Topics in Dynamic Programming

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Declaration

Except where otherwise acknowledged, I certify that this thesis is my original work. The thesis is within the 100,000 word limit set by the Australian National University.

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Abstract

Dynamic programming is an essential tool lying at the heart of many problems in the modern theory of economic dynamics. Due to its versatility in solving dynamic optimization problems, it can be used to study the decisions of households, firms, governments, and other economic agents with a wide range of applications in macroeconomics and finance. Dynamic programming transforms dynamic optimization problems to a class of functional equations, the Bellman equations, which can be solved via appropriate mathematical tools. One of the most important tools is the contraction mapping theorem, a fixed point theorem that can be used to solve the Bellman equation under the usual discounting assumption for economic agents. However, many recent economic models often make alternative discounting assumptions under which contraction no longer holds. This is the primary motivation for the thesis.

This thesis is a re-examination of the standard discrete-time infinite horizon dynamic programming theory under two different discounting specifications: state-dependent discounting and negative discounting. For the case of state-dependent discounting, the standard discounting condition is generalized to an "eventual discounting" condition, under which the Bellman operator is a contraction in the long run, instead of a contraction in one step. For negative discounting, the theory of monotone concave operators is used to derive a unique solution to the Bellman equation; no contraction mapping arguments are required. The core results of the standard theory are extended to these two cases and economic applications are discussed.

Contents

CHAPTER 1

Introduction

1.1. Overview

Dynamic programming is the approach of using recursive methods to solve dynamic optimization problems. It is the building block of the modern theory of economic dynamics with applications spanning multiple fields of economics. It is particularly useful in studying the optimal decisions of households, firms, and governments in a dynamic environment, which forms the basis of research in economic growth, business cycles, asset pricing, fiscal and monetary policies, and other important topics in macroeconomics and finance.

The fundamental idea of dynamic programming is to solve dynamic optimization problems through a functional equation—the *Bellman equation*. This essentially transforms an optimization problem to a fixed point problem, which can be tackled with a wide range of mathematical tools. The Bellman equation is also static in nature and thus much easier to solve than the original dynamic problem. A typical example is a household saving problem, where an agent maximizes lifetime discounted utility of consumption. Under appropriate conditions, the optimal choice of the agent can be recovered as the solution to a single period decision problem.

Standard dynamic programming theory relies heavily on the Banach fixed point theorem, also known as the contraction mapping theorem. A standard assumption is that the *Bellman operator* is a contraction mapping. This restriction, however, affects the applicability of dynamic programming to many recent economic models. The goal of this thesis is to develop extensions in this regard and study related economic applications.

1.2. Structure of the Thesis

Chapter 2 provides technical background on the topics covered in the thesis. It begins with an informal introduction to dynamic programming and moves on to explain the fundamental ideas and theories behind the extensions developed in later chapters. Some applications that can be handled by the extensions are also briefly discussed.

Chapter 3 replaces the constant discount factor in standard dynamic programming with a state-dependent discount process. This specification has been adopted to study a series of empirical phenomena including, for example, the equity premium puzzle, zero lower bound, extreme inequality, and macroeconomic volatility. In some settings, the discount factor is even allowed to temporarily exceed unity, which poses a challenge to the standard theory based on contraction mapping. We obtain a sufficient condition on the discount factor process under which all of the standard optimality results can be recovered. In particular, we show that the Bellman operator is a contraction in the long run, instead of a contraction in one step. We also extend the results to the cases of unbounded rewards and recursive preferences and discuss a range of applications. This chapter is based on a joint paper with my supervisor John Stachurski ("Dynamic programming with state-dependent discounting", published in the *Journal of Economic Theory*, doi: 10.1016/j.jet.2021.105190). The chapter contains mostly my own work, although I have benefited from numerous helpful comments and suggestions from John.

Chapter 4 studies negative discount dynamic programming problems, where the discount factor is larger than one and the Bellman operator is expansive, the diametric opposite of the standard case. We focus on the problem in which an agent minimizes the present value of a sequence of losses over an infinite horizon while assigning greater weight to future losses. We develop a general dynamic programming framework for such noncontractive models based on the theory of monotone concave operators and show that it can be used to recover competitive equilibria in models related to production networks, management layers within firms, and the size distribution of cities. We also give a set of analogous results for the continuous time setting. This chapter is based on the paper "Coase Meets Bellman: Dynamic Programming for Production Chains" jointly with Tomoo Kikuchi, Kazuo Nishimura, and John Stachurski. The chapter only contains content that I have made significant contribution to.

Chapter 5 is an application of the concave operator theory used in Chapter 4 to the production networks model introduced by Kikuchi et al. (2018). We prove the existence, uniqueness, and global stability of equilibrium price, hence improving their results on production networks with multiple upstream partners. We propose and prove the validity of an algorithm for computing the equilibrium price function that is more than ten times faster than value function iteration. The model is then generalized to a stochastic setting that offers richer implications for the distribution of firms in a production network. This chapter is based on a joint paper with Meng Yu ("Equilibrium in production chains with multiple upstream partners", published in the *Journal of Mathematical Economics*, doi: 10.1016/j.jmateco.2019.04.002). The chapter contains my original work except some proofs provided by Meng, which I have included for completeness.

1.3. Literature Review

This section provides background on the two main themes of this thesis: state-dependent discounting and negative discounting in dynamic programs. More closely related studies are reviewed in the introduction of each chapter.

1.3.1. State-Dependent Discounting.

1.3.1.1. *Motivation.* Standard dynamic programming theory assumes that the subjective discount factor of agents is constant and strictly less than one (Bellman, 1957; Stokey and Lucas, 1989). This assumption ensures the applicability of the contraction mapping theorem, which is central to the proof of existence and uniqueness of solutions. However, researchers in economics and finance have increasingly adopted settings where the discount factor varies with the state to either explain empirical phenomena or offer better calibrations.

For example, state-dependent discounting has been shown to help explain some long-standing puzzles in asset pricing. Mehra and Sah (2002) show that small fluctuations in agents' discount factors can have large effect on the volatility of equity prices. Albuquerque et al. (2016) study an asset pricing model in which the discount factor of the representative agent is perturbed by an AR(1) process and show that the resulting demand shocks help explain the equity premium puzzle. Also see Campbell (1986), Albuquerque et al. (2015), and Schorfheide et al. (2018).

State-dependent discounting is also useful in studying wealth inequality. An early example is Krusell and Smith (1998). In related work, Krusell et al. (2009) model the discount process as a three state Markov chain and show how discount factor dispersion helps their heterogeneous agent model match the wealth distribution. Also see Fagereng et al. (2019) and Hubmer et al. (2020).

In labor economics, state-dependent discounting has been adopted to help explain the excess unemployment volatility puzzle discussed in Shimer (2005). For example, Mukoyama (2009) enhances the Diamond–Mortensen–Pissarides model with state dependent discount factors for entrepreneurs and workers, which is then shown to increase unemployment volatility. Related analysis and extensions can be found in Beraja et al. (2016), Hall (2017) and Kehoe et al. (2018).

In addition, state-dependent discounting is often used in studies of macroeconomic volatility. For example, Primiceri et al. (2006) argue that shocks to agents' rates of intertemporal substitution are a key source of macroeconomic fluctuations. Justiniano and Primiceri (2008) study the shifts in the volatility of macroeconomic variables in the US and find that a large portion of consumption volatility can be attributable to the variance in discount factors. Additional research in a similar vein can be found in Justiniano et al. (2010), Justiniano et al. (2011), Christiano et al. (2014), Saijo (2017), and Bhandari et al. (2013).

State-dependent discounting is also an important device in the New Keynesian literature to study the effective lower bound. It usually appears in the form of multiplicative preference shocks to the discount factor. For example, Eggertsson and Woodford (2003) study monetary policy under a zero lower bound constraint for nominal interest rate with preference shocks. Woodford (2011) considers the government expenditure multiplier in a similar environment. Eggertsson (2011) and Christiano et al. (2011) study the effect of fiscal policies at the zero lower bound on interest rates, while Nakata and Tanaka (2020) analyze the term structure of interest rates at the zero lower bound when agents have recursive preferences. See also Correia et al. (2013), Hills and Nakata (2018), Hills et al. (2019) and Williamson (2019). In these papers, it is common to let the discount factor

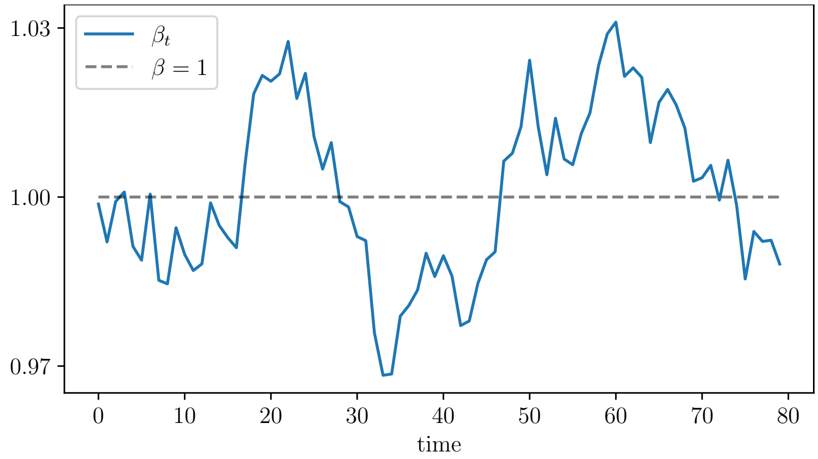


FIGURE 1.1. Simulated time path for $\{\beta_t\}$ in Hills et al. (2019)

temporarily exceed one so that the lower bound on nominal interest rates binds. Figure 1.1 illustrates by showing a simulated time path of $\{\beta_t\}$ in Hills et al. (2019), where the discount factor follows an AR(1) process.

1.3.1.2. *Theoretical Studies.* On the theoretical side, Karni and Zilcha (2000) study the saving behavior of agents with random discount factors in a steady-state competitive equilibrium. Cao (2020) proves the existence of sequential and recursive competitive equilibria in incomplete markets with aggregate shocks in which agents also have state-dependent discount factors. Toda (2019) obtains a necessary and sufficient condition for the existence of a solution to the optimal saving problem with state-dependent discount factors. Also see Ma et al. (2020) for a generalization that exploits a consumption policy operator.

In the mathematical literature, Wei and Guo (2011) study the existence and uniqueness of equilibrium in a general dynamic programming model with state-dependent discount factors. Also see Minjárez-Sosa (2015), Ilhuicatzi-Roldán et al. (2017), and Jasso-Fuentes et al. (2020) for other works along these lines. However, these papers assume that the discount process in the dynamic program is bounded above by one or by some constant less than one. This is too strict for many applications, especially those adopting an AR(1) specification for the discount factor as mentioned above.

1.3.2. Negative Discounting.

1.3.2.1. *Motivation.* Negative discount dynamic programming studies dynamic optimization problems in which the subjective discount rates of agents are negative, or in other words, the subjective discount factors are larger than one. Such preference structures have been documented in Thaler (1981), Loewenstein and Thaler (1989), Loewenstein and Prelec (1991) and Loewenstein and Sicherman (1991). For example, Loewenstein and Prelec (1991) find in a survey that "sequences of outcomes that decline in value are greatly disliked, indicating a negative rate of time preference". Also see Frederick et al. (2002) for a literature review.

Despite its empirical relevance in dynamic settings, negative discount rates arise more naturally in production chain models after the time index is reinterpreted as an index over firms. For example, Kikuchi et al. (2018) embed the ideas on the theory of the firm in Coase (1937) in an equilibrium model with a continuum of price taking firms and represent the equilibrium price as the solution to a discrete-time negative discount dynamic program. The negative discount rate captures transaction costs between firms along the chain. Using a similar formulation in continuous time, Fally and Hillberry (2018) extend the partial equilibrium model in Kikuchi et al. (2018) to an international trade setting. Also see Tyazhelnikov (2019) and Antràs and De Gortari (2020).

1.3.2.2. *Monotone Concave Operator Theory.* As briefly mentioned above, our negative discount dynamic programming framework draws on the theory of concave operators, originally due to Krasnosel'skii (1964, Chapter 6). Our framework is based on a theorem due to Du (1989) for monotone concave operators on arbitrary Banach spaces.

There are other similar techniques that utilize concavity to show uniqueness of the fixed point. Krasnosel'skii (1964) shows that a monotone operator on a positive cone has at most one nonzero fixed point if the operator satisfies a concavity condition (u_0-concave). Marinacci and Montrucchio (2010, 2019) link concavity to contraction in the Thompson metric (Thompson, 1963), which allows one to apply the contraction mapping theorem to operators that are not contractive under the supremum norm. In a similar vein, Marinacci and Montrucchio (2019) establish existence and uniqueness results for monotone operators under a range of weaker concavity conditions using Tarski-type fixed point theorems and the Thompson metric. For similar treatment in the math literature, also see Krasnosel'skii et al. (1972), Krasnosel'skii and Zabreĭko (1984), Guo and Lakshmikantham (1988), Guo et al. (2004), and Zhang (2013).

The monotone concave operator theory has seen some recent success in the economic literature. Lacker and Schreft (1991) study an economy with cash and trade credit as means of payment and show that the equilibrium interest rate is a unique fixed point of a monotone concave operator. Coleman (1991, 2000) studies the equilibrium in a production economy with income tax and proves the existence and uniqueness of consumption function by constructing a monotone concave map. Following this approach, Datta et al. (2002b) prove the existence and uniqueness of equilibrium in a large class of dynamic economies with capital and elastic labor supply. Similar work in the same vein includes Morand and Reffett (2003) and Datta et al. (2002a).

CHAPTER 2

Technical Background

This chapter starts with an informal introduction to dynamic programming for readers who are not familiar with this topic. I will motivate with a standard dynamic optimization problem and state some major results from the literature. Then I will move on to the cases of state-dependent discounting and negative discounting—two main themes of the thesis, and explain the basic ideas behind results in later chapters.

2.1. Dynamic Programming

This section considers an infinite-horizon stochastic dynamic optimization problem where an agent maximizes the expected discounted sum of rewards.

The state of the world consists of a pair (x, z), where x and z represent endogenous and exogenous variables, taking values in spaces X and Z, respectively. Uncertainty is driven by a time-homogeneous Markov process on Z with stochastic kernel Q.[1] Let the initial state (x_0, z_0) be given. In each period, the agent chooses the next period endogenous state subject to a constraint that depends on the current state to maximize lifetime reward. Formally, the agent solves

$$\max_{\{x_t\}_{t=1}^{\infty}} \mathbb{E} \left\{ \sum_{t=0}^{\infty} \beta^t \, r(x_t, z_t, x_{t+1}) \right\} \qquad \text{s.t. } \; x_{t+1} \in \Gamma(x_t, z_t), \; z_{t+1} \sim Q(z_t, \cdot), \tag{2.1}$$

where r is the reward function and Γ is the feasible correspondence. This is a generic additively separable dynamic optimization problem that encompasses many common problems in economics, for example, stochastic optimal growth, optimal consumption saving, etc.[2]

[1] Roughly speaking, if $\{z_t\}$ is a time-homogeneous Markov process with stochastic kernel Q, the distribution of z_{t+1} for any t is given by $Q(z_t, \cdot)$, which depends only on the value of z_t.

[2] One simplification we make here is that the agent can directly choose the next period state. In a more general setting, we can let the agent choose an action that affects the distribution of next period state. The formulation in (2.1) also cannot deal with recursive preferences. A general and abstract framework that can accommodate these complications is discussed in Chapter 3.

Solving (2.1) directly is difficult because it involves choosing an infinite sequence of state-contingent variables $\{x_t\}$. Backward induction is also not applicable since there is an infinite horizon. Dynamic programming, on the other hand, aims to transform the infinite horizon problem into a static problem by exploiting the recursive structure of (2.1).

To see how this works, first note that the maximum value in (2.1) depends only on the initial state, and in each period, the agent faces a sub-problem that is completely characterized by the current state. Hence, it is natural to conjecture that the optimal choice of the agent in each period also depends only on the current state. Let $v^* \colon \mathsf{X} \times \mathsf{Z} \to \mathbb{R}$ be the *value function*, which gives the maximum payoff of the agent for each initial state. We aim to find an optimal policy $\sigma^* \colon \mathsf{X} \times \mathsf{Z} \to \mathsf{X}$ for the agent such that choosing $x_{t+1} = \sigma^*(x_t, z_t)$ in each period gives v^*.

Consider the following functional equation

$$v(x,z) = (Tv)(x,z) := \sup_{x' \in \Gamma(x,z)} \left\{ r(x,z,x') + \beta \mathbb{E}_z v(x',z') \right\}, \tag{2.2}$$

where $\mathbb{E}_z$ represents expectation conditional on z and $\mathbb{E}_z v(x',z') = \int v(x',z') Q(z, dz')$. Equation (2.2) is called the *Bellman equation* due to Bellman (1957) and T is called the *Bellman operator*. The Bellman equation turns out to be closely related to (2.1).

The function v on the right hand side of (2.2) is called the *continuation value function*, which gives the payoff of the agent in the next period. Hence, Tv is the maximum value that can be attained for the agent in the current period given continuation value function v. If the continuation value function is v^*, which implies that agent achieves maximum from the next period onward, it is reasonable to conjecture that $Tv^* = v^*$. In fact, this is one of the main results we aim to prove in dynamic programming. A second main result is that the optimal policy σ^* is given by the maximizer on the right hand side of (2.2) when $v = v^*$. The intuition is that, if a policy achieves maximum in the current period given continuation value v^*, it should also lead to maximum if the agent follows this policy in every period.

In order to prove these results, a crucial step is to show the existence of fixed points of the Bellman operator in a certain function space. A standard assumption is that r is bounded and continuous and $\beta < 1$. Then under some additional regularity conditions, it can be shown that T is a contraction self map on $bc\mathsf{S}$, the space of bounded continuous functions on $\mathsf{S} := \mathsf{X} \times \mathsf{Z}$. Indeed, for any $u, v \in bc\mathsf{S}$, we have

$$\begin{aligned} \|Tu - Tv\| &= \sup_{x \in \mathsf{X}, z \in \mathsf{Z}} |(Tu)(x,z) - (Tv)(x,z)| \\ &\leq \sup_{x \in \mathsf{X}, z \in \mathsf{Z}} \sup_{x' \in \Gamma(x,z)} \beta \left| \mathbb{E}_z u(x', z') - \mathbb{E}_z v(x', z') \right| \leq \beta \|u - v\|, \end{aligned}$$

where $\|\cdot\|$ is the supremum norm. Since $bc\mathsf{S}$ is a complete metric space, the contraction mapping theorem implies that T has a unique fixed point $\bar{v}$ in $bc\mathsf{S}$ and for any $v \in bc\mathsf{S}$, there exists $M > 0$ such that $\|T^n v - \bar{v}\| < M\beta^n$. In other words, the standard assumption not only guarantees the existence and uniqueness of the fixed point, but also a strong form of convergence: uniform convergence with a fixed rate β. This also ensures that we can obtain the unique fixed point via value function iteration starting from any function in $bc\mathsf{S}$.[3]

With these fixed point results, it can be shown that $\bar{v} = v^*$ and a policy function σ^* is optimal if and only if $\sigma^*(x,z) \in \arg\max_{x' \in \Gamma(x,z)} \{r(x,z,x') + \beta \mathbb{E}_z v^*(x', z')\}$ (see, for example, Stokey and Lucas (1989)). The latter property is called the *Bellman's principle of optimality*. Besides these two main results, it can further be proved that v^* is increasing, concave, and continuously differentiable, and that σ^* is single-valued and continuous, under additional assumptions on the primitives.

Recent studies in dynamic programming have relaxed the standard assumptions, especially the boundedness of the reward function, to accommodate more economic applications. For example, Boyd (1990) uses a weighted supremum norm approach to study unbounded rewards and recursive preferences. Alvarez and Stokey (1998) treat dynamic programs with unbounded reward functions that are homogeneous. Rincón-Zapatero and Rodríguez-Palmero (2003), Martins-da Rocha and Vailakis (2010), Matkowski and Nowak (2011) use local contraction methods to deal with unbounded rewards. Also see Durán (2003), Le Van and Vailakis (2005), and Kamihigashi (2007).

[3] Value function iteration is the algorithm of repeatedly applying the Bellman operator on some initial guess.

2.2. State-Dependent Discounting

The standard dynamic programming theory and extensions mentioned above all assume that the subjective discount factor of the agent is a constant $\beta \in (0,1)$. However, as reviewed in Section 1.3, many recent economic applications adopt discount factors that are state-dependent.

To accommodate this case, we replace the constant discount factor of the agent with a discount process that depends on the exogenous state variable. Then problem (2.1) becomes

$$\max_{\{x_t\}_{t=1}^{\infty}} \mathbb{E} \left\{ \sum_{t=0}^{\infty} \left(\prod_{i=0}^{t-1} \beta(z_i) \right) r(x_t, z_t, x_{t+1}) \right\} \quad \text{s.t. } x_{t+1} \in \Gamma(x_t, z_t),\ z_{t+1} \sim Q(z_t, \cdot) \tag{2.3}$$

and the corresponding Bellman equation becomes

$$v(x,z) = (Tv)(x,z) := \sup_{x' \in \Gamma(x,z)} \left\{ r(x,z,x') + \beta(z) \mathbb{E}_z v(x',z') \right\}, \tag{2.4}$$

where $\beta : \mathsf{Z} \to \mathbb{R}_+$ is a bounded continuous function. Now, the discount factor on time-t reward is the product of all past $\beta(z_t)$, and the discount factor on the continuation value function in the Bellman equation is $\beta(z)$, a function of the exogenous state.

Although this is a small change and the new Bellman equation (2.4) is highly similar to (2.2), the argument in the previous section fails to lead to contraction. Indeed, for any $u, v \in bc\mathsf{S}$, we have

$$\|Tu - Tv\| \leq \sup_{z \in \mathsf{Z}} \beta(z) \|u - v\|.$$

We can only show that T is a contraction if $\sup_{z \in \mathsf{Z}} \beta(z) < 1$, which is too restrictive for many applications as discussed in Section 1.3. Intuitively speaking, a uniform upper bound of one on the discount process also appears unnecessarily strong. Problem (2.3) should retain the good properties of (2.1), all other things being equal, if the discount factor only occasionally exceeds one but is less than one "on average".

In Chapter 3, we build on this idea and give a weaker condition—eventual discounting, which requires the existence of some $n \in \mathbb{N}$ such that,

$$r_n^\beta := \sup_{z_0 \in \mathsf{Z}} \mathbb{E}_{z_0} \prod_{i=0}^{n-1} \beta(z_i) < 1. \tag{2.5}$$

The product $\prod_{i=0}^{n-1} \beta(z_i)$ in (2.5) is the discount factor on time-n reward in (2.3). Hence, eventual discounting says that the discount factors on future rewards are eventually less than one. Condition (2.5) also implies $(r_n^\beta)^{1/n} < 1$, which has the interpretation that the geometric average of the discount process is less than one in the long run.

Although T in (2.4) is not necessarily a contraction under eventual discounting, we show in Chapter 3 that the Bellman operator is still well-behaved in the sense that

$$\|T^n u - T^n v\| \leq \sup_{z_0 \in \mathsf{Z}} \mathbb{E}_{z_0} \prod_{i=0}^{t-1} \beta(z_i) \|u - v\|,$$

where T^n is the nth iterate of the Bellman operator. In other words, eventual discounting implies that T^n is eventually a contraction mapping. An extension of the contraction mapping theorem shows that the Bellman operator has a unique fixed point in $bc\mathsf{S}$ and $T^n v$ converges to the fixed point uniformly for all $v \in bc\mathsf{S}$. All standard dynamic programming results can then be recovered building on this result.

To facilitate computation, we connect the eventual discounting condition to the spectral radius of a bounded linear operator.[4] Let L_β be defined by $(L_\beta h)(z) = \beta(z) \int h(z') Q(z, dz')$. We show that eventual discounting holds if and only if the spectral radius of L_β is less than one. This provides us a way to check the eventual discounting condition for different discount specifications. It is particularly useful in applications where the state spaces are discretized, since L_β becomes a matrix and its spectral radius can be readily calculated as the largest absolute value of its eigenvalues.

A range of recent studies consider dynamic optimization problems with state-dependent discounting. For example, in the quantitative model of Hubmer et al. (2020), the discount

[4]The spectral radius of a bounded linear operator L is given by $r(L) := \lim_{n\to\infty} \|L^n\|^{1/n}$ where $\|\cdot\|$ is the operator norm.

factor of households is modeled as an AR(1) process to generate realistic wealth distributions; Christiano et al. (2011) use a two-state discount process for households to study fiscal policy under the zero lower bound constraint. Such problems cannot be handled by the standard dynamic programming theory. In particular, it is not clear if those problems are well defined and well behaved. This issue is covered in our theory as shown above.

In Chapter 3, we also extend our basic results to dynamic programs with unbounded rewards and recursive preferences. We obtain similar eventual discounting and spectral radius conditions for these cases. For example, the eventual discounting condition for dynamic programs with Epstein-Zin preferences also depends on the elasticity of intertemporal substitution. Of particular interest is an application of our extension to the asset pricing model of Albuquerque et al. (2016), in which the representative agent has Epstein-Zin preferences with state-dependent discounting. We find that eventual discounting fails in their model due to high persistence of the discount process and low elasticity of intertemporal substitution. See Section 3.6.2 for details.

2.3. Negative Discounting

For negative discounting, we focus on a special class of dynamic optimization problems in which the agent, instead of maximizing lifetime reward, minimizes a flow of "discounted" losses. The agent solves

$$\min_{\{a_t\}} \sum_{t=0}^{\infty} \beta^t \ell(a_t) \quad \text{s.t. } a_t \geq 0 \text{ for all } t \geq 0 \text{ and } \sum_{t=0}^{\infty} a_t = \hat{x}, \tag{2.6}$$

where a_t is action in period t and ℓ is a loss function. We assume that $\ell(0) = 0$, $\ell' > 0$, $\ell'' > 0$, $\beta > 1$, and $\hat{x} > 0$. We can think of $\hat{x}$ as the amount of tasks that need to be completed by the agent and a_t as the amount completed in period t. The convexity of the loss function provides an incentive for the agent to defer some effort but negative discounting ($\beta > 1$) encourages the agent to finish as soon as possible.

Define a state process $\{x_t\}$ that tracks the remaining tasks by setting $x_0 = \hat{x}$ and $x_{t+1} = x_t - a_t$. Then, the Bellman equation for problem (2.6) is

$$w(x) = (Tw)(x) := \inf_{0 \leq a \leq x} \{\ell(a) + \beta w(x - a)\}. \tag{2.7}$$

There are two crucial differences between problem (2.6) and problem (2.1). First, the extra constraint $\sum_t a_t = \hat{x}$ in (2.6) is not encoded in the Bellman equation. Hence, it is not clear whether (2.7) can characterize the solution to (2.6). Second, the Bellman operator T in (2.7) is not a contraction mapping due to negative discounting. In Chapter 4, we develop a general dynamic programming theory that can handle these two issues. The rest of this section focuses on the solution to the second issue, which relies on a set of fixed point results for monotone concave operators.

The idea behind it is intuitive: imagine an increasing and strictly concave real function f such that $f(u_0) > u_0$ and $f(v_0) < v_0$ with $u_0 < v_0$. Then it must be true that f has a unique fixed point on $[u_0, v_0]$, and by the concavity of f, the fixed point can be computed by successive evaluations of f on any $x \in [u_0, v_0]$. These properties are illustrated in Figure 2.1, where the intersection of f and the 45° line is the fixed point of f and the dashed line segments represent the trajectories of two iterations starting from u_0 and v_0 respectively. Despite not a contraction (as can be seen from the derivative of f), f has very similar fixed point properties to a contraction mapping. In particular, $f^n(x)$ converges to the fixed point for any initial guess in $[u_0, v_0]$.

It turns out that the same idea also works for general functional operators with additional technical assumptions.

For the Bellman operator T in (2.7), we can show that T is increasing and concave.[5] Monotonicity is obvious. For concavity, we have for any $0 \leq \lambda \leq 1$,

$$\begin{aligned}
&\lambda(Tu)(x) + (1-\lambda)(Tv)(x) \\
&= \lambda \inf_{0 \leq a \leq x} \{\ell(a) + \beta u(x-a)\} + (1-\lambda) \inf_{0 \leq a \leq x} \{\ell(a) + \beta u(x-a)\} \\
&\leq \inf_{0 \leq a \leq x} \{\ell(a) + \lambda \beta u(x-a) + (1-\lambda)\beta v(x-a)\}
\end{aligned}$$

[5] We say T is increasing if $Tu \leq Tv$ whenever $u \leq v$ and concave if for all $0 \leq \lambda \leq 1$, $\lambda Tu + (1-\lambda)Tv \leq T(\lambda u + (1-\lambda)v)$.

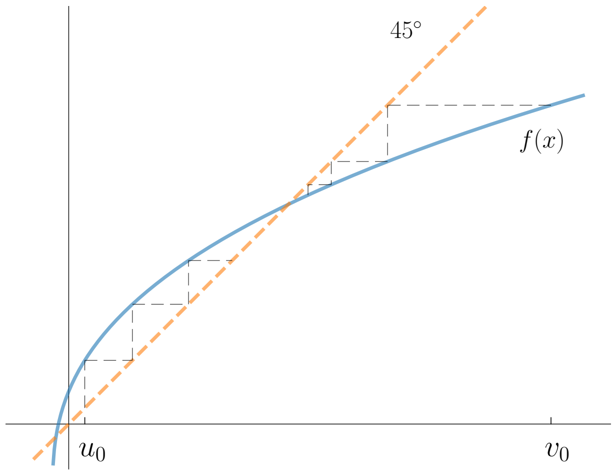


FIGURE 2.1. Fixed Point Properties of an Increasing Concave Function

$$= (T(\lambda u + (1-\lambda)v))(x).$$

Let $\phi(x) = \ell'(0)x$ and $\psi(x) = \ell(x)$.[6] It can be shown that $T\phi > \phi$ and $T\psi < \psi$. By a fixed point theorem for monotone concave operators on Banach spaces, T has a unique fixed point $\bar{w}$ on $[\phi, \psi]$ and $T^n w \to \bar{w}$ for all $w \in [\phi, \psi]$, where $[\phi, \psi]$ is an order interval defined by all w such that $\phi \leq w \leq \psi$.

The idea of deriving fixed point results from monotonicity and concavity/convexity has been adopted in several theoretical studies, albeit not in negative discount settings. For example, Rincón-Zapatero and Rodríguez-Palmero (2003) exploit the monotonicity and convexity properties of the Bellman operator and give conditions for existence and uniqueness of fixed points in the case of unbounded returns. Balbus et al. (2013) study the existence and uniqueness of pure strategy Markovian equilibrium using theories concerning "mixed monotone" operators that are convex in a certain sense. Also see Balbus et al. (2012) for a similar treatment. More recently, this idea has been applied extensively to models with recursive utilities since Marinacci and Montrucchio (2010); other contributions along the same lines include Balbus (2016), Borovička and Stachurski (2020, 2021), Becker and Rincon-Zapatero (2018), Marinacci and Montrucchio (2019), Pavoni et al. (2018), Bloise and Vailakis (2018), and Ren and Stachurski (2018).

[6]These two functions act as upper and lower bounds on candidate value function. Since completing all remaining tasks at once is in the choice set, its value $\ell(x)$ is an upper bound of the minimized value. Regarding the lower bound $\ell'(0)x$, this is the value that could be obtained if $\beta = 1$ (no discounting) and the agent, having no time constraint, subdivided without limit.

The negative discount dynamic programming theory turns out to be suitable for studying a range of competitive equilibrium models that have no direct connection to negative discount rates. To build this connection, we reinterpret the time index in (2.6) as an index over some decision making entities, who are required to complete tasks of amount $\hat{x}$ together sequentially. Specifically, agent t, who is paid to complete tasks of amount x_t, completes a_t and pays agent $t+1$ to complete $x_{t+1} = x_t - a_t$. Due to some friction between agents, the payment to an agent is the total cost for the agent multiplied by β. The following equation illustrates this process:

$$\text{total cost for agent } 0 = \ell(a_0) + \beta \Big[\overbrace{\ell(a_1) + \beta\big(\underbrace{\ell(a_2) + \beta(\ldots)}_{\text{total cost for agent 2}} \big)}^{\text{total cost for agent 1}} \Big].$$

As a result, (2.6) can be thought of as a social planner's problem in which the planner minimizes the total cost of completing tasks of amount $\hat{x}$.

On the other hand, the Bellman equation (2.7) represents a decentralized problem. Let $w^*(\hat{x})$ be the value of (2.6). Then, w^* can be regarded as a price function such that $w^*(x)$ gives the price of completing x. Given w^*, the minimum cost of an agent who is paid to complete x is $\inf_{0 \leq a \leq x} \{\ell(a) + \beta w^*(x-a)\}$. In equilibrium, the total cost of an agent is equal to the price paid to the agent, so the equilibrium price function is the solution to the Bellman equation (2.7).

In Chapter 4, we show that the negative discount dynamic program (2.6) and its variants can be used to study competitive equilibria in models related to production chains, production networks, city hierarchy, and the organization of knowledge within firms. In each model, the equilibrium is the solution to a Bellman equation with negative discounting and the discount factor β represents frictions such as transaction costs, transportation costs, or communication costs.

CHAPTER 3

Dynamic Programming with State-Dependent Discounting

3.1. Introduction

This chapter studies discrete-time dynamic programming over infinite horizons with state-dependent discount factors. Applications of such specifications are prevalent in macroeconomics and finance as reviewed in Section 1.3.

In particular, We replace the constant discount factor β in the standard theory with a discount process $\{\beta_t\}$, so that time t payoff π_t is discounted to present value as $\mathbb{E}_z \prod_{i=0}^{t-1} \beta_i \, \pi_t$ rather than $\beta^t \, \mathbb{E}_z \, \pi_t$. Here z is the initial condition of an exogenous Markov state process that drives evolution of the discount factor. We replace the traditional condition $\beta < 1$ with a weaker "eventual discounting" condition: existence of a $t \in \mathbb{N}$ such that $\sup_{z \in \mathsf{Z}} \mathbb{E}_z \prod_{i=0}^{t-1} \beta_i < 1$. For a finite irreducible state process, this is equivalent to existence of a $t \in \mathbb{N}$ such that $\mathbb{E} \prod_{i=0}^{t-1} \beta_i < 1$, where $\mathbb{E}$ is the unconditional expectation.[1]

We show that, when eventual discounting holds, (i) the value function satisfies the Bellman equation, (ii) an optimal policy exists, (iii) Bellman's principle of optimality holds, and (iv) value function iteration and Howard policy iteration (Howard, 1960) are both convergent. When β_t is constant at $\beta < 1$, eventual discounting holds at $t = 1$, so these results capture the standard theory as a special case.

Note that if $\sup_z \beta_t < 1$, standard contraction mapping arguments can be applied to the Bellman equation. In this sense, our conditions are most relevant when $\beta_t \geq 1$ with positive probability, which is common in the New Keynesian literature among studies of the zero lower bound. For example, Christiano et al. (2011) assume that $\beta = 1.02$ at the beginning of a recession and returns to normal with some probability in each period

[1] As stated above, we assume that the discount factor is driven by an exogenous state process. However, our methods can also be applied to individual agent problems where endogenous aggregates appear in the discount factor, provided that the agent treats these aggregates as external to his or her actions. See, for example, Schmitt-Grohé and Uribe (2003).

afterwards. Also see Eggertsson (2011) and Correia et al. (2013). Fernández-Villaverde et al. (2015) adopts an AR(1) specification for $\log(\beta_t)$ and use a projection method to solve the model, where the largest β is 1.0066. Similarly, Hills et al. (2019) analyze tail risk associated with the effective lower bound on the policy rate in a model where the discount process is a constant multiple of a discretized AR(1) process that regularly generates value of β_t exceeding unity.[2]

To facilitate easy checking of the eventual discounting condition, we give several equivalent conditions, which involve the spectral radius of a discounting operator. We show that eventual discounting is equivalent to the spectral radius being less than one. We provide guidelines on how to calculate the spectral radius for a range of discount specifications. For example, a commonly adopted setting is an AR(1) discount process. We show that, in this case, the eventual discounting condition is more likely to fail when the AR(1) process has higher mean, higher persistence, or higher volatility. When the logarithm of the discount factor follows an AR(1) process, we give an analytical expression that can help us check the eventual discounting condition.

To handle unbounded rewards, we extend two approaches that have been developed previously for the case of constant discounting. The first one treats homogeneous programs in the spirit of Alvarez and Stokey (1998) and Stokey and Lucas (1989, Section 9.3). The second uses a local contraction method pioneered in Rincón-Zapatero and Rodríguez-Palmero (2003) and further developed by Martins-da Rocha and Vailakis (2010) and Matkowski and Nowak (2011). In each case, we show how the eventual discounting condition can be adapted to handle these extensions.

In addition, we study dynamic programming with Epstein–Zin utilities, where rewards are unbounded above and the Bellman operator is not a contraction in the short or long run under standard metrics. To solve the problem, we extend earlier work by Marinacci and Montrucchio (2010), Bloise and Vailakis (2018), and Becker and Rincon-Zapatero (2018), which exploits the monotonicity and concavity of the aggregator, to allow for state-dependent discounting. We show that, in the case of Epstein–Zin utility, the eventual

[2]See Figure 1.1 in Section 1.3 for an illustration. Other studies using an AR(1) specification for the discount process or its logarithm include Nakata (2016), Hubmer et al. (2020), Albuquerque et al. (2016) and Schorfheide et al. (2018).

discounting condition must be adapted to compensate for the role played by elasticity of intertemporal substitution.

As an application of our theory, we discuss asset pricing in an exchange economy of Lucas (1978a) for additively separable as well as Epstein–Zin utilities. We prove the existence and uniqueness of asset prices under different versions of the eventual discounting conditions.

The use of spectral radii connects our work to a strand of literature in finance that studies the long-term factorization of stochastic discount factors using eigenfunctions of valuation operators (see, e.g., Hansen and Scheinkman (2009), Hansen and Scheinkman (2012), and Qin and Linetsky (2017)). Drawing on these ideas, Borovička and Stachurski (2020) and Christensen (2020) connect the spectral radius of valuation operators with existence and uniqueness of recursively defined utilities. However, neither of these papers provides results on optimality or dynamic programming.

Our work is also related to Toda (2019) and Ma et al. (2020), who investigate an income fluctuation problem that features state-dependent discount factors. Their results are specialized to optimal savings with additively separable rewards and do not apply to problems that involve discrete choices, endogenous labor supply, durable goods, or other common features. In contrast, the theory below is developed in a general dynamic programming setting, where the state spaces are arbitrary metric spaces. In addition, their results rely on an consumption policy operator derived from the Euler equation, which is not universally applicable especially in cases of recursive preferences (see, e.g., Albuquerque et al. (2016), Basu and Bundick (2017), Schorfheide et al. (2018), Nakata and Tanaka (2020), or De Groot et al. (2020)).

In the mathematical literature, Schäl (1975) admits state-dependent discounting in discrete time under rather weak conditions, but he directly assumes that expected discounted rewards are finite under any Markov policy. This restricts all primitives in the dynamic program simultaneously and makes the condition impractical for applications.

The rest of this paper is structured as follows. Section 3.2 sets out the model and provides our main results. Section 3.3 gives examples. Section 3.4 reviews our key assumption.

Sections 3.5 and 3.6 treat extensions. Section 3.7 discusses an application in asset pricing. Section 3.8 concludes. All the proofs are in the appendix.

3.2. A Dynamic Program

In what follows, for any metric space Y, the symbols $m\mathsf{Y}$, $bm\mathsf{Y}$ and $bc\mathsf{Y}$ denote the (Borel) measurable, bounded measurable and bounded continuous functions from Y to $\mathbb{R}$ respectively. Unless otherwise stated, the last two spaces are endowed with the supremum norm and this norm is represented by $\|\cdot\|$. In expressions with products below, we adopt the convention that $\prod_{t=0}^{n-1} \beta_t = 1$ whenever $n = 0$.

3.2.1. Framework. The state of the world consists of a pair (x, z), where x and z represent endogenous and exogenous variables. These variables take values in separable metric spaces X and Z respectively. The agent responds to (x, z) by choosing future state x' from $\Gamma(x, z) \subset \mathsf{X}$, where Γ is the *feasible correspondence*. Let $\operatorname{gr}\Gamma$ be the graph of Γ, defined by

$$\operatorname{gr}\Gamma = \{(x, z, x') \in \mathsf{S} \times \mathsf{X} : x' \in \Gamma(x, z)\} \quad \text{where } \mathsf{S} := \mathsf{X} \times \mathsf{Z}. \tag{3.1}$$

Similar to Bertsekas (2013), we combine the remaining elements of the dynamic programming problem into a single *continuation aggregator* H, with the understanding that $H(x, z, x', v)$ is the maximal value that can be obtained from the present time under the continuation value function v, given current state (x, z) and next period state x'. The aggregator H maps each (x, z, x', v) in $\operatorname{gr}\Gamma \times bm\mathsf{S}$ into $\mathbb{R}$ and is assumed to satisfy, for all $v, w \in bm\mathsf{S}$ and all $(x, z, x') \in \operatorname{gr}\Gamma$,

$$H(x, z, x', v) \leq H(x, z, x', w) \text{ whenever } v \leq w. \tag{3.2}$$

This basic monotonicity condition is satisfied in all applications of interest. Bellman's equation takes the form

$$v(x, z) = \sup_{x' \in \Gamma(x,z)} H(x, z, x', v). \tag{3.3}$$

For fixed X and Z, a *dynamic program* $\mathcal{D} = (\Gamma, H)$ consists of a feasible correspondence Γ and a continuation aggregator H.

3.2.2. Feasibility and Optimality. Let $\mathcal{D} = (\Gamma, H)$ be a dynamic program and let Σ be the set of *feasible policies*, defined as all Borel measurable maps σ from S to X such that $\sigma(x, z) \in \Gamma(x, z)$ for each (x, z) in S. Given such σ, let T_σ be the *policy operator* on $bm\mathsf{S}$ given by

$$(T_\sigma v)(x, z) = H(x, z, \sigma(x, z), v). \tag{3.4}$$

Define the *Bellman operator* T on $bm\mathsf{S}$ by

$$(Tv)(x, z) = \sup_{x' \in \Gamma(x,z)} H(x, z, x', v). \tag{3.5}$$

Given v_0 in $bm\mathsf{S}$ and σ in Σ, we can interpret $v_{n,\sigma}(x, z) := (T_\sigma^n v_0)(x, z)$ as the lifetime payoff of an agent who starts at state (x, z), follows policy σ for n periods and uses v_0 to evaluate the terminal state. The *σ-value function* for an infinite-horizon problem is defined here as

$$v_\sigma(x, z) := \lim_{n \to \infty} v_{n,\sigma}(x, z). \tag{3.6}$$

The definition requires that this limit exists and is independent of v_0. Below we impose conditions such that this is always the case.

We define the *value function* corresponding to our dynamic program by

$$v^*(x, z) = \sup_{\sigma \in \Sigma} v_\sigma(x, z) \tag{3.7}$$

at each (x, z) in S. A policy $\sigma^* \in \Sigma$ is called *optimal* if it attains the supremum in (3.7) at each (x, z) in S. We say that *Bellman's principle of optimality holds* when

$$\sigma \in \Sigma \text{ is optimal} \quad \iff \quad \sigma(x, z) \in \operatorname*{arg\,max}_{x' \in \Gamma(x,z)} H(x, z, x', v^*) \text{ for each } (x, z) \text{ in } \mathsf{S}.$$

3.2.3. Assumptions. A dynamic program $\mathcal{D} = (\Gamma, H)$ will be called *regular* if

(a) Γ is continuous, nonempty, and compact valued and

(b) the function $(x, z, x') \mapsto H(x, z, x', v)$ is bounded and measurable on $\operatorname{gr} \Gamma$ for all $v \in bm\mathsf{S}$, and also continuous when $v \in bc\mathsf{S}$.

Most standard cases from the literature are regular, including all dynamic programs with a finite state space.[3] Further discussion of regularity is provided in Section 3.3.

Let $\beta_t = \beta(Z_t) \geq 0$ for some $\beta \in bm\mathsf{Z}$ and Markov process $\{Z_t\}$ on Z with transition kernel Q.[4] Let $\mathbb{E}_z$ represent expectation given $Z_0 = z$. We call (β, Q) *eventually discounting* if $r_n^\beta < 1$ for some $n \in \mathbb{N}$, where

$$r_n^\beta := \sup_{z \in \mathsf{Z}} \mathbb{E}_z \prod_{t=0}^{n-1} \beta_t.$$

Example 3.2.1. If there exists a constant $b \geq 0$ such that $\beta_t \equiv b$ for all $t \geq 0$, then $r_n^\beta = b^n$. Eventual discounting holds if and only if $b < 1$.

Example 3.2.2. If $\{Z_t\}$ is iid, then $r_n^\beta = \prod_{t=0}^{n-1} \mathbb{E}\beta_t = b^n$ where $b := \mathbb{E}\beta_t$. Hence eventual discounting holds if and only if $\mathbb{E}\beta_t < 1$. In particular, higher moments have no influence on eventual discounting unless there is persistence.

Section 3.4 provides an extended discussion of eventual discounting for more sophisticated state processes.

Assumption 3.2.1 (Eventual Contractivity). There is a nonnegative function β in $bc\mathsf{Z}$ and a Feller transition kernel Q on Z such that (β, Q) is eventually discounting and

$$|H(x, z, x', v) - H(x, z, x', w)| \leq \beta(z) \int |v(x', z') - w(x', z')| Q(z, dz') \tag{3.8}$$

for all $v, w \in bm\mathsf{S}$ and $(x, z, x') \in \operatorname{gr} \Gamma$.[5]

The Feller property means that either Z is discrete or the law of motion is continuous.[6]

[3] The continuity and compactness conditions are automatically satisfied when X and Z are finite and endowed with the discrete topology.

[4] That is, $Q(z, B) = \mathbb{P}\{Z_{t+1} \in B \mid Z_t = z\}$ for all $z \in \mathsf{Z}$ and B in the Borel subsets of Z.

[5] Here we implicitly assume that the discount factor is known to the agent at the beginning of each period. Our results hold for alternative timing with slight modifications to (3.8). See Section 3.6.1.

[6] More precisely, we assume that, for any $h \in bc\mathsf{S}$, the function $(x, z) \mapsto \int h(x, z') Q(z, dz')$ is continuous. This holds automatically when Z is countable (under the discrete topology). It also holds if Q is generated by a continuous law of motion, in the sense that $Z_{t+1} = F(Z_t, W_{t+1})$ for some continuous function F and iid sequence $\{W_t\}$. These two cases cover all the applications we consider. Further discussion can be found in Lemma 12.14 of Stokey and Lucas (1989).

3.2.4. Optimality Results. In the statement of the next theorem, a map M from a metric space into itself is called *eventually contracting* if there exists an n in $\mathbb{N}$ such that the n-th iterate M^n is a contraction mapping.[7]

THEOREM 3.2.1. *Let $\mathcal{D}$ be a dynamic program. If $\mathcal{D}$ is regular and Assumption 3.2.1 holds, then the following statements are true:*

(a) *T_σ is eventually contracting on $bm\mathsf{S}$ and T is eventually contracting on $bc\mathsf{S}$.*

(b) *For each feasible policy σ, the lifetime value v_σ is a well defined element of $bm\mathsf{S}$.*

(c) *The value function v^* is finite, continuous, and the only fixed point of T in $bc\mathsf{S}$.*

(d) *At least one optimal policy exists.*

(e) *Bellman's principle of optimality holds.*

In addition, value function and Howard policy iteration converge:

(f) $\lim_{k\to\infty} T^k v = v^*$ *for all* $v \in bc\mathsf{S}$ *and*

(g) $\lim_{k\to\infty} v_{\sigma_k} = v^*$ *when* $\sigma_k \in \Sigma$ *and* $T_{\sigma_k} v_{\sigma_{k-1}} = T v_{\sigma_{k-1}}$ *for all* $k \in \mathbb{N}$.

This theorem extends the core results of dynamic programming theory to the case of state-dependent discounting. In particular, the value function satisfies the Bellman equation, an optimal policy exists, and Bellman's principle of optimality is valid. Value iteration and policy iteration both lead to the value function, so that we have both existence of an optimal policy and means to compute it. The proof of Theorem 3.2.1 can be found in the appendix.

Relative to the results that can be obtained under standard contraction conditions (see, e.g., Bertsekas (2013)), the only significant weakening of the main findings is that T and T_σ are eventually contracting, rather than always contracting in one step. Such an outcome cannot be avoided when values of the discount factor greater than one are admitted.

[7]More precisely, a self-map M on metric space (Y, ρ) is called eventually contracting if there exists an n in $\mathbb{N}$ and a $\lambda < 1$ such that $\rho(M^n y, M^n y') \leq \lambda \rho(y, y')$ for all y, y' in Y.

The eventual discounting condition is, in many cases, not just sufficient but also necessary for the dynamic program to be well defined and the optimality results to hold. Appendix 3.9.7 provides additional discussion.

3.2.5. Blackwell's Condition. Blackwell's sufficient condition for a contraction has a natural analogue in the case of state-dependent discounting. As shown in Proposition 3.9.4, if the Bellman operator satisfies

$$[T(v+c)](x,z) \leq (Tv)(x,z) + \beta(z) \int c(z')Q(z,dz') \qquad ((x,z) \in \mathsf{S})$$

for all $c \in bm\mathsf{Z}_+$ where (β, Q) is eventually discounting, then T is eventually contracting on $bc\mathsf{S}$. As a consequence, T has a unique fixed point in $bc\mathsf{S}$ that is globally attracting under iteration of T. This extends Blackwell's original result,[8] with the caveat that T might not itself be a contraction. Again, this cannot be avoided when β is allowed to take values greater than one.[9]

3.2.6. Monotonicity, Concavity and Differentiability. Next we show that standard results on monotonicity, concavity, and differentiability of the value function (cf, e.g., Stokey and Lucas (1989)) are preserved under state-dependent discounting without additional assumptions on the discount factor process. We assume that X is a convex subset of $\mathbb{R}$ in the discussion below and denote $ibc\mathsf{S}$ the set of functions in $bc\mathsf{S}$ that are increasing and concave in x.

Assumption 3.2.2. For all $v \in ibc\mathsf{S}$ and $z \in \mathsf{Z}$, (i) $x \mapsto H(x,z,x',v)$ is increasing for all $x' \in \Gamma(x,z)$, (ii) $(x,x') \mapsto H(x,z,x',v)$ is strictly concave, (iii) $\Gamma(x,z) \subset \Gamma(y,z)$ for all $x \leq y$, and (iv) the set $\{(x,x') : x' \in \Gamma(x,z)\}$ is convex.

Assumption 3.2.3. The map $x \mapsto H(x,z,x',v)$ is continuously differentiable on $\operatorname{int}\mathsf{X}$ for all $z \in \mathsf{Z}$, $x' \in \operatorname{int}\Gamma(x,z)$, and $v \in ibc\mathsf{S}$.

[8] The original result states that if an operator T is monotone and there exists a $b \in (0,1)$ such that $T(v+c) \leq Tv + bc$ for all $c \geq 0$, then T is a contraction (see, e.g., Stokey and Lucas, 1989, Theorem 3.3).

[9] In fact, when T is an eventual contraction on a Banach space, one can construct a complete metric on the same space under which T is a contraction. See, for example, Krasnosel'skii et al. (1972). Our terminology on contractions in this section refers specifically to the supremum norm.

The following theorem shows that the value function v^* is increasing, strictly concave, and continuously differentiable in x under standard assumptions.[10]

THEOREM 3.2.2. *If $\mathcal{D}$ is regular and Assumptions 3.2.1–3.2.2 hold, then $x \mapsto v^*(x, z)$ is increasing and strictly concave and $x \mapsto \sigma^*(x, z)$ is single-valued and continuous for all $z \in \mathsf{Z}$. If, in addition, Assumption 3.2.3 holds, then $x \mapsto v^*(x, z_0)$ is continuously differentiable at x_0 whenever $x_0 \in \operatorname{int} \mathsf{X}$ with $\sigma^*(x_0, z_0) \in \operatorname{int} \Gamma(x_0, z_0)$ for some z_0, and*

$$v_x^*(x_0, z_0) = H_x(x_0, z_0, \sigma^*(x_0, z_0), v^*).$$

Additional comments on these assumptions and results can be found in the applications.

3.2.7. Optimality over Nonstationary Policies. For the sake of simplicity, we have been restricting our attention to optimality over stationary policies—that is, the agent chooses the same policy in every period. In fact, if we define λ-value function for a sequence of feasible policies $\lambda = (\sigma_0, \sigma_1, \dots)$ by

$$v_\lambda(x, z) := \liminf_{n \to \infty} (T_{\sigma_0} T_{\sigma_1} \dots T_{\sigma_n} v_0)(x, z),$$

we can reformulate the agent's problem taking into account nonstationary policies. The value function over nonstationary policies is defined by

$$\tilde{v}^*(x, z) = \sup_{\{\sigma_t \in \Sigma\}_{t=0}^{\infty}} v_\lambda(x, z), \quad \forall x \in \mathsf{X}, z \in \mathsf{Z}. \tag{3.9}$$

We have the following proposition.

PROPOSITION 3.2.3. *If $\mathcal{D}$ is regular and Assumption 3.2.1 holds, then $v^* = \tilde{v}^*$.*

Proposition 3.2.3 together with Theorem 3.2.1 says that the optimal stationary policy is also optimal over nonstationary policies.

[10] If $\mathcal{D}$ is additively separable, sufficiency of the Euler equations and transversality conditions can also be established, analogous to Section 9.5 of Stokey and Lucas (1989).

3.3. Examples

In this section we discuss examples of dynamic programs with state-dependent discounting in settings where rewards are bounded and the aggregator are additively separable. Extensions to unbounded rewards and recursive preferences are deferred to Sections 3.5 and 3.6.

3.3.1. A Generic Additively Separable Problem. Consider the dynamic program in Section 9.2 of Stokey and Lucas (1989) with the addition of state-dependent discounting. The objective is to maximize

$$\mathbb{E} \sum_{t=0}^{\infty} \prod_{i=0}^{t-1} \beta_i \, F(X_t, Z_t, X_{t+1}) \quad \text{s.t. } X_{t+1} \in \Gamma(X_t, Z_t) \text{ for all } t \geq 0. \tag{3.10}$$

As in Stokey and Lucas (1989), F is assumed to be bounded and continuous on $\operatorname{gr}\Gamma$, while Γ is a continuous, nonempty, and compact-valued correspondence. We set $\beta_t = \beta(Z_t)$ where β is continuous, bounded and nonnegative, while $\{Z_t\}$ is Markov with Feller kernel Q.

We connect this dynamic program to our framework by setting $\mathcal{D} = (\Gamma, H)$ with

$$H(x, z, x', v) := F(x, z, x') + \beta(z) \int v(x', z') Q(z, dz') \tag{3.11}$$

for all $v \in bm\mathsf{S}$. The monotonicity condition (3.2) is clearly satisfied. The function $(x, z, x') \mapsto H(x, z, x', v)$ is bounded and Borel measurable on $\operatorname{gr}\Gamma$ because v and F have these properties, and continuous when v is continuous by the Feller property (see footnote 6). Hence $\mathcal{D}$ is regular.

If (β, Q) is eventually discounting then Assumption 3.2.1 holds, since (3.11) yields

$$|H(x, z, x', v) - H(x, z, x', w)| \leq \beta(z) \left| \int [v(x', z') - w(x', z')] Q(z, dz') \right|,$$

and an application of the triangle inequality gives (3.8).

To connect this application with the definition of optimality given in Section 3.2.2, fix $\sigma \in \Sigma$ and $v \in bm\mathsf{S}$. The policy operator T_σ from (3.4) can be expressed as

$$(T_\sigma v)(x_0, z_0) = F(x_0, z_0, \sigma(x_0, z_0)) + \beta(z_0)\, \mathbb{E}_0\, v(X_1, Z_1) \tag{3.12}$$

where $\{X_t\}$ is generated by $X_{t+1} = \sigma(X_t, Z_t)$, the initial condition is $(X_0, Z_0) = (x_0, z_0)$, and $\mathbb{E}_t$ conditions on $\{Z_i\}_{i \le t}$. If we take T_σ, iterate forward n times and apply the law of iterated expectations, we obtain

$$(T_\sigma^n v)(x_0, z_0) = \mathbb{E}_0 \sum_{t=0}^{n-1} \prod_{i=0}^{t-1} \beta_i F(X_t, Z_t, X_{t+1}) + \mathbb{E}_0 \prod_{i=0}^{n} \beta_i v(X_n, Z_n). \tag{3.13}$$

Recall from (3.6) that, to obtain the value v_σ of the policy σ, we take the limit of (3.13) in n. Eventual discounting implies that the second term vanishes as $n \to \infty$.[11] In the limit we obtain as v_σ the value in (3.10) under the policy σ. Maximizing over σ in Σ yields the optimal policy.

The Bellman operator corresponding to $\mathcal{D}$ is the map T on $bc\mathsf{S}$ defined by

$$(Tv)(x, z) = \max_{x' \in \Gamma(x,z)} \left\{ F(x, z, x') + \beta(z) \int v(x', z') Q(z, dz') \right\}. \tag{3.14}$$

Since the conditions of Theorem 3.2.1 are satisfied, the unique fixed point of T in $bc\mathsf{S}$ is $v^* := \sup_{\sigma \in \Sigma} v_\sigma$, the value function of $\mathcal{D}$. Bellman's principle of optimality applies and an optimal policy can be computed by either value function iteration or Howard's policy iteration algorithm. Monotonicity, concavity and differentiability of v^* can be obtained by imposing the same conditions that Stokey and Lucas (1989) impose on F and Γ and then applying Theorem 3.2.2.

3.3.2. One-Sector Stochastic Optimal Growth. Consider first the one-sector stochastic optimal growth model as found in, say, Stokey and Lucas (1989), except that the discount rate is state-dependent. The agent solves

$$\max_{\{C_t, K_t\}_{t=0}^{\infty}} \mathbb{E} \left\{ \sum_{t=0}^{\infty} \prod_{i=0}^{t-1} \beta(Z_i)\, u(C_t) \right\}$$

[11] This term is dominated by $r_{n+1}^\beta \|v\|$. Hence it suffices to prove that $r_n^\beta \to 0$ as $n \to \infty$. Eventual discounting implies that $r_n^\beta < 1$ for some n, and, as shown in Proposition 3.4.1 below, this in turn gives $\lim_{n\to\infty} (r_n^\beta)^{1/n} < 1$. But then $r_n^\beta \to 0$, as was to be shown.

subject to $C_t = f(K_t, Z_t) - K_{t+1} \geq 0$, where K_t, C_t, and Z_t are consumption, capital, and exogenous shocks at time t, respectively; u is a one-period return function and f is a production function. This problem can be mapped to the framework provided in the previous section by taking capital as the endogenous state and defining the continuation aggregator by

$$H(x, z, x', v) = u(f(x, z) - x') + \beta(z) \int v(x', z') Q(z, dz').$$

The endogenous state space X can be set to $\mathbb{R}_+$ and Z to be some arbitrary metric space. The next period state x' is chosen from the feasible correspondence $\Gamma(x, z) = [0, f(x, z)]$. Note that while the same exogenous state z affects both production and the discount rate under this formulation, there is no loss of generality because the space Z is arbitrary and hence can support multiple independent processes.

If u and f satisfy standard conditions, as in, say, Section 5.1 of Stokey and Lucas (1989), then the dynamic program is regular. The previous section implies that we need only check the eventual discounting condition before applying Theorem 3.2.1. This will be discussed in detail in Section 3.4.

3.3.3. A Household Savings Problem. The dynamic program associated with the household problem in Hubmer et al. (2020) can also be placed within our framework. The continuation aggregator takes the form

$$H(x, z, x', v) = u(R(x, z)x + y(x, z) - x') + \beta(z) \int v(x', z') Q(z, dz') \tag{3.15}$$

where $x \in \mathsf{X} := \mathbb{R}_+$ is current assets, z is a vector of exogenous shocks taking values in $\mathbb{R}^k$, $R(x, z)$ is the gross rate of return on asset holdings (which depends on both exogenous shocks and current asset holdings) and $y(x, z)$ is labor income net of income tax and capital gains tax, as well as a lump sum transfer. The utility function is

$$u(c) := \frac{c^{1-\gamma}}{1-\gamma} \quad \text{where} \quad \gamma > 1. \tag{3.16}$$

Next period assets x' are constrained to lie in

$$\Gamma(x,z) := \{x' \in \mathbb{R} : \bar{x} \leq x' \leq R(x,z)x + y(x,z)\}. \tag{3.17}$$

This problem is not regular because H is not bounded, since u is unbounded below. However, in solving this dynamic program, Hubmer et al. (2020) reduce both the asset space X and the exogenous shock space Z to a finite grid. The aggregator is then bounded and the continuity parts of the regularity condition are automatically satisfied (under the discrete topology). Hence, to show that all of the conclusions of Theorem 3.2.1 apply, we need only verify that eventual discounting holds. This issue is discussed for the parameterization in Hubmer et al. (2020) in Section 3.4 below.

3.3.4. Job Search. Our framework is also able to deal with optimal stopping problems with proper definitions of the primitives. In this section, we demonstrate this using an elementary job search problem in McCall (1970) except that the agent has state-dependent discount factors. The basic structure considered here can be modified to deal with more complicated optimal stopping problems, such as Lucas and Prescott (1974) and Robin (2011).

An unemployed worker searching for a job is given a wage offer every period. He can accept the offer and receive this wage every period forever, or he can choose to receive an unemployment compensation c and wait for another offer next period. Uncertainty is driven by a Markov process on a metric space Z with stochastic kernel Q. The wage offer is given by a function $w : \mathsf{Z} \to \mathbb{R}_+$. Since the discount factors are state dependent, the lifetime utility of accepting an offer at state z is $K(z)w(z)$ where

$$K(z) := \sum_{t=0}^{\infty} \left(\mathbb{E}_z \prod_{i=0}^{t-1} \beta(Z_i) \right), \quad \forall z \in \mathsf{Z}.$$

The Bellman equation is thus

$$v(z) = \max \left\{ w(z)K(z), c + \beta(z) \int_{\mathsf{Z}} v(z')Q(z, dz') \right\}.$$

We can study this problem in a generalized version of the framework in 3.2.1. Instead of directly choosing the next period state, the agent at state $(x, z) \in \mathsf{S}$ chooses an action a from an action space A, which is a separable metric space, subject to $a \in \Gamma(x, z)$. Then $H(x, z, a, v)$ is interpreted as the value obtained by choosing action a under the continuation value v given current state (x, z).

For the job search problem, we let $\mathsf{S} = \mathsf{Z}$, $\mathsf{A} = \{0, 1\}$, $\Gamma \equiv \mathsf{A}$, and

$$H(x, z, a, v) = aw(z)K(z) + (1 - a) \left[c + \beta(z) \int_{\mathsf{Z}} v(z')Q(z, dz') \right].$$

Then $\mathcal{D} = (H, \Gamma)$ is the associated dynamic program. Note that in this setting, the endogenous state space X is redundant. In particular, the Bellman operator defined from $\mathcal{D}$ is

$$\begin{aligned}
(Tv)(z) &= \max_{a \in \{0,1\}} \left\{ aw(z)K(z) + (1 - a) \left[c + \beta(z) \int_{\mathsf{Z}} v(z')Q(z, dz') \right] \right\} \\
&= \max \left\{ w(z)K(z), c + \beta(z) \int_{\mathsf{Z}} v(z')Q(z, dz') \right\}.
\end{aligned}$$

We have the following proposition.

PROPOSITION 3.3.1. *If (i) w and β are bounded and continuous, (ii) (β, Q) is eventually discounting, and (iii) Q has the Feller property, then $\mathcal{D}$ is regular and Assumptions 3.2.1 holds. In particular, the conclusions of Theorems 3.2.1 are valid.*

3.4. The Discount Condition

In this section we discuss tests for the eventual discounting condition and develop intuition regarding its value.

3.4.1. Connection to Spectral Radii.

Given β and Q as in Assumption 3.2.1, let $L_\beta \colon bm\mathsf{Z} \to bm\mathsf{Z}$ be the *discount operator* defined by

$$(L_\beta h)(z) = \beta(z) \int h(z')Q(z, dz') \qquad (h \in bm\mathsf{Z},\ z \in \mathsf{Z}). \tag{3.18}$$

The next proposition shows that we can test Assumption 3.2.1 by computing the spectral radius $r(L_\beta)$ of the operator L_β.[12] In stating it, we set $\beta_t := \beta(Z_t)$ where $\{Z_t\}$ is a Z-valued Markov process generated by Q.

Proposition 3.4.1. *The spectral radius of L_β satisfies $r(L_\beta) = \lim_{n\to\infty}(r_n^\beta)^{1/n}$. Moreover, (β, Q) is eventually discounting if and only if $r(L_\beta) < 1$.*

The expression for $r(L_\beta)$ in Proposition 3.4.1 is obtained through a local spectral radius condition for positive linear operators. It provides both a simple representation of the spectral radius of L_β and a link to eventual discounting. For example, it is immediate from $r(L_\beta) = \lim_{n\to\infty}(r_n^\beta)^{1/n}$ that $r_n^\beta \to 0$ when $r(L_\beta) < 1$. This, in turn, implies that (β, Q) is eventually discounting. The converse implication is more subtle and involves the Markov property. Details are in the appendix.

3.4.2. Finite Exogenous State. Testing eventual discounting is simple when Z is finite. In this case, Q can be represented as a Markov matrix of values Q_{ij}, giving the one-step probability of transitioning from z_i to z_j, and L_β can be represented as the matrix

$$L_\beta := (\beta_i Q_{ij})_{1\le i,j\le N}\,. \tag{3.19}$$

Here $\beta_i := \beta(z_i)$ and N is the number of elements in Z. The spectral radius $r(L_\beta)$ is equal to the dominant eigenvalue of L_β, which is real and nonnegative by the Perron–Frobenius Theorem. In view of Proposition 3.4.1, eventual discounting holds if and only if this eigenvalue is strictly less than unity.

Example 3.4.1. Christiano et al. (2011) consider the case $\beta_t \in \{\beta^\ell, \beta^h\}$ with $\beta^\ell < 1 < \beta^h$. The process $\{\beta_t\}$ stays at β^h with probability p and shifts permanently to β^ℓ with probability $1-p$. Thus, by (3.19),

$$L_\beta = \begin{pmatrix} \beta^\ell & 0 \\ (1-p)\beta^h & p\beta^h \end{pmatrix}.$$

[12] As usual, the spectral radius of a bounded linear operator L from a Banach space B to itself is given by $r(L) := \lim_{n\to\infty}\|L^n\|^{1/n}$, where $\|\cdot\|$ is the operator norm. This limit always exists and is equal to $\inf_{n\in\mathbb{N}}\|L^n\|^{1/n}$. If B is finite dimensional, it equals the maximal modulus of the eigenvalues of L. See, for example, Bühler and Salamon (2018), Theorem 1.5.5.

The eigenvalues are β^ℓ and $p\beta^h$, so $r(L_\beta)$ is the maximum of these values. Since $\beta^\ell < 1$, eventual discounting holds if and only if $p\beta^h < 1$. The condition is violated if the state β^h is too large or too persistent. Christiano et al. (2011) set $\beta^h = 1.02$ and consider $p \leq 0.82$, so eventual discounting is satisfied. Since their household problem can be treated in the same way as in Section 3.3.3, all of our results in Section 3.2.4 apply.

3.4.3. Stationary Spectral Radius. The expression obtained for $r(L_\beta)$ in Proposition 3.4.1 is a geometric mean, and hence is determined by the asymptotic behavior of the discount process. When $\{Z_t\}$ is irreducible, it seems likely that these asymptotics will be independent of the initial condition z. This suggests that the conditional expectation and supremum in the definition of r_n^β can be replaced by the unconditional expectation $\mathbb{E}$ for the stationary process. The next proposition confirms this intuition.

PROPOSITION 3.4.2. *If Z is finite and the exogenous state process $\{Z_t\}$ is irreducible, then $r(L_\beta)$ satisfies the stationary representation*

$$r(L_\beta) = s^\beta \quad \textit{where} \quad s^\beta := \lim_{n\to\infty} (s_n^\beta)^{1/n} \quad \textit{with} \quad s_n^\beta := \mathbb{E} \prod_{t=0}^{n-1} \beta_t. \tag{3.20}$$

Our analysis below shows that this stationary representation is also highly accurate even when Z is infinite, provided that $\{Z_t\}$ is irreducible and sufficiently mean reverting for dependence on initial conditions to die out. This is helpful because the stationary representation of $r(L_\beta)$ sometimes admits analytical solutions that facilitate benchmark calculations and enhance intuition.[13]

3.4.4. Autoregressive Specifications. Some studies adopt discount processes that are autoregressive in levels or logs (e.g., Hubmer et al., 2020; Hills et al., 2019; Nakata, 2016) and then discretize them prior to computation. Such specifications always fit the dynamic programming framework adopted above after discretization.[14] The only remaining issue is whether or not eventual discounting holds. For common reference, all examples

[13]While finiteness of the state space can be weakened, as discussed above, irreducibility is essential. To see this, consider the application in Christiano et al. (2011), where the unique stationary distribution puts all mass on the low state and irreducibility fails. With all mass on the low state we have $s_n^\beta = (\beta^\ell)^n$ for all n, and hence $s^\beta = \beta^\ell$, which differs from $r(L_\beta) = \max\{\beta^\ell, p\beta^h\}$.

[14]Recall that β is assumed to be bounded and continuous in Assumption 3.2.1. Both conditions hold after discretization. (Continuity holds automatically under the discrete topology.)

use the state process

$$Z_{t+1} = \rho Z_t + (1-\rho)\mu + \sigma_\epsilon \epsilon_{t+1}, \quad \{\epsilon_t\} \overset{\text{IID}}{\sim} N(0,1). \tag{3.21}$$

3.4.4.1. *AR(1) in Levels.* We first give examples where β_t is a multiple of Z_t. After following the discretization procedure used by the authors, we calculate the spectral radius of the matrix (3.19).

EXAMPLE 3.4.2. Hubmer et al. (2020) take the AR(1) specification $\beta_t = Z_t$ where $\{Z_t\}$ follows (3.21) with $\rho = 0.992$, $\mu = 0.944$ and $\sigma_\epsilon = 0.0006$ and discretize the process onto a grid of 15 states via Tauchen's method. This gives $r(L_\beta) = 0.9469$, so eventual discounting holds. This is as expected, since the mean μ is substantially less than one and low volatility suggests that the impact of stochastic variation is minor.

EXAMPLE 3.4.3. In Hills et al. (2019), the discount process is $\beta_t = bZ_t$ where $\{Z_t\}$ obeys (3.21). They consider several parameterizations, the most empirically motivated of which is $\mu = 1$, $b = 0.99875$, $\rho = 0.85$ and $\sigma_\epsilon = 0.0062$. Under this parameterization β_t regularly exceeds one, as observed in the simulated process shown in Figure 1.1. Nonetheless, after following their discretization procedure and computing the spectral radius of L_β, we find $r(L_\beta) = 0.9996$, so eventual discounting holds.

EXAMPLE 3.4.4. In a similar setting to Example 3.4.3, Nakata (2016) assumes $\beta_t = bZ_t$ where $\{Z_t\}$ follows (3.21), $\mu = 1$, $b = 0.995$, $\rho = 0.85$, and $\sigma_\epsilon = 0.00395$. The process is discretized onto a grid of 501 points, yielding $r(L_\beta) = 0.9953$.

To illustrate how the stochastic properties of β_t affect the size of $r(L_\beta)$, we take the parameterization in Example 3.4.3 as a benchmark and vary the persistence term ρ and the volatility σ_ϵ. Other parameters are held constant. Figure 3.1 plots the resulting values of $r(L_\beta)$. The figure shows that higher volatility and higher persistence both increase $r(L_\beta)$, leading to a failure of eventual discounting when $r(L_\beta) \geq 1$. Note also that there is a positive interaction between persistence and volatility, with the effect of each parameter enhanced by the other.

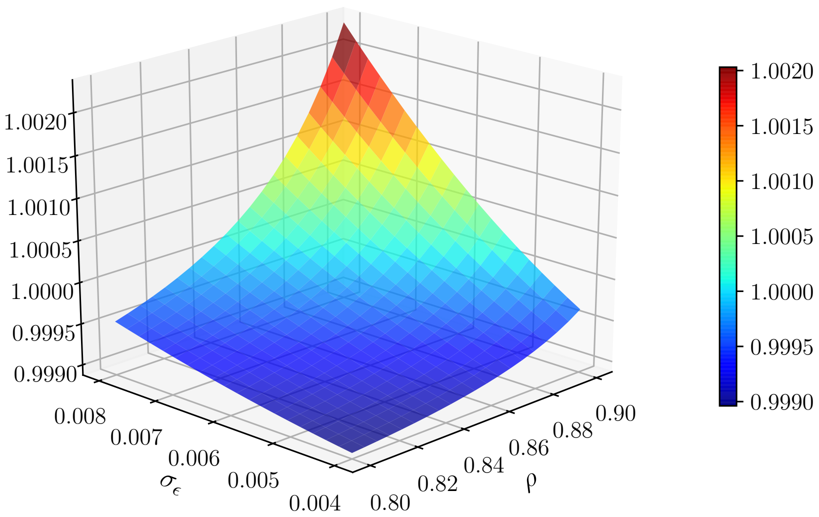


FIGURE 3.1. $r(L_\beta)$ as a function of ρ and σ_ϵ; $\mu = 0.944$

Some further insight can be gained by considering the expected two period discount factor when $\beta_t = Z_t$ and $\{Z_t\}$ is as given in (3.21). Under the stationary distribution, which governs asymptotic outcomes, this evaluates to

$$\mathbb{E}\beta_t\beta_{t+1} = \mu^2 + \rho\frac{\sigma_\epsilon^2}{1-\rho^2}. \tag{3.22}$$

The value in (3.22) depends on the sign of ρ. Positive correlation combined with positive volatility in the state process leads to a value greater than the stationary mean. This is because, under positive correlation, positive deviations from the mean tend to occur consecutively and reinforce each other.

3.4.4.2. *AR(1) in Logs.* Next we set $\beta_t := \exp(Z_t)$ where $\{Z_t\}$ obeys the AR(1) specification (3.21). This specification is arguably more natural than the direct AR(1) approach discussed above due to positivity. While the state space is not finite, irreducibility of $\{Z_t\}$ leads us to conjecture that an approximate version of Proposition 3.4.2 holds, so that the stationary geometric mean $s^\beta = \lim_{n\to\infty}(s_n^\beta)^{1/n}$ for the original process will be close to $r(L_\beta) = \lim_{n\to\infty}(r_n^\beta)^{1/n}$ when the latter is calculated using an appropriately discretized version of the process. As shown in Appendix 3.9.6, for the original process we have

$$s^\beta = \lim_{n\to\infty}(s_n^\beta)^{1/n} = \lim_{n\to\infty}\left(\mathbb{E}\prod_{t=0}^{n-1}\beta_t\right)^{1/n} = \exp\left\{\mu + \frac{\sigma_\epsilon^2}{2(1-\rho)^2}\right\}. \tag{3.23}$$

TABLE 3.1. Comparison of s^β and $r(L_\beta)$ after discretization

Parameters		N=10		N=200	
$\mu = -0.05$	s^β	$r(L_\beta)$	Error	$r(L_\beta)$	Error
$\rho = 0.90,\ \sigma_\epsilon = 0.01$	0.956	0.956	2.5e-05	0.956	1.1e-06
$\rho = 0.90,\ \sigma_\epsilon = 0.02$	0.970	0.970	3.9e-04	0.970	1.8e-05
$\rho = 0.92,\ \sigma_\epsilon = 0.01$	0.959	0.959	7.6e-05	0.959	3.5e-06
$\rho = 0.92,\ \sigma_\epsilon = 0.02$	0.981	0.980	1.2e-03	0.981	5.8e-05
$\rho = 0.94,\ \sigma_\epsilon = 0.01$	0.965	0.964	3.2e-04	0.965	1.5e-05
$\rho = 0.94,\ \sigma_\epsilon = 0.02$	1.006	1.001	4.7e-03	1.005	2.5e-04

Numerical experiments show that the expression on the right hand side of (3.23) provides a good approximation of $r(L_\beta)$ even when the discretization is relatively coarse, and an almost perfect approximation when the discretization is fine. Table 3.1 illustrates by comparing s^β given by (3.23) and $r(L_\beta)$ under two different levels of discretization, for a range of parameter values.[15]

Given this tight relationship between s^β and $r(L_\beta)$, we can use (3.23) to examine how the parameters of the state process affect eventual discounting. Consistent with our previous findings, the expression in (3.23) indicates that $r(L_\beta)$ is increasing in all of the three parameters (although the effect is now exponential). Higher persistence and higher volatility reinforce each other. The impact of ρ is nonlinear and large in the neighborhood of unity.

3.5. Unbounded Rewards

In this section we show that the optimality results presented above extend to a range of unbounded reward settings after suitable modifications. We consider the additively separable aggregator

$$H(x, z, x', v) = u(x, z, x') + \beta(z) \int v(x', z') Q(z, dz'). \tag{3.24}$$

The continuation value function v is in $\mathcal{V}$, which is the set of all *candidate value functions* and varies across applications. As before, $\beta \in bc\mathsf{Z}$ and Q is a Feller transition kernel. The

[15] N is the number of grid points. We use the Rouwenhorst's method for discretization, which has strong asymptotic properties in terms of approximating the distributions of Gaussian AR(1) processes (Kopecky and Suen, 2010). We fix μ because it has no effect on the errors.

feasible correspondence Γ is assumed to be continuous, nonempty, and compact valued. The reward function u is continuous but not necessarily bounded. The Euclidean norm is represented by $|\cdot|$.

3.5.1. Homogeneous Functions. We begin by extending the core results of Alvarez and Stokey (1998) to the case of state-dependent discounting. We consider reward functions that are homogeneous of degree $\theta \in (0, 1]$ and feasible correspondences that are homogeneous of degree one.[16]

ASSUMPTION 3.5.1. X is a convex cone in $\mathbb{R}^k_+$ and $\lambda x' \in \Gamma(\lambda x, z)$ when $(x, z, x') \in \operatorname{gr}\Gamma$ and $\lambda \geq 0$. For each $z \in \mathsf{Z}$, $u(\cdot, z, \cdot)$ is homogeneous of degree θ, and there exists a $B > 0$ such that

$$|u(x, z, x')| \leq B(|x| + |x'|)^\theta \text{ for all } (x, z, x') \in \operatorname{gr}\Gamma.$$

Assumption 3.5.1 follows Alvarez and Stokey (1998). The next assumption generalizes their growth restriction to problems with state-dependent discounting.

ASSUMPTION 3.5.2. There exists an $\alpha \geq 0$ in $bm\mathsf{Z}$ such that $|x'| \leq \alpha(z)|x|$ when $(x, z, x') \in \operatorname{gr}\Gamma$. In addition, for $\{Z_t\}$ generated by Q,

$$\sup_{z \in \mathsf{Z}} \mathbb{E}_z \prod_{t=0}^{n-1} \beta(Z_t)\alpha^\theta(Z_t) < 1 \text{ for some } n \in \mathbb{N}. \tag{3.25}$$

The function α is a state-dependent upper bound on the growth rate of the state variable. Comparing to the eventual discounting condition in Section 3.2.3, the extra term $\alpha^\theta(Z_t)$ in (3.25) reflects the need to take into account the growth restriction when the reward function is homogeneous and unbounded above. If both β and α are constant, then (3.25) reduces to the condition $\alpha^\theta \beta < 1$ used in Alvarez and Stokey (1998).

In household problems where the state is asset holdings, the gross asset return bounds the growth rate of the state. The condition in (3.25) implies that the shocks to the discount factor and asset return have a similar effect on eventual discounting, but their relative importance depends on the degree of homogeneity of the reward function.

[16] Recall that a real-valued f defined on a convex cone C of $\mathbb{R}^k$ is homogeneous of degree θ if $f(\lambda x) = \lambda^\theta f(x)$ for all $\lambda \geq 0$ and $x \in C$.

Let $(h_\theta \mathsf{S}, \|\cdot\|_h)$ be the space of continuous functions on S that are homogeneous of degree θ in x and bounded in the norm defined by

$$\|f\|_h := \sup\{|f(x,z)| : z \in \mathsf{Z},\, x \in \mathsf{X},\, |x| = 1\}. \tag{3.26}$$

Then $h_\theta \mathsf{S}$ is a Banach space (Stokey and Lucas, 1989). To make the problem well defined, we let $v_0 \equiv \mathbf{0}$ so the σ-value function is given by $v_\sigma := \lim_n (T_\sigma \mathbf{0})$.

PROPOSITION 3.5.1. *Let $\mathcal{V} = h_\theta \mathsf{S}$. Under Assumptions 3.5.1–3.5.2, the lifetime value v_σ is well defined and finite on S for any feasible policy σ, the value function v^* is a unique fixed point of T on $\mathcal{V}$, $T^n v \to v^*$ for all $v \in \mathcal{V}$, there exists an optimal policy that is homogeneous of degree one, and the principle of optimality holds.*

EXAMPLE 3.5.1. Consider the household saving problem in Toda (2019) where the exogenous state $\{Z_t\}$ is Markovian on Z with stochastic kernel Q. The asset return R and discount function β are bounded continuous functions of Z_t. The utility function is $u(c) = c^{1-\gamma}/(1-\gamma)$ with $\gamma \in (0,1)$. The budget constraint is $X_{t+1} = R(Z_t)(X_t - C_t) \geq 0$ where X_t is the beginning-of-period wealth and C_t is consumption. The Bellman equation is

$$v(x,z) = \max_{c, x' \geq 0} \left\{ u(c) + \beta(z) \int v(x', z')\, Q(z, dz') \right\} \quad \text{s.t. } \; x' = R(z)(x - c).$$

If we use the constraint to eliminate c in the Bellman equation and let $\Gamma(x,z) = [0, R(z)x]$, then Assumption 3.5.1 is satisfied with $\theta = 1-\gamma$ and $B = 1/(1-\gamma)$. By Proposition 3.4.1, Assumption 3.5.2 holds if $r(L_\alpha) < 1$ with L_α defined by

$$(L_\alpha h)(z) := \beta(z) R^{1-\gamma}(z) \int h(z') Q(z, dz'),$$

where we let the upper bound function $\alpha = R$. This is a direct extension of the results in Toda (2019) to the case of infinite Z. In particular, the condition $r(L_\alpha) < 1$ reduces to the condition in Proposition 1 of Toda (2019) whenever Z is finite.

3.5.2. Local Contractions. Next we adopt a local contraction approach to dynamic programs with state dependent discounting and unbounded rewards, extending methods first developed in Rincón-Zapatero and Rodríguez-Palmero (2003). As in the previous section, the aggregator has the form of (3.24).

Let $c\mathsf{S}$ be all continuous functions on S. Let Z be compact and write $\mathsf{X} = \bigcup_j \operatorname{int} K_j$ where $\{K_j\}$ is a sequence of strictly increasing and compact subsets of X. Let

$$\|f\|_j := \sup_{x \in K_j, z \in \mathsf{Z}} |f(x, z)| \qquad (f \in c\mathsf{S}).$$

Let $c > 1$ and $\{m_j\}$ be an unbounded sequence of increasing positive real numbers. Let $c_m\mathsf{S}$ be all $f \in c\mathsf{S}$ such that

$$\|f\|_m := \sum_{j=1}^{\infty} \frac{\|f\|_j}{m_j c^j} < \infty.$$

The pair $(c_m\mathsf{S}, \|\cdot\|_m)$ forms a Banach space (Matkowski and Nowak, 2011).

Assumption 3.5.3. $\Gamma(x, z) \subset K_j$ for all $x \in K_j$, all $z \in \mathsf{Z}$, and all $j \in \mathbb{N}$, and (β, Q) is eventually discounting in the sense of Section 3.2.3.

Proposition 3.5.2. *Under Assumption 3.5.3, the lifetime value v_σ is well defined and finite on S for any $\sigma \in \Sigma$, there exists a sequence $m_j \uparrow \infty$ such that the value function v^* is the unique fixed point of T on $c_m\mathsf{S}$, $T^n v \to v^*$ for all $v \in c_m\mathsf{S}$, there exists an optimal policy, and the principle of optimality holds.*

Example 3.5.2. Consider a stochastic optimal growth model with state dependent discounting, total production $zf(x)$ and continuous utility u. The feasible correspondence is $\Gamma(x, z) = [0, zf(x)]$. Let $\mathsf{X} = \mathbb{R}_+$ and let $\mathsf{Z} \subset \mathbb{R}_+$ be compact. Suppose $f' > 0$, $f'' < 0$ and $\lim_{x \to \infty} f'(x) = 0$. Let $\{K_j\}$ be an increasing sequence of compact sets covering X such that $\Gamma(x, z) \subset K_j$ for all $x \in K_j$.[17] Assumption 3.5.3 holds and Proposition 3.5.2 can be applied if (β, Q) is eventual discounting.

3.6. Further Extensions

We study two further extensions. Section 3.6.1 studies an alternative discount specification to the framework in Section 3.2. Section 3.6.2 extends our main results to Epstein-Zin preferences with unbounded rewards.

[17] For example, set $K_j := [0, M + j]$ for all $j \in \mathbb{N}$, where M is some large constant.

3.6.1. Alternative Discount Specifications. Discounting methods that differ from the preceding framework can also be analyzed. To illustrate, we consider the shocks to long-run discount factors found in Primiceri et al. (2006), Justiniano et al. (2010), Leeper et al. (2010), and Christiano et al. (2014). Their maximization problems are analogous to the additively separable problem in Section 3.3.1, with the difference that $\prod_{t=0}^{n-1} \beta_t$ is replaced by $b^n Z_n$ for some constant b. While the discount factor $b^n Z_n$ can be expressed as $\prod_{t=0}^{n-1} \beta_t$ after setting $\beta_t := bZ_{t+1}/Z_t$ and $Z_0 = 1$, notice that β_t is not observable until $t+1$. Hence inequality (3.8) cannot be used, since it assumes that β_t is visible at t.

To handle such cases, one option is to replace inequality (3.8) with

$$|H(x, z, x', v) - H(x, z, x', w)| \leq \int \beta(z')|v(x', z') - w(x', z')|Q(z, dz'). \tag{3.27}$$

Inequality (3.27) integrates over $\beta(z')$, supposing that its realization is not observed at the time that x' is chosen. We prove in the appendix that Theorem 3.2.1 extends to this case: the theorem is valid under eventual discounting when (3.27) replaces (3.8).

The set up of Primiceri et al. (2006) and other authors mentioned above satisfies (3.27) after redefining the aggregator and the exogenous state variable.[18] The only question, then, is whether or not eventual discounting holds. The following proposition shows that, in many cases, the answer depends only on the value of b in $\beta_t := bZ_{t+1}/Z_t$. Stochastic components are irrelevant.

PROPOSITION 3.6.1. *If $\beta_t := bZ_{t+1}/Z_t$ for all t and $\{Z_t\}$ is positive and bounded, then eventual discounting holds if and only if $b < 1$.*

The intuition behind Proposition 3.6.1 is that the spectral radius $r(L_\beta)$ equals the asymptotic growth rate of the discount factor process. If $\prod_{t=0}^{n-1} \beta_t = b^n Z_n$ and Z_t is positive and bounded, the asymptotic growth rate is equal to b.

[18]To be specific, let the exogenous state variable be $\tilde{Z}_{t+1} = (Z_{t+1}, Z_t)$. The aggregator then becomes $H(x, z, x', v) = \tilde{F}(x, z, x') + \int \beta(z')v(x', z')\tilde{Q}(z, dz')$, where $\tilde{F}(X_t, \tilde{Z}_t, X_{t+1}) = F(X_t, Z_t, X_{t+1})$, $\beta(\tilde{Z}_{t+1}) = bZ_{t+1}/Z_t$, and $\tilde{Q}$ is the transition kernel on $\tilde{\mathsf{Z}} := \mathsf{Z}^2$ induced by Q.

3.6.2. Epstein-Zin Preferences. Next we extend the preceding results on dynamic programming under state-dependent discounting to settings where lifetime utility is governed by Epstein–Zin preferences. Lifetime utility of an agent satisfies

$$U(C_t, C_{t+1}, \ldots) = \left\{ C_t^{1-1/\psi} + \beta_t \left[\mathbb{E}_t U^{1-\gamma}(C_{t+1}, C_{t+2}, \ldots) \right]^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-1/\psi}}, \tag{3.28}$$

where γ is the relative risk aversion and ψ is the elasticity of intertemporal substitution. The agent maximizes lifetime utility by choosing consumption $\{C_t\}$ subject to $X_{t+1} = R_t(X_t - C_t) \geq 0$. Here X_t is asset holding of the agent at the beginning of time t and R_t is returns. We focus on the empirically relevant case of $\gamma > 1$ and $\psi > 1$, as in, say, Bansal and Yaron (2004), Albuquerque et al. (2016), or Schorfheide et al. (2018). This is the most challenging setting because the usual contraction argument fails and the utility function is unbounded above.

3.6.2.1. *Discounting Continuation Values.* Let $\mathsf{X} = \mathbb{R}_+$, assume that β_t and R_t are functions of the exogenous state, and define the aggregator H by

$$H(x, z, c, v) = \left\{ c^{1-1/\psi} + \beta(z) \left[\int v\left(R(z)(x-c), z'\right)^{1-\gamma} Q(z, dz') \right]^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-1/\psi}}, \tag{3.29}$$

where x, z, and c are asset holding, exogenous state, and consumption, respectively, satisfying $c \in \Gamma(x, z) = [0, x]$.

Assumption 3.6.1. The functions β and R are nonnegative elements of $bm\mathsf{Z}$. In addition, for $\{Z_t\}$ generated by Q, we have

$$\sup_{z \in \mathsf{Z}} \mathbb{E}_z \prod_{t=0}^{n-1} \beta(Z_t)^{1/(1-1/\psi)} R(Z_t) < 1 \text{ for some } n \in \mathbb{N}. \tag{3.30}$$

Assumption 3.6.1 is an eventual discounting condition for the Epstein–Zin case. It is modified to take into account both the underlying growth rate, as in Assumption 3.5.2, and also the role of elasticity of intertemporal substitution. (Intuition and numerical applications are provided below.)

Let $\mathcal{V}$ be all $f \in m\mathsf{S}$ such that $\|f\|_I := \sup_{x \in \mathsf{X}, z \in \mathsf{Z}} |f(x,z)/(1+x)|$ is finite. We show in Appendix 3.9.5.2 that there exists an upper bound function $\hat{v} \in \mathcal{V}$ such that T_σ is a self map on the order interval $[0, \hat{v}] \subset \mathcal{V}$ with the pointwise partial order. Then we show that $v_\sigma := \lim_n (T_\sigma^n \mathbf{0})$ is well defined on the order interval and is a fixed point of T_σ. In addition, if σ satisfies an interiority condition, the fixed point is unique. See Proposition 3.9.6.

Let $\hat{\mathcal{V}}$ be the space of functions in $\mathcal{V}$ that are homogeneous of degree one in x. Our main result for this section is as follows.

Proposition 3.6.2. *If Assumption 3.6.1 holds, then* $\bar{v} := \lim_{n\to\infty} T^n \mathbf{0}$ *is a well defined element of* $\hat{\mathcal{V}}$ *and equal to the value function. There exists an optimal policy* $\sigma^* \in \Sigma$ *that is homogeneous of degree one in* x *and the principle of optimality holds.*

Notice that Proposition 3.6.2 contains no analogue of the eventual contraction condition in Assumption 3.2.1. This is because, as mentioned above, T and T_σ are not contraction mappings under conventional metrics. Instead, the proof uses monotonicity and a form of concavity inherent in Epstein–Zin preferences, combined with fixed point results due to Marinacci and Montrucchio (2010).

3.6.2.2. *Alternative Preference Shocks.* While (3.28) parallels the definitions in, say, Epstein and Zin (1989), Nakata and Tanaka (2020) and De Groot et al. (2020), other studies introduce preference shocks to current consumption (Albuquerque et al., 2016; Schorfheide et al., 2018). In this setting, lifetime utility satisfies

$$U(C_t, C_{t+1}, \ldots) = \left\{ \lambda_t C_t^{1-1/\psi} + b \left[\mathbb{E}_t U^{1-\gamma}(C_{t+1}, C_{t+2}, \ldots) \right]^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-1/\psi}}, \tag{3.31}$$

where $b < 1$ is a fixed constant and $\{\lambda_t\}$ is a preference shock.[19] As we now show, the preceding analysis can be brought to bear on this case as well.

Using homogeneity and dividing both sides of (3.31) by $\lambda_t^{1/(1-1/\psi)}$ yields

$$\tilde{U}_t = \left\{ C_t^{1-1/\psi} + b \left[\mathbb{E}_t \tilde{U}_{t+1}^{1-\gamma} \left(\frac{\lambda_{t+1}}{\lambda_t} \right)^{\frac{1-\gamma}{1-1/\psi}} \right]^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-1/\psi}}, \tag{3.32}$$

[19] Some authors also place an additional term $(1-b)$ before λ_t. This is inconsequential to our optimality results since we can simply redefine λ_t to include $(1-b)$.

where $\tilde{U}_t := U(C_t, C_{t+1}, \ldots)/\lambda_t^{1/(1-1/\psi)}$. If λ_{t+1}/λ_t is measurable with respect to the time-t information set, then (3.32) becomes

$$\tilde{U}_t = \left\{ C_t^{1-1/\psi} + b\delta_t \left[\mathbb{E}_t \tilde{U}_{t+1}^{1-\gamma} \right]^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-1/\psi}}, \tag{3.33}$$

where $\delta_t := \lambda_{t+1}/\lambda_t$. This is the same as the original Koopmans equation in (3.28) with $\beta_t = b\delta_t$.[20] Optimality results from the previous section can now be applied. In particular, Proposition 3.6.2 can be directly applied to the agent's problem in Albuquerque et al. (2016).

3.6.2.3. *Interpretation.* Condition (3.30) is the key restriction required for Proposition 3.6.2 and elasticity of intertemporal substitution plays a role. To illustrate the implications of the condition, we consider the study of Albuquerque et al. (2016), who adopt the specification in (3.31) with $\delta_t := \lambda_{t+1}/\lambda_t$ satisfying $\log \delta_t = \rho \log \delta_{t-1} + \sigma_\epsilon \epsilon_t$. In view of the discussion in Section 3.6.2.2, we can study optimality by applying the eventual discounting condition (3.30) to the transformed representation (3.33). By a result analogous to Proposition 3.4.1, condition (3.30) is equivalent to $r(L_R) < 1$ with L_R defined by

$$(L_R h)(z) := \beta(z)^{1/(1-1/\psi)} R(z) \int h(z') Q(z, dz'). \tag{3.34}$$

One way to obtain insight on the value $r(L_R)$ is to use the stationary approximation $s := \lim_{n\to\infty} s_n^{1/n}$, where $s_n := \mathbb{E} \prod_{t=0}^{n-1} \beta_t^{1/(1-1/\psi)} R_t$. The advantage of the stationary approximation is that, if we specialize to $R(z) \equiv R$, then we obtain the analytical expression

$$s = R \exp\left(\frac{1}{1-1/\psi} \log b + \frac{1}{(1-1/\psi)^2} \frac{\sigma_\epsilon^2}{2(1-\rho)^2} \right). \tag{3.35}$$

(See Appendix 3.9.6 for details.) Analogous to the findings in Section 3.4.4.2 (cf. Table 3.1), this stationary representation closely approximates $r(L_R)$ for a discretized version with moderately fine grid.

The expression in (3.35) sheds light on the role that elasticity of intertemporal substitution plays in eventual discounting. The impact of ψ in (3.35) is not monotone because the

[20]The equivalence between β_t and $b\lambda_{t+1}/\lambda_t$ is demonstrated in De Groot et al. (2020) using the Euler equation in an expected utility setting.

mean term $\log b$ is typically negative, while the volatility term $\sigma_\epsilon^2/(2(1-\rho)^2)$ is positive. Nonetheless, we can understand the impact of ψ by the relative weight placed on the mean and volatility terms: $1/(1-1/\psi)$ enters (3.35) directly for the mean and is squared on the volatility term. Hence, as ψ rises and $1/(1-1/\psi)$ falls, the relative importance of b in determining $r(L_R)$ increases. Conversely, as $\psi \downarrow 1$, the volatility term increasingly dominates.

Intuitively, if ψ is large, then the agent is more willing to shift consumption across time, so the volatility in the discount factor plays a lesser role. Conversely, when ψ is small, consumption cannot shift as freely to compensate for fluctuations in the discount factor. Hence volatility in the discount factor has a large impact on lifetime utility.

3.6.2.4. *Numerical Analysis.* In the applications discussed in Section 3.4.2, discount dynamics are driven by Gaussian AR(1) processes, where standard discretization methods are available and eventual discounting is easy to test. In some recent studies, however, discounting is driven by a Markov process and additional innovations, as in Albuquerque et al. (2016), or stochastic volatility, as in Basu and Bundick (2017). For such cases, one can either use a more sophisticated discretization procedure (see, e.g., Farmer and Toda (2017)) or use Monte Carlo.

To illustrate the Monte Carlo method, we return to the model in Albuquerque et al. (2016) studied above, where the eventual discounting condition is (3.30), or equivalently, $r(L_R) < 1$ with L_R defined in (3.34). An analytical expression was obtained in (3.35) for the case when R_t is constant, but in Albuquerque et al. (2016) this is not the case. Nonetheless, by the strong law of large numbers, we can approximate each s_n by generating m independent simulated paths of $\{\beta_t, R_t\}$ and calculating

$$\hat{s}_n = \frac{1}{m} \sum_{i=1}^{m} \prod_{t=0}^{n-1} \beta_{i,t}^{1/(1-1/\psi)} R_{i,t}. \tag{3.36}$$

Using the parameters in Albuquerque et al. (2016), we find that $\hat{s}_n^{1/n}$ increases with n and exceeds one when n is large, as shown in Table 3.2.[21] This is in line with the analytical

[21] We treat the baseline model in Albuquerque et al. (2016), where $\gamma = 1.516$ and $\psi = 1.4567$. There are three exogenous states: preference shock x_t, log consumption growth Δc_t, and log price consumption ratio z_{ct}. The discount factor is $\beta_t = be^{x_t}$ with $x_t = \rho x_{t-1} + \sigma \epsilon_t$, $b = 0.99795$, $\rho = 0.99132$, and $\sigma =$

TABLE 3.2. Calculate $r(L_R)$ Using Monte Carlo Method

Length of Paths	$n = 100$	$n = 200$	$n = 500$	$n = 1000$
Estimate of $r(L_R)$	1.00355	1.00698	1.01220	1.01321
Standard Error	(0.00004)	(0.00008)	(0.00045)	(0.00054)

expression given by (3.35), which yields $s = 1.0168$ if we fix $R_t \equiv 1$. Hence eventual discounting fails under their parameterization.[22]

3.6.2.5. *The Role of Elasticity of Intertemporal Substitution.* In a New Keynesian model with preference similar to (3.31) studied by Basu and Bundick (2017), De Groot et al. (2018) show that the responses to discount factor shocks explode when the elasticity of intertemporal substitution approaches one, and that this issue disappears if β_t is constant. This matches (3.35). If the volatility term is not zero, then $r(L_R)$ becomes arbitrarily large as ψ approaches one. Hence it appears that the large responses found in De Groot et al. (2018) are the result of an ill-defined household problem that fails the eventual discounting condition. If $\beta_t \equiv b$, then (3.35) becomes $b^{1/(1-\psi)}R$. Letting ψ approach one will push down $r(L_R)$ instead so the issue disappears.

In De Groot et al. (2018), the asymptote in the responses is attributed to the distributional weights on current and future utility not summing to one. They propose an alternative setting where current utility is weighted by $1 - \beta_t$ and future utility is weighted by β_t with $\beta_t < 1$. We show in the appendix that the eventual discounting condition for this specification is the same as Assumption 3.6.1. Since β_t is assumed to be strictly less than one in De Groot et al. (2018), we let $\beta_t \leq b$ for some $b < 1$ and assume fixed returns. Then (3.35) implies that $r(L_R) \leq b^{1/(1-1/\psi)}R$. The previous discussion shows that, in this case, eventual discounting holds when ψ approaches one. This provides an alternative

0.00058631. The logarithm of returns satisfies $r_{t+1} = \kappa_{c0} + \kappa_{c1} z_{ct+1} - z_{ct} + \Delta c_{t+1}$ where $z_{ct} = A_{c0} + A_{c1} x_t$ and $\Delta c_{t+1} = \mu + \sigma_c \epsilon^c_{t+1}$ with $\mu = 0.0015644$ and $\sigma_c = 0.0069004$. The remaining parameters can be solved as detailed in their Internet Appendix, giving $\kappa_{c0} = 0.023108, \kappa_{c1} = 0.99653, A_{c0} = 5.6605$, and $A_{c1} = 82.519$. We run a large number of simulations ($m = 100000$) for each experiment to ensure that $\hat{s}_n$ is close to s_n. The last row lists the standard error for each estimate by calculating the standard deviation of 1000 simulated $\hat{s}_n^{1/n}$ with $\hat{s}_n$ replaced by an approximating normal distribution for computational efficiency.

[22]We have not shown the eventual discounting condition to be necessary in the Epstein–Zin case, so the optimization problem in Albuquerque et al. (2016) might still be well defined. The quantitative exercise in Albuquerque et al. (2016) does not shed light on this issue because they do not solve the agent's optimization problem directly. Instead, they assume that a solution exists and use it to derive asset pricing moments.

explanation of why the model does not produce an asymptote in responses to discount factor shocks.

3.7. Application: Asset Pricing in an Exchange Economy

In this section, we apply our theory to an exchange economy of Lucas (1978a) and study the asset pricing implications of state-dependent discounting.

The economy consists of a representative agent and a productive asset in unit supply.[23] Uncertainty is driven by an exogenous Markov state process $\{Z_t\}$ on Z with a Feller stochastic kernel Q. Given exogenous state z, one unit of asset pays dividend $d(z)$ and is sold at price $p(z)$. The agent maximizes lifetime utility subject to budget constraint

$$C_t + p(Z_t)X_{t+1} \leq d(Z_t)X_t + p(Z_t)X_t \tag{3.37}$$

where C_t is consumption and $X_t \in \mathsf{X} = \mathbb{R}_+$ is asset holding of the agent at time t. In equilibrium, the agent holds one unit of asset and consumes all its dividend in every period. We aim to prove the existence and uniqueness of a price function that supports such an equilibrium.

3.7.1. Additively Separable Preferences. Consider first the case where the agent is an expected utility maximizer with state-dependent discount factors. The Bellman equation is

$$v(x, z) = \max_{x' \in \Gamma(x,z)} \left\{ u(F(x, z, x')) + \beta(z) \int v(x', z')Q(z, dz') \right\}$$

where $F(x, z, x') := d(z)x + p(z)(x - x')$ corresponds to consumption given current-period asset x, next-period asset x', and exogenous state z, and the feasible correspondence is

$$\Gamma(x, z) = \{x' \in [0, \bar{x}] : p(z)x' \leq d(z)x + p(z)x\},$$

where $\bar{x}$ is a number larger than one.

[23]For ease of notation, we consider a single asset. It is straightforward to extend the results to an economy with a finite number of assets.

Assume that $u : X \to \mathbb{R}_+$ is bounded, continuously differentiable, strictly concave, and satisfies $u(0) = 0$. Further assume that (β, Q) is eventually discounting. Then Theorems 3.2.1 and 3.2.2 imply that, given any price function p, there exists a unique solution v^* to the Bellman equation and v^* is increasing, strictly concave, continuously differentiable. Moreover, there exists an optimal policy σ^* such that the envelope condition

$$v_x^*(x, z) = (d(z) + p(z))u'(F(x, z, \sigma^*(x, z)))$$

holds. It follows that in equilibrium,

$$u'(d(z))p(z) = \beta(z) \int [d(z') + p(z')]u'(d(z'))Q(z, dz').$$

Following Lucas (1978a), we define $g(z) := \beta(z) \int d(z')u'(d(z'))Q(z, dz')$ and consider the functional equation

$$f(z) = g(z) + \beta(z) \int f(z')Q(z, dz'). \tag{3.38}$$

If there exists f that satisfies (3.38), then the price function is given by $p(z) = f(z)/u'(d(z))$. Define the operator T_g by

$$(T_g f)(z) := g(z) + \beta(z) \int f(z')Q(z, dz'). \tag{3.39}$$

Then T_g is a self map on $b c\mathsf{Z}$ and is eventually contracting. Hence, there exists a unique $f^* \in b c\mathsf{Z}$ that satisfies (3.38) and the unique price function p^* is given by $p^*(z) = f^*(z)/u'(d(z))$.

3.7.2. Epstein-Zin Preferences. Now suppose that the agent has Epstein-Zin preferences with discount factor shocks as in Section 3.6.2. Write asset returns as

$$R_{t+1} = \frac{d(Z_{t+1}) + p(Z_{t+1})}{p(Z_t)}.$$

Then, the budget constraint (3.37) becomes $\tilde{X}_{t+1} \leq R_{t+1}(\tilde{X}_t - C_t)$, where $\tilde{X}_t := d(Z_t)X_t + p(Z_t)X_t$. Proposition 3.6.2 implies that the value function and optimal policy are homogeneous of degree one in x. Let the value function be $v^*(x, z) = xh(z)$ and optimal consumption be $c^*(x, z) = xc(z)$. It follows from the first order condition that $h(z) = [c(z)]^{-\rho/(1-\rho)}$, where $\rho = 1/\psi$. Plugging it to the Bellman equation gives[24]

$$1 = \beta^\theta(z) \left\{ \int \left(\frac{c^*(x', z')}{c^*(x, z)} \right)^{-\rho\theta} \left(\frac{d(z') + p(z')}{p(z)} \right)^\theta Q(z, dz') \right\}^{1/\theta},$$

where $\theta := (1-\gamma)/(1-\rho)$. In equilibrium, the agent consumes all the dividend and hence $c^*(x, z) = d(z)$ for all $(x, z) \in \mathsf{S}$. Then, a price function p supports the equilibrium if it satisfies

$$p(z) = \beta(z) \left\{ \int \left(\frac{d(z')}{d(z)} \right)^{-\rho\theta} (d(z') + p(z'))^\theta Q(z, dz') \right\}^{1/\theta}. \tag{3.40}$$

Define function f by

$$f(z) := \left(d(z)^{1-\rho} + d(z)^{-\rho} p(z) \right)^\theta.$$

The equilibrium condition (3.40) implies that f satisfies

$$f(z) = (T_d f)(z) := \left\{ d(z)^{1-\rho} + \left(\beta(z)^\theta \int f(z') Q(z, dz') \right)^{1/\theta} \right\}^\theta.$$

Then, finding a equilibrium price function amounts to finding a fixed point of T_d.

Assume that Z is compact and $d, \beta \in bc\mathsf{Z}$ are strictly positive. Then T_d is a self map on $bc\mathsf{Z}_+$. Theorem 1 of Borovička and Stachurski (2020) implies that a necessary and sufficient condition for T_d to have a unique fixed point in $bm\mathsf{Z}_+$ is $r(L_\theta) < 1$, where L_θ is defined by

$$L_\theta h(z) := \beta(z)^\theta \int_{\mathsf{Z}} h(z') Q(z, dz').$$

Let $f^* = L_\theta f^*$. Then the equilibrium price function p^* is given by

$$p^*(z) = d(z)^\rho f^*(z)^{1/\theta} - d(z).$$

3.8. Conclusion

We introduce a weak discounting condition and show that, under this condition, standard infinite horizon dynamic programs with state-dependent discount rates are well defined and well behaved. The value function satisfies the Bellman equation, an optimal policy exists, Bellman's principle of optimality is valid, value function iteration converges and so

[24] Also see Epstein (1988) for a similar derivation in the constant discounting case.

does Howard's policy iteration algorithm. The method can be applied to a broad range of dynamic programming problems, including those with discrete choices, continuous choices and recursive preferences.

We connect eventual discounting to a spectral radius condition and provide guidelines on how to calculate the spectral radius for a range of discount specifications. We show that the condition is more likely to fail when the discount process has higher mean, persistence, or volatility. For models with Epstein–Zin preferences and state-dependent discount factors, the condition also depends on the elasticity of intertemporal substitution.

One natural open question is: how do our results translate into continuous time? It would also be valuable to understand how the results change if discounting depends on endogenous states and actions. Finally, more research is needed on how close to necessary the eventual discounting conditions are for recursive preference models, and especially those involving long run risks, since these models generate realistic asset price processes by driving their parameterizations close to the boundary between stability and instability. These questions are left to future research.

3.9. Appendix

In what follows, we consider the dynamic program described in Section 3.2.1.

3.9.1. Proofs for Section 3.2.

3.9.1.1. *Proof of Theorem 3.2.1.* For each $\sigma \in \Sigma$, let T_σ be defined on $bm\mathsf{S}$ by (3.4). Let T be defined on $bc\mathsf{S}$ by (3.5). We prove part (a) through two lemmas.

LEMMA 3.9.1. *If $\sigma \in \Sigma$, then T_σ is eventually contracting on $bm\mathsf{S}$.*

PROOF. Fix $\sigma \in \Sigma$ and $v \in bm\mathsf{S}$. The map $T_\sigma v$ is Borel measurable on S by the regularity conditions and measurability of σ. It is bounded by the assumption that H is bounded. Hence T_σ is a self-map on $bm\mathsf{S}$. To see that it is eventually contracting, fix (x, z) in S and observe that, by Assumption 3.2.1,

$$|(T_\sigma v)(x, z) - (T_\sigma w)(x, z)| = |H(x, z, \sigma(x, z), v) - H(x, z, \sigma(x, z), w)|$$

$$\leq \beta(z) \int |v(\sigma(x,z),z') - w(\sigma(x,z),z')| Q(z,dz')$$

for any $v, w \in bm\mathsf{S}$. We can write this expression as

$$|T_\sigma v - T_\sigma w| \leq K_\sigma |v - w|, \tag{3.41}$$

where K_σ is the operator defined by

$$(K_\sigma h)(x,z) := \beta(z) \int h(\sigma(x,z),z') Q(z,dz') \qquad (h \in bm\mathsf{S}, z \in \mathsf{Z}).$$

Since $\beta \in bc\mathsf{Z}$, K_σ is a self-map on $bm\mathsf{S}$. Since K_σ is order preserving, we can iterate on (3.41) to obtain $|T_\sigma^n v - T_\sigma^n w| \leq K_\sigma^n |v - w|$ for all $n \in \mathbb{N}$.

Let $\{Z_t\}$ be a Markov process generated by Q and started at z, let $\beta_t = \beta(Z_t)$, and let $\{X_t\}$ be the controlled Markov process generated by $X_{t+1} = \sigma(X_t, Z_t)$ with $(X_0, Z_0) = (x, z)$. We then have $(K_\sigma h)(x,z) = \mathbb{E}_{x,z}\, \beta_0\, h(X_1, Z_1)$ and, iterating on this equation,

$$(K_\sigma^n h)(x,z) = \mathbb{E}_{x,z}\, \beta_0 \beta_1 \cdots \beta_{n-1}\, h(X_n, Z_n) \leq r_n^\beta \|h\|. \tag{3.42}$$

Since $|T_\sigma^n v - T_\sigma^n w| \leq K_\sigma^n |v - w|$, taking the supremum yields $\|T_\sigma^n v - T_\sigma^n w\| \leq r_n^\beta \|v - w\|$. It now follows from the eventual discounting property that T_σ^n is a contraction for some $n \in \mathbb{N}$. Hence T_σ is eventually contracting. □

Lemma 3.9.2. *The operator T is eventually contracting on $bc\mathsf{S}$.*

Proof. Fix $v \in bc\mathsf{S}$. The map Tv is continuous on S by regularity and Berge's Maximum Theorem (Aliprantis and Border, 2006, Theorem 17.31). It is bounded by boundedness of H. Hence T is a self-map on $bc\mathsf{S}$. To see that it is eventually contracting, fix (x, z) in S and observe that, by Assumption 3.2.1,

$$\begin{aligned} |(Tv)(x,z) - (Tw)(x,z)| &\leq \max_{x' \in \Gamma(x,z)} |H(x,z,x',v) - H(x,z,x',w)| \\ &\leq \max_{x' \in \Gamma(x,z)} \beta(z) \int |v(x',z') - w(x',z')|\, Q(z,dz') \end{aligned}$$

for any $v, w \in bc\mathsf{S}$. We can write this expression as

$$|Tv - Tw| \leq K|v - w|, \tag{3.43}$$

where K is the operator on $bc\mathsf{S}$ defined by

$$(Kh)(x,z) := \max_{x' \in \Gamma(x,z)} \beta(z) \int h(x',z') Q(z,dz') \qquad (h \in bc\mathsf{S}, z \in \mathsf{Z}).$$

It follows from regularity and the Feller property (see footnote 6) that $(x',z) \mapsto \int h(x',z')Q(z,dz')$ is continuous. Since $\beta \in bc\mathsf{Z}$, it follows from the maximum theorem that K is a self-map on $bc\mathsf{S}$. Since K is order preserving, we can iterate on (3.43) to obtain $|T^n v - T^n w| \leq K^n |v-w|$ for all $n \in \mathbb{N}$.

Now set $h := |v-w|$, let $\{Z_t\}$ be a Markov process generated by Q with initial condition z and let $\beta_t = \beta(Z_t)$. We then have $(Kh)(x,z) = \max_{x_1 \in \Gamma(x,z)} \mathbb{E}_z \, \beta_0 \, h(x_1, Z_1)$ and hence

$$\begin{aligned}
(K^2 h)(x,z) &= \max_{x_1 \in \Gamma(x,z)} \mathbb{E}_z \, \beta_0 \, (Kh)(x_1, Z_1) \\
&= \max_{x_1 \in \Gamma(x,z)} \mathbb{E}_z \, \beta_0 \max_{x_2 \in \Gamma(x_1, Z_1)} \mathbb{E}_{Z_1} \, \beta_1 \, h(x_2, Z_2) \\
&\leq \|h\| \, \mathbb{E}_z \, \beta_0 \beta_1 .
\end{aligned}$$

More generally, for arbitrary $n \in \mathbb{N}$, we have $(K^n h)(x,z) \leq r_n^\beta \|h\|$. Since $|T^n v - T^n w| \leq K^n h$, taking the supremum gives $\|T^n v - T^n w\| \leq r_n^\beta \|v - w\|$ for all $n \in \mathbb{N}$. It follows from eventual discounting that T^n is a contraction for some $n \in \mathbb{N}$ and hence T is eventually contracting. □

We have an immediate corollary to Lemma 3.9.1 and 3.9.2.

Corollary 3.9.3. *If $v_0 \in bm\mathsf{S}$, the σ-value function v_σ is the unique fixed point of T_σ in $bm\mathsf{S}$ and $T_\sigma^n v \to v_\sigma$ for all $v \in bm\mathsf{S}$. The Bellman operator T has a unique fixed point $\bar{v}$ in $bc\mathsf{S}$ and $T^n w \to \bar{v}$ for all $w \in bc\mathsf{S}$.*

Proof. By Lemma 3.9.1 and a generalized Contraction Mapping Theorem (see, e.g., Cheney, 2013, Section 4.2), T_σ is globally stable on $bm\mathsf{S}$. Hence, if $v_0 \in bm\mathsf{S}$, v_σ is the unique fixed point of T_σ in $bm\mathsf{S}$ and $T_\sigma^n v \to v_\sigma$ for all $v \in bm\mathsf{S}$. The claim for T follows similarly from Lemma 3.9.2. □

Part (b) follows directly from Corollary 3.9.3.

Next we show that $\bar{v}$ given by Corollary 3.9.3 is the value function. First note that $\bar{v} = T\bar{v} \geq T_\sigma \bar{v}$ by definition. Iterating T_σ on both sides and using (3.2), we have $\bar{v} \geq T_\sigma^n \bar{v}$. Taking n to infinity, it follows from Corollary 3.9.3 that $\bar{v} \geq v_\sigma$. Taking the supremum over Σ gives $\bar{v} \geq v^*$.

For the other direction, regularity and the measurable maximum theorem (Aliprantis and Border, 2006, Theorem 18.19) ensure that there exists a $\sigma^* \in \Sigma$ such that $T_{\sigma^*}\bar{v} = T\bar{v}$. Then we have $T_{\sigma^*}\bar{v} = \bar{v}$. Because $\bar{v} \in bc\mathsf{S} \subset bm\mathsf{S}$ and T_{σ^*} has a unique fixed point in $bm\mathsf{S}$ by Corollary 3.9.3, $\bar{v} = v_{\sigma^*}$. By the definition of v^*, we have $v^* \geq v_{\sigma^*} = \bar{v}$. Therefore, $v^* = \bar{v}$ and σ^* is the optimal policy. This proves (c) and (d).

One direction of the Bellman's principle of optimality is implied in the argument above. For the other direction, if a policy σ is optimal, then $v_\sigma = v^*$. It follows from Corollary 3.9.3 that $v^* = T_\sigma v^*$. Since $v^* = \bar{v}$ is the fixed point of T, $T_\sigma v^* = Tv^*$. This proves (e).

Part (f) is valid by Corollary 3.9.3 and the fact that $\bar{v} = v^*$.

For part (g), the following proof is adapted from Bertsekas (2013, Proposition 2.4.1).

Let $\{\sigma_k\} \subset \Sigma$ satisfy $T_{\sigma_k} v_{\sigma_{k-1}} = Tv_{\sigma_{k-1}}$. By definition, $T_{\sigma_k} v_{\sigma_{k-1}} = Tv_{\sigma_{k-1}} \geq T_{\sigma_{k-1}} v_{\sigma_{k-1}} = v_{\sigma_{k-1}}$. By inequality (3.2), applying T_{σ_k} to both sides repeatedly gives $T_{\sigma_k}^n v_{\sigma_{k-1}} \geq Tv_{\sigma_{k-1}} \geq v_{\sigma_{k-1}}$. Taking n to infinity, it follows from Corollary 3.9.3 that $v_{\sigma_k} \geq Tv_{\sigma_{k-1}} \geq v_{\sigma_{k-1}}$. An inductive argument implies that $v^* \geq v_{\sigma_k} \geq T^k v_{\sigma_0}$. Taking k to infinity, Corollary 3.9.3 then implies that $v_{\sigma_k} \to v^*$.

3.9.1.2. *Blackwell's Condition.*

PROPOSITION 3.9.4 (Blackwell's Condition). *Let $\mathcal{D} = (\Gamma, H)$ be a regular dynamic program. If there exists a nonnegative function $\beta \in bc\mathsf{Z}$ and a Feller transition kernel Q on Z such that (β, Q) is eventually discounting and the Bellman operator satisfies*

$$[T(v+c)](x,z) \leq (Tv)(x,z) + \beta(z) \int c(z')Q(z,dz') \tag{3.44}$$

for all $(x,z) \in \mathsf{S}$, $v \in bc\mathsf{S}$, and $c \in bm\mathsf{Z}_+$, then T is eventually contracting on $bc\mathsf{S}$.

Proof of Proposition 3.9.4. For any $v, w \in bc\mathsf{S}$, we have

$$v(x,z) - w(x,z) \leq \sup_{x' \in \mathsf{X}} |v(x',z) - w(x',z)| =: c(z)$$

for all $(x,z) \in \mathsf{S}$, where c is lower semicontinuous (Aliprantis and Border, 2006, Lemma 17.29) and thus $c \in bm\mathsf{Z}_+$. Inequality (3.44) implies that

$$[T(v+c)](x,z) \leq (Tv)(x,z) + \beta(z) \int \sup_{x' \in \mathsf{X}} |v(x',z') - w(x',z')| Q(z,dz').$$

It then follows from (3.2) that

$$(Tv)(x,z) \leq (Tw)(x,z) + \beta(z) \int \sup_{x' \in \mathsf{X}} |v(x',z') - w(x',z')| Q(z,dz').$$

Exchanging the roles of v and w, we have

$$|(Tv)(x,z) - (Tw)(x,z)| \leq \beta(z) \int \sup_{x' \in \mathsf{X}} |v(x',z') - w(x',z')| Q(z,dz').$$

Iterating on the above inequality, it follows from a similar argument to the proof of Lemma 3.9.2 that $\|T^n v - T^n w\| \leq r_n^\beta \|v - w\|$. Since T is a self map on $bc\mathsf{S}$, it follows from eventual discounting that T is eventually contracting. □

3.9.1.3. *Monotonicity, Concavity, and Differentiability.*

Proof of Theorem 3.2.2. Since $ibc\mathsf{S}$ is a closed subset of $bc\mathsf{S}$, it suffices to show that T maps $ibc\mathsf{S}$ to functions in $ibc\mathsf{S}$ that are strictly concave in x. For monotonicity, pick any $z \in \mathsf{Z}$ and $v \in ibc\mathsf{S}$. Then for any $y \geq x$,

$$\begin{aligned}(Tv)(y,z) &= H\left(y,z,\sigma^*(y,z),v\right) \\ &\geq H\left(y,z,\sigma^*(x,z),v\right) \\ &\geq H\left(x,z,\sigma^*(x,z),v\right) = (Tv)(x,z),\end{aligned}$$

where the first inequality holds because $\sigma^*(x,z) \in \Gamma(x,z) \subset \Gamma(y,z)$ and the second inequality holds because H is increasing in x by Assumption 3.2.2. For concavity, pick any x, y satisfying $x \neq y$ and $\theta \in (0,1)$ and define $x_\theta = \theta x + (1-\theta)y$. Then, for any $z \in \mathsf{Z}$

and $v \in ibc\mathsf{S}$,

$$\begin{aligned}\theta(Tv)(x,z) + (1-\theta)(Tv)(y,z) &= \theta H\left(x, z, \sigma^*(x,z), v\right) + (1-\theta) H\left(y, z, \sigma^*(y,z), v\right) \\ &< H\left(x_\theta, z, \theta\sigma^*(x,z) + (1-\theta)\sigma^*(y,z), v\right) \\ &\leq H\left(x_\theta, z, \sigma^*(x_\theta, z), v\right) = (Tv)(x_\theta, z),\end{aligned}$$

where the first inequality holds because $(x, x') \mapsto H(x, z, x', v)$ is strictly concave and the second inequality holds because $\theta\sigma^*(x,z) + (1-\theta)\sigma^*(y,z) \in \Gamma(x_\theta, z)$ by Assumption 3.2.2. The strict concavity of H and the maximum theorem imply that $x \mapsto \sigma^*(x,z)$ is single-valued and continuous.

Now we add Assumption 3.2.3 and consider differentiability. Since $\sigma^*(x_0, z_0) \in \operatorname{int}\Gamma(x_0, z_0)$ and Γ is continuous, there exists an open neighborhood O of x_0 such that $\sigma^*(x_0, z_0) \in \operatorname{int}\Gamma(x, z_0)$ for all $x \in O$. On O we define $W(x) := H\left(x, z_0, \sigma^*(x_0, z_0), v^*\right)$. Then $W(x) \leq v^*(x, z_0)$ on O and $W(x_0) = v^*(x_0, z_0)$. The claim follows then from Assumption 3.2.3 and Benveniste and Scheinkman (1979). □

3.9.1.4. *Optimality over Nonstationary Policies.* Let v_λ and $\tilde{v}^*$ be defined as in Section 3.2.7.

Proof of Proposition 3.2.3. The proof is adapted from Bertsekas (2013, Chapter 2). Since the set of nonstationary policies includes all stationary policies, we have $v^* \leq \tilde{v}^*$. By (3.2) and the definition of T, we have for every $\lambda = (\sigma_0, \sigma_1, \ldots)$,

$$\begin{aligned}v_\lambda(x,z) &= \liminf_{n\to\infty} (T_{\sigma_0} T_{\sigma_1} \ldots T_{\sigma_n} v_0)(x,z) \\ &\leq \lim_{n\to\infty} (T^{n+1} v_0)(x,z).\end{aligned}$$

Since $\mathcal{D}$ is regular and Assumption 3.2.1 is satisfied, we can apply Theorem 3.2.1, which implies that $v_\lambda \leq v^*$ for every λ. Taking supremum over all λ gives $\tilde{v}^* \leq v^*$. This completes the proof. □

3.9.2. Proofs for Section 3.3.

Proof of Proposition 3.3.1. For any $v_1, v_2 \in bc\mathsf{S}$, we have

$$\begin{aligned}|H(x,z,a,v_1) - H(x,z,a,v_2)| &= (1-a)\beta(z)\left|\int_{\mathsf{Z}}(v_1(z') - v_2(z'))Q(z,dz')\right| \\ &\leq \beta(z)\int_{\mathsf{Z}}|v_1(z') - v_2(z')|Q(z,dz').\end{aligned}$$

Since (β, Q) is eventually discounting, Assumption 3.2.1 is satisfied. It is apparent that the feasible correspondence $\Gamma \equiv \{0,1\}$ is continuous, nonempty, and compact valued. It follows from Proposition 3.4.1 and the Cauchy root test that $K(z)$ is well defined and finite for all $z \in \mathsf{Z}$. Since w and β are bounded and continuous and Q is Feller, $\mathcal{D}$ is regular if we can show that K is continuous.

Since Q is Feller, $S_N(z) := \sum_{t=1}^N \mathbb{E}_z \prod_{i=0}^{t-1}\beta(Z_i)$ is bounded and continuous for all $N \in \mathbb{N}$. Since S_N is nonnegative, it follows from Tonelli's theorem that $\lim_{N\to\infty} S_N$ is continuous. Therefore, K is continuous and $\mathcal{D}$ is a regular dynamic program. The proposition then follows from Theorem 3.2.1. □

3.9.3. Proofs for Section 3.4.

Proof of Proposition 3.4.1. Since $\beta \in bc\mathsf{Z}$, L_β defined in (3.18) is a bounded linear operator. It follows from Theorem 1.5.5 of Bühler and Salamon (2018) that $r(L_\beta) := \lim_{n\to\infty}\|L_\beta^n\|^{1/n}$ always exists and is bounded above by $\|L_\beta\| = \sup_z \beta(z)$.

Let $\mathbb{1} \equiv 1$ on Z. For each $z \in \mathsf{Z}$ and $n \in \mathbb{N}$, an inductive argument gives

$$\mathbb{E}_z \prod_{t=0}^{n-1}\beta(Z_t) = L_\beta^n \mathbb{1}(z). \tag{3.45}$$

Thus, eventual discounting can be written as $\|L_\beta^n \mathbb{1}\| < 1$ for some $n \in \mathbb{N}$. Applying Theorem 9.1 of Krasnosel'skii et al. (1972), since (i) L_β is a positive linear operator on $bm\mathsf{Z}$, (ii) the positive cone in this set is solid and normal under the pointwise partial order[25], and (iii) $\mathbb{1}$ lies interior to the positive cone in $bm\mathsf{Z}$, we have

$$r(L_\beta) = \lim_{n\to\infty}\|L_\beta^n \mathbb{1}\|^{1/n} = \lim_{n\to\infty}\left\{\sup_{z\in\mathsf{Z}}\mathbb{E}_z\prod_{t=0}^{n-1}\beta(Z_t)\right\}^{1/n}, \tag{3.46}$$

[25] A cone is solid if it has an interior point; it is normal if $0 \leq x \leq y$ implies that $\|x\| \leq M\|y\|$. The cone of nonnegative functions in $bm\mathsf{Z}$ is both solid and normal.

where the second equality is due to (3.45), nonnegativity of β and the definition of the supremum norm. This confirms the first claim in Proposition 3.4.1. It also follows immediately that $r(L_\beta) < 1$ implies eventual discounting.

To see that the converse is true, suppose there exists an $n \in \mathbb{N}$ such that $r_n^\beta < 1$. Then any $m \in \mathbb{N}$ can be expressed uniquely as $m = kn + i$ for some $k, i \in \mathbb{N}$ with $i < n$. For sufficiently large m, it follows from the Markov property that

$$\begin{aligned}\left(r_m^\beta\right)^{1/m} &= \left\{\sup_{z \in \mathsf{Z}} \mathbb{E}_z \prod_{t=0}^{n-1} \beta(Z_t) \left[\mathbb{E}_{Z_{n-1}} \prod_{t=n}^{m-1} \beta(Z_t)\right]\right\}^{1/m} \\ &\leq \left(r_n^\beta r_{m-n}^\beta\right)^{1/m} \leq \left(r_n^\beta\right)^{k/m} \left(r_i^\beta\right)^{1/m}.\end{aligned}$$

The right hand side is dominated by $(r_n^\beta)^{k/m} M^{1/m}$, where $M := \sup_{i<n} r_i^\beta < \infty$. If $m \to \infty$, then $k/m \to 1/n$, and this term approaches $(r_n^\beta)^{1/n} < 1$. Hence $r(L_\beta) < 1$, as was to be shown. □

Proof of Proposition 3.4.2. The proof of Proposition 3.4.1 uses the fact that $r_{\mathsf{B}}(M) = \lim_{n\to\infty} \|M^n h\|_{\mathsf{B}}^{1/n}$ holds when M is a positive (i.e., order preserving) linear operator on a Banach lattice $(\mathsf{B}, \|\cdot\|_{\mathsf{B}})$ with solid positive cone, r_{B} denotes the spectral radius of a linear operator mapping this Banach lattice to itself, and h is interior to the positive cone (Krasnosel'skii et al., 1972, Theorem 9.1). If Z is finite and $\{Z_t\}$ is irreducible with stationary distribution π, we can take B to be all $h \colon \mathsf{Z} \to \mathbb{R}$ and set $\|h\|_{\mathsf{B}} = \sum_{z \in \mathsf{Z}} |h(z)| \pi(z) =: \mathbb{E}_\pi h$. Under this norm, $\mathbb{1}$ is interior to the positive cone of B because, by irreducibility, $\pi(z) > 0$ for all $z \in \mathsf{Z}$. Applying the above expression for the spectral radius to L_β, as well as the result in (3.45), we obtain

$$r_{\mathsf{B}}(L_\beta) = \lim_{n\to\infty} \|L_\beta^n \mathbb{1}\|_{\mathsf{B}}^{1/n} = \lim_{n\to\infty} \left\{\mathbb{E}_\pi \mathbb{E}_z \prod_{t=0}^{n-1} \beta(Z_t)\right\}^{1/n} = \lim_{n\to\infty} (s_n^\beta)^{1/n}, \tag{3.47}$$

where the last equality uses the law of iterated expectations and the definition of s_n^β in Proposition 3.4.2.

It remains only to show that $r_{\mathsf{B}}(L_\beta) = r(L_\beta)$, where the latter is defined, as before, using the supremum norm (see, e.g., (3.46)). In other words, we need to show that

$$\lim_{n\to\infty} \|L_\beta^n \mathbb{1}\|^{1/n} = \lim_{n\to\infty} \|L_\beta^n \mathbb{1}\|_{\mathsf{B}}^{1/n}. \tag{3.48}$$

On finite dimensional normed linear spaces, any two norms are equivalent (see, e.g., Bühler and Salamon (2018), Theorem 1.2.5), so we can take positive constants c and d with $\|\cdot\| \leq c\|\cdot\|_{\mathsf{B}} \leq d\|\cdot\|$ on B. The equality in (3.48) easily follows and the proof is now complete. □

3.9.4. Proofs for Section 3.5.

3.9.4.1. *Homogeneous Functions.* Let the operators T_σ and T be as defined in (3.4) and (3.5), respectively, with aggregator H given by (3.24). The definition of $h_\theta \mathsf{S}$ is given in Section 3.5.1.

Proof of Proposition 3.5.1. We first show that T is eventually contracting on $\mathcal{V} = h_\theta \mathsf{S}$. Since Assumption 3.5.1 holds, the Feller property implies that T maps $\mathcal{V}$ to itself. Note that for any $v \in \mathcal{V}$, we have $v(x,z) = |x|^\theta v(x/|x|, z)$. It follows from Assumption 3.5.2 that for any $v, w \in \mathcal{V}$,

$$\begin{aligned}
&|(T^n v)(x_0,z_0) - (T^n w)(x_0,z_0)| \\
&\leq \sup_{x_1\in\Gamma(x_0,z_0)} \beta(z_0) \int \left|(T^{n-1}v)(x_1,z_1) - (T^{n-1}w)(x_1,z_1)\right| Q(z_0,dz_1) \\
&\leq \sup_{x_1\in\Gamma(x_0,z_0)} \beta(z_0) \int |x_1|^\theta \left|(T^{n-1}v)\left(\frac{x_1}{|x_1|}, z_1\right) - (T^{n-1}w)\left(\frac{x_1}{|x_1|}, z_1\right)\right| Q(z_0,dz_1) \\
&\leq \sup_{x_1\in\Gamma(x_0,z_0)} \beta(z_0)\alpha^\theta(z_0)|x_0|^\theta \int \left|(T^{n-1}v)\left(\frac{x_1}{|x_1|}, z_1\right) - (T^{n-1}w)\left(\frac{x_1}{|x_1|}, z_1\right)\right| Q(z_0,dz_1).
\end{aligned}$$

An inductive argument gives that

$$\begin{aligned}
&|(T^n v)(x_0,z_0) - (T^n w)(x_0,z_0)| \\
&\leq |x_0|^\theta \sup_{x_1\in\Gamma(x_0,z_0)} \mathbb{E}_{z_0} \prod_{t=0}^{n-1} \beta(z_t)\alpha^\theta(z_t) \left| v\left(\frac{x_n}{|x_n|}, z_n\right) - w\left(\frac{x_n}{|x_n|}, z_n\right)\right| \\
&\leq |x_0|^\theta \left(\mathbb{E}_{z_0} \prod_{t=0}^{n-1} \beta(z_t)\alpha^\theta(z_t)\right) \|v - w\|_h
\end{aligned}$$

where the norm $\|\cdot\|_h$ is defined in (3.26). Therefore, we have

$$\|T^n v - T^n w\|_h \leq \sup_{z_0 \in \mathsf{Z}} \left(\mathbb{E}_{z_0} \prod_{t=0}^{n-1} \beta(z_t)\alpha^\theta(z_t) \right) \|v - w\|_h.$$

By Assumption 3.5.2, T is eventually contracting on $\mathcal{V}$. Hence, T has a unique fixed point $\bar{v}$ on $\mathcal{V}$ and $T^n v \to \bar{v}$ for any $v \in \mathcal{V}$.

Since $T_\sigma v$ is not necessarily in $\mathcal{V}$, we cannot apply the same argument to T_σ. Hence, we prove the remaining results directly. We first show that $v_\sigma := \lim_n (T_\sigma^n \mathbf{0})$ is well defined. It follows from Assumptions 3.5.1 and 3.5.2 that

$$\begin{aligned}
(T_\sigma^n \mathbf{0})(x_0, z_0) &= \sum_{t=0}^{n-1} \mathbb{E}_{z_0} \prod_{i=0}^{t-1} \beta(z_i) u(x_t, z_t, \sigma(x_t, z_t)) \\
&\leq \sum_{t=0}^{n-1} \mathbb{E}_{z_0} \prod_{i=0}^{t-1} \beta(z_i) \left| u(x_t, z_t, \sigma(x_t, z_t)) \right| \\
&\leq \sum_{t=0}^{n-1} \mathbb{E}_{z_0} \prod_{i=0}^{t-1} \beta(z_i) \alpha(z_i)^\theta B(1 + \alpha(z_t))^\theta |x_0| \\
&\leq \sum_{t=0}^{n-1} \mathbb{E}_{z_0} \prod_{i=0}^{t-1} \beta(z_i) \alpha(z_i)^\theta B(1 + \bar{\alpha})^\theta |x_0|
\end{aligned}$$

where $\bar{\alpha} = \sup_{z \in \mathsf{Z}} \alpha(z)$. It follows from Proposition 3.4.1 and the Cauchy root test that the series converges absolutely and hence $v_\sigma(x_0, z_0)$ is finite and well defined.

Next we show that $\bar{v} = v^*$. Since $\bar{v} = T\bar{v}$, we have for any $\sigma \in \Sigma$,

$$\begin{aligned}
\bar{v}(x_0, z_0) &= \max_{x' \in \Gamma(x_0, z_0)} \left\{ u(x_0, z_0, x') + \beta(z_0) \int \bar{v}(x', z_1) Q(z_0, dz_1) \right\} \\
&\geq u(x_0, z_0, \sigma(x_0, z_0)) + \beta(z_0) \int \bar{v}(\sigma(x_0, z_0), z_1) Q(z_0, dz_1).
\end{aligned}$$

It follows from induction that

$$\bar{v}(x_0, z_0) \geq (T_\sigma^n \mathbf{0})(x_0, z_0) + \mathbb{E}_{z_0} \prod_{t=0}^{n-1} \beta(z_t) \bar{v}(x_n, z_n) \tag{3.49}$$

where $\{x_n\}$ is given by σ. Since $\bar{v} \in \mathcal{V}$, we have $\bar{v}(x_n, z_n) \leq \prod_{t=0}^{n-1} \alpha(z_t)^\theta |x_0|^\theta \|\bar{v}\|_h$. Taking n to infinity in (3.49), the last term goes to 0 and thus $\bar{v} \geq v_\sigma$ for all $\sigma \in \Sigma$. By the measurable maximum theorem, we can find $\sigma^* \in \Sigma$ such that $T\bar{v} = T_{\sigma^*}\bar{v}$. A similar argument shows that σ^* achieves the maximum. Therefore, $\bar{v}$ is the value function and σ^* is the optimal policy.

Because $v^* = T_{\sigma^*} v^*$ is homogeneous of degree θ, we have for any $\lambda \geq 0$,

$$v^*(\lambda x, z) = \lambda^\theta v^*(x, z) = \lambda^\theta u(x, z, \sigma^*(x, z)) + \beta(z) \int \lambda^\theta v^*(\sigma^*(x, z), z') Q(z, dz').$$

It follows that $\sigma^*(\lambda x, z) = \lambda \sigma^*(x, z)$, that is, the optimal policy is homogeneous of degree one. □

3.9.4.2. *Local Contractions.* Recall that the operators T_σ and T are as defined in (3.4) and (3.5), respectively, with aggregator H given by (3.24).

Proof of Proposition 3.5.2. Define $u_j(x, z) := \max_{x' \in \Gamma(x,z)} |u(x, z, x')|$ if $x \in K_j$ and $r_j := \sup_{x \in K_j, z \in Z} u_j(x, z)$. Since u is continuous and every K_j is compact, $r_j < \infty$ for all j. For any initial state (x_0, z_0), we can find j such that $x_0 \in K_j$. It follows from Assumption 3.5.3 that $|u(x_t, z_t, x_{t+1})| \leq r_j$ for all $t \in \mathbb{N}$.

Choose any increasing and unbounded $\{m_j\}$ such that $m_j \geq r_j$. Since Q is Feller, Tv is continuous on every K_j for $v \in c_m\mathsf{S}$, where the space $c_m\mathsf{S}$ is defined in Section 3.5.2. It follows from Remark 1(a) of Matkowski and Nowak (2011) that $T : c_m\mathsf{S} \to c\mathsf{S}$.

Since $\Gamma(x, z) \subset K_j$ for all $x \in K_j$, we have on K_j

$$\begin{aligned}
|(T^n v)(x, z) - (T^n w)(x, z)| &\leq \sup_{x' \in \Gamma(x,z)} \beta(z) \int |T^{n-1} v(x', z') - T^{n-1} w(x', z')| Q(z, dz') \\
&\leq \sup_{x' \in K_j} \beta(z) \int |T^{n-1} v(x', z') - T^{n-1} w(x', z')| Q(z, dz') \\
&\leq \beta(z) \|T^{n-1} v - T^{n-1} w\|_j.
\end{aligned}$$

An inductive argument gives

$$|(T^n v)(x, z) - (T^n w)(x, z)| \leq \mathbb{E}_z \prod_{t=0}^{n-1} \beta(Z_t) \|v - w\|_j.$$

Taking the supremum, we have $\|T^n v - T^n w\|_j \leq r_n^\beta \|v - w\|_j$. Since (β, Q) is eventually discounting, T^n is a 0-local contraction for some $n \in \mathbb{N}$.[26] Then it follows from Proposition 1 of Matkowski and Nowak (2011) that T has a unique fixed point $\bar{v}$ in $c_m\mathsf{S}$. It can be proved in the same way that T_σ^n is also a 0-local contraction and hence v_σ is well defined and finite for any initial state. Since we can find σ such that $T_\sigma \bar{v} = T\bar{v}$ by the measurable maximum theorem, the optimality results follow from a similar argument to the proofs of Theorem 3.2.1. □

3.9.5. Proofs for Section 3.6.

3.9.5.1. *Alternative Discount Specifications.* Here we sketch the proof of Theorem 3.2.1 for the alternative timing when the aggregator satisfies (3.27). Let $\{Z_t\}$ be a Markov process generated by Q starting at $z = Z_0$ and let $\beta_t = \beta(Z_{t+1})$. A similar argument to the proof of Lemma 3.9.1 yields $|T_\sigma^n v - T_\sigma^n w| \leq \mathbb{E}_z \prod_{t=1}^n \beta(Z_{t+1}) \|v - w\|$, where $\mathbb{E}_z$ represents expectation conditional on $Z_0 = z$. Taking the supremum gives $\|T_\sigma^n v - T_\sigma^n w\| \leq r_n^\beta \|v - w\|$. Similar result holds for the Bellman operator T. Therefore, both T_σ and T are eventually contracting if $r_n^\beta < 1$ for some $n \in \mathbb{N}$. The rest of the proof remains the same.

Proof of Proposition 3.6.1. Recall that the primitives are redefined as in footnote 18. Then the aggregator satisfies

$$|H(x, z, x', v) - H(x, z, x', w)| \leq \int \beta(z') |v(x', z') - w(x', z')| \tilde{Q}(z, dz').$$

Based on the discussion above, the eventual discounting condition remains the same. It then follows from Proposition 3.4.1 that eventual discounting holds if and only if $r(L_\beta) < 1$ and

$$r(L_\beta) = \lim_{n \to \infty} (r_n^\beta)^{1/n} = \lim_{n \to \infty} \left(\sup_{z \in \tilde{\mathsf{Z}}} \tilde{\mathbb{E}}_z \prod_{t=1}^n \beta(\tilde{Z}_{t+1}) \right)^{1/n}$$

where $\tilde{\mathbb{E}}_z$ represents conditional expectation under $\tilde{Q}$. Since $\tilde{Q}$ is induced by Q and $\beta(\tilde{Z}_{t+1}) = bZ_{t+1}/Z_t$, we can write $r_n^\beta = \sup_{z \in \mathsf{Z}} \mathbb{E}_z b^n Z_t$. Then we have $(b^n z_a)^{1/n} \leq (r_n^\beta)^{1/n} \leq (b^n z_b)^{1/n}$, where z_a and z_b are positive constants such that $z_a < Z_t < z_b$ for all t. Taking $n \to \infty$ gives $r(L_\beta) = b$, so eventual discounting holds if and only if $b < 1$. □

[26] We say an operator $T : c_m\mathsf{S} \to c\mathsf{S}$ is a 0-local contraction if there exists a $\beta \in (0, 1)$ such that $\|Tf - Tg\|_j \leq \beta \|f - g\|_j$ for all $f, g \in c_m\mathsf{S}$ and all $j \in \mathbb{N}$.

3.9.5.2. *Epstein-Zin Preferences.* For ease of notation, we replace $1/\psi$ with ρ in what follows. The definition of $\mathcal{V}$ and $\|f\|_I$ are given in Section 3.6.2. Let the operators T and T_σ be as defined in (3.4) and (3.5), respectively, with aggregator H given by (3.29). Let $\tilde{T}_\sigma$ and $\tilde{T}$ be defined in the same way except that H is replaced by

$$\tilde{H}(x, z, c, v) = \left\{ c^{1-\rho} + \beta(z) \left[\int v\left(R(z)(x-c), z'\right) Q(z, dz') \right]^{1-\rho} \right\}^{\frac{1}{1-\rho}}, \tag{3.50}$$

which is a special case of H when $\gamma = 0$. We first prove a useful lemma.

LEMMA 3.9.5. *$T_\sigma v \leq \tilde{T}_\sigma v$ and $Tv \leq \tilde{T} v$ for all $v \in \mathcal{V}$.*

PROOF. Since $\gamma > 1$, by Jensen's inequality, we have

$$\left[\int v^{1-\gamma}(x, z') Q(z, dz') \right]^{\frac{1}{1-\gamma}} \leq \int v(x, z') Q(z, dz')$$

for all $(x, z) \in \mathsf{S}$ and $v \in \mathcal{V}$. It follows that

$$\begin{aligned}(T_\sigma v)(x, z) &\leq \left\{ \sigma(x,z)^{1-\rho} + \beta(z) \left[\int v\left[R(z)\left(x - \sigma(x,z)\right), z'\right] Q(z, dz') \right]^{1-\rho} \right\}^{\frac{1}{1-\rho}} \\ &= (\tilde{T}_\sigma v)(x, z).\end{aligned}$$

That $Tv \leq \tilde{T} v$ can be shown in a similar way. □

A central result of this section is the following proposition, which guarantees that the σ-value function $v_\sigma = \lim_n (T_\sigma^n \mathbf{0})$ is well defined and a fixed point of T_σ.

PROPOSITION 3.9.6. *Under Assumption 3.6.1, there exists a function $\hat{v} : \mathsf{S} \to \mathbb{R}_+$ given by*

$$\hat{v}(x_0, z_0) := x_0 \left\{ \lim_{n \to \infty} \sum_{t=0}^{n-1} \left[\mathbb{E}_{z_0} \prod_{i=0}^{t-1} \beta(z_i)^{\frac{1}{1-\rho}} R(z_i) \right]^{1-\rho} \right\}^{\frac{1}{1-\rho}} \tag{3.51}$$

such that $\hat{v} \in \mathcal{V}$ and T_σ is a self map on $[0, \hat{v}] \subset \mathcal{V}$. The σ-value function is well defined and is the least fixed point of T_σ on $[0, \hat{v}] \subset \mathcal{V}$. Furthermore, if σ satisfies that $\inf_{z \in \mathsf{Z}} \sigma(x, z)/x > 0$ for all $x > 0$, then v_σ is the unique fixed point of T_σ on $[0, \hat{v}] \subset \mathcal{V}$ and $T_\sigma^n v \to v_\sigma$ for all $v \in [0, \hat{v}] \subset \mathcal{V}$.

We first give two lemmas that are crucial to the proof of Proposition 3.9.6. The first lemma shows that $\hat{v}$ can indeed act as an upper bound function.

LEMMA 3.9.7. *$\hat{v} \in \mathcal{V}$ and $T_\sigma \hat{v} \leq \hat{v}$ for all $\sigma \in \Sigma$.*

PROOF. Let $\hat{v}_n(x_0, z_0) := x_0 A_n(z_0)^{1/(1-\rho)}$ where

$$A_n(z_0) := \sum_{t=0}^{n-1} \left[\mathbb{E}_{z_0} \prod_{i=0}^{t-1} \beta(z_i)^{\frac{1}{1-\rho}} R(z_i) \right]^{1-\rho}.$$

By Proposition 3.4.1 and Assumption 3.6.1, we have

$$\limsup_{n\to\infty} \left[\sup_{z_0 \in \mathsf{Z}} \mathbb{E}_{z_0} \prod_{i=0}^{t-1} \beta(z_i)^{\frac{1}{1-\rho}} R(z_i) \right]^{\frac{1-\rho}{n}} = r(L_R)^{1-\rho} < 1,$$

where L_R is as defined in (3.34). It follows from the root test that $\lim_n A_n$ is well defined and bounded on Z. Hence, $\hat{v} = \lim_n \hat{v}_n$ and it satisfies $\|\hat{v}\|_I = \sup_{x\in\mathsf{X}, z\in\mathsf{Z}} |xA(z)/(1+x)| \leq \sup_{z\in\mathsf{Z}} A(z) < \infty$. Therefore, $\hat{v} \in \mathcal{V}$.

Next, we use the operator $\tilde{T}_\sigma$ defined above to show that $T_\sigma \hat{v} \leq \hat{v}$. Since A_n is increasing in n, by the Monotone Convergence Theorem, we have $\lim_{n\to\infty} (\tilde{T}_\sigma \hat{v}_n)(x_0, z_0) = (\tilde{T}_\sigma \hat{v})(x_0, z_0)$. Write $A_n(z_0) = \sum_{t=0}^{n-1} B_t(z_0)$. Since $\sigma(x, z) \leq x$, it follows that

$$\begin{aligned}
(\tilde{T}_\sigma \hat{v}_n)(x_0, z_0) &\leq x_0 \left\{ 1 + \left[\beta(z_0)^{\frac{1}{1-\rho}} R(z_0) \mathbb{E}_{z_0} A_n(z_1)^{\frac{1}{1-\rho}} \right]^{1-\rho} \right\}^{\frac{1}{1-\rho}} \\
&= x_0 \left\{ 1 + \left[\beta(z_0)^{\frac{1}{1-\rho}} R(z_0) \mathbb{E}_{z_0} \left(\sum_{t=0}^{n-1} B_t(z_1) \right)^{\frac{1}{1-\rho}} \right]^{1-\rho} \right\}^{\frac{1}{1-\rho}}.
\end{aligned}$$

Since $\rho \in (0, 1)$, by the Minkowski inequality, we have

$$(\tilde{T}_\sigma \hat{v}_n)(x_0, z_0) \leq x_0 \left\{ 1 + \sum_{t=0}^{n-1} \left[\beta(z_0)^{\frac{1}{1-\rho}} R(z_0) \mathbb{E}_{z_0} B_t(z_1)^{\frac{1}{1-\rho}} \right]^{1-\rho} \right\}^{\frac{1}{1-\rho}}.$$

Note that the following equation holds

$$\beta(z_0)^{\frac{1}{1-\rho}} R(z_0) \mathbb{E}_{z_0} B_t(z_1)^{\frac{1}{1-\rho}} = B_{t+1}(z_0)^{\frac{1}{1-\rho}}$$

by the Markov property. It follows that

$$(\tilde{T}_\sigma \hat{v}_n)(x_0, z_0) \leq x_0 \left\{ 1 + \sum_{t=1}^{n} B_t(z_0) \right\}^{\frac{1}{1-\rho}} = x_0 A_{n+1}(z_0)^{\frac{1}{1-\rho}} = \hat{v}_{n+1}(x_0, z_0).$$

Taking n to infinity, we have $\tilde{T}_\sigma \hat{v} \leq \hat{v}$. By Lemma 3.9.5, $T_\sigma \hat{v} \leq \hat{v}$. □

LEMMA 3.9.8. *$T_\sigma v \in \mathcal{V}$ for all $\sigma \in \Sigma$ and $v \in \mathcal{V}$.*

PROOF. Evidently $T_\sigma v$ is measurable given $\sigma \in \Sigma$. To see that $T_\sigma v$ is bounded, we have

$$\begin{aligned}(T_\sigma v)(x, z) &\leq \left\{ \sigma(x, z)^{1-\rho} + \beta(z) \left[\int v \left[R(z) \left(x - \sigma(x, z) \right), z' \right] Q(z, dz') \right]^{1-\rho} \right\}^{\frac{1}{1-\rho}} \\ &\leq \left\{ x^{1-\rho} + \beta(z) \|v\|_I^{1-\rho} \left[1 + R(z) x \right]^{1-\rho} \right\}^{\frac{1}{1-\rho}},\end{aligned}$$

where the first inequality follows from Lemma 3.9.5 and the second inequality follows from the fact that $\sigma(x, z) \in [0, x]$ and $|v(x, z)| \leq \|v\|_I(1 + x)$ for all $v \in \mathcal{V}$. Dividing both sides by $(1 + x)$ yields (assuming $\sup_z R(z) > 1$)

$$\|T_\sigma v\|_I \leq \sup_{z \in \mathsf{Z}} \left\{ 1 + \beta(z) \|v\|_I^{1-\rho} R(z)^{1-\rho} \right\}^{\frac{1}{1-\rho}}.$$

Since β and R are bounded, $\|T_\sigma v\|_I < \infty$. □

PROOF OF PROPOSITION 3.9.6. It is apparent that $T_\sigma \mathbf{0} \geq \mathbf{0}$. It follows from Lemma 3.9.7, Lemma 3.9.8, and the monotonicity of T_σ that T_σ is a self map on $[0, \hat{v}] \subset \mathcal{V}$. Let $\{v_n\}$ be a countable chain[27] on $[0, \hat{v}] \subset \mathcal{V}$. Then both $\sup_n v_n$ and $\inf_n v_n$ are measurable and bounded in norm by $\|\hat{v}\|_I$. So $[0, \hat{v}] \subset \mathcal{V}$ is a countably chain complete partially ordered set. For any increasing $\{v_n\} \subset [0, \hat{v}]$, it follows from the Monotone Convergence Theorem that $\sup_n T_\sigma v_n = T_\sigma(\sup_n v_n)$. Hence, T_σ is monotonically sup-preserving. Then, by the Tarski-Kantrovich Theorem,[28] $v_\sigma := \lim_n (T_\sigma^n \mathbf{0})$ is the least fixed point of T_σ on $[0, \hat{v}] \subset \mathcal{V}$.

If σ satisfies that $\inf_{z \in \mathsf{Z}} (\sigma(x, z)/x) > 0$ for all $x > 0$, then there exists an $\alpha > 0$ such that $\sigma(x, z) \geq \alpha x \sup_z A(z) \geq \alpha \hat{v}(x, z)$. Since $T_\sigma \mathbf{0} = \sigma \leq \hat{v}$, $T_\sigma \mathbf{0}$ and $\hat{v}$ are comparable.

[27] A set $C \subset \mathcal{V}$ is called a chain if for every $x, y \in C$, either $x \leq y$ or $y \leq x$.

[28] See, for example, Becker and Rincon-Zapatero (2018) for a version of the theorem and related definitions.

Uniqueness and convergence then follow from Theorems 10 and 11 in Marinacci and Montrucchio (2010). □

Recall from Section 3.6.2 that $\hat{\mathcal{V}}$ is all functions in $\mathcal{V}$ that are homogeneous of degree one in x. The following lemma is useful in the proof of Proposition 3.6.2.

LEMMA 3.9.9. *For any* $v \in \hat{\mathcal{V}}$, $Tv \in \hat{\mathcal{V}}$ *and there exists a* $\sigma \in \Sigma$ *homogeneous in* x *that satisfies* $Tv = T_\sigma v$ *and* $\inf_z \sigma(x,z)/x > 0$ *for all* $x > 0$.

PROOF. Pick $v \in \hat{\mathcal{V}}$ and we can write $v(x,z) = xh(z)$ for some bounded measurable h. Then (3.29) becomes

$$H(x,z,c,v) = \left\{ c^{1-\rho} + \beta(z)R(z)^{1-\rho}(x-c)^{1-\rho} \left[\int h(z')^{1-\gamma} Q(z,dz') \right]^{\frac{1-\rho}{1-\gamma}} \right\}^{\frac{1}{1-\rho}}. \tag{3.52}$$

Since $c \mapsto H(x,z,c,v)$ is continuous and $(x,z) \mapsto H(x,z,c,v)$ is measurable, by the measurable maximum theorem, Tv is measurable and there exists a $\sigma \in \Sigma$ such that $T_\sigma v = Tv$. Since $c \leq x$ in (3.52), a similar argument to the proof of Lemma 3.9.8 shows that Tv is bounded in $\|\cdot\|_I$.

In fact, $\sigma(x,z)$ is the solution of the single variable optimization problem maximizing $c^{1-\rho} + (x-c)^{1-\rho} f(z)$ over $0 \leq c \leq x$ where

$$f(z) := \beta(z)R(z)^{1-\rho} \left[\int h(z')^{1-\gamma} Q(z,dz') \right]^{\frac{1-\rho}{1-\gamma}}.$$

It has closed-form solution $\sigma(x,z) = x/(f(z)^{1/\rho} + 1)$. Therefore, σ is homogeneous in x and thus $Tv = T_\sigma v$ is also homogeneous in x. It follows that $Tv \in \hat{\mathcal{V}}$. Since $f(z)$ is bounded, $\inf_z \sigma(x,z)/x > 0$. □

PROOF OF PROPOSITION 3.6.2. By Lemma 3.9.9, there exists a σ such that $T_\sigma \hat{v} = T\hat{v}$. It follows from Lemma 3.9.7 that $T_\sigma \hat{v} \leq \hat{v}$ and hence $T\hat{v} \leq \hat{v}$. Then the monotonicity of T implies that $Tv \leq \hat{v}$ for all $v \in \hat{\mathcal{V}}$. By Lemma 3.9.9 and the monotonicity of T, $T^n\mathbf{0}$ is an increasing sequence on $\hat{\mathcal{V}}$ bounded above by $\hat{v}$. Therefore, the pointwise limit $\bar{v} := \lim_{n\to\infty}(T^n\mathbf{0})$ is well defined and is also in $[0,\hat{v}] \subset \hat{\mathcal{V}}$.

To see that $\bar{v}$ is the value function, pick any $\sigma \in \Sigma$. Since $T^n\mathbf{0}$ is an increasing sequence converging to $\bar{v}$, $\bar{v} \geq T^n\mathbf{0} \geq T^n_\sigma\mathbf{0}$. Taking n to infinity, it follows from Proposition 3.9.6 that $\bar{v} \geq v_\sigma$. Next we show that $\bar{v}$ can be achieved by a feasible policy. Since $T^n\mathbf{0} \leq \bar{v}$, the monotonicity of T implies that $T^{n+1}\mathbf{0} \leq T\bar{v}$. Taking n to infinity yields $\bar{v} \leq T\bar{v}$. By Lemma 3.9.9, there exists a homogeneous $\sigma^* \in \Sigma$ that satisfies the interiority condition and $T_{\sigma^*}\bar{v} = T\bar{v}$. Then we have $\bar{v} \leq T_{\sigma^*}\bar{v}$ and hence $\bar{v} \leq T^n_{\sigma^*}\bar{v}$ by the monotonicity of T_{σ^*}. Taking n to infinity, it follows from Proposition 3.9.6 that $\bar{v} \leq v_{\sigma^*}$. Since $\bar{v} \geq v_\sigma$ for all $\sigma \in \Sigma$, $\bar{v} = v_{\sigma^*}$. □

For the specification in De Groot et al. (2018) where the lifetime utility satisfies

$$U(C_t, C_{t+1}, \ldots) = \left\{(1-\beta_t)C_t^{1-\rho} + \beta_t \left[\mathbb{E}_t U^{1-\gamma}(C_{t+1}, C_{t+2}, \ldots)\right]^{\frac{1-\rho}{1-\gamma}}\right\}^{\frac{1}{1-\rho}},$$

we can redefine the upper bound function to be

$$\tilde{v}(x_0, z_0) := x_0 \left\{\lim_{n\to\infty} \sum_{t=0}^{n-1} \left[\mathbb{E}_{z_0} \prod_{i=0}^{t-1} \beta(z_i)^{\frac{1}{1-\rho}} R(z_i) \left[1-\beta(z_t)\right]^{\frac{1}{1-\rho}}\right]^{1-\rho}\right\}^{\frac{1}{1-\rho}}.$$

Since $\beta(z_t) < 1$, $\tilde{v}$ is bounded above by $\hat{v}$ in (3.51). Then it can be shown that all the above results hold for the new preference if Assumption 3.6.1 is satisfied. The proof is omitted.

3.9.6. Analytical Expression for the Geometric Mean.

Consider $\beta_t = \exp(\alpha Z_t)$ where $\{Z_t\}$ obeys (3.21). An inductive argument shows that for all $t \geq 1$,

$$Z_t = (1-\rho^t)\mu + \rho^t Z_0 + \sigma_\epsilon(\epsilon_t + \rho\epsilon_{t-1} + \ldots + \rho^{t-1}\epsilon_1). \tag{3.53}$$

It follows that

$$\sum_{t=0}^{n-1} Z_t = \left(n - \frac{\rho(1-\rho^n)}{1-\rho}\right)\mu + \frac{1-\rho^{n+1}}{1-\rho} Z_0 + \sigma_\epsilon\left(\epsilon_n + \frac{1-\rho^2}{1-\rho}\epsilon_{n-1} + \ldots + \frac{1-\rho^n}{1-\rho}\epsilon_1\right).$$

Exploiting the properties of log-normal distributions, we have

$$\mathbb{E}_z \exp\left(\sum_{t=0}^{n-1} Z_t\right) = \exp\left(n\mu - \frac{\rho(1-\rho^n)}{1-\rho}\mu + \frac{1-\rho^{n+1}}{1-\rho} z + \frac{\sigma_\epsilon^2}{2}\sum_{t=1}^{n} m_t\right)$$

where $m_t = (1-\rho^t)^2/(1-\rho)^2$. Using the law of iterated expectations gives

$$\mathbb{E} \exp\left(\sum_{t=0}^{n-1} Z_t\right) = \exp\left((n+1)\mu + \frac{(1-\rho^{n+1})^2\sigma_\epsilon^2}{2(1-\rho)^2(1-\rho^2)} + \frac{\sigma_\epsilon^2}{2}\sum_{t=1}^{n} m_t\right).$$

Since $m_t \to 1/(1-\rho)^2$, $\sum m_t/n \to 1/(1-\rho)^2$. Therefore,

$$\lim_{n\to\infty}\left(\mathbb{E}\prod_{t=0}^{n-1}\beta_t\right)^{1/n} = \exp\left(\alpha\mu + \frac{\alpha^2\sigma_\epsilon^2}{2(1-\rho)^2}\right). \tag{3.54}$$

Setting $\alpha = 1$ gives (3.23). Setting $\mu = \log(b)$ and $\alpha = 1/(1-1/\psi)$ gives (3.35).

3.9.7. Necessity. In many settings, the eventual discounting condition cannot be weakened without violating finite lifetime values. Here we briefly illustrate this point, using the connection to spectral radii provided in Proposition 3.4.1.

Consider a standard dynamic program with lifetime rewards $\mathbb{E}\sum_{t\geq 0}\beta^t\pi_t$ given constant β and reward flow $\{\pi_t\}$. In this setting, $\beta < 1$ cannot be relaxed without imposing specific conditions on rewards. For example, if there are constants $0 < a \leq b$ such that the process $\{\pi_t\}$ satisfies $a \leq \pi_t \leq b$ for all t, then we clearly have[29]

$$\mathbb{E}\sum_{t\geq 0}\beta^t\pi_t < \infty \text{ if and only if } \beta < 1. \tag{3.55}$$

Eventual discounting has the same distinction once we replace the constant β with a process $\{\beta_t\}$ under standard regularity conditions. For example, if Z is compact and $\beta_t = \beta(Z_t)$ for some $\beta \in bc\mathsf{Z}$ and Q-Markov process $\{Z_t\}$, then

$$\mathbb{E}_z\sum_{t\geq 0}\prod_{i=0}^{t-1}\beta_i\,\pi_t < \infty \text{ if and only if } r(L_\beta) < 1. \tag{3.56}$$

To see this, suppose first that $r(L_\beta) < 1$. Since $\pi_t \leq b$, we have

$$\mathbb{E}_z\sum_{t\geq 0}\prod_{i=0}^{t-1}\beta_i\,\pi_t \leq b\sum_{t\geq 0}\mathbb{E}_z\prod_{i=0}^{t-1}\beta_i \leq b\sum_{t\geq 0}\sup_z \mathbb{E}_z\prod_{i=0}^{t-1}\beta_i = b\sum_{t\geq 0} r_t^\beta.$$

By Cauchy's root convergence criterion, the sum $\sum_{t\geq 0} r_t^\beta$ will be finite whenever $\limsup_{t\to\infty}(r_t^\beta)^{1/t} < 1$. This holds when $r(L_\beta) < 1$ by Proposition 3.4.1.

[29]The equivalence in (3.55) is easy to see because, by the Monotone Convergence Theorem, we have $\mathbb{E}\sum_{t\geq 0}\beta^t\pi_t = \sum_{t\geq 0}\beta^t\mathbb{E}\pi_t$ and, moreover, $0 < a \leq \mathbb{E}\pi_t \leq b$.

Now suppose instead that $r(L_\beta) \geq 1$. By compactness of L_β, positivity of the function β from Assumption 3.2.1 and the Krein–Rutman Theorem (see, e.g., Theorem 1.2 in Du (2006)), there exists a positive function $e \in bc\mathsf{Z}$ such that $L_\beta e = r(L_\beta)e$. Choosing $\gamma > 0$ such that $\gamma e \leq 1$, we have

$$\mathbb{E}_z \sum_{t \geq 0} \prod_{i=0}^{t-1} \beta_i \, \pi_t \geq a\gamma \sum_{t \geq 0} L_\beta^t e(z) = a\gamma \sum_{t \geq 0} r(L_\beta)^t e(z)$$

when $Z_0 = z$. Since $e > 0$ and $r(L_\beta) \geq 1$, the sum diverges to infinity.

CHAPTER 4

Negative Discount Dynamic Programming

4.1. Introduction

This chapter focuses on a specific class of negative discount dynamic programming problems, in which the agent minimizes the present value of a sequence of losses with a negative discount rate. Such problems cause technical difficulties because the Bellman operator is expansive, instead of contractive as in the standard dynamic programming theory (Bertsekas, 2017; Stokey and Lucas, 1989).

To address this issue, we develop a general dynamic programming framework that can accommodate such noncontractive models. We recover all standard dynamic programming results under this framework, including convergence of the Bellman operator, existence of an optimal policy, the principle of optimality, and monotonicity, convexity, and differentiability of the value function. Using this framework, we are able to solve the negative discount dynamic program and derive properties of its solution. Furthermore, we characterize the solution with an Euler equation and an envelope condition, which provide economic intuition and are crucial in developing key results in applications. We also give a set of analogous results in the continuous-time setting.

As a second contribution, we show how to apply the theory of negative discount dynamic programming to solve a range of competitive equilibrium problems related to production networks, management layers within firms, and the size distribution of cities. Although the equilibrium concepts in these models are static in nature, the recursive structures in these problems allow us to transform them into dynamic programs. Moreover, each problem features some kind of frictions, which can be represented by a negative discount rate after the transformation.

When deriving fixed point properties of the Bellman operator, we draw on the theory of monotone concave operators reviewed in Section 1.3. In particular, we use the fixed point result given in Du (1989) for monotone concave operators on Banach spaces. In this sense, this paper is related to Ren and Stachurski (2018), who develop a general framework for convex and concave dynamic programs using the same result. However, our focus is on the negative discount dynamic programming problem, which is not discussed in Ren and Stachurski (2018). We also provide new results on monotonicity, convexity and differentiability of solutions. Our general dynamic programming framework is also related to Bloise and Vailakis (2018), who study convex dynamic programs with recursive utilities. Our interiority conditions are slightly stronger but are more suitable for the negative discount dynamic programs we consider. We also provide a full set of optimality results linking Bellman's equation to existence and characterization of optimal solutions.[1]

In terms of applications of our dynamic programming framework, we first connect to the production chain model developed in Kikuchi et al. (2018). Their model of production chains can be transformed to a negative discount dynamic program after the index over firms is reinterpreted as a time index. The negative discount rate corresponds to the transaction costs between upstream and downstream firms. We show that competitive equilibria in a version of the model can be recovered as solutions to the class of negative dynamic programming problems we study. Similar reinterpretations are also applied to the model of knowledge-based hierarchy within an organization treated in Garicano (2000), Garicano and Rossi-Hansberg (2006), and Caliendo and Rossi-Hansberg (2012), and to the city hierarchy model of Hsu et al. (2014). The negative "discount rate" stems from the communication costs between layers of management in the former case and from the transportation costs in the latter case.

The remainder of this paper is structured as follows. In Section 4.2, we introduce a general dynamic programming theory that encompasses the negative discount dynamic programming theory, and discuss its solution. In Section 4.3, we connect this discussion to Coase's theory of the firm and elaborate on the relationship between our model and other

[1]On a technical level, our optimality theory is related to other studies of dynamic programming where the Bellman operator fails to be a contraction, such as Martins-da Rocha and Vailakis (2010) and Rincón-Zapatero and Rodríguez-Palmero (2003). Our methods differ because even the relatively weak local contraction conditions imposed in that line of research fail in our settings. The fixed point results in this paper are related to those found in Kamihigashi et al. (2015), but here we also prove uniqueness of the fixed point, as well as connections to optimality and shape and differentiability properties.

related models. In Section 4.4 we extend our model to expand the scope of applications to more complex networks while in Section 4.5 we show that our model can also be used to understand organization of knowledge within a firm. In Section 4.6, we give analogous results for the continuous-time setting and discuss applications. Most proofs are deferred to the appendix.

4.2. Noncontractive Dynamic Programming

First we provide a general dynamic programming framework suitable for analyzing equilibria in production networks. Throughout this section, we state results in an abstract setting and illustrate them with a fixed example related to minimizing a flow of losses under negative discounting.

4.2.1. Set Up. Given a metric space E, let $\mathcal{R}(E)$ denote the set of real functions on E and let $\mathcal{C}(E)$ be all continuous functions in $\mathcal{R}(E)$. Given $g, h \in \mathcal{R}(E)$, we write $g \leq h$ if $g(x) \leq h(x)$ for all $x \in E$, and $\|f\| := \sup_{x \in E} |f(x)|$.

4.2.1.1. *An Abstract Dynamic Program.* Let X be a compact metric space, referred to as the state space. Let A be a metric space and let Γ be a nonempty, continuous, compact-valued correspondence from X into A. We understand $\Gamma(x)$ as the set of available actions $a \in A$ for an agent facing state x. Let $\operatorname{gr}\Gamma := \{(x, a) : x \in X,\ a \in \Gamma(x)\}$ be all feasible state-action pairs. Let L be an *aggregator function*, which is a map from $\operatorname{gr}\Gamma \times \mathcal{R}(X)$ into $\mathbb{R}$, with the interpretation that $L(x, a, w)$ is lifetime loss associated with current state x, current action a and continuation value function w.

A pair (L, Γ) with these properties is referred to below as an *abstract dynamic program*. The *Bellman operator* associated with such a pair is the operator T defined by

$$(Tw)(x) = \inf_{a \in \Gamma(x)} L(x, a, w) \qquad (w \in \mathcal{R}(X),\ x \in X). \tag{4.1}$$

A fixed point of T in $\mathcal{R}(X)$ is said to satisfy *the Bellman equation*.

4.2.1.2. *Example: Negative Discount DP.* To illustrate the definitions above, we consider an agent who takes action a_t in period t with current loss $\ell(a_t)$. We can interpret a_t as effort and $\ell(a_t)$ as disutility of effort. Her optimization problem is, for some $\hat{x} > 0$,

$$\min_{\{a_t\}} \sum_{t=0}^{\infty} \beta^t \ell(a_t) \quad \text{s.t. } a_t \geq 0 \text{ for all } t \geq 0 \text{ and } \sum_{t=0}^{\infty} a_t = \hat{x}. \tag{4.2}$$

Here and throughout the discussion of this optimization problem, we suppose that

$$\beta > 1, \; \ell(0) = 0, \; \ell' > 0 \; \text{ and } \ell'' > 0. \tag{4.3}$$

The convexity in ℓ encourages the agent to defer some effort. Negative discounting ($\beta > 1$) has the opposite effect.[2] We set $x_{t+1} = x_t - a_t$ with $x_0 = \hat{x}$, so x_t represents "remaining tasks" at the start of time t. The Bellman equation is

$$w(x) = \inf_{0 \leq a \leq x} \{\ell(a) + \beta w(x - a)\}. \tag{4.4}$$

The Bellman operator is

$$Tw(x) = \inf_{0 \leq a \leq x} \{\ell(a) + \beta w(x - a)\}. \tag{4.5}$$

If we set $X := A := [0, \hat{x}]$,

$$L(x, a, w) := \ell(a) + \beta w(x - a) \quad \text{and} \quad \Gamma(x) := [0, x], \tag{ND}$$

then (L, Γ) in (ND) fits the definition of an abstract dynamic program, as given in Section 4.2.1.1, and (4.5) is a special case of (4.1).

[2]The assumption $\ell(0) = 0$ cannot be weakened, since $\ell(0) > 0$ implies that the objective function is infinite. Conversely, with the assumption $\ell(0) = 0$, minimal loss is always finite. Indeed, by choosing the feasible action path $a_0 = \hat{x}$ and $a_t = 0$ for all $t \geq 1$, we get $\sum_{t=0}^{\infty} \beta^t \ell(a_t) \leq \ell(\hat{x})$. Also, given our other assumptions, there is no need to consider the case $\beta \leq 1$ because no solution exists. Because we are minimizing loss, when $\beta < 1$ any proposed solution $\{a_t\}$ can be strictly improved by shifting it one step into the future (set $a'_0 = 0$ and $a'_{t+1} = a_t$ for all $t \geq 0$). Furthermore, if $\beta = 1$, and a solution $\{a_t\}$ exists, then the increments $\{a_t\}$ must converge to zero, and hence there exists a pair a_T and a_{T+1} with $a_T > a_{T+1}$. Since ℓ is strictly convex, the objective $\sum_t \ell(a_t)$ can be reduced by redistributing a small amount ϵ from a_T to a_{T+1}. This contradicts optimality.

4.2.2. Fixed Point Results. The Bellman operator in (4.5) is not a supremum norm contraction, due to the fact that $\beta > 1$.[3] The production chain and network models we consider also have this feature. Hence we introduce a set of conditions for an abstract dynamic program that generate stability properties without requiring contractivity.

4.2.2.1. *Stability Without Contractivity.* Fix an abstract dynamic program (L, Γ) and consider the following assumptions:

A_1. $(x, a) \mapsto L(x, a, w)$ is continuous on $\operatorname{gr}\Gamma$ when $w \in \mathcal{C}(X)$.

A_2. If $u, v \in \mathcal{C}(X)$ with $u \leq v$, then $L(x, a, u) \leq L(x, a, v)$ for all $(x, a) \in \operatorname{gr}\Gamma$.

A_3. Given $\lambda \in (0, 1)$, $u, v \in \mathcal{C}(X)$ and $(x, a) \in \operatorname{gr}\Gamma$, we have

$$\lambda L(x, a, u) + (1 - \lambda) L(x, a, v) \leq L(x, a, \lambda u + (1 - \lambda) v).$$

A_4. There is a ψ in $\mathcal{C}(X)$ such that $T\psi \leq \psi$.

A_5. There is a ϕ in $\mathcal{C}(X)$ and an $\epsilon > 0$ such that $\phi \leq \psi$ and $T\phi \geq \phi + \epsilon(\psi - \phi)$.

Assumptions A_1–A_3 impose some continuity, monotonicity and convexity. Assumptions A_4–A_5 provide upper and lower bounds for the set of candidate value functions.

Although contractivity is not imposed, we can show that the abstract Bellman operator (4.1) is well behaved under A_1–A_5 after restricting its domain to a suitable class of candidate solutions. To this end, let

$$\mathcal{I} := \{f \in \mathcal{C}(X) : \phi \leq f \leq \psi\}.$$

Theorem 4.2.1. *Let (L, Γ) be an abstract dynamic program and let T be the Bellman operator defined in (4.1). If (L, Γ) satisfies A_1–A_5, then*

1. *T has a unique fixed point w^* in $\mathcal{I}$.*
2. *For each $w \in \mathcal{I}$, there exists an $\alpha < 1$ and $M < \infty$ such that*

$$\|T^n w - w^*\| \leq \alpha^n M \quad \text{for all } n \in \mathbb{N}. \tag{4.6}$$

[3] For example, let $w \equiv 1$ and $g \equiv 0$. Then $Tw \equiv \beta > 1$ while $Tg \equiv 0$. One consequence is that, if we take an arbitrary continuous bounded function and iterate with T, the sequence typically diverges. For example, if $w \equiv 1$, then, $T^n w \equiv \beta^n$, which diverges to $+\infty$.

3. *$\pi^*(x) := \arg\min_{a \in \Gamma(x)} L(x, a, w^*)$ is upper hemicontinuous on X.*

Theorem 4.2.1 does not discuss Bellman's principle of optimality. That task is left until Section 4.2.4. Regarding π^*, which has the interpretation of a policy correspondence, an immediate corollary is that π^* is continuous whenever π^* is single-valued on X.

4.2.2.2. *Example: Negative Discount DP.* Let (L, Γ) be defined by (ND) and suppose the assumptions on β and ℓ in (4.3) are true. Let $\phi(x) := \ell'(0)x$ and $\psi(x) := \ell(x)$ be the boundary functions in A_4–A_5. Assumptions A_1–A_3 clearly hold. The condition $T\psi \leq \psi$ in A_4 also holds because, with $\psi = \ell$,

$$T\psi(x) = T\ell(x) = \inf_{0 \leq a \leq x} \{\ell(a) + \beta\ell(x - a)\} \leq \ell(x) + \beta\ell(0) = \ell(x). \tag{4.7}$$

Under the auxiliary assumption $\ell'(0) > 0$, we show that A_5 also holds. The details are in the appendix (see Proposition 4.7.5). Hence, under these assumptions, the conclusions of Theorem 4.2.1 are valid for the Bellman operator T defined in (4.5).

4.2.3. Shape and Smoothness Properties. We now give conditions under which the solution to the Bellman equation associated with an abstract dynamic program possesses additional properties, including monotonicity, convexity and differentiability. In what follows, we assume that X is convex in $\mathbb{R}$ and $\operatorname{gr}\Gamma$ is convex in $X \times A$. We let

1. $\mathcal{C}_i(X)$ be all increasing functions in $\mathcal{C}(X)$ and
2. $\mathcal{C}_c(X)$ be all convex functions in $\mathcal{C}(X)$.

We assume that $\mathcal{I}$ defined above contains at least one element of each set.

4.2.3.1. *Results.* To obtain convexity and differentiability, we impose

Assumption 4.2.1. In addition to A_1–A_5, the abstract dynamic program (L, Γ) satisfies the following conditions:

1. If $w \in \mathcal{C}_c(X)$, then $(x, a) \to L(x, a, w)$ is strictly convex on $\operatorname{gr}\Gamma$.
2. If $a \in \operatorname{int}\Gamma(x)$ and $w \in \mathcal{C}_c(X)$, then $x \to L(x, a, w)$ is differentiable on $\operatorname{int} X$.

We can now state the following result.

THEOREM 4.2.2. *If Tw is strictly increasing for all $w \in \mathcal{C}_i(X)$, then w^* is strictly increasing. If Assumption 4.2.1 holds, then w^* is strictly convex, π^* is single-valued, w^* is differentiable on* $\operatorname{int} X$ *and*

$$(w^*)'(x) = L_x(x, \pi^*(x), w^*) \tag{4.8}$$

whenever $\pi^*(x) \in \operatorname{int} \Gamma(x)$.

4.2.3.2. *Example: Negative Discount DP.* Using Theorem 4.2.2, we can derive properties of the negative discount dynamic program defined in (ND). In particular, we can show that the fixed point w^* of the negative discount Bellman operator T in (4.5) is strictly increasing, strictly convex, and continuously differentiable on $(0, \hat{x})$, and that π^* is single-valued and satisfies the envelope condition

$$(w^*)'(x) = \ell'(\pi^*(x)) \qquad (0 < x < \hat{x}). \tag{4.9}$$

To obtain (4.9) from (4.8), we use the change of variable $y = x - a$ to write

$$w^*(x) = \min_{0 \le a \le x} \{\ell(a) + \beta w^*(x-a)\} = \min_{0 \le y \le x} \{\ell(x-y) + \beta w^*(y)\}.$$

Differentiating the final term with respect to x and evaluating at the optimal choice gives (4.9).

We provide a more detailed proof of (4.9) and proofs of other claims from this section in Proposition 4.7.6 in the appendix. We rely on the convexity and differentiability of ℓ to check Assumption 4.2.1.

4.2.4. The Principle of Optimality. If we consider the implications of the preceding dynamic programming theory, we have obtained existence of a unique solution to the Bellman equation and certain other properties, but we still lack a definition of optimal policies, and a set of results that connect optimality and solutions to the Bellman equation. This section fills these gaps.

Let Π be all $\pi \colon X \to A$ such that $\pi(x) \in \Gamma(x)$ for all $x \in X$. For each $\pi \in \Pi$ and $w \in \mathcal{R}(X)$, define the operator T_π by

$$(T_\pi w)(x) = L(x, \pi(x), w). \tag{4.10}$$

This can be understood as the lifetime loss of an agent following π with continuation value w. Let $\mathcal{M}$ be the set of *(nonstationary) policies*, defined as all $\mu = \{\pi_0, \pi_1, \ldots\}$ such that $\pi_t \in \Pi$ for all t. For stationary policy $\{\pi, \pi, \ldots\}$, we simply refer it as π. Let the *μ-value function* be defined as

$$w_\mu(x) := \limsup_{n\to\infty} (T_{\pi_0} T_{\pi_1} \ldots T_{\pi_n} \phi)(x), \tag{4.11}$$

where ϕ is the lower bound function in $\mathcal{I}$. Note that w_μ is always well defined. The agent's problem is to minimize w_μ by choosing a policy in $\mathcal{M}$. The *value function* $\bar{w}$ is defined by

$$\bar{w}(x) := \inf_{\mu \in \mathcal{M}} w_\mu(x) \tag{4.12}$$

and the *optimal policy* $\bar{\mu}$ is such that $\bar{w} = w_{\bar{\mu}}$. We impose the following assumption.

ASSUMPTION 4.2.2. In addition to A_1–A_5, the abstract dynamic program (L, Γ) satisfies the following conditions:

1. If $(x, a) \in \operatorname{gr}\Gamma$, $v_n \geq \phi$ and $v_n \uparrow v$, then $L(x, a, v_n) \to L(x, a, v)$.
2. There exists a $\beta > 0$ such that, for all $(x, a) \in \operatorname{gr}\Gamma$, $r > 0$ and $w \geq \phi$,

$$L(x, a, w + r) \leq L(x, a, w) + \beta r. \tag{4.13}$$

Part 1 of Assumption 4.2.2 is a weak continuity requirement on the aggregator with respect to the continuation value, similar to Assumption 4 in Bloise and Vailakis (2018). Part 2 of Assumption 4.2.2 is analogous to the Blackwell's condition, with the significant exception that β in (4.13) is not restricted to be less than one.

THEOREM 4.2.3. *If Assumption 4.2.2 holds, then $w^* = \bar{w}$ and an optimal stationary policy exists. Moreover, a stationary policy π is optimal if and only if $T_\pi \bar{w} = T\bar{w}$.*

Theorem 4.2.3 shows that the fixed point of the Bellman operator is the value function and the Bellman's principle of optimality holds. It immediately follows that any selector of π^* in Theorem 4.2.1 is an optimal stationary policy.

4.2.4.1. *Example: Negative Discount DP.* Consider again the negative discount dynamic program (L, Γ) defined in (ND), under the assumptions in (4.3). Both conditions in Assumption 4.2.2 can be verified for (L, Γ). Part 1 of Assumption 4.2.2 is trivial in this setting, since $v_n \uparrow v$ pointwise clearly implies $\ell(a) + \beta v_n(x-a) \to \ell(a) + \beta v(x-a)$ at each $(x,a) \in \operatorname{gr}\Gamma$. Part 2 also holds, since for any $r > 0$ and $w \geq \phi$, we have

$$L(x, a, w + r) = \ell(a) + \beta w(x-a) + \beta r = L(x, a, w) + \beta r.$$

Hence Theorem 4.2.3 applies. In fact, in this setting we can be more explicit, by setting

$$W(x) := \min \left\{ \sum_{t=0}^{\infty} \beta^t \ell(a_t) \; : \; \{a_t\} \in \mathbb{R}_+^\infty \text{ and } \sum_{t=0}^{\infty} a_t = x \right\} \tag{4.14}$$

at each $x \geq 0$. By construction, $W(\hat{x})$ is the minimum cost in (4.2). To connect W and the fixed point w^*, we first show that (4.14) is equivalent to (4.12) and then apply Theorem 4.2.3. The details are in Proposition 4.7.9 in the appendix, which shows that W is the solution to the Bellman equation (4.4), the principle of optimality holds, and there exists a unique solution to (4.2) given by $a_t^* = \pi^*(x_t)$, where the state process is governed by $x_{t+1} = x_t - a_t^*$ and $x_0 = \hat{x}$.

The envelope condition (4.9) now evaluates to

$$W'(x_t) = \ell'(a_t^*) \tag{EN}$$

for all $t \in \mathbb{Z}$, which links marginal value to marginal disutility at optimal action. Furthermore, (EN) implies that the sequence $\{a_t^*\}$ satisfies[4]

$$\ell'(a_{t+1}^*) = \max\left\{ \frac{1}{\beta} \ell'(a_t^*), \; \ell'(0) \right\} \tag{EU}$$

[4]To see this, note that a_t^* solves $\inf_{0 \leq a \leq x_t} \{\ell(a) + \beta w^*(x_t - a)\}$. Since both ℓ and w^* are convex, elementary arguments show that either $\ell'(a_t^*) = \beta (w^*)'(x_t - a_t^*)$ or $a_t^* = x_t$. It follows from (EN) that either $\ell'(a_t^*) = \beta \ell'(a_{t+1}^*)$ or $a_{t+1}^* = 0$, which is equivalent to (EU).

for all $t \in \mathbb{Z}$, which is akin to an Euler equation with a possibly binding constraint. In the applications below we use (EN) and (EU) to aid interpretation and provide economic intuition.

It follows immediately from (EU) that $\{a_t^*\}$ is a decreasing sequence. This concurs with our intuition: future losses are given greater weight than current losses, so $\{a_t^*\}$ declines over time.

4.2.5. Additional Results. Some additional results hold for the specific case of the negative discount dynamic program introduced in Section 4.2.1.2. One result is a strong form of convergence for the Bellman operator T from (4.5). In particular, iteration always converges in finite time. The details are in Proposition 4.7.10 in the appendix. In addition, we can treat the case $\ell'(0) = 0$, which has hitherto been excluded:

PROPOSITION 4.2.4. *When $\ell'(0) = 0$, a feasible sequence $\{a_t^*\}$ solves (4.2) if and only if (EU) holds. This sequence is unique, decreasing, and satisfies $a_t^* > 0$ for all t.*

Proposition 4.2.4 shows that the Euler equation (EU) established above becomes a necessary and sufficient condition for optimality in this case. In fact, (EU) can be reduced to $\beta\ell'(a_{t+1}^*) = \ell'(a_t^*)$ when $\ell'(0) = 0$, which helps us derive analytical solutions for some of the applications below.

As the above results suggest, the set of tasks will be completed in finite time if and only if $\ell'(0) > 0$. Figure 4.1 provides an example, in which $T^n w$ converges to the fixed point w^* in 5 iterations.[5] When the agent never finishes in finite time, the corner solution $a_t^* = 0$ never binds, and the optimality result in Proposition 4.2.4 can be established through elementary arguments. The proof is in the appendix.

4.2.6. Possible Extensions. There are potential extensions to the negative discount dynamic programming problem studied above. Here we suggest a few examples. Detailed analysis of each case is left for future research.

[5]In this example, $\ell(x) = e^{10x} - 1$, $f(x) = \ell'(0)x = 10x$, $\hat{x} = 1$, and $\beta = 2$.

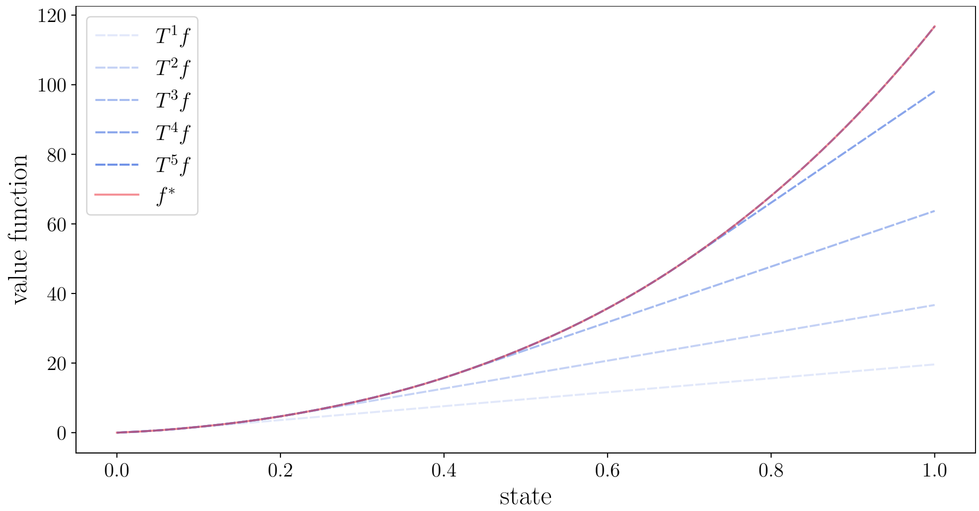


FIGURE 4.1. Convergence in finite steps when $\ell'(0) > 0$.

EXAMPLE 4.2.1. Let the disutility function ℓ and discount factor β also depend on the current state. Then the Bellman equation becomes

$$f(x) = \min_{0 \leq a \leq x} \{\ell(x, a) + \beta(x) f(x - a)\}.$$

EXAMPLE 4.2.2. We can introduce another choice variable t such that the Bellman equation is

$$f(x) = \min_{(a,t) \in \Gamma(x)} \{\ell(a, t) + \beta f(x - a, t)\},$$

where $\Gamma(x)$ gives the set of available actions when facing state x. The model in Section 4.4.2 belongs to this case.

EXAMPLE 4.2.3. Instead of assuming additively separable loss, we can define an aggregator $W : \mathbb{R}^2 \to \mathbb{R}$ such that the Bellman equation becomes

$$f(x) = \min_{0 \leq a \leq x} W(a, f(x - a)).$$

Then, (4.4) is a special case when $W(a, z) := \ell(a) + \beta z$.

4.3. Application: Production Chains

Now we turn to applications of Theorem 4.2.1 motivated by production problems. We begin with a model of linear production chains similar to the one in Kikuchi et al. (2018).

4.3.1. Set Up. Consider a large and competitive market with many price-taking firms, each of which is either inactive or involved in the production of a single good. To produce a unit of this good requires implementing a sequence of tasks, indexed by $s \in [0, 1]$. A production chain is a collection of firms that implements all of these tasks and produces the final good. Firms face no fixed costs or barriers to entry.

Let $c(v)$ be the cost for any one firm of implementing an interval of tasks with length v. We assume that c is increasing, strictly convex, continuously differentiable, and satisfies $c(0) = 0$. (Unlike Kikuchi et al. (2018), we allow $c'(0) = 0$.) Firms face transaction costs, as a wedge between price paid by the buyer and payment received by the seller.[6] For convenience, we assume that the transaction cost falls entirely on the buyers, so that, for a transaction with face value f, the seller receives f and the buyer pays $(1 + \tau)f$, where $\tau > 0$.[7]

Firms are indexed by integers $i \geq 0$. A feasible allocation of tasks across firms is a nonnegative sequence $\{v_i\}$ with $\sum_{i \geq 0} v_i = 1$. We identify firm 0 with the most downstream firm, firm 1 with the second most downstream firm, and so on. Let b_i be the downstream boundary of firm i, so that $b_0 = 1$ and $b_{i+1} = b_i - v_i$ for all $i \in \mathbb{Z}$. Then, profits of the ith firm are

$$\pi_i = p(b_i) - c(v_i) - (1 + \tau)p(b_{i+1}). \tag{4.15}$$

Here $p \colon [0, 1] \to \mathbb{R}_+$ is a price function, with $p(t)$ interpreted as the price of the good at processing stage t.

Definition 4.3.1. Given a price function p and a feasible allocation $\{v_i\}$, let $\{\pi_i\}$ be corresponding profits, as defined in (4.15). The pair $(p, \{v_i\})$ is called an *equilibrium* for the production chain if

1. $p(0) = 0$,
2. $p(s) - c(s - t) - (1 + \tau)p(t) \leq 0$ for any pair s, t with $0 \leq t \leq s \leq 1$, and

[6] This follows Kikuchi et al. (2018) and also studies such as Boehm and Oberfield (2018), where frictions in contract enforcement are treated as a variable wedge between effective cost to the buyer and payment to the supplier.

[7] For example, τf might be the cost of writing a contract for a transaction with face value f. This cost rises in f because more expensive transactions merit more careful contracts. (There are other possible interpretations for τ, some of which are touched on below.)

3. $\pi_i = 0$ for all i.

Condition 1 rules out profits for suppliers of initial inputs, which are assumed for convenience to have zero cost of production. Condition 2 ensures that no firm in the production chain has an incentive to deviate, and that inactive firms cannot enter and extract positive profits. Condition 3 requires that active firms make zero profits, due to free entry and an infinite fringe of potential competitors.

4.3.2. Solution by Dynamic Programming. Note that an equilibrium of the production chain satisfies $p(b_i) = c(v_i) + (1+\tau)p(b_i - v_i)$, which has the same form as the Bellman equation (4.4). Moreover, iterating on this relation yields the price of the final good

$$p(1) = \sum_{i \geq 0} (1+\tau)^i c(v_i), \tag{4.16}$$

which is analogous to (4.2). These facts motivate us to consider a version of the negative discount dynamic program introduced in Section 4.2.1.2 where a (fictitious) agent seeks to minimize $\sum_{i \geq 0}(1+\tau)^i c(a_i)$ subject to $\sum_{i \geq 0} a_i = 1$. In other words, we specialize the problem to one where $\hat{x} = 1$, $\ell = c$ and $\beta = 1 + \tau$. By construction, any feasible action path is also a feasible allocation of tasks in the production chain.

Since the assumptions in Section 4.2.1.2 are satisfied, we know that there exists a unique solution $\{a_i^*\}$. Let W be the corresponding value function given by (4.14). The next proposition shows that the solution to this dynamic program is precisely the competitive equilibrium of the Coasian production chain described above. In view of (4.16), it follows that the equilibrium allocation also gives the minimum price for the final good.

PROPOSITION 4.3.1. *Let $p = W$ and $v_i = a_i^*$ for all $i \in \mathbb{Z}$. Then the pair $(p, \{v_i\})$ is an equilibrium for the production chain.*

One insight from this result is as follows. We know from Section 4.2.3.2 that the price function is continuously differentiable on $(0,1)$ and, for firm with downstream boundary b_i,

$$p'(b_i) = c'(v_i), \tag{4.17}$$

which follows from the envelope condition (EN). Since v_i is the optimal range of tasks implemented in-house by firm i in equilibrium, this is an expression of Coase's key idea: the size of the firm is determined as the scale that equalizes the marginal costs of in-house and market-based operations. This trade-off is further clarified in the Euler equation (EU), which says that each firm achieves optimum when the cost of implementing an additional task equals the cost of purchasing it from a supplier. The Euler equation (EU) also implies that $\{v_i\}$ is decreasing. In other words, firm size increases with downstreamness. This generalizes a finding of Kikuchi et al. (2018).

4.3.3. An Example. Suppose that the range of tasks v implemented by a given firm satisfies $v = f(k, n)$, where k is capital and n is labor. Given rental rate r and wage rate w, the cost function is $c(v) := \min_{k,n}\{rk + wn\}$ subject to $f(k, n) \geq v$. Let us suppose further that, as in Lucas (1978b), the production function has the form $\phi(g(k, n))$, where g has constant returns to scale and ϕ is increasing and strictly concave, with the latter property due to "span-of-control" costs. To generate a closed-form solution, we take $g(k, n) = Ak^\alpha n^{(1-\alpha)}$ and $\phi(x) = x^\eta$, with $0 < \alpha, \eta < 1$. The resulting cost function has the form $c(v) = \kappa v^{1/\eta}$, where κ is a positive constant.

By Proposition 4.3.1, the optimal action path for the fictitious agent corresponds to the equilibrium allocation of tasks across firms, and the value function is the equilibrium price function. Since $c'(0) = 0$, Proposition 4.2.4 applies and the Euler equation (EU) yields $a^*_{i+1} = \theta a^*_i$ for all $i \in \mathbb{Z}$, where $\theta := (1 + \tau)^{\eta/(\eta-1)} < 1$. From $\sum_{i=0}^{\infty} a^*_i = 1$ we obtain $v_i = a^*_i = \theta^i(1 - \theta)$. Substituting this path into (4.14) gives the price function

$$p(x) = W(x) = \kappa\,(1 - \theta)^{(1-\eta)/\eta}\, x^{1/\eta}. \tag{4.18}$$

As anticipated by the theory, p is strictly increasing and strictly convex.

Although this example lies outside the framework of Kikuchi et al. (2018), since $c'(0) = 0$, we have replicated some of their key results. For example, we have found that the size of firms increases from upstream to downstream (recall that upstream firms have larger i), and that the price function is strictly convex due to the costly span of control. Intuitively, firm-level span-of-control costs cannot be eliminated in aggregate due to transaction costs, which force firms to maintain a certain size. This leads to strict convexity of prices. If firms have constant returns to management ($\eta = 1$), then the price function in (4.18) becomes linear.

4.4. Application: Networks

In this section we treat more general network models. Unlike the linear production chains discussed above, agents can interact with multiple partners. As before, our objective is to apply the dynamic programming theory developed in Section 4.2.

4.4.1. Spatial Networks. The distribution of city sizes shows remarkable regularity, as described by the rank-size rule.[8] One early attempt to match the empirical city size distribution is found in the central place theory of Christaller (1933). Hsu (2012) formalizes Christaller's theory in a model where a city hierarchy arises as a market equilibrium, while Hsu et al. (2014) shows that the market equilibrium allocation is identical to the social planner's solution. In this section, we show how a model similar to that of Hsu (2012) can be studied using the dynamic programming theory from Sections 4.2. We then use the Euler equation and envelop condition to gain insights into how a city hierarchy is formed.

Consider a government that opens competition for many developers to build cities to host a continuum of dwellers of measure one. One developer can build a large city that hosts everyone or build a smaller city and assign other developers to build "satellite cities" that host the rest of the population. Further satellites can be built for existing cities until all dwellers are accommodated. This chain of building layers of cities starts with a single developer, who is assigned the whole population. Building satellite cities incurs extra costs that are charged as an ad valorem tax on the payments to the developers. We can think of the extra costs as costs of providing public goods that connect different cities such as roads, electricity, water, telecommunication, etc.

Let developers be paid according to a price $p\colon [0,1] \to \mathbb{R}$, which is a function of the population assigned. Let the cost function of building and expanding a city be $c\colon [0,1] \to$

[8] See Gabaix and Ioannides (2004) and Gabaix (2009) for surveys.

$\mathbb{R}$ and the tax rate be τ. Then, a developer assigned to host s dwellers maximizes profits by solving

$$\max_{0\leq t\leq s}\left\{p(s)-c(s-t)-(1+\tau)kp(t/k)\right\},$$

where $p(s)$ is the payment to the developer, $c(s-t)$ is the cost of building a city of population $s-t$, k is the number of satellite cities, and $(1+\tau)kp(t/k)$ is the cost of assigning population t/k to k satellites. In equilibrium, a city network is formed where every dweller is accommodated and every developer makes zero profits. The equilibrium price function satisfies

$$p(s)=\min_{0\leq t\leq s}\left\{c(s-t)+(1+\tau)kp(t/k)\right\}, \tag{4.19}$$

which is a Bellman equation similar to (4.4) in Section 4.2.1.2. We let $c(s)=s^\gamma$ with $\gamma>1$. To emulate the bifurcation process in Hsu (2012) and Hsu et al. (2014), we let $k=2$.

We can formulate a dynamic programming problem similar to (4.2) and show that the value function satisfies (4.19). Consider now the same problem from the perspective of a social planner who minimizes the total cost of hosting the whole population with value function

$$W(x):=\min_{\{v_i\}}\left\{\sum_{i=0}^{\infty}(1+\tau)^i k^i c(v_i):\{v_i\}\in\mathbb{R}_+^\infty \text{ and } \sum_{i=0}^{\infty}k^i v_i=x\right\},$$

where v_i is the size of cities on layer i. Since the assumptions in Section 4.2.1.2 are satisfied, a similar argument to the proof of Proposition 4.2.4 gives the Euler equation

$$c'(v_i)=(1+\tau)c'(v_{i+1}). \tag{4.20}$$

Using this equation, it can be shown with some algebra that $v_i=\theta^i(1-2\theta)$ if $\theta:=(1+\tau)^{1/(1-\gamma)}<1/2$ and the value function is $W(s)=(1-2\theta)^{\gamma-1}s^\gamma$. It is straightforward to verify that $p=W$ satisfies (4.19). Hence, the minimum value that can be achieved is also the equilibrium price function under which no developer makes positive profits.

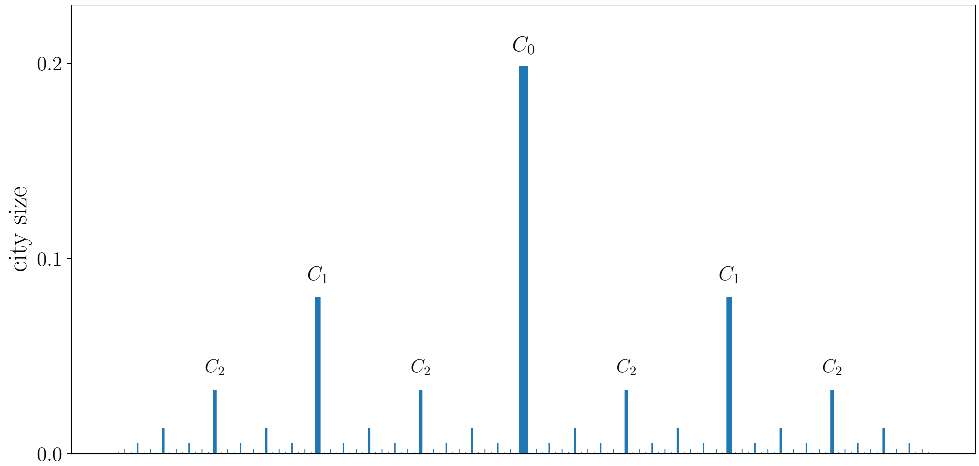


FIGURE 4.2. Illustration of optimal city hierarchy.

The Euler equation (4.20) describes the emergence of optimal city hierarchy where each developer expands a city to accommodate more dwellers until the marginal cost of expanding equals the marginal cost of building and expanding satellite cities. An envelope condition similar to (EN) also holds: if a developer is assigned s dwellers and delegate t dwellers to satellite cities, the equilibrium is reached when $p'(s) = c'(s - t)$. This shows that the marginal value that a city provides must be equal to the marginal cost of accommodating one more city dweller.

Figure 4.2 illustrates the optimal city hierarchy by placing cities according to Hsu (2012) and Hsu et al. (2014), where $\mathcal{C}_i$ represents a city on layer i.[9] It replicates the relative sizes of cities on different layers as in Hsu (2012) and Hsu et al. (2014). Moreover, since the number of cities doubles from one layer to the next, the rank of a city on layer i is 2^i. Hence, the city size distribution generated by our model follows a power law similar to Hsu (2012). In fact, the rank and size of a city satisfy

$$\ln(Rank) = -\frac{\ln(1/2)}{\ln(\theta)} \ln(Size) + C,$$

where C is a constant determined by θ. When θ approaches $1/2$, the slope approaches one, which corresponds to the well-documented rank-size rule.

[9]We set $\gamma = 1.2$ and $\tau = 0.2$.

4.4.2. Snakes and Spiders. Modern production networks are characterized by processes that are both sequential and non-sequential where firms assemble parts in no particular order. There are costs associated with extending each process and changes in those costs affect how production networks are formed. Baldwin and Venables (2013) refer to the sequential process as "snakes" and the non-sequential process as "spider", and analyze how the location of different parts of a production chain is determined by unbundling costs of production across borders. Here we show that the dynamic programming theory developed in Section 4.2 can be used to solve a general production network, which can replicate key results in the literature.

As in Kikuchi et al. (2018) and Yu and Zhang (2019), we consider a generalization of the production chain model in Section 4.3, where each firm can also choose the number of suppliers. The production chain then becomes a combination of "snakes" and "spiders". To account for costs of extending "spiders" we assume that firms bear an additive assembly cost g that is strictly increasing in the number of suppliers, with $g(1) = 0$. Then for a firm at stage s that subcontracts tasks of range t to k suppliers, the profits are

$$p(s) - c(s-t) - g(k) - (1+\tau)kp(t/k),$$

where p is the price function. Having multiple suppliers leads to another trade-off: firms potentially benefit from subcontracting at a lower price but also have to pay additional assembly costs.

We index the layers in the production network by $i \in \mathbb{Z}$ with layer 0 consisting only of the most downstream firm. Let b_i be the downstream boundary of firms on layer i, each producing v_i and having k_i suppliers. Then the boundary of firms on the next layer is given by $b_{i+1} = (b_i - v_i)/k_i$. Similar to Definition 4.3.1, we call the triplet $(p, \{v_i\}, \{k_i\})$ an equilibrium for the production network if (i) $p(0) = 0$, (ii) $p(s) - c(s-t) - g(k) - (1+\tau)kp(t/k) \leq 0$ for all $0 \leq t \leq s \leq 1$ and $k \in \mathbb{N}$, and (iii) $\pi_i = 0$ for all $i \in \mathbb{Z}$ where

$$\pi_i := p(b_i) - c(v_i) - g(k_i) - (1+\tau)kp\left(\frac{b_i - v_i}{k_i}\right). \tag{4.21}$$

As in Section 4.3.2, we seek to find an equilibrium using dynamic programming methods. Let p^* be the solution to the following Bellman equation

$$p(s) = \min_{\substack{0 \le t \le s \\ k \in \mathbb{N}}} \{c(s-t) + g(k) + (1+\tau)kp(t/k)\}\,. \tag{4.22}$$

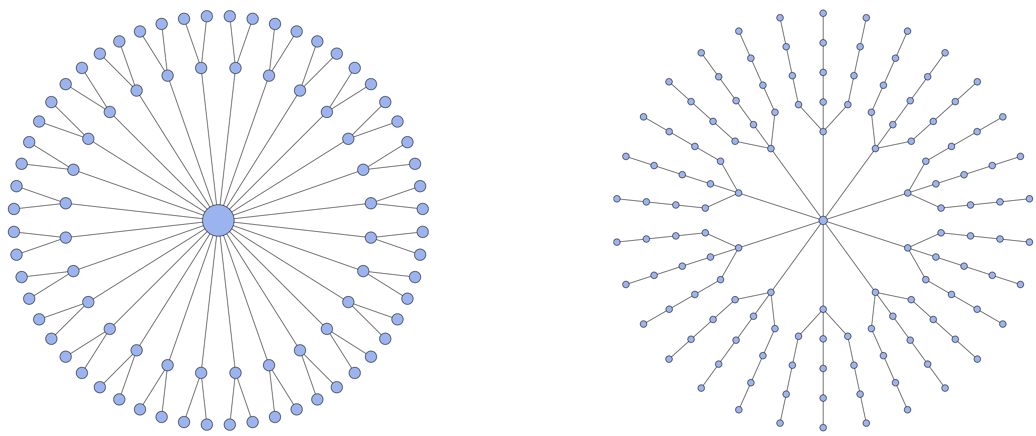
Let $v_i = b_i - t^*(b_i)$ and $k_i = k^*(b_i)$ where $t^*(s)$ and $k^*(s)$ are the minimizers under p^*. Let $\mathcal{I}$ be all continuous p such that $c'(0)s \le p(s) \le c(s)$ for all $s \in [0,1]$. Then we have the following result similar to Proposition 4.3.1.

Proposition 4.4.1. *If $c'(0) > 0$ and $g(k) \to \infty$ as $k \to \infty$, then (4.22) has a unique solution $p^* \in \mathcal{I}$ and $(p^*, \{v_i\}, \{k_i\})$ is an equilibrium for the production network.*

In the appendix, we show that the production network model also fits in our general dynamic programming framework developed in Section 4.2. In particular, assumptions A_1–A_5 hold so that Theorem 4.2.1 can be applied to the Bellman equation (4.22). Therefore, there exists a unique solution p^* that can be computed by value function iteration. We then prove that p^* induces an equilibrium allocation. Theorem 4.2.2 can also be used to show the monotonicity of p^*.

Figure 4.3 plots two production networks with different transaction costs, where each node corresponds to a firm in the network and the one in the center is the most downstream firm.[10] The size of each node is proportional to the size of the firm, represented by the sum of assembly and transaction costs. Figure 4.3 shows that more downstream firms are larger and have more upstream suppliers. Comparing panels (A) and (B), we can see that lower transaction costs increase the number of firms involved in the production network, encouraging the expansion of snakes. This is in line with the model prediction of Baldwin and Venables (2013) that decreasing frictions leads to a finer fragmentation of the production. Also see Tyazhelnikov (2019) and Acemoglu and Azar (2020) for models with similar features.

[10] We set $c(v) = v^{1.5}$ and $g(k) = 0.0001(k-1)^{1.5}$.



(A) $\tau = 0.2$ (B) $\tau = 0.05$

FIGURE 4.3. Examples of production networks.

4.5. Application: Knowledge and Communication

Many firms are characterized by a pyramidal structure, in which employees are organized into management layers with each layer smaller than the previous one. These features have been modeled in the pioneering work by Garicano (2000) and the following literature. The key idea in Garicano's theory of hierarchical organization of knowledge is a trade-off between the cost of acquiring problem solving knowledge and the cost of communicating with others for help. His model highlights the impact that information and communication technology has on organizational design such as the number of management layers and the scope of production of workers in each layer. In this section, we solve a version of Garicano's model using the dynamic programming theory from Section 4.2.

Consider a model where production of a firm requires its employees solve a set of problems denoted by $[0, 1]$. Following Garicano (2000) (Section V.F), we suppose that there is a market for knowledge within the firm and each management layer solves a profit maximization problem. Suppose that employees at management layer i are assigned problems $m_i \in [0, 1]$. They learn to solve z_i at cost $c(z_i)$ and pass on the remainder $m_{i+1} = m_i - z_i$ to the next management layer $i + 1$ for help. This incurs additional communication costs τ that are proportional to the value of problems assigned to layer $i + 1$. Let $p\colon [0, 1] \to \mathbb{R}$ be the function of the value of problems in the internal market. Then, profits of the ith management layer are

$$\pi(m_i, z_i) = p(m_i) - (1+\tau)p(m_i - z_i) - c(z_i),$$

where $p(m_i)$ is the value of problems assigned to layer i, $(1+\tau)p(m_i - z_i)$ is the cost of communicating and assigning unsolved problems to the next layer, and $c(z_i)$ is the cost of learning to solve z_i.

Setting profits to zero and minimizing with respect to m_{i+1} yield the equation

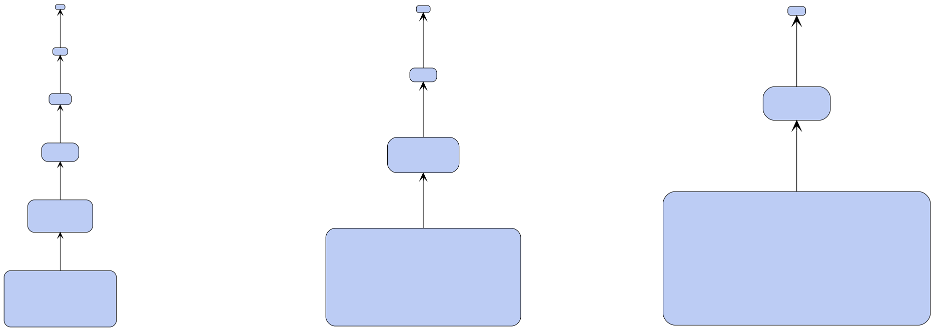
$$p(m_i) = \min_{m_{i+1} \le m_i} \{c(m_i - m_{i+1}) + (1+\tau)p(m_{i+1})\}.$$

This parallels the Bellman equation (4.4) in the negative discount dynamic programming in Section 4.2.1.2.

Let n be the number of employees in a given layer, and suppose that n is related to learning to solve z via $z = f(n)$. In other words, for a given range of problems z, the number of employees required to solve z is $n = f^{-1}(z)$. Assume that f is strictly increasing, strictly concave, and continuously differentiable with $f(0) = 0$, and that $c(z) = wn = wf^{-1}(z)$ for some wage rate w. Then the assumptions in Section 4.2.1.2 are satisfied if we let $\ell = c$ and $\beta = 1 + \tau$. The Euler equation (EU) implies that the optimal sequence $\{z_i\}$ is decreasing, so is the number of employees at each layer as $n_i = c(z_i)/w$. This replicates Garicano's result that the top management layer has the smallest number of employees and each layer below is larger than the one above.

The Euler equation (EU) suggests that the pyramidal structure of the span of control arises in equilibrium where each tier of management acquires knowledge up to the point where the marginal cost of learning to solve problems within the tier equals the marginal cost of communicating and assigning unsolved problems to the next layer. The envelope condition (EN) implies $p'(m_i) = c'(z_i)$, which says that, in equilibrium, the marginal value of problems assigned to a management layer equals the marginal cost of learning to solve problems within the tier.[11]

[11]This result is analogous to (4.17) for the production chain model and reminiscent of Coase's theory of the firm in the context of knowledge organization within a firm.



(A) $\tau = 0.2$ (B) $\tau = 0.4$ (C) $\tau = 0.6$

FIGURE 4.4. Optimal organizational structures.

Figure 4.4 plots the optimal organizational structures of three firms given by the model above.[12] Each node corresponds to one management layer, who asks the layer above for help, and its size is proportional to the number of employees in that layer. As shown in the graphs, each firm has a pyramidal structure and higher communication costs increase the relative knowledge acquisition of lower layers and reduce the number of layers.

4.6. Continuous Time: Theory and Applications

Next we treat dynamic optimization in continuous time, giving results that parallel the discrete time results in Section 4.2. We then show how such results can be connected to production chains and the theory of the firm.

4.6.1. An Infinite Horizon Problem. The continuous time version of problem (4.2) is

$$\min_{a(t)} \int_0^\infty e^{\rho t} \ell(a(t)) dt \tag{4.23}$$

subject to feasibility of $a(t)$, which means that $a\colon \mathbb{R}_+ \to \mathbb{R}_+$ has at most finitely many points of discontinuity and satisfies $\int_0^\infty a(t)dt = \bar{x}$.

ASSUMPTION 4.6.1. The constant ρ is strictly positive, while $\ell\colon \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable, strictly increasing, strictly convex and satisfies $\ell(0) = 0$.

Positivity of ρ indicates that future losses are given more weight than current ones. The conditions on ℓ are identical to those in (4.3).

[12] We set $c(z) = z^{1.2}$ and $m_0 = 1$.

Now let us consider a recursive formulation, to parallel the discrete time results in Section 4.2. Given a control $a(t)$, the state path evolves according to $\dot{x}(t) = -a(t)$ with $x(0) = \bar{x}$ and can be calculated by

$$x(t) = \bar{x} - \int_0^t a(s)ds = \int_t^\infty a(s)ds. \tag{4.24}$$

Similar to (4.14), we can also define the value function by

$$F(x) := \min_{a(t)} \left\{ \int_0^\infty e^{\rho t} \ell(a(t))dt \ : \ a(t) \geq 0 \text{ and } \int_0^\infty a(t)dt = x \right\}, \quad \forall x \in \mathbb{R}_+, \tag{4.25}$$

which is the minimal cost when the amount of tasks to be completed is x. We then have the following continuous time version of Theorem 3.2.1.

Theorem 4.6.1. *If Assumption 4.6.1 holds, then*

1. *there exists a unique feasible solution a^* to (4.23). It satisfies*

$$a^*(t) = \arg\min_{a \geq 0} \left\{ e^{\rho t} \ell(a) + \lambda a \right\} \tag{4.26}$$

where λ is a constant uniquely determined by the feasibility constraint

$$\int_0^\infty a^*(t)dt = \bar{x}. \tag{4.27}$$

2. *The optimal action $a^*(t)$ is decreasing in t. Moreover,*

   C1. if $\ell'(0) = 0$, then $\ell'(a^*(t)) = -\lambda e^{-\rho t}$ and $a^*(t) > 0$ for all t; and

   C2. if $\ell'(0) > 0$, then there is a finite $\bar{T}$ such that $a^*(t) = 0$ for all $t \geq \bar{T}$.

3. *The value function $F(x)$ is differentiable when $x > 0$, and it satisfies*

$$-\rho F(x) = \inf_{a \geq 0} \{\ell(a) - F'(x)a\} \tag{4.28}$$

with boundary condition $F(0) = 0$ and the optimal action a^* at state x satisfies

$$-\rho F(x) = \inf_{a \geq 0} \{\ell(a) - F'(x)a\} = \ell(a^*) - F'(x)a^*. \tag{4.29}$$

Equation (4.26) can be seen as the continuous time Euler equation. For example, if $\ell'(0) = 0$, then, by C1 of Theorem 4.6.1, we have $\ell'(a^*(t)) = -e^{-\rho t}\lambda$, and hence

$$\ell'(a^*(t')) = e^{\rho(t-t')}\ell'(a^*(t)) \quad \text{for any } t' > t.$$

This is a continuous time version of (EU). Furthermore, if a^* is an interior solution in (4.29), we have $F'(x) = \ell'(a^*)$, which is similar to (EN) in the discrete case and gives an envelope-like condition between the value function and the loss function. The proof can be found in Appendix 4.7.5.1

Example 4.6.1. Consider again the case $\ell(x) = x^\gamma$ with $\gamma > 1$, previously considered in discrete time. Since $\ell'(0) = 0$, Theorem 4.6.1 implies that $a^*(t) = (-\lambda e^{-\rho t}/\gamma)^{1/(\gamma-1)}$. Using (4.27) to pin down λ and substituting into the solution gives

$$a^*(t) = \theta \bar{x} e^{-\theta t}. \quad \text{where} \quad \theta := \frac{\rho}{\gamma - 1}. \tag{4.30}$$

Combining (4.24) and (4.30), we have

$$x^*(t) = \int_t^\infty a^*(s) ds = \bar{x} e^{-\frac{\rho}{\gamma-1}t}.$$

Figure 4.5 illustrates the relationship between optimal action $a^*(t)$ and the resulting state path $x^*(t)$ where we set $\gamma = 2$, $\rho = 1$, and $\bar{x} = 1$. Notice that the optimal action $a^*(t)$ decreases over time consistent with Theorem 4.6.1 and the state path is computed according to (4.24).

Now the optimal action can be expressed as $a^* = \rho x^*/(\gamma-1)$, which is always proportional to the state. This relation demonstrates the trade-off between current loss and negatively discounted future losses. If γ is large, the agent will choose to complete a smaller portion of the remaining tasks each time because the loss function $\ell(a)$ grows rapidly with a. On the other hand, if ρ is large, the agent will try to complete the tasks faster because of the greater weight given to future losses indicated by $e^{\rho t}$. Plugging a^* into (4.25) gives the value function

$$F(x) = \left(\frac{\rho}{\gamma - 1}\right)^{\gamma - 1} x^\gamma,$$

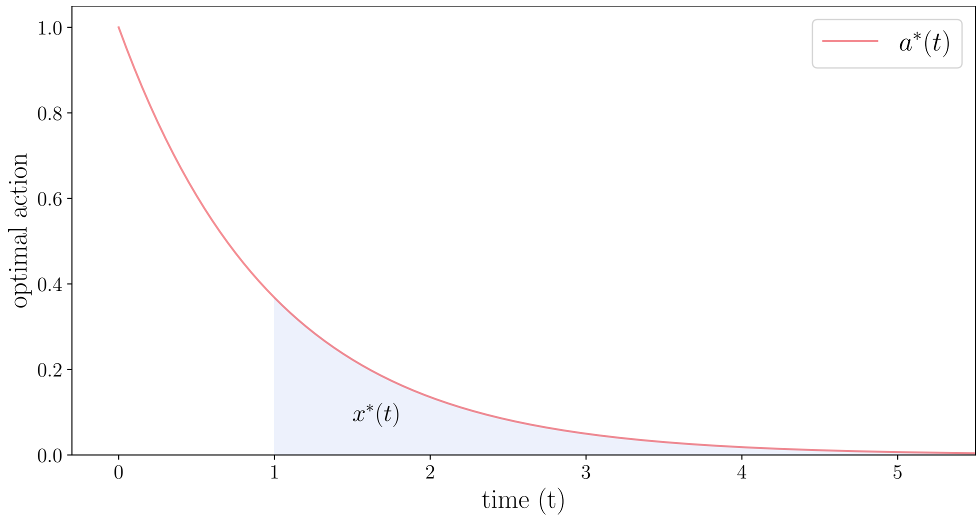


FIGURE 4.5. Optimal action and optimal state path

which is increasing in the amount of tasks x. Also, given x, the value function is increasing in the discount factor ρ. Although the agent will try to finish the tasks faster when ρ increases, minimal cost is still larger.

4.6.2. An Application of the Continuous Time Theory. This section shows how the continuous time negative discount dynamic programming results can be applied to production chains and the theory of the firm. To begin, let us consider a version of the Coasian production chain model in Section 4.3 that is essentially parallel except that there is a continuum of firms.

Firms are indexed by $i \in \mathbb{R}_+$, with $i = 0$ being the most downstream firm. Let $a(i)$ be the range of tasks firm i chooses to implement and let $p(i)$ be the price at which firm i sells the partially completed good. The cost of producing $a(i)$ is $c\,(a(i))$. To finish the final product, $a(\cdot)$ has to satisfy $\int_0^\infty a(i)di = 1$. Buyers bear transaction costs, so firm i pays $\tau\, di \cdot p(i + di)$ for every unit purchased from firm $i + di$ where $\tau > 0$. Hence, the total cost for firm i is

$$c(a(i))di + (1 + \tau\, di) \cdot p(i + di).$$

In equilibrium, firms optimize and make zero profit, so any price function $p(i)$ and optimal production $a^*(i)$ must satisfy[13]

$$\begin{aligned} 0 &= c(a^*(i)) + \tau p(i) + p'(i) \\ &= \min_a \{c(a(i)) + \tau p(i) + p'(i)\}. \end{aligned} \tag{4.31}$$

To utilize our continuous time theory in Section 4.6.1, we define a new price function $F : [0,1] \to \mathbb{R}_+$ by $F(x(i)) := p(i)$, where $x(i)$ corresponds to the stage[14] at which firm i sells its product.

Given differential equation (4.31), the equilibrium price function F, optimal production a^*, and optimal path x^* must satisfy

$$\begin{aligned} 0 &= c(a^*(i)) + \tau F(x^*(i)) - F'(x^*(i))a^*(i) \\ &= \min_a \{c(a(i)) + \tau F(x^*(i)) - F'(x^*(i))a(i)\}, \end{aligned} \tag{4.32}$$

where we use the fact that $(x^*)' = -a^*$. Since we assume that initial inputs have zero cost, a boundary condition $F(0) = 0$ must also be satisfied.

To find the equilibrium price and optimal production function, we first suppose that there is a social planner who minimizes the price of the final good $p(0)$. By solving the differential equation (4.31), we have

$$p(0) = \int_0^\infty e^{\tau i} c(a(i))di + C,$$

where C is any constant. Therefore, the planner solves

$$\min_{a(i)} \int_0^\infty e^{\tau i} c(a(i))di \quad \text{s.t.} \quad a(i) \geq 0 \quad \text{and} \quad \int_0^\infty a(i)di = 1,$$

which is the same as problem (4.23) with $\bar{x} = 1$.

It follows from Theorem 4.6.1 that this problem has exactly one solution, the value function $F(\cdot)$ corresponding to the equilibrium price satisfies the differential equation (4.32) and the boundary condition $F(0) = 0$, and the optimal production $a^*(i)$ is decreasing in i,

[13] Here we write the equilibrium conditions in differential form.

[14] This is parallel to the state path (4.24) in Section 4.6.1. If the firms in the chain produce according to $a(\cdot)$, the stage for firm i can be computed by $x(i) = \int_i^\infty a(j)dj$. For example, the stage for firm 0 is $x(0) = \int_0^\infty a(j)dj = 1$.

suggesting that firm size is increasing in downstreamness. Moreover, by the remarks after Theorem 4.6.1, the solutions also satisfy $F'(x^*) = c'(a^*)$, a similar condition to (4.17) in discrete time, which says that firms expand until the marginal cost of in-house production equals the marginal cost of purchasing from a supplier.

4.6.3. Extension: A Finite Horizon Problem. Next we consider a finite horizon version of the negative discount dynamic programming problem. We then show that this problem also has applications in the study of equilibria in production chains.

The objective is to choose feasible action path a and a terminal date T to solve

$$\min_{a(t),T} \int_0^T e^{\rho t} \ell(a(t))dt + L(T) \quad \text{s.t.} \quad a(t) \geq 0 \quad \text{and} \quad \int_0^T a(t)dt = \bar{x} \tag{MLCF}$$

The terminal cost function L is assumed to be increasing, twice continuously differentiable, and to satisfy $L(t) \to \infty$ as $t \to \infty$. Note that the time horizon itself is also a choice variable.

THEOREM 4.6.2. *Let Assumption 4.6.1 hold and suppose, in addition, that $\ell'(0) = 0$. Let λ and T^* be constants and let*

$$a^*(t) := (\ell')^{-1}(-\lambda e^{-\rho t}). \tag{4.33}$$

If λ and T^* are such that $\int_0^{T^*} a^*(t)dt = \bar{x}$ and $-\lambda a^*(T^*) - e^{\rho T^*} \ell(a^*(T^*)) - L'(T^*) = 0$ both hold, then a^* solves (MLCF) and $a^*(t)$ is decreasing in t.

This theorem gives sufficient conditions for the control function to be optimal in the finite horizon problem (MLCF). Compared with Theorem 4.6.1, there is an additional condition involving $L'(T^*)$ because the agent is also choosing the terminal date in this problem. An application that uses this theorem is discussed below.

In Fally and Hillberry (2018), a production chain for a single final product consists of firms across multiple countries. We restrict our attention to a country that imports an intermediate good and exports a partially finished product after sequential production along the chain. As in Section 4.6.2, firms face transaction costs and diseconomies of scope and are indexed by $i \in \mathbb{R}_+$. Given the import price B and the amount of production to be completed $\bar{x}$, a social planner minimizes the price of the export good by choosing not only the amount each firm produces $a(i)$, but also the total "number" of firms I in this country. Due to transaction costs, the price of the export good is

$$\int_0^I e^{\tau i} c(a(i)) di + Be^{\tau I},$$

where the first part is the total cost along the chain and the second part is from the import price. The social planner's problem is thus

$$\min_{a(i),I} \int_0^I e^{\tau i} c(a(i)) di + Be^{\tau I} \quad \text{s.t.} \quad a(i) \geq 0 \quad \text{and} \quad \int_0^I a(i) di = \bar{x}.$$

To solve the planner's problem, we can directly apply Theorem 4.6.2, which offers sufficient conditions for the optimal solutions. We show in the appendix that all the conditions are satisfied for the parameterizations in Fally and Hillberry (2018), and thus their proposed solution is indeed optimal. Theorem 4.6.2 complements their necessity results in that it provides a way to test the optimality of any solution. It is also able to deal with a wider range of functional forms beyond what is discussed above. A more general version of Theorem 4.6.2 can also be found in Appendix 4.7.5.2.

4.7. Appendix

4.7.1. Proofs for Section 4.2.

Proof of Theorem 4.2.1. By A_1 and Berge's theorem of the maximum, Tw is continuous. Hence T maps $\mathcal{C}(X)$ to itself. It follows directly from A_2 that T is *isotone* on $\mathcal{C}(X)$, in the sense that $u \leq v$ implies $Tu \leq Tv$. Conditions A_4–A_5 and the isotonicity of T imply that, when $\phi \leq w \leq \psi$, we have $\phi \leq T\phi \leq Tw \leq T\psi \leq \psi$. In particular, T is an isotone self-map on $\mathcal{I}$.

The Bellman operator is also concave on $\mathcal{I}$, in the sense that

$$0 \leq \lambda \leq 1 \text{ and } u, v \in \mathcal{I} \text{ implies } \lambda Tu + (1-\lambda)Tv \leq T(\lambda u + (1-\lambda)v). \tag{4.34}$$

Indeed, fixing such λ, u, v and applying A_3, we have

$$\min_{a \in \Gamma(x)} \{\lambda L(x,a,u) + (1-\lambda) L(x,a,v)\} \leq \min_{a \in \Gamma(x)} L(x,a,\lambda u + (1-\lambda) v)$$

for all $x \in X$. Since, for any pair of real valued functions f, g we have $\min_a f(a) + \min_a g(a) \leq \min_a (f(a) + g(a))$, it follows that (4.34) holds.

The preceding analysis shows that T is an isotone concave self-map on $\mathcal{I}$. In addition, by A_4 and A_5, we have $T\psi \leq \psi$ and $T\phi \geq \phi + \epsilon(\psi - \phi)$ for some $\epsilon > 0$. Since $\mathcal{I}$ is an order interval in the positive cone of the Banach space $(\mathcal{C}(X), \|\cdot\|)$, and since that cone is normal and solid, the first two claims in Theorem 4.2.1 are now confirmed via Theorem 2.1.2 of Zhang (2013). The final claim is due to Berge's theorem of the maximum. $\square$

Proof of Theorem 4.2.2. The first part of the theorem follows directly from the fact that $\mathcal{C}_i(X)$ is a closed subspace. The proof is omitted. To prove the strict convexity of w^*, it suffices to show that Tw is strictly convex for all $w \in \mathcal{C}_c(X)$ since $\mathcal{C}_c(X)$ is a closed subspace of $\mathcal{C}(X)$. Pick any $x_1, x_2 \in X$ with $x_1 < x_2$ and any $\lambda \in (0,1)$. Let $x_\lambda = \lambda x_1 + (1-\lambda) x_2$. Pick any $w \in \mathcal{C}_c(X)$ and let $\pi_w \colon X \to A$ be such that $(Tw)(x) = L(x, \pi_w(x), w)$. It follows that

$$\begin{aligned}
\lambda (Tw)(x_1) + (1-\lambda)(Tw)(x_2) &= \lambda L(x_1, \pi_w(x_1), w) + (1-\lambda) L(x_2, \pi_w(x_2), w) \\
&> L(x_\lambda, \lambda \pi_w(x_1) + (1-\lambda)\pi_w(x_2), w) \\
&\geq L(x_\lambda, \pi_w(x_\lambda), w) = (Tw)(x_\lambda),
\end{aligned}$$

where the first inequality holds because $(x,a) \mapsto L(x,a,w)$ is strictly convex and the second inequality holds because $\operatorname{gr}\Gamma$ is convex. Therefore, w^* is strictly convex. Strict convexity of L then implies that π^* is single-valued.

Since $\pi^*(x) \in \operatorname{int}\Gamma(x)$ and Γ is continuous, there exists an open neighborhood D of x such that $\pi^*(x) \in \operatorname{int}\Gamma(y)$ for all $y \in D$. Define $W(y) := L(y, \pi^*(x), w^*)$ for all $y \in D$. Then $W(y) \geq w^*(y)$ for all $y \in D$ and $W(x) = w^*(x)$. Since W is convex and differentiable on D, differentiability of w^* and (4.8) then follow from Benveniste and Scheinkman (1979). $\square$

We say that a dynamic programming problem has the *monotone increase* property if $-\infty < \phi(x) \leq L(x, a, \phi)$ for all $(x, a) \in \operatorname{gr}\Gamma$ and Assumption 4.2.2 are satisfied. We state two useful lemmas from Bertsekas (2013).

LEMMA 4.7.1 (Proposition 4.3.14, Bertsekas (2013)). *Let the monotone increase property hold and assume that the sets*

$$\Gamma_k(x, \lambda) := \{x \in \Gamma(x) \mid L(x, a, T^k\phi) \leq \lambda\}$$

are compact for all $x \in X$, $\lambda \in \mathbb{R}$, and k greater than some integer $\bar{k}$. If $w \in \mathbb{R}^X_+$ satisfies $\phi \leq w \leq \bar{w}$, then $\lim_{n\to\infty} T^n w = \bar{w}$. Furthermore, there exists an optimal stationary policy.

LEMMA 4.7.2 (Proposition 4.3.9, Bertsekas (2013)). *Under the monotone increase property, a stationary policy π is optimal if and only if $T_\pi \bar{w} = T\bar{w}$.*

PROOF OF THEOREM 4.2.3. Theorem 4.2.1 implies that $\lim_{n\to\infty} T^n\phi = w^*$. To prove $w^* = \bar{w}$, it suffices to show that the conditions of Lemma 4.7.1 hold and $\phi \leq \bar{w}$.

It follows from A_5 that $\phi(x) \leq (T\phi)(x) \leq L(x, a, \phi)$ for all $(x, a) \in \operatorname{gr}\Gamma$. Therefore, the monotone increase property is satisfied. Since T is a self-map on $\mathcal{C}(X)$, to check the conditions of Lemma 4.7.1, it suffices to prove that the set

$$\Gamma(x, \lambda) := \{x \in \Gamma(x) \mid L(x, a, w) \leq \lambda\}$$

is compact for any $w \in \mathcal{C}(X)$, $x \in X$, and $\lambda \in \mathbb{R}$. Since $a \mapsto L(x, a, w)$ is continuous by A_1, $L(x, \cdot, w)^{-1}((-\infty, \lambda])$ is a closed set. Since Γ is compact-valued, $\Gamma(x, \lambda)$ is compact. It remains to show that $\phi \leq \bar{w}$. By A_5 and the definition of T, we have for any $\mu = (\pi_0, \pi_1, \ldots) \in \mathcal{M}$, $\phi \leq T^n\phi \leq T_{\pi_0}T_{\pi_1}\ldots T_{\pi_n}\phi$ for all $n \in \mathbb{N}$. Then by definition, $\phi \leq w_\mu$ for all $\mu \in \mathcal{M}$. Taking the infimum gives $\phi \leq \bar{w}$ and there exists an stationary optimal policy. Lemma 4.7.1 then implies that $w^* = \bar{w}$. The principle of optimality follows directly from Lemma 4.7.2. □

4.7.2. Proofs for the Negative Discount Dynamic Program. Let $\mathcal{F}$ be the set of increasing convex functions in $\mathcal{I}$. Throughout the proofs, we regularly use the alternative expression for T given by

$$Tw(x) = \min_{0\le y\le x} \{\ell(x-y) + \beta w(y)\} \tag{4.35}$$

Also, given $w \in \mathcal{F}$, define

$$\pi_w(x) = \arg\min_{0\le a\le x} \{\ell(a) + \beta w(x-a)\}$$

and

$$\sigma_w(x) := \arg\min_{0\le y\le x}\{\ell(x-y) + \beta w(y)\} = x - \pi_w(x). \tag{4.36}$$

These functions are clearly well-defined, unique and single-valued. Let $\sigma = \sigma_{w^*}$ and $\pi = \pi_{w^*}$. Let η be the constant defined by

$$\eta := \max\{0 \le x \le \hat{x} : \ell'(x) \le \beta\ell'(0)\}. \tag{4.37}$$

We begin with several lemmas. The proof of the first lemma is trivial and hence omitted.

Lemma 4.7.3. *We have $\eta > 0$ if and only if $\ell'(0) > 0$. If $\eta < \hat{x}$, then $\ell'(\eta) = \beta\ell'(0)$.*

Lemma 4.7.4. *If $w \in \mathcal{F}$, then $\sigma_w(x) = 0$ if and only if $x \le \eta$.*

Proof. First suppose that $x \le \eta$. Seeking a contradiction, suppose there exists a $y \in (0, x]$ such that $\ell(x-y) + \beta w(y) < \ell(x)$. Since $w \in \mathcal{F}$ we have $w(y) \ge \ell'(0)y$ and hence

$$\beta w(y) \ge \beta\ell'(0)y \ge \ell'(\eta)y.$$

Since $x \le \eta$, this implies that $\beta w(y) \ge \ell'(x)y$. Combining these inequalities gives $\ell(x-y) + \ell'(x)y < \ell(x)$, contradicting convexity of ℓ.

Now suppose that $\sigma_w(x) = 0$. We claim that $x \le \eta$, or, equivalently $\ell'(x) \le \beta\ell'(0)$. To prove $\ell'(x) \le \beta\ell'(0)$, observe that since $w \in \mathcal{F}$ we have $w(y) \le \ell(y)$, and hence

$$\ell(x) \le \ell(x-y) + \beta w(y) \le \ell(x-y) + \beta\ell(y) \quad \text{for all } y \le x.$$

It follows that

$$\frac{\ell(x) - \ell(x-y)}{y} \le \frac{\beta\ell(y)}{y} \quad \text{for all} \quad y \le x.$$

Taking the limit gives $\ell'(x) \leq \beta\ell'(0)$. □

PROPOSITION 4.7.5. *If $\ell'(0) > 0$, then the conclusions of Theorem 4.2.1 are valid for the Bellman operator T in (4.5).*

PROOF OF PROPOSITION 4.7.5. Let $A = X = [0, \hat{x}]$, $\Gamma(x) = [0, x]$ and $L(x, a, w) = \ell(a)+\beta w(x-a)$. Conditions A_1–A_3 obviously hold. Condition A_4 holds since $\min_{0\leq a\leq x}\{\ell(a)+\beta\ell(x-a)\} \leq \ell(x)$. For condition A_5, note that $L(x, a, \phi) = \ell(a) + \beta\ell'(0)(x - a)$. Then $T\phi = \ell$ if $x < \eta$ and $(T\phi)(x) = \ell(\eta) + \beta\ell'(0)(x - \eta)$ if $x \geq \eta$. For $x < \eta$, $T\phi - \phi = \psi - \phi$ so we can choose any $\epsilon \leq 1$. For $x \geq \eta$,

$$\begin{aligned}(T\phi)(x) - \phi(x) &= \ell(\eta) + \beta\ell'(0)(x - \eta) - \ell'(0)x \\ &= \ell(\eta) - \ell'(0)\eta + (\beta - 1)\ell'(0)(x - \eta) \\ &\geq \ell(\eta) - \ell'(0)\eta = (\psi - \phi)(\eta).\end{aligned}$$

Since $\psi - \phi$ is increasing, we can choose any $\epsilon \leq \bar{\epsilon}$ where $(\psi - \phi)(\eta) = \bar{\epsilon}(\psi - \phi)(\hat{x})$. The proposition thus follows from Theorem 4.2.1. □

PROPOSITION 4.7.6. *The fixed point w^* of the negative discount Bellman operator T in (4.5) is strictly increasing, strictly convex, and continuously differentiable on $(0, \hat{x})$. The policy correspondence π^* is single-valued and satisfies $(w^*)'(x) = \ell'(\pi(x))$.*

PROOF OF PROPOSITION 4.7.6. Consider the alternative expression for T in (4.35). Since ℓ is strictly convex, $(x, y) \mapsto \ell(x - y) + \beta w(y)$ is strictly convex for all $w \in \mathcal{C}_c(X)$. Hence, part 1 of Assumption 4.2.1 holds. Evidently Tw is strictly convex for all $w \in \mathcal{F}$.

Next we show that Tw is strictly increasing for all $w \in \mathcal{F}$. Pick any $w \in \mathcal{F}$ and $x_1 \leq x_2$. For ease of notation, let $y_i = \sigma_w(x_i)$ for $i \in \{1, 2\}$. If $y_2 \leq x_1$, then

$$\begin{aligned}(Tw)(x_1) &= \ell(x_1 - y_1) + \beta w(y_1) \\ &\leq \ell(x_1 - y_2) + \beta w(y_2) \\ &< \ell(x_2 - y_2) + \beta w(y_2) = (Tw)(x_2)\end{aligned}$$

where the first inequality holds since y_2 is available when y_1 is chosen and the second inequality holds since ℓ is strictly increasing. If $y_2 > x_1$, we first consider the case of $x_1 + y_2 < x_2$. Then $(Tw)(x_2) > \ell(x_1) + \beta w(y_2) \geq \ell(x_1) \geq (Tw)(x_1)$. For the case of $x_1 + y_2 \geq x_2$, we have $0 \leq y_1' \leq x_1 < y_2$ where $y_1' = x_1 + y_2 - x_2$. Since w is not constant, $w \in \mathcal{F}$ implies that w is strictly increasing. It follows that

$$
\begin{aligned}
(Tw)(x_1) &= \ell(x_1 - y_1) + \beta w(y_1) \\
&\leq \ell(x_1 - y_1') + \beta w(y_1') \\
&< \ell(x_2 - y_2) + \beta w(y_2) = (Tw)(x_2).
\end{aligned}
$$

Therefore, T is a self-map on $\mathcal{F}$ and Tw is strictly increasing and strictly convex for all $w \in \mathcal{F}$. Theorem 4.2.2 then implies that w^* is strictly increasing and strictly convex.

Since ℓ is differentiable, part 2 of Assumption 4.2.1 holds. Theorem 4.2.2 then implies that w^* is differentiable and $(f^*)'(x) = \ell'(x - \sigma(x))$ whenever $\sigma(x)$ is interior. Lemma 4.7.4 implies that $w^*(x) = \ell(x)$ and thus $(f^*)'(x) = \ell'(x)$ when $x \leq \eta$; when $x > \eta$, σ is interior and $(f^*)'(x) = \ell'(x - \sigma(x))$. Since σ is continuous, $(f^*)'$ is continuous. Therefore, w^* is continuously differentiable on $(0, \hat{x})$ and $(f^*)'(x) = \ell'(\pi(x))$. □

The next lemma further characterizes π and σ.

Lemma 4.7.7. *Let $w \in \mathcal{F}$. If x_1, x_2 satisfy $0 < x_1 \leq x_2$, then $\sigma_w(x_1) \leq \sigma_w(x_2)$ and $\pi_w(x_1) \leq \pi_w(x_2)$. Moreover, if $x \geq \eta$, then $\pi_w(x) \geq \eta$; if $x \leq \eta$, then $\pi_w(x) = x$.*

Proof. Pick any $w \in \mathcal{F}$. Since ℓ and w are convex, the maps $(x, a) \mapsto \ell(a) + \beta w(x - a)$ and $(x, y) \mapsto \ell(x - y) + \beta w(y)$ both satisfy the single crossing property. It follows from Theorem 4$'$ of Milgrom and Shannon (1994) that π_w and σ_w are increasing.

For the last claim, since π_w is increasing, Lemma 4.7.4 implies that, if $\eta \leq x$, then $\pi_w(x) \geq \pi_w(\eta) = \eta - \sigma_w(\eta) = \eta$; and if $x \leq \eta$, then $\pi_w(x) = x - \sigma_w(x) = x$. □

The following lemma characterizes the solution to (4.2) and is useful when showing the equivalence between (4.2) and (4.4).

LEMMA 4.7.8. *If $\{a_t\}$ is a solution to (4.2), then $\{a_t\}$ is monotone decreasing and $a_{T+1} = 0$ if and only if $a_T \leq \eta$.*

PROOF. The first claim is obvious, because if $\{a_t\}$ is a solution to (4.2) with $a_t < a_{t+1}$, then, given that $\beta > 1$, swapping the values of these two points in the sequence will preserve the constraint while strictly decreasing total loss. Regarding the second claim, since $\{a_t\}$ is monotone decreasing, it suffices to check the case $a_T > 0$. To this end, suppose to the contrary that $\{a_t\}$ is a solution to (4.2) with $0 < a_T < \eta$ and $a_{T+1} > 0$. Consider an alternative feasible sequence $\{\hat{a}_t\}$ defined by $\hat{a}_T = a_T + \epsilon$, $\hat{a}_{T+1} = a_{T+1} - \epsilon$ and $\hat{a}_t = a_t$ for other t. If we compare the values of these two sequences we get

$$\begin{aligned}\sum_{t=0}^{\infty} \beta^t \ell(a_t) - \sum_{t=0}^{\infty} \beta^t \ell(\hat{a}_t) &= \beta^T [\ell(a_T) - \ell(a_T + \epsilon)] + \beta^{T+1} [\ell(a_{T+1}) - \ell(a_{T+1} - \epsilon)] \\ &= \epsilon \beta^T \left\{ - \frac{\ell(a_T + \epsilon) - \ell(a_T)}{\epsilon} + \beta \frac{\ell(a_{T+1} - \epsilon) - \ell(a_{T+1})}{-\epsilon} \right\}.\end{aligned}$$

The term inside the parenthesis converges to

$$-\ell'(a_T) + \beta \ell'(a_{T+1}) > -\ell'(\eta) + \beta \ell'(0) \geq 0,$$

where the first inequality follows from $a_T \leq \eta$, $a_{T+1} > 0$ and strict convexity of ℓ; and the second inequality is by the definition of η. We conclude that for ϵ sufficiently small, the difference $\sum_{t=0}^{\infty} \beta^t \ell(a_t) - \sum_{t=0}^{\infty} \beta^t \ell(\hat{a}_t)$ is positive, contradicting optimality.

Finally we check the claim $a_{T+1} = 0 \implies a_T \leq \eta$. Note that if $\eta = \hat{x}$ then there is nothing to prove, so we can and do take $\eta < \hat{x}$. Seeking a contradiction, suppose instead that $a_{T+1} = 0$ and $a_T > \eta$. Consider an alternative feasible sequence $\{\hat{a}_t\}$ defined by $\hat{a}_T = a_T - \epsilon$, $\hat{a}_{T+1} = \epsilon$ and $\hat{a}_t = a_t$ for other t. In this case we have

$$\sum_{t=0}^{\infty} \beta^t \ell(a_t) - \sum_{t=0}^{\infty} \beta^t \ell(\hat{a}_t) = \epsilon \beta^T \left\{ \frac{\ell(a_T - \epsilon) - \ell(a_T)}{-\epsilon} - \beta \frac{\ell(\epsilon) - \ell(0)}{\epsilon} \right\}.$$

The term inside the parentheses converges to

$$\ell'(a_T) - \beta \ell'(0) > \ell'(\eta) - \beta \ell'(0) = 0,$$

where the final equality is due to $\eta < \hat{x}$ and Lemma 4.7.3. Once again we conclude that for ϵ sufficiently small, the difference $\sum_{t=0}^{\infty} \beta^t \ell(a_t) - \sum_{t=0}^{\infty} \beta^t \ell(\hat{a}_t)$ is positive, contradicting optimality. □

Proposition 4.7.9. *For the negative discount dynamic program, the sequence $\{a_t^*\}$ defined by $x_0 = \hat{x}$, $x_{t+1} = x_t - \pi^*(x_t)$ and $a_t^* = \pi^*(x_t)$ is the unique solution to (4.2). Moreover, $W = w^*$.*

Proof of Proposition 4.7.9. To show the equivalence between (4.2) and (4.4), we first show that (4.2) is equivalent to $\bar{w} = \inf_{\mu \in \mathcal{M}} w_\mu$ where w_μ is as defined in (4.11). Suppose that the optimal policy is $\mu = (\pi_0, \pi_1, \ldots)$ and we let $\sigma_t(x) = x - \pi_t(x)$. Then we have

$$\bar{w}(\hat{x}) = w_\mu(\hat{x}) = \ell[\pi_0(\hat{x})] + \beta \ell[\pi_1 \sigma_0(\hat{x})] + \beta^2 \ell[\pi_2 \sigma_1 \sigma_0(\hat{x})] + \ldots$$
$$+ \limsup_{t \to \infty} \beta^k \ell'(0) \sigma_{t-1} \sigma_{t-2} \cdots \sigma_0(\hat{x}). \quad (4.38)$$

It is clear that $\bar{w}$ is finite. Therefore, the optimal policy must satisfy $\sigma_t \to 0$, otherwise the last term in (4.38) would go to infinity. Let $a_t = \pi_t \sigma_{t-1} \ldots \sigma_0(\hat{x})$. We claim that $\{a_t\}$ solves (4.2). Suppose not and the solution to (4.2) is $\{a'_t\}$. Then by Lemma 4.7.8, $a'_t = 0$ for all $t > T$ for some T. Thus we can construct a policy μ' that reproduces $\{a'_t\}$ and gives a lower loss. This is a contradiction. Conversely, suppose that the solution to (4.2) is $\{a_t\}$. Using the same argument, we can show that the policy that gives rise to $\{a_t\}$ is an optimal policy. Therefore, $W = \bar{w}$.

Next we show that $w^* = \bar{w}$ using Theorem 4.2.3. That Assumption 4.2.2 holds was shown in Section 4.2.4.1. It follows from Theorem 4.2.3 that $w^* = \bar{w}$, there exists an stationary optimal policy, and the Bellman's principle of optimality holds. Since π^* satisfies $T_{\pi^*} w^* = T w^*$, π^* is a stationary optimal policy.

Theorems 4.2.1 and 4.2.2 implies that π^* is continuous and single-valued. It then follows from the principle of optimality that $\{a_t^*\}$ is the unique solution to (4.2). □

PROPOSITION 4.7.10. *For all $n \in \mathbb{N}$ and increasing convex $w \in \mathcal{I}$, we have*

$$T^n w(x) = w^*(x) \quad \textit{whenever } x \leq n\eta.$$

Proposition 4.7.10 implies uniform convergence in *finite* time. In particular, for $n \geq \hat{x}/\eta$ we have $T^n w = w^*$ everywhere on $[0, \hat{x}]$. Note that this bound $\hat{x}/\eta$ is independent of the initial condition w.

PROOF OF PROPOSITION 4.7.10. It suffices to show that if $f, g \in \mathcal{F}$, then $T^k f = T^k g$ on $[0, k\eta]$. We prove this by induction.

To see that $T^1 f = T^1 g$ on $[0, \eta]$, pick any $x \in [0, \eta]$ and recall from Lemma 4.7.4 that if $h \in \mathcal{F}$ and $x \leq \eta$, then $Th(x) = \ell(x)$. Applying this result to both f and g gives $Tf(x) = Tg(x) = \ell(x)$. Hence $T^1 f = T^1 g$ on $[0, \eta]$ as claimed.

Turning to the induction step, suppose now that $T^k f = T^k g$ on $[0, k\eta]$, and pick any $x \in [0, (k+1)\eta]$. Let $h \in \mathcal{F}$ be arbitrary, let π_h be the h-greedy function, and let $\sigma_h(x) := x - \pi_h(x)$. By Lemma 4.7.7, we have $\pi_h(x) \geq \eta$, and hence

$$\sigma_h(x) \leq x - \eta \leq (k+1)\eta - \eta \leq k\eta.$$

In other words, given function h, the optimal choice at x is less than $k\eta$. Since this is true for both $h = T^k f$ and $h = T^k g$, we have

$$T^{k+1} f(x) = \min_{0 \leq y \leq x} \{\ell(x-y) + \beta T^k f(y)\} = \min_{0 \leq y \leq k\eta} \{\ell(x-y) + \beta T^k f(y)\}.$$

Using the induction step we can now write

$$T^{k+1} f(x) = \min_{0 \leq y \leq k\eta} \{\ell(x-y) + \beta T^k g(y)\} = \min_{0 \leq y \leq x} \{\ell(x-y) + \beta T^k g(y)\}.$$

The last expression is just $T^{k+1} g(x)$, and we have now shown that $T^{k+1} f = T^{k+1} g$ on $[0, (k+1)\eta]$. The proof is complete. □

PROOF OF PROPOSITION 4.2.4. Since $\ell'(0) = 0$, (EU) is equivalent to $\beta \ell'(a^*_{t+1}) = \ell'(a^*_t)$.

Sufficiency. Let $x_0^* = \hat{x}$ and $x_t^* = x_{t-1}^* - a_{t-1}^*$ for $t \geq 1$. Let $\{a_t\}$ be any feasible sequence. Let $x_0 = \hat{x}$ and $x_t = x_{t-1} - a_{t-1}$. It suffices to prove that

$$D := \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [\ell(a_t^*) - \ell(a_t)] \leq 0.$$

Since ℓ is convex, we have

$$\begin{aligned} D &= \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [\ell(x_t^* - x_{t+1}^*) - \ell(x_t - x_{t+1})] \\ &\leq \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \ell'(a_t^*)(x_t^* - x_t - x_{t+1}^* + x_{t+1}). \end{aligned}$$

Since $x_0 = x_0^*$, rearranging gives

$$D \leq \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t (x_{t+1}^* - x_{t+1})[\beta\ell'(a_{t+1}^*) - \ell'(a_t^*)] - \beta^T \ell'(a_T^*)(x_{T+1}^* - x_{T+1}).$$

Since $\beta\ell'(a_{t+1}^*) = \ell'(a_t^*)$, the summation is zero and $\beta^T \ell'(a_T^*) = \ell'(a_0^*)$. We have

$$D \leq -\lim_{T\to\infty} \ell'(a_0^*)(x_{T+1}^* - x_{T+1}).$$

Since $\{a_t\}$ and $\{a_t^*\}$ are feasible, x_{T+1} and x_{T+1}^* go to zero when $T \to \infty$. Therefore, $D \leq 0$.

Existence and Uniqueness. Since $\{a_t^*\}$ is feasible and satisfies $\beta\ell'(a_{t+1}^*) = \ell'(a_t^*)$ for all t, we have

$$\hat{x} = \sum_{t=0}^{\infty} a_t^* = \sum_{t=0}^{\infty} (\ell')^{-1}\left(\frac{1}{\beta^t}\ell'(a_0^*)\right) =: g(a_0^*),$$

where $(\ell')^{-1}$ is well defined on $[0, \lim_{x\to\infty} \ell'(x)]$ because ℓ is increasing, strictly convex, and $\ell'(0) = 0$. Hence, g is well defined on $\mathbb{R}_+$ and $g(a_0^*)$ is continuous and strictly increasing in a_0^*. Since $g(0) = 0$ and $g(\hat{x}) > \hat{x}$, there exists a unique $a_0^* > 0$ such that $\{a_t^*\}$ satisfying $\beta\ell'(a_{t+1}^*) = \ell'(a_t^*)$ is feasible, $a_t^* > 0$ for all t, and $\{a_t^*\}$ is strictly decreasing. That $\{a_t^*\}$ is an optimal solution then follows from the sufficiency part. Since ℓ is strictly convex, the solution is unique.

Necessity. Since we have pinned down a unique solution of (4.2) which satisfies $\beta\ell'(a_{t+1}^*) = \ell'(a_t^*)$, the condition is also necessary. □

4.7.3. Proofs for Section 4.3.

Proof of Proposition 4.3.1. We must verify that $(W, \{a_i^*\})$ satisfies Definition 4.3.1. We first consider the case of $\ell'(0) > 0$. By Propositions 4.7.5 and 4.7.9, the value function W is a solution to the Bellman equation (4.4), and hence satisfies

$$W(s) = \min_{0 \leq v \leq s} \{c(v) + (1+\tau)W(s-v)\} \quad \text{for all } s \in [0,1]. \tag{4.39}$$

By Proposition 4.7.6, it lies in the class $\mathcal{F}$ of increasing, convex and continuous functions $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $c'(0)s \leq f(s) \leq c(s)$ for all $s \in \mathbb{R}_+$. In addition, with $\{x_i\}$ as the optimal state process (see Proposition 4.7.9), we have,

$$W(x_i) = \{c(a_i^*) + (1+\tau)W(x_{i+1})\} \quad \text{for all } i \geq 0. \tag{4.40}$$

We need to show that 1–3 of Definition 4.3.1 hold when $p = W$ and $v_i = a_i^*$ for all $i \in \mathbb{Z}$. Part 1 is immediate because $W \in \mathcal{F}$ and all functions in $\mathcal{F}$ must have this property, while Part 2 follows directly from (4.39). To see that Part 3 of Definition 4.3.1 also holds, let $b_i = x_i$. By the definition of the state process, the sequence $\{b_i\}$ then corresponds to the downstream boundaries of a set of firms obeying task allocation $\{a_i^*\}$. The profits of firm i are $\pi_i = W(b_i) - c(a_i^*) - (1+\tau)W(b_{i+1})$. By (4.40) and $b_i = x_i$, we have $\pi_i = 0$ for all i. Hence Part 3 of Definition 4.3.1 also holds, as was to be shown.

If $\ell'(0) = 0$, part 1 follows from the definition of the value function (4.14). By Proposition 4.2.4, for any t with $0 \leq t \leq 1$, there exists a unique optimal allocation $\{a_{t,j}^*\}$ such that $W(t) = \sum_j \beta^j \ell(a_{t,j}^*)$, and $\sum_j a_{t,j}^* = t$. Since $\{s-t, a_{t,0}^*, a_{t,1}^*, \ldots\}$ is a feasible allocation at stage s with $t \leq s \leq 1$, part 2 follows from the definition of the value function. To see part 3, let $b_0 = 1$ and $b_i = b_{i-1} - a_{i-1}^*$. By Proposition 4.2.4, we have $\ell'(a_i^*) = (1+\tau)\ell'(a_{i+1}^*)$. Since $\sum_{i=j}^{\infty} a_i^* = b_j$ for all j, it follows again from Proposition 4.2.4 that $\{a_i^*\}_{i=j}^{\infty}$ is an optimal allocation for stage b_j. Therefore, $p(b_i) = \sum_{j=0}^{\infty}(1+\tau)^j c(a_{i+j}^*) = c(a_i^*) + (1+\tau)p(b_{i+1})$ for all i. Hence, $\pi_i = 0$ for all i. □

4.7.4. Proofs for Section 4.4.

Proof of Proposition 4.4.1. To study this problem in the framework of Theorem 4.2.1, we set $X = [0, \hat{x}]$, $A = [0, \hat{x}] \times \mathbb{N}$, $\Gamma(x) = [0, x] \times \mathbb{N}$, and

$$L(x, a, w) = c(x - t) + g(k) + (1 + \tau)kp(t/k) \qquad a = (t, k).$$

Since $g(k) \to \infty$ as $k \to \infty$, we can restrict $\Gamma(x)$ to be $[0, x] \times \{1, 2, \ldots, \bar{k}\}$ so that Γ is compact-valued. Under the conditions of Proposition 4.4.1, it can be shown that A_1–A_5 hold with $\psi = c$ and $\phi(s) = c'(0)s$ (see Yu and Zhang (2019)). Then, Theorem 4.2.1 implies that the Bellman equation (4.22) has a unique solution p^* in $\mathcal{I}$, $T^n p \to p^*$ for all $p \in \mathcal{I}$ where

$$(Tp)(s) := \min_{\substack{0 \le t \le s \\ k \in \mathbb{N}}} L(x, a, w),$$

and t^* and k^* exist. We need only verify that $(p^*, \{v_i\}, \{k_i\})$ given by $v_i = b_i - t^*(b_i)$, $k_i = k^*(b_i)$ and $b_{i+1} = (b_i - v_i)/k_i$ is an equilibrium, the definition of which is given in Section 4.4.2.

Since $p^* \in \mathcal{I}$, $p(0) = 0$. Since p^* satisfies (4.22), part (ii) of the definition is also satisfied. To see that part (iii) holds, note that

$$\begin{aligned} p^*(b_i) &= c(b_i - t^*(b_i)) + g(k^*(b_i)) + (1 + \tau)k^*(b_i)p^*\left(\frac{t^*(b_i)}{k^*(b_i)}\right) \\ &= c(v_i) + g(k_i) + (1 + \tau)k_i p^*\left(\frac{b_i - v_i}{k_i}\right). \end{aligned}$$

It follows that $\pi_i = 0$ for all $i \in \mathbb{Z}$ where π_i is as defined in (4.21). This completes the proof. □

4.7.5. Proofs for Continuous Time Theory. Assumption 4.6.1 is imposed throughout.

4.7.5.1. *Proof of Theorem 4.6.1.* In this section, we consider a relatively more general problem:

$$\min_{x(t), a(t)} \int_0^\infty g(t, x(t), a(t))dt \tag{4.41}$$

subject to

$$\dot{x}(t) = f(t, x(t), a(t)), \; x(0) = x_0, \; \lim_{t \to \infty} x(t) = x_1, \text{ and } a(t) \in U \subset \mathbb{R} \; \forall t. \tag{4.42}$$

Throughout the appendix, we assume that f and g are continuously differentiable with respect to x, a, and t, and $a(\cdot)$ is piecewise continuous. Define the Hamiltonian by

$$H(x(t), a(t), \lambda(t), t) = \lambda(t) f(t, x(t), a(t)) - g(t, x(t), a(t)) \tag{4.43}$$

and denote the partial derivatives of H by H_x, H_a, and H_λ. We have the following theorem.[15]

THEOREM 4.7.11. *Consider problem (4.41) subject to (4.42). Assume there exists $(x^*(t), a^*(t))$ such that the cost function is finite. Suppose there exists $(x^*(t), a^*(t))$ satisfying (4.42) and continuously differentiable $\lambda(t)$ such that the following conditions hold:*

1. *$\dot{\lambda}(t) = -H_x(x^*(t), a^*(t), \lambda(t), t)$ except at points of discontinuity of $a^*(t)$;*
2. *$H(x^*(t), a^*(t), \lambda(t), t) = \max_{a\in U} H(x^*(t), a, \lambda(t), t)$ for all t;*
3. *H is jointly concave in x and a;*
4. *U is convex.*

Then $(x^*(t), a^*(t))$ is a solution to problem (4.41). Moreover, if H is strictly concave in x and a, $(x^*(t), a^*(t))$ is a unique solution.

Our continuous time problem (4.23) fits in this framework if we let $g(t, x(t), a(t)) = e^{\rho t}\ell(a(t))$, $f(t, x(t), a(t)) = -a(t)$, $U = [0, \infty)$, $x_0 = \bar{x} > 0$, and $x_1 = 0$. The Hamiltonian is thus

$$H(x(t), a(t), \lambda(t), t) = -\lambda(t)a(t) - e^{\rho t}\ell(a(t)). \tag{4.44}$$

It is easy to check that all the conditions in Theorem 4.7.11 are satisfied as long as there exists a constant λ satisfying (4.26) and (4.27).

We shall prove that such λ indeed exists and is unique when $\ell'(0) = 0$. Since ℓ is increasing, strictly convex, and continuously differentiable, $h := (\ell')^{-1}$ is well defined on an interval $[0, M)$ of $\mathbb{R}_+$, where $M = \lim_{a\to\infty} \ell'(a)$. Moreover, h is continuous, strictly increasing, and ranges from zero to infinity. When $-\lambda e^{-\rho t}$ falls into the domain of h, (4.26) implies that

$$a^*(t; \lambda) = h(-\lambda e^{-\rho t}). \tag{4.45}$$

[15] For more general versions of this sufficiency theorem, see, e.g., Acemoglu (2008).

Because of the properties of h, $\int_0^\infty a^*(t;\lambda)dt$ is strictly increasing and ranges from zero to infinity. Therefore, there exists a unique λ such that (4.26) and (4.27) hold. It then follows from Theorem 4.7.11 that a^* in (4.26) is the unique solution to problem (4.23). Part 1 of Theorem 4.6.1 for $\ell'(0) > 0$ can be proved in a similar way and we leave it to the reader.

When $\ell'(0) = 0$, $-\lambda e^{-\rho t}$ is always in the domain of h. Therefore, a^* is given by (4.45) and is decreasing and strictly positive. When $\ell'(0) > 0$, 0 is not in the domain of h; (4.26) implies that a^* will become zero when t is large enough. This proves part 2 of Theorem 4.6.1.

Define the value function $V : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ for problem (4.41) by

$$V(t,x) = \inf_{x(s),a(s)} \int_t^\infty g(s,x(s),a(s))ds \tag{4.46}$$

subject to

$$\dot{x}(s) = f(s,x(s),a(s)),\ x(t) = x,\ \lim_{s\to\infty} x(s) = x_1,\ \text{and } a(s) \in U \subset \mathbb{R}\ \forall s.$$

We have the following necessary conditions for optimality[16].

THEOREM 4.7.12. *Suppose $V(t,x)$ is differentiable with respect to t and x and there exists $(x^*(t), a^*(t))$ that solves problem (4.41). Then V is the solution to the HJB equation*

$$-V_t(t,x) = \inf_{a\in U} \{g(t,x,a) + V_x(t,x)f(t,x,u)\} \tag{4.47}$$

with boundary condition $\lim_{t\to\infty} V(t,x(t)) = 0$ and $(x^*(t), a^*(t))$ satisfies

$$\begin{aligned} -V_t(t,x^*(t)) &= \inf_{a\in U} \{g(t,x^*(t),a) + V_x(t,x^*(t))f(t,x^*(t),a)\} \\ &= g(t,x^*(t),a^*(t)) + V_x(t,x^*(t))f(t,x^*(t),a^*(t)). \end{aligned} \tag{4.48}$$

For discounted optimal control problems, if we can write $g(t,x,a) = e^{\rho t}g(x,a)$ and $f(t,x,a) = f(x,a)$, we can define the stationary value function by $F(x) := V(0,x)$.

[16] For more general versions of this theorem, see, for example, Bressan and Piccoli (2007), Acemoglu (2008), or Liberzon (2011)

Then we have a stationary version of (4.47):

$$-\rho F(x) = \inf_{a\in U} \{g(x,a) + F'(x)f(x,a)\}. \tag{4.49}$$

The differentiability of F follows from Theorem 7.17 of Acemoglu (2008). From part 1 we know that an optimal control exists, so we can apply Theorem 4.7.12. This concludes part 3 of Theorem 4.6.1.

4.7.5.2. *Production Chains in Continuous Time.* Consider the general problem:

$$\min_{x(t),a(t),T\geq 0} \int_0^T g(t,x(t),a(t))dt + L(T) \tag{4.50}$$

subject to

$$\dot{x}(t) = f(t,x(t),a(t)),\ x(0) = x_0,\ x(T) = x_1,\ \text{and}\ a(t) \in U \subset \mathbb{R}\ \forall t. \tag{4.51}$$

Assume L is twice continuously differentiable and $L(t) \to \infty$ as $t \to \infty$. Define the new Hamiltonian by

$$H(x(t),a(t),\lambda(t),t) = \lambda(t)f(t,x(t),a(t)) - g(t,x(t),a(t)) - \dot{L}(t). \tag{4.52}$$

Since $L(T)$ can become arbitrarily large, we can find a large $\bar{T}$ and choose T from $[0,\bar{T}]$ without loss of generality. We have the following sufficiency theorem.

Theorem 4.7.13 (Seierstad (1984)). *Consider problem (4.50) subject to (4.51) with U bounded. Suppose for each $\delta \leq \bar{T}$ there exists $(x_\delta(t), a_\delta(t))$ satisfying (4.51) and continuously differentiable $\lambda_\delta(t)$ such that the following conditions hold:*

1. $\dot{\lambda}_\delta(t) = -H_x(x_\delta(t), a_\delta(t), \lambda_\delta(t), t)$ *except at points of discontinuity of* $a_\delta(t)$;
2. $H(x_\delta(t), a_\delta(t), \lambda_\delta(t), t) = \max_{a\in U} H(x_\delta(t), a, \lambda_\delta(t), t)$ *for all* t;
3. H *is jointly concave in* x *and* a;
4. U *is convex.*

Moreover, suppose there is no other λ_δ such that the above conditions hold. Then, if there exists T^* such that $H(x_\delta(\delta), a_\delta(\delta), \lambda_\delta(\delta), \delta) \geq 0$ for $\delta < T^*$ and $H(x_\delta(\delta), a_\delta(\delta), \lambda_\delta(\delta), \delta) \leq 0$ for $\delta > T^*$. Then $(x_{T^*}(\cdot), a_{T^*}(\cdot), T^*)$ is a solution to problem (4.50).

In problem (MLCF), $g(t, x, a) = e^{\rho t}\ell(a)$, $f(t, x, a) = -a$, and $\ell'(0) = 0$. Assume $L(t) = Be^{\rho t}$ with $B > 0$ as in Fally and Hillberry (2018). Then, for any fixed δ, we have $a_\delta(t) = h(-\lambda_\delta e^{-\rho t})$ where λ_δ satisfies that $\int_0^\delta a_\delta(t)dt = \bar{x}$. Since for $\lambda \neq \lambda_\delta$, a that minimizes $H(x_\delta(t), a, \lambda, t)$ satisfies $a = h(-\lambda e^{-\rho t}) \neq a_\delta$, there is no other λ_δ such that all the conditions hold. Moreover, we have

$$H(x_\delta(\delta), a_\delta(\delta), \lambda_\delta(\delta), \delta) = -\lambda_\delta a_\delta(\delta) - e^{\rho\delta}\ell(a_\delta(\delta)) - \rho Be^{\rho\delta}$$

$$\frac{d}{d\delta}H(x_\delta(\delta), a_\delta(\delta), \lambda_\delta(\delta), \delta) = -\frac{d\lambda_\delta}{d\delta}a_\delta(\delta) - \rho e^{\rho\delta}\ell(a_\delta(\delta)) - \rho^2 Be^{\rho\delta}$$

where $\frac{d\lambda_\delta}{d\delta} > 0$ is the derivative given by applying the implicit function theorem on $\int_0^\delta h(-\lambda_\delta e^{-\rho t})dt = \bar{x}$. Therefore, $H(x_\delta(\delta), a_\delta(\delta), \lambda_\delta(\delta), \delta)$ is strictly decreasing in δ. If we can find T^* such that $H(x_{T^*}(T^*), a_{T^*}(T^*), \lambda_{T^*}(T^*), T^*) = 0$, then $(x_{T^*}(\cdot), a_{T^*}(\cdot), T^*)$ is optimal. Therefore, the solutions given in Fally and Hillberry (2018) are optimal.

CHAPTER 5

Production Chains with Multiple Upstream Partners

5.1. Introduction

Over the past several centuries, firms have self-organized into ever more complex production networks, spanning both state and international boundaries, and constructing and delivering a vast range of manufactured goods and services. The structures of these networks help determine the efficiency (Levine, 2012; Ciccone, 2002) and resilience (Carvalho, 2007; Jones, 2011; Bigio and La'O, 2016; Acemoglu et al., 2012, 2015a) of the entire economy, and also provide new insights into the directions of trade and financial policies (Baldwin and Venables, 2013; Acemoglu et al., 2015b).

We consider a production chain model introduced by Kikuchi et al. (2018) that examines the formation of such structures. They connect the literature on firm networks and network structure to the underlying theory of the firm by Coase (1937). A single firm at the end of the production chain sells a final product to consumers. The firm can choose to produce the whole product by itself or subcontract a portion of it to possible multiple upstream partners, who then make similar choices until all the remaining production is completed. The main reason for firms to produce more in-house is to save the transaction costs of buying intermediate products from the market. In fact, Coase (1937) regards this as the primary force that brings firms into existence. An opposing force that limits the size of a firm is the costs of organizing production within the firm[1]. A price function governs the choices firms make and is determined endogenously in equilibrium when every firm in the production chain makes zero profit.

Considering that all firms are ex ante identical, a notable feature of this model is its ability to generate a production network with multiple layers of firms different in their

[1]One justification also mentioned in Kikuchi et al. (2018) is that firms usually experience diminishing return to management: when a firm gets bigger it also bears increasing coordination costs. See also Coase (1937), Lucas (1978b), and Becker and Murphy (1992).

sizes and numbers of upstream partners. The source of the heterogeneity lies solely in the transaction costs and firms' different stages in the production chain. This feature provides insights into the formation of potentially more complex structures in a production network. Kikuchi et al. (2018) prove the existence, uniqueness, and global stability[2] of the equilibrium price function restricting every firm to have only one upstream partner. In this case, the resulting production network consists of a single chain.

There are however, several significant weaknesses with the analysis in Kikuchi et al. (2018). First, while they provide comprehensive results on uniqueness of equilibrium prices and convergence of successive approximations in the single upstream partner case, they fail to provide analogous results for the more interesting multiple upstream partner case, presumably due to technical difficulties. Second, their model cannot accurately reflect the data on observed production networks because their networks are always symmetric, with sub-networks at each layer being exact copies of one another. Real production networks do not exhibit this symmetry[3]. Third, they provide no effective algorithm for computing the equilibrium price function in the multiple upstream partner case.

This paper resolves all of the shortcomings listed above. As our first contribution, we extend their existence, uniqueness, and global stability results to the multiple partner case. To avoid the technical difficulties faced in their paper, we employ a different approach utilizing the theory of monotone concave operators, which enables us to give a unified proof for both cases.

Theoretically, the concave operator theory ensures the global stability of the fixed point, so the equilibrium price function can be computed by successive evaluations of the operator. In practice, however, the rates of convergence can be different for different model settings. This leads to unnecessarily long computation time in most cases. As a second contribution, we propose an algorithm that achieves fast computation regardless of parameterizations and is shown to drastically reduce computation time in our simulations.

[2]Mathematically, the equilibrium price function is determined as the fixed point of a Bellman like operator (see Section 5.3). Globally stability means that the fixed point can be computed by successive evaluations of the operator on any function in a certain function space.

[3]For instance, for a mobile phone manufacturer, most subcontractors who supply complicated components like display or CPUs have multiple upstream partners of their own, while those who supply raw materials usually do not (Dedrick et al., 2011; Kraemer et al., 2011).

A third contribution of this paper is that we generalize the model to a stochastic setting. In the original model, the equilibrium firm allocation is symmetric and deterministic: firms at the same stage of production choose the exact same number of upstream partners. In reality, each firm faces uncertainty in the contracting process and cannot always choose the optimal number of partners. We model the number of upstream partners as a Poisson distribution and let the firm choose its parameter, which can be seen as a search effort. Using the same approach, we prove the existence and uniqueness of equilibrium price function as well as the validity of the algorithm. We further use simulations to analyze how production and transaction costs determine the shape of a production network. This generalization provides a new source of heterogeneity in the equilibrium firm allocation and can be a potential channel for future research on size distribution of firms.

Section 5.2 describes the model in detail. Section 5.3 introduces the monotone concave operator theory and gives existence and uniqueness results. The algorithm is described in Section 5.4. Section 5.5 generalizes the model, allowing for stochastic choices of upstream partners. Section 5.6 concludes. All proofs can be found in the Appendix.

5.2. The Model

We study the production chain model with multiple partners in Kikuchi et al. (2018). The chain consists of a single firm at the end of the chain which sells a single final good to consumers and firms at different stages of the production, each of which sells an intermediate good to a downstream firm by producing the good in-house or subcontracting a portion of the production process to possibly multiple upstream firms. We index the stage of production by $s \in X = [0, 1]$ with 1 being the final stage. Each firm faces a price function $p : X \to \mathbb{R}_+$ and a cost function $c : X \to \mathbb{R}_+$. Subcontracting incurs a transaction cost that is proportionate[4] to the price with coefficient $\delta > 1$ for each upstream partner and an additive transaction cost $g : \mathbb{N} \to \mathbb{R}_+$ that is a function of the number of upstream partners. The cost g can be seen as the costs of maintaining partnerships such as legal expenses and communication costs.

[4]Here we follow Kikuchi et al. (2018). This transaction cost can be the cost of gathering information, drafting contract, bargaining, or even tax, all of which tend to increase with the volume of the transaction.

We adopt the same assumptions as in Kikuchi et al. (2018). For the cost function c, we assume that $c(0) = 0$ and it is differentiable, strictly increasing, and strictly convex. In other words, each firm experiences diminishing return to management as mentioned in the introduction. This assumption is needed here because otherwise no firm would want to subcontract its production. We also assume $c'(0) > 0$. For the additive transaction cost function g, we assume that it is strictly increasing, $g(1) = 0$, and $g(k)$ goes to infinity as the number of upstream partners k goes to infinity. To summarize, we have the following two assumptions.

Assumption 5.2.1. The cost function c is differentiable, strictly increasing, and strictly convex. It also satisfies $c(0) = 0$ and $c'(0) > 0$.

Assumption 5.2.2. The additive transaction cost function g is strictly increasing, $g(1) = 0$, and $g(k) \to \infty$ as $k \to \infty$.

Therefore, a firm at stage s solves the following problem:

$$\min_{\substack{t \leq s \\ k \in \mathbb{N}}} \left\{ c(s-t) + g(k) + \delta k p(t/k) \right\}. \tag{5.1}$$

In (5.1), the firm chooses to produce $s - t$ in-house with cost $c(s - t)$ and subcontract t to k upstream partners. Since each subcontractor is in charge of t/k part of the product, this results in a proportionate transaction cost $\delta k p(t/k)$ and an additive transaction cost $g(k)$. Then the firm sells the product to its downstream firm at price $p(s)$.

5.3. Equilibrium

Following Kikuchi et al. (2018), we consider the equilibrium in a competitive market with free entry and free exit. The price adjusts so that in the long run every firm makes zero profit. The equilibrium price function then satisfies

$$p(s) = \min_{\substack{t \leq s \\ k \in \mathbb{N}}} \left\{ c(s-t) + g(k) + \delta k p(t/k) \right\}. \tag{5.2}$$

Let $\mathcal{R}(X)$ be the space of real functions and $\mathcal{C}(X)$ the space of continuous functions on X. Then we can define an operator $T : \mathcal{C}(X) \to \mathcal{R}(X)$ by

$$Tp(s) := \min_{\substack{t \leq s \\ k \in \mathbb{N}}} \{c(s-t) + g(k) + \delta k p(t/k)\}. \tag{5.3}$$

The equilibrium price function is thus determined as the fixed point of the operator T.

5.3.1. Monotone Concave Operator Theory. Before proceeding to our main result, we first introduce a theorem due to Du (1989), which studies the fixed point properties of monotone concave operators on a partially ordered Banach space.

Let E be a real Banach space on which a partial ordering is defined by a cone $P \subset E$, in the sense that $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ but $x \neq y$, we write $x < y$. An operator $A : E \to E$ is called an *increasing* operator if for all $x, y \in E$, $x \leq y$ implies that $Ax \leq Ay$. It is called a *concave* operator if for any $x, y \in E$ with $x \leq y$ and any $t \in [0, 1]$, we have $A(tx + (1-t)y) \geq tAx + (1-t)Ay$. For any $u_0, v_0 \in E$ with $u_0 < v_0$, we can define an order interval by $[u_0, v_0] := \{x \in E : u_0 \leq x \leq v_0\}$. We have the following theorem (see, e.g., Guo et al., 2004, Theorem 3.1.6 or Zhang, 2013, Theorem 2.1.2).

THEOREM 5.3.1 (Du, 1989). *Suppose P is a normal cone*[5]*, $u_0, v_0 \in E$, and $u_0 < v_0$. Moreover, $A : [u_0, v_0] \to E$ is an increasing operator. Let $h_0 = v_0 - u_0$. If A is an concave operator, $Au_0 \geq u_0 + \epsilon h_0$ for some $\epsilon \in (0, 1)$, and $Av_0 \leq v_0$, then A has a unique fixed point x^* in $[u_0, v_0]$. Furthermore, for any $x_0 \in [u_0, v_0]$, $A^n x_0 \to x^*$ as $n \to \infty$.*

This theorem gives a sufficient condition for the existence, uniqueness, and global stability of the fixed point of an operator without assuming it to be a contraction mapping. It is particularly useful in cases where we study a monotone concave operator but the contraction property is hard or impossible to establish. This is the case in our model.

[5] A cone $P \subset E$ is said to be normal if there exists $\delta > 0$ such that $\|x + y\| \geq \delta$ for all $x, y \in P$ and $\|x\| = \|y\| = 1$.

The operator T is not a contraction[6] because the transaction cost coefficient δ is greater than 1, but as will be shown below, T is actually an increasing concave operator.

Based on Theorem 5.3.1, we have the following theorem.

THEOREM 5.3.2. *Let $u_0(s) = c'(0)s$, $v_0(s) = c(s)$, and $[u_0, v_0]$ be the order interval on $\mathcal{C}(X)$ with the usual partial order. If Assumptions 5.2.1 and 5.2.2 hold, then T has a unique fixed point p^* in $[u_0, v_0]$. Furthermore, $T^n p \to p^*$ for any $p \in [u_0, v_0]$.*

This theorem ensures that there exists a unique price function in equilibrium and it can be computed by successive evaluation of the operator T on any function located in that order interval[7]. Furthermore, as is clear in the proof (see Section 5.7.1), the existence of the minimizers $t^*(s)$ and $k^*(s)$ can also be proved, although they might not be single valued for some s.

5.3.2. Properties of the Solution. In the case where each firm can only have one upstream partner, the equilibrium price function is strictly increasing and strictly convex (Kikuchi et al., 2018). In this model, however, complications arise since firms at different stages might choose to have different numbers of upstream partners. In fact, the equilibrium price is usually piece-wise convex due to this fact. An example[8] of the equilibrium price function is plotted in Figure 5.1 where $c(s) = e^{10s} - 1$, $g(k) = \beta(k - 1)$ with $\beta = 50$, and $\delta = 10$. As is shown in the plot, the price function as a whole is not convex, but it is piece-wise convex with each piece corresponding to a choice of k. Monotonicity of p^* remains true.

PROPOSITION 5.3.3. *The equilibrium price function $p^* : X \to \mathbb{R}_+$ is strictly increasing.*

[6]To be more rigorous, T is not a contraction under the supremum norm, but it might be a contraction in some other complete metric. In fact, Bessaga (1959) proves a partial converse of the Contraction Mapping Theorem, which ensures that under certain conditions there exists a complete metric in which T is a contraction. Also see Leader (1982); for the construction of such metrics, see Janos (1967) and Williamson and Janos (1987). For an application of this theorem in the economic literature, see Balbus et al. (2013). We wish to thank an anonymous referee for referring us to this literature.

[7]For the choice of the order interval we also follow Kikuchi et al. (2018).

[8]The parameterization here is merely chosen to highlight the shape of the price function and is not economically realistic. The price is computed using a faster algorithm introduced in Section 5.4 with $m = 5000$ grid points instead of successive evaluation of T.

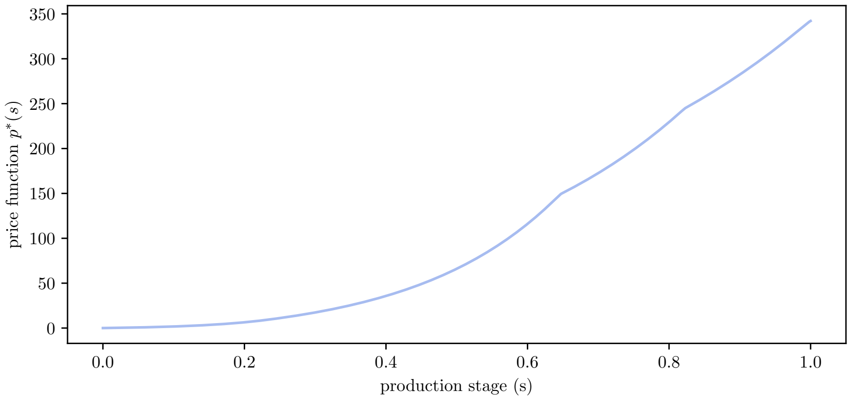


FIGURE 5.1. An example of equilibrium price function.

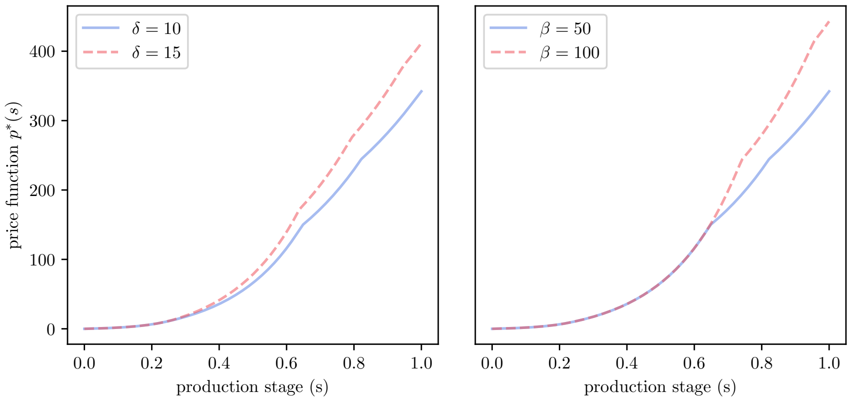


FIGURE 5.2. Equilibrium price function when $c(s) = e^{10s} - 1$ and $g(k) = \beta(k-1)$.

As for comparative statics, we have some basic results also present in Kikuchi et al. (2018) about the effect of changing transaction costs on the equilibrium price function. If either transaction cost (δ or g) increases, the equilibrium price function also increases.

PROPOSITION 5.3.4. *If $\delta_a \leq \delta_b$, then $p_a^* \leq p_b^*$. Similarly, if $g_a \leq g_b$, $p_a^* \leq p_b^*$.*

In Figure 5.2, we plot how the equilibrium price function changes when transaction cost increases. The baseline model setting is the same as Figure 5.1. We can see that if δ or β increases, the equilibrium price function also increases.

5.4. Computation

To compute an approximation to the equilibrium price function given δ, c, and g, one possibility is to take a function in $[u_0, v_0]$ and iterate with T. However, in practice we can only approximate the iterates, and, since T is not a contraction mapping the rate of convergence can be unsatisfactory for some model settings. On the other hand, as we now show, there is a fast, non-iterative alternative that is guaranteed to converge.

Let $G = \{0, h, 2h, ..., 1\}$ for fixed h. Given G, we define our approximation p to p^* via the recursive procedure in Algorithm 1. In the fourth line, the evaluation of $p(s)$ is by setting

$$p(s) = \min_{\substack{t \leq s-h \\ k \in \mathbb{N}}} \{c(s-t) + g(k) + \delta k p(t/k)\}. \tag{5.4}$$

In line five, the linear interpolation is piecewise linear interpolation of grid points $0, h, 2h, \ldots, s$ and values $p(0), p(h), p(2h), \ldots, p(s)$.

The procedure can be implemented because the minimization step on the right-hand side of (5.4), which is used to compute $p(s)$, only evaluates p on $[0, s-h]$, and the values of p on this set are determined by previous iterations of the loop. Once the value $p(s)$ has been computed, the following line extends p from $[0, s-h]$ to the new interval $[0, s]$. The process repeats. Once the algorithm completes, the resulting function p is defined on all of $[0, 1]$ and satisfies $p(0) = 0$ and (5.4) for all $s \in G$ with $s > 0$.

Now consider a sequence of grids $\{G_n\}$, and the corresponding functions $\{p_n\}$ defined by Algorithm 1. Let $G_n = \{0, h_n, 2h_n, \ldots, 1\}$ with $h_n = 2^{-n}$. In this setting we have the following result, the proof of which is given in Section 5.7.2.

THEOREM 5.4.1. *If Assumptions 5.2.1 and 5.2.2 hold, then $\{p_n\}$ converges to p^* uniformly.*

The main advantage of this algorithm is that, for any chosen number of grid points, the number of minimization operations required is fixed, and we can improve the accuracy of this algorithm by increasing the number of grid points. For the iteration method, however,

Algorithm 1 Construction of p from $G = \{0, h, 2h, ..., 1\}$

```
p(0) ← 0
s ← h
while s ≤ 1 do
    evaluate p(s) via equation (5.4)
    define p on [0, s] by linear interpolation of p(0), p(h), p(2h), ..., p(s)
    s ← s + h
end while
```

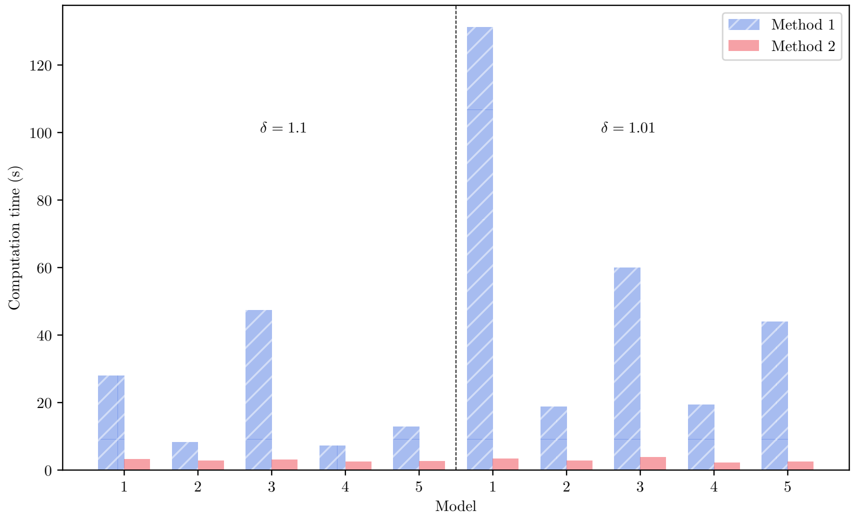


FIGURE 5.3. Computation time comparison for the two methods.

the rate of convergence is different for different model settings and to achieve the same accuracy it usually requires longer computation time.

In Figure 5.3, we plot the computation time[9] of successive iterations of T with $p_0 = c$ (method 1) and Algorithm 1 (method 2) for ten different model settings when the number of grid points is set to be $m = 1000$. The first and last five models are the same[10] except $\delta = 1.1$ for the former and $\delta = 1.01$ for the latter. In each model, we also compute an accurate price function using Algorithm 1 with a very large number of grid points ($m = 50000$) and compare it with results from both methods when $m = 1000$. We find that the error from method 2 is comparable or smaller than that from method 1 in each model. The algorithm achieves more accurate results at a much faster speed. As we can see in Figure 5.3, method 2 completes the computation in around 3 seconds in each model while the computation time of method 1 ranges from 7 seconds to more than 2 minutes. The speed difference is especially drastic when δ is close to 1, since it takes T more iterations to converge with smaller δ but the number of operations for the algorithm is fixed. In model 1 with $\delta = 1.01$, the algorithm is 40 times faster than successive iterations of T!

[9]The computations were conducted on a XPS 13 9360 laptop with i7-7500U CPU. The program only utilizes a single core.

[10]The cost function c and additive transaction cost function g for the five models are: (1) $c(s) = e^{10s} - 1$, $g(k) = k - 1$; (2) $c(s) = e^s - 1$, $g(k) = 0.01(k-1)$; (3) $c(s) = e^{s^2} - 1$, $g(k) = 0.01(k-1)$; (4) $c(s) = s^2 + s$, $g(k) = 0.01(k-1)$; (5) $c(s) = e^s + s^2 - 1$, $g(k) = 0.05(k-1)$.

5.5. Stochastic Choices

So far we have discussed the case in which each firm can choose the optimal number of upstream partners according to (5.1). In reality, however, firms usually face uncertainty when choosing their partners. The result is that some firms might choose fewer or more partners than what is optimal. For instance, a firm might not be able to choose a certain number of upstream partners due to regulation or failure to arrive at agreements with potential partners. Conversely, the upstream partners of a firm might experience supply shocks and fail to meet production requirements, causing it to sign more partners than what is optimal and bear more transaction costs. In this section, we model this scenario and incorporate uncertainty into each firm's optimization problem.

We assume that each firm chooses an amount of "search effort" λ and the resulting number of upstream partners follows a Poisson distribution[11] with parameter λ that starts from $k = 1$. In other words, the probability of having k partners is

$$f(k; \lambda) = \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!}$$

when $\lambda > 0$. We also assume that when $\lambda = 0$, $\text{Prob}(n = 1) = 1$, that is, each firm can always choose to have only one upstream partner with certainty. For example, if a firm chooses to exert effort $\lambda = 2.5$, the probabilities of it ending up with 1, 2, 3, 4, 5 partners are, respectively: 0.08, 0.2, 0.26, 0.21, 0.13. One characteristic of the Poisson distribution is that both its mean and variance increase with λ, which makes it suitable for our model since the more partners a firm aims for, the more uncertainty there will be in the contracting process.

Hence, a firm at stage s solves the following problem:

$$\min_{\substack{t \leq s \\ \lambda \geq 0}} \left\{ c(s-t) + \mathbb{E}_k^{\lambda} \left[g(k) + \delta k p(t/k) \right] \right\} \tag{5.5}$$

[11]Note that in the usual sense, if a random variable X follows the Poisson distribution, X takes values in nonnegative integers. Here we shift the probability function so that k starts from 1.

where $\mathbb{E}_k^\lambda$ stands for taking expectation of k under the Poisson distribution with parameter λ. Specifically,

$$\mathbb{E}_k^\lambda \left[g(k) + \delta k p(t/k)\right] = \sum_{k=1}^{\infty} \left[g(k) + \delta k p(t/k)\right] f(k; \lambda).$$

Similar to Section 5.3, we can define another operator $\tilde{T} : \mathcal{C}(X) \to \mathcal{R}(X)$ by

$$\tilde{T}p(s) := \min_{\substack{t \leq s \\ \lambda \geq 0}} \left\{ c(s-t) + \mathbb{E}_k^\lambda \left[g(k) + \delta k p(t/k)\right] \right\}. \tag{5.6}$$

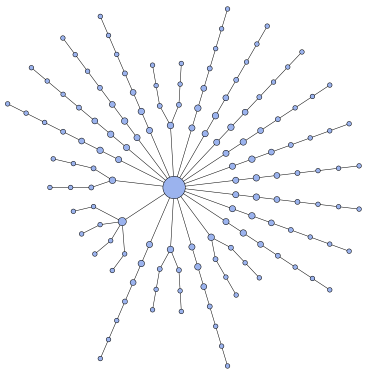
As will be shown in Section 5.7.3, all of the above results still apply in the stochastic case and we summarize them in the following theorem.

THEOREM 5.5.1. *Let $u_0(s) = c'(0)s$, $v_0(s) = c(s)$. If Assumptions 5.2.1 and 5.2.2 hold, then the operator $\tilde{T}$ has a unique fixed point $\tilde{p}^*$ in $[u_0, v_0]$ and $\tilde{T}^n p \to \tilde{p}^*$ for any $p \in [u_0, v_0]$. Furthermore, $\tilde{p}_n$ from Algorithm 1 converges to $\tilde{p}^*$ uniformly.*

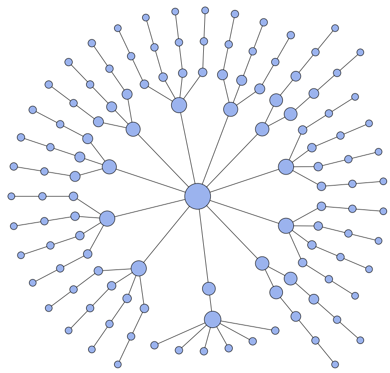
By Theorem 5.5.1, there exists a unique equilibrium price function $\tilde{p}^*$ and we can compute it either by successive evaluation of $\tilde{T}$ or by Algorithm 1. The algorithm is particularly useful here since it now takes much longer time to complete one minimization operation with firms choosing continuous values of λ instead of discreet values of k.

Similarly, there exist minimizers t^* and λ^* so that firm at any stage s has an optimal choice $t^*(s)$ and $\lambda^*(s)$. With the optimal choice functions, we can compute an equilibrium firm allocation recursively as in Kikuchi et al. (2018). Specifically, we start at the most downstream firm at $s = 1$ and compute its optimal choices t^* and λ^*. Next, we pick a realization of k according to the Poisson distribution with parameter λ^* and repeat the process for each of its upstream firm at $s' = t^*/k$. The whole process ends when all the most upstream firms choose to carry out the remaining production process by themselves. Note that due to the stochastic nature of this model, each simulation will give a different firm allocation.

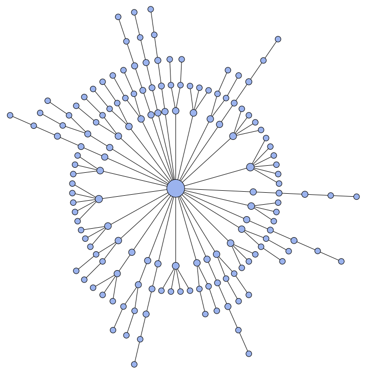
In Figure 5.4, we plot some production networks for different model parameterizations using the above approach. Each node represents a firm and the one at the center is the firm



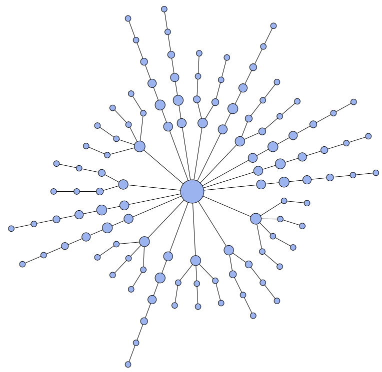
(A) $\beta = 0.0005$, $\delta = 1.05$, $\theta = 1.2$



(B) $\beta = 0.0005$, $\delta = 1.1$, $\theta = 1.2$



(C) $\beta = 0.0001$, $\delta = 1.05$, $\theta = 1.2$



(D) $\beta = 0.0005$, $\delta = 1.05$, $\theta = 1.15$

FIGURE 5.4. Production networks with stochastic choices of upstream partners

at $s = 1$. The size of each node is proportionate to the size[12] of the corresponding firm. The cost function is set to be $c(s) = s^\theta$ and the additive transaction cost is $g(k) = \beta(k-1)^{1.5}$. Compared with production networks in Kikuchi et al. (2018), the graphs here are no longer symmetric since even firms on the same layer can have different realized numbers of upstream partners and thus different firm sizes. The prediction that downstream firms are larger and tend to have more subcontractors, on the other hand, is also valid in our networks.

Comparing (a) and (b), an increase in transaction cost makes firms in (b) outsource less and produce more in-house, resulting in fewer layers in the production network. Similarly, comparing (c) with (a), a decrease in additive transaction costs encourages firms at each level to find more subcontractors. The results are more but smaller firms at each level and fewer layers in the network. Comparing (d) with (a), the difference is a decrease in

[12]Here the firm size is calculated using its value added $c(s - t^*) + g(k)$ where k is a realization of the Poisson distribution with parameter λ^*.

curvature of the cost function c, which makes outsourcing less appealing. The firms in (d) tend to produce more in-house, resulting in a production network of fewer layers.

5.6. Conclusion

In this paper, we extend the production chain model of Kikuchi et al. (2018) to more realistic settings, in which each firm can have multiple upstream partners and face uncertainty in the contracting process. We prove the uniqueness of equilibrium price for these extensions and propose a fast algorithm for computing the price function that is guaranteed to converge.

The key to proving uniqueness of equilibrium price in this model is the theory of monotone concave operators, which gives sufficient conditions for existence, uniqueness, and convergence. This theory has been proven useful in finding equilibria in a range of economic models as mentioned in the introduction and can potentially be applied to more problems where contraction property is hard to establish.

Our model also has some predictions regarding the shape of production networks and the size distribution of firms. In our extended model with uncertainty, we generate a series of production networks (Figure 5.4) under different model settings. A notable observation from this exercise is that increasing the proportionate transaction cost δ or decreasing the additive transaction cost g will reduce the number of layers in a network. In the former case, the cost of market transactions increases; this encourages vertical integration and hence leads to larger firms along each chain. In the latter case, the cost of maintaining multiple partners decreases; this discourages lateral integration and leads to more firms in each layer. This prediction can potentially be tested with suitable choice of proxies for δ and g.

Another observation is that different model settings lead to different size distributions of firms. For example, smaller δ seems to lead to more extreme differences in firm sizes as shown in the comparison between (a) and (b) in Figure 5.4. The underlying mechanism is unclear in our model, which provides a possible channel for future research.

A notable feature of our model is that firms are ex-ante identical but ex-post heterogeneous in equilibrium in terms of sizes, positions in a network, and number of subcontractors. However, the cost function c and transaction costs δ and g are assumed to be fixed throughout this paper. Introducing heterogeneity into these costs might offer richer implications for firm distribution and industry policies. We also leave this possibility for future research.

5.7. Appendix

5.7.1. Proofs from Section 5.3. Let $U = \mathbb{N} \times [0, 1]$ equipped with the Euclidean metric in $\mathbb{R}^2$ and X be equipped with the Euclidean metric in $\mathbb{R}$. To simplify notation, we can write T as

$$Tp(s) = \min_{(k,t)\in\Theta(s)} f_p(s, k, t)$$

where $\Theta : X \to U$ is a correspondence defined by $\Theta(s) = \mathbb{N} \times [0, s]$, and $f_p(s, k, t) = c(s - t) + g(k) + \delta kp(t/k)$.

Lemma 5.7.1. *$Tp \in \mathcal{C}([0, 1])$ for all $p \in \mathcal{C}([0, 1])$.*

Proof. We use Berge's theorem to prove continuity. By Assumption 5.2.2, we can restrict Θ to be $\Theta(s) = \{1, 2, \dots, \bar{k}\} \times [0, s]$ for some large $\bar{k} \in \mathbb{N}$. Then Θ is compact-valued.

To see Θ is upper hemicontinuous, note $\Theta(s)$ is closed for all $s \in X$. Since the graph of Θ is also closed, by the Closed Graph Theorem (see, e.g., Aliprantis and Border, 2006, p. 565), Θ is upper hemicontinuous on X.

To check for lower hemicontinuity, fix $s \in X$. Let V be any open set intersecting $\Theta(s) = \{1, 2, \dots, \bar{k}\} \times [0, s]$. Then it is easy to see that we can find a small $\epsilon > 0$ such that $\Theta(s') \cap V \neq \emptyset$ for all $s' \in [s - \epsilon, s + \epsilon]$. Hence Θ is lower hemicontinuous on X.

Because $p \in \mathcal{C}([0, 1])$, f_p is jointly continuous in its three arguments. By Berge's theorem, Tp is continuous on X. □

Note that by Berge's theorem, the minimizers t^* and k^* exist and are upper hemicontinuous.

LEMMA 5.7.2. *T is increasing and concave.*

PROOF. It is apparent that T is increasing. To see T is concave, let $p, q \in \mathcal{C}([0,1])$ and $\alpha \in (0,1)$. Then we have

$$\begin{aligned}
\alpha Tp(s) + (1-\alpha)Tq(s) &= \min_{(k,t)\in\Theta(s)} \alpha f_p(s,k,t) + \min_{(k,t)\in\Theta(s)} (1-\alpha) f_q(s,k,t) \\
&\leq \min_{(k,t)\in\Theta(s)} \{\alpha f_p(s,k,t) + (1-\alpha) f_q(s,k,t)\} \\
&= \min_{(k,t)\in\Theta(s)} \{c(s-t) + g(k) + \delta k\left[\alpha p(t/k) + (1-\alpha) q(t/k)\right]\} \\
&= \min_{(k,t)\in\Theta(s)} f_{\alpha p + (1-\alpha) q}(s,k,t) \\
&= T\left[\alpha p + (1-\alpha) q\right](s)
\end{aligned}$$

which completes the proof. □

LEMMA 5.7.3. *$Tu_0 \geq u_0 + \epsilon(v_0 - u_0)$ for some $\epsilon \in (0,1)$.*

PROOF. Define $\bar{s} := \max\{0 \leq s \leq 1 : c'(s) \leq \delta c'(0)\}$. Then we have

$$\begin{aligned}
Tu_0(s) &= \min_{(k,t)\in\Theta(s)} f_{u_0}(s,k,t) \\
&= \min_{(k,t)\in\Theta(s)} \{c(s-t) + g(k) + \delta c'(0) t\} \\
&= \min_{t \leq s} \{c(s-t) + \delta c'(0) t\} \\
&= \begin{cases} c(\bar{s}) + \delta c'(0)(s - \bar{s}), & \text{if } s \geq \bar{s} \\ c(s), & \text{if } s < \bar{s} \end{cases}
\end{aligned}$$

Since $Tu_0(s) > u_0(s)$ for all s except at 0, we can find $\epsilon \in (0,1)$ such that $Tu_0 \geq u_0 + \epsilon(v_0 - u_0)$. □

LEMMA 5.7.4. *$Tv_0 \leq v_0$.*

Proof. Choose $k = 1$ and $t = 0$. We have $Tv_0(s) \leq c(s-0) + g(1) + \delta c(0) = c(s) = v_0(s)$. □

Proof of Theorem 5.3.2. Since $P = \{f \in \mathcal{C}(X) : f(x) \geq 0 \text{ for all } x \in X\}$ is a normal cone, the theorem follows from the previous lemmas and Theorem 5.3.1. □

Proof of Proposition 5.3.3. We first show that T maps a strictly increasing function to a strictly increasing function. Suppose $p \in [u_0, v_0]$ and is strictly increasing. Pick any $s_1, s_2 \in [0,1]$ with $s_1 < s_2$. Let t^* and k^* be the minimizers of T. To simplify notation, let $t_1 \in t^*(s_1)$, $t_2 \in t^*(s_2)$, $k_1 \in k^*(s_1)$, and $k_2 \in k^*(s_2)$. If $t_2 \leq s_1$, then we have

$$\begin{aligned} Tp(s_2) &= c(s_2 - t_2) + g(k_2) + \delta k_2 p(t_2/k_2) \\ &> c(s_1 - t_2) + g(k_2) + \delta k_2 p(t_2/k_2) \\ &\geq Tp(s_1). \end{aligned}$$

If $s_1 < t_2 \leq s_2$, then $t_2 + s_1 - s_2 \leq s_1$. Since p is strictly increasing, we have

$$\begin{aligned} Tp(s_2) &= c\left(s_1 - (t_2 + s_1 - s_2)\right) + g(k_2) + \delta k_2 p(t_2/k_2) \\ &> c\left(s_1 - (t_2 + s_1 - s_2)\right) + g(k_2) + \delta k_2 p\left((t_2 + s_1 - s_2)/k_2\right) \\ &\geq Tp(s_1). \end{aligned}$$

Since $c \in [u_0, v_0]$, by Theorem 5.3.2, $T^n c \to p^*$ as $n \to \infty$. Furthermore, since c is strictly increasing, it follows from the above result that p^* is strictly increasing. □

Proof of Proposition 5.3.4. If $\delta_a \leq \delta_b$, then $T_a p \leq T_b p$ for any $p \in [u_0, v_0]$. Since T is increasing by Lemma 5.7.2, we have $T_a^n p \leq T_b^n p$ for any $p \in [u_0, v_0]$ and any $n \in \mathbb{N}$. Then by Theorem 5.3.2, $p_a^* \leq p_b^*$. The same arguments applies if $g_a \leq g_b$. □

5.7.2. Proof of Theorem 5.4.1.

Lemma 5.7.5. *The function p_n is increasing for every n.*

Proof. As p_n is piecewise linear, we shall prove it by induction. Since $p_n(0) = 0$ and $p_n(h_n) = c(h_n)$, p_n is increasing on $[0, h_n]$. Suppose it is increasing on $[0, s]$ for some $s = h_n, 2h_n, \ldots, 1 - h_n$, then we have

$$\begin{aligned} p_n(s + h_n) &= \min_{t \le s,\ k \in \mathbb{N}} \{c(s + h_n - t) + g(k) + \delta k p_n(t/k)\} \\ &= c(s + h_n - t^*) + g(k^*) + \delta k^* p_n(t^*/k^*) \end{aligned}$$

where t^* and k^* are the minimizers. If $t^* \le s - h_n$, it follows from the monotonicity of c that

$$\begin{aligned} p_n(s + h_n) &\ge c(s - t^*) + g(k^*) + \delta k^* p_n(t^*/k^*) \\ &\ge \min_{t \le s - h_n,\ k \in \mathbb{N}} \{c(s - t) + g(k) + \delta k p_n(t/k)\} \\ &= p_n(s). \end{aligned}$$

If $t^* \in (s - h_n, s]$, then $s + h_n - t^* \ge h_n$. Because p_n is increasing on $[0, s]$, we have

$$\begin{aligned} p_n(s + h_n) &\ge c[s - (s - h_n)] + g(k^*) + \delta k^* p_n[(s - h_n)/k^*] \\ &\ge \min_{t \le s - h_n,\ k \in \mathbb{N}} \{c(s - t) + g(k) + \delta k p_n(t/k)\} \\ &= p_n(s), \end{aligned}$$

which completes the proof. □

Lemma 5.7.6. *The sequence* $\{p_n\}_{n=1}^{\infty}$ *is uniformly bounded and equicontinuous.*

Proof. To see $\{p_n\}$ is uniformly bounded, note that for each n,

$$\begin{aligned} p_n(s + h_n) &= \min_{t \le s,\ k \in \mathbb{N}} \{c(s + h_n - t) + g(k) + \delta k p_n(t/k)\} \\ &\le c(s + h_n) + g(1) + \delta p_n(0) \\ &= c(s + h_n) \le c(1) \end{aligned}$$

for all $s = 0, h_n, \ldots, 1 - h_n$.

Due to Lemma 5.7.5, to see $\{p_n\}$ is equicontinuous, it suffices to show that there exists $K > 0$ such that $p_n(s+h_n) - p_n(s) \leq Kh_n$ for all $n \in \mathbb{N}$ and all $s = 0, h_n, 2h_n, \ldots, 1-h_n$. Fix such n and s. If $s = 0$, $p_n(h_n) - p_n(0) = c(h_n) \leq c'(1)h_n$. If $s \geq h_n$, denote the minimizers in the definition of $p_n(s)$ by t^* and k^*, i.e.,

$$\begin{aligned} p_n(s) &= \min_{t \leq s-h_n,\, k \in \mathbb{N}} \{c(s-t) + g(k) + \delta k p_n(t/k)\} \\ &= c(s-t^*) + g(k^*) + \delta k^* p_n(t^*/k^*). \end{aligned}$$

Since $t^* \leq s$, it follows that

$$\begin{aligned} p_n(s+h_n) &= \min_{t \leq s,\, k \in \mathbb{N}} \{c(s+h_n-t) + g(k) + \delta k p_n(t/k)\} \\ &\leq c(s+h_n-t^*) + g(k^*) + \delta k^* p_n(t^*/k^*). \end{aligned}$$

Hence,

$$\begin{aligned} p_n(s+h_n) - p_n(s) &\leq c(s+h_n-t^*) + c(s-t^*) \\ &\leq c'(1)h_n, \end{aligned}$$

which completes the proof. □

Lemma 5.7.7. *There exists a uniformly convergent subsequence of $\{p_n\}$. Furthermore, every uniformly convergent subsequence of $\{p_n\}$ converges to a fixed point of T.*

Proof. Lemma 5.7.6 and the Arzelà-Ascoli theorem imply that p_n has a uniformly convergent subsequence. To simplify notation, let $\{p_n\}$ be such a subsequence and converge uniformly to $\bar{p}$. Because p_n are continuous, $\bar{p}$ is continuous. By Berge's theorem,

$$T\bar{p}(s) = \min_{t \leq s,\, k \in \mathbb{N}} \{c(s-t) + g(k) + \delta k \bar{p}(t/k)\}$$

is also continuous. To see $\bar{p}$ is a fixed point of T, it is sufficient to show that $\bar{p}$ and $T\bar{p}$ agree on the dyadic rationals $\cup_n G_n$, i.e.,

$$\lim_{n \to \infty} \min_{\substack{t \leq s-h_n \\ k \in \mathbb{N}}} \{c(s-t) + g(k) + \delta k p_n(t/k)\} = \min_{t \leq s,\, k \in \mathbb{N}} \{c(s-t) + g(k) + \delta k \bar{p}(t/k)\}$$

for every $s \in \cup_n G_n$.

Fix $\epsilon > 0$. Since $p_n \to \bar{p}$ uniformly, there exists $N_1 \in \mathbb{N}$ such that $n > N_1$ implies that

$$p_n(x) > \bar{p}(x) - \epsilon/(\delta\bar{k})$$

for all $x \in [0,1]$ where $\bar{k}$ is the upper bound on the possible values of k. It follows that for $n > N_1$ we have

$$\begin{aligned}\min_{\substack{t\leq s-h_n\\k\in\mathbb{N}}} \{c(s-t)+g(k)+\delta k p_n(t/k)\} &> \min_{\substack{t\leq s-h_n\\k\in\mathbb{N}}} \{c(s-t)+g(k)+\delta k\bar{p}(t/k)\} - \epsilon\\ &\geq \min_{\substack{t\leq s\\k\in\mathbb{N}}} \{c(s-t)+g(k)+\delta k\bar{p}(t/k)\} - \epsilon.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \min_{\substack{t\leq s-h_n\\k\in\mathbb{N}}} \{c(s-t)+g(k)+\delta k p_n(t/k)\} \geq \min_{t\leq s,\, k\in\mathbb{N}} \{c(s-t)+g(k)+\delta k\bar{p}(t/k)\}.$$

For the other direction, there exists $N_2 \in \mathbb{N}$ such that $n > N_2$ implies that

$$p_n(x) < \bar{p}(x) + \epsilon/(2\delta\bar{k})$$

for all $x \in [0,1]$. Then for $n > N_2$ we have

$$\min_{\substack{t\leq s-h_n\\k\in\mathbb{N}}} \{c(s-t)+g(k)+\delta k p_n(t/k)\} < \min_{\substack{t\leq s-h_n\\k\in\mathbb{N}}} \{c(s-t)+g(k)+\delta k\bar{p}(t/k)\} + \epsilon/2.$$

Since $c, g, \bar{p}$ are continuous and $h_n \to 0$, we can choose N_3 such that $n > N_3$ implies that

$$\min_{\substack{t\leq s-h_n\\k\in\mathbb{N}}} \{c(s-t)+g(k)+\delta k\bar{p}(t/k)\} < \min_{\substack{t\leq s\\k\in\mathbb{N}}} \{c(s-t)+g(k)+\delta k\bar{p}(t/k)\} + \epsilon/2.$$

Hence, for $n > \max\{N_2, N_3\}$ we have

$$\min_{\substack{t\leq s-h_n\\k\in\mathbb{N}}} \{c(s-t)+g(k)+\delta k p_n(t/k)\} < \min_{\substack{t\leq s\\k\in\mathbb{N}}} \{c(s-t)+g(k)+\delta k\bar{p}(t/k)\} + \epsilon.$$

This implies

$$\lim_{n\to\infty} \min_{\substack{t\leq s-h_n\\k\in\mathbb{N}}} \{c(s-t)+g(k)+\delta k p_n(t/k)\} \leq \min_{t\leq s,\, k\in\mathbb{N}} \{c(s-t)+g(k)+\delta k\bar{p}(t/k)\}.$$

Therefore, $\bar{p} = T\bar{p}$. □

Lemma 5.7.8. *Every uniformly convergent subsequence of* $\{p_n\}$ *converges to* p^*.

Proof. Let $\{p_n\}$ be the subsequence that converges uniformly to $\bar{p}$. By Theorem 5.3.2, to see $\bar{p} = p^*$, it suffices to show that $\bar{p}$ is continuous and $c'(0)x \leq \bar{p}(x) \leq c(x)$ for all $x \in [0,1]$. Continuity is satisfied by the fact that each p_n is continuous and $p_n \to \bar{p}$ uniformly. To show the second one, we again prove this holds on $\cup_n G_n$, and it is sufficient to show that $c'(0)s \leq p_n(s) \leq c(s)$ for all $s \in G_n$ and all $n \in \mathbb{N}$. It is apparent that $p_n(s) \leq c(s)$ (choose $t = 0$ and $k = 1$). We show $p_n(s) \geq c'(0)s$ by induction. Suppose $p_n(x) \geq c'(0)x$ for all $x \leq s$. Then we have

$$\begin{aligned}
p_n(s + h_n) &= \min_{t \leq s,\, k \in \mathbb{N}} \{c(s + h_n - t) + g(k) + \delta k p_n(t/k)\} \\
&\geq \min_{t \leq s,\, k \in \mathbb{N}} \{c'(0)(s + h_n - t) + g(k) + \delta c'(0)t\} \\
&= \min_{t \leq s} \{c'(0)(s + h_n - t + \delta t)\} \\
&= c'(0)(s + h_n).
\end{aligned}$$

Since $p_n(0) = 0 \geq c'(0) \cdot 0$, it follows that $p_n(s) \geq c'(0)s$. This concludes the proof. □

5.7.3. Proof of Theorem 5.5.1. Similar to Section 5.7.1, we can write the operator $\tilde{T}$ in (5.6) as

$$\tilde{T}p(s) = \min_{(\lambda,t) \in \tilde{\Theta}(s)} \left\{c(s - t) + \mathbb{E}_k^{\lambda}\left[g(k) + \delta k p(t/k)\right]\right\}$$

where $\tilde{\Theta}(s) = [0, \infty) \times [0, s]$. Upon close inspection, all of the above lemmas still hold for $\tilde{T}$ if we can restrict $\tilde{\Theta}(s)$ to be a compact set. To be more specific, Lemma 5.7.2 and 5.7.5 can be proved in the exact same way; Lemma 5.7.3, 5.7.4, 5.7.6, and 5.7.8 hold since each firm can choose $k = 1$ with probability 1; Lemma 5.7.7 and 5.7.1 need the compactness of $\tilde{\Theta}(s)$. To avoid redundancy, we omit the proofs and shall only show that there exists an upper bound on the choice set of λ.

Let ν be the median of the Poisson distribution and denote the ceiling of ν (i.e., the least integer greater than or equal to ν) by $\bar{\nu}$. Then we have

$$\sum_{k=\bar{\nu}}^{\infty} f(k;\lambda) \geq \frac{1}{2}$$

by definition. It follows that the expectation of $g(k)$

$$\begin{aligned}
\mathbb{E}_k^{\lambda} g(k) &= \sum_{k=1}^{\infty} g(k) f(k;\lambda) \\
&\geq \sum_{k=\bar{\nu}}^{\infty} g(k) f(k;\lambda) \\
&\geq g(\bar{\nu}) \sum_{k=\bar{\nu}}^{\infty} f(k;\lambda) \\
&\geq \frac{1}{2} g(\bar{\nu})
\end{aligned}$$

where the second inequality follows from Assumption 5.2.2. Choi (1994) gives bounds[13] for the median of the Poisson distribution:

$$\lambda - \ln 2 \leq \nu - 1 < \lambda + \frac{1}{3}.$$

So we have

$$\mathbb{E}_k^{\lambda} g(k) \geq \frac{1}{2} g(\bar{\nu}) \geq \frac{1}{2} g(\nu) \geq \frac{1}{2} g(\lambda - \ln 2 + 1).$$

Therefore, we can find $\bar{\lambda}$ such that $\mathbb{E}_k^{\lambda} g(k) \geq c(1)$ for all $\lambda \geq \bar{\lambda}$ and hence $\Theta(s)$ is essentially $[0, \bar{\lambda}] \times [0, s]$ which is a compact set.

[13]Since in our model k starts from 1, we write $\nu - 1$ in the inequality.

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