\alpha' \in A$ with $\alpha \leqslant \alpha'$ such that $f_\alpha p_{\alpha\alpha'} = g_\alpha p_{\alpha\alpha'}$.*

For any topological space X there exist an inverse system in $\mathcal{HCW}$ associated with X [11], and so there is a shape theory for all topological spaces because the abstract theory of shape yields that there is a shape theory for $\mathcal{HTOP}$ iff every topological space has an expansion, i.e., iff $\mathcal{HCW}$ is a so-called "dense" subcategory of $\mathcal{HTOP}$.

On the categorical approach to shape theory of S. Mardešić [9] for each topological space X it is introduced a new comma category $\mathcal{W}^X$, whose objects are homotopy classes $f : X \to Q$ and whose morphisms are the following commutative triangles where $Q, Q' \in \mathrm{Ob}(\mathcal{HCW})$. Then a shape map $f : X \to Y$

$$\begin{tikzcd} & X \arrow[dl, "f'"'] \arrow[dr, "f"] & \\ Q' & & Q \arrow[ll, "\mu"] \end{tikzcd}$$

is defined as a covariant functor $f : \mathcal{W}^Y \to \mathcal{W}^X$ which keeps morphisms μ fixed.

Definition 2.2 ([10]). *An inverse system $\{X_\alpha, p_{\alpha\alpha'}, A\}$ in $\mathcal{HCW}$ is called movable if*

(*) *for every $\alpha \in A$, there exists an $\alpha' \in A$, $\alpha' \geqslant \alpha$ such that for all $\alpha'' \in A$, $\alpha'' \geqslant \alpha$, there exists a homotopy class $r^{\alpha'\alpha''} : X_{\alpha'} \to X_{\alpha''}$ such that*
$$p_{\alpha\alpha'} = p_{\alpha\alpha''} r^{\alpha'\alpha''}.$$

The topological space X is called movable if there exist an inverse system $\{X_\alpha, p_{\alpha\alpha'}, A\}$ in $\mathcal{HCW}$ which associated with X and which is movable.

The reader is referred to the book by S. Mardešić and J. Segal [9] for general information about shape theory.

3. The movable categories

Let K be an arbitrary category and K' any subcategory of the category K.

Definition 3.1. *We say that a subcategory K' is movable in a category K, if for any object $X \in Ob(K')$ there exists an object $Y \in Ob(K')$ and a morphism $f \in K'(Y,X)$ such that for any object $Z \in Ob(K')$ and any morphism $g \in K'(Z,X)$ there is a morphism $h \in K(Y,Z)$ which make the following diagram commutative*

$$\begin{tikzcd} & Y \arrow[dl, "f"'] \arrow[dd, "h"] \\ X & \\ & Z \arrow[ul, "g"] \end{tikzcd}$$

Definition 3.2. *We say that a category is movable if it is movable in itself.*

Definition 3.3 ([2]). *It is said that K is a category with zero-morphisms if for any pair (A,B) of objects from a category K there exist morphisms $o_{BA} : A \to B$ which, for all morphisms $\nu : B \to C$ and $u : D \to A$, where C and D are objects of the category K, satisfy the following equalities*

$$\nu o_{BA} = o_{CA}, \qquad o_{BA} u = o_{BD}.$$

Definition 3.4 ([2]). *An object $O \in Ob(K)$ is called initial if for any object $X \in Ob(K)$ the set $Mor_K(O,X)$ consists of a single morphism.*

Proposition 3.5. *Any category K with zero-morphisms is movable.*

Proof. Let $X \in \mathrm{Ob}(K)$ be an arbitrary object. It appears that for the object we seek (see definition 3.1), we may take any object $Y \in ob(K)$ and for the morphism $f \in K(Y,X)$ it is necessary to take a zero-morphism $o_{XY} : Y \to X$. Indeed, let $g \in K(Z,X)$ be an arbitrary morphism. It is clear that zero-morphism $o_{ZY} : Y \to Z$ is the morphism we seek, that is $g o_{ZY} = o_{XY}$, which follows from definition 3.3. ∎

Proposition 3.6. *Any category K with initial objects is movable.*

Proof. Let $X \in \mathrm{Ob}(K)$ be any object. Let us consider the initial object O of the category K. We denote by u_X the single morphism from the object O to the object X. Now it is not difficult to note that the object O and the morphism $u_X : O \to X$ satisfies the condition of definition 3.1. Indeed: let $Y \in ob(K)$ be any object and $g : Y \to X$ be any morphism of the category K. It is clear that the single morphism $u_Y : O \to Y$ satisfies the condition $u_X = g \circ u_Y$. $\square$

4. The movability of topological spaces

Theorem 4.1. *The topological space X is movable if and only if the category $\mathcal{W}^X$ is movable.*

This theorem is a simple reformulation of the following theorem.

Theorem 4.2. *The topological space X is movable if and only if the following condition is satisfied.*

(*) *For any $Q \in Ob(\mathcal{HCW})$ and any homotopy class $f : X \to Q$ there exist $Q' \in Ob(\mathcal{HCW})$ and homotopy classes $f' : X \to Q'$, $\eta : Q' \to Q$, satisfying $f = \eta \circ f'$, such that for any $Q'' \in Ob(\mathcal{HCW})$ and homotopy classes $f'' : X \to Q''$, $\eta' : Q'' \to Q$, satisfying the condition $f = \eta' \circ f''$, there exist a homotopy class $\eta'' : Q' \to Q''$ which satisfies the condition $\eta = \eta' \circ \eta''$ (Diagram 1).*

$$
\begin{tikzcd}
Q & & \\
& X \arrow[ul, "f"'] \arrow[dl, "f'"] \arrow[r, "f''"] & Q'' \arrow[ull, "\eta'"'] \\
Q' \arrow[uu, "\eta"] \arrow[urr, "\eta''"'] & &
\end{tikzcd}
$$

Diagram 1.

Proof. Let condition $(*)$ be satisfied. We must prove that X is movable. Let us consider an inverse system $\{X_\alpha, p_{\alpha\alpha'}, A\}$ in $\mathcal{HCW}$ which is associated with the topological space X.

Let $\alpha \in A$ be any element and $p_\alpha : X \to X_\alpha$ be the natural projection. By $(*)$ for the homotopy class $p_\alpha : X \to X_\alpha$ let $Q' \in \mathrm{Ob}(\mathcal{HCW})$, and $f' : X \to Q'$, $\eta : Q' \to X_\alpha$ are homotopy classes satisfying the condition $\eta \circ f' = p_\alpha$ (Diagram 2).

Since inverse system $\{X_\alpha, p_{\alpha\alpha'}, A\}$ is associated with X, there exists $\tilde{\alpha} \in A$, $\tilde{\alpha} \geqslant \alpha$ and $\tilde{f}' : X_{\tilde{\alpha}} \to Q'$ such that

$$f' = \tilde{f}' \circ p_{\tilde{\alpha}}. \tag{4.1}$$

It is not difficult to verify that

$$p_{\alpha\tilde{\alpha}} \circ p_{\tilde{\alpha}} = \eta \circ \tilde{f}' \circ p_{\tilde{\alpha}}. \tag{4.2}$$

Indeed:

$$\eta \circ \tilde{f}' \circ p_{\tilde{\alpha}} = \eta \circ f' = p_\alpha = p_{\alpha\tilde{\alpha}} \circ p_{\tilde{\alpha}}.$$

From the equality (4.2) and the definition 1 we infer the existence of an index $\alpha' \in A$, $\alpha' \geqslant \tilde{\alpha}$ for which

$$p_{\alpha\tilde{\alpha}} \circ p_{\tilde{\alpha}\alpha'} = \eta \circ \tilde{f}' \circ p_{\tilde{\alpha}\alpha'}. \tag{4.3}$$

The obtained index $\alpha' \in A$ satisfies the condition of the movability of inverse system $\{X_\alpha, p_{\alpha\alpha'}, A\}$. Indeed, let $\alpha'' \in A$, $\alpha'' \geqslant \alpha$ be any element. For the homotopy classes $p_{\alpha\alpha''} : X_{\alpha''} \to X_\alpha$ and $p_{\alpha''} : X \to X_{\alpha''}$ (with the condition $p_\alpha = p_{\alpha\alpha''} \circ p_{\alpha''}$) there exist a homotopy class $\eta'' : Q' \to X_{\alpha''}$, which satisfies the equality

$$\eta = p_{\alpha\alpha''} \circ \eta''. \tag{4.4}$$

(see the condition $(*)$). Now it is easy to see that $g = \eta'' \circ \tilde{f}' \circ p_{\tilde{\alpha}\alpha'}$ is the homotopy class we seek, i. e. the following condition is satisfied:

$$p_{\alpha\alpha'} = p_{\alpha\alpha''} \circ g. \tag{4.5}$$

Indeed:

$$p_{\alpha\alpha'} = p_{\alpha\tilde{\alpha}} \circ p_{\tilde{\alpha}\alpha'} = \eta \circ \tilde{f}' \circ p_{\tilde{\alpha}\alpha'} = p_{\alpha\alpha''} \circ \eta'' \circ \tilde{f}' \circ p_{\tilde{\alpha}\alpha'} = p_{\alpha\alpha''} \circ g.$$

$$\begin{tikzcd}
 & & Q' \arrow[dl, "\eta"'] \arrow[ddd, "\eta''"] \\
X_{\tilde{\alpha}} \arrow[urr, "\tilde{f}'"] \arrow[d, "p_{\alpha\tilde{\alpha}}"'] & X_{\alpha'} \arrow[l, "p_{\tilde{\alpha}\alpha'}"] & \\
X_\alpha & X \arrow[l, "p_\alpha"'] \arrow[u, "p_{\alpha'}"] \arrow[uur] \arrow[dr, "p_{\alpha''}"] & \\
 & & X_{\alpha''} \arrow[ull, "p_{\alpha\alpha''}"]
\end{tikzcd}$$

Diagram 2.

Now we must prove the converse. Let X be a movable topological space and some inverse system $\{X_\alpha, p_{\alpha\alpha'}, A\}$ associated with X . Let us prove that the condition $(*)$ is satisfied. To this end, consider any homotopy class $f : X \to Q$ (Diagram 3). From the association of the inverse system $\{X_\alpha, p_{\alpha\alpha'}, A\}$ with the space X follows that there exist an index $\alpha \in A$ and a homotopy class $f_\alpha : X_\alpha \to Q$ such that

$$f = f_\alpha \circ p_\alpha. \tag{4.6}$$

For the index $\alpha \in A$ let us consider an index $\alpha' \in A$, $\alpha' \geqslant \alpha$, which satisfies the condition of movability of the inverse system $\{X_\alpha, p_{\alpha\alpha'}, A\}$. From (4.6) we get

$$f = f_\alpha \circ p_{\alpha\alpha'} \circ p_{\alpha'}. \tag{4.7}$$

Now let us prove that $X_{\alpha'}$, the homotopy classes $p_\alpha : X \to X_{\alpha'}$ and $f_\alpha \circ p_{\alpha\alpha'} : X_{\alpha'} \to Q$ satisfy condition $(*)$. Indeed, let $Q'' \in \mathrm{Ob}(\mathcal{HCW})$ and $f'' : X \to Q''$, $\eta' : Q'' \to Q$ homotopy classes, which satisfy the condition

$$f = \eta' \circ f''. \tag{4.8}$$

For the homotopy class $f'' : X \to Q''$ there exist an index $\alpha'' \in A$, $\alpha'' \geqslant \alpha$ and a homotopy class $\tilde{f}'' : X_{\alpha''} \to Q''$ that

$$f'' = \tilde{f}'' \circ p_{\alpha''}. \tag{4.9}$$

It is clear that

$$f_\alpha \circ p_{\alpha\alpha''} \circ p_{\alpha''} = \eta' \circ \tilde{f}'' \circ p_{\alpha''}.$$

Therefore, according to the definition 1 of "association", we can find an index $\alpha''' \in A$, $\alpha''' \geqslant \alpha''$ such that

$$f_\alpha \circ p_{\alpha\alpha''} \circ p_{\alpha''\alpha'''} = \eta' \circ \tilde{f}'' \circ p_{\alpha''\alpha'''}. \tag{4.10}$$

By the movability of the inverse system $\{X_\alpha, p_{\alpha\alpha'}, A\}$, we can select the homotopy class $g : X_{\alpha'} \to X_{\alpha'''}$ satisfying the condition

$$p_{\alpha\alpha'} = p_{\alpha\alpha''} \circ p_{\alpha''\alpha'''} \circ g. \tag{4.11}$$

$$\begin{tikzcd}
 & X_\alpha \arrow[dl, "f_\alpha"'] & & X_{\alpha'} \arrow[ll, "p_{\alpha\alpha'}"'] \arrow[dr, "g"] & \\
Q & & X \arrow[ul, "p_\alpha"] \arrow[ur, "p_{\alpha'}"'] \arrow[ll, "f"'] \arrow[rr, "p_{\alpha'''}"] \arrow[dl, "f''"] \arrow[dr, "p_{\alpha''}"'] & & X_{\alpha'''} \arrow[dl, "p_{\alpha''\alpha'''}"] \\
 & Q'' \arrow[ul, "\eta'"] & & X_{\alpha''} \arrow[ll, "\tilde{f}''"] &
\end{tikzcd}$$

Diagram 3.

Let us define $\eta'' = \tilde{f}'' \circ p_{\alpha''\alpha'''} \circ g$. It is remains to note that the homotopy class $\eta'' : X_{\alpha'} \to Q''$ satisfies the condition

$$f_\alpha \circ p_{\alpha\alpha'} = \eta' \circ \eta''. \tag{4.12}$$

Indeed:

$$\eta' \circ \eta'' = \eta' \circ \tilde{f}'' \circ p_{\alpha''\alpha'''} \circ g = f_\alpha \circ p_{\alpha\alpha''} \circ p_{\alpha''\alpha'''} \circ g = f_\alpha \circ p_{\alpha\alpha'}.$$

$\square$

Remark 4.3. The condition $(*)$ of Theorem 2 one can consider as a definition of movability of topological space.

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