

MOVABLE CATEGORIES

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ABSTRACT. The notion of movability for metrizable compacta was introduced by K.Borsuk [1]. In this paper we define the notion of a movable category and prove that the movability of a topological space X coincides with the movability of a suitable category, which is generated by the topological space X (i.e., the category \mathcal{W}^X , defined by S.Mardešić [9]).

1. INTRODUCTION

The notion of movability was introduced, for metrizable compacta, by K. Borsuk [1]. For the more general cases this notion was extended by S. Mardešić and J. Segal [10], J. Segal [12], P. Shostak [13]. In the equivariant theory of shape this notion was studied in the works of the author of present article [4, 5, 6, 7, 8] and of Z. Čerin [3].

It is necessary to note that in the works mentioned above the movability of topological spaces was defined by means of neighborhoods of the given space (embedded as closed set in a certain AR -space) or by means of inverse systems, depending on the approach to shape theory used. However, the categorical approach to shape theory of S. Mardešić [9] lacks a suitable categorical definition of movability.

In this article we define the notion of movable category and prove that the movability of a topological space is equivalent to the movability of a certain category.

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2. BASIC NOTIONS AND CONVENTIONS CONCERNING SHAPE AND MOVABILITY

Let \mathcal{HTOP} denote the homotopy category of topological spaces and homotopy classes of maps and \mathcal{HCW} the full subcategory of \mathcal{HTOP} whose objects are all topological spaces having the homotopy type of a CW-complex.

DEFINITION 2.1 (K. Morita [11]). *An inverse system $\{X_\alpha, p_{\alpha\alpha'}, A\}$ in \mathcal{HCW} is called **associated** with or an **expansion** of a topological space X if there are homotopy classes $p_\alpha : X \rightarrow X_\alpha$ for $\alpha \in A$ such that the following conditions are satisfied.*

- (1) $p_{\alpha\alpha'} p_{\alpha'} = p_\alpha$, if $\alpha < \alpha'$.
- (2) For any homotopy class $f : X \rightarrow Q$ with $Q \in \text{Ob}(\mathcal{HCW})$, there exists $\alpha \in A$ and a homotopy class $f_\alpha : X_\alpha \rightarrow Q$ such that $f = f_\alpha p_\alpha$.
- (3) For $\alpha \in A$ and for homotopy classes $f_\alpha, g_\alpha : X_\alpha \rightarrow Q$ with $Q \in \text{Ob}(\mathcal{HCW})$ such that $f_\alpha p_\alpha = g_\alpha p_\alpha$, there exist $\alpha' \in A$ with $\alpha \leq \alpha'$ such that $f_\alpha p_{\alpha\alpha'} = g_\alpha p_{\alpha\alpha'}$.

For any topological space X there exist an inverse system in \mathcal{HCW} associated with X [11], and so there is a shape theory for all topological spaces because the abstract theory of shape yields that there is a shape theory for \mathcal{HTOP} iff every topological space has an expansion, i.e., iff \mathcal{HCW} is a so-called "dense" subcategory of \mathcal{HTOP} .

On the categorical approach to shape theory of S. Mardešić [9] for each topological space X it is introduced a new comma category \mathcal{W}^X , whose objects are homotopy classes $f : X \rightarrow Q$ and whose morphisms are the following commutative triangles where $Q, Q' \in \text{Ob}(\mathcal{HCW})$. Then a shape map $f : X \rightarrow Y$

$$\begin{array}{ccc}
 & X & \\
 f' \swarrow & & \searrow f \\
 Q' & \xleftarrow{\mu} & Q
 \end{array}$$

is defined as a covariant functor $f : \mathcal{W}^Y \rightarrow \mathcal{W}^X$ which keeps morphisms μ fixed.

DEFINITION 2.2 ([10]). *An inverse system $\{X_\alpha, p_{\alpha\alpha'}, A\}$ in \mathcal{HCW} is called **movable** if*

- (*) for every $\alpha \in A$, there exists an $\alpha' \in A$, $\alpha' \geq \alpha$ such that for all $\alpha'' \in A$, $\alpha'' \geq \alpha$, there exists a homotopy class $r^{\alpha'\alpha''} : X_{\alpha'} \rightarrow X_{\alpha''}$ such that

$$p_{\alpha\alpha'} = p_{\alpha\alpha''} r^{\alpha'\alpha''}.$$

The topological space X is called **movable** if there exist an inverse system $\{X_\alpha, p_{\alpha\alpha'}, A\}$ in \mathcal{HCW} which associated with X and which is movable.

The reader is referred to the book by S. Mardešić and J. Segal [9] for general information about shape theory.

3. THE MOVABLE CATEGORIES

Let K be an arbitrary category and K' any subcategory of the category K .

DEFINITION 3.1. *We say that a subcategory K' is movable in a category K , if for any object $X \in \text{Ob}(K')$ there exists an object $Y \in \text{Ob}(K')$ and a morphism $f \in K'(Y, X)$ such that for any object $Z \in \text{Ob}(K')$ and any morphism $g \in K'(Z, X)$ there is a morphism $h \in K(Y, Z)$ which make the following diagram commutative*

$$\begin{array}{ccc}
 & & Y \\
 & f \swarrow & \downarrow h \\
 X & & Z \\
 & g \searrow & \\
 & &
 \end{array}$$

DEFINITION 3.2. *We say that a category is movable if it is movable in itself.*

DEFINITION 3.3 ([2]). *It is said that K is a category with zero-morphisms if for any pair (A, B) of objects from a category K there exist morphisms $o_{BA} : A \rightarrow B$ which, for all morphisms $\nu : B \rightarrow C$ and $u : D \rightarrow A$, where C and D are objects of the category K , satisfy the following equalities*

$$\nu o_{BA} = o_{CA}, \quad o_{BA} u = o_{BD}.$$

DEFINITION 3.4 ([2]). *An object $O \in \text{Ob}(K)$ is called initial if for any object $X \in \text{Ob}(K)$ the set $\text{Mor}_K(O, X)$ consists of a single morphism.*

PROPOSITION 3.5. *Any category K with zero-morphisms is movable.*

PROOF. Let $X \in \text{Ob}(K)$ be an arbitrary object. It appears that for the object we seek (see definition 3.1), we may take any object $Y \in \text{ob}(K)$ and for the morphism $f \in K(Y, X)$ it is necessary to take a zero-morphism $o_{XY} : Y \rightarrow X$. Indeed, let $g \in K(Z, X)$ be an arbitrary morphism. It is clear that zero-morphism $o_{ZY} : Y \rightarrow Z$ is the morphism we seek, that is $g o_{ZY} = o_{XY}$, which follows from definition 3.3. \square

PROPOSITION 3.6. *Any category K with initial objects is movable.*

PROOF. Let $X \in \text{Ob}(K)$ be any object. Let us consider the initial object O of the category K . We denote by u_X the single morphism from the object O to the object X . Now it is not difficult to note that the object O and the morphism $u_X : O \rightarrow X$ satisfies the condition of definition 3.1. Indeed: let $Y \in \text{ob}(K)$ be any object and $g : Y \rightarrow X$ be any morphism of the category K . It is clear that the single morphism $u_Y : O \rightarrow Y$ satisfies the condition $u_X = g \circ u_Y$. \square

4. THE MOVABILITY OF TOPOLOGICAL SPACES

THEOREM 4.1. *The topological space X is movable if and only if the category \mathcal{W}^X is movable.*

This theorem is a simple reformulation of the following theorem.

THEOREM 4.2. *The topological space X is movable if and only if the following condition is satisfied.*

- (*) *For any $Q \in \text{Ob}(\mathcal{HCW})$ and any homotopy class $f : X \rightarrow Q$ there exist $Q' \in \text{Ob}(\mathcal{HCW})$ and homotopy classes $f' : X \rightarrow Q'$, $\eta : Q' \rightarrow Q$, satisfying $f = \eta \circ f'$, such that for any $Q'' \in \text{Ob}(\mathcal{HCW})$ and homotopy classes $f'' : X \rightarrow Q''$, $\eta' : Q'' \rightarrow Q$, satisfying the condition $f = \eta' \circ f''$, there exist a homotopy class $\eta'' : Q' \rightarrow Q''$ which satisfies the condition $\eta = \eta' \circ \eta''$ (Diagram 1).*

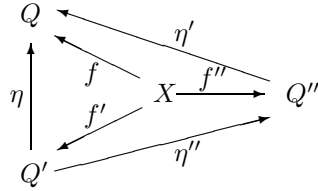


DIAGRAM 1.

PROOF. Let condition (*) be satisfied. We must prove that X is movable. Let us consider an inverse system $\{X_\alpha, p_{\alpha\alpha'}, A\}$ in \mathcal{HCW} which is associated with the topological space X .

Let $\alpha \in A$ be any element and $p_\alpha : X \rightarrow X_\alpha$ be the natural projection. By (*) for the homotopy class $p_\alpha : X \rightarrow X_\alpha$ let $Q' \in \text{Ob}(\mathcal{HCW})$, and $f' : X \rightarrow Q'$, $\eta : Q' \rightarrow X_\alpha$ are homotopy classes satisfying the condition $\eta \circ f' = p_\alpha$ (Diagram 2).

Since inverse system $\{X_\alpha, p_{\alpha\alpha'}, A\}$ is associated with X , there exists $\tilde{\alpha} \in A$, $\tilde{\alpha} \geq \alpha$ and $\tilde{f}' : X_{\tilde{\alpha}} \rightarrow Q'$ such that

$$(4.1) \quad f' = \tilde{f}' \circ p_{\tilde{\alpha}}.$$

It is not difficult to verify that

$$(4.2) \quad p_{\alpha\tilde{\alpha}} \circ p_{\tilde{\alpha}} = \eta \circ \tilde{f}' \circ p_{\tilde{\alpha}}.$$

Indeed:

$$\eta \circ \tilde{f}' \circ p_{\tilde{\alpha}} = \eta \circ f' = p_{\alpha} = p_{\alpha\tilde{\alpha}} \circ p_{\tilde{\alpha}}.$$

From the equality (4.2) and the definition 1 we infer the existence of an index $\alpha' \in A$, $\alpha' \geq \tilde{\alpha}$ for which

$$(4.3) \quad p_{\alpha\tilde{\alpha}} \circ p_{\tilde{\alpha}\alpha'} = \eta \circ \tilde{f}' \circ p_{\tilde{\alpha}\alpha'}.$$

The obtained index $\alpha' \in A$ satisfies the condition of the movability of inverse system $\{X_{\alpha}, p_{\alpha\alpha'}, A\}$. Indeed, let $\alpha'' \in A$, $\alpha'' \geq \alpha$ be any element. For the homotopy classes $p_{\alpha\alpha''} : X_{\alpha''} \rightarrow X_{\alpha}$ and $p_{\alpha''} : X \rightarrow X_{\alpha''}$ (with the condition $p_{\alpha} = p_{\alpha\alpha''} \circ p_{\alpha''}$) there exist a homotopy class $\eta'' : Q' \rightarrow X_{\alpha''}$, which satisfies the equality

$$(4.4) \quad \eta = p_{\alpha\alpha''} \circ \eta''.$$

(see the condition (*)). Now it is easy to see that $g = \eta'' \circ \tilde{f}' \circ p_{\tilde{\alpha}\alpha'}$ is the homotopy class we seek, i. e. the following condition is satisfied:

$$(4.5) \quad p_{\alpha\alpha'} = p_{\alpha\alpha''} \circ g.$$

Indeed:

$$p_{\alpha\alpha'} = p_{\alpha\tilde{\alpha}} \circ p_{\tilde{\alpha}\alpha'} = \eta \circ \tilde{f}' \circ p_{\tilde{\alpha}\alpha'} = p_{\alpha\alpha''} \circ \eta'' \circ \tilde{f}' \circ p_{\tilde{\alpha}\alpha'} = p_{\alpha\alpha''} \circ g.$$

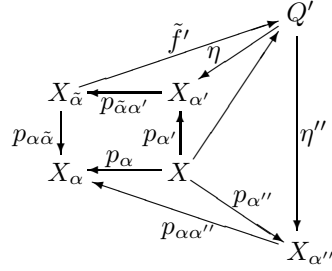


DIAGRAM 2.

Now we must prove the converse. Let X be a movable topological space and some inverse system $\{X_{\alpha}, p_{\alpha\alpha'}, A\}$ associated with X . Let us prove that the condition (*) is satisfied. To this end, consider any homotopy class $f : X \rightarrow Q$ (Diagram 3). From the association of the inverse system $\{X_{\alpha}, p_{\alpha\alpha'}, A\}$ with the space X follows that there exist an index $\alpha \in A$ and a homotopy class $f_{\alpha} : X_{\alpha} \rightarrow Q$ such that

$$(4.6) \quad f = f_{\alpha} \circ p_{\alpha}.$$

For the index $\alpha \in A$ let us consider an index $\alpha' \in A$, $\alpha' \geq \alpha$, which satisfies the condition of movability of the inverse system $\{X_{\alpha}, p_{\alpha\alpha'}, A\}$. From (4.6) we get

$$(4.7) \quad f = f_{\alpha} \circ p_{\alpha\alpha'} \circ p_{\alpha'}.$$

Now let us prove that $X_{\alpha'}$, the homotopy classes $p_{\alpha} : X \rightarrow X_{\alpha'}$ and $f_{\alpha} \circ p_{\alpha\alpha'} : X_{\alpha'} \rightarrow Q$ satisfy condition (*). Indeed, let $Q'' \in \text{Ob}(\mathcal{HCW})$ and $f'' : X \rightarrow Q''$, $\eta' : Q'' \rightarrow Q$ homotopy classes, which satisfy the condition

$$(4.8) \quad f = \eta' \circ f''.$$

For the homotopy class $f'' : X \rightarrow Q''$ there exist an index $\alpha'' \in A$, $\alpha'' \geq \alpha$ and a homotopy class $\tilde{f}'' : X_{\alpha''} \rightarrow Q''$ that

$$(4.9) \quad f'' = \tilde{f}'' \circ p_{\alpha''}.$$

It is clear that

$$f_{\alpha} \circ p_{\alpha\alpha''} \circ p_{\alpha''} = \eta' \circ \tilde{f}'' \circ p_{\alpha''}.$$

Therefore, according to the definition 1 of "association", we can find an index $\alpha''' \in A$, $\alpha''' \geq \alpha''$ such that

$$(4.10) \quad f_{\alpha} \circ p_{\alpha\alpha''} \circ p_{\alpha''\alpha'''} = \eta' \circ \tilde{f}'' \circ p_{\alpha''\alpha'''}$$

By the movability of the inverse system $\{X_{\alpha}, p_{\alpha\alpha'}, A\}$, we can select the homotopy class $g : X_{\alpha'} \rightarrow X_{\alpha'''}$ satisfying the condition

$$(4.11) \quad p_{\alpha\alpha'} = p_{\alpha\alpha''} \circ p_{\alpha''\alpha'''} \circ g.$$

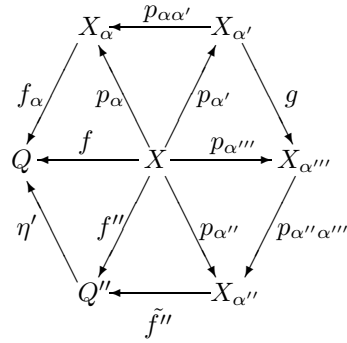


DIAGRAM 3.

Let us define $\eta'' = \tilde{f}'' \circ p_{\alpha''\alpha'''} \circ g$. It remains to note that the homotopy class $\eta'' : X_{\alpha'} \rightarrow Q''$ satisfies the condition

$$(4.12) \quad f_{\alpha} \circ p_{\alpha\alpha'} = \eta' \circ \eta''.$$

Indeed:

$$\eta' \circ \eta'' = \eta' \circ \tilde{f}'' \circ p_{\alpha''\alpha'''} \circ g = f_{\alpha} \circ p_{\alpha\alpha''} \circ p_{\alpha''\alpha'''} \circ g = f_{\alpha} \circ p_{\alpha\alpha'}.$$

□

REMARK 4.3. The condition (*) of Theorem 2 one can consider as a definition of movability of topological space.

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