Weyl Pseudodifferential Calculus and the Heisenberg Group in New Settings

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Declaration

I hereby declare that the material in this thesis is my own original work except where stated otherwise.

The content of Chapter 1 has been published in *Indagationes Mathematicae* [Har19]. The content of Chapter 2 has been accepted for publication in the *Illinois Journal of Mathematics* and is available in preprint form on the arXiv at [Har20].

Chapters 3 and 4 represent ongoing work. The work in Chapter 3 is expected to be ready for publication in 2021.

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"Mathematical!"

- Finn the Human,
Adventure Time

Abstract

We examine both Weyl pseudodifferential calculus and the Heisenberg group in new settings, through two published papers and a chapter representing work towards a construction of Haar bases of unimodular LC groups. We finish with some remarks about an extension of both Heisenberg group and pseudodifferential calculus techniques to non-Abelian settings, detailing possible links with representation theory and the Langlands program.

The first of the two papers achieves the following

"We give a simple proof of the fact that the classical Ornstein-Uhlenbeck operator L is R-sectorial of angle $\arcsin |1 - 2/p|$ on $L^p(\mathbb{R}^n, \exp(-|x|^2/2)dx)$ (for $1 < p < \infty$). Applying the abstract holomorphic functional calculus theory of Kalton and Weis, this immediately gives a new proof of the fact that L has a bounded H^∞ functional calculus with this optimal angle."

In the second paper,

"We construct a Weyl pseudodifferential calculus tailored to studying boundedness of operators on weighted L^p spaces over $\mathbb{R}^d$ with weights of the form $\exp(-\phi(x))$, for $\phi(x)$ a C^2 function, a setting in which the operator associated to the weighted Dirichlet form typically has only holomorphic functional calculus. A symbol class giving rise to bounded operators on L^p is determined, and its properties analysed. This theory is used to calculate an upper bounded on the H^∞ angle of relevant operators, and deduces known optimal results in some cases. Finally, the symbol class is enriched and studied under an algebraic viewpoint."

The construction of Haar bases on unimodular LC groups proceeds via the tools of fractal tilings. After a review of these concepts we prove the key results required to obtain a "good" Haar basis, namely that the boundary of the relevant tilings is of measure 0. We explain how the constructed Haar bases can be used to study Fourier multipliers in such settings, detailing future work.

Contents

Introduction

This thesis consists of four chapters with a division into two main parts. The first two chapters deal with pseudodifferential operator theories in settings in which classical pseudodifferential operator techniques are impossible to apply. The third and fourth chapter represent some developments and ideas related to non-commutative harmonic analysis, namely Haar bases and Heisenberg-like groups.

Chapters 1 and 2 present work developed in tandem. Both of these chapters are concerned with the development of pseudodifferential calculus theories which are suitable for the study of certain Witten Laplacians on $\mathbb{R}^d$. That is, we study operators of the form

$$Lf(x) = -\Delta f(x) + \nabla\phi(x) \cdot \nabla f(x)$$

on function spaces over $\mathbb{R}^d$ equipped with measure weighted by $\exp(-\phi(x))$, for some $\phi : \mathbb{R}^d \to \mathbb{R}$ continuously twice differentiable and approximately quadratic positive semi-definite. The underlying metric measure spaces in such settings are typically non-doubling, which results in significant technical difficulties. The results of García-Cuerva, Mauceri, Meda, Sjögren, and Torrea in [GCMM$^+$01] show that in the Gaussian case $\phi(x) = \frac{x^2}{2}$ in which L is called the Ornstein-Uhlenbeck operator, said operator has strictly holomorphic functional calculus on L^p for $p \in (1, \infty), p \neq 2$. Thus to study such operators via a pseudodifferential calculus, symbol classes necessarily require some holomorphic nature, and this eliminates most classical pseudodifferential techniques.

Despite these complications, van Neerven and Portal in [vNP18] have recovered $L^p - L^q$ boundedness and analytic extension properties of the semigroup generated by the Ornstein-Uhlenbeck operator via the use of a Weyl pseudodifferential calculus tailored to the specific weight $\exp\left(-\frac{x^2}{2}\right)$, with the observation that passage through a Weyl pseudodifferential calculus "splits the analytic and algebraic difficulties". The result of this splitting is that the analytic arguments required in [vNP18] are considerably simpler than those required in previous proofs of the recovered results.

Chapter 1 represents a natural evolution of the work in [vNP18], presenting a dramatically simplified proof of the result of [GCMM$^+$01] via the Weyl pseudodifferential calculus of [vNP18] (although the pseudodifferential calculus is hidden in the proof). Chapter 1 has been published in *Indagationes Mathematicae* as [Har19].

Chapter 2 presents a general theory of Weyl pseudodifferential calculus viable for weighted settings. The Weyl pseudodifferential calculus is defined in such a setting (Definition 2.2). The chapter culminates in the definition of a symbol class $HS_0(M)$ which gives rise to L^p bounded operators (Definition 2.9 and Theorem 8), analogous to Hörmander's symbol class in classical pseudodifferential operator theory. We find that some kind of holomorphic nature is present in such symbols (Remark 5), as is to be expected based on the results of [GCMM$^+$01]. The structure of $HS_0(M)$ prevents the use of classical pseudodifferential operator techniques but is well-suited to algebraic techniques, through which we recover and generalise the main theorem of [GCMM$^+$01] in Theorem 10. Chapter 2 has been uploaded to the arXiv [Har20], and is accepted for publication in the *Illinois Journal of Mathematics*.

Chapter 3 represents the construction of Haar bases on certain locally compact Hausdorff groups and possible applications of the theory, and proceeds via fractal tiling and martingale methods. The chapter starts by recalling the definition of the classical Haar basis on $\mathbb{R}$ along with the topics of multiresolution analysis, fractal tilings, unconditionality, and their link to the classical Haar basis.

We abstract the required structure from $\mathbb{R}$ to the setting of general unimodular metrisable LC groups, and provide in Theorems 3.20, 3.26 and 3.28 the key regularity results necessary to produce a Haar basis (under two mild and mostly group-theoretic assumptions which almost certainly always hold). The proof of Theorem 3.20 is likely the most mathematically beautiful in this thesis. The key L^p properties of the Haar basis are proven, which follow immediately from the method of construction and the use of martingale methods.

The chapter concludes with some examples, discussion around the removal of the assumptions, comparison to existing work, and planned future work. The examples suggest that such Haar basis are significantly "rough" in the non-Abelian setting, in stark contrast to the Abelian setting. The main difference between what is achieved in Chapter 3 and existing work in this field is that we make no use of Lie/nilpotency assumptions or techniques, and we make significant use of martingale techniques. As for further work, we expect very similar methods to those used in [Pet00], [Hyt08], [Hyt10] will allow us to study L^p behaviour of Fourier multipliers via the constructed Haar bases, and we expect that a full wavelet theory can be developed with our Haar bases as the first step with interesting implications of the observed roughness of Haar bases in the non-Abelian setting.

Chapter 4 begins by noting the significance of the Heisenberg group in the results achieved in Chapters 1 and 2. We recall some results about Heisenberg groups of general LCA groups and the significance of such groups in harmonic analysis. Motivated by this significance, we detail the possible implications of having a theory of Heisenberg-like groups in the non-Abelian setting and some of our ongoing investigations in this direction, with interesting possibilities in representation theory and the Langlands program.

Chapter 1

Optimal Holomorphic Functional Calculus for the Ornstein-Uhlenbeck operator

1.1 Introduction

The Ornstein-Uhlenbeck operator appears in many areas of mathematics: as the number operator of quantum field theory, the analogue of the Laplacian in the Malliavin calculus, the generator of the transition semigroup associated with the simplest mean-reverting stochastic process (the Ornstein-Uhlenbeck process), or as the operator associated with the classical Dirichlet form on $\mathbb{R}^d$ equipped with the Gaussian measure $d\mu = (2\pi)^{-\frac{d}{2}} e^{-|x|^2/2} dx$. For the sake of this paper, the Ornstein-Uhlenbeck operator will be defined via the Ornstein-Uhlenbeck semigroup $\{T_t\}_{t>0}$ whose action on $f \in L^p(\mu)$ is

$$T_t f(x) = \int_{\mathbb{R}^d} M_t(x,y) f(y) dy, \text{ for } x \in \mathbb{R}^d$$

where $M_t : \mathbb{R}^{2d} \to \mathbb{R}$ is given by

$$(x,y) \mapsto \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{1}{1-e^{-2t}} \right)^{\frac{d}{2}} \exp\left(-\frac{1}{2} \frac{|e^{-t}x - y|^2}{(1-e^{-2t})} \right), \tag{1.1}$$

the Mehler kernel.

Let us recall the basic properties of the Ornstein-Uhlenbeck semigroup used in this article. For each $p \in [1,\infty]$ and each $t > 0$, the map $f \mapsto T_t f$ is bounded $L^p(\mu) \to L^p(\mu)$, with operator norm at most 1, and is a positive operator. For

$p \in [1,\infty)$, $T_t : L^p(\mu) \to L^p(\mu)$ is a C_0 semigroup, i.e. as $t \to 0$, $T_t \to I$ strongly and $T_tT_s = T_{t+s}$ for all $t, s > 0$. For a proof of these preliminary facts, see for example Theorem 2.5 of [UR19]. It should be noted that although the Ornstein-Uhlenbeck semigroup arises in many different areas of mathematics, these basic properties can be proven solely with use of the explicit kernel and elementary techniques. It is a simple calculation to show that T_t is bounded with norm 1 on both $L^\infty(\mu)$ and $L^1(\mu)$, from which interpolation can be used to deduce boundedness with norm 1 on $L^p(\mu)$ for $p \in [1,\infty]$. Positivity follows from non-negativity of the Mehler kernel. Strong continuity of the semigroup follows as in typical proofs of the strong continuity of the classical heat semigroup, and the semigroup property follows from a somewhat tedious exercise in integrating Gaussian functions. It should be noted that by using other representations of the Ornstein-Uhlenbeck semigroup, such as a spectral multiplier for the multivariate Hermite ONB of $L^2(\mu)$ or through a different representation via an integral kernel, one may prove some of these results even more simply, however the difficulty then becomes showing that all these representations for the Ornstein-Uhlenbeck semigroup are equivalent (for example, see [NN18]). We consider the generator of the Ornstein-Uhlenbeck semigroup on $L^p(\mu)$, $p \in [1,\infty)$, whose negative we shall call the Ornstein-Uhlenbeck operator and denote by L. This operator is a closed densely-defined unbounded operator on $L^p(\mu)$, $p \in [1,\infty)$, which uniquely determines T_t. Thus from here on, we will use the notation $\exp(-tL)$ for the operator T_t, on any of these spaces.

This paper presents a new proof of the following theorem.

Theorem 1. For $p \in (1,\infty)$, the Ornstein-Uhlenbeck operator has a bounded $H^\infty(\Sigma_{\theta_p})$ functional calculus on $L^p(\mu)$, where $\sin(\theta_p) = \left|1 - \frac{2}{p}\right|$.

See [5] for the theory of the H^∞ functional calculus. That L has a bounded H^∞ functional calculus (of some angle $\theta < \pi$) follows from general results in the theory of the H^∞ functional calculus (for example, Theorem 10.7.13 of [HvNVW17] states that any generator of an analytic semigroup on an L^p space for $p \in (1,\infty)$ which is a positive contraction semigroup for real time has a bounded H^∞ functional calculus of some angle less than $\frac{\pi}{2}$). The difficulty in Theorem 1 is to prove the boundedness of the calculus with precisely the optimal angle θ_p.

Theorem 1 was originally proven by García-Cuerva, Mauceri, Meda, Sjögren and Torrea in [GCMM$^+$01], also proving that θ_p is optimal. They use Mauceri's abstract multiplier theorem to reduce the problem to precisely estimating $u \mapsto ||L^{iu}||$. To do so, they express L^{iu} as an integral of the semigroup, using a carefully chosen contour of integration. They then consider the kernels of operators corresponding to different parts of the contour, and decompose them into a local and global part. To treat the global parts they then use a range of subtle kernel estimates.

In [CD17], Carbonaro and Dragičević reproved and extended the result of Theorem 1 to treat arbitrary generators of symmetric contraction semigroups on an L^p space over a σ-finite measure space. Note that as they work on abstract L^p spaces, their result gives dimension independent estimates working over $\mathbb{R}^d$. For their proof, they first reduce the problem to proving a bilinear embedding for the semigroup, with constants depending optimally on the angle θ_p. They then use the Bellman function method, controlling the bilinear form by an optimally (depending on p) chosen function. This function turns out to be a known Bellman function introduced by Nazarov and Treil, but just proving that it has the right properties is a highly non-trivial task.

In contrast, the proof presented in this paper is based on the well-known result that in L^p spaces, the optimal angle of the H^∞ functional calculus of an operator is equal to its optimal angle of R-sectoriality (see [HvNVW17] for the theory of R-sectoriality, and its Theorem 10.7.13 for a proof of the stated result). Our proof that the latter is equal to θ_p uses Theorem 10.3.3 of [HvNVW17], which states an equivalence between an operator A being R-sectorial of angle $\theta < \frac{\pi}{2}$ and $-A$ being the generator of an analytic semigroup of angle $\frac{\pi}{2} - \theta$ which is R-bounded on each smaller sector. To deduce R-boundedness of the Ornstein-Uhlenbeck semigroup on such sectors, a standard result on R-boundedness of integral operators with radially decaying kernels is employed (Proposition 8.2.3 of [HvNVW17]). This key step only requires simple manipulations of the kernel for the Ornstein-Uhlenbeck semigroup. It is based on an approach designed by van Neerven and Portal in [vNP18], where they recover classical results about the Ornstein-Uhlenbeck semigroup in a very direct manner. Their idea is to separate algebraic difficulties from analytic difficulties by considering a non-commutative functional calculus of the Gaussian position and momentum operators (the Weyl calculus). Using this calculus, one sees how to modify the kernels in a way that makes their analysis straightforward. A posteriori, the use of the Weyl calculus can be removed, and the proof can be read as a simple computation exploiting the change of time parameter $t \mapsto \frac{1-e^{-t}}{1+e^{-t}}$ (which has been used by many authors before).

Throughout the paper, we make use of the following notation. The function $\phi : \mathbb{R}^d \to \mathbb{R}$ will have action $x \mapsto \frac{x^2}{2}$. The standard Gaussian measure μ on $\mathbb{R}^d$

will thus be written with density $d\mu = (2\pi)^{-\frac{d}{2}} e^{-\phi(x)} dx$. The Lebesgue measure on $\mathbb{R}^d$ will be denoted by λ. As we only ever work over $\mathbb{R}^d$ with Borel σ-algebra, the measurable space over which we consider Lebesgue spaces will be dropped from the notation. For $\theta \in [0, \pi]$, we will write Σ_θ for the open sector $\{z \in \mathbb{C}\backslash\{0\}; |\arg(z)| < \theta\}$.

1.2 R-Sectoriality of L

To simplify things, for the rest of the article we will assume that $p \in (1, \infty)$ is fixed. Similarly, all concepts of boundedness and R-boundedness will be on either $L^p(\mu)$ or $L^p(\lambda)$ without explicit mention of the space, the measure being clear from context.

Lemma 1.1. *M_t has the alternate form for $t > 0$ and $x, y \in \mathbb{R}^d$,*

$$\begin{aligned} M_t(x,y) = &\frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{1}{1 - e^{-2t}} \right)^{\frac{d}{2}} \exp\left(-s_t \left(\frac{x+y}{2\sqrt{2}} \right)^2 - \frac{1}{4s_t} \left(\frac{x-y}{\sqrt{2}} \right)^2 \right) \\ &\times \exp\left(\frac{1}{2} (\phi(x) - \phi(y)) \right), \end{aligned}$$

where $s_t = \frac{1-e^{-t}}{1+e^{-t}}$.

Proof. We will rearrange the exponent from Equation (1.1) and show that it is equal to the exponent given above for all $x, y \in \mathbb{R}^d$ and $t > 0$, as that is all that has changed between the two representations. For each $t > 0, x, y \in \mathbb{R}^d$ we have

$$
\begin{aligned}
-\frac{1}{2}\frac{|e^{-t}x-y|^2}{(1-e^{-2t})} &= -\frac{1}{2}\frac{|e^{-t}x-y|^2}{(1-e^{-2t})} - \frac{1}{4}(x^2-y^2) + \frac{1}{4}(x^2-y^2) \\
&= -\frac{1}{2}\frac{|e^{-t}x-y|^2}{(1-e^{-2t})} - \frac{1}{4}(x^2-y^2) + \frac{1}{2}(\phi(x)-\phi(y)) \\
&= -\frac{1}{2(1-e^{-2t})}\left(|e^{-t}x-y|^2 + \frac{(1-e^{-2t})}{2}(x^2-y^2)\right) \\
&+ \frac{1}{2}(\phi(x)-\phi(y)) \\
&= -\frac{1}{2(1-e^{-2t})}\left(e^{-2t}x^2 - 2e^{-t}xy + y^2 + \frac{(1-e^{-2t})}{2}(x^2-y^2)\right) \\
&+ \frac{1}{2}(\phi(x)-\phi(y)) \\
&= -\frac{1}{2(1-e^{-2t})}\left(\frac{1}{2}\big(1+e^{-2t}\big)x^2 - 2e^{-t}xy + \frac{1}{2}\big(1+e^{-2t}\big)y^2\right) \\
&+ \frac{1}{2}(\phi(x)-\phi(y)) \\
&= -\frac{1}{8(1-e^{-2t})} \\
&\times \Big[\big((1+e^{-t})^2+(1-e^{-t})^2\big)x^2 + 2\big((1-e^{-t})^2-(1+e^{-t})^2\big)xy \\
&+\big((1+e^{-t})^2+(1-e^{-t})^2\big)y^2\Big] + \frac{1}{2}(\phi(x)-\phi(y)) \\
&= -\frac{1}{8(1-e^{-2t})}\big((1-e^{-t})^2(x+y)^2 + (1+e^{-t})^2(x-y)^2\big) \\
&+ \frac{1}{2}(\phi(x)-\phi(y)) \\
&= -\left(\frac{1-e^{-t}}{1+e^{-t}}\left(\frac{x+y}{2\sqrt{2}}\right)^2 + \frac{1}{4}\frac{1+e^{-t}}{1-e^{-t}}\left(\frac{x-y}{\sqrt{2}}\right)^2\right) \\
&+ \frac{1}{2}(\phi(x)-\phi(y)) \\
&= -\left(s_t\left(\frac{x+y}{2\sqrt{2}}\right)^2 + \frac{1}{4s_t}\left(\frac{x-y}{\sqrt{2}}\right)^2\right) + \frac{1}{2}(\phi(x)-\phi(y)).
\end{aligned}
$$

□

The next definition, albeit a simple one, forms the backbone of the rest of our arguments.

Definition 1.2. Define the (multiple of an) isometry $U_p : L^p(\mu) \to L^p(\lambda)$ by

$$U_p f(x) = f(x) \exp\left(-\frac{\phi(x)}{p}\right), \text{ for } x \in \mathbb{R}^d.$$

As explained previously, we need only show that the Ornstein-Uhlenbeck semigroup has an analytic extension to a sector of the correct angle, and that it is R-bounded on each smaller sector. We will in fact show a lot more with no more effort. We shall work with the reparametrisation of the kernel of the semigroup in terms of s_t from Lemma 2.10. The function $t \mapsto s_t$ is analytic and can clearly be analytically extended to the domain $\mathbb{C} \backslash i\pi(2\mathbb{Z}+1)$. We will consider the analytic extension $z \mapsto s_z$ on domains of the form

$$E := \left\{z \in \mathbb{C}; s_z \in \Sigma_{\frac{\pi}{2}-\theta_p}; z \notin i\pi\mathbb{Z}\right\} \tag{1.2}$$

where $\sin(\theta_p) = M_p := \left|1 - \frac{2}{p}\right|$. We will show the Ornstein-Uhlenbeck semigroup extends to an analytic semigroup on the domain E. Moreover, we will simultaneously show that the Ornstein-Uhlenbeck semigroup is R-bounded on sets of the form

$$E_{\epsilon,\delta} = \left\{ \begin{array}{c} z \in \mathbb{C}; |\Re(s_z)|^2/|s_z|^2 = \cos^2(\arg(s_z)) > M_p^2 + \epsilon; \\ \operatorname{dist}(z, i\pi(2\mathbb{Z}+1)) > \delta; z \notin 2i\pi\mathbb{Z} \end{array} \right\} \tag{1.3}$$

for all $\epsilon, \delta > 0$. Note that, in terms of the reparametrisation s_z, these sets are open sectors of angle $\frac{\pi}{2} - \theta_p$ or less, with certain points removed. We claim that $\Sigma_{\frac{\pi}{2}-\theta_p} \subset E$, and that for all $\epsilon' > 0$ there exists $\epsilon, \delta > 0$ such that $\Sigma_{\frac{\pi}{2}-\theta_p-\epsilon'} \subset E_{\epsilon,\delta}$ (see [vNP18] for details of this calculation). These results combined will imply that the maximal domain of analyticity of the Ornstein-Uhlenbeck semigroup contains the sector $\Sigma_{\frac{\pi}{2}-\theta_p}$, and that it is R-bounded on each smaller sector, which combined with the procedure outlined in the introduction will show at least that the Ornstein-Uhlenbeck operator is R-sectorial of the desired angle.

Theorem 2. For $p \in (1, \infty)$, the Ornstein-Uhlenbeck operator on $L^p(\mu)$ is R-sectorial of angle θ_p, where $\sin(\theta_p) = M_p := \left|1 - \frac{2}{p}\right|$.

Proof. To determine (R-)boundedness of the analytic extension of $\exp(-tL)$ on $L^p(\mu)$ we conjugate by the (multiple of an) isometry $U_p : L^p(\mu) \to L^p(\lambda)$, and work with $U_p \exp(-tL) U_p^{-1}$ on $L^p(\lambda)$. As (multiples of) isometries preserve (R-)boundedness, $\exp(-tL)$ has an analytic extension to $z \in \mathbb{C}$ if and only if $U_p \exp(-tL) U_p^{-1}$ does, and both families of operators will be R-bounded on the same subdomains of the domain of analyticity. Using the integral kernel of Lemma

2.10 and the explicit form of the isometry U_p from Definition 2.5, we find the integral representation for $f \in L^p(\lambda)$:

$$U_p \exp(-tL)U_p^{-1} f = \left(x \mapsto \int_{\mathbb{R}^d} k_t(x,y) f(y) dy \right),$$

with

$$\begin{aligned} k_t(x,y) = & \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{1}{1-e^{-2t}} \right)^{\frac{d}{2}} \exp\left(-s_t \left(\frac{x+y}{2\sqrt{2}} \right)^2 - \frac{1}{4s_t} \left(\frac{x-y}{\sqrt{2}} \right)^2 \right) \\ & \times \exp\left(\left(\frac{1}{2} - \frac{1}{p} \right) (\phi(x) - \phi(y)) \right) \end{aligned}$$

and $s_t = \frac{1-e^{-t}}{1+e^{-t}}$. If $U_p \exp(-tL)U_p^{-1}$ were to have an analytic extension $U_p \exp(-zL)U_p^{-1}$ for z in some domain containing $[0,\infty)$, uniqueness theory of analytic functions implies that $U_p \exp(-zL)U_p^{-1}$ would also have an integral representation, with kernel

$$\begin{aligned} k_z(x,y) = & \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{1}{1-e^{-2z}} \right)^{\frac{d}{2}} \exp\left(-s_z \left(\frac{x+y}{2\sqrt{2}} \right)^2 - \frac{1}{4s_z} \left(\frac{x-y}{\sqrt{2}} \right)^2 \right) \\ & \times \exp\left(\left(\frac{1}{2} - \frac{1}{p} \right) (\phi(x) - \phi(y)) \right), \end{aligned}$$

where $s_z = \frac{1-e^{-z}}{1+e^{-z}}$. To understand why this must be the case, we can act $U_p \exp(-tL)U_p^{-1}$ on some $f \in L^p(\mu)$, and then pair with some $g \in (L^p(\mu))^* = L^{p'}(\mu)$ to obtain a function $\mathbb{R}^+ \to \mathbb{C}$, $t \mapsto \langle U_p \exp(-tL)U_p^{-1} f, g \rangle$. This function will have an analytic extension to the set of z for which the operator with integral kernel $k_z(x,y)$ is bounded on $L^p(\mu)$, and standard uniqueness results for $\mathbb{C}$-valued analytic functions implies that the analytic extension will be given by the operator with integral kernel $k_z(x,y)$ applied to f and paired with g. Thus $U_p \exp(-tL)U_p^{-1}$ would have as weak-analytic extension the operator with integral kernel $k_z(x,y)$, to the set of z for which this is bounded on $L^p(\mu)$. By the equivalence of strong-analytic and weak-analytic Banach space valued functions (see, for example, Chapter VII §3, Exercise 4 of [Con90]), the claim follows. (There is a slight notational issue here, in that the definition of an analytic semigroup on a Banach space X is only ever analytic in the strong operator topology, such that the functions $z \mapsto \exp(-zL)f$ are X-valued norm-analytic functions, for each $f \in X$).

We will now work on bounding $k_z(x,y)$. We start by assuming that $z \in E$ (see Equation (1.2)). Note that this implies $\Re(s_z) > 0$ and $1 - e^{-2z} \neq 0$. Then we have:

$$\begin{aligned}|k_z(x,y)| &\leq \frac{1}{(2\pi)^{\frac{d}{2}}}\left|\frac{1}{1-e^{-2z}}\right|^{\frac{d}{2}} \exp\left(-\Re(s_z)\left(\frac{x+y}{2\sqrt{2}}\right)^2 - \frac{1}{4}\Re\left(\frac{1}{s_z}\right)\left(\frac{x-y}{\sqrt{2}}\right)^2\right) \\ &\quad\times \exp\left(\left(\frac{1}{2}-\frac{1}{p}\right)(\phi(x)-\phi(y))\right) \\ &\leq \frac{1}{(2\pi)^{\frac{d}{2}}}\left|\frac{1}{1-e^{-2z}}\right|^{\frac{d}{2}} \\ &\quad\times \exp\left(-\Re(s_z)\left(\frac{x+y}{2\sqrt{2}}\right)^2 + M_p\frac{1}{4}|x^2-y^2| - \frac{1}{4}\Re\left(\frac{1}{s_z}\right)\left(\frac{x-y}{\sqrt{2}}\right)^2\right) \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}}\left|\frac{1}{1-e^{-2z}}\right|^{\frac{d}{2}} \\ &\quad\times \exp\left(-\Re(s_z)\left(\frac{x+y}{2\sqrt{2}}\right)^2 + M_p\left|\frac{x+y}{2\sqrt{2}}\right|\left|\frac{x-y}{\sqrt{2}}\right| - \frac{1}{4}\Re\left(\frac{1}{s_z}\right)\left(\frac{x-y}{\sqrt{2}}\right)^2\right)\end{aligned}$$

For notational simplicity, let $u = \left|\frac{x+y}{2\sqrt{2}}\right|$ and $k = \left|\frac{x-y}{\sqrt{2}}\right|$. Then rewriting in terms of u and k and completing the square in u gives

$$\begin{aligned}|k_z(x,y)| &\leq \frac{1}{(2\pi)^{\frac{d}{2}}}\left|\frac{1}{1-e^{-2z}}\right|^{\frac{d}{2}} \exp\left(-\Re(s_z)u^2 + M_p uk - \frac{1}{4}\Re\left(\frac{1}{s_z}\right)k^2\right) \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}}\left|\frac{1}{1-e^{-2z}}\right|^{\frac{d}{2}} \\ &\quad\times \exp\left(-\left(\sqrt{\Re(s_z)}u - \frac{M_p}{2\sqrt{\Re(s_z)}}k\right)^2 - \frac{1}{4}\left(\Re\left(\frac{1}{s_z}\right) - \frac{M_p^2}{\Re(s_z)}\right)k^2\right).\end{aligned}$$

So

$$\begin{aligned}|k_z(x,y)| &\leq \frac{1}{(2\pi)^{\frac{d}{2}}}\left|\frac{1}{1-e^{-2z}}\right|^{\frac{d}{2}} \exp\left(-\frac{1}{4}\left(\Re\left(\frac{1}{s_z}\right) - \frac{M_p^2}{\Re(s_z)}\right)k^2\right) \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}}\left|\frac{1}{1-e^{-2z}}\right|^{\frac{d}{2}} \exp\left(-\frac{1}{4}\left(\Re\left(\frac{1}{s_z}\right) - \frac{M_p^2}{\Re(s_z)}\right)\left(\frac{x-y}{\sqrt{2}}\right)^2\right). \\ &= \frac{1}{(2\pi)^{\frac{d}{2}}}\left|\frac{1}{1-e^{-2z}}\right|^{\frac{d}{2}} \exp\left(-\frac{1}{8}\left(\Re\left(\frac{1}{s_z}\right) - \frac{M_p^2}{\Re(s_z)}\right)(x-y)^2\right).\end{aligned}$$

For each $z \in E$, let $g_z : \mathbb{R}^d \to \mathbb{R}$ be the function

$$x \mapsto \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \frac{1}{1 - e^{-2z}} \right|^{\frac{d}{2}} \exp\left(-\frac{1}{8} \left(\Re\left(\frac{1}{s_z} \right) - \frac{M_p^2}{\Re(s_z)} \right) x^2 \right).$$

Then we have that for all $z \in E$, $f \in L^p(\lambda)$ and a.e. $x \in \mathbb{R}^d$

$$\left| \left(U_p \exp(-zL) U_p^{-1} f \right)(x) \right| \leq (g_z * |f|)(x).$$

Therefore, provided the family of convolution operators $f \in L^p(\lambda) \mapsto g_z * f$ is (R-)bounded for z in (a subset of) E, we will have proven, by domination and isometry, that $\exp(-zL)$ is (R-)bounded on (the same subset of) E (to see that domination implies R-boundedness, see Proposition 8.1.10 of [HvNVW17], and note that in the proof of said proposition the fixed positive operator can be replaced by an R-bounded family of positive operators). For $z \in E$, we find

$$\Re\left(\frac{1}{s_z} \right) - \frac{M_p^2}{\Re(s_z)} = \frac{\Re(s_z)}{|s_z|^2} - \frac{M_p^2}{\Re(s_z)} = \frac{1}{\Re(s_z)} \left(\frac{|\Re(s_z)|^2}{|s_z|^2} - M_p^2 \right) > 0,$$

since $\Re(s_z) > 0$ and $|\Re(s_z)|^2/|s_z|^2 = \cos^2(\arg(s_z)) > M_p^2$ by definition of E (since $\cos\left(\frac{\pi}{2} - \theta_p\right) = \sin(\theta_p) = M_p$). So for $z \in E$, $g_z \in L^1(\lambda)$ and so by Young's convolution inequality, convolution by g_z is a bounded operator on $L^p(\lambda)$ with operator norm at most $||g_z||_{L^1(\lambda)}$. Now we will focus on sets of the form $E_{\epsilon,\delta}$ for some fixed $\epsilon, \delta > 0$ (see Equation (1.3)). We will show that

$$\sup_{z \in E_{\epsilon,\delta}} \int_{\mathbb{R}^d} \sup_{|y|>|x|} |g_z(y)| dx < \infty,$$

from which we can apply Proposition 8.2.3 of [HvNVW17] to find that the family of convolution operators $\{g_z *\}_{z \in E_{\epsilon,\delta}}$ is R-bounded on $L^p(\lambda)$. Noting that each g_z

is radially decaying and positive, the quantity to bound is

$$\begin{aligned}
\sup_{z\in E_{\epsilon,\delta}} \int_{\mathbb{R}^d} \sup_{|y|>|x|} |g_z(y)|dx &= \sup_{z\in E_{\epsilon,\delta}} \int_{\mathbb{R}^d} g_z(x)dx \\
&= \sup_{z\in E_{\epsilon,\delta}} \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{\frac{d}{2}}}\left|\frac{1}{1-e^{-2z}}\right|^{\frac{d}{2}} \\
&\quad \times \exp\left(-\frac{1}{8}\left(\Re\left(\frac{1}{s_z}\right)-\frac{M_p^2}{\Re(s_z)}\right)x^2\right)dx \\
&= \sup_{z\in E_{\epsilon,\delta}} \frac{1}{(2)^{\frac{d}{2}}}\left|\frac{1}{1-e^{-2z}}\right|^{\frac{d}{2}}\left(\frac{1}{8}\left(\Re\left(\frac{1}{s_z}\right)-\frac{M_p^2}{\Re(s_z)}\right)\right)^{-\frac{d}{2}} \\
&\leq \sup_{z\in E_{\epsilon,\delta}} 2^d\left|\frac{1}{1-e^{-2z}}\right|^{\frac{d}{2}}\left(\frac{\epsilon}{\Re(s_z)}\right)^{-\frac{d}{2}} \\
&= \sup_{z\in E_{\epsilon,\delta}} \epsilon^{-\frac{d}{2}}2^d\left(\left|\frac{\Re(s_z)}{1-e^{-2z}}\right|\right)^{\frac{d}{2}} \\
&\leq \sup_{z\in E_{\epsilon,\delta}} \epsilon^{-\frac{d}{2}}2^d\left(\frac{|s_z|}{|1-e^{-z}||1+e^{-z}|}\right)^{\frac{d}{2}} \\
&= \sup_{z\in E_{\epsilon,\delta}} \epsilon^{-\frac{d}{2}}2^d\left(\frac{\left|\frac{1-e^{-z}}{1+e^{-z}}\right|}{|1-e^{-z}||1+e^{-z}|}\right)^{\frac{d}{2}} \\
&= \sup_{z\in E_{\epsilon,\delta}} \epsilon^{-\frac{d}{2}}2^d\left(\frac{1}{|1+e^{-z}|}\right)^{d} \\
&< \infty
\end{aligned}$$

since z is bounded away from $(2\mathbb{Z}+1)i\pi$. So the family of convolution operators $\{g_z*\}_{z\in E_{\epsilon,\delta}}$ is R-bounded. By pointwise domination, $U_p\exp(-zL)U_p^{-1}$ is bounded for $z\in E$, and is R-bounded on subsets $E_{\epsilon,\delta}\subset E$ of the form (1.3). Hence by isometric equivalence, $\exp(-zL)$ shares the same properties. Hence the claim follows from the discussion preceding this proof. $\square$

Chapter 2

A Weyl Pseudodifferential Calculus associated with exponential weights on $\mathbb{R}^d$

2.1 Introduction

We construct a Weyl pseudodifferential calculus on typically non-doubling measure spaces over $\mathbb{R}^d$, with an aim to study the L^p behaviour of the natural analogue of the Laplacian in such contexts. Typically, this analogue of the Laplacian is such that bounded spectral multipliers on L^p have to be of holomorphic type for $p \neq 2$. From a PDE point of view, the operators we consider are perturbations of the Laplacian by an unbounded drift term, including the classical finite-dimensional Ornstein-Uhlenbeck operator.

In the paper [vNP18], van Neerven and Portal introduce a Gaussian Weyl calculus to study the classical Ornstein-Uhlenbeck operator $L = -\Delta + x \cdot \nabla$ on $L^p\Big(\mathbb{R}^d, (2\pi)^{-\frac{d}{2}} \exp\Big(-\frac{x^2}{2}\Big)dx\Big)$. This approach retrieves important known results about the Ornstein-Uhlenbeck semigroup, such as boundedness $L^p \to L^q$ and optimal domains of holomorphy for the semigroup, using analytic arguments as simple as Schur estimates. With classical approaches, such properties are difficult to prove.

The classical Ornstein-Uhlenbeck operator can be written in terms of a pair of operators P (momentum) and Q (position), satisfying the Heisenberg commutation relations. The Weyl calculus examined by van Neerven and Portal in [vNP18] is a certain choice of joint functional calculus for this pair (Q, P), that is, a way to assign to a suitable function $a : \mathbb{R}^{2d} \to \mathbb{C}$ a bounded operator $a(Q, P)$. It was their philosophy that studying L via studying the joint functional calculus was more natural, as it separates the strong algebraic properties of the pair (Q, P) from the analytic issues found in proving properties of L directly. Essentially, studying L directly is forgetting that it has its roots in an algebraically rich setting. The approach used by van Neerven and Portal in [vNP18] is well-adapted to studying the standard Ornstein-Uhlenbeck semigroup, thanks to an exact formula for the semigroup in the (Q, P)-calculus.

To generalise the theory developed in [vNP18], van Neerven and Portal consider in [vNP20] pairs of d-tuples of operators on a Banach space, which generate uniformly bounded groups satisfying certain commutation relations, which they refer to as a Weyl pair. This path of generalisation has been fruitful, and proves that the sum of squares of all $2d$ operators comprising a Weyl pair with a bounded Weyl calculus has a bounded Hörmander functional calculus (under some mild conditions).

However, in the Gaussian case, the natural position and momentum operators generate uniformly bounded groups on L^p if and only if $p = 2$. In fact, $\exp(i\xi P)$ is bounded on L^p for $p \neq 2$ if and only if $\xi = 0$, in which case $\exp(i0P)$ is the identity. Thus the theory developed in [vNP20] cannot possibly be applied in Gaussian situations. In this paper we consider such cases.

We will work on measure spaces of the form $(\mathbb{R}^d, \mathcal{B}, \mu)$, where $\mathcal{B}$ is the Borel σ-algebra and $d\mu = \exp(-\phi(x))dx$ for $\phi \in C^2(\mathbb{R}^d)$ approximately quadratic (see Remark 2). We introduce a generalised Weyl pair (Q, P) associated with such a measure via a specific unitary equivalence on $L^2(\mu)$ to the standard Weyl pair on $\mathbb{R}^d$ equipped with Lebesgue measure. Typically, the pair (Q, P) will not generate uniformly bounded groups on $L^p(\mu)$ for $p \neq 2$. Thus, our generalised Weyl calculus will be developed as an extension theory: we work on $L^p(\mu) \cap L^2(\mu)$ and find conditions under which we have a bounded extension to $L^p(\mu)$.

For a function $M : \mathbb{R}^d \to \mathbb{R}^d$, we introduce the normed vector space $HS_0(M) \subset L^\infty\left(\mathbb{R}^{2d}\right)$ of Holomorphic Strip symbols. The main theorem of this paper (Theorem 8) proves that for a correctly chosen function M based on p and ϕ, $a \in HS_0(M)$ implies that $a(Q, P)$ extends from $L^p(\mu) \cap L^2(\mu)$ to a bounded operator on $L^p(\mu)$, with norm bounded by $||a||_{HS_0(M)}$. We prove that symbols in $HS_0(M)$ have a certain holomorphic extendability property, which is reminiscent of the optimal functional calculus result for the classical Ornstein-Uhlenbeck operator in the Gaussian case. We also deduce a simple condition on a set $A \subset HS_0(M)$ which implies that $\{a(Q, P); a \in A\}$ is R-bounded on $L^p(\mu)$.

This theory is used to give a short proof of an upper bound on the optimal angle of the bounded functional calculus result for the Ornstein-Uhlenbeck operator associated with ϕ of the form $\phi(x) = xN(x) + lx$, where $N : \mathbb{R}^d \to \mathbb{R}^d$ is a positive definite real-symmetric linear operator, and $l \in \mathbb{R}^d$. This result returns the known optimal angle for the classical Ornstein-Uhlenbeck operator, for which $N = \frac{1}{2}I, l = 0$. The proof depends on knowing the Weyl symbol for the semigroup generated by the Ornstein-Uhlenbeck operator, which we provide. This explicit knowledge of the symbol in the classical case is also crucial to the results of the paper [vNP18].

The space $HS_0(M)$ is then examined further, with an aim to find symbols for the semigroup generated by the relevant Ornstein-Uhlenbeck operator outside of the quadratic case. We show that $HS_0(M)$ can be expanded and endowed with a certain product to form the unital Banach algebra $(HS(M), \#)$, with $HS(M) \subset L^\infty(\mathbb{R}^{2d})$ and $\#$ the Moyal product, in which case the generalised Weyl calculus can be extended in a natural way such that it becomes a contractive Banach algebra homomorphism $HS(M) \to B(L^p(\mu))$.

Some ideas are then presented, about how the symbol class $HS(M)$ and its properties as a Banach algebra of functions could be used to study Ornstein-Uhlenbeck like operators, such as determining symbols for relevant semigroups and a factorisation of the functional calculus through the space $HS(M)$.

2.2 Initial Definitions

The work of van Neerven and Portal in [vNP20] develops a Weyl calculus for $2d$-tuples of operators satisfying the following definition:

Definition 2.1. A $2d$-tuple $(Q, P) = (Q_1, \ldots, Q_d, P_1, \ldots, P_d)$ of closed, densely defined operators on a Banach space is called a Weyl pair if each operator generates a uniformly bounded group $\exp(ix_iQ_i), \exp(i\xi_iP_i)$ which satisfy the integrated canonical commutation relations

$$\begin{gathered}\exp(ix_iQ_i)\exp(ix_jQ_j) = \exp(ix_jQ_j)\exp(ix_iQ_i), \forall i, j = 1, ...d \\ \exp(i\xi_iP_i)\exp(i\xi_jP_j) = \exp(i\xi_jP_j)\exp(i\xi_iP_i), \forall i, j = 1, ...d \\ \exp(ixQ)\exp(i\xi P) = \exp(-ix\xi)\exp(i\xi P)\exp(ixQ).\end{gathered}$$

In which case we define

$$\exp(i(xQ + \xi P)) = \exp(\frac{1}{2}ix\xi)\exp(ixQ)\exp(i\xi P), \forall x, \xi \in \mathbb{R}^d.$$

This definition of $\exp(i(xQ+\xi P))$ for Weyl pairs can be motivated by the Baker-Campbell-Hausdorff formula, for example, noting that formally differentiating the integrated commutation relations produces the formal commutation relation "$[Q_i, P_j] = i\delta_{ij}I$". It is easy to check that this definition does make the set of operators $\{\lambda \exp(i(xQ+\xi P)); x, \xi \in \mathbb{R}^d . \lambda \in \mathbb{C}, |\lambda| = 1\}$ into a (non-commutative) group, in fact a representation of the $(2d+1)$-dimensional Heisenberg group. From here, the Weyl calculus is defined as:

Definition 2.2. For a Weyl pair (Q, P) on a Banach space, we define the bounded operator $a(Q, P)$ for $a \in \mathcal{S}(\mathbb{R}^{2d})$ via the formula

$$a(Q, P) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \mathcal{F}a(x, \xi) \exp(i(xQ+\xi P)) dx d\xi,$$

where $\mathcal{F}$ denotes the Fourier transform, normalised such that equality holds above with Q and P replaced by elements of $\mathbb{R}^d$. The map $a \mapsto a(Q, P)$ is called the Weyl pseudodifferential calculus, or the joint functional calculus, for the Weyl pair (Q, P).

That this definition makes sense follows from boundedness of the set of operators $\exp(i(xQ+\xi P))$ and integrability of the Schwartz function $\mathcal{F}a$. As an example of a Weyl pair, consider the position and momentum operators (X, D) on the Euclidean Lebesgue space $L^p(\lambda)$ ($p \in [1, \infty]$) given by $X_i f(x) = x_i f(x), D_i f(x) = -i\frac{\partial}{\partial x_i} f(x)$ equipped with their natural domains. These generate the groups of phase shift and translation respectively, both of which are bounded. That they satisfy the integrated canonical commutation relations is a simple exercise, which once checked shows that (X, D) on $L^p(\lambda)$ are a Weyl pair, for any value of $p \in [1, \infty]$. The pair (X, D) will be referred to as the standard pair, and their Weyl calculus is the standard Weyl calculus (see for example [H$\ddot{7}$9]).

The body of the work of [vN,P] follows from the next theorem, displaying the semigroup generated by $\frac{1}{2}(Q^2+P^2-d)$ in the joint functional calculus, which turns out to be the correct expression for the Ornstein-Uhlenbeck operator in the case they consider. This formula has been known by physicists for the standard pair $(Q, P) = (X, D)$ (in which case $\frac{1}{2}(X^2+D^2-d)$ is the harmonic oscillator operator minus $d/2$ times the identity), and relies heavily on the simple algebraic nature of $\frac{1}{2}(Q^2+P^2)$ in terms of (Q, P).

Theorem 3. For $t > 0$, let $a_t : \mathbb{R}^{2d} \to \mathbb{C}$ be the function

$$(x, \xi) \mapsto \left(\frac{2}{1+e^{-t}}\right)^d \exp(-s_t(x^2+\xi^2)),$$

where $s_t = \frac{1-e^{-t}}{1+e^{-t}}$. Then $\{a_t(Q,P)\}_{t>0}$ is the semigroup generated by $\frac{1}{2}(Q^2+P^2-d)$, on its natural domain inherited from the domains of Q and P.

In this paper we wish to develop a different side to the Weyl calculus, in which we do not require the bounded group assumption of Definition 2.1. The main reason for this is because the key example we wish to apply the Weyl calculus to - the original motivation of study for van Neerven and Portal in [vNP18] - is the natural position and momentum operator pair in the standard Gaussian weighted spaces $L^p\Big(\mathbb{R}^d, \exp\Big(-\frac{x^2}{2}\Big)dx\Big)$, in which the momentum operator P does not satisfy the bounded group generation property unless $p = 2$. However, in this case we can define the Weyl calculus on L^2 via Definition 2.2 and then determine conditions under which $a(Q,P)$ extends to a bounded operator on L^p. The cases we will consider will be based over $\mathbb{R}^d$ equipped with Borel measures of the following form.

Definition 2.3. A function $\phi \in C^2(\mathbb{R}^d)$ will be referred to as a potential. associated with a potential ϕ is the Borel measure μ on $\mathbb{R}^d$ with $d\mu = \exp(-\phi(x))dx$.

We will generally assume ϕ to be a twice differentiable function throughout the paper. Although the initial definitions work for any C^2 function ϕ, to obtain boundedness of operators we will later have to restrict to ϕ which is approximately quadratic (see Remark 2). To be able to relate things to the standard pair, we proceed via unitary equivalence:

Definition 2.4. For $c > 0$, define the (multiples of) unitary transformations

$$\tilde{U}_2 = A \circ E : L^2(\mu) \to L^2(\lambda)$$

where

$$\begin{aligned} E : L^2(\mu) &\to L^2(\lambda) \\ f &\mapsto \Big(x \mapsto f(x)\exp(-\tfrac{\phi(x)}{2})\Big) \end{aligned}$$

$$\begin{aligned} A : L^2(\lambda) &\to L^2(\lambda) \\ f &\mapsto (x \mapsto f(cx)) \end{aligned}$$

Definition 2.5. Define the isometry $U_p : L^p(\mu) \to L^p(\lambda)$ by

$$f \mapsto \left(x \mapsto f(x)\exp(-\frac{\phi(x)}{p})\right)$$

The generalised Weyl pairs we will consider in this context will be

$$\begin{aligned} Q_i &= \tilde{U}_2^{-1} \circ X_i \circ \tilde{U}_2 \\ P_i &= \tilde{U}_2^{-1} \circ D_i \circ \tilde{U}_2 \end{aligned}$$

with domains $\tilde{U}_2^{-1} D(X_i)$, $\tilde{U}_2^{-1} D(D_i)$ respectively. That (Q, P) satisfy the requirements of Definition 2.1 on $L^2(\mu)$ follows from their unitary equivalence to the standard pair. It is a simple exercise to determine the action of Q and P on their domains. They act as

$$\begin{aligned} (Q_i f)(x) &= \tfrac{x_i}{c} f(x) \\ (P_i f)(x) &= -ic\Big(\tfrac{\partial}{\partial x_i} - \tfrac{1}{2}\tfrac{\partial \phi}{\partial x_i}\Big) f(x). \end{aligned}$$

The integral kernel of operators $a(Q, P)$ can be calculated explicitly via unitary equivalence to the standard Weyl Calculus (see, for example, [Ḧ79]), leading to the formula:

Theorem 4. For $a \in \mathcal{S}(\mathbb{R}^{2d})$, Q, P as above and $f \in L^2(\mu)$, we have for all $y \in \mathbb{R}^d$

$$\begin{aligned} (a(Q, P)f)(y) = \frac{1}{(2\pi)^d c^d} \int_{\mathbb{R}^{2d}} & a\left(\frac{x+y}{2c}, \xi\right) \exp\left(-i\xi\left(\frac{x-y}{c}\right)\right) \\ & \times \exp\left(\frac{1}{2}(\phi(y) + \phi(x))\right) f(x) d\xi d\mu(x). \end{aligned}$$

Remark 1. The inclusion of c is to satisfy physicists. What many physicists would consider as THE Ornstein-Uhlenbeck operator, and what we shall refer to as the classical Ornstein-Uhlenbeck operator, is related to the choice $\phi(x) = \frac{x^2}{2}$. In its relationship to Fock spaces, a scaling is introduced to make the classical Ornstein-Uhlenbeck operator appear more symmetric in some sense, which corresponds to taking $c = \sqrt{2}$. However, as we shall see later (take for example Definition 2.16), in many ways it seems more natural to take $c = 1$. We will not include the subscript c as part of the notation, but it will always be lurking in the background ready to be set to 1 or $\sqrt{2}$. Note that the un-tilde'd U_2 falls under Definition 2.5, and does not depend on c.

To ensure technical correctness, we need some information about the domains of important operators. The next theorem provides a suitable p-independent core for our functional calculus

Theorem 5. The space $\mathcal{C}_\phi = U_2^{-1} C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mu)$ for all $p \in [1, \infty)$, and elements of $\mathcal{C}_\phi$ are in $C_c^2(\mathbb{R}^d)$.

Proof. Regularity follows from the chain rule and the regularity of ϕ. To show density in $L^p(\mu)$ we will show density of $U_p\mathcal{C}_\phi$ in $L^p(\lambda)$, employing the isometry of Definition 2.5. Note that $U_p\mathcal{C}_\phi$ contains every function of the form $g\exp\Big(\Big(\frac{1}{2}-\frac{1}{p}\Big)\phi\Big)$ for $g \in C_c^\infty(\mathbb{R}^d)$, which are all bounded since $\exp\Big(\Big(\frac{1}{2}-\frac{1}{p}\Big)\phi\Big)$ is continuous and hence bounded on the compact set $\mathrm{supp}(g)$. Let $X \subset L^p(\lambda)$ be those functions which have compact support. Note that X is dense in $L^p(\lambda)$ and $U_p\mathcal{C}_\phi$ is contained in X, so if we can show $U_p\mathcal{C}_\phi$ is dense in X then we are done. To this effect, fix some $f \in X$. Since f is compactly supported, $\exp\Big(-\Big(\frac{1}{2}-\frac{1}{p}\Big)\phi\Big)$ attains a maximum on $\mathrm{supp}(f)$, and so $F(\cdot) = f(\cdot)\exp\Big(-\Big(\frac{1}{2}-\frac{1}{p}\Big)\phi(\cdot)\Big) \in L^p(\lambda)$. Since $C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\lambda)$, we can choose a sequence $\{g_n\} \subset C_c^\infty(\mathbb{R}^d)$ (furthermore, each with support $\mathrm{supp}(g_n) \subset 2\mathrm{supp}(f)$, say) such that $||F-g_n||_{L^p(\lambda)} \to 0$. Thus we have:

$$\begin{aligned}
||f(\cdot)-g_n(\cdot)&\exp\left(\left(\frac{1}{2}-\frac{1}{p}\right)\phi(\cdot)\right)||^p_{L^p(\lambda)} \\
&= \int_{2\mathrm{supp}(f)} \left| f(x) - g_n(x)\exp\left(\left(\frac{1}{2}-\frac{1}{p}\right)\phi(x)\right)\right|^p dx \\
&= \int_{2\mathrm{supp}(f)} \exp\left(\left(\frac{p}{2}-1\right)\phi(x)\right)|F(x)-g_n(x)|^p dx \\
&\leq C\int_{2\mathrm{supp}(f)} |F(x)-g_n(x)|^p dx \\
&\leq C||F-g_n||^p_{L^p(\lambda)} \\
&\to 0.
\end{aligned}$$

So we are done. □

The formula in Theorem 4 and the formula for the semigroup generated by $\frac{1}{2}(Q^2+P^2-d)$ (Theorem 3) is what allowed van Neerven and Portal to deduce such significant results for the classical Ornstein-Uhlenbeck semigroup via the Weyl calculus in [vNP20]. The explicit formula for kernels of $a(Q,P)$ will allow us to work on boundedness on $L^p(\mu)$ of more general symbols.

We will now define the Ornstein-Uhlenbeck operator in our setting. We consider the Dirichlet form $E(f,g) = \int_{\mathbb{R}^d} \nabla f \nabla \overline{g} d\mu$ with domain $\mathcal{C}_\phi$. It follows from the general theory of Dirichlet forms (see [FOT11]) that the operator on $L^2(\mu)$ associated with the form E is positive and generates a positive contraction C_0-semigroup, which extends to a positive contraction C_0-semigroup on $L^p(\mu)$ for all $p \in (1,\infty)$. We call this semigroup the Ornstein-Uhlenbeck semigroup and denote it by $\exp(-tL)$, and its generator the Ornstein-Uhlenbeck operator which we will denote by L. It is a simple computation to find that on $f \in \mathcal{C}_\phi$, L has action

$$Lf(x) = -\Delta f(x) + \nabla\phi(x) \cdot \nabla f(x). \tag{2.1}$$

Due to our set-up, we have the fact:

Corollary 2.6. *For any potential $\phi \in C^2$ and $p \in (1, \infty)$, the corresponding Ornstein-Uhlenbeck operator has a bounded H^∞ functional calculus of some angle less than $\pi/2$.*

This follows from Theorem 10.7.13 of [HvNVW17], as below.

Theorem 6. Suppose (Ω, m) is a measure space (σ-algebra omitted). If an unbounded operator T on $L^p(\Omega, m)$, $p \in (1, \infty)$ generates a positive contraction semigroup, then T has a bounded H^∞ functional calculus of some angle less than $\pi/2$.

After developing the generalised Weyl calculus associated with the generalised Weyl pair (Q, P) as defined above, we will aim to study the Ornstein-Uhlenbeck operator via the generalised Weyl calculus. In [vNP18], the formal expression for the classical Ornstein-Uhlenbeck operator $L = \frac{1}{2}(Q^2+P^2-d)$ was very important. This is no longer true in our case. We have as a replacement the following theorem, representing L associated with ϕ in the generalised Weyl calculus of the pair (Q, P) associated with ϕ (at least formally).

Theorem 7. Take Q, P as above, and let $h : \mathbb{R}^{2d} \to \mathbb{R}$ be the function taking value $h(x, \xi) = \frac{\xi^2}{c^2} + C_\phi(x)$, where $C_\phi(x) = \frac{1}{4}|\nabla\phi(cx)|^2 - \frac{1}{2}\Delta\phi(cx)$. For $f \in \mathcal{C}_\phi$, define $h(Q, P)f \in L^1_{loc}(\mathbb{R}^d)$ by $((x, \xi) \mapsto C_\phi(x))(Q, P)$ interpreted as a multiplication operator by $C_\phi(x/c)$ and $\left((x, \xi) \mapsto \frac{\xi^2}{c^2}\right)(Q, P)$ interpreted as $\frac{1}{c^2}P^2$. Then $Lf = h(Q, P)f$.

Proof. Fix $f \in \mathcal{C}_\phi$. Let $H = U_2 L U_2^{-1}$. Note that $U_2 f =: g \in C_c^\infty(\mathbb{R}^d)$, so we want to consider H acting on $g \in C_c^\infty(\mathbb{R}^d)$. Calculating $(Hg)(x)$ for all $x \in \mathbb{R}^d$ gives:

$$(U_2^{-1} g)(x) = g(x/c) \exp\left(\frac{\phi(x)}{2}\right).$$

So (suppressing the summation over k until the last equality)

$$\begin{aligned}
(LU_2^{-1}g)(x) &= \left(\left(-(\partial_k)^2 + \partial_k\phi(\cdot)\partial_k\right)g(\cdot/c)\exp\left(\frac{\phi(\cdot)}{2}\right)\right)(x) \\
&= \left(-\partial_k\left(\frac{1}{c}\partial_k g(\cdot/c)\exp\left(\frac{\phi(\cdot)}{2}\right) + \frac{1}{2}g(\cdot/c)\exp\left(\frac{\phi(\cdot)}{2}\right)\partial_k\phi(\cdot)\right)\right)(x) \\
&+ \partial_k\phi(x)\left(\frac{1}{c}\partial_k g(x/c)\exp\left(\frac{\phi(x)}{2}\right) + \frac{1}{2}g(x/c)\exp\left(\frac{\phi(x)}{2}\right)\partial_k\phi(x)\right) \\
&= -\frac{1}{c^2}\partial_k^2 g(x/c)\exp\left(\frac{\phi(x)}{2}\right) - \frac{1}{2c}\partial_k g(x/c)\exp\left(\frac{\phi(x)}{2}\right)\partial_k\phi(x) \\
&- \frac{1}{2c}\partial_k g(x/c)\exp\left(\frac{\phi(x)}{2}\right)\partial_k\phi(x) - \frac{1}{4}g(x/c)\exp\left(\frac{\phi(x)}{2}\right)(\partial_k\phi(x))^2 \\
&- \frac{1}{2}g(x/c)\exp\left(\frac{\phi(x)}{2}\right)\partial_k^2\phi(x) \\
&+ \frac{1}{c}\partial_k g(x/c)\exp\left(\frac{\phi(x)}{2}\right)\partial_k\phi(x) + \frac{1}{2}g(x/c)\exp\left(\frac{\phi(x)}{2}\right)(\partial_k\phi(x))^2 \\
&= -\frac{1}{c^2}\partial_k^2 g(x/c)\exp\left(\frac{\phi(x)}{2}\right) + \frac{1}{4}g(x/c)\exp\left(\frac{\phi(x)}{2}\right)(\partial_k\phi(x))^2 \\
&- \frac{1}{2}g(x/c)\exp\left(\frac{\phi(x)}{2}\right)\partial_k^2\phi(x) \\
&= \left(-\frac{1}{c^2}\Delta g(x/c) + \left(\frac{1}{4}|\nabla\phi(x)|^2 - \frac{1}{2}\Delta\phi(x)\right)g(x/c)\right)\exp\left(\frac{\phi(x)}{2}\right).
\end{aligned}$$

So

$$\begin{aligned}
(Hg)(x) &= -\frac{1}{c^2}\Delta g(x) + \left(\frac{1}{4}|\nabla\phi(cx)|^2 - \frac{1}{2}\Delta\phi(cx)\right)g(x) \\
&= \left(\left(-\frac{1}{c^2}\Delta + C_\phi(\cdot)\right)g(\cdot)\right)(x).
\end{aligned}$$

This can be expressed directly in the standard Weyl calculus as $h(X,D)$, for $h(x,\xi) := \frac{\xi^2}{c^2} + C_\phi(x)$ with the same interpretation as in the statement of this theorem. But $h(Q,P)f = U_2^{-1}h(X,D)U_2 f = Lf$, and so we are done. $\square$

2.3 L^p bounds on Weyl Pseudodifferential Operators

In this section we investigate properties of the generalised Weyl calculus associated with a potential ϕ, insofar as they relate to the functional calculus for the corresponding Ornstein-Uhlenbeck operator. In a later section we will return to the study of the symbol calculus.

Our symbols will correspond to certain functions $a \in L^\infty(\mathbb{R}^{2d})$, which shall be denoted $a(x,\xi)$ for $x,\xi \in \mathbb{R}^d$. To define our symbol class, we need a few intermediate definitions.

Definition 2.7. Define the Banach space of L^1 dominated functions $\mathcal{D}_{\infty,1} = \{g \in L^\infty(\mathbb{R}^d; L^1(\mathbb{R}^d)); \exists G \in L^1(\mathbb{R}^d), |g(x,k)| < G(k) \text{ for a.e. } x,k \in \mathbb{R}^d\}$, equipped with the norm

$$||g||_{\mathcal{D}_{\infty,1}} = \inf\big\{||G||_{L^1(\mathbb{R}^d)}; G \in L^1(\mathbb{R}^d), |g(x,k)| < G(k) \text{ for a.e. } x,k \in \mathbb{R}^d\big\}.$$

We won't prove that $\mathcal{D}_{\infty,1}$ is a Banach space, although it is easy. Subadditivity and homogeneity of $||\cdot||_{\mathcal{D}_{\infty,1}}$ is obvious. That $||\cdot||_{\mathcal{D}_{\infty,1}}$ is positive definite can be seen by noting it is bounded below by the $L^\infty(\mathbb{R}^d; L^1(\mathbb{R}^d))$ norm. Checking that $\mathcal{D}_{\infty,1}$ is complete is a standard exercise in telescoping sums, and using the fact that $L^\infty(\mathbb{R}^d; L^1(\mathbb{R}^d))$ is complete.

Definition 2.8. For $a \in L^\infty(\mathbb{R}^{2d})$, define $I_a : \mathbb{R}^d \to \mathcal{S}'(\mathbb{R}^d)$, via the action at $x \in \mathbb{R}^d$ and for $\varphi \in \mathcal{S}(\mathbb{R}^d)$ as

$$\langle I_a(x), \varphi\rangle = \int_{\mathbb{R}^d} a(x,\xi)\varphi(\xi)d\xi.$$

We will make use of the Fourier transform $\mathcal{F}$ acting on tempered distributions $\sigma \in \mathcal{S}'(\mathbb{R}^d)$, which we normalise such that $\langle \mathcal{F}\sigma, \varphi\rangle = \langle \sigma, \mathcal{F}^*\varphi\rangle$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, where

$$\mathcal{F}^*\varphi(k) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \varphi(\xi)\exp(i\xi k)d\xi.$$

Now we can define our symbol class.

Definition 2.9. Fix $M : \mathbb{R}^d \to \mathbb{R}^d$. The space $HS_0(M)$ (standing for Holomorphic Strip) is a subspace of $L^\infty(\mathbb{R}^{2d})$, with

$$HS_0(M) = \left\{ \begin{array}{c} a \in L^\infty(\mathbb{R}^{2d}); \exists! g_a \in \mathcal{D}_{\infty,1}, \forall x \in \mathbb{R}^d, \forall\varphi \in \mathcal{S}(\mathbb{R}^d), \\ \langle \mathcal{F}(I_a(x)), \varphi\rangle = (2\pi)^{\frac{d}{2}} \int_{\mathbb{R}^d} \exp(-|M(x)k|)g_a(x,k)\varphi(k)dk \end{array} \right\}$$

and norm defined by

$$||a||_{HS_0(M)} = ||g_a||_{\mathcal{D}_{\infty,1}}.$$

For $a \in HS_0(M)$, we define $\mathcal{F}_2 a$ to be the measurable function $\mathbb{R}^{2d} \to \mathbb{C}$ with action

$$(x,k) \mapsto (2\pi)^{\frac{d}{2}} \exp(-|M(x)k|)g_a(x,k).$$

In some sense, $\mathcal{F}_2 a(x,k)$ is the "Fourier transform in the second (ξ) variable" with k the variable dual to ξ, which is our reason for using this notation. It is easy to verify that if $a(x,\cdot) \in L^1(\mathbb{R}^d)$ for each $x \in \mathbb{R}^d$, then $\mathcal{F}_2 a$ is indeed the Fourier transform in the second variable for each fixed x. However, not all symbols we will consider will be integrable in the second variable, which makes our definition via the space of tempered distributions useful.

We will also often refer to the $g_a \in \mathcal{D}_{\infty,1}, G_a \in L^1(\mathbb{R}^d)$ associated with $a \in HS_0(M)$, by which we mean the unique such g_a as seen in Definition 2.9, and G_a dominating g_a as in Definition 2.7. The main reason for using such a complicated definition is that the integral kernel of $a(Q,P)$ for $a \in \mathcal{S}(\mathbb{R}^{2d})$ from Theorem 4 is closely related to the Fourier transform in the second variable of the Schwartz function a, as in the following lemma.

Lemma 2.10. *Fix a potential $\phi \in C^2$, and let (Q,P) be the associated generalised Weyl pair. For $a \in \mathcal{S}(\mathbb{R}^{2d})$, the integral kernel of the operator $a(Q,P)$ is given by*

$$k(y,x) = \frac{1}{(2\pi)^{d/2}c^d}\mathcal{F}_2 a\left(\frac{x+y}{2c}, \frac{x-y}{c}\right)\exp\left(\frac{1}{2}(\phi(y)-\phi(x))\right).$$

Proof. From Theorem 4, we have for $f \in \mathcal{C}_\phi$, $y \in \mathbb{R}^d$,

$$\begin{aligned}(a(Q,P)f)(y) &= \frac{1}{(2\pi)^d c^d}\int_{\mathbb{R}^{2d}} a\left(\frac{x+y}{2c},\xi\right)\exp\left(-i\xi\left(\frac{x-y}{c}\right)\right)\\ &\quad\times\exp\left(\frac{1}{2}(\phi(y)+\phi(x))\right)f(x)d\xi d\mu(x)\\ &= \frac{1}{(2\pi)^d c^d}\int_{\mathbb{R}^{2d}} a\left(\frac{x+y}{2c},\xi\right)\exp\left(-i\xi\left(\frac{x-y}{c}\right)\right)\\ &\quad\times\exp\left(\frac{1}{2}(\phi(y)-\phi(x))\right)f(x)d\xi dx\\ &= \frac{1}{(2\pi)^{d/2}c^d}\int_{\mathbb{R}^d}\mathcal{F}_2 a\left(\frac{x+y}{2c},\frac{x-y}{c}\right)\exp\left(\frac{1}{2}(\phi(y)-\phi(x))\right)f(x)dx,\end{aligned}$$

noting that the order of integration can be changed, and the integration in ξ carried out, as everything converges absolutely. □

We make use of Lemma 2.10 to extend the generalised Weyl calculus to $HS_0(M)$, as in the following definition:

Definition 2.11. For any function $M : \mathbb{R}^d \to \mathbb{R}^d$, and any potential $\phi \in C^2$, define for $a \in HS_0(M)$ the operator $a(Q,P) : \mathcal{C}_\phi \to L^1_{loc}(\lambda)$ via the action

$$(a(Q,P)f)(y) = \frac{1}{(2\pi)^{d/2}c^d}\int_{\mathbb{R}^d}\mathcal{F}_2 a\left(\frac{x+y}{2c},\frac{x-y}{c}\right)\exp\left(\frac{1}{2}(\phi(y)-\phi(x))\right)f(x)dx.$$

To ensure this definition makes sense, we check that for $f \in \mathcal{C}_\phi$, $a(Q,P)f \in L^1_{loc}(\mathbb{R}^d)$.

Proof. Using the fact that $a \in HS_0(M)$ for some $M : \mathbb{R}^d \to \mathbb{R}^d$, there exists a $G \in L^1(\mathbb{R}^d)$ such that $|\mathcal{F}_2 a(x,k)| \leq G(k)$ for a.e. $x, k \in \mathbb{R}^d$. Hence we find for a.e. $y \in \mathbb{R}^d$

$$\begin{aligned} |(a(Q,P)f)(y)| &= \frac{1}{(2\pi)^{d/2}c^d}\left|\int_{\mathbb{R}^d} \mathcal{F}_2 a\left(\frac{x+y}{2c}, \frac{x-y}{c}\right)\exp\left(\frac{1}{2}(\phi(y)-\phi(x))\right)f(x)dx\right| \\ &\leq \frac{1}{(2\pi)^{d/2}c^d}\int_{\mathrm{supp}(f)} G\left(\frac{x-y}{c}\right)\exp\left(\frac{1}{2}(\phi(y)-\phi(x))\right)|f(x)|dx \end{aligned}$$

Since $f \in \mathcal{C}_\phi$, supp(f) is compact and $||f||_{L^\infty(\lambda)}$ is bounded. Since ϕ is continuous, it will be bounded on supp(f), by $C > 0$ say. So

$$\begin{aligned} |(a(Q,P)f)(y)| &\leq \frac{1}{(2\pi)^{d/2}c^d}\exp(C/2)||f||_{L^\infty(\lambda)}\exp(\phi(y)/2)\int_{\mathrm{supp}(f)} G\left(\frac{x-y}{c}\right)dx \\ &\leq \frac{1}{(2\pi)^{d/2}}\exp(C/2)||f||_{L^\infty(\lambda)}||G||_{L^1(\lambda)}\exp(\phi(y)/2). \end{aligned}$$

Since ϕ is continuous, $(y \mapsto \exp(\phi(y)/2)) \in L^1_{loc}(\lambda)$, and thus so is $a(Q,P)f$. □

We now show our main theorem: that for $a \in HS_0(M)$ for the correct M, $a(Q,P)$ extends to a bounded operator on $L^p(\mu)$. The correct M is as follows.

Definition 2.12. A pair (M, ϵ) consisting of a measurable function $M : \mathbb{R}^d \to \mathbb{R}^d$ and a number $\epsilon \geq 0$ is a called a valid growth pair for $\phi \in C^2(\mathbb{R}^d)$ and $p \in [1, \infty]$ if for all $x, y \in \mathbb{R}^d$,

$$\left|\left(\left|\frac{1}{2}-\frac{1}{p}\right||\phi(x)-\phi(y)| - \left|\left(\frac{x-y}{c}\right)M\left(\frac{x+y}{2c}\right)\right|\right)\right| \leq \epsilon.$$

Theorem 8. Fix a potential $\phi \in C^2$, let (Q,P) be the associated generalised Weyl pair, and fix $p \in [1,\infty]$. Suppose there exists a valid growth pair (M,ϵ) for ϕ and p. Then for $a \in HS_0(M)$ the operator $a(Q,P)$, defined as in Definition 2.11, extends to a bounded operator on $L^p(\mu)$ and

$$||a(Q,P)||_{B(L^p(\mu))} \leq e^\epsilon ||a||_{HS_0(M)}.$$

That is, the generalised Weyl calculus extends to a bounded linear map

$$HS_0(M) \to B(L^p(\mu)).$$

Proof. Let $U_p : L^p(\mu) \to L^p(\lambda)$ be the isometry from Definition 2.5. Then $a(Q,P)$ has a bounded extension to $L^p(\mu)$ if and only if $U_p a(Q,P)U_p^{-1}$ has a bounded extension to $L^p(\lambda)$, in which case $||a(Q,P)||_{B(L^p(\mu))} = ||U_p a(Q,P)U_p^{-1}||_{B(L^p(\lambda))}$. We will use a Young's convolution inequality argument to show that under our conditions $U_p a(Q,P)U_p^{-1}$ extends boundedly to $L^q(\lambda)$ for all $q \in [1,\infty]$ with norm at most $e^\epsilon ||a||_{HS_0(M)}$. We then remove the isometries on L^p to obtain the desired result.

From Definition 2.11, $U_p a(Q,P)U_p^{-1}$ can be expressed as an integral operator on $L^q(\lambda)$ with kernel $k : \mathbb{R}^{2d} \to \mathbb{C}$ where for $x, y \in \mathbb{R}^d$,

$$k(y,x) = \frac{1}{(2\pi)^{d/2}c^d}\mathcal{F}_2 a\left(\frac{x+y}{2c}, \frac{x-y}{c}\right)\exp\left(\left(\frac{1}{2}-\frac{1}{p}\right)(\phi(y)-\phi(x))\right).$$

Using the definition of $\mathcal{F}_2 a$ and $HS_0(M)$, we find

$$\begin{aligned}
k(y,x) &= \frac{1}{(2\pi)^{d/2}c^d}\mathcal{F}_2 a\left(\frac{x+y}{2c}, \frac{x-y}{c}\right)\exp\left(\left(\frac{1}{2}-\frac{1}{p}\right)(\phi(y)-\phi(x))\right) \\
&= \frac{1}{c^d} g_a\left(\frac{x+y}{2c}, \frac{x-y}{c}\right) \\
&\quad \times \exp\left(\left(\frac{1}{2}-\frac{1}{p}\right)(\phi(y)-\phi(x)) - \left|\left(\frac{x-y}{c}\right)M\left(\frac{x+y}{2c}\right)\right|\right),
\end{aligned}$$

for a unique $g_a \in \mathcal{D}_{\infty,1}$. Our assumption implies

$$\left(\frac{1}{2}-\frac{1}{p}\right)(\phi(x)-\phi(y)) - \left|\left(\frac{x-y}{c}\right)M\left(\frac{x+y}{2c}\right)\right| \le \epsilon,$$

which we incorporate to find

$$\begin{aligned}
|k(y,x)| &\le e^\epsilon \frac{1}{c^d}\left|g_a\left(\frac{x+y}{2c}, \frac{x-y}{c}\right)\right| \\
&\le e^\epsilon \frac{1}{c^d} G_a\left(\frac{x-y}{c}\right),
\end{aligned}$$

for some $G_a \in L^1(\mathbb{R}^d)$. Young's convolution inequality implies extendability and that

$$\begin{aligned}
||U_p a(Q,P)U_p^{-1}||_{B(L^q(\lambda))} &\le e^\epsilon \frac{1}{c^d}\int_{\mathbb{R}^d} G_a\left(\frac{x}{c}\right)dx \\
&= e^\epsilon \int_{\mathbb{R}^d} G_a(x)dx.
\end{aligned}$$

Taking infimum over all G_a dominating g_a in the sense of Definition 2.7 gives $||U_p a(Q,P)U_p^{-1}||_{B(L^q(\lambda))} \le e^\epsilon ||a||_{HS_0(M)}$. Removing the isometries on L^p we obtain our desired result. $\square$

Remark 2 (Existence of a Valid Growth Pair)**.** If $p = 2$, the function taking value 0, and $\epsilon = 0$ is a valid growth pair for any $\phi \in C^2(\mathbb{R}^d)$. If $\phi(x) = xN(x)+lx+\epsilon(x)$, where N is a linear map $\mathbb{R}^d \to \mathbb{R}^d$, $l \in \mathbb{R}^d$, and ϵ is a bounded C^2 function, then M can be taken to be an affine function of x with real-symmetric linear part, depending only on c, p, N and l. We will not make such an assumption until Section 2.5, and so we will keep M as a general measurable function from $\mathbb{R}^d$ to itself unless otherwise specified.

Note that for ϕ of the form $x \mapsto xN(x) + lx + \epsilon(x)$ with $\epsilon(x)$ a bounded C^2 function, an operator will be bounded on $L^p(\mu)$ if and only if it is bounded on $L^p(\tilde{\mu})$, where $\tilde{\mu}$ is associated with $\tilde{\phi}(x) = xN(x) + lx$, as the two measures are equivalent and the Radon-Nikodym derivative of one with respect to the other is $\exp(\pm\epsilon(x))$, which is bounded above and below by positive constants. Thus boundedness of operators in our generalised Weyl calculus should only depend on c, p, N and l. However, the operators which we should care about (such as the relevant Ornstein-Uhlenbeck operator L), will depend on all of ϕ, not just its unbounded terms.

Remark 3 (Extension)**.** It should be noted that the way this functional calculus is defined is quite different to the standard methods. Rather than a convergence lemma or a density argument, we have found an integral operator expression for "nice" symbols, and then extended to a large class of symbols for which the integral representation can be made sense of. Thus when we say that the generalised Weyl calculus extends to $HS_0(M)$, we mean both that for $a \in HS_0(M)$, $a(Q, P) : \mathcal{C}_\phi \to L^1_{loc}(\lambda)$ as defined in Definition 2.11 extends uniquely to a bounded operator on $L^p(\mu)$ (by density of $\mathcal{C}_\phi$ as in Theorem 5), and also that if $a \in \mathcal{S}(\mathbb{R}^{2d}) \cap HS_0(M)$, then the expressions for $a(Q, P)$ from Definitions 2.2 and 2.11 agree (by Lemma 2.10).

Remark 4. Suppose that $a : \mathbb{R}^{2d} \to \mathbb{C}$ is such that for each $x \in \mathbb{R}^d$, $a(x, \xi + i\eta)$ can be extended to a holomorphic function for η in $B(0, |M(x)|)$, and such that there exists a constant $K > 0$ such that for any multiindex α with $|\alpha| \leq (d + 1)$ we have

$$\sup_{u\in\mathbb{R}^d, \eta\in B(0,|M(u)|)} \int_{\mathbb{R}^d} \left|\partial_\xi^{(\alpha)} a(u, \xi + i\eta)\right| d\xi \leq K.$$

Then $a \in HS_0(M)$. However, this integrability condition is much stronger than what we actually require for admission into $HS_0(M)$. Although, as we shall see, some "pseudo-holomorphic" nature is always apparent for symbols in $HS_0(M)$.

Remark 5 (Holomorphic Nature of $HS_0(M)$)**.** It is well-known that if a function $f \in L^\infty(\mathbb{R}^d)$ has Fourier transform of the form

$$(\mathcal{F}f)(\xi) = (2\pi)^{\frac{d}{2}} \exp(-a|\xi|)g(\xi)$$

for some $a > 0$, $g \in L^1(\mathbb{R}^d)$, then f almost everywhere agrees with the restriction to $\mathbb{R}^d$ of a function holomorphic on the cylinder $\{\xi + i\eta; |\eta| < a\}$ (this can be verified via a change of contour in the integral expression for the inverse Fourier transform). Furthermore, this holomorphic extension has a continuous extension to the closure of the cylinder, and the supremum norm of said continuous extension is bounded by $||g||_{L^1(\mathbb{R}^d)}$.

There is an analogous statement for elements of $HS_0(M)$. Fix $a \in HS_0(M)$, $x, \xi \in \mathbb{R}^d$ and $t \in [-1, 1]$. Then we find

$$\begin{aligned}
\frac{1}{(2\pi)^{\frac{d}{2}}} &\left| \int_{\mathbb{R}^d} \mathcal{F}_2 a(x,k) \exp(ik(\xi + itM(x)))dk \right| \\
&\leq \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |\mathcal{F}_2 a(x,k)| \exp(-tkM(x)))dk \\
&= \int_{\mathbb{R}^d} \exp(-|kM(x)| - tkM(x))|g_a(x,k)|dk \\
&\leq \inf \int_{\mathbb{R}^d} G_a(k)dk \\
&\leq ||g_a||_{\mathcal{D}_{\infty,1}} \\
&= ||a||_{HS_0(M)},
\end{aligned}$$

where the infimum is over all G_a dominating g_a as in Definition 2.7. By the Fourier inversion formula, the above agrees ξ-almost everywhere with $a(x, \xi)$ when $t = 0$. We thus take the above as a definition of an extension of a to the set

$$D_M = \left\{(x, \xi + i\eta) \in \mathbb{R}^d \times \mathbb{C}^d; \xi \in \mathbb{R}^d, \exists t \in [-1, 1] \text{ s.t. } \eta = tM(x).\right\}$$

We denote this extension at x, ξ, η as $a(x, \xi + i\eta)$. For each fixed x, continuity as a function of ξ and t follows by the dominated convergence theorem. If $d = 1$, for fixed x this extension is holomorphic as a function of $\xi + i\eta \in \mathbb{C}$, which also follows from the DCT.

If $d \geq 2$ and fixed $x \in \mathbb{R}^d$, it does not make sense to speak of holomorphy of $a(x, \cdot)$ due to its domain not being an open subset of $\mathbb{C}^d$. However, for fixed $x \in \mathbb{R}^d$, $(\xi, t) \mapsto a(x, \xi + itM(x))$ will be a real-analytic function on $\mathbb{R}^d \times (-1, 1)$ and will satisfy some modified form of the Cauchy-Riemann equations.

That functions $a \in HS_0(M)$ possess for each fixed $x \in \mathbb{R}^d$ such a "pseudo-holomorphic" extension to the strip $\{\xi \in \mathbb{R}^d, \exists t \in [-1,1] \text{ s.t. } \eta = tM(x)\}$ is where the name Holomorphic Strip originated.

Remark 6 (Comparison to the standard symbol classes)**.** We should compare the symbol class $HS_0(M)$ to standard symbol classes giving rise to bounded operators through the Weyl calculus. If we take $\phi(x) = 0$ our space of functions is $L^p(\lambda)$ and our generalised Weyl pair is the standard one, in which there are many known symbol classes giving rise to bounded operators (see for instance [SMP93], Chapter 6). These classes typically assume boundedness and decay in the ξ variable of sufficiently many derivatives of $a(x, \xi)$, and allow for some singular integral operators. If $\phi(x) = 0$ we can take $M = 0$, in which case $HS_0(0)$ will be the space of functions whose Fourier transform in the second variable is dominated by an integrable function, thus not including singular integral operators. This implies boundedness, continuity in ξ and decay of $a(x, \xi)$ (by the Riemann-Lebesgue Lemma), but does not give a rate of decay or any differentiability. Similarly, in the case $p = 2$ but $\phi(x) \neq 0$, our generalised Weyl calculus is unitarily equivalent to the standard Weyl calculus, and we can again take $M = 0$.

Alternatively, when $\phi(x) \neq 0$ and $p \neq 2$, we find the relevant M is non-zero and so by Remark 5, symbols in $HS_0(M)$ must have pseudo-holomorphic extendability condition. In this case, there is no isometry back to $L^p(\lambda)$, mapping the associated Weyl pair to the standard Weyl pair. At first sight, this seems infinitely worse than the standard symbol classes. However, this may be the best that can be done. The classical Ornstein-Uhlenbeck operator, associated with $\phi(x) = \frac{x^2}{2}$, is known to only have bounded functional calculus which is holomorphic (see [GCMM$^+$01]), and so if we expect to be able to study the functional calculus of the classical Ornstein-Uhlenbeck operator via the associated generalised Weyl calculus we should be forced into accepting some sort of holomorphic extendability condition on the symbols which give rise to bounded operators.

For the specific case $\phi(x) = \frac{x^2}{2}$, we can factorise the exponential term

$$\exp\left(\left(\frac{1}{2} - \frac{1}{p}\right)(\phi(y) - \phi(x))\right)$$

of the integral kernel of $U_p \circ a(Q, P) \circ U_p^{-1}$ into a function of $\frac{x+y}{2}$ and $x - y$. Using this, for any $U_p \circ a(Q, P) \circ U_p^{-1}$ for (Q, P) associated with $\phi(x)$, we can find a symbol $\tilde{a}$ such that $U_p \circ a(Q, P) \circ U_p^{-1} = \tilde{a}(X, D)$, where (X, D) is the standard Weyl pair. Thus we could derive boundedness of $a(Q, P)$ by checking when $\tilde{a}$ satisfies the standard symbol estimates of classical pseudodifferential operator theory. However, by carrying out this calculation formally, we find that for $x, \xi \in \mathbb{R}^d$, $\tilde{a}(x,\xi) = a\Big(\frac{x}{c}, c\xi + ic\Big(\frac{1}{2} - \frac{1}{p}\Big)x\Big)$, defined as in Remark 5. So for such an argument to work, we would need standard symbol estimates on the "boundary" of this extension. It is clear that this method would still lead to some strong restrictions on symbols.

We can push the techniques used to prove Theorem 8 ever so slightly to prove the following R-boundedness theorem. See [HvNVW17] for the theory of R-boundedness.

Theorem 9. Fix $p \in (1,\infty)$, a potential ϕ, and suppose there exists a valid growth pair (M,ϵ) for ϕ and p. Let $A \subset HS_0(M)$, and $G \subset \mathcal{D}_{\infty,1}$ be the set of g_a corresponding to each $a \in A$ as in Definition 2.9. Suppose that for each $g_a \in G$ we can choose a dominating $G_a \in L^1(\mathbb{R}^d)$ such that the supremum over our selections of the quantity

$$\int_{\mathbb{R}^d} \operatorname*{ess\,sup}_{|y|\geq|x|} |G_a(y)| dx$$

is finite. Then $A(Q,P) = \{a(Q,P), a \in A\}$ is R-bounded on $L^p(\mu)$.

Proof. We apply the same technique as was used in Theorem 8, introducing the isometries $U_p : L^p(\mu) \to L^p(\lambda)$ from Definition 2.5. We will show that the set $U_p \circ A(Q,P) \circ U_p^{-1}$ is R-bounded on $L^q(\lambda)$ for all $q \in (1,\infty)$, specifically for $q = p$, in which case we obtain R-boundedness of $A(Q,P)$ on $L^p(\mu)$ by removing the isometries. As in Theorem 8, the integral kernel of an operator $U_p \circ a(Q,P) \circ U_p^{-1}$ is given by

$$k_a(y,x) = \frac{1}{(2\pi)^{d/2}c^d}\mathcal{F}_2 a\left(\frac{x+y}{2c}, \frac{x-y}{c}\right)\exp\left(\left(\frac{1}{2} - \frac{1}{p}\right)(\phi(y) - \phi(x))\right)$$

and so for all $x, y \in \mathbb{R}^d$,

$$\begin{aligned} |k_a(y,x)| &\leq \frac{1}{c^d}\left|g_a\left(\frac{x+y}{2c}, \frac{x-y}{c}\right)\right|\exp\left(-\left|\left(\frac{x-y}{c}\right)M\left(\frac{x+y}{2c}\right)\right|\right) \\ &\quad \times \exp\left(\left(\frac{1}{2} - \frac{1}{p}\right)(\phi(y) - \phi(x))\right) \\ &\leq e^\epsilon \frac{1}{c^d} G_a\left(\frac{x-y}{c}\right) \end{aligned}$$

So each $U_p \circ a(Q,P) \circ U_p^{-1}$ has kernel dominated by a convolution, namely convolution against $e^\epsilon \frac{1}{c^d} G_a\left(\frac{\cdot}{c}\right)$. Hence R-boundedness of the set of convolution operators

$\{f \mapsto e^{\epsilon}\frac{1}{c^d}G_a\left(\frac{\cdot}{c}\right) * f\}$ on $L^q(\mathbb{R}^d)$ will imply R-boundedness of $U_p \circ A(Q,P) \circ U_p^{-1}$ on $L^q(\mathbb{R}^d)$ (see Proposition 8.1.10 of [HvNVW17], and note that in the proof of said proposition the fixed positive operator can be replaced by an R-bounded family of positive operators). Applying Proposition 8.2.3 of [HvNVW17] and our assumptions shows that $\{f \mapsto e^{\epsilon}\frac{1}{c^d}G_a\left(\frac{\cdot}{c}\right) * f\}$ is R-bounded on $L^q(\lambda)$ for all $q \in (1,\infty)$. Therefore, $A(Q,P)$ is R-bounded on $L^p(\mu)$. $\square$

2.4 An Application

In this section we will use our generalised Weyl calculus developed in the previous section to show that for ϕ of the form $\phi(x) = xN(x) + lx$, where $N : \mathbb{R}^d \to \mathbb{R}^d$ is a positive-definite real-symmetric linear operator and $l \in \mathbb{R}^d$ is a vector, the associated Ornstein-Uhlenbeck operator has bounded $H^\infty(\Sigma_{\theta_p})$ functional calculus on $L^p(\mu)$, where $\sin(\theta_p) = M_p := |1 - \frac{2}{p}|$. This result generalises that of [GCMM+01], which shows that the given angle is optimal for the bounded H^∞ functional calculus of the classical Ornstein-Uhlenbeck operator (corresponding to N as half the identity and $l = 0$). In this proof, the use of the generalised Weyl calculus can be a posteriori removed, leading to a strikingly simple proof in the classical case (see [Har19]). We include the argument here to show that Theorem 8 has important consequences despite the simplicity of its proof and the complexity of the definition of $HS_0(M)$.

Theorem 10. For $p \in (1,\infty)$, $\phi(x) = xN(x) + lx$ where $N : \mathbb{R}^d \to \mathbb{R}^d$ is a positive definite real-symmetric linear operator and vector $l \in \mathbb{R}^d$, the associated Ornstein-Uhlenbeck operator has bounded $H^\infty(\Sigma_{\theta_p})$ functional calculus on $L^p(\mu)$, where $\sin(\theta_p) = M_p := |1 - \frac{2}{p}|$.

Our proof is based on the well-known result that in L^p spaces, if an operator is known to have a bounded H^∞ functional calculus of some angle, the optimal angle of the H^∞ functional calculus of the operator is equal to its optimal angle of R-sectoriality (see [HvNVW17] for the theory of R-sectoriality, and its Theorem 10.7.13 for a proof of the stated result). We have already seen that any of the Ornstein-Uhlenbeck operators considered in this paper automatically have H^∞ functional calculus of some angle (see Corollary 2.6), so all that we need to do is optimise the angle. Our proof that the angle of R-sectoriality of the relevant Ornstein-Uhlenbeck operator is equal to θ_p uses Theorem 10.3.3 of [HvNVW17], which states an equivalence between an operator A being R-sectorial of angle $\theta < \frac{\pi}{2}$ and $-A$ being the generator of an analytic semigroup of angle $\frac{\pi}{2} - \theta$ which is R-bounded on each smaller sector. We prove the required analytic extendability and R-boundedness of the relevant Ornstein-Uhlenbeck semigroup by using the following generalisation of Theorem 3 to transfer to a study of the generalised Weyl calculus, and Theorem 9 to deduce the required R-boundedness result.

Theorem 11. Suppose $\phi(x) = xN(x) + lx$ for a positive semi-definite real-symmetric linear map $N : \mathbb{R}^d \to \mathbb{R}^d$ and vector $l \in \mathbb{R}^d$. Then for the associated Ornstein-Uhlenbeck operator L we have for all $t > 0$ (initially defined as a map $\mathcal{C}_\phi \to L^1_{loc}(\mathbb{R}^d)$)

$$\exp(-tL) = a_t^{N,l}(Q, P)$$

where

$$a_t^{N,l}(x,\xi) = \det\bigl(\cosh(tN)^{-1}\exp(tN)\bigr) \\ \times \exp\left(-\frac{1}{c^2}\xi N_t(\xi) - \left(cN(x) + \frac{l}{2}\right) N_t\left(cN(x) + \frac{l}{2}\right)\right)$$

and

$$N_t = N^{-1}\tanh(tN),$$

with all functions of N interpreted via the functional calculus of a real-symmetric operator on $\mathbb{R}^d$ with the standard inner product (if $0 \in \sigma(N)$, we set N_t to act as multiplication by t on the 0-eigenspace, which can be motivated by noting that for fixed $t > 0$, $n \mapsto \frac{\tanh tn}{n}$ has a unique entire analytic extension, with value t at $n = 0$).

Proof. Note that L with domain $\mathcal{C}_\phi$ is unitarily equivalent to $-\frac{1}{c^2}\Delta + C_\phi(x)$ with domain $C_c^\infty(\mathbb{R}^d)$, where $C_\phi(x) = \frac{1}{4}|\nabla\phi(cx)|^2 - \frac{1}{2}\Delta\phi(cx)$, as is seen in the proof of Theorem 7. Note that as $\phi(x)$ is a second degree polynomial in the components of x, $C_\phi(x)$ will also be a second degree polynomial in the components of x. The work of Hörmander in [Hör18] gives an explicit representation for the classical Weyl symbol for the semigroup generated by operators of the form $-\Delta + V(x)$ where V is a degree two polynomial in the components of x. Using Hörmander's formula in our particular case and noting the joint unitary equivalence of L with $-\frac{1}{c^2}\Delta + C_\phi(x)$ and $a(Q, P)$ with $a(X, D)$, we obtain the claimed expression for $a_t^{N,l}$. □

To deduce the desired H^∞ functional calculus result, we need only show that the relevant Ornstein-Uhlenbeck semigroup has an analytic extension to a sector of the correct angle, and that it is R-bounded on each smaller sector. We will in fact show a lot more with no more effort. The function $t \mapsto N_t$ is analytic and can clearly be extended to $\mathbb{C}\setminus\bigcup_{n\in\sigma(N)}\frac{i\pi}{n}(2\mathbb{Z}+1)$. We will consider the analytic extension $z\mapsto N_z$ on domains of the form

$$E^N := \{z\in\mathbb{C};\sigma(N_z)\subset\Sigma_{\frac{\pi}{2}-\theta_p}\}\setminus\bigcup_{n\in\sigma(N)}\frac{i\pi}{n}\left(\mathbb{Z}+\frac{1}{2}\right) \tag{2.2}$$

where $\sin(\theta_p)=M_p:=\left|1-\frac{2}{p}\right|$. We will show the Ornstein-Uhlenbeck semigroup extends to an analytic semigroup on the domain E. Moreover, we will show that the Ornstein-Uhlenbeck semigroup is R-bounded on sets of the form

$$E^N_{\epsilon,\delta} := \left\{\begin{array}{c} z\in\mathbb{C};\sigma(\cos^2(\arg(N_z)))\in(M_p^2+\epsilon,\infty), \\ \operatorname{dist}\Big(z,\Big(\bigcup_{n\in\sigma(N)}\frac{i\pi}{2n}\mathbb{Z}\Big)\setminus\{0\}\Big)>\delta \end{array}\right\} \tag{2.3}$$

for all $\epsilon,\delta>0$. The condition on the spectrum of the cosine of the argument of N_z is a rephrasing of the spectrum of N_z being contained in a sector slightly smaller than Σ_{θ_p}, which is a useful form for the proof to come. Note the condition on the distance to $\Big(\bigcup_{n\in\sigma(N)}\frac{i\pi}{2n}\mathbb{Z}\Big)\setminus\{0\}$ ensures we remain uniformly away from the poles and zeroes of N_z, besides $z=0$. We claim that $\Sigma_{\frac{\pi}{2}-\theta_p}\subset E^N$ for any N, and that for all $\epsilon'>0$ there exists $\epsilon,\delta>0$ such that $\Sigma_{\frac{\pi}{2}-\theta_p-\epsilon'}\subset E^N_{\epsilon,\delta}$ (see [vNP18] for details of this calculation in the case N is a multiple of the identity, and note that the general case follows by taking intersections over the eigenvalues of N). These results combined will imply that the maximal domain of analyticity of the Ornstein-Uhlenbeck semigroup contains the sector $\Sigma_{\frac{\pi}{2}-\theta_p}$, and that it is R-bounded on each smaller sector.

Theorem 12. Suppose $\phi(x)=xN(x)+lx$ for a positive semi-definite real-symmetric linear map $N:\mathbb{R}^d\to\mathbb{R}^d$ and vector $l\in\mathbb{R}^d$. For $p\in(1,\infty)$, the associated Ornstein-Uhlenbeck semigroup has an analytic extension on $L^p(\mu)$ to the domain E^N. Furthermore, if N is positive definite, this extension is R-bounded on each domain $E^N_{\epsilon,\delta}$ for all $\epsilon,\delta>0$.

Proof. Let $z\in E^N$. By Theorem 11, $\exp(-zL)=a_z(Q,P)$ (dropping N,l from the notation), where

$$\begin{aligned} a_z(x,\xi) &= \det\big(\cosh(zN)^{-1}\exp(zN)\big) \\ &\quad\times\exp\left(-\frac{1}{c^2}\xi N_z(\xi)-\left(cN(x)+\frac{l}{2}\right)N_z\left(cN(x)+\frac{l}{2}\right)\right) \end{aligned}$$

and $N_z = N^{-1}\tanh(zN)$. Computing $\mathcal{F}_2 a_z(x,k)$ gives:

$$\mathcal{F}_2 a_z(x,k) = 2^{-\frac{d}{2}} c^d \det\Big(N_z^{-\frac{1}{2}} \cosh(zN)^{-1} \exp(zN)\Big)$$
$$\times \exp\bigg(-\frac{c^2}{4} k N_z^{-1}(k) - \Big(cN(x) + \frac{l}{2}\Big) N_z \Big(cN(x) + \frac{l}{2}\Big)\bigg)$$

For $\phi(x) = xN(x) + lx$ and fixed $p \in (1,\infty)$, we claim $M : \mathbb{R}^d \to \mathbb{R}^d$, $x \mapsto M_p\big(c^2 N(x) + \frac{cl}{2}\big)$, and $\epsilon = 0$ are a valid growth pair. To see this, we have for all $x, y \in \mathbb{R}^d$

$$\begin{aligned}\left|\frac{1}{2} - \frac{1}{p}\right| |\phi(x) - \phi(y)| &= \frac{M_p}{2} |xN(x) + lx - yN(y) - ly| \\ &= \frac{M_p}{2} |(x-y)N(x+y) + l(x-y)| \\ &= \left|\Big(\frac{x-y}{c}\Big) M_p \Big(c^2 N\Big(\frac{x+y}{2c}\Big) + \frac{cl}{2}\Big)\right|.\end{aligned}$$

Using this, we can rewrite $\mathcal{F}_2 a_z(x,k)$ as

$$\mathcal{F}_2 a_z(x,k) = 2^{-\frac{d}{2}} c^d \det\Big(N_z^{-\frac{1}{2}} \cosh(zN)^{-1} \exp(zN)\Big)$$
$$\times \exp\Big(-\frac{c^2}{4} k N_z^{-1}(k) - \frac{1}{c^2 M_p^2} M(x) N_z(M(x))\Big)$$

We will show that a_z satisfies the conditions of admission into $HS_0(M)$ for this specific M, from which Theorem 8 gives boundedness on $L^p(\mu)$ of $a_z(Q,P)$. The decomposition of a_z as in Definition 2.9 has

$$g_{a_z}(x,k) = \frac{c^d}{2^d \pi^{\frac{d}{2}}} \det\Big(N_z^{-\frac{1}{2}} \cosh(zN)^{-1} \exp(zN)\Big)$$
$$\times \exp\Big(-\frac{c^2}{4} k N_z^{-1}(k) + |kM(x)| - \frac{1}{c^2 M_p^2} M(x) N_z(M(x))\Big)$$

We wish to show $g_{a_z} \in \mathcal{D}_{\infty,1}$, so we must dominate in x by an integrable function in k. Letting ι be the sign of $kM(x)$, we find by completing the square in $M(x)$:

$$\begin{aligned}
|g_{a_z}(x,k)| &= \frac{c^d}{2^d\pi^{\frac{d}{2}}}\Big|\det\Big(N_z^{-\frac{1}{2}}\cosh(zN)^{-1}\exp(zN)\Big)\Big| \\
&\exp\Big(-\frac{c^2}{4}k\Re(N_z^{-1})(k)+\iota kM(x)-\frac{1}{c^2M_p^2}M(x)\Re(N_z)(M(x))\Big) \\
&= \frac{c^d}{2^d\pi^{\frac{d}{2}}}\Big|\det\Big(N_z^{-\frac{1}{2}}\cosh(zN)^{-1}\exp(zN)\Big)\Big| \\
&\exp\Big(-\frac{c^2}{4}k\Re(N_z^{-1})(k)+\frac{c^2M_p^2}{4}k\Re(N_z)^{-1}(k) \\
&-\Big(\frac{1}{cM_p}\Re(N_z)^{\frac{1}{2}}(M(x))-\frac{\iota cM_p}{2}\Re(N_z)^{-\frac{1}{2}}(k)\Big)^2\Big) \\
&= \frac{c^d}{2^d\pi^{\frac{d}{2}}}\Big|\det\Big(N_z^{-\frac{1}{2}}\cosh(zN)^{-1}\exp(zN)\Big)\Big| \\
&\exp\Big(-\frac{c^2}{4}k\big(\Re(N_z^{-1})-M_p^2\Re(N_z)^{-1}\big)(k) \\
&-\Big(\frac{1}{cM_p}\Re(N_z)^{\frac{1}{2}}(M(x))-\frac{\iota cM_p}{2}\Re(N_z)^{-\frac{1}{2}}(k)\Big)^2\Big),
\end{aligned}$$

where by $\left(\frac{1}{cM_p}\Re(N_z)^{\frac{1}{2}}(M(x))-\frac{\iota cM_p}{2}\Re(N_z)^{-\frac{1}{2}}(k)\right)^2$ we mean the inner product of the contents of the brackets with itself, which is non-negative. So we may take as dominating function

$$G_{a_z}(k)=\frac{c^d}{2^d\pi^{\frac{d}{2}}}\Big|\det\Big(N_z^{-\frac{1}{2}}\cosh(zN)^{-1}\exp(zN)\Big)\Big|\exp\Big(-\frac{c^2}{4}\sigma_z^{\min}k^2\Big)$$

Where $\sigma_z^{\min}$ denotes the lowest eigenvalue of $\big(\Re(N_z^{-1})-M_p^2\Re(N_z)^{-1}\big)$. For G_{a_z} to be integrable, we require $\sigma_z^{\min}>0$, or equivalently $\Re(N_z^{-1})-M_p^2\Re(N_z)^{-1}$ to be positive definite. As $\Re(N_z),\Im(N_z)$ are both in the functional calculus of the single self-adjoint operator N, they commute and so we find

$$\Re(N_z^{-1})=\Re(N_z)(N_z^*N_z)^{-1}.$$

Using this, we find

$$\Re(N_z^{-1})-M_p^2\Re(N_z)^{-1}=\big(\Re(N_z)^2(N_z^*N_z)^{-1}-M_p^2I\big)\Re(N_z)^{-1}. \tag{2.4}$$

But $\sigma(N_z)$ is a finite subset of $\Sigma_{\frac{\pi}{2}-\theta_p}$ so $\sigma(\Re(N_z)^{-1})$ is a finite subset of $(0,\infty)$, and so this is a product of commuting positive definite operators and is hence positive definite, noting $\Re(N_z)^2(N_z^*N_z)^{-1} = \cos^2(\arg(N_z)) > M_p^2 I$. As integrability of G_{a_z} implies boundedness of $a_z(Q,P)$, we find that $\exp(-zL)$ has a holomorphic extension as a $B(L^p(\mu))$-valued function from the domain $\mathbb{R}^+$ to the domain E^N.

Next we wish to investigate the semigroup for sets $E^N_{\epsilon,\delta} \subset E$, for all $\epsilon, \delta > 0$ (see (2.3)). For $z \in E^N_{\epsilon,\delta}$, we find that the dominating functions $G_{a_z}(k)$ are radially decaying for each z, and so the bound in Theorem 9 becomes checking finiteness of:

$$\begin{aligned}
&\sup_{z\in E^N_{\epsilon,\delta}} \int_{\mathbb{R}^d} G_{a_z}(k)dk \\
&= \sup_{z\in E^N_{\epsilon,\delta}} \int_{\mathbb{R}^d} \frac{c^d}{2^d\pi^{\frac{d}{2}}}\left|\det\Big(N_z^{-\frac{1}{2}}\cosh(zN)^{-1}\exp(zN)\Big)\right| \exp\Big(-\frac{c^2}{4}\sigma_z^{\min}k^2\Big)dk \\
&= \sup_{z\in E^N_{\epsilon,\delta}} \frac{c^d}{2^d\pi^{\frac{d}{2}}}\left|\det\Big(N_z^{-\frac{1}{2}}\cosh(zN)^{-1}\exp(zN)\Big)\right|\pi^{\frac{d}{2}}\Big(\frac{c^2}{4}\sigma_z^{\min}\Big)^{-\frac{d}{2}} \\
&= \sup_{z\in E^N_{\epsilon,\delta}} \left|\det\Big(\big(\sigma_z^{\min}N_z\big)^{-\frac{1}{2}}\cosh(zN)^{-1}\exp(zN)\Big)\right| \\
&= \sup_{z\in E^N_{\epsilon,\delta}} \left|\det\big(\sigma_z^{\min}N_z\big)\right|^{-\frac{1}{2}}\left|\det\big(\cosh(zN)^{-1}\exp(zN)\big)\right|.
\end{aligned}$$

As the spectrum of N is contained in the positive real line and $E^N_{\epsilon,\delta}$ is at least a distance of δ from the purely imaginary periodic poles of $\cosh(zN)^{-1}$, we find $|\cosh(zN)^{-1}\exp(zN)|$ is uniformly bounded for $z \in E^N_{\epsilon,\delta}$. Thus the second determinant above is uniformly bounded. As we have assumed $z \in E^N_{\epsilon,\delta}$, we find that $\big(\Re(N_z)^2(N_z^*N_z)^{-1} - M_p^2 I\big) = \big(\cos^2(\arg(N_z)) - M_p^2 I\big) > \epsilon I > 0$, and so by Equation 2.4, $\sigma_z^{\min}$ is bounded below by ϵ times the lowest eigenvalue of $(\Re(N_z))^{-1}$ which is the same as ϵ divided by the largest eigenvalue of $\Re(N_z)$. So expressing $\det\big(\sigma_z^{\min}N_z\big)$ as the product of its eigenvalues, the supremum over $z \in E^N_{\epsilon,\delta}$ of $\left|\det\big(\sigma_z^{\min}N_z\big)\right|^{-\frac{1}{2}}$ will be finite if and only if the ratio of largest eigenvalue of $\Re(N_z)$ and smallest eigenvalue of N_z is uniformly bounded. We find by the spectral mapping theorem, that this ratio of eigenvalues is of the form

$$\frac{m}{n}\frac{\Re(\tanh(zn))}{|\tanh(zm)|},$$

where $n, m \in \sigma(N)$. As we have assumed N is positive definite, $n, m > 0$. Thus we find

$$\begin{aligned}
\left|\frac{m}{n}\frac{\Re(\tanh(zn))}{|\tanh(zm)|}\right| &\le \frac{m}{n}\left|\frac{\tanh(zn)}{\tanh(zm)}\right| \\
&= \frac{m}{n}\left|\frac{(e^{2zn}-1)(e^{2zm}+1)}{(e^{2zn}+1)(e^{2zm}-1)}\right|.
\end{aligned}$$

We denote this final bound $C_{n,m}(z)$. As $|\Re(z)| \to \infty$, $C_{n,m}(z)$ converges to m/n uniformly in $\Im(z)$, so we may restrict to a set $\overline{E^n_{\epsilon,\delta}} \cap \{|\Re(z)| \leq C\}$. On such a set, z is uniformly distant from the zeroes of $(e^{2zn}+1)(e^{2zm}-1)$, except $z = 0$. However,

$$\begin{aligned}
\lim_{z\to 0} \frac{m}{n} \frac{(e^{2zn}-1)(e^{2zm}+1)}{(e^{2zn}+1)(e^{2zm}-1)} &= \frac{m}{n} \lim_{z\to 0} \frac{(e^{2zn}-1)}{(e^{2zm}-1)} \\
&= \frac{m}{n} \lim_{z\to 0} \frac{2ne^{2zn}}{2me^{2zm}} \\
&= 1.
\end{aligned}$$

So $C_{n,m}(z)$ is bounded near 0. Away from zero and with bounded real part, $C_{n,m}(z)$ is the product of two functions periodic in $\Im(z)$,

$$m\left|\frac{(e^{2zm}+1)}{(e^{2zm}-1)}\right| \text{ and } \frac{1}{n}\left|\frac{(e^{2zn}-1)}{(e^{2zn}+1)}\right|,$$

whose poles z remains distant from. Using periodicity, boundedness is equivalent to boundedness on a compact set for each periodic function individually. However, both are continuous, and thus bounded. Thus $C_{n,m}(z)$ is uniformly bounded on $E^N_{\epsilon,\delta}$, and thus so is the relevant ratio of eigenvalues. As there are only finitely many choices for $m, n \in \sigma(N)$, we find that the relevant ratio of eigenvalues is uniformly bounded. Hence

$$\sup_{z\in E^N_{\epsilon,\delta}} \int_{\mathbb{R}^d} G_{a_z}(k)dk < \infty,$$

and so we apply Theorem 9 to deduce that $\{a_z(Q,P); z \in E^N_{\epsilon,\delta}\}$ is R-bounded on $L^p(\mu)$. □

Remark 7. Both the domain $E^{\frac{1}{2}I}$, and the union of all domains of the form of $E^{\frac{1}{2}I}_{\epsilon,\delta}$ are exactly the classical Epperson region, which is known to be the largest domain on which the classical Ornstein-Uhlenbeck semigroup has a bounded analytic extension on $L^p(\mu)$ (see for example, [Epp89]). The set E^N is thus an analogue of the Epperson region, for certain variants of the classical Ornstein-Uhlenbeck operator. By examining how $E^{\frac{1}{2}I}_{\epsilon,\delta}$ fill out $E^{\frac{1}{2}I}$ as $\epsilon, \delta \to 0$, it can be seen that the Ornstein-Uhlenbeck semigroup is R-bounded if and only if it is uniformly bounded. This implies that the angle of sectoriality and R-sectoriality of the classical Ornstein-Uhlenbeck operator agree.

2.5 The Symbol Class $HS(M)$

The first thing we wish to do is enrich the symbol class $HS_0(M)$ with an identity. In fact, without much more effort we can easily include symbols corresponding to anything in the Borel functional calculus of the position operators Q.

Definition 2.13. Let $B \subset L^\infty(\mathbb{R}^{2d})$ be the sub-Banach space

$$\{b \in L^\infty(\mathbb{R}^{2d}); b(x,\xi) = b(x,0), \text{ for a.e. } x, \xi \in \mathbb{R}^d\}.$$

Lemma 2.14. $HS_0(M) \cap B = \{0\}$.

Proof. We will calculate $\mathcal{F}(I_b(x))$ for $b \in B$. Fixing some $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we have:

$$\begin{aligned}
\langle \mathcal{F}(I_b(x)), \varphi \rangle &= \langle I_b(x), \mathcal{F}^* \varphi \rangle \\
&= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^{2d}} b(x,\xi)\varphi(k) \exp(i\xi k) d\xi dk \\
&= (2\pi)^{-\frac{d}{2}} b(x,0) \int_{\mathbb{R}^{2d}} \varphi(k) \exp(i\xi k) d\xi dk \\
&= (2\pi)^{\frac{d}{2}} b(x,0)\varphi(0) \\
&= \left\langle (2\pi)^{\frac{d}{2}} b(x,0)\delta_0, \varphi \right\rangle.
\end{aligned}$$

Where δ_0 is the Dirac distribution. The second last equality follows from noting that $(2\pi)^{-d} \int_{\mathbb{R}^{2d}} \varphi(k) \exp(i\xi k) d\xi dk$ is the evaluation at 0 of the inverse Fourier transform of the Fourier transform of φ. This clearly shows that b does not satisfy the requirements of admission into $HS_0(M)$ unless $b = 0$. $\square$

Definition 2.15. Fix $M : \mathbb{R}^d \to \mathbb{R}^d$. The space $HS(M)$ (standing for Holomorphic Strip) is a subspace of $L^\infty(\mathbb{R}^{2d})$, with

$$HS(M) = HS_0(M) \oplus B$$

with the norm of $a = a_0 + a_b$, $a_0 \in HS_0(M), a_b \in B$ defined by

$$||a||_{HS(M)} = ||a_0||_{HS_0(M)} + ||a_b||_{L^\infty(\mathbb{R}^{2d})}.$$

(Note that this norm is well-defined as $HS_0(M) \cap B = \{0\}$, as in Lemma 2.14).

Due to the generality of symbols in B, we can no longer use the formula of Theorem 4 as a definition for the operator associated with a symbol in B. We thus provide an explicit extension of the generalised Weyl calculus to B (as a contraction with respect to the $HS(M)$ norm and operator norm), motivated by our intuition as to how things should work. With this definition, the generalised Weyl calculus will extend to a bounded map $HS(M) \mapsto B(L^p(\mu))$.

Definition 2.16. Define the extension of the generalised Weyl calculus to $a = a_b \in B$ via the action

$$(a(Q,P)f)(y) = a_b(y/c)f(y).$$

This is natural because, formally, for $a = a_b \in B$ and $f \in C_\phi$, so

$$\exp\left(-\frac{1}{2}\phi(\cdot)\right)f(\cdot) \in \mathcal{S}(\mathbb{R}^d),$$

we have

$$\begin{aligned}(a(Q,P)f)(y) &= \frac{1}{(2\pi)^d c^d}\int_{\mathbb{R}^{2d}} a\left(\frac{x+y}{2c},\xi\right)\exp\left(-i\xi\left(\frac{x-y}{c}\right)\right)\\ &\quad\times \exp\left(\frac{1}{2}(\phi(y)+\phi(x))\right)f(x)d\xi d\mu(x)\\ &= \frac{1}{c^d}\exp\left(\frac{1}{2}\phi(y)\right)\int_{\mathbb{R}^d} a_b\left(\frac{x+y}{2c}\right)\delta_0\left(\frac{x-y}{c}\right)\\ &\quad\times \exp\left(-\frac{1}{2}\phi(x)\right)f(x)dx\\ &= \frac{1}{c^d}\exp\left(\frac{1}{2}\phi(y)\right)\left\langle a_b\left(\frac{\cdot+y}{2c}\right)\delta_0\left(\frac{\cdot-y}{c}\right),\exp\left(-\frac{1}{2}\phi(\cdot)\right)f(\cdot)\right\rangle\\ &= a_b\left(\frac{2y}{2c}\right)\exp\left(\frac{1}{2}(\phi(y)-\phi(y))\right)f(y)\\ &= a_b(y/c)f(y).\end{aligned}$$

Where δ_0 is the Dirac distribution. This extension is clearly contractive as a map $B \to B(L^p(\mu))$. Combining this with Theorem 8, we have:

Theorem 13. Fix $\phi \in C^2(\mathbb{R}^d)$ and $p \in [1,\infty]$, and suppose there exists a valid growth pair (M,ϵ) for ϕ and p. Then the generalised Weyl calculus extends uniquely to a linear map $HS(M) \to B(L^p(\mu))$ and we have

$$||a(Q,P)||_{B(L^p(\mu))} \leq e^\epsilon ||a||_{HS(M)}.$$

Exactly as we have shown for $HS_0(M)$, we also get for free that symbols in $HS(M)$ have some holomorphic nature.

Remark 8 (Holomorphic Nature of $HS(M)$)**.** Exactly as in Remark 5, for any symbol $a \in HS(M)$ and $x \in \mathbb{R}^d$, $a(x,\xi+i\eta)$ has an extension as a function of $\xi+i\eta$ for $\eta = tM(x), t \in (-1,1)$, and the essential range of $a(x,\xi+i\eta)$ on domain $\{(x,\xi+i\eta) \in \mathbb{R}^d \times \mathbb{C}^d; \exists t \in (-1,1) \text{ s.t. } \eta = tM(x)\}$ is bounded by $||a||_{HS(M)}$. This follows trivially, as $a(x,\xi) = a_0(x,\xi)+a_b(x)$ and $a_0 \in HS_0(M)$ has the given extendability, while the constant function $\xi \mapsto a_b(x)$ has an entire and bounded (and constant) extension.

A similar R-boundedness theorem also holds.

Theorem 14. Fix $\phi \in C^2(\mathbb{R}^d)$ and $p \in [1, \infty]$, and suppose there exists a valid growth pair (M, ϵ) for ϕ and p. Let $A \subset HS(M)$, and let $G \subset \mathcal{D}_{\infty,1}$, $A_B \subset B$ be the sets of g_a, a_b corresponding to each $a \in A$ as in the decomposition $HS(M) = HS_0(M) \oplus B$ in Definition 2.15. Suppose that for each $g_a \in G$ we can choose dominating $G_a \in L^1(\mathbb{R}^d)$ such that the supremum over our selections of the quantity

$$\int_{\mathbb{R}^d} \operatorname*{ess\,sup}_{|y| \geq |x|} |G_a(y)| dx$$

is finite. Also suppose that

$$\sup_{a_b \in B} ||a_b||_{L^\infty(\mathbb{R}^d)} < \infty.$$

Then $A(Q, P) = \{a(Q, P), a \in A\}$ is R-bounded on $L^p(\mu)$.

Proof. First note that $A \subset A_0 + A_B$, where A_0 and A_B are the projections of A onto $HS_0(M)$ and B respectively, so by subadditivity of R-boundedness it suffices to check that $A_0(Q, P)$ and $A_B(Q, P)$ are R-bounded on $L^p(\mu)$. That $A_0(Q, P)$ is R-bounded follows from Theorem 9. Since everything in A_B is in B, Definition 2.16 implies that $A_B(Q, P)$ consists of multiplication operators $f(x) \mapsto a_b(x/c)f(x)$. Then since $\sup_{a_b \in B} ||a_b||_{L^\infty(\mathbb{R}^d)} < \infty$, $B(Q, P)$ is R-bounded on $L^q(\mu)$ for all $q \in (1, \infty)$, and hence on $L^p(\mu)$ (see, for example, Example 8.1.9 of [HvNVW17]). $\square$

We wish to study the space $HS(M)$ - equipped with a natural product (to be defined below) - to gain knowledge about operators on $L^p(\mu)$ related to Q and P. To do so, we need to verify some facts about $HS(M)$.

Theorem 15. For any measurable $M : \mathbb{R}^d \to \mathbb{R}^d$, $HS(M)$ is complete.

Proof. Note that $HS(M)$ is a direct sum of the spaces $HS_0(M)$ and B, so providing both of these spaces are complete, we will be done. That B is complete is obvious. Let $\{a_n\} \subset HS_0(M)$ be a Cauchy sequence, with corresponding sequence $\{g_n\} \subset \mathcal{D}_{\infty,1}$

Then since a_n is $HS_0(M)$-Cauchy, we find $\{g_n\}$ is $\mathcal{D}_{\infty,1}$-Cauchy, and hence has a limit $g \in \mathcal{D}_{\infty,1}$, say. Let $a \in L^\infty(\mathbb{R}^{2d})$ be for each $x, \xi \in \mathbb{R}^d$,

$$a(x, \xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} (2\pi)^{\frac{d}{2}} g(x, k) \exp(-|M(x)k|) \exp(ik\xi) dk$$

Note this is well-defined for each x, ξ as $|\exp(-|M(x)k|)\exp(ik\xi)| \leq 1$ and $|g(x,k)| < G(k)$ for some $G \in L^1(\mathbb{R}^d)$, and hence $|a(x,\xi)|$ is bounded by $||G||_{L^1(\mathbb{R}^d)}$. By the Fourier inversion formula on $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{F}(I_a(x)) \in \mathcal{S}'(\mathbb{R}^d)$ is given by integration against $(2\pi)^{\frac{d}{2}} g(x,k)\exp(-|M(x)k|)$, and so $a \in HS_0(M)$. It is clear that $||a_n - a||_{HS_0(M)} \to 0$ by construction. So $HS_0(M)$ is complete, and hence $HS(M)$ is complete. □

In the study of the standard Weyl calculus, there is a bilinear product $\mathcal{S}(\mathbb{R}^{2d}) \times \mathcal{S}(\mathbb{R}^{2d}) \to \mathcal{S}(\mathbb{R}^{2d})$, known as the Moyal product and denoted #, which makes the Weyl calculus into an algebra homomorphism, I.e. such that for all $a_1, a_2 \in \mathcal{S}(\mathbb{R}^{2d})$, $a_1(X,D)a_2(X,D) = (a_1 \# a_2)(X,D)$. We wish to define a similar product on $HS(M)$, making the functional calculus an algebra homomorphism. Note that as our generalised Weyl pairs (Q,P) are unitarily equivalent to the standard pair (X,D) on L^2, any such product should agree with the Moyal product on $HS(M) \cap \mathcal{S}(\mathbb{R}^{2d})$, and hence we will also refer to such a product on $HS(M)$ as the Moyal product and denote it #.

In the classical case, the Moyal product is either written in terms of an oscillatory integral involving $a_1(x,\xi)$ and $a_2(x,\xi)$, or as an asymptotic formula involving derivatives of a_1 and a_2. We will avoid both of these expressions by deducing what the product must be from the relation $a_1(Q,P)a_2(Q,P) = (a_1 \# a_2)(Q,P)$. As both the definition of $HS(M)$ and the generalised Weyl calculus for (Q,P) are written in terms of $\mathcal{F}_2 a$ instead of a explicitly, we will find a formula for the Moyal product in terms of $\mathcal{F}_2$ of the symbols.

Theorem 16. For $a^1 = a_0^1 + a_b^1, a^2 = a_0^2 + a_b^2 \in HS(M) = HS_0(M) \oplus B$, the Moyal product $a^1 \# a^2 \in L^\infty(\mathbb{R}^{2d})$ is given as the sum of $a_0^1 \# a_b^2 \in HS_0(M)$, $a_b^1 \# a_0^2 \in HS_0(M)$, $a_b^1 \# a_b^2 \in B$, and $a_0^1 \# a_0^2 \in L^\infty(\mathbb{R}^{2d})$ where

1. $\mathcal{F}_2(a_0^1 \# a_0^2) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \mathcal{F}_2 a_0^1\big(x + \frac{v-k}{2}, v\big)\mathcal{F}_2 a_0^2\big(x + \frac{v}{2}, k - v\big)dv$.

2. $\mathcal{F}_2(a_0^1 \# a_b^2)(x,k) = \mathcal{F}_2 a_0^1(x,k) a_b^2\big(x + \frac{k}{2}\big)$.

3. $\mathcal{F}_2(a_b^1 \# a_0^2)(x,k) = a_b^1\big(x - \frac{k}{2}\big)\mathcal{F}_2 a_0^2(x,k)$.

4. $(a_b^1 \# a_b^2)(x) = a_b^1(x) a_b^2(x)$.

Furthermore, suppose $M : \mathbb{R}^d \to \mathbb{R}^d$ is an affine function with real-symmetric linear part. Then $a_0^1 \# a_0^2 \in HS_0(M)$, and the Moyal product is a Banach algebra product on $HS(M)$, such that for all $a^1, a^2 \in HS(M)$

$$||a^1 \# a^2||_{HS(M)} \leq ||a^1||_{HS(M)} ||a^2||_{HS(M)}.$$

Proof. We will only prove the first and second formula, as the third follows in almost the same way as the second, and the fourth is apparent from Definition 2.16. Recall Definition 2.11, which states for $a \in HS_0(M)$,

$$(a(Q,P)f)(y) = \frac{1}{(2\pi)^{d/2}c^d}\int_{\mathbb{R}^d} \mathcal{F}_2 a\left(\frac{x+y}{2c}, \frac{x-y}{c}\right)\exp\left(\frac{1}{2}(\phi(y)-\phi(x))\right)f(x)dx.$$

Thus we find

$$\begin{aligned}\big(a_0^1(Q,P)a_0^2(Q,P)f\big)(y) = {} & \frac{1}{(2\pi)^d c^{2d}}\int_{\mathbb{R}^{2d}} \mathcal{F}_2 a_0^1\left(\frac{z+y}{2c}, \frac{z-y}{c}\right)\exp\left(\frac{1}{2}(\phi(y)-\phi(z))\right) \\ & \mathcal{F}_2 a_0^2\left(\frac{x+z}{2c}, \frac{x-z}{c}\right)\exp\left(\frac{1}{2}(\phi(z)-\phi(x))\right)f(x)dxdz \\ = {} & \frac{1}{(2\pi)^{d/2}c^d}\int_{\mathbb{R}^d}\left(\frac{1}{(2\pi)^{d/2}c^d}\int_{\mathbb{R}^d}\mathcal{F}_2 a_0^1\left(\frac{z+y}{2c}, \frac{z-y}{c}\right)\right. \\ & \left.\mathcal{F}_2 a_0^2\left(\frac{x+z}{2c}, \frac{x-z}{c}\right)dz\right)\exp\left(\frac{1}{2}(\phi(y)-\phi(x))\right)f(x)dx\end{aligned}$$

and by making a change of variables $v = (z-y)/c$ we find

$$\begin{aligned}&\frac{1}{(2\pi)^{d/2}c^d}\int_{\mathbb{R}^d}\mathcal{F}_2 a_0^1\left(\frac{z+y}{2c}, \frac{z-y}{c}\right)\mathcal{F}_2 a_0^2\left(\frac{x+z}{2c}, \frac{x-z}{c}\right)dz \\ &= \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\mathcal{F}_2 a_0^1\left(\frac{x+y}{2c} - \frac{1}{2}\frac{x-y}{c} + \frac{v}{2}, v\right)\mathcal{F}_2 a_0^2\left(\frac{x+y}{2c} + \frac{v}{2}, \frac{x-y}{c} - v\right)dv.\end{aligned}$$

By comparing with Definition 2.11, it can be seen that this confirms the formula for $\mathcal{F}_2(a_0^1 \# a_0^2)(x,k)$.

For the second term, we find

$$\begin{aligned}\big(a_0^1(Q,P)a_b^2(Q,P)f\big)(y) = {} & \frac{1}{(2\pi)^{d/2}c^d}\int_{\mathbb{R}^d}\mathcal{F}_2 a_0^1\left(\frac{x+y}{2c}, \frac{x-y}{c}\right)a_b^2\left(\frac{x}{c}\right) \\ & \times \exp\left(\frac{1}{2}(\phi(y)-\phi(x))\right)f(x)dx\end{aligned}$$

and

$$\mathcal{F}_2 a_0^1\left(\frac{x+y}{2c}, \frac{x-y}{c}\right)a_b^2\left(\frac{x}{c}\right) = \mathcal{F}_2 a_0^1\left(\frac{x+y}{2c}, \frac{x-y}{c}\right)a_b^2\left(\frac{x+y}{2c} + \frac{1}{2}\frac{x-y}{c}\right),$$

which, after comparison with Definition 2.11, confirms the stated formula.

Note that by boundedness of a_b^1, a_b^2, the three products besides $a_0^1 \# a_0^2$ lie in the spaces as given above. To see that the formula for $\mathcal{F}_2(a_0^1 \# a_0^2)$ implies

$a_0^1 \# a_0^2 \in L^\infty(\mathbb{R}^{2d})$, we have

$$\begin{aligned}
&\left|\mathcal{F}_2(a_0^1\#a_0^2)(x,k)\right| \leq (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \left|\mathcal{F}_2 a_0^1\left(x+\frac{v-k}{2}, v\right)\mathcal{F}_2 a_0^2\left(x+\frac{v}{2}, k-v\right)\right| dv \\
&= (2\pi)^{\frac{d}{2}} \int_{\mathbb{R}^d} \exp\left(-\left|M\left(x+\frac{v-k}{2}\right)v\right| - \left|M\left(x+\frac{v}{2}\right)(k-v)\right|\right) \\
&\quad \times \left|g_{a_0^1}\left(x+\frac{v-k}{2}, v\right) g_{a_0^2}\left(x+\frac{v}{2}, k-v\right)\right| dv \\
&\leq (2\pi)^{\frac{d}{2}} \int_{\mathbb{R}^d} \left|g_{a_0^1}\left(x+\frac{v-k}{2}, v\right) g_{a_0^2}\left(x+\frac{v}{2}, k-v\right)\right| dv \\
&\leq (2\pi)^{\frac{d}{2}} \int_{\mathbb{R}^d} G_{a_0^1}(v) G_{a_0^2}(k-v) dv,
\end{aligned}$$

which is in $L^1(\mathbb{R}^d)$ as the convolution of two L^1 functions. Hence $\mathcal{F}_2(a_0^1\#a_0^2) \in \mathcal{D}_{\infty,1}$, and so $a_0^1\#a_0^2 \in L^\infty(\mathbb{R}^{2d})$ as the partial inverse Fourier transform of a function dominated by an L^1 function (as in the proof of Theorem 15).

We now suppose that M is affine with real-symmetric linear part, say $M(x) = \tilde{M}(x) + \ell$ where $\tilde{M}$ is linear and real-symmetric, and $\ell \in \mathbb{R}^d$. Then

$$\begin{aligned}
&|M(x)k| - \left|M\left(x+\frac{v-k}{2}\right)v\right| - \left|M\left(x+\frac{v}{2}\right)(k-v)\right| \\
&= \left|\tilde{M}(x)k + \ell k\right| - \left|\tilde{M}(x)v + \tilde{M}\left(\frac{v-k}{2}\right)v + \ell v\right| \\
&\quad - \left|\tilde{M}(x)k + \ell k + \tilde{M}\left(\frac{v}{2}\right)k - \tilde{M}(x)v - \tilde{M}\left(\frac{v}{2}\right)v - \ell v\right| \\
&= \left|\tilde{M}(x)k + \ell k\right| - \left|\tilde{M}(x)v + \tilde{M}\left(\frac{v-k}{2}\right)v + \ell v\right| \\
&\quad - \left|\tilde{M}(x)k + \ell k - \tilde{M}(x)v - \tilde{M}\left(\frac{v-k}{2}\right)v - \ell v\right|,
\end{aligned}$$

which is of the form $|A+B| - |A| - |B|$ for $A, B \in \mathbb{R}^d$, and is hence less than or equal to 0 by the triangle inequality. Thus we find

$$
\begin{aligned}
&\left|\mathcal{F}_2(a_0^1 \# a_0^2)(x,k)\right| \\
&\leq (2\pi)^{\frac{d}{2}} \int_{\mathbb{R}^d} \exp\left(|M(x)k| - \left|M\left(x + \frac{v-k}{2}\right)v\right| - \left|M\left(x + \frac{v}{2}\right)(k-v)\right|\right) \\
&\exp(-|M(x)k|)\left|g_{a_0^1}\left(x + \frac{v-k}{2}, v\right) g_{a_0^2}\left(x + \frac{v}{2}, k-v\right)\right| dv \\
&\leq (2\pi)^{\frac{d}{2}} \exp(-|M(x)k|) \int_{\mathbb{R}^d} \left|g_{a_0^1}\left(x + \frac{v-k}{2}, v\right) g_{a_0^2}\left(x + \frac{v}{2}, k-v\right)\right| dv \\
&\leq (2\pi)^{\frac{d}{2}} \exp(-|M(x)k|) \int_{\mathbb{R}^d} G_{a_0^1}(v) G_{a_0^2}(k-v) dv \\
&= (2\pi)^{\frac{d}{2}} \exp(-|M(x)k|) G_{a_0^1} * G_{a_0^2}(k)
\end{aligned}
$$

Where $*$ denotes the convolution. Note that $G_{a_0^1} * G_{a_0^2} \in L^1(\mathbb{R}^d)$ as $G_{a_0^1}, G_{a_0^2} \in L^1(\mathbb{R}^d)$, so $a_0^1 \# a_0^2 \in HS_0(M)$. Further, $||G_{a_0^1} * G_{a_0^2}||_{L^1(\mathbb{R}^d)} \leq ||G_{a_0^1}||_{L^1(\mathbb{R}^d)} ||G_{a_0^2}||_{L^1(\mathbb{R}^d)}$ so $||a_0^1 \# a_0^2||_{HS_0(M)} \leq ||a_0^1||_{HS_0(M)} ||a_0^2||_{HS_0(M)}$.

We check the other terms of $a^1 \# a^2$.

$$
\begin{aligned}
\mathcal{F}_2(a_0^1 \# a_b^2)(x,k) &= \mathcal{F}_2 a_0^1(x,k) a_b^2\left(x + \frac{k}{2}\right) \\
&= (2\pi)^{\frac{d}{2}} a_b^2\left(x + \frac{k}{2}\right) g_1(x,k) \exp(-M(x)|k|),
\end{aligned}
$$

with $a_b^2\left(x + \frac{k}{2}\right) g_1(x,k) \in \mathcal{D}_{\infty,1}$ with norm bounded by $||g_1||_{\mathcal{D}_{\infty,1}} ||a_b^2||_{L^\infty(\mathbb{R}^{2d})}$.

$$
\begin{aligned}
\mathcal{F}_2(a_b^1 \# a_0^2)(x,k) &= a_b^1(x - \frac{k}{2}) \mathcal{F}_2 a_0^2(x,k) \\
&= (2\pi)^{\frac{d}{2}} a_b^1\left(x - \frac{k}{2}\right) g_2(x,k) \exp(-M(x)|k|),
\end{aligned}
$$

with $a_b^1\left(x - \frac{k}{2}\right) g_2(x,k) \in \mathcal{D}_{\infty,1}$ with norm bounded by $||a_b^1||_{L^\infty(\mathbb{R}^{2d})} ||g_2||_{\mathcal{D}_{\infty,1}}$.

$$
(a_b^1 \# a_0^2)(x) = a_b^1(x) a_b^2(x),
$$

with $a_b^1(x) a_b^2(x) \in B$ with L^∞ norm bounded by $||a_b^1||_{L^\infty(\mathbb{R}^{2d})} ||a_b^2||_{L^\infty(\mathbb{R}^{2d})}$. Putting these together and using subadditivity of the $HS_0(M)$ norm gives

$$\begin{aligned}
||a^1\#a^2||_{HS(M)} &= ||a_0^1\#a_0^2 + a_0^1\#a_b^2 + a_b^1\#a_0^2||_{HS_0(M)} + ||a_0^1\#a_0^2||_{L^\infty(\mathbb{R}^{2d})} \\
&\leq ||a_0^1\#a_0^2||_{HS_0(M)} + ||a_0^1\#a_b^2||_{HS_0(M)} + ||a_b^1\#a_0^2||_{HS_0(M)} \\
&\quad + ||a_0^1\#a_0^2||_{L^\infty(\mathbb{R}^{2d})} \\
&\leq ||a_0^1||_{HS_0(M)}||a_0^2||_{HS_0(M)} + ||a_0^1||_{HS_0(M)}||a_b^2||_{L^\infty(\mathbb{R})} \\
&+ ||a_b^1||_{L^\infty(\mathbb{R})}||a_0^2||_{HS_0(M)} + ||a_0^1||_{L^\infty(\mathbb{R})}||a_0^2||_{L^\infty(\mathbb{R}^{2d})} \\
&= \big(||a_0^1||_{HS_0(M)} + ||a_b^1||_{L^\infty(\mathbb{R})}\big)\big(||a_0^2||_{HS_0(M)} + ||a_0^2||_{L^\infty(\mathbb{R}^{2d})}\big) \\
&= ||a^1||_{HS(M)}||a^2||_{HS(M)}
\end{aligned}$$

□

We have the easily verified lemma and corollary:

Lemma 2.17. *Assuming M is affine, the Moyal product is associative, and has as identity the constant function $1(x, \xi) = 1$.*

Corollary 2.18. *Fix $\phi \in C^2(\mathbb{R}^d)$ and $p \in [1, \infty]$, and suppose there exists a valid growth pair (M, ϵ) for ϕ and p, with M affine with real-symmetric linear part. Then $(HS(M), \#)$ is a unital Banach algebra, and the generalised Weyl calculus $HS(M) \to B(L^p(\mu))$, $a \mapsto a(Q, P)$ is a bounded Banach algebra homomorphism with norm at most e^ϵ.*

This corollary makes our symbol class $HS(M)$ very distinct from the standard symbol classes of pseudodifferential calculus, and more like a single operator functional calculi. This suggests we really have the “right” norm for symbols, or at least something very close. We can hypothesise that Corollary 2.18 will allow us to get closer to bounded functional calculus for L via softer Banach algebra techniques.

2.6 Concluding Remarks

2.6.1 Semigroup Generation in $HS(M)$

The application of the generalised Weyl calculus developed in Section 2.4 was only possible because the symbol a_t for the Ornstein-Uhlenbeck semigroup, such that $a_t(Q, P) = \exp(-tL)$, was known explicitly for ϕ of the form $\phi(x) = xN(x) + lx$. In this subsection we present some ideas about how one might be able to determine or prove existence of the symbol for the semigroup in cases other than such polynomials.

If we suppose that $a_t(Q, P) = \exp(-tL)$ for some symbol $a_t \in HS(M)$ and that we can extend the Moyal product to include products with h, of Theorem 7 with $h(Q, P) = L$, then the semigroup properties of $\exp(-tL)$ correspond to the following properties of a_t

$$\begin{cases} \frac{da_t}{dt} = -h \# a_t, t > 0 \\ \qquad a_0 = 1. \end{cases} \tag{2.5}$$

We have replaced a (strong) ODE in the Banach algebra $B(L^p(\mu))$ with an ODE in the Banach algebra $HS(M)$. Since $HS(M)$ is also a space of functions, we have a lot of explicit tools to solve for the symbol a_t, such as taking ansatz. By taking a good ansatz, the above ODE can be solved explicitly when $\phi(x) = xN(x) + lx$, which is what was done by the author in determining the formula in Theorem 11. This was also essentially the method used by Hörmander in [Hör18], whose proof we refer to in the proof of our formula, although the method was discovered independently.

This method also lends itself to non-quadratic ϕ, or perturbation, as an abstract semigroup generation problem in a Banach algebra. Noting Remark 2, we know that any ϕ for which the theory presented in this paper is applicable must typically be a bounded C^2 perturbation of a potential of the form $xN(x) + lx$, and that the relevant $HS(M)$ class of symbols is the same for ϕ and $xN(x) + lx$. Thus it would be natural to consider the symbol for the semigroup of the relevant Ornstein-Uhlenbeck operator as a perturbation of the symbol for the semigroup for the Ornstein-Uhlenbeck associated with $xN(x) + lx$.

2.6.2 Banach Algebra Techniques

The idea of the $HS(M)$-valued ODE in Equation 2.5 could be taken even further. If we consider $h\#$ as an unbounded operator on the Banach space $HS(M)$ with an appropriate domain, we could ask when does $h\#$ have a bounded H^∞ functional calculus, I.e. a bounded Banach algebra homomorphism $H^\infty(\Omega) \to B(HS(M))$, $f \mapsto f(h\#)$, for some domain $\Omega \subset \mathbb{C}$. Supposing such a bounded Banach algebra homomorphism exists, abstract theory of Banach algebras and the "associative" nature of the unbounded operator $h\#$ (namely, that $h\#(a^1\#a^2) = (h\#a^1)\#a^2$) will imply that the image of the homomorphism will lie in the subspace of $B(HS(M))$ naturally identifiable with $HS(M)$ (as a Banach algebra $\mathcal{A}$ is always contained in $B(\mathcal{A})$). Making this identification, we can then compose with our generalised Weyl calculus, another bounded Banach algebra homomorphism, to obtain a bounded Banach algebra homomorphism $H^\infty(\Omega) \to B(L^p(\mu))$. As resolvents of $h\#$ would map to resolvents of L under this homomorphism, it must be a bounded H^∞ functional calculus for L on $L^p(\mu)$. Diagrammatically, this process is as follows:

$$\begin{array}{ccccc} H^\infty(\Omega) & \longrightarrow & HS(M) \subset B(HS(M)) & \longrightarrow & B(L^p) \\ \Psi & & \Psi & & \Psi \\ f & \longmapsto & f(h\#) & \longmapsto & f(h\#)(Q,P) = f(L) \end{array}$$

Thus we would find that the bounded H^∞ functional calculus of L factors through $HS(M)$, and so we would know that symbols for the functional calculus for L exist in $HS(M)$, even if we can't write them explicitly. There are some interesting complications to this approach, such that the domain of $h\#$ cannot possibly be dense as its domain cannot include the subspace generated by the identity (or $B \subset HS(M)$, for that matter), while an assumption of dense domain is common in the literature of the H^∞ functional calculus.

This leads us naturally to consider the spectrum of a symbol in $HS(M)$. We have seen in Remark 5 that symbols $a \in HS(M)$ have a "pseudo-holomorphic" extension to the domain $D_M = \{(x, \xi + i\eta) \in \mathbb{R}^d \times \mathbb{C}^d ; \exists t \in (-1,1) \text{ s.t. } \eta = tM(x)\}$, and that the image of this holomorphic extension is contained in the closed complex disc of radius $||a||_{HS(M)}$. It is true that the spectrum of an element of a Banach algebra is always contained in the set of complex numbers of modulus less than or equal to the norm of the element. Also, when we look at the multiplication operator subspace $B \subset HS(M)$, it is clear that the spectrum of an element is the essential range of the element, as an L^∞ function of x. Similarly, if $M = 0$ then $HS(M)$ will contain the Banach algebra $L^1(\mathbb{R}^d)$ with convolution as product (corresponding to those symbols which do not depend on x, in which case the Moyal product degenerates to convolution in k of the partial Fourier transforms). It is true for $L^1(\mathbb{R}^d)$ with convolution as product, that the spectrum of an element is the range of its Fourier transform. Thus we could hypothesise that for $a \in HS(M)$,

$$\sigma_{HS(M)}(a) := \{\lambda \in \mathbb{C}; (a - \lambda) \text{ is not invertible in } HS(M)\}$$

is related to the set

$$\mathrm{EssRan}(a(x, \xi + i\eta)),$$

where the essential range is taken for the extension discussed in Remark 5, over the domain $D_M = \{(x, \xi + i\eta) \in \mathbb{R}^d \times \mathbb{C}^d ; \exists t \in (-1,1) \text{ s.t. } \eta = tM(x)\}$.

While it might seem reasonable to think that the spectrum of a symbol is exactly the essential range of this extension of the symbol, that is most likely not true due to the following example. Consider one of the cases presented in Section 2.4, for $\phi(x) = \frac{x^2}{2}$, the unbounded operator $h\#$ with $h(x,\xi) = \frac{1}{2}(x^2 + \xi^2 - d)$ and $h(Q,P) = L$ the classical Ornstein-Uhlenbeck operator. In this case, a family of symbols for the semigroup generated by L was found, which is uniformly bounded in $HS(M)$ for real time, and so $h\#$ generates (in some sense) a uniformly bounded semigroup in/on $HS(M)$ for real time. Thus $h\#$ should have spectrum with real part bounded below by 0. However, the range of h over the relevant domain for the analytic extension will always include the point $-\frac{d}{2}$, which should cause the semigroup to blow up for large time.

If there was some relationship between the spectrum of a symbol in $HS(M)$ and the range of its holomorphic extension, it would show that if $M \neq 0$, the only elements of $HS(M)$ with real spectrum are elements of B which take real values. Assuming our argument about factorisation of the H^∞ functional calculus through $HS(M)$ holds true, this could be seen as a more explicit reason as to why the classical Ornstein-Uhlenbeck operator has only holomorphic functional calculus on $L^p(\mu)$, for $p \neq 2$. Namely, the symbol $h(x,\xi) = \frac{1}{2}(x^2 + \xi^2 - d)$ for the classical Ornstein-Uhlenbeck operator will have non-real spectrum as an element of $HS(M)$, even though the classical Ornstein-Uhlenbeck operator has only real spectrum.

Chapter 3

Haar Bases and Fractal Tilings of LC Groups

3.1 Introduction

3.1.1 Haar Basis and Daubechies Wavelets on $\mathbb{R}$

The Haar Basis on $\mathbb{R}$

The Haar basis on $\mathbb{R}$ consists of the functions $\psi_{n,l}^{\mathrm{Haar}} : \mathbb{R} \to \mathbb{C}$ for $n, l \in \mathbb{Z}$, given by

$$\psi_{n,l}^{\mathrm{Haar}}(t) = \begin{cases} 2^{\frac{n}{2}}; & t \in [2^{-n}l, 2^{-n}(l+1/2)) \\ -2^{\frac{n}{2}}; & t \in [2^{-n}(l+1/2), 2^{-n}(l+1)) \\ 0; & \text{otherwise.} \end{cases}$$

Alternatively, the Haar basis is the set of all integer translations and L^2-normalised pullbacks under scaling by $2^n, n \in \mathbb{Z}$ of the Haar mother wavelet $\psi_{0,0}$:

$$\psi_{0,0}^{\mathrm{Haar}}(t) = \begin{cases} 1; & t \in [0, 1/2) \\ -1; & t \in [1/2, 1) \\ 0 & \text{otherwise} \end{cases}$$

The Haar basis is a ubiquitous tool in harmonic analysis on $\mathbb{R}$. Not only does it form an orthonormal basis of $L^2(\mathbb{R})$, it also forms an unconditional basis of $L^p(\mathbb{R})$ (see [HvNVW16]). This unconditionality makes the Haar basis essential for L^p analysis, in which equal bandwidth Fourier decompositions fail to be unconditional (as opposed to the exponentially growing bandwidth decompositions in the Littlewood-Paley decomposition). Furthermore, that the Haar basis consists of discretely valued and compactly supported functions makes it a useful tool in computer simulation. The main issue one could claim of the Haar basis is that there is no differential regularity, a problem solved via more general wavelet theories.

Multiresolution Analysis and Daubechies Wavelets

The Haar basis is a simple example of a multiresolution analysis, and many of its useful properties follow from this more general theory. Over $L^2(\mathbb{R})$, a mutiresolution analysis is a sequence of closed subspaces $\{V_n\}_{n\in\mathbb{Z}}$ which are nested,

$$V_n \subset V_{n-1},$$

with

$$\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R}),$$

and

$$\overline{\bigcap_{n\in\mathbb{Z}} V_n} = \{0\}.$$

Furthermore, the subspaces are related via

$$f \in V_n \iff f(2^n \cdot) \in V_0,$$

and are invariant under discrete translations

$$f \in V_0 \iff f(\cdot - l) \in V_0 \ \forall l \in \mathbb{Z}.$$

Lastly, we require that there exists a scaling function $\phi_0 \in V_0$ such that $\{\phi_0(\cdot - l); l \in \mathbb{Z}\}$ is an orthonormal basis for V_0.

The picture one should have is that the spaces are giving increasingly high resolution pictures as $n \to -\infty$ with "pixel" given by the scaling function ϕ_0.

From a multiresolution analysis, one can show that the set of orthogonal complements W_n (such that $V_{n-1} = V_n \oplus W_n$) are mutually orthogonal, have a similar scaling structure under pullback by multiplication by powers of 2, and have orthonormal bases generated by dilations and translations of a single L^2 normalised function $\psi_{0,0}$; the **mother wavelet**. From

$$L^2(\mathbb{R}) = \overline{\bigcup_{n\in\mathbb{Z}} V_n} = \bigoplus_{n\in\mathbb{Z}} W_n,$$

one finds that the set of integer translations and normalised dilations by powers of 2 of $\psi_{0,0}$ form an orthonormal basis for $L^2(\mathbb{R})$. Such a basis is called a **wavelet basis**.

This theory has been developed by Daubechies, Meyer and others (for example, see the standard text [Dau92]). Daubechies has produced a series of compactly supported wavelet bases of increasing regularity which have found significant applications in both analysis and industry. The construction involves many interesting areas of mathematics, including the interplay between Fourier analysis on $\mathbb{R}$ and its symmetric spaces, and polynomial algorithms. The least regular of this series of wavelet bases exactly recovers the Haar basis of $\mathbb{R}$, corresponding to the scaling function taken as the characteristic function of $[0, 1]$.

3.1.2 Fractal Tilings

The theory of fractals as fixed points of iterated function systems was developed in the work of Hutchinson [Hut81] and has found many applications within the sciences at large. This theory can be "inverted" to produce the fractal tilings, as unified in the work of Barnsley and Vince [BV14]. In this section we will explain the components of both of these theories relevant to the current goal. For the sake of this chapter, fractal will be synonymous with "attractor of a contractive IFS".

Throughout this section, X will denote a complete metric space with distance function d.

Definition 3.1. A contractive iterated functions system (IFS) $\mathbb{F}$ is a finite set of strict contractions $f_i : X \to X$, $i = 1, \ldots, n$, i.e. functions satisfying

$$d(f_i(x), f_i(y)) \leq \lambda d(x, y),$$

for some fixed $0 \leq \lambda < 1$.

The main existence theorem of [Hut81] is an example of a contraction mapping argument, with respect to the Hausdorff metric.

Definition 3.2. The Hausdorff metric δ on the set $\mathbb{B}$ of non-empty closed bounded subsets of X is defined to be

$$\delta(A, B) = \sup\{d(a, B), d(b, A); a \in A, b \in B\},$$

where $d(a, B) = \inf\{d(a, b); b, \in B\}$.

Theorem 3.3. $(\mathbb{B}, \delta)$ *is a complete metric space.*

By considering a contractive IFS $\mathbb{F}$ as a function on $\mathbb{B}$ via the action $\mathbb{F}(A) := \bigcup_{i=1}^{n} f_i(A)$ (the Hutchinson operator) and showing that this constitutes a δ-contractive map, Hutchinson proves

Theorem 3.4. *Given a contractive IFS on the complete metric space X, there exists a unique closed, bounded set $K \subset X$ such that*

$$K = \bigcup_{i=1}^{n} f_i(K).$$

Furthermore, K is compact, and given any $A \in \mathbb{B}$, $\lim_{m\to\infty} \delta(\mathbb{F}^m(A), K) = 0$.

The subset K provided by the above theorem is called the **attractor** of the contractive IFS $\mathbb{F}$.

By visualising the fixed point property of attractors, one can see that such fractals are made up of (possibly overlapping) distorted images of themselves, which are themselves made up of distorted images of themselves, a process which continues all the way down. By reversing the procedure as in [BV14], one can obtain a (fractal) tiling instead, per the following definition:

Definition 3.5. A tile in a metric space X is a nonempty compact subset. A tiling of a subset $S \subset X$ is a set of non-overlapping tiles whose union is S (i.e. tiles with disjoint interiors). A tiling of all of X is called a full tiling.

Similarly,

Definition 3.6. An attractor K of a contractive IFS $\mathbb{F} = \{f_i\}_{i=1}^{n}$ is called overlapping if for some $i \neq j, f_i(K)$ and $f_j(K)$ are overlapping. Otherwise K is called non-overlapping. A non-overlapping attractor is either totally disconnected, $f_i(K) \cap f_j(K) = \emptyset$ for $i \neq j$, or just touching, $\partial f_i(K) \cap \partial f_j(K) \neq \emptyset$ for some $i \neq j \in \{1, \ldots, n\}$.

Note if K is non-overlapping and has non-empty interior then K must be just-touching.

Now suppose that each function in the contractive IFS $\mathbb{F}$ has a (continuous) inverse. Suppose further that the unique attractor K of $\mathbb{F}$ has non-empty interior. Then Barnsley and Vince in Theorem 3.8 of [BV14] prove that X can be tiled by copies of K (this is a particularly simple case which suffices for our purposes; Barnsley and Vince prove similar tiling results under the assumption of a non-contractive point-fibred IFS). The picture in this simplified case is roughly as follows:

Choose a point k_0 in the interior of K. Then there exists an open ball $B(k_0, \epsilon)$ around k_0 contained in K. The inverses of functions in $\mathbb{F}$ must be strictly dilating, and so any preimage of K under any finite sequences of functions in $\mathbb{F}$ contains an open ball, the radius of which depends exponentially on sequence length. By making a careful choice of a sequence of inverses the centre of said balls can be chosen to not move very much, so the open balls cover X. Taking the union as $M \to \infty$ and using the fact that K is the attractor of $\mathbb{F}$ to write the relevant pre-image as a union of images of K under compositions of functions in $\mathbb{F}$ and their inverses provides a tiling of all of X.

In the sequel, we will make this sketch explicit in the particular cases we care about.

3.1.3 Unconditionality and Conditional Expectations

We next take a very brief look at unconditionality and conditional expectations, restricting to only those parts of the theory relevant to our needs (including making use of slightly more restrictive definitions than need be). Throughout this section, we let X be a Banach space and $\mathcal{I}$ denote a countably infinite set. Unless otherwise mentioned, the results in this section are all classical and can be found in [HvNVW16], Chapters 3 and 4.

It is a common result shown to people early in their studies of real analysis that if a sequence of real numbers $\{x_n\}_{n\in\mathbb{Z}}$ is conditionally summable (i.e. $\lim_{N\to\infty} \sum_{n=1}^{N} x_n$ exists) but not absolutely summable (i.e. $\lim_{N\to\infty} \sum_{n=1}^{N} |x_n|$ does not exist), then for every real number x there is some reordering $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$\lim_{N\to\infty} \sum_{n=1}^{N} x_{\sigma(n)} = x.$$

If a sequence is absolutely summable, then the limit is invariant under such permutations. In higher (infinite) dimensional spaces, the notion of unconditional summability lies between conditional summability and absolute summability, and still allows for some controlled cancellation.

Definition 3.7 (Definition 4.1.2 of [HvNVW16], paraphrased)**.** A sequence $\{x_i\}_{i\in\mathcal{I}} \subset X$ is called **unconditionally summable** with sum $x \in X$ if for every $\epsilon > 0$ there exists a finite set $F_\epsilon \subset \mathcal{I}$ such that if $F \subset \mathcal{I}$ is a finite set containing F_ϵ then

$$||x - \sum_{i \in F} x_i|| < \epsilon.$$

In this case we write "$\sum_{i \in I} x_i = x$ unconditionally".

There are many equivalent characterisations of unconditional summability, we have chosen the one simplest for our interests (see [HvNVW16] for details). It is clear from the definition that over L^p for $p \in [1, \infty)$, if a sequence of functions $\{f_i\}_{i \in \mathbb{N}}$ have supports which are disjoint up to a set of measure 0, then if the partial sums $\sum_{i=1}^{N} f_i$ converge to some f then it must be the case $\sum_{i \in \mathbb{N}} f_i = f$ unconditionally in L^p. Similarly, we have

Definition 3.8 (Definition 4.1.8 of [HvNVW16]). A **pre-decomposition** of X is a family $\{D_i\}_{i \in \mathcal{I}}$ of bounded projections (i.e. $D_i^2 = D_i$) on X satisfying

$$D_i D_j = 0 \text{ whenever } i \neq j.$$

A pre-decomposition is called an unconditional decomposition if

$$\sum_{i \in I} D_i x = x$$

unconditionally for all $x \in X$. If each D_i has one-dimensional image, we call such a set an unconditional basis (which we say is normalised if each projection has norm 1).

Remark 9. The previous definition does not exactly reproduce a basis in the usual sense, but is rather the set of projections associated with a basis. We make this association throughout.

Examples include (projections onto) orthonormal bases in Hilbert spaces. Significant non-examples include the Fourier basis on L^p of the unit circle for $p \neq 2$, and equal bandwidth Fourier projections on $L^p(\mathbb{R})$ for $p < 2$. One of the most significant properties of unconditional decompositions is the following.

Theorem 3.9 (Proposition 4.1.10 of [HvNVW16], paraphrased). *A pre-decomposition $\{D_i\}_{i \in \mathcal{I}}$ is an unconditional decomposition if and only if*

- *the sums $\sum_{i \in F} D_i x$, where $x \in X$ and $F \subset \mathcal{I}$ is finite, are dense in X.*

- *For any $a = \{a_i\}_{i\in\mathcal{I}} \in \ell^\infty(\mathcal{I})$ and $x \in X$, the sequence $\{a_i D_i x\}_{i\in\mathcal{I}}$ is unconditionally summable, and there exists some constant $C > 0$ independent of a such that the operator*

$$x \mapsto M_a(x) := \sum_{i\in I} a_i D_i x$$

is bounded with norm $||M_a|| \leq C||a||_{\ell^\infty}$.

Thus unconditional decompositions provide enough "independence" that ℓ^∞ multiplier operators can be defined and controlled, with significant applications in the study of functional calculus/spectral multipliers. We will make use of the following lemma, which is immediately true from the definitions presented thus far.

Lemma 3.10. *Suppose $\{D_n\}_{n\in\mathbb{Z}}$ is an unconditional/orthonormal decomposition of X. Given a normalised unconditional/orthonormal basis for $D_n(X)$ for each $n \in \mathbb{Z}$, the union is an unconditional/orthonormal basis for X.*

We now turn to conditional expectations as a source of unconditional decompositions. We let $(S, \mathcal{A}, \mu)$ denote a measure space.

Definition 3.11. A family of sub σ-algebras $\{\mathcal{F}_n\}_{n\in\mathbb{Z}}$ of $\mathcal{A}$ is called a **filtration** if $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all $n \in \mathbb{Z}$. A filtration is called σ-finite if μ is σ-finite on each $\mathcal{F}_n$.

In the setting of probability spaces, a filtration can be thought of as the events whose probabilities can be measured at time n. Through analogy with conditional expectations, one may "condition" over events which have already happened to learn something (probabilistic) about the future. This leads to the following definition

Definition 3.12 (Definition 2.6.4 of [HvNVW16], paraphrased)**.** Let $\mathcal{F} \subset \mathcal{A}$ be a sub σ-algebra. A function $g \in L^1_{\text{loc}}(S, \mathcal{F}, \mu)$ is said to be a **conditional expectation with respect to** $\mathcal{F}$ of $f \in L^1_{\text{loc}}(S, \mathcal{A}, \mu)$ if for all $A \in \mathcal{F}$

$$\int_A g d\mu = \int_A f d\mu.$$

Roughly in words, g is a function defined in terms of the "knowable events" which has the same expectation as f on any such knowable event. By combining many of the results from Chapter 2 of [HvNVW16] (albeit in much less generality than they are given), we obtain

Theorem 3.13. *If μ is σ-finite on $\mathcal{F}$, then conditional expectations exist and are unique for all $f \in L^p(S, \mathcal{A}, \mu)$ for any $p \in [1, \infty)$. We denote the unique conditional expectation of such an f as $\mathbb{E}(f|\mathcal{F})$. The map $f \mapsto \mathbb{E}(f|\mathcal{F})$ is linear and contractive on $L^p(S, \mathcal{A}, \mu)$ for any $p \in [1, \infty)$.*

The following theorem provides the link back to unconditionality.

Theorem 3.14 (Burkholder, Theorem 4.1.11 of [HvNVW16], paraphrased)**.** *Let $(S, \mathcal{A}, \mu)$ be a measure space with σ-finite filtration $\{\mathcal{F}_n\}_{n\in\mathbb{Z}}$ such that the σ-algebra generated by $\bigcup_{n\in\mathbb{Z}} \mathcal{F}_n$ is $\mathcal{A}$. Then the martingale difference sequence $\{D_n\}_{n\in\mathbb{Z}}$ defined by*

$$D_n = \mathbb{E}(\cdot|\mathcal{F}_{n+1}) - \mathbb{E}(\cdot|\mathcal{F}_n)$$

is an unconditional/orthogonal decomposition for $L^p(S, \mathcal{A}, \mu)$ for all $p \in (1, \infty)/p = 2$.

That the martingale difference sequence consists of projections is due to the filtration property $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and properties of conditional expectations. Density of the images of $\{D_n\}_{n\in\mathbb{Z}}$ follows from the so-called martingale convergence theorem and the assumption that the σ-algebra generated by $\bigcup_{n\in\mathbb{Z}} \mathcal{F}_n$ is $\mathcal{A}$.

The following characterisation of the images of $\{D_n\}_{n\in\mathbb{Z}}$ will be of significant use to us. We do not prove the following lemma as we have taken a very shallow look into the theory of conditional expectations, although it follows directly from the definitions.

Lemma 3.15. *Suppose that $\{\mathcal{F}_n\}_{n\in\mathbb{Z}}$ is a σ-finite filtration such that the σ-algebra generated by $\bigcup_{n\in\mathbb{Z}} \mathcal{F}_n$ is $\mathcal{A}$. Suppose further that each $\mathcal{F}_n$ is generated by a countable set T_n such that every intersection between pairs of distinct elements of T_n has measure 0. Then the image of D_n on $L^p(S, \mathcal{A}, \mu)$ can be characterised as those L^p functions which are a.e. constant on elements of T_{n+1} and have integral zero on every element of T_n.*

3.1.4 The Haar Basis on $\mathbb{R}$ from a Fractal Tiling and associated Conditional Expectation

From a choice of invertible contractive IFS on $\mathbb{R}$ and martingale techniques we can recover the Haar basis and some of its significant harmonic-analytic properties.

Consider the invertible contractive IFS $\mathbb{F}^{\mathrm{Haar}} := \{f_{0+2\mathbb{Z}}, f_{1+2\mathbb{Z}}\}$ on $\mathbb{R}$ equipped with the Euclidean distance, where

$$f_{0+2\mathbb{Z}}(t) = \frac{1}{2}t,\ f_{1+2\mathbb{Z}}(t) = \frac{1}{2}(t+1).$$

It is immediately obvious that these maps are both contractive with contraction parameter $\frac{1}{2}$. It can be easily verified that the attractor of $\mathbb{F}^{\text{Haar}}$ is the unit interval $[0,1]$, which has non-empty interior and is just-touching.

Thus the method of Barnsley and Vince can be applied, resulting in the tiling of $\mathbb{R}$ by the intervals $T_0^{\text{Haar}} = \{[l, l+1]; l \in \mathbb{Z}\}$. By utilising the fractal nature, we get a scale of similar tilings $T_n^{\text{Haar}} = \{[2^{-n}l, 2^{-n}(l+1)]; l \in \mathbb{Z}\}$ for each $n \in \mathbb{Z}$. We include the negation in n to match standard notation used for filtrations.

We next use martingale methods to proceed from tiling to Haar basis. For each $n \in \mathbb{Z}$, denote by $\mathcal{F}_n^{\text{Haar}}$ the σ-algebra generated by the tiling T_n^{Haar}, and note that since each tile in T_n^{Haar} is the union of (two) tiles in $T_{(n+1)}^{\text{Haar}}$, we find $\mathcal{F}_n^{\text{Haar}} \subset \mathcal{F}_{n+1}^{\text{Haar}}$. Thus, by definition, $\{\mathcal{F}_n^{\text{Haar}}\}_{n\in\mathbb{Z}}$ is a filtration. Furthermore, the σ-algebra generated by $\bigcup_{n\in\mathbb{Z}} \mathcal{F}_n^{\text{Haar}}$ is $\mathcal{B}(\mathbb{R})$, the Borel σ-algebra of $\mathbb{R}$. The latter equality can be deduced by noting that due to contractivity of $\mathbb{F}^{\text{Haar}}$, any bounded open ball is both contained in and contains the interior of a tile in one of the tilings T_n^{Haar}, and the interior of tiles in T_n^{Haar} are clearly contained in $\mathcal{F}_n^{\text{Haar}}$.

Let $D_n^{\text{Haar}} = \mathbb{E}(\cdot|\mathcal{F}_{n+1}^{\text{Haar}}) - \mathbb{E}(\cdot|\mathcal{F}_n^{\text{Haar}})$ denote the sequence of associated martingale difference operators. By Theorem 3.14, for $p \in (1,\infty)/p = 2$ the decomposition

$$L^p(\mathbb{R}, \mathcal{B}(\mathbb{R})) = \bigoplus_{n\in\mathbb{Z}} D_n(L^p(\mathbb{R}, \mathcal{B}(\mathbb{R})))$$

is unconditional/orthogonal. By Lemma 3.15 the image of D_n on L^p consists of those L^p functions a.e. constant on tiles in T_{n+1}^{Haar} which have integral zero on any tile in T_n^{Haar}. From this characterisation, it can be seen that the span of the subset of the Haar basis with support of measure 2^{-n} is dense in the image of D_n. Combined with the fact such a subset of the Haar basis has supports overlapping on measure 0 sets, the projections associated with the given subset of the Haar basis is an unconditional/orthonormal basis for the image of D_n (normality due to our choice of scaling). Finally, noting the projection operators

$$f(t) \mapsto \psi_{n,l}^{\text{Haar}}(t) \int_{\mathbb{R}} f(s)\psi_{n,l}^{\text{Haar}}(s)ds$$

are normalised on L^p for all $p \in [1,\infty)$, Lemma 3.10 implies that (the projections associated with) the Haar basis is an unconditional/orthonormal basis for $L^p(\mathbb{R}, \mathcal{B}(\mathbb{R})$ for $p \in (1,\infty)/p = 2$.

3.2 Fractal Tilings of LC Groups

Fractal tilings of $\mathbb{R}^d$ based on integer matrices which act as dilations with respect to the Euclidean norm and translations by the integer lattice have been investigated by Vince in [Vin00], and Gelbrich in [Gel94] generalised from the integer lattice to any crystallographic group, which still produces a tiling of $\mathbb{R}^d$. In this section, we instead investigate certain tilings of groups other than $\mathbb{R}^d$.

3.2.1 Notation and Initial Properties

The previous section showed that the construction of the Haar basis and the proof of its unconditionality can essentially be reduced to the construction of the fractal tilings $\{T_n^{\mathrm{Haar}}\}_{n\in\mathbb{Z}}$, which in turn follow from the IFS $\mathbb{F}^{\mathrm{Haar}}$ and properties of its attractor.

The key property of the IFS $\mathbb{F}^{\mathrm{Haar}}$ and its attractor $[0,1]$ which makes the Haar basis correspond to a multiresolution analysis is the fact that each tiling T_n^{Haar} consists of translates of a single tile, rather than the more general homeomorphic images one may expect from fractal tiling theory, and that the different tilings are related by dilations.

We now try to replicate this for metrisable LC groups. Let G be such a group with identity e. We equip G with a right G-invariant metric d inducing the topology on G, that is, a metric $d : G \times G \to [0,\infty)$ such that for all $f, g, h \in G$,

$$d(gf, hf) = d(g, h).$$

It is a theorem of Struble [Str74] that such a metric exists. Local compactness implies that (G, d) is a complete metric space. Furthermore, we suppose that G is unimodular, and we fix a bi-invariant Haar measure μ on G and equip G with $\mathcal{B}(G)$, its Borel σ-algebra.

As a replacement for the integer translations arising in multiresolution analysis and the classical Haar basis, we suppose that G has a countable lattice $L \subset G$ (a lattice is a closed discrete subgroup such that G/L is compact, and hence the projection $G \to G/L$ is a local homeomorphism). No normality is assumed of L. As a replacement for dilation by 2, we suppose that G has a continuous automorphism $\phi : G \to G$ which is a strict dilation with respect to the metric d with dilation factor $C > 1$ (that is, $d(\phi(g), \phi(h)) \geq Cd(g, h)$ for all $g, h \in G$), such that $\phi(L) \subset L$. We fix this notation for the rest of this chapter.

Recall that every two non-trivial right Haar measures are positive multiples of each other. Noting that the pullback $\phi^*\mu$ of μ by ϕ is a non-trivial right Haar measure, there must exist some positive real number, which we denote $|\phi|$, such that $\phi^*\mu = |\phi|\mu$.

Remark 10. It has been brought to the attention of the author during the review period of this thesis that work of Müller-Römer in [MR76] and Siebert in [Sie86] shows that any locally compact group admitting a contractive automorphism (i.e. the inverse of a strictly dilating automorphism) must in fact be a simply connected Lie group whose Lie algebra admits a positive grading (and is hence nilpotent and unimodular). Thus we could simplify things significantly by using this structure from the beginning. However, we continue to work without making direct use of the Lie/simply connected/nilpotent structure as the eventual aim is to apply these methods to homogeneous spaces rather than locally compact groups, and it seems that the assumptions can be significantly weakened in the homogeneous space case. For example, one can clearly construct a Haar basis on the circle by using the covering map $\mathbb{R} \mapsto \mathbb{R}/\mathbb{Z}$, despite the circle not being simply connected and possessing no strictly dilating automorphism. The full generality at which the constructions of this chapter can be applied will be developed in future work.

From this initial setup, we have the following observations:

Lemma 3.16. *The index $|L : \phi(L)|$ is greater than* 1*.*

Proof. As L is discrete and closed, there exists a closest element $l \in L \setminus \{e\}$ to e. As ϕ is an automorphism it fixes e. Suppose $l \in \phi(L)$. Then there exists $l' \in L$ such that $\phi(l') = l$, so $d(l,e) = d(\phi(l'),e) > Cd(l',e) > d(l',e)$, contradicting minimality. Hence there is at least one non-trivial coset in $L/\phi(L)$, and so $|L : \phi(L)| > 1$. □

Lemma 3.17. *The metric d on G is unbounded, hence G is not compact and $\mu(G) = \infty$, and L is not finite. Furthermore, for each positive number M, L contains an infinite set of points each of which at least distance M apart.*

Proof. The initial claim follows from the fact that ϕ is a strict dilation with respect to d. The next two claims follow immediately, and the fourth follows from compactness of G/L. Note that since L is discrete there is some minimum distance M_0 between points of L, so there are infinitely many points of L (in fact, all of L) at least distance $M_0/2$ apart. By repeatedly applying ϕ and using the facts that ϕ is a strict dilation and $\phi(L) \subset L$, the last claim follows. □

Lemma 3.18. *$\phi(L)$ is a lattice, and the map $\tilde{\phi} : G/L \to G/\phi(L)$ defined by $\tilde{\phi}(gL) = \phi(g)\phi(L)$ is a homeomorphism.*

Proof. Discreteness and closedness of $\phi(L)$ is immediate from the fact that ϕ is a strict dilation, and hence the projection $G \to G/\phi(L)$ is a local homeomorphism. We claim the map $\tilde{\phi} : G/L \to G/\phi(L)$ defined by $\tilde{\phi}(gL) = \phi(g)\phi(L)$ is well defined, and is a continuous surjection. Noting ϕ is invertible, the fact $\tilde{\phi}$ is well-defined and injective follows from the observation $gL = hL \iff g^{-1}h \in L \iff \phi(g)^{-1}\phi(h) \in \phi(L) \iff \phi(g)\phi(L) = \phi(h)\phi(L)$. Similarly, given any $g\phi(L) \in G/\phi(L)$, we can choose $\phi^{-1}(g)L \in G/L$ and find $\tilde{\phi}(\phi^{-1}(g)L) = g\phi(L)$. Continuity of $\tilde{\phi}$ follows from the fact that the projections $G \to G/L, G \to G/\phi(L)$ are local homeomorphisms, and the fact that ϕ is continuous. Hence $G/\phi(L) = \tilde{\phi}(G/L)$ is the continuous image of a compact space and is thus compact. That $\tilde{\phi}$ is a homeomorphisms follows from the fact it is a continuous bijection between compact spaces. □

The following lemma is seemingly our only reason for taking G unimodular, but it plays a significant role in the regularity of our upcoming tilings.

Lemma 3.19. *G/L possesses a non-zero finite left G-invariant Borel measure $\mu_{G/L}$. The measure $\mu_{G/L}$ is such that for measurable sets $U \subset G$ on which the natural projection $\pi_L : G \to G/L$ restricted to U is injective, $\mu(U) = \mu_{G/L}(\pi_L(U))$. Similarly, there exists a left G-invariant Borel measure $\mu_{G/\phi(L)}$ on $G/\phi(L)$ with analogous properties.*

Proof. This follows from our assumption that G is unimodular, the fact that both L and $\phi(L)$ are discrete and hence unimodular, and Theorem 3 of Chapter 7, Section 2.6 of [BB04]. □

The following theorem is the key technical hurdle in obtaining well-behaved tilings.

Theorem 3.20. *$|L : \phi(L)| = |\phi|$, and hence $|L : \phi(L)| < \infty$.*

The idea of the proof is to use an a priori misbehaved "tile" T, which is nonetheless "fractal" up to translations by L and $\phi(L)$, to compute the measure of $G/\phi(L)$ in two different ways. It may help to follow the proof with a picture of what happens for the Haar IFS $\mathbb{F}^{\text{Haar}}$.

Proof. Consider the natural quotient maps $\pi_L : G \to G/L$, $\pi_{\phi(L)} : G \to G/\phi(L)$, which are local homeomorphisms as $L, \phi(L)$ are lattices. Since $\pi_L, \pi_{\phi(L)}$ are local homeomorphisms with countable fibres they are both bi-measurable (image and pre-image of Borel measurable sets are Borel measurable).

We construct a measurable fundamental domain for G/L, i.e. a measurable set $T \subset G$ such that $\pi_L|_T : T \to G/L$ is a bijection, and hence $\mu(T) = \mu_{G/L}(G/L) \in (0, \infty)$ as per Lemma 3.19. As π_L is a local homeomorphism and G/L is compact, we can pick a finite open cover $\{U_i\}$ for G/L such that π_L has a continuous inverse on the closure of each U_i.

As L is infinite and discrete and there are only finitely many compact sets $\overline{U_i}$, we can choose continuous right inverses $s_i : \overline{U_i} \to G$ to $\pi_L|_{\pi_L^{-1}(\overline{U_i})}$ for each i. Any right translation by elements of L commutes with the projection π_L. Furthermore, the image of each s_i is compact, and hence we can right translate each s_i such that the images are disjoint (noting the last part of 3.17, and the fact we have chosen a right invariant distance). By cutting intersections, we obtain a finite disjoint measurable collection of sets $\{V_i\}$ with $V_i \subset U_i$ for each i which cover G/L. Finally, set $T = \bigsqcup_i s_i(V_i)$. By construction π_L restricted to T is bijective, and so $\mu(T) = \mu_{G/L}(G/L)$ by Lemma 3.19.

We next use the fundamental domain T to construct two different fundamentals domains for $G/\phi(L)$, the comparison of which will prove the theorem.

The first of these is $\phi(T)$. Measurability follows since ϕ is a homeomorphism and hence bi-measurable. Bijectivity of $\pi_{\phi(L)}|_{\phi(T)}$ follows from bijectivity of the three maps $\pi_L|_T$, ϕ, and $\tilde{\phi}$ (see Lemma 3.18), and the fact that $\pi_{\phi(L)} \circ \phi = \tilde{\phi} \circ \pi_L$.

For the second, choose a set of representatives L' for each left coset $L/\phi(L)$, of which there are countably many since L is countable. We impose the further restriction that $T\ell \cap T\ell' = \emptyset$ whenever ℓ, ℓ' are distinct members of L', which is possible due to the Lemma 3.17 and the fact that T is contained in a compact set. I claim that $\tilde{T} = \bigsqcup_{\ell \in L'} T\ell$ is also a measurable fundamental domain for $G/\phi(L)$. Measurability is clear as the countable union of measurable sets. For bijectivity, we can use the fact $\phi(L) \subset L$ to decompose left L-cosets into left $\phi(L)$-cosets, finding

$$G = \bigsqcup_{t \in T} tL = \bigsqcup_{t \in T} \bigsqcup_{\ell \in L'} t\ell\phi(L) = \bigsqcup_{\tilde{t} \in \tilde{T}} \tilde{t}\phi(L),$$

where the first equality holds as T is a measurable fundamental domain for G/L. Hence $\tilde{T}$ is a measurable fundamental domain for $G/\phi(L)$.

Using right invariance of μ and the fact $\tilde{T} = \bigsqcup_{\ell \in L'} T\ell$ is a countable disjoint union, we find

$$\begin{aligned}
\mu_{G/\phi(L)}(G/\phi(L)) &= \mu(T) \\
&= \mu\left(\bigsqcup_{\ell \in L'} T\ell\right) \\
&= \sum_{\ell \in L'} \mu(T\ell) \\
&= \sum_{\ell \in L'} \mu(T) \\
&= |L'|\mu(T) \\
&= |L : \phi(L)|\mu(T).
\end{aligned}$$

But using the other fundamental domain $\phi(T)$, we find

$$\mu_{G/\phi(L)}(G/\phi(L)) = \mu(\phi(T)) = |\phi|\mu(T).$$

As $\mu(T) = \mu_{G/L}(G/L) \in (0, \infty)$, this implies $|L : \phi(L)| = |\phi|$.

□

3.2.2 The Tiling and its Regularity

We now have enough knowledge to introduce a contractive IFS and prove properties of its attractor.

Definition 3.21. Fix a choice of representatives $L' \subset L$ for the finite set of left cosets $L/\phi(L)$. The IFS $\mathbb{F}^{L'}$ is the finite set of functions $f_\ell : G \to G$ for $\ell \in L'$ with action

$$f_\ell(g) = \phi^{-1}(g\ell).$$

Lemma 3.22. *$\mathbb{F}^{L'}$ is a contractive IFS.*

Proof. As ϕ is a strictly dilating automorphism with dilation parameter $C > 1$, ϕ^{-1} is a strictly contractive automorphism with contraction parameter $C^{-1} < 1$. Recalling that d is right G-invariant, we have for all $g, h \in G$, $\ell \in L'$

$$\begin{aligned}
d(f_\ell(g), f_\ell(h)) &= d\big(\phi^{-1}(g\ell), \phi^{-1}(h\ell)\big) \\
&= d\big(\phi^{-1}(g)\phi^{-1}(\ell), \phi^{-1}(h)\phi^{-1}(\ell)\big) \\
&= d\big(\phi^{-1}(g), \phi^{-1}(h)\big) \\
&\leq C^{-1} d(g, h),
\end{aligned}$$

so f_ℓ is a strict contraction. □

Thus $\mathbb{F}^{L'}$ has a unique compact attractor, which we denote $K^{L'}$. Due to the ongoing nature of this research we make the following definition and assumptions, which are almost surely provable and both hold in every case investigated.

Definition 3.23. A choice of representatives $L' \subset L$ for the finite set of left cosets $L/\phi(L)$ is **admissible** if

- $e \in L'$
- There is some distinguished $\ell_0 \in L'$ which is in the interior of $\phi(K^{L'})$ (equivalently, $\phi^{-1}(\ell_0)$ is an interior point of $K^{L'}$). We call such ℓ_0 the **centre** of L'.
- $K^{L'}$ is non-overlapping, and (hence) $K^{L'}$ is just-touching.

Assumption 3.24 (Admissible choice exists)**.** *There exists an admissible choice of representatives.*

We also make a group-theoretic assumption.

Assumption 3.25 (Centre-shifted ϕ-nary expansion)**.** *Given an admissible choice of representatives L' with centre ℓ_0, any element in $l \in L$ can be written as*

$$\mathit{Expand}_n(\ell_1, \dots \ell_n) := \ell_n \phi(\ell_{n-1}) \phi^2(\ell_{n-2}) \dots \phi^{n-1}(\ell_1) \phi^{n-1}\big(\ell_0^{-1}\big)$$

for some $n \in \mathbb{N}$ and $\ell_1, \dots \ell_n \in L'$. For each fixed n, $\mathit{Expand}_n : (L')^n \to L$ is injective.

Note injectivity does not depend on the choice of centre but surjectivity does. In the case of $\mathbb{F}^{\mathrm{Haar}}$, both $\{0, 1\}$ and $\{-1, 0\}$ are admissible choices of representatives. The attractors are respectively $[0, 1]$ and $[-1, 0]$, which respectively contain $1/2, -1/2$ as interior points, and so 1 and -1 are the only possible respective centres. The second assumption with $L' = \{0, 1\}$ is equivalent to the fact that every natural number less than 2^n has a unique binary expansion, and so the integers in $\{-2^{n-1}, \dots, 2^{n-1} - 1\}$ have a unique expansion as above with n bits, and such sets cover $\mathbb{Z}$.

We fix L' to be admissible with a choice of centre $\ell_0 \in L'$, and now investigate the regularity of its attractor and tiling. From Assumption 3.24, we may use the method of Barnsley and Vince sketched in the introduction to produce a tiling from $\mathbb{F}^{L'}$ and $K^{L'}$. We make a very specific choice of dilating sequence, which makes this a very regular tiling.

Theorem 3.26. *G is tiled by $T_0^{L'}$, the set of right L-translates of $K^{L'}$, that is*

$$G = \bigcup_{l \in L} K^{L'} l$$

and the union is non-overlapping.

Proof. The inverse of f_{ℓ_0} is $g \mapsto \phi(g)\ell_0^{-1}$. From the fact

$$K^{L'} = \bigcup_{\ell \in L'} \phi^{-1}\Big(K^{L'}\ell\Big),$$

we find

$$\begin{aligned} f_{\ell_0}^{-1}\Big(K^{L'}\Big) &= \phi\Big(K^{L'}\Big)\ell_0^{-1} \\ &= \phi\left(\bigcup_{\ell \in L'} \phi^{-1}\Big(K^{L'}\ell\Big)\right)\ell_0^{-1} \\ &= \bigcup_{\ell \in L'} K^{L'}\ell\ell_0^{-1}, \end{aligned}$$

and the union is just-touching. As $e \in L'$ by admissibility, we can dilate via $f_e^{-1} = \phi$. By an easy induction argument,

$$\phi^{n-1}\Big(f_{\ell_0}^{-1}\Big(K^{L'}\Big)\Big) = \bigcup_{\ell_1, \ldots \ell_n \in L'} K^{L'}\mathrm{Expand}_n(\ell_1, \ldots \ell_n)$$

where the union is just-touching. By Assumption 3.25, we thus hit every element of L.

Note that

$$\phi^{n-1}\big(f_{\ell_0}^{-1}(\phi^{-1}(\ell_0))\big) = \phi^{n-1}\big(\ell_0\ell_0^{-1}\big) = e$$

for all $n \geq 1$. As $\phi^{-1}(\ell_0)$ is an interior point of $K^{L'}$ (due to admissibility), there exists an open ball of positive radius centred at $\phi^{-1}(\ell_0)$ contained in $K^{L'}$. Hence e is an interior point of $\phi^{n-1}\big(f_{\ell_0}\big(K^{L'}\big)\big)$ for all $n \geq 1$, and due to the fact ϕ is a strict dilation, each $\phi^{n-1}\big(f_{\ell_0}\big(K^{L'}\big)\big)$ contains an open ball centred at e with positive radius which grows exponentially as n increases. Thus the sets $\big\{\phi^{n-1}\big(f_{\ell_0}\big(K^{L'}\big)\big)\big\}_{n \geq 1}$ cover all of G. Using the above representation of such sets as unions of right L-translates of $K^{L'}$, we conclude

$$G = \bigcup_{l \in L} K^{L'} l$$

and the union is non-overlapping. $\square$

Remark 11. The outcome of the above construction did not depend on a specific choice of centre, only that one exists.

Lemma 3.27. *The tiling $G = \bigcup_{l \in L} K^{L'} l$ is locally finite, i.e. for each tile there are finitely many tiles which intersect it non-trivially.*

Proof. Due to right L-invariance of the tiling, the result will follow for every tile if we show it for a single tile. Note ℓ_0 is an interior point of $K^{L'}$ (by admissibility), $K^{L'}$ is bounded, and ϕ^{-1} is a strict contraction. Thus there must exist some $n \in \mathbb{N}$ such that $\phi^{-n}\big(K^{L'}\big)\ell_0$ is contained in a ball around ℓ_0 in the interior of $K^{L'}$. So $K^{L'}\phi^n(\ell_0)$ is contained in a ball in the interior of $\phi^n\big(K^{L'}\big)$, and so can intersect no more translates of $K^{L'}$ than are contained in $\phi^n\big(K^{L'}\big)$, of which there are finitely many (based on similar reasoning to in the proof of the previous theorem). $\square$

We next move to regularity, depending critically on Theorem 3.20.

Theorem 3.28. *The boundary of $K^{L'}$ has measure zero, $\mu\big(\partial K^{L'}\big) = 0$.*

Proof. Note that since $K^{L'}$ is compact, L is countable, and $G = \bigcup_{l \in L} K^{L'} l$ as a just-touching union, the boundary of $K^{L'}$ is the countable union

$$\partial K^{L'} = \bigcup_{l \in L \setminus \{e\}} K^{L'} \cap K^{L'} l.$$

Due to Lemma 3.27, only finitely many of the translates intersect $K^{L'}$, and so we can choose some large n such that all relevant translates and $K^{L'}$ lie within $f_{\ell_0}^{-n} K^{L'}$. We find

$$\begin{aligned}
\mu\Big(f_{\ell_0}^{-n}\Big(K^{L'}\Big)\Big) &= \mu\Big(\phi^n\Big(K^{L'}\Big)\phi^n\big(\ell_0^{-1}\big)\phi^{n-1}\big(\ell_0^{-1}\big)\dots\phi\big(\ell_0^{-1}\big)\ell_0^{-1}\Big) \\
&= \mu\Big(\phi^n\Big(K^{L'}\Big)\Big) \\
&= |\phi|^n \mu\Big(K^{L'}\Big).
\end{aligned}$$

While by right invariance and subadditivity of μ,

$$\begin{aligned}
\mu\Big(f_{\ell_0}^{-n}K^{L'}\Big) &= \mu\Bigg(\bigcup_{\ell_1,\dots\ell_n\in L'} K^{L'}\mathrm{Expand}_n(\ell_1,\dots\ell_n)\Bigg)\\
&\leq \sum_{\ell_1,\dots\ell_n\in L'} \mu\Big(K^{L'}\mathrm{Expand}_n(\ell_1,\dots\ell_n)\Big)\\
&= \sum_{\ell_1,\dots\ell_n\in L'} \mu\Big(K^{L'}\Big)\\
&= |L'|^n\mu\Big(K^{L'}\Big)\\
&= |L:\phi(L)|^n\mu\Big(K^{L'}\Big).
\end{aligned}$$

By Assumption 3.25, specifically that $\mathrm{Expand}_n : (L')^n \to L$ is injective, equality above can only occur if every pairwise intersection has measure 0. Theorem 3.20 tells us $|\phi| = |L : \phi(L)|$, and so we have equality by comparison to the earlier computation. Thus every pairwise intersection $K^{L'} \cap K^{L'}l$ has measure zero. As there are only countably (finitely) many such intersections and $\partial K^{L'}$ is contained within them, $\mu\big(\partial K^{L'}\big) = 0$ as claimed.

□

From this we obtain the intuitively satisfying corollary:

Corollary 3.29. *Up to a set of measure zero which is contained in $\partial K^{L'}$, $K^{L'}$ is a fundamental domain for G/L.*

Remark 12. This is not the end of the story on regularity of $K^{L'}$. We expect that $K^{L'}$ is in fact a regular closed set (the closure of its interior) based on the filtration properties mentioned at the start of the next section. We also expect that our requirement that L' be an admissible choice of representatives will result in $K^{L'}$ being connected (or possibly even path-connected in the case G is locally path-connected). This will be explored in future, along with proving Assumptions 3.24 and 3.25.

3.3 The Haar Basis and properties

We can finally introduce our Haar Basis. Due to the nature of this section as one long construction, we keep most of the proofs in-text divided up amongst a few definitions.

Definition 3.30. For $n \in \mathbb{Z}$, let

$$T_n^{L'} = \left\{\phi^{-n}\left(K^{L'}l\right); l \in L\right\} = \left\{\phi^{-n}\left(K^{L'}\right)l'; l' \in \phi^{-n}(L)\right\} = \left\{\phi^{-n}(K); K \in T_0^{L'}\right\}$$

It follows immediately from the fact ϕ is a homeomorphism and Theorem 3.26 that for each $n \in \mathbb{Z}$, $T_n^{L'}$ is a tiling of G. It follows from the fact

$$K^{L'} = \bigcup_{\ell \in L'} \phi^{-1}\left(K^{L'}\ell\right)$$

that every tile in $T_n^{L'}$ is a finite union of tiles in $T_{n+1}^{L'}$. We can thus form a well-defined filtration.

Definition 3.31. Define $\{\mathcal{F}_n^{L'}\}_{n\in\mathbb{Z}}$ to be the filtration with $\mathcal{F}_n^{L'}$ the σ-algebra generated by $T_n^{L'}$.

That ϕ is a strict dilation and $K^{L'}$ is bounded with non-empty interior implies that the σ-algebra generated by $\bigcup_{n\in\mathbb{Z}} \mathcal{F}_n^{L'}$ is $\mathcal{B}(G)$.

Consider the martingale difference operators $D_n^{L'} = \mathbb{E}\left(\cdot|\mathcal{F}_{n+1}^{L'}\right) - \mathbb{E}\left(\cdot|\mathcal{F}_n^{L'}\right)$. Note that each tiling is locally finite (Lemma 3.27), that the measure of the boundary of each tile is 0 (Theorem 3.28) and hence the measure of the intersection of any two tiles is 0, and the fact there are countably many tiles. Thus we may apply Lemma 3.15, to deduce that the image of $D_n^{L'}$ on L^p can be characterised as those L^p functions which are a.e. constant on tiles in $T_{n+1}^{L'}$, with integral 0 over each tile in $T_n^{L'}$.

Consider the $n = 0$ case, then, up to a set of measure 0, a function f in the image of $D_0^{L'}$ on L^p must be of the form

$$f = \sum_{l \in L} c_l 1\left(\phi^{-1}\left(K^{L'}l\right)\right),$$

for some $\{c_l\} \in \ell^p(L)$, where $1\left(\phi^{-1}\left(K^{L'}l\right)\right)$ denotes the indicator function of $\phi^{-1}\left(K^{L'}l\right)$. Due to the fractal nature of our tiling, every tile in $T_0^{L'}$ is of the form

$$K^{L'}l' = \left(\bigcup_{\ell \in L'} \phi^{-1}\left(K^{L'}\ell\right)\right)l'$$

for some $l' \in L$. Hence for the integral of f over each tile in $T_0^{L'}$ to be 0, we have

$$\begin{aligned}
0 &= \int_{K^{L'}l'} \sum_{l \in L} c_l 1\Big(\phi^{-1}\Big(K^{L'}l\Big)\Big) d\mu \\
&= \int_{\left(\bigcup_{\ell \in L'} \phi^{-1}\left(K^{L'}\ell\right)\right)l'} \sum_{l \in L} c_l 1\Big(\phi^{-1}\Big(K^{L'}l\Big)\Big) d\mu \\
&= \sum_{\ell \in L'} \int_{\phi^{-1}\left(K^{L'}\ell\right)l'} \sum_{l \in L} c_l 1\Big(\phi^{-1}\Big(K^{L'}l\Big)\Big) d\mu \\
&= \sum_{\ell \in L'} \left(\sum_{l \in L} c_l \mu\Big(\phi^{-1}\Big(K^{L'}l\Big) \cap \phi^{-1}\Big(K^{L'}\ell\Big)l'\Big)\right) \\
&= \sum_{\ell \in L'} c_{\ell\phi(l')} \mu\Big(\phi^{-1}\Big(K^{L'}\ell\Big)l'\Big) \\
&= \mu\Big(\phi^{-1}\Big(K^{L'}\Big)\Big) \sum_{\ell \in L'} c_{\ell\phi(l')}
\end{aligned}$$

where we have used the fact that the intersection of non-equal tiles has measure 0 and μ is right G-invariant. Since $\mu\big(K^{L'}\big) \neq 0$, this shows $f \in D_0^{L'}(L^p(G, \mathcal{B}(G)))$ if and only if

$$f = \sum_{l \in L} c_l 1\Big(\phi^{-1}\Big(K^{L'}l\Big)\Big),$$

for some $\{c_l\} \in \ell^p(L)$ and for all $l' \in L$,

$$\sum_{\ell \in L'} c_{\ell\phi(l')} = 0.$$

Let V_e be the set of such f which are supported on $K^{L'} = \bigcup_{\ell \in L'} \phi^{-1}\big(K^{L'}\ell\big)$, we thus only have the condition

$$\sum_{\ell \in L'} c_\ell = 0.$$

Note that the dimension of V_e is $|L'| - 1 = |L : \phi(L)| - 1 = |\phi| - 1$ as the kernel of a non-trivial linear functional on a space of dimension $|L'|$ (and the chain of equalities courtesy of Theorem 3.20). As each tile $\phi^{-1}\big(K^{L'}\ell\big)$ has the same measure

$$\mu\Big(\phi^{-1}\Big(K^{L'}\ell\Big)\Big) = |\phi|^{-1}\mu\Big(K^{L'}\ell\Big) = |\phi|^{-1}\mu\Big(K^{L'}\Big)$$

and their intersections have measure 0, the L^2 inner product restricted to V_e is $|\phi|^{-\frac{1}{2}}\mu\big(K^{L'}\big)^{\frac{1}{2}}$ times the standard dot product of the column vector of constants $\{c_\ell\}$ seen as an element of $\mathbb{C}^{(|\phi|-1)}$.

Definition 3.32. We call a choice of orthonormal basis $\{\psi_{0,e}^d\}_{d=1}^{|\phi|-1}$ for V_e a set of **mother wavelets** for $\mathbb{F}^{L'}$.

Definition 3.33. For $n \in \mathbb{Z},\ l \in L,\ d = 1, \dots |\phi| - 1$, define $\psi_{n,l}^d : G \to \mathbb{C}$ by

$$\psi_{n,l}^d(g) = |\phi|^{\frac{n}{2}} \psi_{0,e}^d\big(\phi^n\big(gl^{-1}\big)\big).$$

Definition 3.34. We introduce the notation $P_{n,l}^d$ for the projection onto $\psi_{n,l}^d$,

$$f(g) \mapsto \psi_{n,l}^d(g) \int_G \overline{\psi_{n,l}^d(h)} f(h) d\mu(h).$$

Theorem 3.35. *Fix a set of mother wavelets for $\mathbb{F}^{L'}$. For each $n \in \mathbb{Z}$, $\{P_{n,l}^d; l \in L, d = 1, \dots, |\phi| - 1\}$ is an unconditional/orthonormal basis for $D_n^{L'}(L^p(G, \mathcal{B}(G)))$ for all $p \in (1, \infty)/p = 2$. Furthermore, $\{P_{n,l}^d; n \in \mathbb{Z}, l \in L, d = 1, \dots, |\phi| - 1\}$ is an unconditional/orthonormal basis for $L^p(G, \mathcal{B}(G))$ for all $p \in (1, \infty)/p = 2$.*

Proof. We prove the first statement in the case $n = 0$, from which we will deduce the rest. As we have seen, $D_0^{L'}(L^p(G, \mathcal{B}(G)))$ is the set of functions f for which

$$f = \sum_{l \in L} c_l 1\Big(K^{L'} l\Big),$$

for some $\{c_l\} \in \ell^p(L)$ and for all $l' \in L$,

$$\sum_{\ell \in L'} c_{\ell \phi(l')} = 0.$$

It is clear from this characterisation that $D_0^{L'}(L^p(G, \mathcal{B}(G)))$ is the closure of the direct sum of right $\phi(L)$-translates of V_e. As the support of functions in V_e are contained within $\phi\big(K^{L'}\big)$ and the right $\phi(L)$-translates of $\phi\big(K^{L'}\big)$ tile G with intersections of measure 0, the decomposition of $D_0^{L'}(L^p(G, \mathcal{B}(G)))$ into right $\phi(L)$-translates of V_e is unconditional/orthogonal. As V_e is finite dimensional and we have chosen $\{\psi_{0,e}^d\}_{d=1}^{|\phi|-1}$ to be orthonormal, $\{\psi_{0,e}^d\}_{d=1}^{|\phi|-1}$ is an unconditional/orthonormal basis for V_e. Hence the set of projections corresponding to right $\phi(L)$-translates of $\{\psi_{0,e}^d\}_{d=1}^{|\phi|-1}$ are an unconditional/orthonormal basis for $D_0^{L'}(L^p(G, \mathcal{B}(G)))$, and this is exactly the set $\{P_{0,l}^d; l \in L, d = 1, \dots, |\phi| - 1\}$.

We next extend to $n \neq 0$. Recall the characterisation of $D_n^{L'}(L^p(G; \mathcal{B}(G))$ as those L^p functions which are a.e. constant on tiles in $T_n^{L'}$, with integral 0 over each tile in $T_{n-1}^{L'}$. Combined with the fact that $T_n^{L'} = \big\{\phi^{-n}(K); K \in T_0^{L'}\big\}$, this implies that $f(g) \mapsto |\phi|^{\frac{n}{2}} f(\phi^n(g))$ is a multiple of an isometry $D_0^{L'}(L^p(G; \mathcal{B}(G)) \to D_n^{L'}(L^p(G; \mathcal{B}(G))$ for all p. The choice of normalisation makes it unitary on for $p = 2$. Conjugating $\{P^d_{0,l}; l \in L, d = 1, \ldots, |\phi| - 1\}$ by the multiple of an isometry/unitary produces a normalised unconditional/orthonormal basis for $D_n^{L'}(L^p(G; \mathcal{B}(G))$. A quick calculation verifies that such maps are exactly $\{P^d_{n,l}; l \in L, d = 1, \ldots, |\phi| - 1\}$. We have thus proven the first statement.

Finally we apply Theorem 3.14 and Lemma 3.10 to deduce the second statement.

□

3.4 Examples and Concluding Remarks/Further Ideas

3.4.1 Easy Example: $\mathbb{R}$ with a different dilating automorphism

We take $G = \mathbb{R}$ with its usual group structure and Euclidean metric, take as lattice $L = \mathbb{Z}$, but replace the dilating automorphism from the Haar IFS $\mathbb{F}^{\text{Haar}}$ with $\phi(x) = 3x$. It can be quickly verified that the only possible admissible choices of representatives are $\{0, 1, 2\}, \{-1, 0, 1\}$ and $\{-2, -1, 0\}$, with corresponding attractors $[0, 1], [-1/2, 1/2]$ and $[-1, 0]$. For $\{0, 1, 2\}$ and $\{-2, -1, 0\}$ either of the non-zero elements can be chosen as centre, while for $\{-1, 0, 1\}$ any of the three points satisfies the requirements to be a centre. Any combination of the choices will then satisfy Assumption 3.25.

3.4.2 Easy Example: $\mathbb{R}^2$ and $\mathbb{R}^d$

For one tiling, we take

- $G = \mathbb{R}^2$ with its usual LC group structure.
- $L = \mathbb{Z}^2$
- $\phi = 2I$
- $L' = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$.

The tile obtained is the unit square $K^{L'} = [0, 1] \times [0, 1]$, and we note after the fact that $(1, 1)$ is a centre for L' (in fact, the only possible centre). In this case it is obvious that we get a tiling by $K^{L'}$ which is very well-behaved, so we won't bother verifying Assumption 3.25.

As an alternative, we can keep everything the same except swap to $L' = \{(0,0),(0,1),(1,1),(1,2)\}$. In this case $K^{L'}$ is a quadrilateral with vertices given by the elements of L', and $(1,2)$ is a centre (again the only possible).

In fact, it can be quickly verified that if we choose any two $\ell_1, \ell_2 \in L \setminus \phi(L)$ representatives for distinct cosets in $L/\phi(L)$ which are as close as possible to e, then $L' = \{e, \ell_1, \ell_2, \ell_1 + \ell_2\}$ is an admissible choice of representatives, the corresponding attractor is the quadrilateral with vertices L', and $\ell_1 + \ell_2$ is the unique centre.

This is readily generalised to $\mathbb{R}^d$, in which case we pick for d cosets in $L/\phi(L)$ a closest possible representative $\ell_1, \dots \ell_d$ to e, and then take take

$$L' = \left\{ \sum_{i=1}^{d} c_i \ell_i ; c_i \in \{0,1\} \right\}.$$

The attractor will be the parallelepiped with vertices L', and the unique centre is $\sum_{i=1}^{d} \ell_i$.

3.4.3 Non-Trivial Example: The simply-connected Heisenberg Group $\mathbb{H}_3(\mathbb{R})$

The following example is of a very special nature, in that the group G is a homogeneous Lie group (hence simply-connected, graded, nilpotent). Such Lie groups have by definition a particularly nice choice of dilating automorphisms ϕ_r for every $r \in (1, \infty)$, and there always exists a right invariant metric d inducing the topology on G such that ϕ_r scales distances, $d(\phi_r(g), \phi_r(h)) = rd(g,h)$ (an example of a Carnot-Carathéodory metric). Fractal tilings of a certain sub-class of such groups has been investigated by Strichartz in [Str92], which includes the following example.

We take $\mathbb{H}_3(\mathbb{R})$ the simply-connected Heisenberg group, which is given by

$$\mathbb{H}_3(\mathbb{R}) := \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} ; a, b, c \in \mathbb{R} \right\},$$

with matrix multiplication. It is easily verified that if we denote the matrix

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

as $M(a,b,c)$, then $M(a,b,c)M(a',b',c') = M(a+a', b+b', c+c'+ab')$. From here, we see that the identity is $M(0,0,0)$, and $M(a,b,c)^{-1} = M(-a,-b,-c+ab)$. The group $\mathbb{H}_3(\mathbb{R})$ is unimodular (in fact nilpotent), and in the coordinates used above left/right Haar measure agrees with Lebesgue measure on $(a,b,c) \in \mathbb{R}^3$.

We use the right $\mathbb{H}_3(\mathbb{R})$-invariant distance which is given in the coordinates above by

$$d((a,b,c),(a',b',c')) = \left(\left((a-a')^2 + (b-b')^2 \right)^2 + (c'-c+b(a-a'))^2 \right)^{\frac{1}{4}}.$$

The verification that this is indeed right invariant and defines a distance function is left to the reader (This distance is associated with the Korányi norm, see [FR16] for details).

From here, we make the following choices:

- We take $L = \mathbb{H}_3(\mathbb{Z}) = \left\{ \begin{bmatrix} 1 & m & p \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}; m,n,p \in \mathbb{Z} \right\}$, which is discrete with respect to the distance d and is a lattice as $\{M(a,b,c); a,b,c \in [0,1]\}$ is compact and surjects onto the quotient $\mathbb{H}_3(\mathbb{R})/L$
- We take $\phi : \mathbb{H}_3(\mathbb{R}) \to \mathbb{H}_3(\mathbb{R})$ to be $\phi(M(a,b,c)) = M(2a,2b,4c)$, which is easily checked to be an automorphism and a strict dilation, in fact $d((2a,2b,4c),(2a',2b',4c')) = 2d((a,b,c),(a',b',c'))$. Furthermore, $\phi(L) \subset L$.
- We take $L' := \{M(m,n,p); m,n \in \{0,1\}, p \in \{0,1,2,3\}\}$ as a choice of representatives for $L/\phi(L)$.

Some images of (approximations to) the resulting attractor $K^{L'}$ are included in Figure 3.1, from which one can see $M(1,1,2)$ is the unique centre (all other elements of L' are on the boundary). Note that the tile has a partially rough boundary, the presence of which we will discuss later on.

We now verify that Assumption 3.25 holds in this case with the specific choice of centre $M(1,1,2)$, which will imply that $K^{L'}$ tiles G under right translation by L.

Lemma 3.36. *Assumption 3.25 holds for the choices above.*

Proof. We use the notation $\ell_i = M(m_i, n_i, p_i)$. First we claim

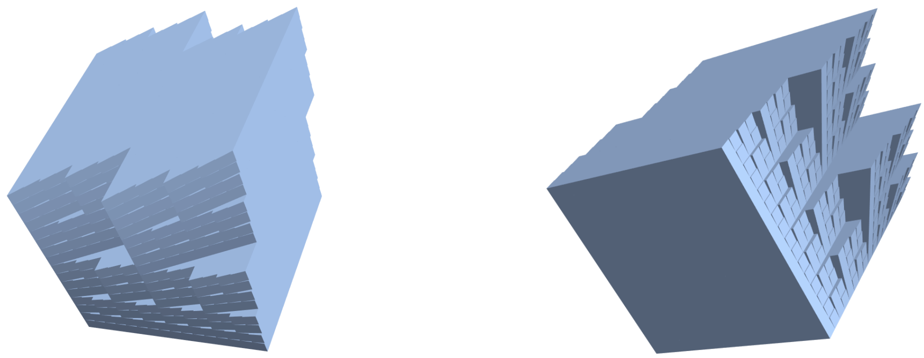

Figure 3.1: The attractor of the IFS considered in Subsection 3.4.3, as a subset of $\mathbb{R}^3$ in the coordinates $(a, b, c) \in \mathbb{R}^3$. Note the relevant translations are not those of $\mathbb{R}^3$ and the relevant distance is not the Euclidean one. The rough face is in the direction corresponding to coordinate c, and so one can see that the only possible centre for L' is $M(1, 1, 2)$, all other points lying on the boundary. This is also the only figure in this entire thesis.

$$\mathrm{Expand}_w(\ell_1, \ldots, \ell_w)\phi^{w-1}(\ell_0) = M(M_w, N_w, P_w)$$

where

$$M_w = \sum_{k=1}^{w} 2^{w-k} m_k, \;\; N_w = \sum_{k=1}^{w} 2^{w-k} n_k,$$

and

$$P_w = \sum_{k=1}^{w} 4^{w-k} p_k + \sum_{k=2}^{w} 4^{w-k} m_k \left(\sum_{j=1}^{k-1} 2^{k-j} n_j \right)$$

Note all three formulae hold if $w = 1$ trivially. If we assume all three formulae hold for some $w \geq 1$, then we have

$$\begin{aligned}
\mathrm{Expand}_{w+1}(\ell_1, \ldots, \ell_w, \ell_{w+1})\phi^w(\ell_0) &= \ell_{w+1}\phi\big(\mathrm{Expand}_w(\ell_1, \ldots, \ell_w)\phi^{w-1}(\ell_0)\big) \\
&= M(m_{w+1}, n_{w+1}, p_{w+1})M(2M_w, 2N_w, 4P_w) \\
&= M(m_{w+1} + 2M_w, n_{w+1} + 2N_w, \\
&\qquad p_{w+1} + 4P_w + 2m_{w+1}N_w)
\end{aligned}$$

By substitution, we find $m_{w+1} + 2M_w = M_{w+1}$, $n_{w+1} + 2N_w = N_{w+1}$, and $p_{w+1} + 4P_w + 2m_{w+1}N_w = P_{w+1}$. Hence by induction the claim is true.

From the formulae proven above, we can see $((m_1,\dots,m_w),(n_1,\dots,n_w)) \mapsto (M_w, N_w)$ is injective and has image all the pairs of natural numbers in

$$\{0,\dots,2^w-1\}.$$

Due to non-negativity of every $N_{w'}$ for $w' \le w$ for any fixed choice of $(m_1,\dots,m_w)$, $(n_1,\dots,n_w)$, combined with the fact $(p_1,\dots,p_w) \mapsto \sum_{k=1}^{w} 4^{w-k} p_k$ bijects onto $\{0,\dots 4^w-1\}$, we thus find $\mathrm{Expand}_w(\cdot,\dots,\cdot)\phi^{w-1}(\ell_0)$ bijects onto the set of all triplets (M_w, N_w, P_w) of natural numbers with

$$M_w, N_w \in \{0,\dots,2^w-1\} \text{ with expansions } M_w = \sum_{k=1}^{w} 2^{w-k} m_k,\, N_w = \sum_{k=1}^{w} 2^{w-k} n_k,$$

and

$$P_w \in \{P_{M_w,N_w},\dots,4^w-1+P_{M_w,N_w}\},$$

where

$$P_{M_w,N_w} := \sum_{k=2}^{w} 4^{w-k} m_k \left(\sum_{j=1}^{k-1} 2^{k-j} n_j\right).$$

As right multiplication by $\phi^{w-1}(\ell_0)$ is invertible, we thus find for each $w \in \mathbb{N}$, Expand_w is injective. To check that the union over w of the images of Expand_w covers all of L, we right multiply the above image by

$$\phi^{w-1}\big(\ell_0^{-1}\big) = M\big(2^{w-1},2^{w-1},4^{w-1}*2\big)^{-1} = M\big(-2^{w-1},-2^{w-1},-4^{w-1}\big)$$

to find that the image of $\mathrm{Expand}_w(\ell_1,\dots,\cdot)$ corresponds to the set of all triplets $\left(\tilde{M},\tilde{N},\tilde{P}\right)$ with

$$\tilde{M},\tilde{N} \in \{-2^{w-1},\dots,2^{w-1}-1\}$$

and

$$\tilde{P} \in \left\{-4^{w-1}+P_{\tilde{M}+2^{w-1},\tilde{N}+2^{w-1}},\dots,3*4^{w-1}-1+P_{\tilde{M}+2^{w-1},\tilde{N}+2^{w-1}}\right\}.$$

As w varies these sets indeed cover $\mathbb{Z}^3$, so we have verified Assumption 3.25. $\square$

3.4.4 Removal of the Assumptions

As seen in all the examples considered, both Assumptions 3.24 and 3.25 hold. It is highly expected that existence of an admissible choice of representatives always holds, the proof of which we expect will involve examining the interplay between the three quotient maps $G \to G/L$, $G \to G/\phi(L)$, $G/\phi(L) \to G/L$ and the homeomorphism $\tilde{\phi} : G/L \to G/\phi(L)$.

As seen in the examples, we could hypothesise that an admissible choice of representatives can be made by starting with some collection L'' of representatives closest to e of a few of the cosets in $L/\phi(L)$, and taking L' to be a set generated by products of elements of L''. It seems from the examples that a possible choice of centre may then be an element of L' which is maximally factorisable with respect to elements of L''.

Injectivity of the expansion maps $\text{Expand}_n : (L')^n \to L$, which we have already noted in Assumption 3.25 is independent of the choice of centre, is expected to hold directly from some group theoretic argument, and is seen as being essentially equivalent to existence of a well-behaved division algorithm. However, that the images of Expand_n cover L will critically rely on properties of the centre.

3.4.5 Study of Fourier Multipliers using Haar Bases

In studying Fourier multipliers on $\mathbb{R}^d$ (i.e. translation invariant operators), the most natural decomposition to use from the point of view of spectral theory is the Fourier decomposition, which corresponds to decomposing the Fourier transform of a function into a sum of functions each supported on a collection of sets $\{T_i\}$ with measure zero intersection.

One issue with this approach in the L^p, $p < 2$ setting is that equal bandwidth Fourier decomposition (eg taking the above $\{T_i\}$ to be a tiling of $\mathbb{R}^d$ by translates of a single cube) turns out to not be an unconditional one (see [HvNVW16]). From Theorem 3.9, this excludes such a decomposition from being helpful for studying boundedness of Fourier multipliers in such settings. By instead taking the sets in $\{T_i\}$ to be exponentially growing annuli $T_i = \{x \in \mathbb{R}; |x| \in [2^{i-1}, 2^i]\}$, then unconditionality does hold for the corresponding decomposition (this is an example of a Littlewood-Paley decomposition; see for example Chapter 5 of [HvNVW16] for details). The issue now is that such a decomposition does not allow for much freedom in Fourier multipliers (they must act as a constant multiple of the identity on increasingly large subspaces corresponding to the exponentially increasing bandwidth decompositions).

Another issue with the Fourier basis (present even in the $p = 2$ setting) arises in computer implementation: the Fourier transform of a function with compact support never has compact support. Thus every part of a Fourier decomposition with each element of $\{T_i\}$ compact must have non-compact support. This has obvious implications in computer implementation, making both the computer of Fourier decompositions and the storage of said decomposition difficult.

The Haar basis, on the other hand, is unconditional on L^p for all $p \in (1, \infty)$ and consists only of compactly supported functions. The issue then, is that the Haar basis is clearly not invariant under translations and so seems a priori ill-suited to the study of Fourier multipliers.

Work by Petermichl in [Pet00], (see also work by Bourgain [Bou86] and Nazarov-Treil-Volberg [NTV03], among others), uses an averaging procedure to represent some certain "hard" Fourier multipliers (Hilbert transform) via the use of the Haar basis, which initiated significant research into such methods. In essence, even though the Haar basis isn't itself translation invariant, each subset with support of measure an integer multiple of 2^{-n} is invariant with respect to translations by $2^{-n}\mathbb{Z}$. Hence the relevant group of translations is actually $\mathbb{R}/2^{-n}\mathbb{Z}$, which is compact. To single out the Hilbert transform among all Fourier multipliers, a characterisation of the Hilbert transform in terms of its invariance properties was used. Rather than just invariance under translation, the Hilbert transform is also invariant under dilations. As was the case for translations, the Haar basis is also invariant under a large subgroup of dilations, with only a compact group of relevant dilations. By averaging over both the relevant compact groups of dilation and translation, the Hilbert transform is recovered from a smart starting choice of operator on the Haar basis.

The work of Hytönen in [Hyt08], [Hyt10], along with many others, builds upon Petermichl's method. Hytönen provides L^p boundedness of Calderón-Zygmund operators and a proof of the A_2 conjecture using a method of averaging over translations of operators defined in terms of the Haar basis (with both multiplier and shift components).

We expect that very similar methods can be used in our setting to study L^p boundedness of Fourier multipliers on LC groups via the use of the Haar bases developed in this chapter. For the case $G = \mathbb{H}_3(\mathbb{R})$, L^p boundedness of Riesz transforms associated with sub-Laplacians has been studied by others (see for example [ST12] and the references therein). Riesz transforms in such settings can be characterised in terms of invariance properties (Theorem 1.4 of [ST12]), and so we expect to be able to use the methods developed in [Pet00] to express such operators in terms of the Haar bases constructed in this chapter and study their L^p boundedness via unconditionality of such bases. Notably, compactness of G/L and self-similarity of our tilings under dilations provides room for a similar averaging argument. Following this, we expect similar arguments to those in [Hyt10] will provide boundedness of operators similar to those in the Calderón-Zygmund class.

One aspect we expect to play a role in this development is the non-trivial roughness of the boundary of our tiling. It is expected that optimal boundedness of Fourier multipliers can be related to roughness of the tile boundary, which itself is present due to the underlying group being non-Abelian. This would be an incredibly satisfying link from a mathematical point of view, combining fractal geometry, functional analysis and group theory.

3.4.6 Comparison to Existing Works

Fractal tilings have been a popular topic of study for many decades now, and they have found applications in many different areas of mathematics. Unlike our tilings, there are earlier works which intersect the work presented here on a set of strictly positive measure, which we now discuss. There has been significant work in this area so we do not suppose to cover it all, but following the references below leads to the general picture that a lot of work has been put towards studying wavelets on homogeneous Lie groups.

The previously-mentioned work of Strichartz [Str92] investigates tilings of simply-connected rationally-graded nilpotent Lie groups, a subclass of the class of homogeneous Lie groups. Within this study it is shown very explicitly that such groups have a lattice L and dilation ϕ satisfying the requirements we have put forth (although Strichartz chooses a specific set of representatives rather than working with any admissible choice), and it is shown that the resulting tile has boundary of measure 0. For specific examples very similar to $\mathbb{H}_3(\mathbb{R})$, it is shown the boundary of tiles decomposes into parts which have strictly differing Hausdorff dimension based on a homogeneous distance function (the dimension is explicitly calculated and found to be integral), a quantitative version of our observation that there is a rough part and smooth part to the tile in Figure 3.1. This is explained in the simplest case as arising from the grading on the Lie algebra and the fact that curves parallel or perpendicular to each graded component have different Hausdorff dimensions. With the further restriction of a stratification structure on the Lie groups considered, Strichartz also investigates asymptotic heat flow associated with fractal measures arising from his tilings.

In the series of works [LP97], [Pen02], [LL06], [LLW09], Liu, Liu, Peng and Wang (individually and together) use methods similar to those presented in this chapter to construct Haar bases on $\mathbb{H}_3(\mathbb{R})$, and show that such a construction provides a unitary basis on L^2. They develop the theory further towards smooth wavelets via representation theory, remaining mostly in the L^2 setting (with some study of L^p convergence of so-called cascade sequences which are related to multiresolution analysis). Building on this series, Li and Yang in [YL15] produce a wavelet basis for the Heisenberg group, with some relationship to the lattice and dilation considered in our example for $\mathbb{H}_3(\mathbb{R})$, by essentially tensoring wavelet bases on $\mathbb{R}$, and characterise various function spaces in terms of such wavelets.

Fractals on Carnot groups (another certain subclass of simply-connected nilpotent Lie groups similar to those studied in [Str92]) have been investigated by Balogh, Berger, Monti and Tyson among others, for example in [BBMT10], with research which seems to focus mostly on regularity/dimensional considerations. There has been much other work in the area by the listed authors and others such as Führ, Lemarié, Maggioni and Mayeli.

The work presented in this chapter thus differs in two major ways from the existing literature. The first of these differences is that the general theory developed here only makes use of topological/metric space structure, as opposed to the referenced papers which make heavy use of nilpotency to restrict attention to polynomial functions on the relevant Lie algebra and to apply the Baker-Campbell-Hausdorff formula, for example. Hence there is hope that the methods developed in this chapter could be applied to homogeneous spaces related to a more general class of LC groups, a hope currently under investigation by the author. The other main difference is that we have made use of martingale methods (filtrations, conditional expectations, Burkholder's theorem), which allows the theory developed to automatically apply to L^p spaces for $p \in (1, \infty)$.

3.4.7 Further Ideas

Besides those already mentioned, we have the following further ideas which would make interesting topics of further investigation:

1. As seen in the examples, $K^{L'}$ was always piecewise linear for G Abelian, while for G non-Abelian $K^{L'}$ had distinctly rough boundary. In [Str92], Strichartz has calculated the Hausdorff dimension of the boundary of fractal tilings of groups closely related to $\mathbb{H}_3(\mathbb{R})$ and explains that such roughness should always be present in tilings of such groups. Is this true outside of the cases Strichartz has studied, or outside of the setting of simply-connected nilpotent Lie groups more generally?

2. That we have chosen to use admissible choices of representatives for $L/\phi(L)$ seems to lead to tilings which are very regular from the point of view of both

topology and measure theory. Can we prove or quantify a statement similar to "*An admissible choice of representatives leads to the "most regular" tiling possible, and the regularity is independent of exactly which admissible choice is made*". Similarly, if we fix a ϕ and let L vary while enforcing the use of an admissible choice of representatives, how does the inherent regularity depend on the choice of L? Such a choice seems to make no difference in the examples examined in this chapter.

3. Can the methods developed in this chapter be used to construct Haar bases on homogeneous spaces as mentioned in Remark 10, in which it seems the initial assumptions can be weakened considerably?

4. As the Haar basis is the first step in the theory of Daubechies wavelet bases [Dau92], can we similarly produce increasingly regular compactly supported wavelet bases which form a multiresolution analysis with respect to translation by L and dilating automorphism ϕ, of which the bases constructed in this chapter are a first step? It is expected such a construction would rely on the interplay between the representation theory of G and Fourier analysis on the homogeneous spaces G/L, for which an answer to the previous question would be pertinent.

Chapter 4

Future Directions: Heisenberg groups outside the Abelian Setting

In Chapters 1 and 2 we have witnessed the strength of maintaining a tie to the Heisenberg group when studying pseudodifferential operators. In the following few sections we examine the Heisenberg group itself and its deep interaction with Fourier analysis (in the general setting of LCA groups).

We finish this thesis by presenting some ideas and ongoing research about how a construction of a Heisenberg group-like object in the non-Abelian setting could be used to study such settings, work in constructing group-group dualities in which Heisenberg groups could take form, and ideas about how to look at the Langlands duality for algebraic reductive groups in such a way that could conceivably produce a Heisenberg group.

Unless otherwise specified, the quoted results in this chapter may be found in Folland's *Abstract Harmonic Analysis*, [Fol95].

4.1 The Abelian Case

4.1.1 The Pontryagin dual and Heisenberg group of an LCA group

Recall the definition of the Pontryagin dual group of a Hausdorff locally compact Abelian (LCA) group G:

Definition 4.1. The Pontryagin dual group $\hat{G}$ of an LCA group G is the LCA group which consists of the continuous homomorphisms $G \to S^1 =: \{z \in \mathbb{C}; |z| = 1\}$, with group operation given by pointwise multiplication and topology given by uniform convergence on compact sets.

That the Pontryagin dual is in fact LCA requires proof, which can be found in [Fol95]. The Pontryagin dual acts very much like the dual of a reflexive Banach space, in that the natural inclusion of G into $\hat{\hat{G}}$ is an isomorphism.

Theorem 4.2 (Pontryagin Duality)**.** *G and $\hat{\hat{G}}$ are canonically (homeomorphically) isomorphic via the evaluation map $g \mapsto (\hat{G} \ni \phi \mapsto \phi(g) \in S^1)$.*

There are many properties LCA groups can have which hold if and only if their Pontryagin dual have a corresponding property. As $\hat{\hat{G}} \cong G$, these pairs of properties are reflexive. We note some of them below.

1. Finite/finite.
2. Compact/discrete.
3. Torsion-free and discrete/connected and compact.
4. Torsion/profinite.
5. Lie/compactly generated.
6. Second countable/Second countable.
7. Separable/metrizable.

From here we can construct a Heisenberg group, exactly as for the case $G = \mathbb{R}^d$. Note that we equip G with its Borel σ-algebra and a Haar measure (which is automatically bi-invariant as we are only considering Abelian groups), but omit both from the notation.

Definition 4.3. The Heisenberg group $H(G)$ of an LCA group G is the subgroup of $U(L^2(G))$ generated by the translation action of G given by $g \cdot f(h) = f(g^{-1}h)$, and the action of $\hat{G}$ given by point-wise multiplication $\phi \cdot f(h) = \phi(h)f(h)$.

In the specific case that $G = \mathbb{R}^d$, we may identify $\hat{\mathbb{R}^d}$ with $\mathbb{R}^d$ by choosing an inner product $\langle \cdot, \cdot \rangle$, and associating to $\xi \in \mathbb{R}^d$ the homomorphism $\phi_\xi \in \hat{\mathbb{R}^d}$, $\phi_\xi(x) = \exp(i\langle \xi, x \rangle)$. Making this identification, the Heisenberg group of $\mathbb{R}^d$ is isomorphic to $\mathbb{R}^{2d} \times S^1$, with product

$$(x, \xi, \omega)(x', \xi', \omega') = \left(x + x', \xi + \xi', \omega\omega' \exp\left(\frac{i}{2}(\langle x, \xi' \rangle - \langle x', \xi \rangle) \right) \right),$$

and is a Lie group with the standard smooth structure of $\mathbb{R}^{2d} \times S^1$.

Upon obtaining a definition for the Heisenberg group of an LCA group, we can construct pseudodifferential calculus on G exactly as for the Euclidean case: we weight and integrate over the action of the Heisenberg group. From this point of view, pseudodifferential calculus (and hence Fourier multipliers) have a clear link to the Heisenberg group.

A much deeper link is now examined. The Mackey-Stone-von Neumann theorem (Theorem 2 of [Mac49]) describes all unitary irreducible representations of such Heisenberg groups.

Theorem 4.4 (Mackey-Stone-von Neumann)**.** *Suppose the LCA group G is separable. Up to unitary equivalence, the strongly continuous irreducible unitary representations of $H(G)$ with centre acting non-trivially are determined by the action of the centre. Furthermore, the defining representation of $H(G)$ on $L^2(G)$ is strongly continuous, unitary and irreducible.*

More precisely, on any irreducible representation the centre must act by multiples of the identity (which follows from Schur's theorem), and thus a representation of $H(G)$ produces a homomorphism $Z(H(G)) \to S^1$. Providing this homomorphism is non-trivial, the above theorem states that the homomorphism determines the isomorphism class of the whole representation. The special case $G = \mathbb{R}^d$ is known as the Stone-von Neumann Theorem.

We can combine this rigidity of the representation theory to deduce the existence of the Fourier transform on such LCA groups and its properties. Note that the proof of the Mackey-Stone-von Neumann theorem does not rely on existence of the Fourier transform, so this does not result in a circular construction.

First note that due to Pontryagin duality, $G \cong \hat{\hat{G}}$, we can determine that $H(G) \cong H\left(\hat{G}\right)$. Under this isomorphism, the defining representations of $H(G)$ on $L^2(G)$ and $H\left(\hat{G}\right) \cong H(G)$ on $L^2\left(\hat{G}\right)$ are both irreducible and have the same action of the centre. Hence

Corollary 4.5. *There exists a unitary map $\mathcal{F} : L^2(G) \to L^2\left(\hat{G}\right)$ which intertwines the action of $H(G)$, $H\left(\hat{G}\right) \cong H(G)$ on either space.*

One can verify that the intertwining property above implies that the map $\mathcal{F}$ swaps products and convolutions, and swaps translation and multiplication by characters. Using these facts, one finds that $\mathcal{F}$ is exactly the Fourier transform on G

Proposition 4.6. *The map $\mathcal{F} : L^2(G) \to L^2\left(\hat{G}\right)$ is the Fourier transform, and is given by the explicit expression (for f in some suitable dense subspace)*

$$\mathcal{F}(f)(\phi) = C \int_G f(g)\overline{\phi(g)}d\mu(g),$$

where C is a normalisation constant.

Remark 13. Although it holds essentially by definition of the Pontryagin dual group, we make very explicit the following basic fact about how the Fourier transform can be used to explore the representation theory of G. As the Fourier transform intertwines the representations of $H(G)$ on $L^2(G)$ and $L^2\left(\hat{G}\right)$, the decompositions of $L^2(G)$ and $L^2\left(\hat{G}\right)$ into $G \subset H(G)$ invariant subspaces must be respected by the Fourier transform. Restricting the corresponding representations to $G \subset H(G)$, we find that the G-invariant subspaces on the $L^2\left(\hat{G}\right)$ side are much easier to classify than those in $L^2(G)$, since G acts by multiplication on $L^2\left(\hat{G}\right)$.

Remark 14 (Metaplectic Representation and the Harmonic Oscillator)**.** In the case $G = \mathbb{R}^d$, the fact $\hat{(\mathbb{R}^d)} \cong \mathbb{R}^d$ means that we can push the (Mackey-)Stone-von Neumann theorem much further to obtain the metaplectic representation. Out of this, we automatically obtain some of the special properties of the harmonic oscillator $H = -\Delta + x^2$ on $L^2(\mathbb{R}^d)$.

Let $Sp(\mathbb{R}^d)$ denote the symplectic group, automorphisms of $H(\mathbb{R}^d)$ which fix the centre. There are many "non-diagonal" automorphisms in $Sp(\mathbb{R}^d)$ due to the fact $\hat{(\mathbb{R}^d)} \cong \mathbb{R}^d$ (i.e. automorphisms which don't restrict to an automorphism of $G, \hat{G} \subset H(G)$ separately). If we let $\pi : H(\mathbb{R}^d) \to U(L^2(\mathbb{R}^d))$ denote the standard representation, it follows by the (Mackey-)Stone-von Neumann Theorem that for all $\Psi \in Sp(\mathbb{R}^d)$, there exists a $U(\Psi) \in U(L^2(\mathbb{R}^d))$ such that for all $h \in H(\mathbb{R}^d)$,

$$\pi(\Psi(h)) = U(\Psi)\pi(h)U(\Psi)^{-1}.$$

The map $\Psi \mapsto U(\Psi)$ is only defined up to multiplication by unit complex numbers (as these are exactly the unitary operators which commute with all $\pi(h)$ by irreducibility). The map $\Psi \mapsto U(\Psi)$ is hence a projective representation. It turns out that this projective representation can be lifted to an honest representation of the double cover of $Sp(\mathbb{R}^d)$, which we denote $Mp(\mathbb{R}^d)$, the metaplectic group. This representation of $Mp(\mathbb{R})$ on $L^2(\mathbb{R})$ is known as the metaplectic, Segal-Shale-Weil, or oscillator, representation.

Consider $\Psi \in Sp(\mathbb{R}^d)$ with $\Psi(k, \eta, \omega) = (\eta, -k, \omega)$ (in the coordinates introduced earlier). This automorphism corresponds to the ever-present automorphism

$H(G) \cong H\left(\hat{G}\right)$ and $U(\Psi)$ is the usual Fourier transform on $L^2(\mathbb{R}^d)$. Note that Ψ is an involution but it is a well known fact that the Fourier transform on $\mathbb{R}^d$ is of order 4: this discrepancy is evidence of the fact we are forced to take a double cover.

Restricting to $d = 1$, $Sp(\mathbb{R})$ contains $SO(2)$ as a maximal compact subgroup, acting as rotation on the (k, η) part of $(k, \eta, \omega) \in H(\mathbb{R})$. Restricting the metaplectic representation to the lift of this compact subgroup, we find that the generator of the action is exactly i times the Harmonic oscillator, $H = -\Delta + x^2$. Thus H is self-adjoint as $\exp(iH)$ is unitary, H has eigenvalues differing by integers as an integer multiple of $2\pi iH$ must exponentiate to the identity, and that these eigenvalues are half-integer is related to the fact that we pass through the double cover. Furthermore, the existence of the creation and annihilation operators follows from finding a copy of $SL(2, \mathbb{R})$ inside $Sp(\mathbb{R})$.

More information about the metaplectic representation can be found in [Fol16].

4.2 The non-Abelian Case

4.2.1 Fourier decomposition and multipliers

We say a group is "nice" if it is required to satisfy any of a long list of requirements which we are not going deep enough into the theory to describe, such as Type I, liminal, post-liminal, etc.

For a compact LCA group G with normalised Haar measure, the Pontryagin dual $\hat{G}$ is discrete and constitutes an orthonormal basis of $L^2(G)$, an analogue of Fourier series on S^1. For each $\phi \in \hat{G}$, span(ϕ) is an irreducible G-invariant subspace under the representation $(g \cdot f)(h) = f(g^{-1}h)$, as $\phi(g^{-1}h) = \phi(g)^{-1}\phi(h) \in$ span(ϕ). Similarly, each $\phi \in \hat{G}$ can be seen as an irreducible unitary representation of G on $\mathbb{C}$, with g acting as multiplication by $\phi(g)$.

Each of the above observations generalises in a reasonable way to non-Abelian compact LC groups, in the unitary dual and Peter-Weyl theorem.

Definition 4.7. Let G be a "nice" LC group. The (left/right) unitary dual $\hat{G}$ of G is the set of equivalence classes of (left/right) irreducible unitary representations of G.

The unitary dual is certainly not a group, but it can be embedded into the set of equivalence classes of all unitary representations of G which is an Abelian monoid under tensor products. The unitary dual can be equipped with a natural topology, although this topology can be bad if G is not "nice" (for example, if G isn't compact or Abelian then $\hat{G}$ is typically not Hausdorff). If G is Abelian, the unitary dual is a group under the operation of tensor product of representations, and is naturally isomorphic to the Pontryagin dual.

Theorem 17 (Peter-Weyl)**.** Let G be a compact LC group with normalised (left/ right) Haar measure μ. Then every (left/right) irreducible unitary representation of G is finite dimensional, and there are countably many such equivalence classes of representations. The set of matrix elements $\rho_{i,j}$ of each equivalence class of (left/right) irreducible unitary representations ρ of G scaled by $\sqrt{\dim(\rho)}$ forms an orthonormal basis of $L^2(G, \mu)$. The subspace of $L^2(G)$ spanned by each column/row of the matrix elements of irreducible unitary representation ρ is (left/right) G-invariant and irreducible, and the action of G on such a subspace is equivalent to ρ/ρ^*. I.e.

$$L^2(G) \cong \bigoplus_{[\rho]\in\hat{G}} V_{\rho^*} \otimes V_\rho \cong \bigoplus_{[\rho]\in\hat{G}} HS(V_\rho) \cong \bigoplus_{[\rho]\in\hat{G}} V_\rho^{\dim(V_\rho)}$$

Where the direct sum is over equivalence classes of irreducible unitary representations $\rho : G \to U(V_\rho)$, and $HS(V_\rho)$ denotes the space of Hilbert-Schmidt operators on V_ρ equipped with the Hilbert-Schmidt norm. Specifically, the decomposition

$$L^2(G) \cong \bigoplus_{[\rho]\in\hat{G}} HS(V_\rho)$$

corresponds to the jointly left/right irreducible G-invariant subspaces of $L^2(G)$.

There are extensions of the Peter-Weyl theorem to classes of non-compact LC groups, analogous to the Fourier transform on $\mathbb{R}$, except that irreducible unitary representations may be infinite dimensional, and the direct sum is replaced by a direct integral of Hilbert spaces (this also requires the unitary dual to have a measurable structure, which can be pathological if G is not sufficiently "nice"). The closest analogue is that for unimodular "nice" LC groups, but unimodularity can be removed at the price of requiring some kind of extra symmetrisation in the above formula.

Under the decomposition granted by the Peter-Weyl theorem, Fourier multipliers are defined as below

Definition 4.8. Let G be a compact LC group, $\hat{G}$ its unitary dual. A (left/right) Fourier multiplier on G is an operator $T \in B(L^2(G))$ such that T respects the decomposition $L^2(G) \cong \bigoplus_{[\rho]\in\hat{G}} HS(V_\rho)$ and there exists an assignment $\hat{G} \ni [\rho] \mapsto \sigma_T(\rho) \in B(V_\rho)$ such that under the given decomposition T acts as (left/right) multiplication by $\sigma_T(\rho)$ on $HS(V_\rho)$. The assignment σ_T is called the (left/right) symbol of the Fourier multiplier.

The left and right Fourier multipliers can be combined into two-sided Fourier multipliers, but these are not so easy to define in terms of a symbol (the naïve approach of defining two-sided Fourier multipliers to have both a left and right symbol does not quite work because the collection of such operators is not closed under addition). The two-sided Fourier multipliers are however closer to what is found in the Abelian case, in that they can be characterised as elements of the bicommutant of the joint left/right regular representations of G, analogous to the statement that Fourier multipliers on $\mathbb{R}^d$ are exactly the translation invariant operators.

4.2.2 Current pseudodifferential calculi on LC groups

Some current pseudodifferential calculi on suitably nice LC groups mirror those on manifolds, taking a phase space/Kohn-Nirenberg approach. Having found a suitable dual to G in the unitary dual $\hat{G}$, pseudodifferential operators are defined to have symbols as operator-valued functions on $G \times \hat{G}$, analogous to the view that pseudodifferential operators on $\mathbb{R}^d$ live in phase space, the (co)tangent bundle $T^*\mathbb{R}^d$. Such theories have been developed by Fischer, Mantoiu, Ruzhansky and Turunen among others, see for example [RT10], [FR14].

A particularly fruitful approach of M. Mantoiu and M. Ruzhansky in [MR17] emulates the Weyl system approach, defining a Weyl system $G \times \hat{G} \ni (g, \rho) \mapsto W(g, \rho) \in U(L^2(G, V_\rho))$ which satisfies relations analogous to the reduced Heisenberg group $\{(k, \eta, 1)\} \subset H(\mathbb{R}^d)$ (when extended to an assignment from G times the Abelian monoid generated by $\hat{G}$ contained in the Abelian monoid of all equivalence classes of unitary representations of G under tensor product, so that a sense of composition can be defined). Pseudodifferential operators are then defined as operators associated to sesquilinear forms built out of (operator-weighted) sums of the Weyl system.

However, all of these approaches remain asymmetric outside of the Abelian setting, due to the fact that $\hat{G}$ is not a group. As we have seen that maintaining links to the Heisenberg group provides a strong algebraic setting with which to do pseudodifferential calculus, we aim instead to start from a Heisenberg-like group.

4.3 Duality and Heisenberg groups of LC groups

Motivated by the fact that the Heisenberg group $H(G)$ of an LCA group G produces both a symmetric pseudodifferential operator theory and through the rigidity of its representation theory implies the existence of the Fourier transform, we would like to develop a theory of Heisenberg-like groups for non-Abelian LC groups. The first step to do so is the development of a group-to-group duality theory for non-Abelian LC groups, which should be (typically) reflexive. Upon defining a reasonable dual group G^* to G, we would next need a reasonable space $\mathcal{H}_G$ on which both act with non-trivial interaction (e.g. there is obviously a joint action on $L^2(G \times G^*)$, but this does not produce any interaction between the group and its dual). Ideally we can take $\mathcal{H}_G = L^2(G)$, but we expect this may not be possible outside the Abelian setting. The Heisenberg group $H(G)$ would then be defined to be the group generated by the actions of G and G^* on $\mathcal{H}_G$. Providing there is some analogue of the Mackey-Stone-von Neumann theorem for such Heisenberg-like groups and something similar to $H(G) \cong H(G^*)$ holds, we would then be granted a Fourier transform intertwining the representations of $H(G)$ on $\mathcal{H}_G$ and $H(G) \cong H(G^*)$ on $\mathcal{H}_{G^*}$, with similar implications to the representation theory of G, G^* as in the discussion in Remark 13.

4.3.1 One approach to duality of LC groups and why it fails

The following work was done towards defining a duality theory for LC groups, although it was realised that this could not provide a satisfying Heisenberg group (the method detailed below will either be non-reflexive or only result in $H(G) \cong H\left(\hat{G}\right)$ if G is LCA). However, we will explain later on that this was in fact based on somewhat similar ideas to Langlands duality for reductive algebraic groups (via the Geometric Satake equivalence).

For G non-Abelian, there may exist no non-trivial continuous homomorphisms to S^1. To avoid this issue, we reinterpret the Pontryagin dual of an LCA group as symmetries rather than homomorphisms.

Theorem 18. For G an LCA group, the action of $\hat{G}$ on any $L^p(G)$ space, $p \in [1, \infty]$ by pointwise multiplication is faithful, preserves every $L^p(G)$ norm, and acts as automorphisms of the convolution algebra. I.e. for $\phi \in \hat{G}$, $f, g \in L^1(G)$,

$$\phi \cdot (f * g) = (\phi \cdot f) * (\phi \cdot g).$$

Hence we can see the Pontryagin dual as a subgroup of

$$\left(\bigcap_{1\le p\le\infty} \mathrm{Isom}(L^p(G))\right) \cap \mathrm{Aut}\big(L^1(G), *\big),$$

where $\mathrm{Isom}(X)$ denotes the group of isometries of Banach space X. In the case $G = \mathbb{R}$ we can already see that this inclusion is proper, as $U : L^p(\mathbb{R}) \to L^p(\mathbb{R})$, $(Uf)(x) = f(-x)$ does not belong to $\hat{G}$ but does belong to the other.

For algebraic convenience, we note that any operator in

$$\left(\bigcap_{1\le p\le\infty} \mathrm{Isom}(L^p(G))\right) \cap \mathrm{Aut}\big(L^1(G), *\big)$$

has a unique extension to an automorphism of $\mathrm{vN}(G)$, the von Neumann algebra generated by $L^1(G)$ acting via convolution on $L^2(G)$. The algebra $\mathrm{vN}(G)$ has benefits over the Banach algebra $(L^1(G), *)$, in that G naturally embeds into $\mathrm{vN}(G)$ via the (left/right) regular representation, and so we will work with $\mathrm{vN}(G)$ rather than $(L^1(G), *)$.

Every component of the above theorem has a clear analogue if G is non-Abelian, the only necessary change being that we need to choose to use either the left or right Haar measure and convolution. The above theorem gives a starting point to finding a possible dual for non-Abelian LC groups which is itself a group, and hence a path to a Heisenberg group in such settings.

We find a simpler expression for the group $\left(\bigcap_{1\le p\le\infty} \mathrm{Isom}(L^p(G))\right) \cap \mathrm{Aut}(\mathrm{vN}(G))$ within which we wish to find a suitable dual group G^*. Noting that $G \hookrightarrow vN(G)$, any $\Psi \in \mathrm{Aut}(\mathrm{vN}(G))$ will map G into the group of invertible elements of $\mathrm{vN}(G)$. It is true that G is a subgroup of the group of units of $\mathrm{vN}(G)$, and in fact any non-zero multiple of any element of G is a unit, but it turns out to be very difficult problem to determine if this is the entire group of units.

A very similar question, determining the group of units in the group algebra of a torsion-free group, is open and progress is considered very difficult. The unit conjecture of Kaplansky [Kap70] conjectures that the units of the group algebra are exactly the units of the ground field times the elements of the group.

However, that we consider the intersection $\left(\bigcap_{1\le p\le\infty} \mathrm{Isom}(L^p(G))\right) \cap \mathrm{Aut}(\mathrm{vN}(G))$ rather than $\mathrm{Aut}(\mathrm{vN}(G))$ allows us to avoid requiring a proof of Kaplansky's unit conjecture. Lamperti's Corollary 3.1 of [Lam58] states the following:

Corollary 4.9. *Suppose that U is a linear transformation of functions measurable on measure space (X, F, μ) which preserve L^p norms for two different values of*

$p \in [1, \infty]$. Then there exists a measure preserving set isomorphism $M : F \to F$ and function $h : X \to \mathbb{C}$ with $|h(x)| = 1$ a.e. on $M(X)$, such that

$$Uf(x) = h(x)Mf(x).$$

It follows that U preserves all L^p norms.

Where a measure-preserving set isomorphism is a map $M : F \to F$, defined up to sets of μ measure zero such that for all $A, A_1, \ldots \in F$

1. $M(X \backslash A) = M(X) \backslash M(A)$
2. $M(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} M(A_n)$
3. $\mu(M(A)) = \mu(A)$,

and Mf is defined by extension from $M\chi_A = \chi_{M(A)}$ for characteristic functions $\chi_A, A \in F$.

It is easy to verify from this corollary that if $X = G$, F is the Borel σ-algebra and μ is left or right Haar measure, then the only such maps which also extend to automorphisms of $\mathrm{vN}(G)$ (i.e. algebraic automorphisms which are continuous in the topology on $\mathrm{vN}(G)$) have M (induced by) a continuous automorphism of G, and $h : G \to S^1$ a homomorphism.

Hence we have the following theorem

Theorem 4.10.

$$\left(\bigcap_{1 \leq p \leq \infty} \mathit{Isom}(L^p(G)) \right) \cap \mathrm{Aut}(\mathrm{vN}(G)) \cong \mathrm{Aut}^{\mathrm{mp}}(G) \ltimes \mathrm{Hom}(G, S^1) \cong \underset{S^1}{\mathrm{Aut}}^{\mathrm{mp}}(G \times S^1),$$

where $\mathrm{Aut}^{\mathrm{mp}}(G)$ denotes the set of (left/right) Haar measure preserving continuous automorphisms of G, the semidirect product is the one implied by the above paragraph, and ${\mathrm{Aut}^{\mathrm{mp}}}_{S^1}(G \times S^1)$ denotes continuous (left/right) Haar measure preserving automorphisms of $G \times S^1$ which act trivially on $\{e_G\} \times S^1$.

Thus, in the same way that the Pontryagin dual of an LCA group is typically seen as a set of certain homomorphisms from G to S^1 (equivalently, automorphisms of $G \times S^1$ of the form $(g, \omega) \mapsto (g, h(g)\omega)$), we can see our hypothetical G^* as a subgroup of ${\mathrm{Aut}^{\mathrm{mp}}}_{S^1}(G \times S^1)$. The extra freedom of permitting automorphisms of G allows us to avoid the problem of not having enough homomorphisms $G \to S^1$.

We can now see that either reflexivity or $H(G) \cong H(G^*)$ must fail in general if we define the dual group as in this section, and define the Heisenberg group as the group of operators generated by the left/right regular representation of G and the action of G^* on $L^2(G)$. If we suppose G is non-Abelian, then for reflexivity to be possible we would need G^* to be non-Abelian. Hence under the above identification, G^* must not be contained within $\mathrm{Hom}(G, S^1) \subset \mathrm{Aut}^{\mathrm{mp}}(G) \ltimes \mathrm{Hom}(G, S^1)$, i.e. there exists some $(\phi, h) \in G^*$ under the above identification with ϕ non-trivial. The action of $(\phi, h) \in G^*$ on $L^2(G)$ is given by

$$(\phi, h) \cdot f(g) = h(g) f(\phi(g)).$$

Comparing to left translation, we find

$$L_k \cdot ((\phi, h) \cdot f(g)) = L_k \cdot h(g) f(\phi(g)) = h(k)^{-1} h(g) f\big(\phi(k)^{-1}\phi(g)\big)$$

while

$$(\phi, h) \cdot (L_k \cdot f(g)) = (\phi, h) \cdot f\big(k^{-1}g\big) = h(g) f\big(k^{-1}\phi(g)\big).$$

These two expressions differ by a global multiplication by $h(k)^{-1}$ and a left translation by $\phi(k)^{-1}k$, so the commutator $[L_k, (\phi, h)]$ lies within $G \times S^1$ (seen as operators on $L^2(G)$ via left translation and multiplication by a phase). Thus $G \times S^1 \subset H(G)$ is normal and since there is some $(\phi, h) \in G^*$ with ϕ non-trivial (and thus some $k \in G$ with $\phi(k)^{-1}k \neq e$), $G^* \times S^1 \subset H(G)$ is not normal. Hence it is impossible for $H(G)$ to be isomorphic to $H(G^*)$ in any canonical sense, as normality is preserved under isomorphisms.

For this reason, we have stopped investigating such a setup and instead turn to a well-developed form of group-to-group duality.

4.3.2 Langlands Duality and the geometric Satake equivalence

The Langlands program represents an enormous area of current research, which in very short terms includes the aim of providing an algebraic parametrisation of finite dimensional representations over certain fields of algebraic reductive groups. This is analogous to how for an LCA group G, the points of the Pontryagin dual $\hat{G}$ parametrise the irreducible unitary representations, albeit much more complicated.

One of the first components of the Langlands program is the Langlands dual group ${}^L G$. An algebraic reductive group can be characterised by combinatorial data known as a root datum in a bijective fashion, and the class of possible root datum has a clear reflexive duality operation on it. Roughly, the Langlands dual group ${}^LG$ of G is defined to be the group which has root datum dual to that of G. Langlands duality is inherently reflexive. Part of the Langlands program aims to relate the (algebraic) geometry of ${}^LG$ to the representation theory of G.

This combinatorial construction for ${}^LG$ does not provide a clear space $\mathcal{H}_G$ on which G and ${}^LG$ act. However, work of Mirkovic and Vilonen in [MV04] has produced a geometric and non-combinatorial construction of the Langlands dual group known as the geometric Satake equivalence, out of which I hope to be able to find a reasonable choice for $\mathcal{H}_G$. This construction proceeds by developing a certain monoidal category $\big(P_{G(\mathcal{O})}(\mathcal{G}\mathrm{r}_G, k), *\big)$ out of geometric objects related to G and noting that the constructed monoidal category satisfies enough requirements to be equivalent to the category $(\mathrm{Rep}_k(H), \otimes)$ of finite dimensional representations over field k of some other algebraic reductive group H (i.e. an algebraic version of Tannakian reconstruction can be applied a la Deligne-Milne and Rivano [DM82]). A lot of work is then put in to show that this other group H is in fact the Langlands dual group ${}^LG$.

Without going into too many details, the objects in the monoidal category $\big(P_{G(\mathcal{O})}(\mathcal{G}\mathrm{r}_G, k), *\big)$ are examples of perverse sheaves, some of which behave very similarly to vector-valued Dirac measures or surface measures which are "locally constant" in some sense. The monoidal product $*$ is in fact a convolution on a space related to G. From this point of view, such a geometric equivalence makes sense from the point of view of LCA groups, as per Remark 13 and Theorem 18.

The complication in using the geometric Satake equivalence to instantly produce a Heisenberg group out of a group and its Langlands dual is that the objects of the category $P_{G(\mathcal{O})}(\mathcal{G}\mathrm{r}_G, k)$ are not exactly functions (or more accurately, perverse sheaves) on G itself, but on the related space $\mathcal{G}\mathrm{r}_G$. It is unclear to me, due to my lack of experience in this subject, if we can make G act on this other space by something analogous to left/right translations. I am currently investigating the possibility of swapping the category $P_{G(\mathcal{O})}(\mathcal{G}\mathrm{r}_G, k)$ with another category which has objects which "live on" G instead, albeit with more complicated behaviour (no longer "locally constant").

Another hope I have from working on this is a development of a Langlands-style duality for a class of LC groups, where we would instead work with unitary complex representations rather than finite dimensional representations over certain fields. The idea here is that the geometric Satake equivalence may have an analytic/measure-theoretic counterpart as per the previous paragraph, and if such an analytic variant exists we may be able to extend it to reasonably well-behaved LC groups. This would require a version of Tannaka-Krein duality and reconstruction for non-compact LC groups/ a priori infinite dimensional unitary representations, of which I have not been able to locate in the literature. One of the particularly nice things about such a possibility is that the geometric Satake equivalence tells you something about ${}^{L}G$ from knowledge of G, rather than the other way around. This is a not particularly interesting observation in the world of reductive algebraic groups, as Langlands duality is inherently reflexive. However, even if an extension of Langlands duality to LC groups were to not be reflexive in general, there would still be benefits to having a pre-dual, as is the case for Banach spaces and von Neumann algebras.

Bibliography

[BB04] N. Bourbaki and S.K. Berberian. *Integration II: Chapters 7–9.* Actualités scientifiques et industrielles. Springer Berlin Heidelberg, 2004.

[BBMT10] Zoltán M. Balogh, Reto Berger, Roberto Monti, and Jeremy T. Tyson. Exceptional sets for self-similar fractals in Carnot groups. *Math. Proc. Cambridge Philos. Soc.*, 149(1):147–172, 2010.

[Bou86] Jean Bourgain. Vector-valued singular integrals and the H^1-BMO duality. In *Probability theory and harmonic analysis (Cleveland, Ohio, 1983)*, volume 98 of *Monogr. Textbooks Pure Appl. Math.*, pages 1–19. Dekker, New York, 1986.

[BV14] Michael Barnsley and Andrew Vince. Fractal tilings from iterated function systems. *Discrete & Computational Geometry*, 51, 2014.

[CD17] Andrea Carbonaro and Oliver Dragičević. Functional calculus for generators of symmetric contraction semigroups. *Duke Math. J.*, 166(5):937–974, 2017.

[Con90] J.B. Conway. *A Course in Functional Analysis.* Graduate texts in mathematics. Springer, 1990.

[Dau92] Ingrid Daubechies. *Ten Lectures on Wavelets.* Society for Industrial and Applied Mathematics, USA, 1992.

[DM82] P. Deligne and J.S. Milne. Tannakian Categories. In *Hodge Cycles, Motives, and Shimura Varieties*, pages 101–228. LNM 900, 1982.

[Epp89] Jay B. Epperson. The hypercontractive approach to exactly bounding an operator with complex Gaussian kernel. *J. Funct. Anal.*, 87(1):1–30, 1989.

[Fol95] G.B. Folland. *A course in abstract harmonic analysis*. Studies in Advanced Mathematics. CRC Press Inc., 1995.

[Fol16] Gerald B. Folland. *Harmonic Analysis in Phase Space*. Princeton University Press, Princeton, 2016.

[FOT11] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet Forms and Symmetric Markov Processes*. De Gruyter studies in mathematics. De Gruyter, 2011.

[FR14] Véronique Fischer and Michael Ruzhansky. A pseudo-differential calculus on graded nilpotent Lie groups. In *Fourier Analysis - Pseudo-differential Operators, Time-Frequency Analysis and Partial Differential Equations*, Trends in Mathematics, pages 107–132, Switzerland, 2014. Springer International Publishing.

[FR16] V. Fischer and M. Ruzhansky. *Quantization on Nilpotent Lie Groups*. Progress in Mathematics. Springer International Publishing, 2016.

[GCMM+01] José García-Cuerva, Giancarlo Mauceri, Stefano Meda, Peter Sjögren, and José Luis Torrea. Functional calculus for the Ornstein-Uhlenbeck operator. *J. Funct. Anal.*, 183(2):413–450, 2001.

[Gel94] Götz Gelbrich. Self-similar periodic tilings on the Heisenberg group. *J. Lie Theory*, 4:31–37, 1994.

[Ḧ79] L. Hörmander. The Weyl calculus of pseudo-differential operators. *Communications on Pure and Applied Mathematics*, 32(3):359–443, 1979.

[Har19] Sean Harris. Optimal angle of the holomorphic functional calculus for the Ornstein-Uhlenbeck operator. *Indag. Math. (N.S.)*, 30(5):854–861, 2019.

[Har20] Sean Harris. A Weyl pseudodifferential calculus associated with exponential weights on $\mathbb{R}^d$, 2020. Accepted for publication by *Illinois Journal of Mathematics*, preprint at arXiv:2001.04572.

[Hör18] L. Hörmander. *Unpublished Manuscripts: from 1951 to 2007*. Springer International Publishing, 2018.

[Hut81] John Hutchinson. Fractals and self similarity. *Indiana University Mathematics Journal*, 30:713–747, 1981.

[HvNVW16] Tuomas Hytönen, Jan van Neerven, Mark Veraar, and Lutz Weis. *Analysis in Banach Spaces Volume I: Martingales and Littlewood-Paley Theory.* Springer International Publishing, 2016.

[HvNVW17] Tuomas Hytönen, Jan van Neerven, Mark Veraar, and Lutz Weis. *Analysis in Banach Spaces Volume II: Probabilistic Methods and Operator Theory.* Springer International Publishing, 2017.

[Hyt08] Tuomas Hytönen. On Petermichl's dyadic shift and the Hilbert transform. *Comptes Rendus Mathematique*, 346(21):1133 – 1136, 2008.

[Hyt10] Tuomas Hytönen. The sharp weighted bound for general Calderon-Zygmund operators. *Annals of Mathematics*, 175, 2010.

[Kap70] Irving Kaplansky. Problems in the theory of rings revisited. *The American Mathematical Monthly*, 77(5):445–454, 1970.

[Lam58] J. Lamperti. On the isometries of certain function-spaces. *Pacific Journal of Mathematics*, 8:459–466, 1958.

[LL06] Heping Liu and Yu Liu. Convergence of cascade sequence on the Heisenberg group. *Proc. Amer. Math. Soc.*, 134(5):1413–1423, 2006.

[LLW09] Heping Liu, Yu Liu, and Haihui Wang. Multiresolution analysis, self-similar tilings and Haar wavelets on the Heisenberg group. *Acta Math. Sci. Ser. B (Engl. Ed.)*, 29(5):1251–1266, 2009.

[LP97] Heping Liu and Lizhong Peng. Admissible wavelets associated with the Heisenberg group. *Pacific J. Math.*, 180(1):101–123, 1997.

[Mac49] George W. Mackey. A theorem of Stone and von Neumann. *Duke Math. J.*, 16(2):313–326, 06 1949.

[MR76] P.R. Müller-Römer. Kontrahierende Erweiterungen und kontrahierbare Gruppen. *Journal für die reine und angewandte Mathematik*, 0283_0284:238–264, 1976.

[MR17] Mark Mantoiu and Michael Ruzhansky. Pseudo-differential operators, Wigner transform and Weyl systems on type I locally compact groups. *Documenta Mathematica*, 22:1539–1592, 2017.

[MV04] I. Mirkovic and K. Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Annals of Mathematics*, 166:95–143, 2004.

[NN18] David Nualart and Eulalia Nualart. *Introduction to Malliavin Calculus*. Institute of Mathematical Statistics Textbooks. Cambridge University Press, 2018.

[NTV03] F. Nazarov, S. Treil, and A. Volberg. The Tb-theorem on non-homogeneous spaces. *Acta Math.*, 190(2):151–239, 2003.

[Pen02] Lizhong Peng. Wavelets on the Heisenberg group. In *Geometry and nonlinear partial differential equations (Hangzhou, 2001)*, volume 29 of *AMS/IP Stud. Adv. Math.*, pages 123–131. Amer. Math. Soc., Providence, RI, 2002.

[Pet00] Stefanie Petermichl. Dyadic shift and a logarithmic estimate for Hankel operators with matrix symbol. *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*, 330:455–460, 2000.

[RT10] Michael Ruzhansky and Ville Turunen. *Pseudo-differential operators and symmetries : background analysis and advanced topics*, volume 2. Birkhäuser, 2010.

[Sie86] Eberhard Siebert. Contractive Automorphisms on Locally Compact Groups. *Mathematische Zeitschrift*, 191:73–90, 1986.

[SMP93] E.M. Stein, T.S. Murphy, and Princeton University Press. *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals*. Monographs in harmonic analysis. Princeton University Press, 1993.

[ST12] P. K. Sanjay and Sundaram Thangavelu. Revisiting Riesz transforms on Heisenberg groups. *Rev. Mat. Iberoam.*, 28(4):1091–1108, 2012.

[Str74] Raimond A. Struble. Metrics in locally compact groups. *Compositio Mathematica*, 28(3):217–222, 1974.

[Str92] Robert S. Strichartz. Self-similarity on nilpotent Lie groups. In *Geometric analysis (Philadelphia, PA, 1991)*, volume 140 of *Contemp. Math.*, pages 123–157. Amer. Math. Soc., Providence, RI, 1992.

[UR19] Wilfredo Urbina-Romero. *Gaussian Harmonic Analysis*. Springer International Publishing, 2019.

[Vin00] Andrew Vince. Digit tiling of Euclidean space. *Directions in Mathematical Quasicrystals*, pages 329–370, 01 2000.

[vNP18] Jan van Neerven and Pierre Portal. The Weyl calculus with respect to the Gaussian measure and restricted L^p-L^q boundedness of the Ornstein-Uhlenbeck semigroup in complex time. *Bull. Soc. Math. France*, 146(4):691–712, 2018.

[vNP20] Jan van Neerven and Pierre Portal. The Weyl calculus for group generators satisfying the canonical commutation relations. *J. Operator Theory*, 83(2):253–298, 2020.

[YL15] Qixiang Yang and Pengtao Li. Regular orthogonal basis on Heisenberg group and application to function spaces. *Math. Methods Appl. Sci.*, 38(15):3163–3182, 2015.