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A GENERALIZATION OF A RESULT ON MAXIMUM MODULUS OF POLYNOMIALS

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Abstract. For an arbitrary entire function $f(z)$, let

$$M(f,d) = \max_{|z|=d} |f(z)|.$$

It is known that if the geometric mean of the moduli of the zeros of a polynomial $p(z)$ of degree n is at least 1, and $M(p,1) = 1$, then for $R > 1$

$$M(p,R) \leq \begin{cases} \frac{R}{2} + \frac{1}{2}, & \text{if } n = 1, \\ \frac{R^n}{2} + \frac{(3+2\sqrt{2})R^{n-2}}{2}, & \text{if } n \geq 2. \end{cases}$$

We have obtained a generalization of this result, by assuming the geometric mean of the moduli of the zeros of the polynomial to be at least k, $(k > 0)$.

1. Introduction and statement of result

For a polynomial $p(z)$ of degree n, we have, as a simple consequence [4, Part III, Chapter 6, Problem no. 269] of maximum modulus principle

Theorem 1.1. *If $p(z)$ is a polynomial of degree n such that $M(p,1) = 1$, then for $R > 1$*

$$M(p,R) \leq R^n. \tag{1.1}$$

Equality holds in (1.1) for $p(z) = az^n$, with $|a| = 1$.

Ankeny and Rivlin [1] considered a restricted class of polynomials and obtained the following refinement

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Theorem 1.2. *If the moduli of the zeros of a polynomial $p(z)$ of degree n are all ≥ 1 and $M(p,1) = 1$, then for $R > 1$*

$$M(p,R) \leq \frac{R^n + 1}{2}. \tag{1.2}$$

Equality holds in (1.2) for $p(z) = (bz^n + d)/2$, with $|b| = |d| = 1$.

Frappier and Rahman [3] in a somewhat different context, obtained a similar type of result for a broader class of polynomials and proved

Theorem 1.3. *If the geometric mean of the moduli of the zeros of a polynomial $p(z)$ of degree n is at least 1 and $M(p,1) = 1$, then for $R > 1$*

$$M(p,R) \leq \begin{cases} \frac{R}{2} + \frac{1}{2}, & n = 1, \\ \frac{R^n}{2} + \frac{(3+2\sqrt{2})R^{n-2}}{2}, & n \geq 2. \end{cases}$$

In this note, we have obtained a generalization of Theorem 1.3, by assuming the geometric mean of the moduli of the zeros of the polynomial $p(z)$ to be at least k, $(k > 0)$. More precisely, we prove

Theorem 1.4. *If the geometric mean of the moduli of the zeros of a polynomial $p(z)$ of degree n is at least k, $(k > 0)$, and $M(p,1) = 1$, then for $R > 1$*

$$M(p,R) \leq \begin{cases} \frac{R}{1+k} + \frac{k}{1+k}, & n = 1, \\ \frac{R^n}{1+k^n} + \frac{R^{n-2}}{4}\left[(5+k^n) + \frac{1}{1+k^n}\sqrt{D}\right], & n \geq 2. \end{cases} \tag{1.3}$$

where

$$D = k^{4n} + 4k^{3n} + 30k^{2n} + 52k^n + 41.$$

Equality holds in (1.3_1) for $p(z) = (z+k)/(1+k)$.

2. Lemmas

For the proof of the theorem, we require following lemmas.

Lemma 2.1. *If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n such that $M(p,1) = 1$, then*

$$|a_0| + |a_n| \leq 1.$$

This lemma is due to Visser [5].

Lemma 2.2. *If $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n such that $M(p,1) = 1$, then*

$$2|a_0| \cdot |a_n| + \sum_{k=0}^{n} |a_k|^2 \leq 1.$$

This lemma is due to van der Corput and Visser [2].

3. Proof of Theorem 1.4

If

$$p(z) = a_0 + a_1 z,$$

then

$$\frac{M(p,R)}{M(p,1)} = \frac{|a_0| + |a_1|R}{|a_0| + |a_1|} \leq \frac{R+k}{1+k},$$

thereby proving the theorem for this particular case. Therefore we now assume that

$$n \geq 2,$$

and

$$\begin{aligned} p(z) &= a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_0, \\ &= a_n z^n + a_{n-1} z^{n-1} + r(z). \end{aligned} \tag{3.1}$$

As the geometric mean of the moduli of the zeros of the polynomial is at least k, we have

$$|a_0| \geq k^n |a_n|, \tag{3.2}$$

and therefore, by Lemma 2.1

$$\alpha := |a_n| \leq \frac{1}{1+k^n}. \tag{3.3}$$

Further, by Lemma 2.2, we have

$$(|a_0| + |a_n|)^2 + |a_{n-1}|^2 \leq 1,$$

which, by (3.2) and (3.3), implies

$$(k^n \alpha + \alpha)^2 + |a_{n-1}|^2 \leq 1,$$

i.e.

$$|a_{n-1}| \leq \sqrt{\{1 - \alpha^2 (1+k^n)^2\}}. \tag{3.4}$$

Using (3.3) and (3.4), we can now say that

$$\begin{aligned} |a_n z^n + a_{n-1} z^{n-1}| &\leq \alpha |z|^n + |z|^{n-1} \sqrt{\{1 - \alpha^2 (1+k^n)^2\}} \\ &\leq \frac{1}{1+k^n} |z^n| + \frac{(1+k^n) + \alpha (1+k^n)^2}{4} |z|^{n-2}, \end{aligned} \tag{3.5}$$

by (3.3). And, by (3.1)

$$r(z) = p(z) - a_n z^n - a_{n-1} z^{n-1}$$

is a polynomial, of degree at most $(n-2)$, with

$$M(r,1) \leq 1 + \alpha + \sqrt{\{1 - \alpha^2 (1+k^n)^2\}},$$

(by (3.3) and (3.4)), thereby implying, by Theorem 1.1, for $R > 1$

$$M(r,R) \leq \left[1 + \alpha + \sqrt{\{1 - \alpha^2 (1+k^n)^2\}}\right] R^{n-2}.$$

Hence, by (3.1) and (3.5), we have, for $R > 1$

$$M(p,R) \leq \frac{R^n}{1+k^n} + \left[\frac{5+k^n}{4} + \alpha\left\{1+\frac{(1+k^n)^2}{4}\right\} + \sqrt{\{1-\alpha^2(1+k^n)^2\}}\right]R^{n-2},$$

from which, the inequality (1.3_2) follows, on finding the maximum value of the function

$$\phi(\alpha) = \alpha\left\{1+\frac{(1+k^n)^2}{4}\right\} + \sqrt{\{1-\alpha^2(1+k^n)^2\}},$$

on the interval $[0, 1/(1+k^n)]$. This completes the proof of Theorem 1.4.

References

[1] N. C. Ankeny and T. J. Rivlin, *On a theorem of S.Bernstein*, Pacific J. Math. **5** (1955), 849–852.

[2] J. G. van der Corput and C. Visser, *Inequalities concerning polynomials and trigonometric polynomials*, Nederl. Akad. Wetensch. Proc. **49**, 383–392 (Indag. Math. **8** (1946), 238–247).

[3] C. Frappier and Q. I. Rahman, *On an inequality of S. Bernstein*, Canad. J. Math. **34** (1982), 932–944.

[4] G. Polya and G. Szegö, Problems and Theorems in analysis, Vol. 1. Springer-Verlag, Berlin-Heidelberg, 1972.

[5] C. Visser, *A simple proof of certain inequalities concerning polynomials*, Nederl. Akad. Wetensch. Proc. **47**, 276–281 (Indag. Math. **7** (1945), 81–86).

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