Time Averaged Consensus in a Direct Coupled Coherent Quantum Observer Network

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Abstract—**This paper considers the problem of constructing a direct coupling quantum observer for a closed linear quantum system. The proposed distributed observer consists of a network of quantum harmonic oscillators and it is shown that the observer network converges to a consensus in a time averaged sense in which each element of the observer estimates the specified output of the quantum plant. An example and simulations are included to illustrate the properties of the observer network.**

I. INTRODUCTION

A number of papers have recently considered the problem of constructing a coherent quantum observer for a quantum system; see [1]–[4]. In the coherent quantum observer problem, a quantum plant is coupled to a quantum observer which is also a quantum system. The quantum observer is constructed to be a physically realizable quantum system so that the system variables of the quantum observer converge in some suitable sense to the system variables of the quantum plant.

In the papers [1], [2], [4], the quantum plant under consideration is a linear quantum system. In recent years, there has been considerable interest in the modeling and feedback control of linear quantum systems; e.g., see [5]–[8]. Such linear quantum systems commonly arise in the area of quantum optics; e.g., see [9], [10]. For such linear quantum system models an important class of quantum control problems are referred to as coherent quantum feedback control problems; e.g., see [5], [6], [11]–[20]. In these coherent quantum feedback control problems, both the plant and the controller are quantum systems and the controller is typically to be designed to optimize some performance index. The coherent quantum observer problem can be regarded as a special case of the coherent quantum feedback control problem in which the objective of the observer is to estimate the system variables of the quantum plant.

In some of the previous papers on quantum observers such as [1]–[3], the coupling between the plant and the observer is via a field coupling. This leads to an observer structure of the form shown in Figure 1. This enables a one way connection between the quantum plant and the quantum observer. Also, since both the quantum plant and the quantum observer are open quantum systems, they are both subject to quantum noise.
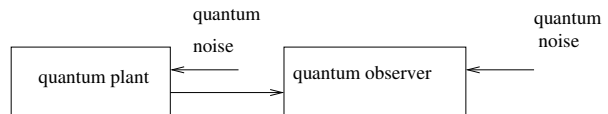
Fig. 1: Coherent Observer Structure with Field Coupling.

Fig. 2: Coherent Observer Structure with Direct Coupling.

However in the paper [13], a coherent quantum control problem is considered in which both field coupling and direct coupling is considered between the quantum plant and the quantum controller. In this paper, we explore the construction of a coherent quantum observer in which there is only direct coupling between quantum plant and the quantum observer. Furthermore, both the quantum plant and the quantum observer are assumed to be closed quantum systems which means that they are not subject to quantum noise and are purely deterministic systems. This leads to an observer structure of the form shown in Figure 2. It is shown that for the case being considered, a quantum observer can be constructed to estimate some but not all of the system variables of the quantum plant. Also, the observer variables converge to the plant variables in a time averaged sense rather than a quantum expectation sense such as considered in the papers [1], [2].

In this paper, we consider the construction of a direct coupling quantum observer for a linear quantum plant and consider the case in which the quantum observer has the structure of an observer network make up of a collection of observer elements. This observer network is constructed so that the output of each observer element converges to the output of the quantum plant in a time averaged sense. This means that there is a consensus of the observer network element in estimating the output of the quantum plant. In recent years, there has been significant interest in controlling networks of multi-agent systems to achieve a consensus among the agents; e.g., see [21]–[25]. In particular, some authors have looked at the problem of consensus in distributed estimation problems; e.g., see [26], [27]. Furthermore, issues of consensus have been considered in networked quantum systems; see [28]–[32]. This work is motivated by the fact that it is becoming increasingly possible for quantum control experiments

to involve the networked interconnection of many quantum elements and these quantum networks will have important applications in problems such as quantum communication and quantum information processing. Also, many macroscopic systems can be regarded as consisting of a large quantum network. These issues motivate the direct coupled coherent quantum observer network problem being considered in this paper.

The results presented in this paper build on some of the results presented in the preliminary conference papers [33]–[35]). However, the results presented here provide a significant generalization compared to the results of [33]–[35]. In particular, in this paper we allow for a non-zero Hamiltonian for the quantum plant, whereas in the papers [33]–[35], the plant Hamiltonian was assumed to be zero. Also, in the paper [33], the quantum observer did not have a network structure and corresponds to a special case of the current paper in which the quantum observer network has only a single element. In addition, the paper [34], restricts attention to quantum observer networks having a simple chain structure and for which the quantum plant and each element of the quantum observer network contains only a single mode. Finally, the paper [35] considers the case in which the quantum plant is a single qubit rather than a quantum linear system as considered in this paper. Also, it is assumed in [35] that each element of the quantum observer network contains only a single mode.

In addition to the papers [33]–[35], a number of other conference papers have considered problems related to the current problem. The paper [36] considers the case in which the quantum plant is a single qubit and the quantum observer is a single mode quantum linear system. The paper [37] considers the problem of an experimental implementation of the results of [33]. The paper [38] considers the problem of an experimental implementation of the results of [33] with the modification that the quantum observer allows for a measurement of its output using Homodyne detection. The paper [39] considers a modification of the results of [33] to allow for a reduced order quantum observer. The paper [40] modifies the approach of [34] to allow for a chain structured observer network which would be more straightforward to implement experimentally than the approach proposed in [34].

II. QUANTUM SYSTEMS

In the quantum observer network problem under consideration, both the quantum plant and the quantum observer network are linear quantum systems; see also [5], [13], [41]. We will restrict attention to closed linear quantum systems which do not interact with an external environment. The quantum mechanical behavior of a linear quantum system is described in terms of the system *observables* which are self-adjoint operators on an underlying infinite dimensional complex Hilbert space $\mathfrak{H}$. The commutator of two scalar operators x and y on $\mathfrak{H}$ is defined as $[x, y] = xy - yx$. Also, for a vector of operators x on $\mathfrak{H}$, the commutator of x and a scalar operator y on $\mathfrak{H}$ is the vector of operators $[x, y] = xy - yx$, and the commutator of x and its adjoint $x^\dagger$ is the matrix of operators

$$[x, x^\dagger] \triangleq xx^\dagger - (x^\# x^T)^T,$$

where $x^\# \triangleq (x_1^* \ x_2^* \ \cdots \ x_n^*)^T$ and * denotes the operator adjoint.

The dynamics of the closed linear quantum systems under consideration are described by non-commutative differential equations of the form

$$\dot{x}(t) \ = \ Ax(t); \quad x(0) = x_0 \qquad (1)$$

where A is a real matrix in $\mathbb{R}^{n \times n}$, and $x(t) = [\ x_1(t) \ \ldots \ x_n(t) \ ]^T$ is a vector of system observables; e.g., see [5]. Here n is assumed to be an even number and $\frac{n}{2}$ is the number of modes in the quantum system.

The initial system variables $x(0) = x_0$ are assumed to satisfy the *commutation relations*

$$[x_j(0), x_k(0)] = 2i\Theta_{jk}, \quad j, k = 1, \ldots, n, \qquad (2)$$

where Θ is a real skew-symmetric matrix with components Θ_{jk}. In the case of a single quantum harmonic oscillator, we will choose $x = (x_1, x_2)^T$ where $x_1 = q$ is the position operator, and $x_2 = p$ is the momentum operator. The commutation relations are $[q, p] = 2i$. In general, the matrix Θ is assumed to be of the form

$$\Theta = \text{diag}(J, J, \ldots, J) \qquad (3)$$

where J denotes the real skew-symmetric 2×2 matrix

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The system dynamics (1) are determined by the system Hamiltonian which is a which is a self-adjoint operator on the underlying Hilbert space $\mathfrak{H}$. For the linear quantum systems under consideration, the system Hamiltonian will be a quadratic form $\mathcal{H} = \frac{1}{2}x(0)^T R x(0)$, where R is a real symmetric matrix. Then, the corresponding matrix A in (1) is given by

$$A = 2\Theta R \qquad (4)$$

where Θ is defined as in (3); e.g., see [5]. In this case, the system variables $x(t)$ will satisfy the *commutation relations* at all times:

$$[x(t), x(t)^T] = 2i\Theta \text{ for all } t \geq 0. \qquad (5)$$

That is, the system will be *physically realizable*; e.g., see [5].

Remark 1. *Note that that the Hamiltonian $\mathcal{H}$ is preserved in time for the system (1). Indeed, $\dot{\mathcal{H}} = \frac{1}{2}\dot{x}^T R x + \frac{1}{2}x^T R \dot{x} = -x^T R \Theta R x + x^T R \Theta R x = 0$ since R is symmetric and Θ is skew-symmetric.*

Quantum Plant

In our proposed direct coupling coherent quantum observer network, the quantum plant is a linear quantum system of the form (1) described by the non-commutative differential equations

$$\begin{aligned} \dot{x}_p(t) \ &= \ A_p x_p(t); \quad x_p(0) = x_{0p}; \\ z_p(t) \ &= \ C_p x_p(t) \end{aligned} \qquad (6)$$

where $z_p(t)$ denotes the vector of system variables to be estimated by the observer network and $A_p \in \mathbb{R}^{n_p \times n_p}$, $C_p \in \mathbb{R}^{m_p \times n_p}$. It is assumed that n_p is even. It is also assumed
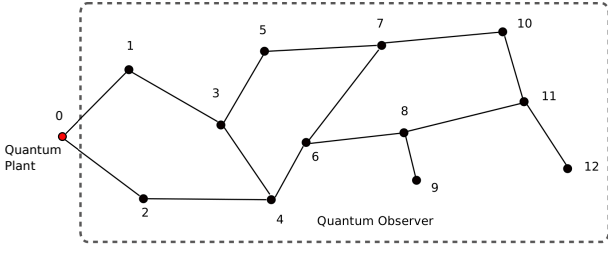
Fig. 3: The graph $(\mathcal{G}, E)$ for a typical quantum observer network.

that this quantum plant corresponds to a plant Hamiltonian $\mathcal{H}_p = \frac{1}{2} x_p(0)^T R_p x_p(0)$. It follows from (4) that $A_p = 2\Theta_p R_p$ where the matrix Θ_p is of the form (3).

Quantum Observer Network

We now describe the linear quantum system of the form (1) which will correspond to the quantum observer network; see also [5], [13], [41]. This system is described by non-commutative differential equations of the form

$$
\begin{aligned}
\dot{x}_o(t) &= A_o x_o(t); \quad x_o(0) = x_{0o}; \\
z_o(t) &= C_o x_o(t)
\end{aligned} \tag{7}
$$

where the observer output $z_o(t)$ is the observer network estimate vector and $A_o \in \mathbb{R}^{n_o \times n_o}$, $C_o \in \mathbb{R}^{\frac{n_o}{2} \times n_o}$. Also, $x_o(t)$ is a vector of self-adjoint non-commutative system variables; e.g., see [5]. We assume the observer network order n_o is an even number. We also assume that the plant variables commute with the observer variables. The system dynamics (7) are determined by the observer system Hamiltonian which is a self-adjoint operator on the underlying Hilbert space for the observer. For the quantum observer network under consideration, this Hamiltonian is given by a quadratic form: $\mathcal{H}_o = \frac{1}{2} x_o(0)^T R_o x_o(0)$, where R_o is a real symmetric matrix. Then, the corresponding matrix A_o in (7) is given by

$$
A_o = 2\Theta_o R_o \tag{8}
$$

where Θ_o is of the form (3). Furthermore, we will assume that the quantum observer network has a graph structure with N nodes and is coupled to the quantum plant as illustrated in Figure 3.

The combined plant observer system is described by a connected graph $(\mathcal{G}, E)$ which has $N+1$ nodes with node 0 corresponding to the quantum plant and the remaining nodes, labelled $1, 2, \ldots, N$, corresponding to the observer elements. This corresponds to an observer Hamiltonian of the form

$$
\begin{aligned}
\mathcal{H}_o &= \frac{1}{2} x_o(0)^T R_o x_o(0) \\
&= \frac{1}{2} \sum_{i=1}^{N} x_{oi}(0)^T R_{oi} x_{oi}(0) \\
&\quad + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} x_{oi}(0)^T R_{cij} x_{oj}(0)
\end{aligned}
$$

where the vector of observer system variables x_o is partitioned according to each element of the quantum observer network

as follows

$$
x_o = \begin{bmatrix} x_{o1} \\ x_{o2} \\ \vdots \\ x_{oN} \end{bmatrix}.
$$

We assume that the variables for each element of the quantum observer network commute with the variables of all other elements of the quantum observer network; i.e.,

$$
[x_{oi}, x_{oj}^T] = 0 \quad \forall \ i \neq j.
$$

Also, we partition the matrix Θ_o as

$$
\Theta_o = \begin{bmatrix} \Theta_{o1} & & & \\ & \Theta_{o2} & \mathbf{0} & \\ & \mathbf{0} & \ddots & \\ & & & \Theta_{oN} \end{bmatrix} \tag{9}
$$

where each matrix Θ_{oi} is also of the form (3).

We define a coupling Hamiltonian which defines the coupling between the quantum plant and the quantum observer network:

$$
\mathcal{H}_c = \frac{1}{2} \sum_{i=1}^{N} \left(x_p(0)^T R_{c0i} x_{oi}(0) + x_{oi}(0)^T R_{c0i}^T x_p(0) \right).
$$

Furthermore, we write

$$
z_o = \begin{bmatrix} z_{o1} \\ z_{o2} \\ \vdots \\ z_{oN} \end{bmatrix}
$$

where

$$
z_{oi} = C_{oi} x_{oi} \text{ for } i = 1, 2, \ldots, N.
$$

Then

$$
C_o = \begin{bmatrix} C_{o1} & & & \\ & C_{o2} & \mathbf{0} & \\ & \mathbf{0} & \ddots & \\ & & & C_{oN} \end{bmatrix}.
$$

Note that $R_{oi} \in \mathbb{R}^{n_{oi} \times n_{oi}}$, $R_{cij} \in \mathbb{R}^{n_{oi} \times n_{oj}}$, $C_{oi} \in \mathbb{R}^{m_p \times n_{oi}}$, and each matrix R_{oi} is symmetric for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N$. In addition, $R_{c0j} \in \mathbb{R}^{n_p \times n_{oi}}$ for $j = 1, 2, \ldots, N$. Also, the matrices R_{cij} for $i = 0, 1, \ldots, N$, $j = 1, 2, \ldots, N$ are such that $R_{cij} \neq 0$ if and only if $(i, j) \in E$, the set of edges for the graph $(\mathcal{G}, E)$.

The augmented quantum linear system consisting of the quantum plant and the quantum observer network is described

by the total Hamiltonian

$$
\begin{aligned}
\mathcal{H}_a &= \mathcal{H}_p + \mathcal{H}_c + \mathcal{H}_o \\
&= \frac{1}{2} x_p(0)^T R_p x_p(0) + \frac{1}{2} \sum_{i=1}^{N} x_{oi}(0)^T R_{oi} x_{oi}(0) \\
&\quad + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} x_{oi}(0)^T R_{cij} x_{oj}(0) \\
&\quad + \frac{1}{2} \sum_{i=1}^{N} \left(x_p(0)^T R_{c0i} x_{oi}(0) + x_{oi}(0)^T R_{c0i}^T x_p(0) \right) \\
&= \frac{1}{2} x_a(0)^T R_a x_a(0) \qquad\qquad\qquad (10)
\end{aligned}
$$

where

$$
x_a = \begin{bmatrix} x_p \\ x_{o1} \\ x_{o2} \\ \vdots \\ x_{oN} \end{bmatrix},
$$

$$
R_a = \begin{bmatrix}
R_p & R_{c01} & R_{c02} & \dots & R_{c0N} \\
R_{c01}^T & R_{o1} & R_{c12} & \dots & R_{c1N} \\
R_{c02}^T & R_{c12}^T & R_{o2} & \dots & R_{c2N} \\
\vdots & & & \ddots & \vdots \\
R_{c0N}^T & R_{c1N}^T & R_{c2N}^T & \dots & R_{oN}
\end{bmatrix}. \quad (11)
$$

Then using (4), it follows that the augmented quantum linear system is described by the equations

$$
\begin{bmatrix} \dot{x}_p(t) \\ \dot{x}_{o1}(t) \\ \dot{x}_{o2}(t) \\ \vdots \\ \dot{x}_{oN}(t) \end{bmatrix} = A_a \begin{bmatrix} x_p(t) \\ x_{o1}(t) \\ x_{o2}(t) \\ \vdots \\ x_{oN}(t) \end{bmatrix};
$$

$$
\begin{aligned}
z_p(t) &= C_p x_p(t); \\
z_o(t) &= C_o x_o(t) \qquad\qquad (12)
\end{aligned}
$$

where $A_a = 2\Theta_a R_a$,

$$
\Theta_a = \begin{bmatrix} \Theta_p & 0 \\ 0 & \Theta_o \end{bmatrix} \qquad (13)
$$

and

$$
C_o = \begin{bmatrix} C_{o1} & & & \\ & C_{o2} & 0 & \\ & 0 & \ddots & \\ & & & C_{oN} \end{bmatrix}.
$$

We now formally define the notion of a direct coupled linear quantum observer network.

Definition 1. *The matrices R_{oj}, R_{cij}, C_{oj} for $i = 0, 1, \dots, N$, $j = 1, 2, \dots, N$ and the graph $(\mathcal{G}, E)$ define a linear quantum observer network achieving time-averaged consensus convergence for the quantum plant (6) if the corresponding augmented linear quantum system (12) is such that*

$$
\lim_{T \to \infty} \frac{1}{T} \int_0^T \left(\begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} z_p(t) - z_o(t) \right) dt = 0. \qquad (14)
$$

III. CONSTRUCTING A DIRECT COUPLING COHERENT QUANTUM OBSERVER NETWORK

We now describe the construction of a direct coupled linear quantum observer network. We assume that $m_p = \frac{n_p}{2}$ and the matrix C_p is of the form $C_p = \alpha_0^T$ where

$$
\alpha_0 = \begin{bmatrix} \alpha_{01} & 0 & & \\ 0 & \alpha_{02} & & 0 \\ & & \ddots & \\ & 0 & & \alpha_{0m_p} \end{bmatrix} \in \mathbb{R}^{n_p \times m_p} \qquad (15)
$$

and $\alpha_{0i} \in \mathbb{R}^{2 \times 1}$ for $i = 1, 2, \dots, m_p$. This assumption means that the plant variables to be estimated include only one quadrature for each mode of the plant. Also, we assume

$$
\|\alpha_{0i}\| = \alpha^2
$$

for $i = 1, 2, \dots, m_p$. Corresponding to the form (15), we can partition the vector of plant variables as

$$
x_p = \begin{bmatrix} x_{p1} \\ x_{p2} \\ \vdots \\ x_{pm_p} \end{bmatrix} \qquad (16)
$$

where each x_{pi} is a 2 by 1 vector of plant variables for $i = 1, 2, \dots, m_p$.

In addition, we assume that R_p is of the form

$$
R_p = \alpha_0 M \alpha_0^T \qquad (17)
$$

where $M = M^T$. It that A_p in (6) is of the form

$$
A_p = 2\Theta_p \alpha_0 M \alpha_0^T.
$$

Hence, it follows from (6) that

$$
\dot{z}_p(t) = C_p 2\Theta_p \alpha_0 M \alpha_0^T x_p(t) = 2\alpha_0^T \Theta_p \alpha_0 M \alpha_0^T x_p(t).
$$

However,

$$
\alpha_0^T \Theta_p \alpha_0 = \begin{bmatrix}
\alpha_{01}^T J \alpha_{01} & 0 & & \\
0 & \alpha_{02}^T J \alpha_{02} & & 0 \\
& & \ddots & \\
& 0 & & \alpha_{0m_p}^T J \alpha_{0m_p}
\end{bmatrix}
$$
$$
= 0 \qquad (18)
$$

since J is skew-symmetric. Therefore

$$
\dot{z}_p(t) = 0.
$$

That is, the vector of plant variables to be estimated $z_p(t)$ will remain fixed if the plant is not coupled to the observer network. However, when the plant is coupled to the quantum observer network this may no longer be the case. We will show that if the quantum observer is suitably designed, the plant quantity to be estimated $z_p(t)$ will remain fixed and the condition (14) will be satisfied.

We assume that each element of the observer network is of dimension n_p and that the vector of observer variables x_{oi} can also be partitioned as in (16) as

$$
x_{oi} = \begin{bmatrix} x_{oi1} \\ x_{oi2} \\ \vdots \\ x_{oim_p} \end{bmatrix} \qquad (19)
$$

for $i = 0, 1, \ldots, N$. Here, each x_{oij} is a 2 by 1 vector of observer variables. We also suppose that the matrices R_{cij}, R_{oi} for $i = 0, 1, \ldots, N$, $j = 1, 2, \ldots, N$ are of the form

$$R_{cij} = \alpha_{ij}\beta_{ij}^T, \quad R_{oi} = \omega_i I \tag{20}$$

where $\alpha_{ij} \in \mathbb{R}^{n_{oi} \times m_p}$, $\beta_{ij} \in \mathbb{R}^{n_{oj} \times m_p}$ and $\omega_i > 0$ for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N$. Also, we assume that

$$R_{c0j} = \alpha_{0j}\beta_{0j}^T \quad \text{where } \alpha_{0j} = \alpha_0 = C_p^T \in \mathbb{R}^{n_p \times m_p} \tag{21}$$

for $j = 1, 2, \ldots, N$ such that $(0, j) \in E$. In addition, note that $\alpha_{ij} = 0$ and $\beta_{ij} = 0$ for $(i, j) \notin E$. Furthermore, we assume

$$C_{oi} = C_p = \alpha_0^T \tag{22}$$

for $i = 1, 2, \ldots, N$.

We will show that these assumptions imply that the quantity $z_p(t) = C_p x_p(t)$ will be constant for the augmented quantum system (12). Indeed, the total Hamiltonian (10) will be given by

$$
\begin{aligned}
\mathcal{H}_a &= \frac{1}{2}x_p(0)^T \alpha_0 M \alpha_0^T x_p(0) + \frac{1}{2}\sum_{i=1}^{N}\omega_i x_{oi}(0)^T x_{oi}(0) \\
&\quad + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} x_{oi}(0)^T \alpha_{ij}\beta_{ij}^T x_{oj}(0) \\
&\quad + \sum_{j=1}^{N} x_p(0)^T \alpha_{0j}\beta_{0j}^T x_{oj}(0).
\end{aligned}
$$

We will show that these assumptions imply that the quantity $z_p(t) = C_p x_p(t)$ will be constant for the augmented quantum system (12). Indeed, it follows from (11), (12), (13) that

$$
\begin{aligned}
\dot{x}_p(t) &= 2\Theta_p R_p x_p(t) + 2\sum_{i=1}^{N}\Theta_p R_{c0i} x_{oi}(t) \\
&= \Theta_p \alpha_0 M \alpha_0^T x_p(t) + 2\sum_{i=1}^{N}\Theta_p \alpha_0 \beta_{0i}^T x_{oi}(t).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\dot{z}_p(t) &= 2C_p \Theta_p \alpha_0 M \alpha_0^T x_p(t) + 2\sum_{i=1}^{N}C_p\Theta_p \alpha_0 \beta_{0i}^T x_{oi}(t) \\
&= 2\alpha_0^T \Theta_p \alpha_0 M \alpha_0^T x_p(t) + 2\sum_{i=1}^{N}\alpha_0^T \Theta_p \alpha_0 \beta_{0i}^T x_{oi}(t).
\end{aligned}
$$

However, it follows from (18) that $\alpha_0^T \Theta_p \alpha_0 = 0$ and hence,

$$\dot{z}_p(t) = 0.$$

Therefore

$$z_p(t) = z_p(0) = z_p \tag{23}$$

for all $t \geq 0$.

Also, it follows from (9), (11), (12), (13) that

$$
\begin{aligned}
\dot{x}_{oj}(t) &= 2\omega_j \Theta_{oj} x_{oj}(t) + \Theta_{oj}\sum_{i=1}^{N}\beta_{ij}\alpha_{ij}^T x_{oi}(t) \\
&\quad + \Theta_{oj}\sum_{i=1}^{N}\alpha_{ji}\beta_{ji}^T x_{oi}(t) + 2\Theta_{oj}\beta_{0j}z_p \tag{24}
\end{aligned}
$$

for $j = 1, 2, \ldots, N$.

To construct a suitable quantum observer network, we will further assume that

$$\alpha_{ij} = \alpha_0, \quad \beta_{ij} = -\mu_{ij}\alpha_0 \tag{25}$$

for $i = 1, \ldots, N$, $j = 1, 2, \ldots, N$ where $(i, j) \in E$. Here,

$$\mu_{ij} = \mu_{ji} > 0. \tag{26}$$

Also, we will assume that

$$\beta_{0j} = -\mu_{0j}\alpha_0 \tag{27}$$

for $j = 1, 2, \ldots, N$ where $(0, j) \in E$.

In order to construct suitable values for the quantities μ_{ij} and ω_i so that (14) is satisfied, we will require that

$$
\begin{aligned}
2\omega_j \Theta_{oj}\alpha_0 &- \sum_{(i,j)\in E, i>0}\mu_{ij}\Theta_{oj}\alpha_0\alpha_0^T\alpha_0 \\
&- \sum_{(i,j)\in E, i>0}\mu_{ij}\Theta_{oj}\alpha_0\alpha_0^T\alpha_0 - 2\Theta_{oj}\mu_{0j}\alpha_0\alpha_0^T\alpha_0 = 0 \tag{28}
\end{aligned}
$$

for $j = 1, 2, \ldots, N$. This condition is equivalent to

$$\omega_j = \sum_{(i,j)\in E, i>0}\mu_{ij}\alpha^2 + \mu_{0j}\alpha^2 \tag{29}$$

for $(0, j) \in E$ and

$$\omega_j = \sum_{(i,j)\in E, i>0}\mu_{ij}\alpha^2 \tag{30}$$

for $(0, j) \notin E$.

Then, we define

$$\tilde{x}_{oj}(t) = x_{oj}(t) - \frac{1}{\alpha^2}\alpha_0 z_p$$

for $j = 1, 2, \ldots, N$. It follows from (28) and (24) that

$$
\begin{aligned}
\dot{\tilde{x}}_{oj}(t) &= 2\omega_j \Theta_{oj}\tilde{x}_{oj}(t) + \Theta_{oj}\sum_{i=1}^{N}\beta_{ij}\alpha_{ij}^T\tilde{x}_{oi}(t) \\
&\quad + \Theta_{oj}\sum_{i=1}^{N}\alpha_{ji}\beta_{ji}^T\tilde{x}_{oi}(t) \\
&= 2\omega_j\Theta_{oj}\tilde{x}_{oj}(t) - 2\sum_{(i,j)\in E, i>0}\mu_{ij}\Theta_{oj}\alpha_0\alpha_0^T\tilde{x}_{oi}(t)
\end{aligned}
$$

for $j = 1, 2, \ldots, N$.

We now write this equation as

$$
\begin{bmatrix}
\dot{\tilde{x}}_{o1}(t) \\
\dot{\tilde{x}}_{o2}(t) \\
\vdots \\
\dot{\tilde{x}}_{oN}(t)
\end{bmatrix}
= A_o
\begin{bmatrix}
\tilde{x}_{o1}(t) \\
\tilde{x}_{o2}(t) \\
\vdots \\
\tilde{x}_{oN}(t)
\end{bmatrix} \tag{31}
$$

where A_o is an $N \times N$ block matrix with blocks

$$
a_{oij} = \begin{cases}
2\omega_i\Theta_{oj} & \text{for } i = j, \\
-2\mu_{ij}\Theta_{oj}\alpha_0\alpha_0^T & \text{for } i \neq j \text{ and } (i, j) \in E, \\
0 & \text{otherwise}
\end{cases}
$$

for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N$. Also, A_o is as given in (8) where R_o is a symmetric $N \times N$ block matrix with blocks

$$
r_{oij} = \begin{cases} \omega_i I & \text{for } i = j, \\ -\mu_{ij}\alpha_0\alpha_0^T & \text{for } i \neq j \text{ and } (i,j) \in E, \\ 0 & \text{otherwise} \end{cases}
$$

for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N$.

To show that the above candidate quantum observer network leads to the satisfaction of the condition (14), we note that

$$
\tilde{x}_o = \begin{bmatrix} \tilde{x}_{o1} \\ \tilde{x}_{o2} \\ \vdots \\ \tilde{x}_{oN} \end{bmatrix}
$$

satisfies (31). Hence, if we can show that

$$
\lim_{T \to \infty} \frac{1}{T} \int_0^T \tilde{x}_o(t) dt = 0 \tag{32}
$$

then it will follow from

$$
\begin{aligned}
C_o \frac{1}{\alpha^2} &\begin{bmatrix} \alpha_0 \\ \alpha_0 \\ \vdots \\ \alpha_0 \end{bmatrix} z_p \\
&= \frac{1}{\alpha^2} \begin{bmatrix} \alpha_0^T & & \\ & \alpha_0^T & 0 \\ & 0 & \ddots \\ & & & \alpha_0^T \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_0 \\ \vdots \\ \alpha_0 \end{bmatrix} z_p \\
&= \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} z_p
\end{aligned} \tag{33}
$$

that (14) is satisfied.

We now show that the symmetric matrix R_o is positive-definite.

Lemma 1. *The matrix R_o is positive definite.*

Proof: In order to establish this lemma, let

$$
x_o = \begin{bmatrix} x_{o1} \\ x_{o2} \\ \vdots \\ x_{oN} \end{bmatrix}
$$

be a non-zero real vector. Then

$$
\begin{aligned}
x_o^T R_o x_o &= \sum_{i=1}^{N} \omega_i \|x_{oi}\|^2 \\
&\quad - \sum_{(i,j) \in E, i>0, j>0} \mu_{ij} x_{oi}^T \alpha_0 \alpha_0^T x_{oj} \\
&= \sum_{i=1}^{N} \omega_i \|x_{oi}\|^2 \\
&\quad - \sum_{(i,j) \in E, i>0, j>0} \mu_{ij} \sum_{k=1}^{m_p} x_{oik}^T \alpha_{0k} x_{ojk}^T \alpha_{0k} \\
&\geq \sum_{i=1}^{N} \omega_i \|x_{oi}\|^2 \\
&\quad - \sum_{(i,j) \in E, i>0, j>0} \mu_{ij} \sum_{k=1}^{m_p} \|x_{oik}\| \|x_{ojk}\| \alpha^2
\end{aligned} \tag{34}
$$

using (15), (19) and the Cauchy-Schwarz inequality. We now define

$$
\check{x}_{oi} = \begin{bmatrix} \|x_{oi1}\| \\ \|x_{oi2}\| \\ \vdots \\ \|x_{oim_p}\| \end{bmatrix}
$$

for $i = 1, 2, \ldots, N$. Again using the Cauchy-Schwarz inequality, it follows that

$$
\begin{aligned}
x_o^T R_o x_o &\geq \sum_{i=1}^{N} \omega_i \|\check{x}_{oi}\|^2 \\
&\quad - \sum_{(i,j) \in E, i>0, j>0} \mu_{ij} \|\check{x}_{oi}\| \|\check{x}_{oj}\| \alpha^2 \\
&= \sum_{i=1}^{N} \omega_i \|\check{x}_{oi}\|^2 \\
&\quad - \sum_{(i,j) \in E, i>0, j>0} \tilde{\mu}_{ij} \|\check{x}_{oi}\| \|\check{x}_{oj}\|
\end{aligned} \tag{35}
$$

where

$$
\tilde{\mu}_{ij} = \mu_{ij}\alpha^2
$$

for $0 = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N$. Thus, (35) implies

$$
x_o^T R_o x_o \geq \check{x}_o^T \tilde{R}_o \check{x}_o
$$

where

$$
\check{x}_o = \begin{bmatrix} \|\check{x}_{o1}\| \\ \|\check{x}_{o2}\| \\ \vdots \\ \|\check{x}_{oN}\| \end{bmatrix}
$$

and $\tilde{R}_o$ is a symmetric $N \times N$ matrix with elements defined by

$$
\tilde{r}_{oij} = \begin{cases} \omega_i & \text{for } i = j, \\ -\tilde{\mu}_{ij} & \text{for } i \neq j \text{ and } (i,j) \in E, \\ 0 & \text{otherwise} \end{cases}
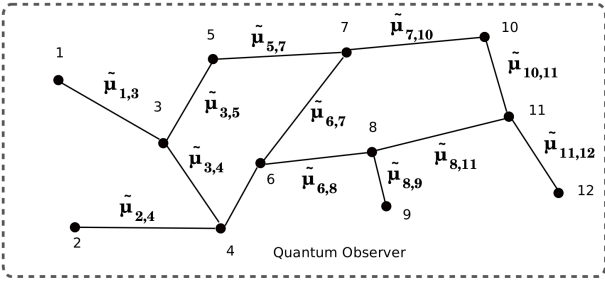$$

Fig. 4: The weighted graph $(\tilde{\mathcal{G}}, \tilde{E})$ corresponding to the graph $(\mathcal{G}, E)$ in Figure 3.

for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N$.

Now the vector $\tilde{x}_o$ will be non-zero if and only if the vector x_o is non-zero. Hence, the matrix R_o will be positive-definite if we can show that the matrix $\tilde{R}_o$ is positive-definite. In order to establish this fact, we first note that (29) and (30) imply that

$$\omega_j \;=\; \sum_{(i,j) \in E, i>0} \tilde{\mu}_{ij} + \tilde{\mu}_{0j}$$

for $(0, j) \in E$ and

$$\omega_j = \sum_{(i,j) \in E, i>0} \tilde{\mu}_{ij}$$

for $(0, j) \notin E$. Hence, we can write

$$\tilde{R}_o \;=\; \tilde{R}_{o1} + \tilde{R}_{o2}$$

where $\tilde{R}_{o1}$ is a symmetric $N \times N$ matrix with elements defined by

$$\tilde{r}_{o1ij} = \begin{cases} \sum_{(k,j) \in E, k>0} \tilde{\mu}_{kj} & \text{for } i = j, \\ -\tilde{\mu}_{ij} & \text{for } i \neq j \text{ and } (i,j) \in E, \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N$. Also, $\tilde{R}_{o2}$ is a diagonal $N \times N$ matrix with elements defined by

$$\tilde{r}_{o2ij} = \begin{cases} \tilde{\mu}_{0j} & \text{for } i = j \text{ and } (0,j) \in E, \\ 0 & \text{otherwise} \end{cases}$$

It follows that the matrix $\tilde{R}_{o2}$ is positive semidefinite.

Now the matrix $\tilde{R}_{o1}$ is the Laplacian matrix for the weighted graph $(\tilde{\mathcal{G}}, \tilde{E})$ obtained by removing node 0 from the graph $(\mathcal{G}, E)$ along with the associated edges. Then each edge $(i, j) \in \tilde{E}$ is given a weight $\tilde{\mu}_{ij}$; e.g., see Figure 4 which shows the weighted graph $(\tilde{\mathcal{G}}, \tilde{E})$ which would correspond to the graph $(\mathcal{G}, E)$ shown in Figure 3.

It follows that the matrix $\tilde{R}_{o1}$ is positive-semidefinite with null space of the following form:

$$\mathcal{N}(\tilde{R}_{o1}) = \text{span}\{f_1, f_2, \ldots, f_m\}$$

where m is the number of connected components of the graph $(\tilde{\mathcal{G}}, \tilde{E})$. Also, each of the vectors $f_1, f_2, \ldots, f_m$ are vectors whose elements are either zeros or ones. For the vector f_k, the elements of this vector which are ones correspond to the nodes in the graph $(\tilde{\mathcal{G}}, \tilde{E})$ in the kth connected component.

The fact that $\tilde{R}_{o1} \geq 0$ and $\tilde{R}_{o2} \geq 0$ implies that $\tilde{R}_o \geq 0$. In order to show that $\tilde{R}_o > 0$, suppose that x is a non-zero vector in $\mathcal{N}(\tilde{R}_o)$. It follows that

$$x^T \tilde{R}_o x = x^T \tilde{R}_{o1} x + x^T \tilde{R}_{o2} x = 0.$$

Since $\tilde{R}_{o1} \geq 0$ and $\tilde{R}_{o2} \geq 0$, x must be contained in the null space of $\tilde{R}_{o1}$ and the null space of $\tilde{R}_{o2}$. Therefore x must be of the form

$$x = \sum_{k=1}^{m} \gamma_k f_k$$

where not all $\gamma_k = 0$. However, since the graph $(\mathcal{G}, E)$ is connected, it follows that there must be at least one branch $(0, j) \in E$ to a node in each of the connected components in the graph $(\tilde{\mathcal{G}}, \tilde{E})$. Then

$$x^T \tilde{R}_{o2} x = \sum_{(0,j) \in E} \tilde{\mu}_{0,j} \gamma_{k(j)}^2 = 0$$

where $k(j)$ corresponds to the node of the connected component in $(\tilde{\mathcal{G}}, \tilde{E})$ which the branch $(0, j)$ connects to. Since each $\tilde{\mu}_{0,j} > 0$, it follows that

$$\gamma_{k(j)} = 0$$

for all $(0, j) \in E$. Furthermore, since each connected component in $(\tilde{\mathcal{G}}, \tilde{E})$ has at least one branch $(0, j) \in E$ connected to it, it follows that $\gamma_1 = \gamma_2 \ldots = \gamma_m = 0$. However, this contradicts the assumption that not all $\gamma_k = 0$. Thus, we can conclude that the matrix $\tilde{R}_o$ is positive definite and hence, the matrix R_o is positive definite. This completes the proof of the lemma. ∎

We now verify that the condition (14) is satisfied for the quantum observer network under consideration. We recall from Remark 1 that the quantity $\frac{1}{2}\tilde{x}_o(t)^T R_o \tilde{x}_o(t)$ remains constant in time for the linear system:

$$\dot{\tilde{x}}_o = A_o \tilde{x}_o = 2\Theta R_o \tilde{x}_o.$$

That is

$$\frac{1}{2}\tilde{x}_o(t)^T R_o \tilde{x}_o(t) = \frac{1}{2}\tilde{x}_o(0)^T R_o \tilde{x}_o(0) \quad \forall t \geq 0. \qquad (36)$$

However, $\tilde{x}_o(t) = e^{2\Theta R_o t}\tilde{x}_o(0)$ and $R_o > 0$. Therefore, it follows from (36) that

$$\sqrt{\lambda_{min}(R_o)}\|e^{2\Theta R_o t}\tilde{x}_o(0)\| \leq \sqrt{\lambda_{max}(R_o)}\|\tilde{x}_o(0)\|$$

for all $\tilde{x}_o(0)$ and $t \geq 0$. Hence,

$$\|e^{2\Theta R_o t}\| \leq \sqrt{\frac{\lambda_{max}(R_o)}{\lambda_{min}(R_o)}} \qquad (37)$$

for all $t \geq 0$.

Now since Θ and R_o are non-singular,

$$\int_0^T e^{2\Theta R_o t} dt = \frac{1}{2}e^{2\Theta R_o T} R_o^{-1}\Theta^{-1} - \frac{1}{2}R_o^{-1}\Theta^{-1}$$

and therefore, it follows from (37) that

$$
\begin{aligned}
\frac{1}{T}\| \int_0^T e^{2\Theta R_o t} dt\| \\
= \quad & \frac{1}{T}\|\frac{1}{2}e^{2\Theta R_o T}R_o^{-1}\Theta^{-1} - \frac{1}{2}R_o^{-1}\Theta^{-1}\| \\
\leq \quad & \frac{1}{2T}\|e^{2\Theta R_o T}\|\|R_o^{-1}\Theta^{-1}\| \\
& + \frac{1}{2T}\|R_o^{-1}\Theta^{-1}\| \\
\leq \quad & \frac{1}{2T}\sqrt{\frac{\lambda_{max}(R_o)}{\lambda_{min}(R_o)}}\|R_o^{-1}\Theta^{-1}\| \\
& + \frac{1}{2T}\|R_o^{-1}\Theta^{-1}\| \\
\rightarrow \quad & 0
\end{aligned}
$$

as $T \rightarrow \infty$. Hence,

$$
\begin{aligned}
\lim_{T\to\infty}\frac{1}{T}\| \int_0^T \tilde{x}_o(t)dt\| \\
= \quad & \lim_{T\to\infty}\frac{1}{T}\| \int_0^T e^{2\Theta R_o t}\tilde{x}_o(0)dt\| \\
\leq \quad & \lim_{T\to\infty}\frac{1}{T}\| \int_0^T e^{2\Theta R_o t}dt\|\|\tilde{x}_o(0)\| \\
= \quad & 0.
\end{aligned}
$$

This implies

$$
\lim_{T\to\infty}\frac{1}{T}\int_0^T \tilde{x}_o(t)dt = 0
$$

and hence, it follows from (31) and (33) that

$$
\lim_{T\to\infty}\frac{1}{T}\int_0^T z_o(t)dt = \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} z_p.
$$

Also, (23) implies

$$
\lim_{T\to\infty}\frac{1}{T}\int_0^T \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} z_p(t)dt = \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix} z_p.
$$

Therefore, condition (14) is satisfied. Thus, we have established the following theorem.

Theorem 1. *Consider a quantum plant of the form (6) where R_p is of the form (17). Then the matrices R_{oi}, R_{cij}, C_{oi}, R_{oi} for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N$ and the connected graph $(\mathcal{G}, E)$ will define a direct coupled quantum observer network achieving time-averaged consensus convergence for this quantum plant if the conditions (20), (21), (22), (25), (27), (26), (29), (30) are satisfied.*

Remark 2. *The quantum observer network constructed above is determined by the choice of the positive parameters $\tilde{\mu}_{ij}$ for $i = 0, 1, \ldots, N$, $j = 1, 2, \ldots, N$. A number of possible choices for these parameters could be considered. One choice is to choose all of these parameters to be the same as $\tilde{\mu}_{ij} = \omega_0$ for $i = 0, 1, \ldots, N$, $j = 1, 2, \ldots, N$ where $\omega_0 > 0$ is a*
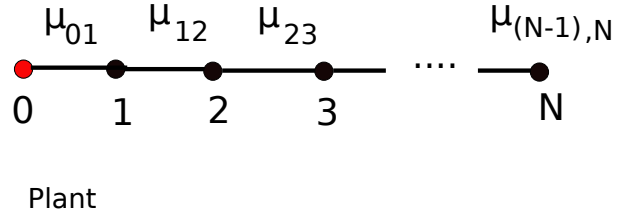

Fig. 5: Quantum Observer Network.

frequency parameter. Another possible approach is to choose the parameters μ_{ij} for $i = 0, 1, \ldots, N$, $j = 1, 2, \ldots, N$ randomly with a uniform distribution on a suitable frequency interval.

IV. ILLUSTRATIVE EXAMPLE

We now present some numerical simulations to illustrate the direct coupled quantum observer network described in the previous section. We choose the quantum plant to have $n_p = 2$, $R_p = 0$ and $C_p = [1 \quad 0]$. That is, the variable to be estimated by the quantum observer is the position operator of the quantum plant; i.e., $z_p(t) = q_p(t)$ where $x_p(t) = \begin{bmatrix} q_p(t) \\ p_p(t) \end{bmatrix}$.

For the quantum observer network, we choose a chain structured network of the form shown in Figure 5 where the number of observer elements is $N = 5$; see also [34]. In this quantum observer network, each element is of order two. We choose the parameters $\tilde{\mu}_{01}, \tilde{\mu}_{12}, \ldots, \tilde{\mu}_{N-1,N}$ so that $\tilde{\mu}_{(k-1),k} = k\omega_0$ for $k = 1, 2, \ldots, N$ where $\omega_0 = 1$. Also, the parameters ω_j, are defined by equations (29), (30) for $j = 1, 2, \ldots, N$. Then the corresponding quantum observer network is defined by equations (20), (25).

The augmented plant-observer system is described by the equations (12), (11). Then, we can write

$$
x_a(t) = \Phi(t)x_a(0)
$$

where

$$
\Phi(t) = e^{A_a t}.
$$

Thus, the plant variable to be estimated $z_p(t)$ is given by

$$
\begin{aligned}
z_p(t) &= e_1 C_a \Phi(t)x_a(0) \\
&= \sum_{i=1}^{2N+2} e_1 C_a \Phi_i(t)x_{ai}(0)
\end{aligned}
$$

where

$$
C_a = \begin{bmatrix} C_p & 0 \\ 0 & C_o \end{bmatrix},
$$

e_1 is the first unit vector in the standard basis for $\mathbb{R}^{N+1}$, $\Phi_i(t)$ is the ith column of the matrix $\Phi(t)$ and $x_{ai}(0)$ is the ith component of the vector $x_a(0)$. We plot each of the quantities $e_1 C_a \Phi_1(t), e_1 C_a \Phi_2(t), \ldots, e_1 C_a \Phi_{2N+2}(t)$ in Figure 6.

From this figure, we can see that $e_1 C_a \Phi_1(t) \equiv 1$ and $e_1 C_a \Phi_2(t) \equiv 0$, $e_1 C_a \Phi_2(t) \equiv 0$, $\ldots$, $e_1 C_a \Phi_{2N+2}(t) \equiv 0$, and $z_p(t)$ will remain constant at $z_p(0)$ for all $t \geq 0$.
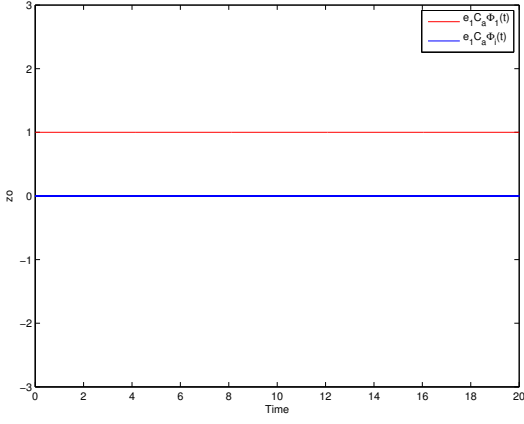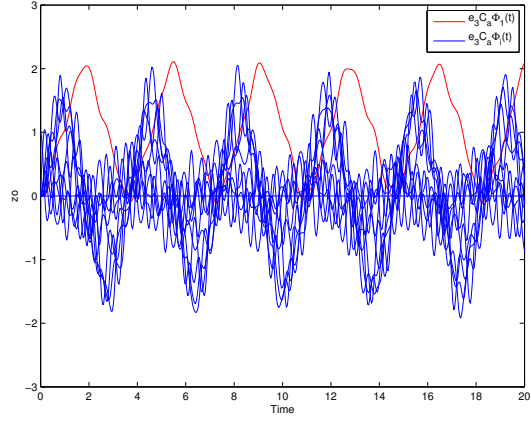
Fig. 6: Coefficients defining $z_p(t)$.

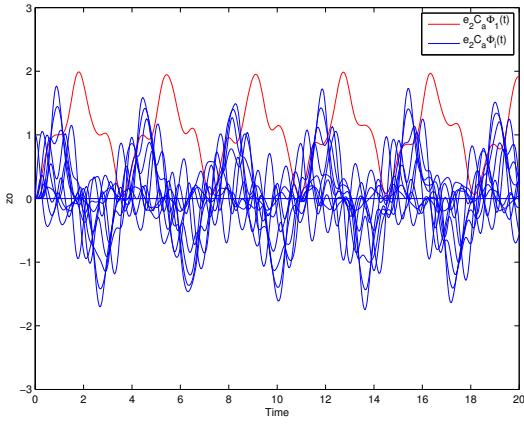Fig. 8: Coefficients defining $z_{o2}(t)$.

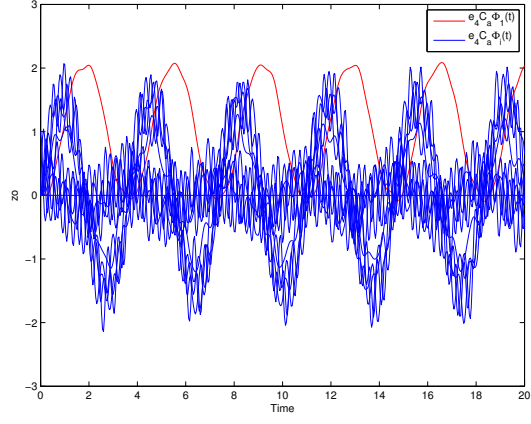
Fig. 7: Coefficients defining $z_{o1}(t)$.

Fig. 9: Coefficients defining $z_{o3}(t)$.

We now consider the output variables of the quantum observer network $z_{oi}(t)$ for $i = 1, 2, \ldots, N$ which are given by

$$z_{oi}(t) = \sum_{j=1}^{2N+2} e_{i+1} C_a \Phi_j(t) x_{aj}(0)$$

where e_{i+1} is the $(i + 1)$th unit vector in the standard basis for $\mathbb{R}^{N+1}$. We plot each of the quantities $e_{i+1} C_a \Phi_1(t), e_{i+1} C_a \Phi_2(t), \ldots, e_{i+1} C_a \Phi_{2N+2}(t)$ in Figures 7 - 11.

Also, we can consider the spatial average obtained by averaging over each of the distributed observer outputs:

$$z_{os}(t) = \frac{1}{N} \sum_{i=1}^{N} z_{oi}(t) = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{2N+2} e_{i+1} C_a \Phi_j(t) x_{aj}(0).$$

Then we plot each of the quantities $\frac{1}{N} \sum_{i=1}^{N} e_{i+1} C_a \Phi_1(t)$, $\frac{1}{N} \sum_{i=1}^{N} e_{i+1} C_a \Phi_2(t), \ldots, \frac{1}{N} \sum_{i=1}^{N} e_{i+1} C_a \Phi_{2N+2}(t)$ in Figure 12.

To illustrate the time average convergence property of the quantum observer (14), we now plot the quantities $\frac{1}{T} \int_0^T e_{i+1} C_a \Phi_1(t) dt$, $\frac{1}{T} \int_0^T e_{i+1} C_a \Phi_2(t) dt$, $\ldots, \frac{1}{T} \int_0^T e_{i+1} C_a \Phi_{2N+2}(t) dt$ for $i = 1, 2, \ldots, N$ in Figures 13-17. These quantities determine the averaged value of the

ith observer output

$$z_{oi}^{ave}(T) = \frac{1}{T} \int_0^T \sum_{j=1}^{2N+2} e_{i+1} C_a \Phi_j(t) x_{aj}(0) dt$$

for $i = 1, 2, \ldots, N$.

From these figures, we can see that for each $i = 1, 2, \ldots, N$, the time average of $z_{oi}(t)$ converges to $z_p(0)$ as $t \to \infty$. That is, the distributed quantum observer reaches a time averaged

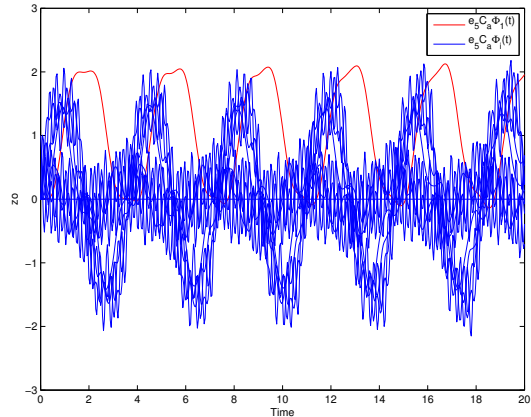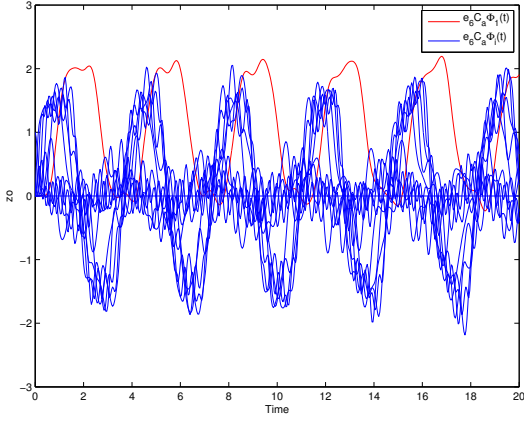Fig. 10: Coefficients defining $z_{o4}(t)$.
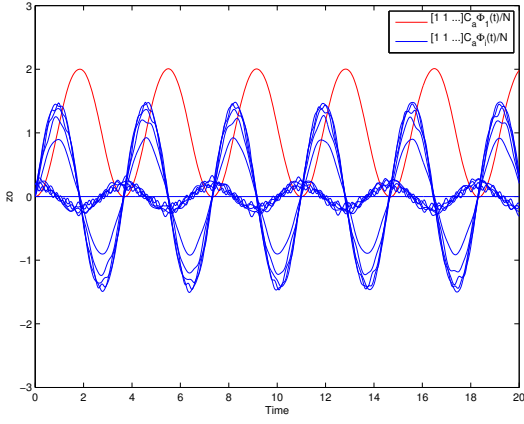
Fig. 11: Coefficients defining $z_{o5}(t)$.

Fig. 14: Coefficients defining the time average of $z_{o2}(t)$.
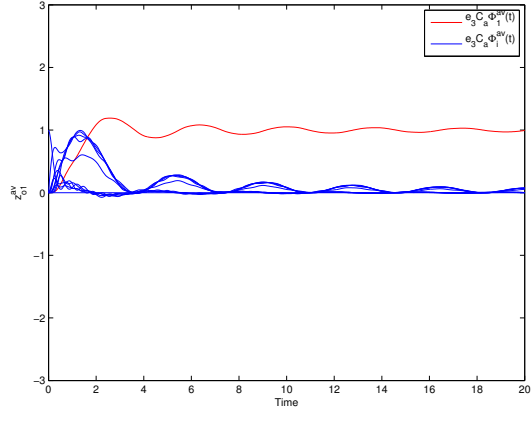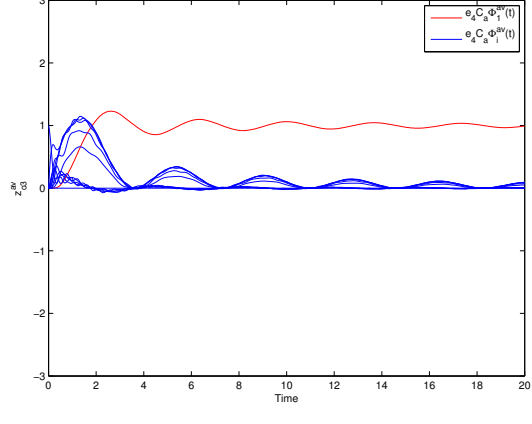

Fig. 12: Coefficients defining $z_{os}(t)$.

Fig. 15: Coefficients defining the time average of $z_{o3}(t)$.

consensus corresponding to the output of the quantum plant which is to be estimated.

simulations to investigate the consensus behavior of the direct coupling observer network.

V. CONCLUSIONS

In this paper we have considered the construction of a direct coupling observer network for a closed quantum linear system in order to achieve a time averaged consensus convergence. We have also presented an illustrative example along with

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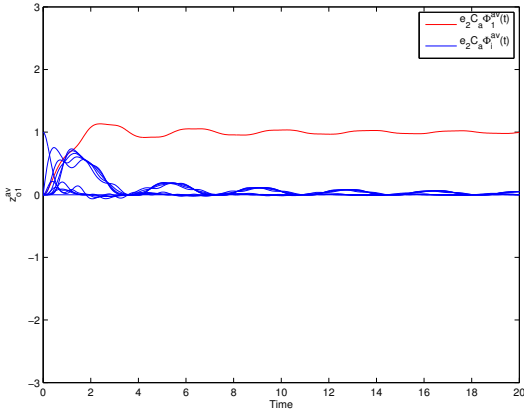
Fig. 13: Coefficients defining the time average of $z_{o1}(t)$.
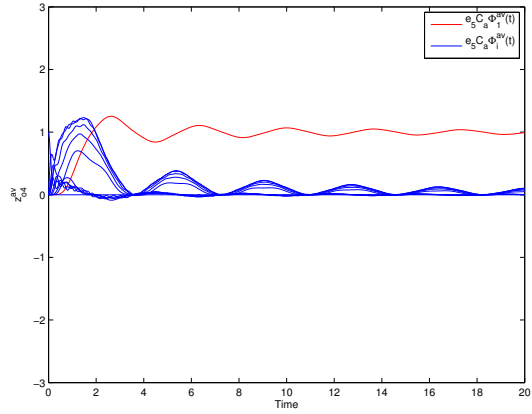

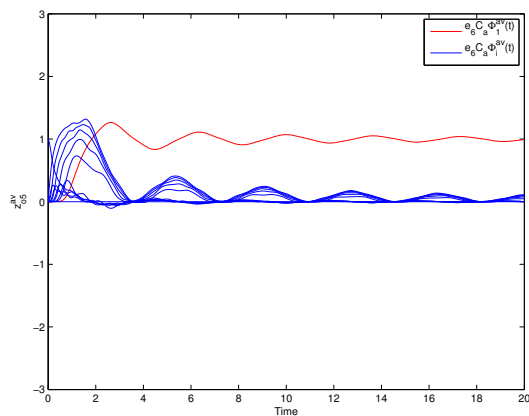Fig. 16: Coefficients defining the time average of $z_{o4}(t)$.

Fig. 17: Coefficients defining the time average of $z_{o5}(t)$.

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