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Evolution of Social Power in Social Networks with Dynamic Topology

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***Abstract*—The recently proposed DeGroot-Friedkin model describes the dynamical evolution of individual social power in a social network that holds opinion discussions on a sequence of different issues. This paper revisits that model, and uses nonlinear contraction analysis, among other tools, to establish several novel results. First, we show that for a social network with constant topology, each individual's social power converges to its equilibrium value exponentially fast, whereas previous results only concluded asymptotic convergence. Second, when the network topology is dynamic (i.e., the relative interaction matrix may change between any two successive issues), we show that each individual exponentially forgets its initial social power. Specifically, individual social power is dependent only on the dynamic network topology, and initial (or perceived) social power is forgotten as a result of sequential opinion discussion. Last, we provide an explicit upper bound on an individual's social power as the number of issues discussed tends to infinity; this bound depends only on the network topology. Simulations are provided to illustrate our results.**

***Index Terms*—opinion dynamics, social networks, social power, nonlinear contraction analysis, discrete-time, dynamic topology**

I. Introduction

SOCIAL network analysis is the study of a group of social actors (individuals or organisations) who interact according to a social connection or relationship. Study of social networks has spanned several decades [1], [2] and across several scientific communities. Recently, and in part due to lessons learned and tools developed from research on multi-agent systems [3], the systems and control community has taken an interest in social network analysis.

Of particular interest in this context is the problem of "opinion dynamics", which is the study of how individuals in a social network interact and exchange their opinions on an issue or topic. A critical aspect is to develop models which simultaneously capture observed social phenomena and are simple enough to be analysed, particularly from a system-theoretic point of view. The seminal works of [4], [5] proposed a discrete-time opinion pooling/updating rule, now known as the French-DeGroot (or simply DeGroot) model. A continuous-time counterpart, known as the Abelson model, was proposed in [6]. These opinion updating rules are closely related to consensus algorithms for coordinating autonomous multi-agent systems [7], [8]. The Friedkin-Johnsen model [9], [10] extended the French-DeGroot model by introducing the concept of a "stubborn individual", i.e., an individual who remains attached to its initial opinion. This helped to model *social cleavage* [2], a phenomenon where opinions tend towards separate clusters. Other models which attempt to explain social cleavage include the Altafini model with negative/antagonistic interactions [11]–[14] and the Hegelsmann-Krause bounded confidence model [15], [16]. Simultaneous opinion discussion on multiple, logically interdependent topics was studied with a multidimensional Friedkin-Johnsen model [17], [18].

The concept of *social power* or *social influence* has been integral throughout the development of these models. Indeed, French Jr's seminal paper [4] was an attempt to quantitatively study an individual's social power in a group discussion. Broadly speaking, in the context of opinion dynamics, individual social power is the amount of influence an individual has on the overall opinion discussion. Individuals which maximise the spread of an idea or rumour in diffusion models were identified in [19]. The social power of an individual in a group can change over time as group members interact and are influenced by each other. Recently, the DeGroot-Friedkin model was proposed in [20] to study the dynamic evolution of an individual's social power as a social network discusses opinions on a sequence of issues. In this paper, we present several major, novel results on the DeGroot-Friedkin model. In Section II, we provide a precise mathematical formulation of the model, but a brief description is given shortly to better motivate the study, and elucidate the contributions of the paper.

The discrete-time DeGroot-Friedkin model [20] is a two-stage model. In the first stage, individuals update their opinions on a particular issue, and in the second stage, each individual's level of self-confidence for the next issue is updated. For a given issue, the social network discusses opinions using the DeGroot opinion updating model, which has been empirically shown to outperform Bayesian learning methods in the modelling of social learning processes [21]. The row-stochastic opinion update matrix used in the DeGroot model is parametrised by two sets of variables. The first is individual

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social powers, which are the diagonal entries of the opinion update matrix (i.e. the weight an individual places on its own opinion). The second is the *relative interaction matrix*, which is used to scale the off-diagonal entries of the opinion update matrix to ensure that, for any given values of individual social powers, the opinion update matrix remains row-stochastic. In the original model [20], the relative interaction matrix was assumed to be constant over all issues, and constant throughout the opinion discussion on any given issue. Under some mild conditions on the relative interaction matrix, the opinions reach a consensus on every issue.

At the end of the period of discussion of an issue, i.e., when opinions have effectively reached a consensus, each individual undergoes a sociological process of *self-appraisal* (detailed in the seminal work [22]) to determine its impact or influence on the final consensus value of opinion. Such a mechanism is well accepted as a hypothesis [23], [24] and has been empirically validated [25]. Immediately before discussion on the next issue, each individual self-appraises and updates its individual social power (the weight an individual places on its own opinion) according to the impact or influence it had on discussion of the previous issue. In updating its individual social power, an individual also updates the weight it accords its neighbours' opinions, by scaling using the relative interaction matrix, to ensure that the opinion updating matrix for the next issue remains row-stochastic. The primary objective of the DeGroot-Friedkin model is to *study the dynamical evolution of the individual social powers over the sequence of issues, with self-appraisal occurring after each issue.*

The model is centralised in the sense that individuals are able to observe and detect their impact relative to every other individual in the opinion discussions process, which indicates that the DeGroot-Friedkin model is best suited for networks of small or moderate size. Such networks are found in many decision making groups such as boards of directors, government cabinets or jury panels. A distributed model, referred to as the "modified DeGroot-Friedkin model", was studied in discrete-time in [26], [27]. In continuous-time, [28] studied a model referred to as the "distributed DeGroot-Friedkin model". Dynamic topology, but restricted to doubly-stochastic relative interaction matrices, was studied in [27].

A. Contributions of This Paper

This paper significantly expands on the original DeGroot-Friedkin model in several respects, which we now detail.

1) A novel approach based on nonlinear contraction analysis [29] is used to conclude an exponential convergence property for non-autocratic social power configurations of the DeGroot-Friedkin model, when the social network has dynamic topology, i.e. the relative interaction matrix is issue-varying. Originally, [20] used LaSalle's Invariance Principle to establish global asymptotic stability for constant topology. Exponential convergence was conjectured but not proved; Lefschetz fixed point theory was used in [30] to prove this conjecture. However, [30] only obtained a local convergence result.
2) Configurations where all social power is held be a single autocratic individual are explicitly shown to be unstable, or asymptotically stable, but not exponentially so, with the associated conditions identified.
3) We extend a "contraction-like" result in [20], and use this to establish an upper bound on the individual social power at equilibrium, dependent only on the relative interaction matrix. The ordering of individuals' equilibrium social powers can be determined [20], but numerical values for nongeneric network topologies cannot be determined.
4) We establish an upper bound on the convergence rate for a class of relative interaction matrices.
5) Results 3) and 4) are extended to cover dynamic topology.

By dynamic topology, we mean relative interaction matrices which are different between issues, *but remain constant during the period of discussion for any given issue*. Relative interaction matrices encode trust or relationship strength between individuals in a network, and in Section II-C, we give reasons why the topology might be dynamic, and provide examples in support. It is assumed that the relative interaction matrices do not vary in a manner dependent on the social powers of the individuals, but can otherwise vary arbitrarily.

In more detail, we show that the individual social powers converge exponentially fast to a unique trajectory (as opposed to unique stationary values for constant interactions). Specifically, every individual forgets its initial social power estimate (initial condition) for each issue exponentially fast. During the discussion of any one issue, and as the number of issues discussed tends to infinity, individuals' social powers are determined only by the relative interaction matrices of the previous issues. This paper therefore concludes that a social network described by the DeGroot-Friedkin model is *self-regulating* in the sense that, even on dynamic topologies, sequential discussion combined with reflected self-appraisal removes perceived social power (initial estimates of social power). True social power is determined by topology. Periodically-varying topologies are presented as a special case. A conference paper [31] by the authors studied the special case of periodically-varying topology and proved the existence of periodic trajectories, but did not provide a convergence proof.

B. Structure of the Rest of the Paper

Section II introduces mathematical notations, nonlinear contraction analysis and the DeGroot-Friedkin model. Section III uses nonlinear contraction analysis to study the original DeGroot-Friedkin model. Dynamic topologies are studied in Section IV. Simulations are presented in Section V, and concluding remarks are given in Section VI.

II. Background and Problem Statement

We begin by introducing some mathematical notations used in the paper. Let $\mathbf{1}_n$ and $\mathbf{0}_n$ denote, respectively, the $n \times 1$ column vectors of all ones and all zeros. For a vector $\boldsymbol{x} \in \mathbb{R}^n$, $0 \preceq \boldsymbol{x}$ and $0 \prec \boldsymbol{x}$ indicate component-wise inequalities, i.e., for all $i \in \{1, \dots, n\}$, $0 \leq x_i$ and $0 < x_i$, respectively. The n-simplex is $\Delta_n = \{\boldsymbol{x} \in \mathbb{R}^n : 0 \preceq \boldsymbol{x}, \mathbf{1}_n^\top \boldsymbol{x} = 1\}$. The canonical basis of $\mathbb{R}^n$ is given by $\mathbf{e}_1, \dots, \mathbf{e}_n$. Define $\widetilde{\Delta}_n = \Delta_n \backslash \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and $\text{int}(\Delta_n) = \{\boldsymbol{x} \in \mathbb{R}^n : 0 \prec \boldsymbol{x}, \mathbf{1}_n^\top \boldsymbol{x} = 1\}$. The 1-norm and

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infinity-norm of a vector, and their induced matrix norms, are denoted by $\|\cdot\|_1$ and $\|\cdot\|_\infty$, respectively. For the rest of the paper, we shall use the terms "node", "agent", and "individual" interchangeably. We shall also interchangeably use the words "self-weight", "social power", and "individual social power".

A square matrix with all entries nonnegative is called *row-stochastic* (respectively *doubly stochastic*) if its row sums all equal 1 (respectively if its row and column sums all equal 1).

A. Graph Theory

The interaction between individuals in a social network is modelled using a weighted directed graph, denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{C})$. Each individual corresponds to a node in the finite, nonempty set of nodes $\mathcal{V} = \{v_1, \ldots, v_n\}$. The set of ordered edges is $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. We denote an ordered edge as $e_{ij} = (v_i, v_j) \in \mathcal{E}$, and because the graph is directed, in general, e_{ij} and e_{ji} may not both exist. An edge e_{ij} is said to be outgoing with respect to v_i and incoming with respect to v_j. The presence of an edge e_{ij} connotes that individual j learns of, and takes into account, the opinion value of individual i when updating its own opinion. The incoming and outgoing neighbour sets of v_i are respectively defined as $\mathcal{N}_i^+ = \{v_j \in \mathcal{V} : e_{ji} \in \mathcal{E}\}$ and $\mathcal{N}_i^- = \{v_j \in \mathcal{V} : e_{ij} \in \mathcal{E}\}$. The relative interaction matrix $\boldsymbol{C} \in \mathbb{R}^{n\times n}$ is associated with $\mathcal{G}$, the relevance of which is explained below. The matrix $\boldsymbol{C}$ has nonnegative entries c_{ij}, termed "relative interpersonal weights" in [20]. The entries of $\boldsymbol{C}$ have properties such that $0 < c_{ij} \leq 1 \Leftrightarrow e_{ji} \in \mathcal{E}$ and $c_{ij} = 0$ otherwise. It is assumed that $c_{ii} = 0$ (i.e., there are no self-loops), and we impose the restriction that $\sum_{j\in\mathcal{N}_i^+} c_{ij} = 1$ (i.e., $\boldsymbol{C}$ is a row-stochastic matrix). The word "relative" therefore refers to the fact that c_{ij} can be considered as a percentage of the total weight or trust individual i places on individual j compared to all of individual i's incoming neighbours.

A directed path is a sequence of edges of the form $(v_{p_1}, v_{p_2}), (v_{p_2}, v_{p_3}), \ldots$ where $v_{p_i} \in \mathcal{V}, e_{p_i p_{i+1}} \in \mathcal{E}$. Node i is reachable from node j if there exists a directed path from v_j to v_i. A graph is said to be strongly connected if every node is reachable from every other node. The relative interaction matrix $\boldsymbol{C}$ is irreducible if and only if the associated graph $\mathcal{G}$ is strongly connected. If $\boldsymbol{C}$ is irreducible, then it has a unique left eigenvector $\boldsymbol{\gamma}^\top \succ 0$ satisfying $\boldsymbol{\gamma}^\top \mathbf{1}_n = 1$, associated with the eigenvalue 1 (Perron-Frobenius Theorem, see [32]). Henceforth, we call $\boldsymbol{\gamma}^\top$ the *dominant left eigenvector of* $\boldsymbol{C}$.

B. The DeGroot-Friedkin Model

We define $\mathcal{S} = \{0, 1, 2, \ldots\}$ to be the set of indices of sequential issues which are being discussed by the social network. For a given issue $s \in \mathcal{S}$, the social network discusses it using the discrete-time DeGroot consensus model (with constant weights throughout the discussion of the issue). At the end of the discussion (i.e. when the DeGroot model has effectively reached steady state), each individual undergoes reflected self-appraisal, with "reflection" referring to the fact that self-appraisal occurs following the completion of discussion on the particular issue s. Each individual then updates its own self-weight, and discussion begins on the next issue $s+1$ (using the DeGroot model but now with adjusted weights).

Remark 1 (Time-scales)**.** *The DeGroot-Friedkin model assumes the opinion dynamics process operates on a different time-scale than that of the reflected appraisal process. This allows for a simplification in the modelling and is reasonable if we consider that having separate time-scales merely implies that the social network reaches a consensus on opinions on one issue before beginning discussion on the next issue. The distributed DeGroot-Friedkin model studies the case when the time-scales are comparable [26]. However, the analysis of the distributed model is much more involved, and has not yet reached the same level of understanding as the original model.*

We next explain the mathematical modelling of the opinion dynamics for an issue and the updating of self-weights from one issue to the next.

1) DeGroot Consensus of Opinions: For each issue $s \in \mathcal{S}$, individual i updates its opinion $y_i(s, \cdot) \in \mathbb{R}$ at time $t+1$ as

$$y_i(s, t+1) = w_{ii}(s) y_i(s, t) + \sum_{j\neq i}^{n} w_{ij}(s) y_j(s, t)$$

where $w_{ii}(s)$ is the self-weight individual i places on its own opinion and $w_{ij}(s)$ is the weight placed by individual i on the opinion of its neighbour individual j. Note that $\forall\, i, j$, $w_{ij}(s) \in [0, 1]$ is constant for any given s. As will be made apparent below, $\sum_{j=1}^n w_{ij} = 1$, which implies that individual i's new opinion value $y_i(s, t+1)$ is a convex combination of its own opinion and the opinions of its neighbours at the current time instant. The opinion dynamics for the entire social network can be expressed as

$$\boldsymbol{y}(s, t+1) = \boldsymbol{W}(s) \boldsymbol{y}(s, t) \tag{1}$$

where $\boldsymbol{y}(s, t) = [y_1(s, t), \ldots, y_n(s, t)]^\top$ is the vector of opinions of the n individuals in the network at time instant t. This model was studied in [4], [5] with $\mathcal{S} = \{0\}$ (i.e., only one issue was discussed), and with individuals who remember their initial opinions $y_i(s, 0)$ [9], [10].

Let the self-weight (individual social power) of individual i be denoted by $x_i(s) = w_{ii}(s) \in [0, 1]$ (the i^{th} diagonal entry of $\boldsymbol{W}(s)$) [20], with the individual social power vector given as $\boldsymbol{x}(s) = [x_1, \ldots, x_n]^\top$. For a given issue s, the influence matrix $\boldsymbol{W}(s)$ is defined as

$$\boldsymbol{W}(s) = \boldsymbol{X}(s) + (\boldsymbol{I}_n - \boldsymbol{X}(s))\boldsymbol{C} \tag{2}$$

where $\boldsymbol{C}$ is the relative interaction matrix associated with the graph $\mathcal{G}$, and the matrix $\boldsymbol{X}(s) \doteq \mathrm{diag}[\boldsymbol{x}(s)]$. From the fact that $\boldsymbol{C}$ is row-stochastic with zero diagonal entries, (2) implies that $\boldsymbol{W}(s)$ is a row-stochastic matrix. It has been shown in [20] that $\boldsymbol{W}(s)$ defined as in (2) ensures that for any given s, there holds $\lim_{t\to\infty} \boldsymbol{y}(s, t) = (\boldsymbol{\zeta}(s)^\top \boldsymbol{y}(s, 0))\mathbf{1}_n$. Here, $\boldsymbol{\zeta}(s)^\top$ is the unique nonnegative left eigenvector of $\boldsymbol{W}(s)$ associated with the eigenvalue 1, normalised such that $\mathbf{1}_n^\top \boldsymbol{\zeta}(s) = 1$. That is, the opinions converge to a constant consensus value.

Next, we describe the model for the updating of $\boldsymbol{W}(s)$ (specifically $w_{ii}(s)$ via a reflected self-appraisal mechanism).

Kronecker products may be used if each individual has simultaneous opinions on p unrelated topics, $\boldsymbol{y}_i \in \mathbb{R}^p, p \geq 2$. Simultaneous discussion of p logically interdependent topics is treated in [17], [18] under the assumption that $\mathcal{S} = \{0\}$.

2) Friedkin's Self-Appraisal Model for Determining Self-Weight: The Friedkin component of the model proposes a method for updating the individual self-weights, $\boldsymbol{x}(s)$. We assume the starting self-weights $x_i(0) \geq 0$ satisfy $\sum_i x_i(0) = 1$.[1] At the end of the discussion of issue s, the self-weight vector updates as

$$\boldsymbol{x}(s+1) = \boldsymbol{\zeta}(s) \tag{3}$$

Note that $\boldsymbol{\zeta}(s)^\top \mathbf{1}_n = 1$ implies that $\boldsymbol{x}(s) \in \Delta_n$, i.e., $\sum_{i=1}^n x_i(s) = 1$ for all s. From (2), and because $\boldsymbol{C}$ is row-stochastic, it is apparent that by adjusting $w_{ii}(s+1) = \zeta_i(s)$, individual i also scales $w_{ij}(s+1), j \neq i$ using c_{ij} to be $(1-w_{ii}(s+1))c_{ij}$ to ensure that $\boldsymbol{W}(s)$ remains row-stochastic.

Remark 2 (Social Power)**.** *The precise motivation behind using* (3) *to update* $\boldsymbol{x}(s)$ *is detailed in [20], but we provide a brief overview here in the interest of making this paper self-contained. As discussed in Subsection II-B1, for any given* s*, there holds* $\lim_{t\to\infty} \boldsymbol{y}(s,t) = (\boldsymbol{\zeta}(s)^\top \boldsymbol{y}(s,0))\mathbf{1}_n$*. In other words, for any given issue* s*, the opinions reach a consensus value* $\boldsymbol{\zeta}(s)^\top \boldsymbol{y}(s,0)$ *equal to a convex combination of the individuals' initial opinion values* $\boldsymbol{y}(s,0)$ *for that issue. The elements of* $\boldsymbol{\zeta}(s)^\top$ *are the convex weights. For a given issue* s*,* $\zeta_i(s)$ *is therefore a precise manifestation of individual* i*'s social power or influence in the social network, as it is a measure of the ability of individual* i *to control the outcome of a discussion [1]. The reflected self-appraisal mechanism therefore describes an individual* 1) *observing how much power it had on the discussion of issue* s *(the nonnegative quantity* $\zeta_i(s)$*), and* 2) *for the next issue* $s+1$*, adjusting its self-weight to be equal to this power, i.e.,* $x_i(s+1) = w_{ii}(s+1) = \zeta_i(s)$*.*

Lemma 2.2 of [20] showed that the system (3) is equivalent to the discrete-time system

$$\boldsymbol{x}(s+1) = \boldsymbol{F}(\boldsymbol{x}(s)) \tag{4}$$

where the nonlinear map $\boldsymbol{F}(\boldsymbol{x}(s))$ is defined as

$$\boldsymbol{F}(\boldsymbol{x}(s)) = \begin{cases} \mathbf{e}_i & \text{if } \boldsymbol{x}(s) = \mathbf{e}_i \text{ for any } i \\ \alpha(\boldsymbol{x}(s)) \begin{bmatrix} \frac{\gamma_1}{1-x_1(s)} \\ \vdots \\ \frac{\gamma_n}{1-x_n(s)} \end{bmatrix} & \text{otherwise} \end{cases} \tag{5}$$

with $\alpha(\boldsymbol{x}(s)) = 1/\sum_{i=1}^n \frac{\gamma_i}{1-x_i(s)}$ where $\boldsymbol{\gamma} = [\gamma_1, \ldots, \gamma_n]^\top$ is the dominant left eigenvector of $\boldsymbol{C}$. Note that $\sum_i F_i = 1$, where F_i is the i^{th} entry of $\boldsymbol{F}$. We now introduce an assumption *which will be invoked throughout the paper*.

Assumption 1. *The matrix* $\boldsymbol{C} \in \mathbb{R}^{n\times n}$*, with* $n \geq 3$*, is irreducible, row-stochastic, and has zero diagonal entries. Irreducibility of* $\boldsymbol{C}$ *implies, and is implied by, the strongly connectedness of the graph* $\mathcal{G}$ *associated with* $\boldsymbol{C}$*.*

This assumption was in place in [20] by and large throughout its development. Dynamic topology involving reducible $\boldsymbol{C}$ is a planned future work of the authors. A special topology studied in [20] is termed "star topology", the definition and relevance of which follow.

Definition 1 (Star topology)**.** *A strongly connected graph*[2] $\mathcal{G}$ *is said to have star topology if* $\exists$ *a node* v_i*, called the centre node, such that every edge of* $\mathcal{G}$ *is either to or from* v_i*.*

The irreducibility of $\boldsymbol{C}$ implies that a star $\mathcal{G}$ must include edges in both directions between the centre node v_i and every other node $v_j, j \neq i$. We now provide a lemma and a theorem (the key result of [20]) regarding the convergence of $\boldsymbol{F}(\boldsymbol{x}(s))$ as $s \to \infty$, and a fact useful for analysis throughout the paper.

Lemma 1 ([20, Lemma 3.2])**.** *Suppose that* $n \geq 3$*, and suppose further that* $\mathcal{G}$ *has star topology, which without loss of generality has centre node* v_1*. Let* $\boldsymbol{C}$ *satisfy Assumption 1. Then,* $\forall\, \boldsymbol{x}(0) \in \widetilde{\Delta}_n$*,* $\lim_{s\to\infty} \boldsymbol{x}(s) = \mathbf{e}_1$*.*

This implies that $\forall\, \boldsymbol{x}(0) \in \widetilde{\Delta}_n$, a network with star topology converges to an "autocratic configuration" where centre individual 1 holds all of the social power.

Fact 1. *[20, Lemma 2.3] Suppose that* $n \geq 3$ *and let* $\boldsymbol{\gamma}^\top$*, with entries* γ_i*, be the dominant left eigenvector of* $\boldsymbol{C} \in \mathbb{R}^{n\times n}$*, satisfying Assumption 1. Then,* $\|\boldsymbol{\gamma}\|_\infty = 0.5$ *if and only if* $\boldsymbol{C}$ *is associated with a star topology graph, and in this case* $\gamma_i = 0.5$ *where* i *is the centre node; otherwise,* $\|\boldsymbol{\gamma}\|_\infty < 0.5$*.*

Theorem 1 ([20, Theorem 4.1])**.** *For* $n \geq 3$*, consider the DeGroot-Friedkin dynamical system* (4) *with* $\boldsymbol{C}$ *satisfying Assumption 1. Assume further that the digraph* $\mathcal{G}$ *associated with* $\boldsymbol{C}$ *does not have star topology. Then,*

(i) For all initial conditions $\boldsymbol{x}(0) \in \widetilde{\Delta}_n$*, the self-weights* $\boldsymbol{x}(s)$ *converge to* $\boldsymbol{x}^*$ *as* $s \to \infty$*, where* $\boldsymbol{x}^* \in int(\Delta_n)$ *is the unique fixed point satisfying* $\boldsymbol{x}^* = \boldsymbol{F}(\boldsymbol{x}^*)$*.*
(ii) There holds $x_i^* < x_j^*$ *if and only if* $\gamma_i < \gamma_j$ *for any* i, j*, where* γ_i *is the* i^{th} *entry of the dominant left eigenvector* $\boldsymbol{\gamma}$*. There holds* $x_i^* = x_j^*$ *if and only if* $\gamma_i = \gamma_j$*.*
(iii) The unique fixed point $\boldsymbol{x}^*$ *is determined only by* $\boldsymbol{\gamma}$*, and is independent of the initial conditions.*

Remark 3. *Since the DeGroot model was introduced in [5], many different opinion dynamics models have been proposed, of increasing sophistication. We covered some in the introduction. However, the DeGroot model continues to be of relevance; the recent paper [33] applied the DeGroot model to show how discussion over social networks could improve the "wisdom of crowd" effect, then experimentally validated the results. In future, one could replace* (1) *with other opinion dynamics models. The key difficulty will be in defining the influence of the individuals, currently captured by* $\boldsymbol{\zeta}(s)^\top$ *(see Remark 2) and obtaining a system* (4) *with an analysable map* $\boldsymbol{F}$*. For the Friedkin-Johnsen model, experimental and*

[1] The assumption that $\sum_i x_i(0) = 1$ is not strictly required, as we will prove in Section IV that if $0 \leq x_i(0) < 1, \forall i$ and $\exists\, j : x_j(0) > 0$, then the system will remain inside the simplex Δ_n for all $s \geq 1$.

[2] While it is possible to have a star graph that is not strongly connected, this paper, similarly to [20], deals only with strongly connected graphs.

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simulation results [34] are available but theoretical study has proved to be extremely difficult, with limited results [35].

C. Quantitative Aspects of the Dynamic Topology Problem

In the introduction, we discussed in qualitative terms that we are seeking to study the evolution, and in particular the convergence properties, of social power in dynamically changing social networks. Quantitative details on the problem of interest are now provided. Specifically, we will consider dynamic relative interaction matrices $\boldsymbol{C}(s)$ which are *issue-driven or individual-driven*. Having properly introduced the DeGroot-Friedkin model, it is appropriate for us to expand on this motivation.

Issue-driven: Consider a government cabinet that meets to discuss the issues of defence, economic growth, social security programs and foreign policy. Each minister (individual in the cabinet) has a specialist portfolio (e.g. defence) and perhaps a secondary portfolio (e.g. foreign policy). While every minister will partake in the discussion of each issue, the weights $c_{ij}(s)$ will change. For example, if minister i's portfolio is on defence, then $c_{ji}(s_{\text{defence}})$ will be high as other ministers j place more trust on minister i's opinion. On the other hand, $c_{ji}(s_{\text{security}})$ will be low. It is then apparent that $\boldsymbol{C}(s_{\text{defence}}) \neq \boldsymbol{C}(s_{\text{security}})$ in general. This motivates the incorporation of *issue-dependent or issue-driven* topology change into the DeGroot-Friedkin model.

Individual-driven: Dynamic relative interaction matrices are a natural way of describing *network structural changes over time*. For many reasons, relationships may form, change, or die out. Consider individual i and individual j in a network, with $c_{ij}(0) > 0$, and suppose that $y_i(0, s) \gg y_j(0, s)$ over, say, 5 issues, i.e. individual j consistently holds an initial opinion vastly different from individual i. Then, i may decide that j is an extremist not worth listening to, and set $c_{ij}(6) = 0$. This is similar to the concept of homophily, which assumes that individuals interact only with others who hold similar views, and appears in the Hegselmann-Krause model [15], [16].

The two examples above are different from each other, but both equally provide motivation for *dynamic* topology. We assume that $\forall\, s$, $\boldsymbol{C}(s)$ satisfies Assumption 1. Given that $\boldsymbol{C}(s)$ is dynamic, the opinion dynamics for each issue is then given by $\boldsymbol{y}(s, t+1) = \boldsymbol{W}(s)\boldsymbol{y}(s, t)$ where

$$\boldsymbol{W}(s) = \boldsymbol{X}(s) + (\boldsymbol{I}_n - \boldsymbol{X}(s))\boldsymbol{C}(s) \tag{6}$$

which records the fact that $\boldsymbol{C}(s)$ is dynamic, in distinction to (2). Precise details of the adjustments to the model arising from dynamic $\boldsymbol{C}$ are left for Section IV. We are now ready to formulate the key objective of this paper as follows.

Objective. *To study the dynamic evolution (including convergence) of $\boldsymbol{x}(s)$ over a sequence of discussed issues by using the DeGroot model* (1) *for opinion discussion, where $\boldsymbol{W}(s)$ is given in* (6)*, with the reflected self-appraisal mechanism* (3) *used to update $\boldsymbol{x}(s)$.*

D. Contraction Analysis for Nonlinear Systems

We now present results on nonlinear contraction analysis of discrete-time systems, first exposed in the now classic [29, Section 5]. This will be used to obtain a fundamental convergence result for the original DeGroot-Friedkin model. The analysis framework will enable an extension to the study of issue-varying $\boldsymbol{C}$.

Consider a deterministic discrete-time system of the form

$$\boldsymbol{x}(k+1) = \boldsymbol{f}_k(\boldsymbol{x}(k), k) \tag{7}$$

with $n \times 1$ state vector $\boldsymbol{x}$ and $n \times 1$ vector-valued function $\boldsymbol{f}$. It is assumed that $\boldsymbol{f}$ is smooth, by which we mean that any required derivative or partial derivative exists, and is continuous. The associated virtual[3] dynamics is

$$\delta\boldsymbol{x}(k+1) = \frac{\partial \boldsymbol{f}_k}{\partial \boldsymbol{x}(k)} \delta\boldsymbol{x}(k)$$

Define the transformation $\delta\boldsymbol{z}(k) = \boldsymbol{\Theta}_k(\boldsymbol{x}(k), k)\delta\boldsymbol{x}(k)$ where $\boldsymbol{\Theta}_k(\boldsymbol{x}(k), k) \in \mathbb{R}^{n\times n}$ is uniformly nonsingular. More specifically, uniform nonsingularity means that there exist a real number $\kappa > 0$ and a matrix norm $\|\cdot\|'$ such that $\kappa < \|\boldsymbol{\Theta}_k(\boldsymbol{x}(k), k)\|' < \kappa^{-1}$ holds for all $\boldsymbol{x}$ and k. If the uniformly nonsingular condition holds, then exponential convergence of $\delta\boldsymbol{z}$ to $\boldsymbol{0}_n$ implies, and is implied by, exponential convergence of $\delta\boldsymbol{x}$ to $\boldsymbol{0}_n$. The transformed virtual dynamics can be computed as

$$\delta\boldsymbol{z}(k+1) = \boldsymbol{F}(k)\delta\boldsymbol{z}(k) \tag{8}$$

where $\boldsymbol{F}(k) = \boldsymbol{\Theta}_{k+1}(\boldsymbol{x}(k+1), k+1)\frac{\partial \boldsymbol{f}_k}{\partial \boldsymbol{x}(k)}\boldsymbol{\Theta}_k(\boldsymbol{x}(k), k)^{-1}$ is the transformed Jacobian.

Definition 2 (Generalised Contraction Region)**.** *Given the discrete-time system* (7)*, a region of the state space is called a generalised contraction region with respect to the metric $\|\boldsymbol{x}\|_{\boldsymbol{\Theta},1} = \|\boldsymbol{\Theta}_k(\boldsymbol{x}(k), k)\boldsymbol{x}(k)\|_1$ if in that region, $\|\boldsymbol{F}(k)\|_1 < 1 - \eta$ holds for all k, where $\eta > 0$ is an arbitrarily small constant.*

Note that here we are in fact working with the 1-norm metric in the variable space $\delta\boldsymbol{z}$ which in turn leads to a *weighted* 1-norm in the variable space $\delta\boldsymbol{x}$. Here, the weighting matrix is $\boldsymbol{\Theta}_k(\boldsymbol{x}(k), k)$ and the weighted 1-norm is well defined over the entire state space because $\boldsymbol{\Theta}$ is uniformly nonsingular.

Theorem 2. *Given the system* (7)*, consider a tube of constant radius with respect to the metric $\|\boldsymbol{x}\|_{\boldsymbol{\Theta},1}$, centred at a given trajectory of* (7)*. Any trajectory, which starts in this tube and is contained at all times in a generalised contraction region, remains in that tube and converges exponentially fast to the given trajectory as $k \to \infty$.*

Furthermore, global exponential convergence to the given trajectory is guaranteed if the whole state space is a generalised contraction region with respect to the metric $\|\boldsymbol{x}\|_{\boldsymbol{\Theta},1}$.

The now classic paper [29] focused on contraction in the Euclidean metric $\|\boldsymbol{x}\|_{\boldsymbol{\Theta},2} = \|\boldsymbol{\Theta}_k(\boldsymbol{x}(k), k)\boldsymbol{x}(k)\|_2$. However, norms other than the Euclidean norm can be studied because the solutions of (8) can be *superimposed*. This is because (8) around a specific trajectory $\boldsymbol{x}(k)$ represents a linear time-varying system in $\delta\boldsymbol{z}$ coordinates [29, Section 3.7]. In the

[3]The term "virtual" is taken from [29]; $\delta\boldsymbol{x}$ is a virtual, i.e. infinitesimal, displacement.

paper, we require use of the 1-norm metric because the 2-norm metric does not deliver a convergence result. We provide a sketch of the proof here, modified for the 1-norm metric, and refer the reader to [29] for precise details.

Proof. In a generalised contraction region, there holds

$$\|\delta \boldsymbol{z}(k+1)\|_1 = \|\boldsymbol{F}(k)\delta\boldsymbol{z}(k)\|_1$$
$$\|\delta \boldsymbol{z}(k+1)\|_1 < (1-\eta)\|\delta\boldsymbol{z}(k)\|_1$$

since $\|\boldsymbol{F}(k)\|_1 < 1-\eta$ holds for all k inside the generalised contraction region[4]. This implies that $\lim_{k\to\infty}\delta\boldsymbol{z}(k) = \boldsymbol{0}_n$ exponentially fast, which in turn implies that $\lim_{k\to\infty}\delta\boldsymbol{x}(k) = \boldsymbol{0}_n$ exponentially fast due to uniform nonsingularity of $\boldsymbol{\Theta}_k(\boldsymbol{x}(k),k)$. The definition of $\delta\boldsymbol{x}$ then implies that *any two infinitesimally close trajectories* of (7) converge to each other exponentially fast.

The distance between two points, P_1 and P_2, with respect to the metric $\|\cdot\|_{\boldsymbol{\Theta},1}$ is defined as the shortest path length between P_1 and P_2, i.e., the smallest path integral $\int_{P_1}^{P_2}\|\delta\boldsymbol{z}\|_1 = \int_{P_1}^{P_2}\|\delta\boldsymbol{x}\|_{\boldsymbol{\Theta},1}$. A tube centred about a trajectory $\boldsymbol{x}_1(k)$ and with radius R is then defined as the set of all points whose distances to $\boldsymbol{x}_1(k)$ with respect to $\|\cdot\|_{\boldsymbol{\Theta},1}$ are strictly less than R.

Let $\boldsymbol{x}_2(k) \neq \boldsymbol{x}_1(k)$ be any trajectory that starts inside this tube, separated from $\boldsymbol{x}_1(k)$ by a finite distance with respect to the metric $\|\cdot\|_{\boldsymbol{\Theta},1}$. Suppose that the tube is contained at all times in a generalised contraction region. The fact that $\lim_{k\to\infty}\|\delta\boldsymbol{x}(k)\|_{\boldsymbol{\Theta},1} = 0$ then implies that $\lim_{k\to\infty}\int_{\boldsymbol{x}_1(k)}^{\boldsymbol{x}_2(k)}\|\delta\boldsymbol{x}(k)\|_{\boldsymbol{\Theta},1} = 0$ exponentially fast. That is, given the trajectories $\boldsymbol{x}_2(k)$ and $\boldsymbol{x}_1(k)$, separated by a finite distance with respect to the metric $\|\cdot\|_{\boldsymbol{\Theta},1}$, $\boldsymbol{x}_2(k)$ converges to $\boldsymbol{x}_1(k)$ exponentially fast. Global convergence is obtained by setting $R=\infty$. □

Corollary 1. *If the contraction region is convex, then all trajectories converge exponentially fast to a unique trajectory.*

Proof. This immediately follows because any finite distance between two trajectories shrinks exponentially in the convex region. □

III. Contraction Analysis for Constant $\boldsymbol{C}$

In this section, before we address dynamic topology in Section IV, we derive a convergence result for the constant DeGroot-Friedkin model (4) (i.e., $\boldsymbol{C}$ is constant for all $s\in\mathcal{S}$) using nonlinear contraction analysis methods as detailed in Section II-D. The framework built using nonlinear contraction analysis is then applied in the next section to the DeGroot-Friedkin Model with dynamic topology.

In order to obtain a convergence result, we make use of two properties of $\boldsymbol{F}(\boldsymbol{x}(s))$ established in [20], but beyond these two properties, the analysis method is novel.

Property 1. *The map $\boldsymbol{F}(\boldsymbol{x}(s))$ in* (5) *is continuous on Δ_n.*

If $\mathcal{G}$ does not have star topology, then the following *contraction-like* property holds [20, pp. 390, Appendix F].

[4]We need $\eta>0$ to eliminate the possibility that $\lim_{k\to\infty}\|\boldsymbol{F}(k)\|_1 = 1$, which would not result in exponential convergence.

Property 2. *Define the set $\mathcal{A} = \{\boldsymbol{x}\in\Delta_n : 1-r\geq x_i\geq 0, \forall\, i\in\{1,\dots,n\}\}$, where $r\ll 1$ is a small strictly positive scalar. Then, for the system* (4)*, there exists a sufficiently small r such that $x_i(s)\leq 1-r$ implies $x_i(s+1)<1-r$, for all i.*

By choosing r sufficiently small, it follows that $\boldsymbol{x}(s)\in\mathcal{A}, \forall\, s>0$. In other words, $\boldsymbol{F}(\mathcal{A})\subset\mathcal{A}$. We term this a *contraction-like* property so as not to confuse the reader with our main result; this property establishes a contraction only near the boundary of the simplex Δ_n.

As a consequence of the above two properties, one can easily show, using Brouwer's Fixed Point Theorem (as shown in [20]), that there exists *at least* one fixed point $\boldsymbol{x}^* = \boldsymbol{F}(\boldsymbol{x}^*)$ in the convex compact set $\mathcal{A}$. In [20], a method involving multiple inequalities is used to show that the fixed point $\boldsymbol{x}^*$ is unique. This is done separately to the convergence proof. In the following proof, we are able to establish exponential convergence to a fixed point, and as a consequence of the method used, immediately prove that it is unique. Lastly, we present a third, easily verifiable property.

Property 3. *For the system* (4)*, if $\boldsymbol{x}(s_1)\in\widetilde{\Delta}_n$ for some $s_1<\infty$, then $\boldsymbol{x}(s)\in int(\Delta)_n$ for all $s>s_1$.*

Proof. Since $\boldsymbol{x}(s_1)\in\widetilde{\Delta}_n$, $\exists\, j : x_j(s_1)>0$. In addition, $\gamma_i>0, \forall\, i$ because $\boldsymbol{C}$ is irreducible. It follows that $\alpha(\boldsymbol{x}(s_1))>0$, and $x_i(s_1+1)>0,\forall i$. Thus, $\boldsymbol{x}(s)\in \mathrm{int}(\Delta_n)\forall s>s_1$. □

A. Fundamental Contraction Analysis

We now state a fundamental convergence result of the system (4). In the original work [20], LaSalle's Invariance Principle was used to prove an asymptotic convergence result. This paper strengthens the result by establishing exponential convergence. In the following proof, we say a property holds uniformly if the property holds for all $\boldsymbol{x}(s)\in\mathcal{A}$.

Theorem 3. *Suppose that $n\geq 3$ and suppose further that the network $\mathcal{G}$ does not have star topology, and has a constant relative interaction matrix $\boldsymbol{C}$ satisfying Assumption 1. Then, the system* (4)*, with initial conditions $\boldsymbol{x}(0)\in\widetilde{\Delta}_n$, converges exponentially fast to a unique equilibrium point $\boldsymbol{x}^*\in int(\Delta_n)$.*

Proof. Consider any given initial condition $\boldsymbol{x}(0)\in\widetilde{\Delta}_n$. According to Property 2, $\boldsymbol{x}(s)\in\mathcal{A},\forall\, s>0$ for a sufficiently small r. It remains for us to study the system (4) for $\boldsymbol{x}(s)\in\mathcal{A}$; in the following analysis, we assume that $s>0$. The concepts and terminology of Section II-D will be heavily utilised.

Define the Jacobian of $\boldsymbol{F}(\boldsymbol{x}(s))$ at the s^{th} issue as $\boldsymbol{J_F}(\boldsymbol{x}(s)) = \{\frac{\partial F_i}{\partial x_j}(\boldsymbol{x}(s))\}$. We obtain

$$\frac{\partial F_i}{\partial x_i}(\boldsymbol{x}(s)) = \frac{\gamma_i\alpha(\boldsymbol{x}(s))}{(1-x_i(s))^2} - \frac{\gamma_i^2\alpha(\boldsymbol{x}(s))^2}{(1-x_i(s))^3} \tag{9}$$

$$\frac{\partial F_i}{\partial x_j}(\boldsymbol{x}(s)) = -\frac{\gamma_i\gamma_j\alpha(\boldsymbol{x}(s))^2}{(1-x_i(s))(1-x_j(s))^2}, \quad j\neq i \tag{10}$$

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From (4) and (5), it is straightforward to verify that the Jacobian entries, as given above, can be expressed as

$$\frac{\partial F_i}{\partial x_i}(\boldsymbol{x}(s)) = x_i(s+1)\frac{1-x_i(s+1)}{1-x_i(s)}$$
$$\frac{\partial F_i}{\partial x_j}(\boldsymbol{x}(s)) = -\frac{x_i(s+1)x_j(s+1)}{1-x_j(s)}, \quad j \neq i$$

which establishes the relation between the Jacobian entries and the social power of issue $s+1$. Our reason for doing so will become clear shortly. Accordingly, we have the following virtual dynamics

$$\delta\boldsymbol{x}(s+1) = \boldsymbol{J_F}(\boldsymbol{x}(s))\delta\boldsymbol{x}(s)$$

Note that $\boldsymbol{J_F}(\boldsymbol{x}(s))$ is well defined and uniformly continuous because $x_i(s) < 1-r, \forall\, i, s$, thus enabling nonlinear contraction analysis to be used.

Specifically, consider the transformed virtual displacement

$$\delta\boldsymbol{z}(s) = \boldsymbol{\Theta}(\boldsymbol{x}(s))\delta\boldsymbol{x}(s) \tag{11}$$

where $\boldsymbol{\Theta}(\boldsymbol{x}(s)) = \text{diag}[1/(1-x_i(s))]$ is a diagonal matrix with the i^{th} diagonal element being $1/(1-x_i(s))$, i.e. $\boldsymbol{\Theta}$ is dependent on the state $\boldsymbol{x}(s)$ but not dependent explicitly on s.

Property 2 establishes that $1 > 1-x_i(s) > r > 0$, which in turn implies that $\boldsymbol{\Theta}(\boldsymbol{x}(s))$ is uniformly nonsingular, with $\lambda_{\min}\big(\boldsymbol{\Theta}(\boldsymbol{x}(s))\big) > 1$ and $\lambda_{\max}\big(\boldsymbol{\Theta}(\boldsymbol{x}(s))\big) < 1/r$. In other words, $\kappa < \|\boldsymbol{\Theta}(\boldsymbol{x}(s))\|_1 < \kappa^{-1}$ for some $\kappa > 0$, $\forall\, \boldsymbol{x}(s) \in \mathcal{A}$, as required in Section II-D.

The transformed virtual dynamics is given by

$$\begin{aligned}\delta\boldsymbol{z}(s+1) &= \boldsymbol{\Theta}(\boldsymbol{x}(s+1))\boldsymbol{J_F}(\boldsymbol{x}(s))\boldsymbol{\Theta}(\boldsymbol{x}(s))^{-1}\delta\boldsymbol{z}(s) \\ &= \bar{\boldsymbol{H}}(\boldsymbol{x}(s))\delta\boldsymbol{z}(s)\end{aligned} \tag{12}$$

where $\bar{\boldsymbol{H}}(\boldsymbol{x}(s)) = \boldsymbol{\Theta}(\boldsymbol{F}(\boldsymbol{x}(s)))\boldsymbol{J_F}(\boldsymbol{x}(s))\boldsymbol{\Theta}(\boldsymbol{x}(s))^{-1}$ is the Jacobian associated with the transformed virtual dynamics. By denoting $\bar{\boldsymbol{\Phi}}(\boldsymbol{x}(s)) = \boldsymbol{J_F}(\boldsymbol{x}(s))\boldsymbol{\Theta}(\boldsymbol{x}(s))^{-1}$, one can write $\bar{\boldsymbol{H}}(\boldsymbol{x}(s)) = \boldsymbol{\Theta}(\boldsymbol{F}(\boldsymbol{x}(s)))\bar{\boldsymbol{\Phi}}(\boldsymbol{x}(s))$.

The matrix $\bar{\boldsymbol{\Phi}}(\boldsymbol{x}(s))$ is computed in (13) below, and note that it can be considered as being solely dependent on $\boldsymbol{x}(s+1) = \boldsymbol{F}(\boldsymbol{x}(s))$. Therefore, we let $\boldsymbol{\Phi}(\boldsymbol{x}(s+1)) = \bar{\boldsymbol{\Phi}}(\boldsymbol{x}(s))$. For brevity, we drop the argument $\boldsymbol{x}(s+1)$ where there is no ambiguity and write simply $\boldsymbol{\Phi}$. Note that for each row i, $\phi_{ii} = x_i(s+1)\big(1-x_i(s+1)\big)$ and $\phi_{ij} = -x_i(s+1)x_j(s+1)$ where ϕ_{ij} is the $(i,j)^{th}$ element of $\boldsymbol{\Phi}$. Because $s > 0$, Properties 2 and 3 establish that $0 < x_i(s) < 1-r, \forall\, i$. It follows that all diagonal entries of $\boldsymbol{\Phi}$ are uniformly strictly positive and all off-diagonal entries of $\boldsymbol{\Phi}$ are uniformly strictly negative. Notice that $\boldsymbol{\Phi} = \boldsymbol{\Phi}^\top$. Lastly, for any row i, there holds

$$\sum_{j=1}^{n}\phi_{ij} = x_i(s+1)\big[1-x_i(s+1) - \sum_{j=1,j\neq i}^{n} x_j(s+1)\big] = 0$$

because $x_i(s+1)+\sum_{j=1,j\neq i}^{n} x_j(s+1) = 1$. In other words, $\boldsymbol{\Phi}$ has row and column sums equal to 0. We thus conclude that $\boldsymbol{\Phi}$ is the weighted Laplacian associated with an undirected, *completely connected*[5] graph with edge weights which vary with $\boldsymbol{x}(s+1)$. The edge weights, $-\phi_{ij}$, are uniformly lower bounded away from zero and upper bounded away from 1. This implies that $0 = \lambda_1(\boldsymbol{\Phi}) < \lambda_2(\boldsymbol{\Phi}) \leq \ldots \leq \lambda_n(\boldsymbol{\Phi}) < \infty$ [32], i.e., $\boldsymbol{\Phi}$ is uniformly positive semidefinite with a single eigenvalue at 0, with the associated eigenvector $\mathbf{1}_n$.

Since $\bar{\boldsymbol{\Phi}}(\boldsymbol{x}(s)) = \boldsymbol{\Phi}(\boldsymbol{x}(s+1))$ and $\boldsymbol{\Theta}(\boldsymbol{x}(s+1)) = \boldsymbol{\Theta}(\boldsymbol{F}(\boldsymbol{x}(s)))$, we note that $\bar{\boldsymbol{H}}(\boldsymbol{x}(s))$ can be considered as depending solely on $\boldsymbol{x}(s+1)$. Letting $\boldsymbol{H}(\boldsymbol{x}(s+1)) = \bar{\boldsymbol{H}}(\boldsymbol{x}(s))$, we complete the calculation $\boldsymbol{H}(\boldsymbol{x}(s+1)) = \boldsymbol{\Theta}(\boldsymbol{x}(s+1))\boldsymbol{\Phi}(\boldsymbol{x}(s+1))$ to obtain that, for any $i \in \{1,\ldots,n\}$,

$$h_{ii}(\boldsymbol{x}(s+1)) = x_i(s+1)$$
$$h_{ij}(\boldsymbol{x}(s+1)) = -\frac{x_i(s+1)x_j(s+1)}{1-x_i(s+1)}, \quad j \neq i$$

where $h_{ij}(\boldsymbol{x}(s+1))$ is the $(i,j)^{th}$ element of $\boldsymbol{H}(\boldsymbol{x}(s+1))$. For brevity, and when there is no risk of ambiguity, we drop the argument $\boldsymbol{x}(s+1)$ and simply write $\boldsymbol{H}$. We note that the diagonal entries and off-diagonal entries of $\boldsymbol{H}(\boldsymbol{x}(s+1))$ are uniformly strictly positive and uniformly strictly negative, respectively. Notice that $\boldsymbol{\Phi}\mathbf{1}_n = \mathbf{0}_n \Rightarrow \boldsymbol{H}\mathbf{1}_n = \boldsymbol{\Theta}(\boldsymbol{x}(s+1))\boldsymbol{\Phi}(\boldsymbol{x}(s+1))\mathbf{1}_n = \mathbf{0}_n$. In other words, each row of $\boldsymbol{H}$ sums to zero. It follows that $\boldsymbol{H}$ is the weighted Laplacian matrix associated with a directed, *completely connected* graph with edge weights which vary with $\boldsymbol{x}(s+1)$. The edge weights, $-h_{ij}$, are uniformly upper bounded away from infinity and lower bounded away from zero. It is well known that if a directed graph contains a directed spanning tree, the associated Laplacian matrix has a single eigenvalue at 0, and all other eigenvalues have positive real parts [8].

With $\boldsymbol{\Theta}(\boldsymbol{x}(s+1))$ uniformly symmetric and positive definite, and $\boldsymbol{\Phi}(\boldsymbol{x}(s+1))$ uniformly symmetric and positive semidefinite, it follows from [36, Corollary 7.6.2] that all eigenvalues of $\boldsymbol{H} = \boldsymbol{\Theta\Phi}$ are real. Combining with the above analysis, we conclude that $\boldsymbol{H}$ has a single zero eigenvalue and *all other eigenvalues are strictly positive and real*. By observing that $\text{trace}(\boldsymbol{H}) = \sum_{i=1}^{n} x_i(s+1) = 1 = \sum_{i=1}^{n}\lambda_i(\boldsymbol{H})$, we conclude that $\max_i\big(\lambda_i(\boldsymbol{H})\big) < 1$ uniformly, since $n \geq 3$.

We now establish the stronger result that $\|\boldsymbol{H}\|_1 < 1$ uniformly. Observe that $\|\boldsymbol{H}\|_1 < 1$ if and only if, for all $i \in \{1,\ldots,n\}$, there holds $\sum_{j=1}^{n}|h_{ji}| < 1$, or equivalently,

$$x_i + \sum_{j=1,j\neq i}^{n}\left(\frac{x_i}{1-x_j}\right)x_j < 1 \tag{14}$$

and notice that we have dropped the time argument $s+1$ for brevity. From the fact that $x_i > 0, \forall\, i$ (recall $\alpha(\boldsymbol{x}(s)) > 0$), and $n \geq 3$, we obtain $x_i + x_j < 1 \Rightarrow x_i/(1-x_j) < 1$ for all $j \neq i$. Combining this with the fact that $x_i + \sum_{j=1,j\neq i}^{n} x_j = 1$, we immediately verify that (14) holds for all i. Because $\mathcal{A}$ is bounded, this implies that $\|\boldsymbol{H}\|_1 < 1-\eta$ for some $\eta > 0$ and all $\boldsymbol{x}(s) \in \mathcal{A}$. Recalling the transformed virtual dynamics in (12), we conclude that

$$\|\delta\boldsymbol{z}(s+1)\|_1 = \|\boldsymbol{H}(\boldsymbol{x}(s+1))\delta\boldsymbol{z}(s)\|_1 < (1-\eta)\|\delta\boldsymbol{z}(s)\|_1$$

We thus conclude that the transformed virtual displacement $\delta\boldsymbol{z}$ converges to zero exponentially fast. Recall the definition of $\delta\boldsymbol{z}(s)$ in (11), and the fact that $\boldsymbol{\Theta}(\boldsymbol{x}(s))$ is uniformly nonsingular. It follows that $\delta\boldsymbol{x}(s) \to \mathbf{0}_n$ exponentially, $\forall\, \boldsymbol{x}(s) \in \mathcal{A}$.

[5]By completely connected, we mean that there is an edge going from every node i to every other node j.

$$\bar{\mathbf{\Phi}}(\boldsymbol{x}(s)) = \begin{bmatrix} x_1(s+1)\frac{1-x_1(s+1)}{1-x_1(s)} & -\frac{x_1(s+1)x_2(s+1)}{1-x_2(s)} & \dots & -\frac{x_1(s+1)x_n(s+1)}{1-x_n(s)} \\ -\frac{x_1(s+1)x_2(s+1)}{1-x_1(s)} & x_2(s+1)\frac{1-x_2(s+1)}{1-x_2(s)} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{x_1(s+1)x_n(s+1)}{1-x_1(s)} & -\frac{x_2(s+1)x_n(s+1)}{1-x_2(s)} & \dots & x_n(s+1)\frac{1-x_n(s+1)}{1-x_n(s)} \end{bmatrix} \times \begin{bmatrix} 1-x_1(s) & & \\ & \ddots & \\ & & 1-x_n(s) \end{bmatrix}$$
$$= \begin{bmatrix} x_1(s+1)\big(1-x_1(s+1)\big) & -x_1(s+1)x_2(s+1) & \dots & -x_1(s+1)x_n(s+1) \\ -x_1(s+1)x_2(s+1) & x_2(s+1)\big(1-x_2(s+1)\big) & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ -x_1(s+1)x_n(s+1) & -x_2(s+1)x_n(s+1) & \dots & x_n(s+1)\big(1-x_n(s+1)\big) \end{bmatrix} \quad (13)$$

We have thus established that $\mathcal{A}$ is a *generalised contraction region* in accordance with Definition 2. Because $\mathcal{A}$ is compact and convex, we conclude from Theorem 2 and Corollary 1 that *all trajectories* of $\boldsymbol{x}(s+1) = \boldsymbol{F}(\boldsymbol{x}(s))$ with $\boldsymbol{x}(0) \in \Delta_n$, converge exponentially *to a single trajectory*. According to Brouwer's Fixed Point Theorem, there is at least one fixed point $\boldsymbol{x}^* = \boldsymbol{F}(\boldsymbol{x}^*) \in \text{int}(\Delta_n)$, *which is a trajectory of* $\boldsymbol{x}(s+1) = \boldsymbol{F}(\boldsymbol{x}(s))$. It then immediately follows that all trajectories of $\boldsymbol{x}(s+1) = \boldsymbol{F}(\boldsymbol{x}(s))$ converge exponentially to *a unique fixed point* $\boldsymbol{x}^* \in \text{int}(\Delta_n)$ (recall Property 3). □

Corollary 2 (Vertex Equilibrium)**.** *The fixed point* $\mathbf{e}_i$ *of the map* $\boldsymbol{F}(\boldsymbol{x})$ *in* (5) *is unstable if* $\gamma_i < 1/2$*. If* $\gamma_i = 1/2$*, i.e.,* v_i *is the centre node of a star graph, then the fixed point* $\mathbf{e}_i$ *is asymptotically stable, but is not exponentially stable.*

Proof. Without loss of generality, consider $i = 1$. Observe that

$$F_1(\boldsymbol{x}) = \frac{\gamma_1}{\gamma_1 + \sum_{k=2}^n \frac{(1-x_1)\gamma_k}{1-x_k}}$$
$$F_j(\boldsymbol{x}) = \frac{\gamma_j(1-x_1)}{(1-x_j)(\gamma_1 + \sum_{k=2}^n \frac{\gamma_k(1-x_1)}{1-x_k})}, \forall j \neq 1$$

and it is evident that these expressions are analytic in x_1 for all $\boldsymbol{x} \in \Delta_n$. The same is then necessarily true of all their derivatives. We conclude that $\boldsymbol{F}$ is continuous and smooth. This simplifies and extends the proof in [20, Lemma 2.2], which established continuity but not smoothness.

At $\boldsymbol{x} = \mathbf{e}_1$, the expressions above yield that $\boldsymbol{F}(\mathbf{e}_1) = \mathbf{e}_1$, and differentiating the expressions yields a value for the Jacobian at $\boldsymbol{x} = \mathbf{e}_1$ in which $\frac{\partial F_1}{\partial x_1} = \frac{1-\gamma_1}{\gamma_1}$, $\frac{\partial F_i}{\partial x_1} = -\frac{\gamma_i}{\gamma_1}$, $\frac{\partial F_i}{\partial x_j} = 0$ for all $i, j \neq 1$. It follows that $\boldsymbol{J}_{\boldsymbol{F}(\boldsymbol{x})}$ has a single eigenvalue at $(1-\gamma_1)/\gamma_1$ and all other eigenvalues are 0. If $\gamma_1 < 1/2$, then $(1-\gamma_1)/\gamma_1 > 1$ and the fixed point $\mathbf{e}_1$ is unstable. If $\gamma_1 = 1/2$, then $\boldsymbol{J}_{\boldsymbol{F}(\boldsymbol{x})}$ has a single eigenvalue at 1. A discrete-time counterpart to [37, Theorem 4.15] (converse Lyapunov theorem) then rules out $\boldsymbol{e}_1$ as an *exponentially stable* fixed point of $\boldsymbol{F}(\boldsymbol{x})$ (asymptotic stability was established in Lemma 1). We omit the proof of the discrete-time counterpart to Theorem 4.15 of [37] due to space limitations. □

Remark 4. *When we first analyse* $\boldsymbol{H}$*, we establish that* $\forall\, i$*,* $\lambda_i(\boldsymbol{H})$ *is real, nonnegative and less than 1. This tells us that the trajectories of* (4) *about* $\boldsymbol{x}^*$ *are not oscillatory in nature. It also follows that the spectral radius of* $\boldsymbol{H}$*, given by* $\rho(\boldsymbol{H})$*, is strictly less than 1. According to [36], there exists a submultiplicative matrix norm* $\|\cdot\|'$ *such that* $\|\boldsymbol{H}\|' < 1$*. However, we must recall that* $\boldsymbol{H}(\boldsymbol{x}(s+1))$ *is in fact a nonconstant matrix which changes over the trajectory of the system* (4)*. It is not immediately obvious, and in fact is not a consequence of the eigenvalue property, that a single submultiplicative matrix norm* $\|\cdot\|''$ *exists such that* $\|\boldsymbol{H}\|'' < 1$ ***for all*** $\boldsymbol{x} \in \mathcal{A}$*. Existence of such a norm* $\|\cdot\|''$ *would establish the desired stability property.*

In fact, the system $\delta\boldsymbol{z}(s+1) = \boldsymbol{H}(\boldsymbol{x}(s+1))\delta\boldsymbol{z}(s)$*, with* $\boldsymbol{H} \in \mathcal{M}$*,* $\mathcal{M} = \{\boldsymbol{H}(\boldsymbol{x}(s+1)) : \boldsymbol{x}(s+1) \in \mathcal{A}\}$*, can be considered as a discrete-time linear switching system with state* $\delta\boldsymbol{z}$*, and thus under arbitrary switching, the system is stable if and only if the* ***joint spectral radius*** *is less than* 1*, that is* $\rho(\mathcal{M}) = \lim_{k\to\infty} \max_i\{\|\boldsymbol{H}_{i_1} \dots \boldsymbol{H}_{i_k}\|^{1/k} : \boldsymbol{H}_i \in \mathcal{M}\} < 1$ *[38]. This is of course a more restrictive condition than simply requiring that* $\rho(\boldsymbol{H}_i) < 1$*. It is known that even when* $\mathcal{M}$ *is finite, computing* $\rho(\mathcal{M})$ *is NP-hard [39] and the question "*$\rho(\mathcal{M}) \leq 1$*?" is an undecidable problem [38]. The problem is made even more difficult because in this paper, the set* $\mathcal{M}$ *is not finite. We were therefore motivated to prove the stronger, and nontrivial, result that* $\|\boldsymbol{H}\|_1 < 1, \forall\, \boldsymbol{x} \in \mathcal{A}$*.*

Remark 5. *For* $\delta\boldsymbol{z}$ *given in* (11)*, we are able to obtain* $z_i = -\ln(1-x_i)$ *where* z_i *is the* i^{th} *element of* $\boldsymbol{z}(\boldsymbol{x}(s))$*. However, we did not present the above convergence arguments by first defining* $\boldsymbol{z}(\boldsymbol{x}(s))$ *and then seeking to study* $\boldsymbol{z}(s+1) = \boldsymbol{G}(\boldsymbol{z}(s))$*. This is because our proof arose from considering* $\boldsymbol{x}(s+1) = \boldsymbol{F}(\boldsymbol{x}(s))$ *using nonlinear contraction analysis [29], which studies stability via differential concepts. It was through* (11) *that we integrated*[6] *to obtain* $z_i = -\ln(1-x_i)$*. Moreover, it will be observed in the sequel that by conducting analysis on the transformed Jacobian using nonlinear contraction theory, we are able to straightforwardly deal with issue-varying* $\boldsymbol{C}(s)$*.*

Remark 6. *Our simulations showed that sometimes* $|\lambda_{\max}(\boldsymbol{J_F})| > 1$*, which implies that it is not always possible to find a matrix norm* $\|\cdot\|'$ *such that* $\|\boldsymbol{J_F}(\boldsymbol{x}(s))\|' < 1$ *uniformly. This is what motivated our approach of using non-linear contraction analysis, with transformation* $\mathbf{\Theta}$*. Note that [29] specifically discusses* contraction in the Euclidean metric

[6] In general, the entries of $\mathbf{\Theta}$ may have expressions which do not have analytic antiderivatives, and thus an analytic $\boldsymbol{z}(\boldsymbol{x}(s), s)$ cannot always be found, but $\delta\boldsymbol{z}(s)$ can always be defined.

$\|\delta z\|_2 = \|\Theta \delta x\|_2$, which requires $\lambda_{\max}\big(H(x)^\top H(x)\big) < 1$ to hold uniformly, guaranteeing that $\|\delta z(s)\|_2 \to 0$ exponentially fast. However, our simulations showed that $\lambda_{\max}\big(H(x)^\top H(x)\big)$ was often larger than 1. This motivated us to consider contraction in the absolute sum metric, with appropriate adjustments to the proof presented in Section II-D. Such an approach is alluded to in [29, Section 3.7].

B. Extending the Contraction-like Analysis

In this subsection, we provide a result which significantly expands Property 2 by providing an explicit value for r and introduces a stronger *contraction-like result* that is also applicable to social networks with star topology, unlike Property 2.

Lemma 2. *Suppose that $n \geq 3$, $x(0) \in \widetilde{\Delta}_n$, and the social network $\mathcal{G}$ is strongly connected. Define*

$$r_j = \frac{1 - 2\gamma_j}{1 - \gamma_j} \tag{15}$$

where γ_j is the j^{th} entry of $\gamma^\top$, the dominant left eigenvector of the relative interaction matrix C. If $\gamma_j < 1/2$, which implies that $r_j > 0$, then for any $0 < r \leq r_j$, there holds

$$x_j \leq 1 - r \Rightarrow F_j(x) < 1 - r \tag{16}$$

where $F_j(x)$ is the j^{th} entry of $F(x)$ given in (5). If $\gamma_j = 1/2$, then $\nexists r > 0 : r \leq r_j$, and thus (16) does not hold.

Proof. It was shown that for $x(0) \in \widetilde{\Delta}_n$, there holds $x_i(s) > 0, \forall\, i$ and $s > 0$. Consider then $s > 0$. Suppose that $x_j \leq 1 - r$ and $\gamma_j < 1/2$. Then, with $r \leq r_j$, there holds

$$\begin{aligned} F_j(x) &= \alpha(x)\frac{\gamma_j}{1 - x_j} \\ &= \frac{1}{\frac{\gamma_j}{1-x_j}\Big(1 + \frac{\sum_{k\neq j}^n \gamma_k/(1-x_k)}{\gamma_j/(1-x_j)}\Big)}\frac{\gamma_j}{1-x_j} \\ &= \frac{1}{1 + \frac{\sum_{k\neq j}^n \gamma_k/(1-x_k)}{\gamma_j/(1-x_j)}} \leq \frac{1}{1 + \sum_{k\neq j}^n \frac{r}{\gamma_j}\frac{\gamma_k}{(1-x_k)}} \end{aligned} \tag{17}$$

because $r \leq 1 - x_j$. Because $1 - x_k < 1$, we obtain $\gamma_k/(1 - x_k) > \gamma_k$, which implies that the right hand side of (17) obeys

$$\frac{1}{1 + \sum_{k\neq j}^n \frac{r}{\gamma_j}\frac{\gamma_k}{(1-x_k)}} < \frac{1}{1 + \sum_{k\neq j}^n \frac{\gamma_k r}{\gamma_j}} \tag{18}$$

$$= \frac{1}{1 + \frac{(1-\gamma_j)r}{\gamma_j}} = \frac{\gamma_j}{\gamma_j + (1-\gamma_j)r} \tag{19}$$

with the first equality obtained by noting that $\sum_{k\neq j}^n \gamma_k = 1 - \gamma_j$ according to the definition of γ. It follows from (17) and (19) that

$$\begin{aligned} 1 - r - F_j(x) &> 1 - r - \frac{\gamma_j}{\gamma_j + (1-\gamma_j)r} \\ &= \frac{\gamma_j + (1-\gamma_j)r - r\gamma_j - (1-\gamma_j)r^2 - \gamma_j}{\gamma_j + (1-\gamma_j)r} \\ &= \frac{r(1-2\gamma_j) - r^2(1-\gamma_j)}{\gamma_j + (1-\gamma_j)r} \\ &= \frac{r(1-\gamma_j)\left[\frac{1-2\gamma_j}{1-\gamma_j} - r\right]}{\gamma_j + (1-\gamma_j)r} \end{aligned}$$

Substituting in r_j from (15) then yields

$$1 - r - F_j(x) > \frac{r(1-\gamma_j)(r_j - r)}{\gamma_j + (1-\gamma_j)r} \geq 0 \tag{20}$$

because $r_j \geq r$. In other words, $1 - r > F_j(x)$, which completes the proof. □

This contraction-like result is now used to establish an upper bound on the social power of an individual at equilibrium. We stress here that no general results appear to exist for analytical computation of the vector x^* given $\gamma^\top$. Results exist for some special cases, though, such as for doubly stochastic C and for $\mathcal{G}$ with star topology [20]. While we do not provide an explicit equality relating x_i^* to γ_i, we do provide an explicit *inequality*.

Corollary 3 (Upper bound on x_i^*)**.** *Suppose that $n \geq 3$ and $x(0) \in \widetilde{\Delta}_n$. Suppose further that $\mathcal{G}$ is strongly connected, and is not a star graph. Then, the i^{th} entry of the unique fixed point x^* of F, given in (5), satisfies $x_i^* < \gamma_i/(1 - \gamma_i)$.*

Proof. Lemma 2 establishes that, for any $j \in \{1, \ldots, n\}$, if $x_j \geq 1 - r_j$, then the map will always contract in that $F_j(x(s)) < x_j$. This is proved as follows. Suppose that $x_j \geq 1 - r_j$. Define $r = 1 - x_j$, which satisfies $r \leq r_j$ as in Lemma 2. Then, we have $F_j(x) < 1 - r = x_j$. It is then straightforward to conclude that the map $F(x)$ continues to contract towards the centre of the simplex Δ_n until $x_i(s) < 1 - r_i, \forall\, i$, where r_i is given by (15).

Suppose that $x_j^* \geq 1 - r_j = \gamma_j/(1 - \gamma_j)$. According to the arguments in the paragraph above, we have $F_j(x^*) < 1 - r_j \leq x_j^*$. On the other hand, the definition of x^* as a fixed point of F implies that $x_j^* = F_j(x^*)$, which leads to a contradiction. Therefore, $x_j^* < 1 - r_j = \gamma_j/(1 - \gamma_j)$ as claimed. □

Note that this result is separate from the result of Theorem 3, which deals with convergence to x^*. Here, we established an upper bound for the values of the entries of the unique fixed point x^*, i.e., the social power at equilibrium, given γ.

We mention three specific conclusions following from these two results. First, in relation to the transient behaviour of $x(s)$: Lemma 2 indicates that for any i and with $x_i(0)$ small, the peak overshoot of $x_i(s)$ above x_i^* is bounded as $x_i(s) \leq \gamma_i/(1-\gamma_i)$. Second, suppose that $\mathcal{G}$ has star topology with centre node v_1. Then, $\gamma_1 = 0.5$ according to Fact 1, and thus x_1 does not decrease according to Lemma 2. This is consistent with the findings in [20], i.e., Lemma 1. Last, suppose that $\mathcal{G}$ is strongly connected and that $\gamma_i < 1/3, \forall\, i \in \{1, \ldots, n\}$. Then, according to Corollary 3, no individual will have more than half of the total social power at equilibrium, i.e., $x_i^* < 1/2, \forall\, i \in \{1, \ldots, n\}$. This provides a sufficient condition on the social network topology to ensure that no individual has a dominating presence in the opinion discussion.

Remark 7. *[Tightness of the Bound] The tightness of the bound $x_i^* < \gamma_i/(1 - \gamma_i)$ increases as γ_k decreases $\forall\, k \neq i$. This is in the sense that the ratio $x_i^*(1 - \gamma_i)/\gamma_i$ approaches 1 from below as γ_k decreases $\forall\, k \neq i$. We draw this conclusion by noting that in order to obtain (18), we make use of the inequality $1 - x_k < 1$. From the fact that $\lim_{x_k \to 0} 1 - x_k = 1$, and because the contraction-like property of Lemma 2 holds* *for $x_k \geq \gamma_k/(1-\gamma_k)$, we conclude that the tightness of the bound $x_i^* < \gamma_i/(1-\gamma_i)$ increases as γ_k decreases $\forall\, k \neq i$. If there is a single individual i with $\gamma_i \gg \gamma_k, \forall\, k \neq i$, one can accurately estimate x_i^*, e.g. if $\gamma_i \geq 1/3 \gg \gamma_k, \forall\, k \neq i$, we say with high confidence that individual i will hold more than half of the total social power at equilibrium, i.e., $x_i^* \geq 0.5$.*

C. Convergence Rate for a Set of C Matrices

We now present a result on the convergence rate for a constant $\boldsymbol{C}$ which is in a subset of all possible $\boldsymbol{C}$ matrices.

Lemma 3 (Convergence Rate)**.** *Suppose that $\boldsymbol{C} \in \mathcal{L}$, where $\mathcal{L} = \{\boldsymbol{C} \in \mathbb{R}^{n\times n} : \gamma_i < 1/3, \forall\, i,\ n \geq 3\}$[7] and γ_i is the i^{th} entry of the dominant left eigenvector $\boldsymbol{\gamma}^\top$ associated with $\boldsymbol{C}$. Then, with $\boldsymbol{x}(0) \in \widetilde{\Delta}_n$, there exists a finite s_1 such that for all $s \geq s_1$, the system (4) contracts to its unique equilibrium point $\boldsymbol{x}^*$ with a convergence rate obeying*

$$\|\boldsymbol{x}^* - \boldsymbol{x}(s+1)\|_1 \leq (2\beta - \epsilon)\|\boldsymbol{x}^* - \boldsymbol{x}(s)\|_1 \tag{21}$$

where $2\beta - \epsilon < 1 - \eta$, with $\beta = \max_i \gamma_i/(1-\gamma_i) < 1/2$ and ϵ, η being sufficiently small positive constants.

Proof. From Corollary 3, we conclude that $x_i^* < \beta_i$ where $\beta_i = \gamma_i/(1-\gamma_i) < 1/2$. Defining $\beta = \max_i \beta_i$, we conclude that $x_i^* \leq \beta - \epsilon_1$ for all i, where ϵ_1 is an arbitrarily small positive constant. From Lemma 2, we know that $x_i(s)$ is always decreasing for $x_i(s) > \beta - \epsilon_1$. This means that there exists a strictly positive ϵ satisfying $\epsilon/2 < \epsilon_1$ and $s_1 < \infty$ such that $x_i(s) \leq \beta - \epsilon/2$ for all $s \geq s_1$. In other words, $x_i(s)$ will be no greater than $\beta - \epsilon/2$ in a finite number of issues after $s = 0$; we use this fact below to upper bound the norm of the untransformed Jacobian.

The Jacobian $\boldsymbol{J}_{\boldsymbol{F}(\boldsymbol{x}(s))}$ has column sum equal to zero. We obtain this fact by observing that, for any i,

$$\begin{aligned}
&\frac{\partial F_i}{\partial x_i} + \sum_{j=1, j\neq i}^{n} \frac{\partial F_j}{\partial x_i} \\
&\quad = x_i(s+1)\frac{1 - x_i(s+1)}{1 - x_i(s)} - \sum_{j=1, j\neq i}^{n} \frac{x_i(s+1)x_j(s+1)}{1 - x_i(s)} \\
&\quad = \frac{x_i(s+1)}{1 - x_i(s)}\left[1 - x_i(s+1) - \sum_{j=1, j\neq i}^{n} x_j(s+1)\right] = 0
\end{aligned}$$

because $x_i(s+1) + \sum_{j=1, j\neq i}^{n} x_j(s+1) = 1$ by definition. Note also that the diagonal entries of the Jacobian are strictly positive and for $s \geq s_1$, there holds $\partial F_i/\partial x_i \leq \beta - \epsilon/2, \forall\, i$. This is because $x_i(1-x_i) \leq (\beta - \epsilon/2)(1 - \beta + \epsilon/2)$ for $x_i \leq \beta - \epsilon/2 < 0.5$ and $1/(1-x_i) \leq 1/(1 - \beta + \epsilon/2)$. Combining the column sum property and the fact that the off-diagonal entries of the Jacobian are strictly negative, we conclude that for $s \geq s_1$, there holds $\|\boldsymbol{J}_{\boldsymbol{F}(\boldsymbol{x}(s))}\|_1 = 2\max_i \partial F_i/\partial x_i \leq 2\beta - \epsilon < 1 - \eta$ where η is an arbitrarily small positive constant. The inequality in (21) follows straightforwardly. □

The quantity $2\beta - \epsilon$, which is a Lipschitz constant associated with the map $\boldsymbol{F}$, is an upper bound on the convergence rate of the system as in (21). By assuming $\gamma_i < 1/3, \forall\, i$, we are able to work directly with the Jacobian $\boldsymbol{J}_{\boldsymbol{F}}$, as opposed to the transformed Jacobian $\boldsymbol{H}$. It is in general much more difficult to compute an upper bound on $\|\boldsymbol{H}\|_1$ using $\boldsymbol{\gamma}$ and Corollary 3 when $\exists\, i : \gamma_i \geq 1/3$. Note that $\mathcal{L}$ includes many of the topologies likely to be encountered in social networks. Topologies for which $\exists\, i : \gamma_i \geq 1/3$ will have an individual who holds more than half the social power at equilibrium. Such topologies are more reflective of autocracy-like or dictatorship-like networks, as opposed to a group of equal peers.

[7] According to Fact 1, $\mathcal{L}$ does not contain any $\boldsymbol{C}$ whose associated graph has a star topology.

IV. Dynamic Relative Interaction Topology

In this section, we will explore the evolution of individual social power when the relative interaction topology is *issue- or individual-driven*, i.e., $\boldsymbol{C}(s)$ is a function of s. Motivations have been discussed in detail in Sections I and II. In our earlier work [31], we provided analysis on the special case of periodically varying $\boldsymbol{C}(s)$, showing the existence of a periodic trajectory. This section provides complete analysis for general switching $\boldsymbol{C}(s)$ and extends the periodic special case in [31]. We now place a mild restriction on how $\boldsymbol{C}(s)$ varies, and explain why we do so in the sequel.

Assumption 2. *For any $s_1 \geq 0$, the entries of $\boldsymbol{C}(s_1)$, given as $c_{ij}(s_1)$, are independent of $\boldsymbol{x}(s)$, $\forall\, s \leq s_1$.*

Assumption 2 ensures that $\boldsymbol{\gamma}_{\sigma(s)}$ is independent of the state $\boldsymbol{x}(s)$, because $\boldsymbol{C}$ is independent of $\boldsymbol{x}(s)$. Notice that both the issue-driven and individual-driven examples in Section II-C satisfy Assumption 2. Almost all issue-driven dynamics $\boldsymbol{C}(s)$ satisfy the assumption because the sequence of $\boldsymbol{C}(s)$ depends only on the sequence of issues. That is, for analysis purposes the sequence of $\boldsymbol{C}(s)$ is considered to be determined before discussion begins on $s = 0$, but individual i may not necessarily know the sequence of $c_{ij}(s)$ a priori.

However, a situation where individual i adjusts $c_{ij}(s)$ to be larger because i observed that individual j had large $\zeta_j(s-1)$, does not satisfy the assumption. For social network models with state-dependent parameters, limiting behaviour depends critically on the function relating the parameters to the state; different functional dependencies result in different limiting behaviour. Since we are studying social systems, this functional dependence must necessarily reflect sociology or social psychology concepts; it is beyond the scope of this paper to propose such functional dependence. Thus, Assumption 2 is in place, and investigation of how individuals might determine $\boldsymbol{C}(s)$ based on $\boldsymbol{x}(s), \ldots, \boldsymbol{x}(0)$ is left for future work.

Suppose that for a given social network with $n \geq 3$ individuals, there is a finite set $\mathcal{C}$ of P possible relative interaction matrices, defined as $\mathcal{C} = \{\boldsymbol{C}_p \in \mathbb{R}^{n\times n} : p \in \mathcal{P}\}$ where $\mathcal{P} = \{1, 2, \ldots, P\}$. We assume that Assumption 1 holds for all $\boldsymbol{C}_p$, $p \in \mathcal{P}$. For simplicity, we assume that $\nexists p$ such that the graph $\mathcal{G}_p$ associated with $\boldsymbol{C}_p$ has star topology. Let $\sigma(s) : [0, \infty) \to \mathcal{P}$ be a piecewise constant switching signal, determining the dynamic switching as $\boldsymbol{C}(s) = \boldsymbol{C}_{\sigma(s)}$. Under Assumption 2, for any $s_1 \geq 0$, $\sigma(s_1)$ is independent of the state $\boldsymbol{x}(s)$, for all $s < s_1$. The DeGroot-Friedkin model with dynamic relative interaction matrices is thus

$$\boldsymbol{x}(s+1) = \boldsymbol{F}_{\sigma(s)}(\boldsymbol{x}(s)) \tag{22}$$

where the nonlinear map $\boldsymbol{F}_p(\boldsymbol{x}(s))$ for $p \in \mathcal{P}$, is defined as

$$\boldsymbol{F}_p(\boldsymbol{x}(s)) = \begin{cases} \mathbf{e}_i & \text{if } \boldsymbol{x}(s) = \mathbf{e}_i \text{ for any } i \\ \alpha_p(\boldsymbol{x}(s)) \begin{bmatrix} \frac{\gamma_{p,1}}{1-x_1(s)} \\ \vdots \\ \frac{\gamma_{p,n}}{1-x_n(s)} \end{bmatrix} & \text{otherwise} \end{cases} \tag{23}$$

where $\alpha_p(\boldsymbol{x}(s)) = 1/\sum_{i=1}^{n} \frac{\gamma_{p,i}}{1-x_i(s)}$ and $\gamma_{p,i}$ is the i^{th} entry of the dominant left eigenvector of $\boldsymbol{C}_p$, $\boldsymbol{\gamma}_p = [\gamma_{p,1}, \gamma_{p,2}, \ldots, \gamma_{p,n}]^\top$. Note that the derivation for (23) is a straightforward extension of the derivation (5) using Lemma 2.2 in [20], from constant $\boldsymbol{C}$ to $\boldsymbol{C}(s) = \boldsymbol{C}_{\sigma(s)}$. We therefore omit this step.

Remark 8. *Analysis using the usual techniques for switched systems is difficult for the system* (22)*. For arbitrary switching, one might typically seek to find a common Lyapunov function, i.e., one which would establish convergence for any fixed value of $p \in \mathcal{P}$. This, however, appears to be difficult (if not impossible) for* (22)*. In the constant $\boldsymbol{C}$ case studied in [20], the convergence result relied on* 1) *a Lyapunov function which was dependent on the unique equilibrium point $\boldsymbol{x}^*$, and* 2) *LaSalle's Invariance Principle. Both* 1) *and* 2) *are invalid when analysing* (22)*. In the case of* 1)*, the system* (22) *does not have a unique equilibrium point $\boldsymbol{x}^*$ but rather a unique trajectory $\boldsymbol{x}^*(s)$ (as will be made clear in the sequel). In the case of* 2)*, LaSalle's Invariance Principle is not applicable to general non-autonomous systems.*

A. Convergence for Arbitrary Switching

We now state the main result of this section, the proof of which turns out to be fairly straightforward. This is a consequence of the analysis framework arising from the techniques used in the proof of Theorem 3. Note that in the theorem statement immediately below, a relaxation of the initial conditions is made; we no longer require $\sum_i x_i(0) = 1$. A social interpretation of this is given in Remark 9 just following the theorem, and an interpretation of the theorem itself is given in Remark 10.

Theorem 4. *Suppose Assumption 2 holds, and that $\nexists p$ such that $\boldsymbol{C}_p \in \mathcal{C}$ is associated with a star topology graph, with $\mathcal{C}$ defined above* (22)*. Then, for system* (22)*, a) there exists a unique trajectory $\boldsymbol{x}^*(s) \in int(\Delta_n)$ determined solely by $\boldsymbol{\gamma}_{\sigma(s)}$, and b) for all initial conditions satisfying $0 \leq x_i(0) < 1, \forall i$ and $\exists j : x_j(0) > 0$, there holds $\lim_{s\to\infty} [\boldsymbol{x}(s) - \boldsymbol{x}^*(s)] = \boldsymbol{0}_n$ exponentially fast. If $\boldsymbol{x}(0) = \mathbf{e}_i$ for some i, then $\boldsymbol{x}(s) = \mathbf{e}_i, \forall s$.*

Proof. First, observe that if $\boldsymbol{x}(0) = \mathbf{e}_i$ for some i, then (23) leads to the conclusion that $\boldsymbol{x}(s) = \mathbf{e}_i$ for all s.

Next, it is straightforward to conclude that Property 1, as stated at the beginning of Section III, holds for each map $\boldsymbol{F}_p$. With initial conditions $x_i(0) < 1$, the map $\boldsymbol{F}_{\sigma(0)}(\boldsymbol{x}(s)) \neq \mathbf{e}_i$ for any i. We also easily verify that with these initial conditions, the matrix $\boldsymbol{W}(0)$ is row-stochastic, irreducible and aperiodic, which implies that the opinions converge for $s = 0$ as in the constant $\boldsymbol{C}$ case. Because $\boldsymbol{C}(0)$ is irreducible, this implies that $\gamma_{\sigma(0),i} > 0$ for all i, and we conclude that $\alpha_{\sigma(0)}(\boldsymbol{x}(0)) > 0$ because $\exists j : x_j(0) > 0$. We thus conclude that $\boldsymbol{x}(1) = \boldsymbol{F}_{\sigma(0)}(\boldsymbol{x}(0)) \succ 0$, i.e., for issue $s = 1$, every individual's social power/self-weight is strictly positive, and the sum of the weights is 1.

Moreover, because $\boldsymbol{C}_p$ is irreducible $\forall p$, this implies that for any p, there holds $\gamma_{p,i} > 0$ for all i. It follows that for $s \geq 1$, $\alpha_{\sigma(s)}(\boldsymbol{x}(s)) > 0$, which in turn guarantees that $\boldsymbol{x}(s+1) = \boldsymbol{F}_{\sigma(s)}(\boldsymbol{x}(s)) \succ 0$, i.e., $\boldsymbol{x}(s) \in \text{int}(\Delta_n)$ for all $s > 0$. This satisfies the requirements [20] on $\boldsymbol{x}(s)$ which ensures that $\forall s$, $\boldsymbol{W}(s)$ is row-stochastic, irreducible, and aperiodic, which implies that opinions converge for every issue.

Denote the i^{th} entry of $\boldsymbol{F}_p$ by $F_{p,i}$. Regarding Property 2, stated at the beginning of Section III, for each map $\boldsymbol{F}_p$, define the set $\mathcal{A}_p(r_p) = \{\boldsymbol{x} \in \Delta_n : 1 - r_p \geq x_i \geq 0, \forall i \in \{1, \ldots, n\}\}$, where $0 < r_p \ll 1$ is sufficiently small such that $x_i(s) \leq 1 - r_p$ for all i, which implies that $F_{p,i}(\boldsymbol{x}(s)) = x_i(s+1) < 1 - r_p$. Define $\bar{\mathcal{A}} = \{\boldsymbol{x} \in \Delta_n : 1 - \bar{r} \geq x_i \geq 0, \forall i \in \{1, \ldots, n\}\}$ where $\bar{r} = \min_p r_p$. Because $\boldsymbol{F}_p(\bar{\mathcal{A}}) \subset \bar{\mathcal{A}}$, it follows that $\cup_{p=1}^{P} \mathcal{A}_p \subset \bar{\mathcal{A}}$, and that for the system (22), for all $s > 0$, $\boldsymbol{x}(s) \in \bar{\mathcal{A}}$.

Denoting the Jacobian for the system (22) at issue s as $\boldsymbol{J}_{\boldsymbol{F}_{\sigma(s)}} = \{\frac{\partial F_{\sigma(s),i}}{\partial x_j}\}$, and because from Assumption 2 we have that $\boldsymbol{\gamma}_{\sigma(s)}$ is independent of $\boldsymbol{x}(s)$, we obtain

$$\frac{\partial F_{\sigma(s),i}}{\partial x_i}(\boldsymbol{x}(s)) = \frac{\gamma_{\sigma(s),i}\alpha_{\sigma(s)}(\boldsymbol{x}(s))}{(1-x_i(s))^2} - \frac{\left[\gamma_{\sigma(s),i}\alpha_{\sigma(s)}(\boldsymbol{x}(s))\right]^2}{(1-x_i(s))^3}$$

Similarly, we obtain, for $j \neq i$,

$$\frac{\partial F_{\sigma(s),i}}{\partial x_j}(\boldsymbol{x}(s)) = -\frac{\gamma_{\sigma(s),i}\gamma_{\sigma(s),j}\left[\alpha_{\sigma(s)}(\boldsymbol{x}(s))\right]^2}{(1-x_i(s))(1-x_j(s))^2}$$

Comparing to (9) and (10), we note that the Jacobian of the non-autonomous system (22) with map (23) is expressible in the same form as the Jacobian of the original system (4) with map (5). More precisely, it can be expressed in a form which is dependent on the trajectory of the system, and not explicitly dependent on s. Using the same transformation of $\delta\boldsymbol{z}$ given in (11) with the same $\boldsymbol{\Theta}(\boldsymbol{x}(s))$, we obtain the exact same transformed virtual dynamics (12), expressed as

$$\delta\boldsymbol{z}(s+1) = \boldsymbol{H}(\boldsymbol{x}(s+1))\delta\boldsymbol{z}(s) \tag{24}$$

and it was shown in the proof of Theorem 3 that, for some arbitrarily small $\eta > 0$, there holds $\|\boldsymbol{H}\|_1 < 1 - \eta$ for all $\boldsymbol{x}(s) \in \bar{\mathcal{A}}$, independent of $p \in \mathcal{P}$. It follows that $\delta\boldsymbol{x}(s) \to \boldsymbol{0}_n$ exponentially fast for all $\boldsymbol{x}(s) \in \bar{\mathcal{A}}$. We thus conclude that $\bar{\mathcal{A}}$ is a generalised contraction region. Then, because $\bar{\mathcal{A}}$ is compact and convex, it follows from Theorem 2 and Corollary 1 that all trajectories of $\boldsymbol{x}(s+1) = \boldsymbol{F}_{\sigma(s)}(\boldsymbol{x}(s))$ converge exponentially *to a single trajectory*, which we denote by $\boldsymbol{x}^*(s)$. The trajectory $\boldsymbol{x}^*(s)$ depends only on $\boldsymbol{\gamma}_{\sigma(s)}$, i.e. it is independent of $\boldsymbol{x}(0)$. We also established earlier that $\boldsymbol{x}^*(s) \in \text{int}(\Delta_n)$.

Finally, following the same analysis as in [20, pp.393], one can show that $\lim_{s\to\infty} \boldsymbol{\zeta}(s) = \boldsymbol{x}^*(s)$ and $\lim_{s\to\infty} \boldsymbol{W}(\boldsymbol{x}(s)) = \boldsymbol{X}^*(s) + (\boldsymbol{I}_n - \boldsymbol{X}^*(s))\boldsymbol{C}(s) = \boldsymbol{W}(\boldsymbol{x}^*(s))$. ☐

The above result implies that the system (22), with initial conditions satisfying $0 \leq x_i(0) < 1, \forall\, i$ and $\exists\, j : x_j(0) > 0$, converges to a unique trajectory $\boldsymbol{x}^*(s)$ as $s \to \infty$. For convenience in future discussions and presentation of results, we shall call this the *unique limiting trajectory of* (22). This is a limiting trajectory in the sense that $\lim_{s\to\infty} \boldsymbol{x}(s) = \boldsymbol{x}^*(s)$.

Remark 9 (Relaxation of the initial conditions)**.** *Theorem 4 contains a mild relaxation of the initial conditions of the original DeGroot-Friedkin model, and provides a more reasonable interpretation from a social context. One can consider $x_i(0)$ as individual i's estimate of i's social power (or perceived social power) when the social network is first formed and before discussion begins on issue $s = 0$. The original model requires $\boldsymbol{x}(0) \in \widetilde{\Delta}_n$ to avoid an autocratic system (an autocratic system is where $\boldsymbol{x}(s) = \mathbf{e}_i$ for some i, i.e., an individual holds all the social power). However, this is unrealistic because one cannot expect individuals to have estimates such that $\sum_i x_i(0) = 1$. On the other hand, we do show that the unique limiting trajectory satisfies further, as already commented, $\sum x_i(1) = 1$, and then easily $\sum x_i(k) = 1, \forall k > 1$ and $\boldsymbol{x}^*(s) \in int(\Delta_n)$, i.e., $x_i^*(s) > 0, \forall\, i$ and $\sum_i x_i^*(s) = 1, \forall\, s$. This holds as long as no individual i estimates its social power to be autocratic $(x_i(0) = 1)$ and at least one individual estimates its social power to be strictly positive $(\exists\, j : x_j(0) > 0)$.*

B. Contraction-Like Property with Arbitrary Switching

We now extend Lemma 2, Corollary 3 and Lemma 3 to the case of dynamic relative interaction matrices.

Lemma 4. *Consider the system* (22) *and suppose that $0 \leq x_i(0) < 1, \forall\, i$ and $\exists k \in \{1, \ldots, n\} : x_k(0) > 0$. Define*

$$\bar{r}_j = \frac{1 - 2\bar{\gamma}_j}{1 - \bar{\gamma}_j}, \quad j \in \{1, \ldots, n\} \tag{25}$$

where $\bar{\gamma}_j = \max_{p\in\mathcal{P}} \gamma_{p,j}$, $\mathcal{P} = \{1, \ldots, P\}$ with $P < \infty$, and $\gamma_{p,j}$ is the j^{th} entry of the dominant left eigenvector $\boldsymbol{\gamma}_p$ of $\boldsymbol{C}_p$. Then, for any $0 < r \leq \bar{r}_j$ and $p \in \mathcal{P}$, there holds

$$x_j \leq 1 - r \Rightarrow F_{p,j}(\boldsymbol{x}) < 1 - r \tag{26}$$

where $F_{p,j}(\boldsymbol{x})$ is the j^{th} entry of $\boldsymbol{F}_p(\boldsymbol{x})$ defined in (23).

Proof. The lemma is proved by straightforwardly checking that, for the given definition of $\bar{r}_j$, the result in Lemma 2 holds separately for every map $\boldsymbol{F}_p$, $p \in \mathcal{P}$. In other words, for all i, p, $x_i(s) \leq 1 - r \Rightarrow F_{p,i}(\boldsymbol{x}(s)) < 1 - r\,, \forall\, r \leq \bar{r}_i$. □

Corollary 4 (Upper bound on $x_i^*(s)$)**.** *Consider the system* (22)*, and suppose $0 \leq x_i(0) < 1, \forall\, i$ and $\exists j \in \{1, \ldots, n\} : x_j(0) > 0$. Then, there holds $x_i^*(s) \leq \bar{\gamma}_i/(1 - \bar{\gamma}_i), \forall\, s$, where $\bar{\gamma}_j = \max_{p\in\mathcal{P}} \gamma_{p,j}$, $\mathcal{P} = \{1, \ldots, P\}$ with $P < \infty$, and $x_i^*(s)$ is the i^{th} entry of the unique limiting trajectory $\boldsymbol{x}^*(s)$.*

Proof. The proof is a straightforward extension of the proof of Corollary 3, and is therefore not included here. □

Lemma 5 (Convergence Rate for Dynamic Topology)**.** *Let $\mathcal{P} = \{1, \ldots, P\}$ with $P < \infty$. For all $p \in \mathcal{P}$, suppose that $\boldsymbol{C}_p \in \mathcal{L}$ where $\mathcal{L} = \{\boldsymbol{C}_p \in \mathbb{R}^{n\times n} : \gamma_{p,i} < 1/3, \forall i\}$ and $\gamma_{p,i}$ is the i^{th} entry of $\boldsymbol{\gamma}_p$. Then, there exists a finite s_1 such that, for all $s \geq s_1$, the system* (22) *contracts to its unique limiting trajectory $\boldsymbol{x}^*(s)$ with a convergence rate obeying*

$$\|\boldsymbol{x}^*(s) - \boldsymbol{x}(s+1)\|_1 \leq (2\bar{\beta} - \epsilon)\|\boldsymbol{x}^*(s) - \boldsymbol{x}(s)\|_1 \tag{27}$$

where $\bar{\beta} = \max_p \max_i \gamma_{p,i}/(1 - \gamma_{p,i}) < 1/2$ and ϵ, η are sufficiently small positive constants.

Proof. The proof is a straightforward extension of the proof of Lemma 3, by recalling from Theorem 4 proof that the Jacobian takes on the same form. We omit the minor details. □

Remark 10 (Self-Regulation)**.** *Exponential convergence to a unique trajectory $\boldsymbol{x}^*(s)$ can be considered from another point of view as the system* (22) forgetting its initial conditions at an exponential rate*, and is a powerful notion. It implies that sequential discussion of topics combined with reflected self-appraisal is a method of "self-regulation" for social networks, even in the presence of dynamic topology. Consider an individual i who is extremely arrogant, e.g. $x_i(0) = 0.99$. However, individual i is not likeable and others tend to not trust i's opinions on any issue, e.g. $c_{ji}(s) \ll 1\,, \forall\, j, s$. Then, $\gamma_i(s) \ll 1$ because $\boldsymbol{\gamma}(s)^\top = \boldsymbol{\gamma}(s)^\top \boldsymbol{C}(s)$ implies $\gamma_i(s) = \sum_{j\neq i} \gamma_j(s)c_{ji}(s)$. According to Corollary 4, $x_i^*(s) \ll 1$, and individual i exponentially loses its social power. An interesting future extension would be to expand on the reflected self-appraisal by modelling individual* ***behaviour****. For example, we can consider $x_i(s + 1) = \phi_i(\zeta_i(s))$ where $\phi_i(\cdot)$ may capture arrogance or humility when self-appraising.*

We also conclude that, for large s, any individual wanting to have an impact on the discussion of topic $s+1$ should focus on ensuring it has a large impact on discussion of the prior topic s. This concept can be applied to e.g. [40].

C. Periodically Varying Topology

In this subsection, we investigate an interesting, special case of issue-dependent topology, that of periodically varying $\boldsymbol{C}(s)$ which satisfies Assumption 1 for all s. Preliminary analysis and results were presented in [31] without convergence proofs. We now provide a complete analysis by utilising Theorem 4.

Motivation for Periodic Variations: Consider *Example 1* in Section II-C of a government cabinet that meets to discuss the issues of defence, economic growth, social security programs and foreign policy. Since these issues are vital to the smooth running of the country, we expect the issues to be discussed *regularly and repeatedly*. Regular meetings on the same set of issues for decision making/governance/management of a country or company then points to periodically varying $\boldsymbol{C}(s)$, i.e., social networks with periodic topology.

The system (22), with periodically switching $\boldsymbol{C}(s)$, can be described by a switching signal $\sigma(s)$ of the form $\sigma(0) = P$, and for $s \geq 1$, $\sigma(Pq + p) = p$,[8] where $P < \infty$ is the period length, $p \in \mathcal{P} = \{1, 2, \ldots, P\}$ and $q \in \mathbb{Z}_{\geq 0}$ is any nonnegative integer. Note that in general, $\boldsymbol{C}_i \neq \boldsymbol{C}_j, \forall\, i, j \in \mathcal{P}$ and $i \neq j$. Theorem 4 immediately allows us to conclude that system (22) with periodic switching converges exponentially fast to its unique limiting trajectory $\boldsymbol{x}^*(s)$. This subsection's

[8]Note that any given $s \in \mathcal{S}$ can be uniquely expressed by a given fixed positive integer P, a nonnegative integer q, and positive $p \in \mathcal{P}$, as shown.

key contribution is to use a transformation to obtain additional, useful information on the limiting trajectory.

For simplicity, we shall begin analysis by assuming that $\mathcal{P} = \{1, 2\}$, i.e., there are two different $\boldsymbol{C}$ matrices, and the switching is of period 2. In the sequel, we show that analysis for $\mathcal{P} = \{1, 2, \dots, P\}$, with arbitrarily large but finite P, is a simple recursive extension on the analysis for $\mathcal{P} = \{1, 2\}$. For the two matrices case, we obtain

$$\boldsymbol{x}(s+1) = \begin{cases} \boldsymbol{F}_1(\boldsymbol{x}(s)) & \text{if } s \text{ is odd} \\ \boldsymbol{F}_2(\boldsymbol{x}(s)) & \text{if } s \text{ is even} \end{cases} \tag{28}$$

We now seek to transform the periodic system into a time-invariant system. Define a new state $\boldsymbol{y} \in \mathbb{R}^{2n}$ (note that this is not the opinion state given in Section II-B1) as

$$\boldsymbol{y}(2q) = \begin{bmatrix} \boldsymbol{y}_1(2q) \\ \boldsymbol{y}_2(2q) \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}(2q) \\ \boldsymbol{x}(2q+1) \end{bmatrix} \tag{29}$$

and study the evolution of $\boldsymbol{y}(2q)$ for $q \in \{0, 1, 2, \dots\}$. Note that

$$\boldsymbol{y}(2(q+1)) = \begin{bmatrix} \boldsymbol{y}_1(2(q+1)) \\ \boldsymbol{y}_2(2(q+1)) \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}(2(q+1)) \\ \boldsymbol{x}(2(q+1)+1) \end{bmatrix} \tag{30}$$

In view of the fact that $\boldsymbol{x}(2(q+1)) = \boldsymbol{F}_1(\boldsymbol{x}(2q+1))$ and $\boldsymbol{x}(2(q+1)+1) = \boldsymbol{F}_2(\boldsymbol{x}(2q+2))$ for any $q \in \{0, 1, 2, \dots\}$, we obtain

$$\boldsymbol{y}(2(q+1)) = \begin{bmatrix} \boldsymbol{F}_1(\boldsymbol{x}(2q+1)) \\ \boldsymbol{F}_2(\boldsymbol{x}(2q+2)) \end{bmatrix} \tag{31}$$

Similarly, notice that $\boldsymbol{x}(2q+1) = \boldsymbol{F}_2(\boldsymbol{x}(2q))$ and $\boldsymbol{x}(2q+2) = \boldsymbol{F}_1(\boldsymbol{x}(2q+1))$ for any $q \in \{0, 1, 2, \dots\}$. From this, for $q \in \{0, 1, 2, \dots\}$, we obtain that

$$\boldsymbol{y}(2(q+1)) = \begin{bmatrix} \boldsymbol{F}_1\Big(\boldsymbol{F}_2(\boldsymbol{y}_1(2q))\Big) \\ \boldsymbol{F}_2\Big(\boldsymbol{F}_1(\boldsymbol{y}_2(2q))\Big) \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_1(\boldsymbol{y}_1(2q)) \\ \boldsymbol{G}_2(\boldsymbol{y}_2(2q)) \end{bmatrix} \tag{32}$$

for the time-invariant nonlinear composition functions $\boldsymbol{G}_1 = \boldsymbol{F}_1 \circ \boldsymbol{F}_2$ and $\boldsymbol{G}_2 = \boldsymbol{F}_2 \circ \boldsymbol{F}_1$. We can thus express the periodic system (28) as the nonlinear time-invariant system

$$\boldsymbol{y}(2q+2) = \bar{\boldsymbol{G}}(\boldsymbol{y}(2q)) \tag{33}$$

where $\bar{\boldsymbol{G}} = [\boldsymbol{G}_1^\top, \boldsymbol{G}_2^\top]^\top$.

Theorem 5. *There exists a unique periodic sequence $\boldsymbol{x}^*(s)$ for the system* (28)*, with map $\boldsymbol{F}_p$ given in* (23) *for $p = 1, 2$, obeying*

$$\boldsymbol{x}^*(s) = \begin{cases} \boldsymbol{y}_1^* & \text{if } s \text{ is odd} \\ \boldsymbol{y}_2^* & \text{if } s \text{ is even} \end{cases} \tag{34}$$

where $\boldsymbol{y}_1^* \in int(\Delta_n)$ and $\boldsymbol{y}_2^* \in int(\Delta_n)$ are the unique fixed points of, respectively, $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$, which are defined above (33)*. For all initial conditions $0 \leq x_i(0) < 1, \forall\, i$ and $\exists\, j : x_j(0) > 0$, $\lim_{s\to\infty}[\boldsymbol{x}(s) - \boldsymbol{x}^*(s)] = \boldsymbol{0}_n$ exponentially fast.*

Proof. As mentioned above, one can immediately apply Theorem 4 to show $\lim_{s\to\infty} \boldsymbol{x}(s) = \boldsymbol{x}^*(s)$. This proof therefore focuses on using the time-invariant transformation to show that $\boldsymbol{x}^*(s)$ has the properties described in the theorem statement.

Part 1: In this part, we prove that the map $\boldsymbol{G}_i$, $i = 1, 2$ has at least one fixed point. Firstly, we proved in Theorem 4 that the system (22), with initial conditions $0 \leq x_i(0) < 1, \forall\, i$ and for at least one j, $x_j(0) > 0$, will have $\boldsymbol{x}(s) \in \text{int}(\Delta_n)$ for all $s > 0$, which implies that $\boldsymbol{x}^*(s) \in \text{int}(\Delta_n)$. Let $p \in \{1, 2\}$. In the proof of Corollary 2, we proved that $\boldsymbol{F}_p$ is continuous on Δ_n. The composition of two continuous functions is continuous, which means both $\boldsymbol{G}_1 = \boldsymbol{F}_1 \circ \boldsymbol{F}_2 : \Delta_n \mapsto \Delta_n$ and $\boldsymbol{G}_2 = \boldsymbol{F}_2 \circ \boldsymbol{F}_1 : \Delta_n \mapsto \Delta_n$ are continuous.

The proof of Theorem 4 also showed that for all p, $\boldsymbol{F}_p \in \bar{\mathcal{A}}$ where $\bar{\mathcal{A}} = \{\boldsymbol{x} \in \Delta_n : 1 - \bar{r} \geq x_i \geq 0, \forall\, i \in \{1, \dots, n\}\}$ and $\bar{r}$ is some small strictly positive constant. For the system (28) with $p = 1, 2$, it follows that $\boldsymbol{F}_1(\bar{\mathcal{A}}) \subset \bar{\mathcal{A}} \Rightarrow \boldsymbol{F}_2(\boldsymbol{F}_1(\bar{\mathcal{A}})) \subset \bar{\mathcal{A}}$, which implies that $\boldsymbol{G}_1(\bar{\mathcal{A}}) \subset \bar{\mathcal{A}}$. Similarly, $\boldsymbol{G}_2(\bar{\mathcal{A}}) \subset \bar{\mathcal{A}}$. Brouwer's fixed-point theorem then implies that there exists at least one fixed point $\boldsymbol{y}_1^* \in \bar{\mathcal{A}}$ such that $\boldsymbol{y}_1^* = \boldsymbol{G}_1(\boldsymbol{y}_1^*)$ (respectively $\boldsymbol{y}_2^* \in \bar{\mathcal{A}}$ such that $\boldsymbol{y}_2^* = \boldsymbol{G}_2(\boldsymbol{y}_2^*)$) because $\boldsymbol{G}_1$ (respectively $\boldsymbol{G}_2$) is a continuous function on the compact, convex set $\mathcal{A}$. The arguments in *Part 1* appeared in [31], but proofs were omitted due to space limitations.

Part 2: In this part, we prove that the unique limiting trajectory of (28) obeys (34). Let $\boldsymbol{y}_1^*$ be a fixed point of $\boldsymbol{G}_1$. We will show below that $\boldsymbol{y}_1^*$ is in fact unique. Observe that $\boldsymbol{y}_1^* = \boldsymbol{F}_2(\boldsymbol{F}_1(\boldsymbol{y}_1^*))$. Define $\boldsymbol{y}_2^* = \boldsymbol{F}_1(\boldsymbol{y}_1^*)$. We thus have $\boldsymbol{y}_1^* = \boldsymbol{F}_2(\boldsymbol{y}_2^*)$. Observe that $\boldsymbol{F}_1(\boldsymbol{y}_1^*) = \boldsymbol{F}_1(\boldsymbol{F}_2(\boldsymbol{y}_2^*))$, which implies that $\boldsymbol{y}_2^* = \boldsymbol{F}_1(\boldsymbol{F}_2(\boldsymbol{y}_2^*)) = \boldsymbol{G}_2(\boldsymbol{y}_2^*)$. *In other words, $\boldsymbol{y}_2^*$ is a fixed point of $\boldsymbol{G}_2$ (but at this stage we have not yet proved its uniqueness).*

We now prove uniqueness. Theorem 4 allows us to conclude that *all trajectories* of (28) converge exponentially fast to a unique limiting trajectory $\boldsymbol{x}^*(s) \in \text{int}(\Delta_n)$. It follows, from (33) and the definition of $\boldsymbol{y}(2q)$, that for all $s \geq 0$, (34) *is a trajectory* of the system (28); the critical point here is that (34) *holds for all s*. Combining these arguments, it is clear that (34) is precisely *the unique limiting trajectory.*

Last, we show that $\boldsymbol{y}_1^*$ and $\boldsymbol{y}_2^*$ are the unique fixed point of $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$, respectively. To this end, suppose that, to the contrary, at least one of $\boldsymbol{y}_1^*$ and $\boldsymbol{y}_2^*$ is not unique. Without loss of generality, suppose in particular that $\boldsymbol{y}_1' \neq \boldsymbol{y}_1^*$ is any other fixed point of $\boldsymbol{G}_1$. Then, $\boldsymbol{y}_2' = \boldsymbol{F}_1(\boldsymbol{y}_1')$ is a fixed point of $\boldsymbol{G}_2$, and

$$\boldsymbol{x}(s) = \begin{cases} \boldsymbol{y}_1' & \text{if } s \text{ is odd} \\ \boldsymbol{y}_2' & \text{if } s \text{ is even} \end{cases} \tag{35}$$

is a trajectory of (28) that holds for all $s \geq 0$, and is *different from the trajectory* (34) *because* $\boldsymbol{y}_1' \neq \boldsymbol{y}_1^*$. On the other hand, Theorem 4 implies that all trajectories of (28) converge exponentially fast *to a unique limiting trajectory*, which is a contradiction. Thus, $\boldsymbol{y}_1^*$ and $\boldsymbol{y}_2^*$ are the unique fixed point of $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$, respectively, and (28) converges exponentially fast to the unique limiting trajectory (34). □

We now provide the straightforward generalisation to periodically switching topology $\boldsymbol{C}(s) = \boldsymbol{C}_{\sigma(s)}$, where $\sigma(s)$ is of the form $\sigma(0) = P$, and for $s \geq 1$, $\sigma(Pq + p) = p$. Here, $2 \leq P < \infty$, $p \in \mathcal{P} = \{1, 2, \dots, P\}$ and $q \in \mathbb{Z}_{\geq 0}$. The periodic DeGroot-Friedkin model is described by

$$\boldsymbol{x}(s+1) = \begin{cases} \boldsymbol{F}_P(\boldsymbol{x}(s)) & \text{for } s = 0 \\ \boldsymbol{F}_p(\boldsymbol{x}(s = Pq + p)) & \text{for all } s \geq 1 \end{cases} \tag{36}$$

A transformation of (36) to a time-invariant system is achieved by following a procedure similar to the one detailed for $p = 2$. A new state variable $\boldsymbol{y} \in \mathbb{R}^{Pn}$ is defined as

$$\boldsymbol{y}(Pq) = \begin{bmatrix} \boldsymbol{y}_1(Pq) \\ \boldsymbol{y}_2(Pq) \\ \vdots \\ \boldsymbol{y}_P(Pq) \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}(Pq) \\ \boldsymbol{x}(Pq+1) \\ \vdots \\ \boldsymbol{x}(Pq+P-1) \end{bmatrix} \tag{37}$$

and we study the evolution of $\boldsymbol{y}(Pq)$ for $q \in \{0, 1, \ldots\}$. It follows that

$$\boldsymbol{y}_p(P(q+1)) = \boldsymbol{x}(P(q+1)+p-1)\,,\ \forall\, p \in \mathcal{P}$$

Following the logic in the 2 period case, but with the precise steps omitted, we obtain

$$\begin{aligned} \boldsymbol{y}(P(q+1)) &= \begin{bmatrix} \boldsymbol{F}_{P-1}(\boldsymbol{F}_{P-2}(\ldots(\boldsymbol{F}_P(\boldsymbol{y}_1(Pq))))) \\ \boldsymbol{F}_P(\boldsymbol{F}_{P-1}(\ldots(\boldsymbol{F}_1(\boldsymbol{y}_2(Pq))))) \\ \vdots \\ \boldsymbol{F}_{P-2}(\boldsymbol{F}_{P-1}(\ldots(\boldsymbol{F}_P(\boldsymbol{y}_{P-1}(Pq))))) \end{bmatrix} \\ &= \bar{\boldsymbol{G}}(\boldsymbol{y}(Pq)) \end{aligned} \tag{38}$$

where $\bar{\boldsymbol{G}}(\boldsymbol{y}) = [\boldsymbol{G}_1(\boldsymbol{y}_1), \boldsymbol{G}_2(\boldsymbol{y}_2), \ldots, \boldsymbol{G}_P(\boldsymbol{y}_P)]^\top$. This leads to the following generalisation of Theorem 5.

Theorem 6. *There exists a unique periodic sequence $\boldsymbol{x}^*(s)$ for the system* (36)*, with map $\boldsymbol{F}_p$ given in* (23) *for $p = 1, 2, \ldots, P$, which for any nonnegative integer q, obeys*

$$\boldsymbol{x}^*(Pq+p-1) = \boldsymbol{y}_p^*, \text{ for all } p \in \{1, 2, \ldots, P\} \tag{39}$$

where $\boldsymbol{y}_p^* \in int(\Delta_n)$ is the unique fixed point of $\boldsymbol{G}_p$ defined in (38)*. For all initial conditions satisfying $0 \leq x_i(0) < 1, \forall\, i$ and for at least one j, $x_j(0) > 0$, $\lim_{s\to\infty}[\boldsymbol{x}(s) - \boldsymbol{x}^*(s)] = \boldsymbol{0}_n$ exponentially fast.*

Proof. The proof is obtained by recursively applying the same techniques used in the proof of Theorem 5. We therefore omit the details. □

Note that Lemmas 4 and 5 and Corollary 4 are all applicable to the periodic system (36) because (36) is just a special case of the general switching system (22).

D. Convergence to a Single Point

We conclude Section IV by showing that if the set of switching matrices has a special property, then the unique limiting trajectory $\boldsymbol{x}^*(s)$ is in fact a stationary point.

Theorem 7. *Suppose that the relative interaction matrix of the DeGroot-Friedkin model switches as $\boldsymbol{C}(s) = \boldsymbol{C}_{\sigma(s)} \in \mathcal{K}(\widetilde{\boldsymbol{\gamma}})$, where $\mathcal{K}(\widetilde{\boldsymbol{\gamma}}) = \{\boldsymbol{C}_p \in \mathbb{R}^{n\times n} : \boldsymbol{\gamma}_p = \widetilde{\boldsymbol{\gamma}}, \forall\, p \in \mathcal{P} = \{1, 2, \ldots, P\}\}$, is the set of $\boldsymbol{C}$ all having the same dominant left eigenvector $\widetilde{\boldsymbol{\gamma}}^\top$, with P finite*[9]*. Then, the system* (22)*, with initial conditions $0 \leq x_i(0) < 1, \forall i$ and $\exists j : x_j(0) > 0$, converges exponentially fast to a unique point $\boldsymbol{x}^* \in int(\Delta_n)$.*

There holds $x_i^* < x_j^* \Leftrightarrow \widetilde{\gamma}_i < \widetilde{\gamma}_j$, for any i, j, where $\widetilde{\gamma}_i$ and x_i^* are the i^{th} entry of $\widetilde{\boldsymbol{\gamma}}$ and $\boldsymbol{x}^*$, respectively. There holds $x_i^* = x_j^* \Leftrightarrow \widetilde{\gamma}_i = \widetilde{\gamma}_j$.

[9]A well-known set is $\mathcal{K}(\boldsymbol{1}_n/n)$, the set of $n \times n$ doubly-stochastic $\boldsymbol{C}$.

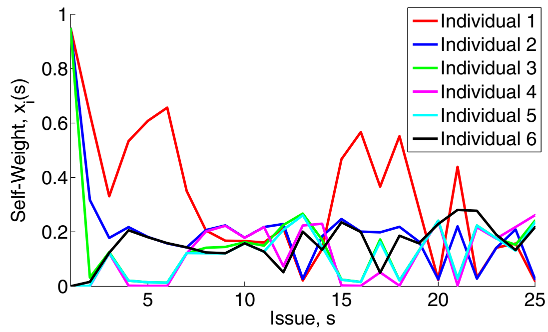

Figure 1. Evolution of individuals' social powers $\boldsymbol{x}(s)$ for initial condition set $\widehat{\boldsymbol{x}}(0)$.

Proof. The map $\boldsymbol{F}_{\sigma(s)}$ is parametrised simply by the vector $\boldsymbol{\gamma}_{\sigma(s)}$. Under the stated condition of $\boldsymbol{C}(s) = \boldsymbol{C}_{\sigma(s)} \in \mathcal{K}(\widetilde{\boldsymbol{\gamma}})$, the map $\boldsymbol{F}_{\sigma(s)}$ is time-invariant. The result in Theorem 3 is then used to complete the proof. □

V. Simulations

In this section, we provide a short simulation for a network with 6 individuals to illustrate our key results. The set of topologies is given as $\mathcal{C} = \{\boldsymbol{C}_1, \ldots, \boldsymbol{C}_5\}$, i.e., $\mathcal{P} = \{1, 2, \ldots, 5\}$. The switching signal $\sigma(s)$ is generated such that for any given s, there is equal probability that $\sigma(s) = p, \forall\, p \in \mathcal{P}$. The precise numerical forms of $\boldsymbol{C}_p$ are omitted due to space limitations, and included along with additional figures in the ArXiv version of this paper [41].

Figure 1 shows the evolution of individual social power over a sequence of issues for the system as described in the above paragraph, initialised from a set of initial conditions, $\widehat{\boldsymbol{x}}(0)$. For each individual, with $\bar{\gamma}_i = \max_{p\in\mathcal{P}} \gamma_{p,i}$, we computed $\bar{\gamma}_1 = 0.474, \bar{\gamma}_2 = 0.237, \bar{\gamma}_3 = 0.244, \bar{\gamma}_4 = 0.244, \bar{\gamma}_5 = 0.244, \bar{\gamma}_6 = 0.239$. Note that $\sum_i \bar{\gamma}_i \neq 1$ in general due to the definition of $\bar{\gamma}_i$. According to Corollary 4, we have $\boldsymbol{x}^*(s) \preceq [0.9, 0.311, 0.323, 0.323, 0.323, 0.314]$. This is precisely what is shown in Fig. 1. Since only $\bar{\gamma}_1 > 1/3$, we observe that after the first 10 or so issues, only $x_1^*(s) > 0.5$, i.e., only individual 1 can hold more than half the social power in the limit. Note that $x_4^*(s) > 0$, although for several issues, $x_4(s)$ is close to 0.

Figure 2 compares the social power for selected individuals (1,3,6) with two different sets of initial conditions. The solid and dotted lines correspond to initial condition set $\widetilde{\boldsymbol{x}}(0)$ and $\widehat{\boldsymbol{x}}(0)$, respectively, with $\widetilde{\boldsymbol{x}}(0) \neq \widehat{\boldsymbol{x}}(0)$. Notice that individuals $1, 3, 6$ have large perceived social power for the initial condition set $\widehat{\boldsymbol{x}}(0)$ but low perceived social power for the initial condition set $\widetilde{\boldsymbol{x}}(0)$. Through sequential discussion and reflected self-appraisal, the initial conditions are exponentially forgotten and the plot shows convergence to the same unique limiting trajectory $\boldsymbol{x}^*(s)$ by about $s = 10$. Simulations for periodically-varying topology are available in [31].

VI. Conclusion

In this paper, we have presented several novel results on the DeGroot-Friedkin model. For the original model, convergence

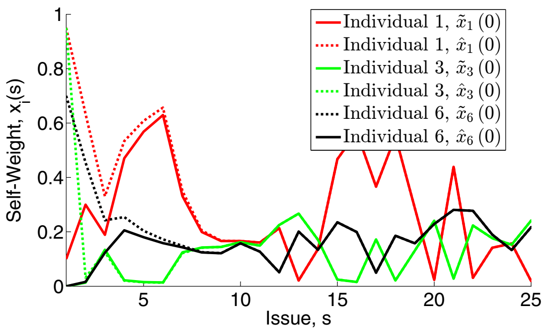

Figure 2. Evolution of selected individuals' social powers $x_i(s)$: a comparison of different initial condition sets $\widehat{\boldsymbol{x}}(0)$ and $\widetilde{\boldsymbol{x}}(0)$.

to the unique equilibrium point has been shown to be exponentially fast. The nonlinear contraction analysis framework allowed for a straightforward extension to dynamic topologies. The key conclusion of this paper is that, according to the DeGroot-Friedkin model, sequential opinion discussion, combined with reflected self-appraisal between any two successive issues, removes perceived (initial) individual social power at an exponential rate. True social power in the limit is determined by the network topology, i.e., interpersonal relationships and their strengths. An upper bound on each individual's limiting social power depends explicitly on only the network topology.

A number of questions remain. Firstly, we aim to relax the graph topology assumption from strongly connected (i.e., the relative interaction matrix is irreducible) to containing a directed spanning tree (i.e., the relative interaction matrix is reducible). Moreover, one may consider a graph whose union over a set of issues is strongly connected, but for each issue, the graph is not strongly connected. Stubborn individuals (i.e., the Friedkin-Johnsen model) should be incorporated; only partial results are currently available [35]. Effects of noise and other external inputs should be studied, as well as the concept of personality affecting the reflected self-appraisal mechanism (as mentioned in Remark 10).

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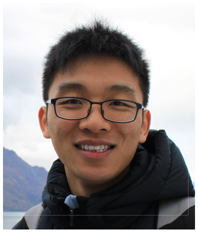

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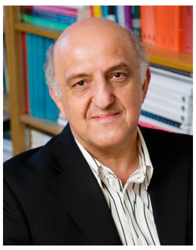

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