Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 50-1 (2017) 11902–11907

On the Analysis of the DeGroot-Friedkin Model with Dynamic Relative Interaction Matrices ⋆

Mengbin Ye *, **Ji Liu** **, **Brian D.O. Anderson** *,***,****
Changbin Yu *,***, and **Tamer Başar** **

* *Research School of Engineering, Australian National University, Canberra, A.C.T. 2601, Australia*
** *University of Illinois at Urbana-Champaign, USA*
*** *Hangzhou Dianzi University, Hangzhou, Zhejiang, China*
**** *Data61-CSIRO (formerly NICTA Ltd.)*
E-mail: {mengbin.ye, brian.anderson, brad.yu}@anu.edu.au, {jiliu, basar1}@illinois.edu

Abstract: This paper analyses the DeGroot-Friedkin model for evolution of the individuals' social powers in a social network when the network topology varies dynamically (described by dynamic relative interaction matrices). The DeGroot-Friedkin model describes how individual social power (self-appraisal, self-weight) evolves as a network of individuals discuss opinions on a sequence of issues. We seek to study dynamically changing relative interactions because interactions may change depending on the issue being discussed. Specifically, we study relative interaction matrices which vary periodically with respect to the issues. This may reflect a group of individuals, e.g. a government cabinet, that meet regularly to discuss a set of issues sequentially. It is shown that individuals' social powers admit a periodic solution. Initially, we study a social network which varies periodically between two relative interaction matrices, and then generalise to an arbitrary number of relative interaction matrices.

Keywords: social and behavioural sciences, multi-agent systems, social networks, networked systems, opinion dynamics

1. INTRODUCTION

In the past decade and a half, the systems and control community has conducted extensive research on multi-agent systems. A multi-agent system is comprised of multiple interacting agents. Problems such as formation control, distributed optimisation, consensus based coordination and robotics have been intensively studied, see (Cao et al., 2013; Knorn et al., 2016) for two overviews.

On the other hand, the control community has recently turned to study multi-agent systems that appear in the social sciences. Specifically, a social network consisting of groups of people interacting with their acquaintances can be considered from one point of view as a multi-agent system. The emergence of social media platforms such as Facebook, Instagram and Twitter has only increased the relevance of research on social networks.

A problem of particular interest is "opinion dynamics", which studies how opinions within a social network may evolve as individuals discuss an issue, e.g. religion or politics. The classical DeGroot model (DeGroot, 1974) is closely related to the consensus process (Jadbabaie et al., 2003; Shi and Johansson, 2013). Other models include the Friedkin-Johnsen model (Friedkin and Johnsen, 1990), the Altafini model (Altafini, 2013; Altafini and Lini, 2015), and Hegselmann-Krause model (Hegselmann and Krause, 2002; Etesami and Başar, 2015). The DeGroot-Friedkin model proposed and analysed in (Jia et al., 2015) is a two-stage model for *multi-issue* discussions, where issues are discussed sequentially. For each issue, the DeGroot model is used to study the opinion dynamics; each individual updates its own opinion based on a convex combination of its opinion and those of its neighbours. The coefficients of the convex combination are determined by 1) the individual's self-weight (which represents self-appraisal, self-confidence) and 2) the weights assigned by the individual to its neighbours (which might be a measure of trust or friendship). At the beginning of each new issue, a reflected appraisal mechanism is used by each individual to update its own self-weight. This mechanism takes into account the individual's influence and impact on the discussion of opinions on the prior issue. From one perspective, an individual's self-weight is a representation of that individual's social power in the social network.

The key contribution of this paper is the study of the DeGroot-Friedkin model with issue-varying interactions among the individuals. Because interactions in the

⋆ This work was supported by the Australian Research Council (ARC) under the ARC grants DP-130103610 and DP-160104500, by the National Natural Science Foundation of China (grant 61375072), and by Data61-CSIRO (formerly NICTA). The work of Liu and Başar was supported in part by Office of Naval Research (ONR) MURI Grant N00014-16-1-2710, and in part by NSF under grant CCF 11-11342.

10.1016/j.ifacol.2017.08.1426

DeGroot-Friedkin model are modelled by a matrix termed the "*relative interaction matrix*", we will be investigating relative interaction matrices which *dynamically change* between issues but remain constant during each issue. In particular, we will investigate periodic issue-varying interactions as to be explained shortly. As an extension of the DeGroot-Friedkin model, a modified version was proposed and analysed in (Xu et al., 2015; Chen et al., 2015, 2017). Time-varying interactions on this modified DeGroot-Friedkin model was studied in (Xia et al., 2016). On the other hand, there have been no results studying issue-varying interactions for the original DeGroot-Friedkin model proposed in (Jia et al., 2015) (which assumed a constant relative interaction matrix during each discussion and between all issues).

This paper considers issue-dependent interactions that change periodically. Periodically changing interactions may occur if a group meets regularly to discuss the same set of issues. For example, consider a government cabinet that meets regularly to discuss several different issues, e.g. defence, finance, and social security. Because different ministers will have different portfolios and specialisations, it is likely that the weights assigned by an individual to its neighbours (used in the opinion dynamics component of the DeGroot-Friedkin model) will change depending on the issue at hand. Initially, we consider the situation where the social network switches periodically between two different interaction topologies. We show that the self-weight of each individual in the social network has a periodic solution where each individual always has strictly positive self-weight that is less than one. This result is then generalised to multiple periodically switching interaction topologies.

The remainder of the paper is organised as follows. Section 2 provides mathematical notation and introduces the DeGroot-Friedkin model. Section 3 considers interactions which change periodically with issues. Simulations are presented in Section 4 and the conclusion of the paper is presented in Section 5.

2. BACKGROUND AND PROBLEM STATEMENT

We begin by introducing some mathematical notations used in the paper. Let $\mathbf{1}_n$ and $\mathbf{0}_n$ denote, respectively, the $n \times 1$ column vectors of all ones and all zeros. For a vector $\boldsymbol{x} \in \mathbb{R}^n$, $0 \preceq \boldsymbol{x}$ and $0 \prec \boldsymbol{x}$ indicate component-wise inequalities, i.e., for all $i \in \{1, 2, \ldots, n\}$, $0 \leq x_i$ and $0 < x_i$, respectively. Let Δ_n denote the n-simplex, the set which satisfies $\{\boldsymbol{x} \in \mathbb{R}^n : 0 \preceq \boldsymbol{x}, \mathbf{1}_n^\top \boldsymbol{x} = 1\}$. The canonical basis of $\mathbb{R}^n$ is given by $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Define $\widetilde{\Delta}_n = \Delta_n \backslash \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and $\text{int}(\Delta_n) = \{\boldsymbol{x} \in \mathbb{R}^n : 0 \prec \boldsymbol{x}, \mathbf{1}_n^\top \boldsymbol{x} = 1\}$. For the rest of the paper, we shall use the terms "node", "agent", and "individual" interchangeably. We shall also interchangeably use the words "self-weight" and "individual social power".

An $n \times n$ matrix is called a *row-stochastic matrix* if its entries are all nonnegative and its row sums all equal 1.

2.1 Graph Theory

The interaction between agents in a social network is modelled using a weighted directed graph, denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Each individual agent is a node in the finite, nonempty set of nodes $V = \{v_1, \ldots, v_n\}$. The set of ordered edges is $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. We denote an ordered edge as $e_{ij} = (v_i, v_j) \in \mathcal{E}$, and because the graph is directed, in general the assumption $e_{ij} = e_{ji}$ does not hold. An edge e_{ij} is outgoing with respect to v_i and incoming with respect to v_j. The presence of an edge e_{ij} connotes that individual j's opinion is influenced by the opinion of individual i (the precise nature of the influence will be made clear in the sequel). The incoming and outgoing neighbour set of v_i are respectively defined as $\mathcal{N}_i^+ = \{v_j \in \mathcal{V} : e_{ji} \in \mathcal{E}\}$ and $\mathcal{N}_i^- = \{v_j \in \mathcal{V} : e_{ij} \in \mathcal{E}\}$. The relative interaction matrix $\boldsymbol{C} \in \mathbb{R}^{n \times n}$ associated with $\mathcal{G}$ has nonnegative entries c_{ij}, termed "relative interpersonal weights" in (Jia et al., 2015). The entries of $\boldsymbol{C}$ have properties such that $0 < c_{ij} \leq 1 \Leftrightarrow e_{ji} \in \mathcal{E}$ and $c_{ij} = 0$ otherwise. It is assumed that $c_{ii} = 0$ (i.e. with no self-loops), and we impose the restriction that $\sum_{j \in \mathcal{N}_i^+} c_{ij} = 1$ (i.e. that $\boldsymbol{C}$ is a row-stochastic matrix).

A directed path is a sequence of edges of the form $(v_{p_1}, v_{p_2}), (v_{p_2}, v_{p_3}), \ldots$ where $v_{p_i} \in \mathcal{V}, e_{ij} \in \mathcal{E}$. Node i is reachable from node j if there exists a directed path from v_j to v_i. A graph is said to be strongly connected if every node is reachable from every other node. The relative interaction matrix $\boldsymbol{C}$ is irreducible if and only if the associated graph $\mathcal{G}$ is strongly connected. If $\boldsymbol{C}$ is irreducible then it has a unique left eigenvector $\boldsymbol{c}^\top$ associated with the eigenvalue 1, with the property $\boldsymbol{c}^\top \mathbf{1}_n = 1$ (Perron-Frobenius Theorem, see (Godsil et al., 2001)). Henceforth, we shall call this left eigenvector $\boldsymbol{c}^\top$, the *dominant left eigenvector of* $\boldsymbol{C}$.

2.2 Modelling the Dynamics of the Social Network

The discrete-time DeGroot-Friedkin model is comprised of a consensus model and a mechanism for updating self-appraisal (the precise meaning of self-appraisal will be made clear in the sequel). We define $\mathcal{S} = \{1, 2, 3, \ldots\}$ to be the set of indices of sequential issues which are being discussed by the social network. For a given issue s, the social network discusses the issue using the discrete-time DeGroot consensus model. At the end of the discussion (i.e. when the DeGroot model has effectively reached steady state), each individual judges its impact on the discussion (self-appraisal). The individual then updates its own self-weight and discussion begins on the next issue $s + 1$.

DeGroot Consensus of Opinions For each issue $s \in \mathcal{S}$, each agent updates its opinion $y_i(s, \cdot) \in \mathbb{R}$ at the $t + 1^{th}$ time instant as

$$y_i(s, t+1) = w_{ii}(s) y_i(s, t) + \sum_{j \in \mathcal{N}_i^+, j \neq i}^{n} w_{ij}(s) y_j(s, t) \quad (1)$$

where $w_{ii}(s)$ is the self-weight individual i places on its own opinion and w_{ij}'s are the weights given by agent i to the opinions of its neighbour individual j. The opinion dynamics for the entire social network may be expressed as

$$\boldsymbol{y}(s, t+1) = \boldsymbol{W}(s) \boldsymbol{y}(s, t) \quad (2)$$

where $\boldsymbol{y}(s, t) = [y_1(s, t) \ \cdots \ y_n(s, t)]^\top$ is the vector of opinions of the $n+1$ agents in the network at time instant

t. This model was first proposed in (DeGroot, 1974) with $\mathcal{S} = 1$ (i.e. only one issue was discussed). Below, we provide the model for the updating of $\boldsymbol{W}(s)$ (and specifically $w_{ii}(s)$ via a reflected self-appraisal mechanism).

Friedkin's Self-Appraisal Model for Determining Self-Weight The Friedkin component of the model proposes a method for updating the self-weight (self-appraisal, self-confidence or self-esteem) of agent i, which is denoted by $x_i(s) = w_{ii}(s) \in [0,1]$ (the i^{th} diagonal term of $\boldsymbol{W}(s)$) (Jia et al., 2015). Define the vector $\boldsymbol{x}(s) = [x_1(s) \; \cdots \; x_n(s)]^\top$ as the vector of self-weights for the social network, with starting self-weight $\boldsymbol{x}(1) \in \Delta_n$. The influence matrix $\boldsymbol{W}(s)$ can be expressed as

$$\boldsymbol{W}(s) = \boldsymbol{X}(s) + (\boldsymbol{I}_n - \boldsymbol{X}(s))\boldsymbol{C} \tag{3}$$

where $\boldsymbol{C}$ is the relative interaction matrix associated with the graph $\mathcal{G}$ and $\boldsymbol{X}(s) \doteq diag[\boldsymbol{x}(s)]$. From the fact that $\boldsymbol{C}$ is row-stochastic with zero diagonal entries, (3) implies that $\boldsymbol{W}(s)$ is a row-stochastic matrix. The self-weight vector $\boldsymbol{x}(s)$ is updated at the end of issue s as

$$\boldsymbol{x}(s+1) = \boldsymbol{w}(s) \tag{4}$$

where $\boldsymbol{w}(s)$ is the dominant left eigenvector of $\boldsymbol{W}(s)$ with the properties that $\boldsymbol{1}_n^\top \boldsymbol{w}(s) = 1$ and $\boldsymbol{w}(s) \succeq 0$ (Jia et al., 2015). This implies that $\boldsymbol{x}(s) \in \Delta_n$ for all s.

In (Jia et al., 2015), the DeGroot-Friedkin model was studied under the assumption that $\boldsymbol{C}$ was constant for all t and all s. In this paper, we investigate the model when $\boldsymbol{C}(s)$ varies between issues. We assume that each agent's opinion, $y_i(s,t)$, is a scalar for simplicity. The results can readily be applied to the scenario where each agent's opinion state is a vector $\boldsymbol{y}_i \in \mathbb{R}^p, p \geq 2$, by using Kronecker products.

It is shown in [Lemma 2.2, (Jia et al., 2015)] that the system (4), with $\boldsymbol{C}$ independent of s, is equivalent to

$$\boldsymbol{x}(s+1) = \boldsymbol{F}(\boldsymbol{x}(s)) \tag{5}$$

where the nonlinear vector-valued function $\boldsymbol{F}(\boldsymbol{x}(s))$ is defined as

$$\boldsymbol{F}(\boldsymbol{x}(s)) = \begin{cases} \mathbf{e}_i & \text{if } x_i(s) = \mathbf{e}_i, \text{for any } i \\ \alpha(\boldsymbol{x}(s)) \begin{bmatrix} \dfrac{c_1}{1-x_1(s)} \\ \vdots \\ \dfrac{c_n}{1-x_n(s)} \end{bmatrix} & \text{otherwise} \end{cases} \tag{6}$$

with $\alpha(\boldsymbol{x}(s)) = 1/\sum_{i=1}^n \frac{c_i}{1-x_i(s)}$ and where c_i is the i^{th} entry of the dominant left eigenvector $\boldsymbol{c}^\top$ of the relative interaction matrix $\boldsymbol{C}$.

Theorem 1. (Theorem 4.1, (Jia et al., 2015)). For $n \geq 3$, consider the DeGroot-Friedkin dynamical system (5) with a relative interaction matrix $\boldsymbol{C}$ that is row-stochastic, irreducible, and has zero diagonal entries. Assume that the digraph $\mathcal{G}$ associated with $\boldsymbol{C}$ does not have star topology [1] and define $\boldsymbol{c}^\top$ as the dominant left eigenvector of $\boldsymbol{C}$. Then,

(i) For all initial conditions $\boldsymbol{x}(1) \in \widetilde{\Delta}_n$, the self-weights $\boldsymbol{x}(s)$ converge to $\boldsymbol{x}^*$ as $s \to \infty$. Here, $\boldsymbol{x}^* \in \widetilde{\Delta}_n$ is the unique fixed point satisfying $\boldsymbol{x}^* = \boldsymbol{F}(\boldsymbol{x}^*)$.
(ii) There holds $x_i^* < x_j^*$ if and only if $c_i < c_j$, for any i, j, where c_i is the i^{th} entry of the dominant left eigenvector $\boldsymbol{c}$. There holds $x_i^* = x_j^*$ if and only if $c_i = c_j$.
(iii) The unique fixed point $\boldsymbol{x}^*$ is determined only by $\boldsymbol{c}^\top$, and is independent of the initial conditions.

2.3 Problem Formulation

This paper studies the extended DeGroot-Friedkin model where $\boldsymbol{C}$ is allowed to change when moving from one issue to the next. For a given s, however, $\boldsymbol{C}$ is assumed to remain constant for all t. Thus, the relative interactions among the individuals, i.e. $\boldsymbol{C}(s)$, may change between issues, but remains constant for all t for a given issue. We will consider alternative situations corresponding to alternative assumptions. We leave the details of these assumptions to their corresponding future sections.

To facilitate our analysis, we make the following two assumptions *which will hold in all models considered in this paper.*

Assumption 1. The graph $\mathcal{G}$ does not have star topology, the relative interaction matrix $\boldsymbol{C}(s)$ is irreducible, and $n \geq 3$.

Assumption 2. The initial conditions of the DeGroot-Friedkin model dynamics (5) satisfy $\boldsymbol{x}(1) \in \widetilde{\Delta}_n$.

Note that Assumption 1 requires $n \geq 3$, because for $n = 2$, any irreducible $\boldsymbol{C}(s)$ is doubly stochastic and corresponds to a star topology. Assumption 2 ensures that no individual begins with self-weight equal to 1 (autocratic configuration).

3. PERIODIC SWITCHING

In this section, we investigate the situation where $\boldsymbol{C}(s)$ changes periodically. In order to simplify the problem, we make the following assumption.

Assumption 3. The social network switches between two relative interaction matrices, $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$, where both matrices are irreducible, row-stochastic, but not necessarily doubly stochastic. More specifically, the social network switches between $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ periodically, with period $T = 2$, as given by

$$\boldsymbol{C}(s) = \begin{cases} \boldsymbol{C}_1 & \text{if } s \text{ is odd} \\ \boldsymbol{C}_2 & \text{if } s \text{ is even} \end{cases} \tag{7}$$

Note that for a constant $\boldsymbol{C}$, simulations show that convergence of $\boldsymbol{x}(s)$ to $\boldsymbol{x}^*$ typically occurs after only a few issues (Jia et al., 2015) (i.e. exponential convergence is conjectured). In light of this, we are therefore interested in periodic switching with a fast switching between issues (e.g. a sequence $\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_1, \boldsymbol{C}_2, \ldots$) as opposed to a slow switching between issues (e.g. $\boldsymbol{C}_1, \ldots, \boldsymbol{C}_1, \boldsymbol{C}_2, \ldots, \boldsymbol{C}_2$).

3.1 Transformation into a Time-Invariant System

Under Assumption 3, the update of the self-weights occurs as $\boldsymbol{x}(s+1) = \boldsymbol{F}(\boldsymbol{x}(s), s)$, where we now acknowledge the

[1] A graph $\mathcal{G}$ is said to have star topology if there exists a node i such that every edge of $\mathcal{G}$ is either to or from node i.

fact that $\boldsymbol{F}(\boldsymbol{x}(s), s)$ is an explicit function of time. More specifically, and in accordance with (6), we have

$$\boldsymbol{x}(s+1) = \begin{cases} \boldsymbol{F}_1(\boldsymbol{x}(s)) & \text{if } s \text{ is odd} \\ \boldsymbol{F}_2(\boldsymbol{x}(s)) & \text{if } s \text{ is even} \end{cases} \tag{8}$$

The function $\boldsymbol{F}_p$, for $p = 1, 2$, is

$$\boldsymbol{F}_p(\boldsymbol{x}(s)) = \begin{cases} \mathbf{e}_i & \text{if } x_i(s) = \mathbf{e}_i, \text{for any } i \\ \alpha_p(\boldsymbol{x}(s)) \begin{bmatrix} \frac{c_{1,p}}{1-x_1(s)} \\ \vdots \\ \frac{c_{n,p}}{1-x_n(s)} \end{bmatrix} & \text{otherwise} \end{cases} \tag{9}$$

with $\alpha_p(\boldsymbol{x}(s)) = 1/\sum_{i=1}^n \frac{c_{i,p}}{1-x_i(s)}$. Here $c_{i,p}$ is the i^{th} element of the dominant left eigenvector $\boldsymbol{c_p}^\top$ associated with the relative interaction matrix $\boldsymbol{C}_p$.

We now define a new state $\boldsymbol{y} \in \mathbb{R}^{2n}$ as

$$\boldsymbol{y}(2s) = \begin{bmatrix} \boldsymbol{y}_1(2s) \\ \boldsymbol{y}_2(2s) \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}(2s-1) \\ \boldsymbol{x}(2s) \end{bmatrix} \tag{10}$$

and study the evolution of $\boldsymbol{y}(2s)$ for every $s \in \mathcal{S}$. Note that

$$\boldsymbol{y}(2s+2) = \begin{bmatrix} \boldsymbol{y}_1(2s+2) \\ \boldsymbol{y}_2(2s+2) \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}(2s+1) \\ \boldsymbol{x}(2s+2) \end{bmatrix} \tag{11}$$

In view of the fact that $\boldsymbol{x}(2s+1) = \boldsymbol{F}_2(\boldsymbol{x}(2s))$ and $\boldsymbol{x}(2s+2) = \boldsymbol{F}_1(\boldsymbol{x}(2s+1))$ for any s, we obtain

$$\boldsymbol{y}(2s+2) = \begin{bmatrix} \boldsymbol{F}_2(\boldsymbol{x}(2s)) \\ \boldsymbol{F}_1(\boldsymbol{x}(2s+1)) \end{bmatrix} \tag{12}$$

Similarly, by noticing that $\boldsymbol{x}(2s) = \boldsymbol{F}_1(\boldsymbol{x}(2s-1))$ and $\boldsymbol{x}(2s+1) = \boldsymbol{F}_2(\boldsymbol{x}(2s))$ for any s, we obtain

$$\boldsymbol{y}(2s+2) = \begin{bmatrix} \boldsymbol{F}_2\Big(\boldsymbol{F}_1(\boldsymbol{x}(2s-1))\Big) \\ \boldsymbol{F}_1\Big(\boldsymbol{F}_2(\boldsymbol{x}(2s))\Big) \end{bmatrix} \tag{13}$$

$$= \begin{bmatrix} \boldsymbol{F}_2\Big(\boldsymbol{F}_1(\boldsymbol{y}_1(2s))\Big) \\ \boldsymbol{F}_1\Big(\boldsymbol{F}_2(\boldsymbol{y}_2(2s))\Big) \end{bmatrix} \tag{14}$$

$$= \begin{bmatrix} \boldsymbol{F}_3(\boldsymbol{y}_1(2s)) \\ \boldsymbol{F}_4(\boldsymbol{y}_2(2s)) \end{bmatrix} \tag{15}$$

for the time-invariant nonlinear composition functions $\boldsymbol{F}_3 = \boldsymbol{F}_2 \circ \boldsymbol{F}_1$ and $\boldsymbol{F}_4 = \boldsymbol{F}_1 \circ \boldsymbol{F}_2$.

We can thus express the periodic system (8) as the nonlinear time-invariant system

$$\boldsymbol{y}(2s+2) = \bar{\boldsymbol{F}}(\boldsymbol{y}(2s)) \tag{16}$$

where $\bar{\boldsymbol{F}} = [\boldsymbol{F}_3^\top, \boldsymbol{F}_4^\top]^\top$, and seek to study the equilibrium of this system. More specifically, suppose that $\boldsymbol{y}^* = [\boldsymbol{y}_1^{*\top} \ \boldsymbol{y}_2^{*\top}]^\top$ is an equilibrium of the system (16). In the following subsection, we show that $\boldsymbol{F}_3$ and $\boldsymbol{F}_4$ are continuous. It is then straightforward to see that if $\lim_{k\to\infty} \boldsymbol{y}(k) \to \boldsymbol{y}^*$, then $\boldsymbol{x}(s)$ is an asymptotic periodic sequence where

$$\boldsymbol{x}(s) = \begin{cases} \boldsymbol{F}_3(\boldsymbol{y}_1^*) & \text{if } s \text{ is odd} \\ \boldsymbol{F}_4(\boldsymbol{y}_2^*) & \text{if } s \text{ is even} \end{cases} \tag{17}$$

Define y_i (respectively $\bar{F}_i$) as the i^{th} element of the vector $\boldsymbol{y}$ (respectively $\bar{\boldsymbol{F}}$). From the above, some manipulation allows us to obtain, for $i = 1, \dots, n$,

$$\bar{F}_i(\boldsymbol{y}_1(2s)) = \alpha_2(\boldsymbol{F}_1(\boldsymbol{y}_1(2s))) \frac{c_{i,2}}{1 - \alpha_1(\boldsymbol{y}_1(2s)) \frac{c_{i,1}}{1-y_i(2s)}} \tag{18}$$

where $\alpha_1(\boldsymbol{y}_1(2s)) = 1/\sum_{j=1}^n \frac{c_{j,1}}{1-y_j(2s)}$ and

$$\alpha_2(\boldsymbol{F}_1(\boldsymbol{y}_1(2s))) = \frac{1}{\sum_{p=1}^n \frac{c_{p,2}}{1-\alpha_1(\boldsymbol{y}_1(2s))\frac{c_{p,1}}{1-y_p(2s)}}} \tag{19}$$

3.2 Existence of a Periodic Sequence

In this subsection, we establish properties of the function $\bar{\boldsymbol{F}}$. More specifically, we detail properties of $\boldsymbol{F}_3(\boldsymbol{y}_1(2s))$. Because $\boldsymbol{F}_3(\boldsymbol{y}_1(2s))$ is similar in form to $\boldsymbol{F}_4(\boldsymbol{y}_2(2s))$, a similar proof can be used to establish similar properties for $\boldsymbol{F}_4(\boldsymbol{y}_2(2s))$. Due to space limitations, the proofs for Lemma 2 and Theorem 3 are provided in an extended version of this paper, available on ArXiv (Ye et al., 2017).

Lemma 2. The following properties of $\boldsymbol{F}_3(\boldsymbol{y}_1(2s))$ hold.

- P1 The quantity $\alpha_2(\boldsymbol{F}_1(\boldsymbol{y}_1(2s))) > 0$ if $\boldsymbol{y}_1(2s) \in \widetilde{\Delta}_n$, for any s.
- P2 If $\boldsymbol{y}_1(2s) = \mathbf{e}_i$ for any i, then $\boldsymbol{F}_3(\boldsymbol{y}_1(2s)) = \mathbf{e}_i$. In other words, the n vertices of Δ_n are fixed points of $\boldsymbol{F}_3$.
- P3 The function $\boldsymbol{F}_3(\boldsymbol{y}_1(2s)) : \Delta_n \to \Delta_n$ is continuous.
- P4 There exists at least one fixed point in $\text{int}(\Delta_n)$.

Lemma 2 states that $\boldsymbol{F}_3$ and $\boldsymbol{F}_4$ each have at least one fixed point, which we denote by $\boldsymbol{y}_1^*$ and $\boldsymbol{y}_2^*$ respectively. We will leave the study of whether the fixed points are unique or not, as well as analysis of convergence to the fixed points for future work. We now state the following theorem which establishes the periodic behaviour of (8).

Theorem 3. Suppose that Assumption 3 holds.

- T1 Suppose further that for some $s_1 \in \mathcal{S}$, there holds $\boldsymbol{x}(2s_1 - 1) = \boldsymbol{y}_1^*$, where $\boldsymbol{y}_1^* \in \text{int}(\Delta_n)$ is any fixed point of $\boldsymbol{F}_3$. Then, for all $s \geq 2s_1 - 1$, there holds
$$\boldsymbol{x}(s) = \begin{cases} \boldsymbol{y}_1^* & \text{if } s \text{ is odd} \\ \boldsymbol{y}_2^* & \text{if } s \text{ is even} \end{cases} \tag{20}$$
where $\boldsymbol{y}_2^* \in \text{int}(\Delta_n)$ is a fixed point of $\boldsymbol{F}_4$.
- T2 Suppose now that, instead of T1, there holds for some $s_1 \in \mathcal{S}$, $\boldsymbol{x}(2s_1) = \boldsymbol{y}_2^*$, where $\boldsymbol{y}_2^* \in \text{int}(\Delta_n)$ is any fixed point of $\boldsymbol{F}_3$. Then, (20) holds for all $s \geq 2s_1$, with $\boldsymbol{y}_1^* \in \text{int}(\Delta_n)$ being a fixed point of $\boldsymbol{F}_3$.

Note that the above result establishes that a periodic sequence exists, but convergence to this sequence has not been established. We conjecture that $\boldsymbol{F}_3$ does in fact have a unique fixed point (i.e. a unique periodic sequence for $\boldsymbol{x}(s)$) and that any $\boldsymbol{y}_1(0) \in \widetilde{\Delta}_n$ will converge to the unique $\boldsymbol{y}_1^*$. We conjecture a similar result for $\boldsymbol{F}_4$. In Section 4, we provide simulations in support of these two conjectures.

Remark 4. Theorem 3 leads to an interesting conclusion. Consider the case where, at some point in the evolution of the system trajectory, we have $\boldsymbol{x}(s) = \boldsymbol{y}_1^*$ or $\boldsymbol{y}_2^*$ (e.g. the self-weights are initialised as $\boldsymbol{x}(1) = \boldsymbol{y}_1^*$). Then, the self-weights will exhibit a periodic sequence. Furthermore, for each individual in the network, that individual's self-weight/social power is never zero.

Remark 5. Notice that in the Theorem 3, we did not require the fixed points of $\boldsymbol{F}_3$ and $\boldsymbol{F}_4$ to be unique.

Suppose that there are two distinct fixed points of $\boldsymbol{F}_3$, which we label $\boldsymbol{y}^*_{1,a}$ and $\boldsymbol{y}^*_{1,b}$. The theorem concludes that if $\boldsymbol{x}(2s) = \boldsymbol{y}^*_{1,a}$ for some s, then the system (8) will exhibit a periodic sequence. If on the other hand $\boldsymbol{x}(2s) = \boldsymbol{y}^*_{1,b}$ for some s, the system (8) will also exhibit a periodic sequence, *but different from the sequence involving* $\boldsymbol{y}^*_{1,a}$.

3.3 Generalisation to M Topologies

We now generalise the above framework to the case where the social network switches between M different topologies. The following assumption is now placed on the social network instead of Assumption 3.

Assumption 4. The social network switches between $M \geq 3$ relative interaction matrices in the following manner. For issue $s = M(q-1)+p$, where $q \in \mathbb{Z}_{>0}$ and $p \in \{1, 2, \dots, M\}$ [2], the relative interaction matrix $\boldsymbol{C}(s)$ is given by

$$\boldsymbol{C}(M(q-1)+p) = \boldsymbol{C}_p \tag{21}$$

The matrices $\boldsymbol{C}_p$ are all irreducible, row-stochastic and in general $\boldsymbol{C}_i \neq \boldsymbol{C}_j, \forall\, i, j \in \{1, 2, \dots, M\}$.

With the above Assumption 4, the update of the self-weights is given by

$$\boldsymbol{x}(M(q-1)+p+1) = \boldsymbol{F}_p(\boldsymbol{x}(M(q-1)+p)) \tag{22}$$

for all $q \in \mathbb{Z}_{>0}$ and any $p \in \{1, 2, \dots, M\}$. The function $\boldsymbol{F}_p$ is given in (9), but now for $p = 1, 2, \dots, M$. Following the steps in subsection 3.1, we now show the generalised transformation of the time-varying system with M different topologies to a time-invariant nonlinear system.

A new state variable $\boldsymbol{y} \in \mathbb{R}^{Mn}$ is defined as

$$\boldsymbol{y}(Mq) = \begin{bmatrix} \boldsymbol{y}_1(Mq) \\ \boldsymbol{y}_2(Mq) \\ \vdots \\ \boldsymbol{y}_M(Mq) \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}(M(q-1)+1) \\ \boldsymbol{x}(M(q-1)+2) \\ \vdots \\ \boldsymbol{x}(M(q-1)+M) \end{bmatrix} \tag{23}$$

and we study the evolution of $\boldsymbol{y}(Mq)$ for every strictly positive integer q. It follows that

$$\boldsymbol{y}(M(q+1)) = \begin{bmatrix} \boldsymbol{y}_1(M(q+1)) \\ \boldsymbol{y}_2(M(q+1)) \\ \vdots \\ \boldsymbol{y}_M(M(q+1)) \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}(Mq+1) \\ \boldsymbol{x}(Mq+2) \\ \vdots \\ \boldsymbol{x}(Mq+M) \end{bmatrix} \tag{24}$$

Following the logic in subsection 3.1, but with the precise steps omitted due to space limitations, we obtain that

$$\boldsymbol{y}(M(s+1)) = \begin{bmatrix} \boldsymbol{F}_M(\boldsymbol{F}_{M-1}(\dots(\boldsymbol{F}_1(\boldsymbol{y}_1(Mq))))) \\ \boldsymbol{F}_1(\boldsymbol{F}_M(\dots(\boldsymbol{F}_2(\boldsymbol{y}_2(Mq))))) \\ \vdots \\ \boldsymbol{F}_{M-1}(\boldsymbol{F}_{M-2}(\dots(\boldsymbol{F}_M(\boldsymbol{y}_M(Mq))))) \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_1(\boldsymbol{y}_1(Mq)) \\ \boldsymbol{G}_2(\boldsymbol{y}_2(Mq)) \\ \vdots \\ \boldsymbol{G}_M(\boldsymbol{y}_M(Mq)) \end{bmatrix} \tag{25}$$

Note that each $\boldsymbol{G}_M$ is a time-invariant nonlinear function. Due to the complexity of each $\boldsymbol{G}_i$, we do not reproduce their expressions here, but their forms are similar to the expressions in (18) - (19). The transformed nonlinear system is expressed as

$$\boldsymbol{y}(M(q+1)) = \bar{\boldsymbol{G}}(\boldsymbol{y}(Mq)) \tag{26}$$

The generalisations of Lemma 2 and Theorem 3 are now given below.

Lemma 6. The following properties of $\bar{\boldsymbol{G}}(\boldsymbol{y}(Mq))$ hold, for any $p \in \{1, 2, \dots, M\}$:

- P1 The quantity $\alpha_j > 0, \forall\, j \in \{1, 2, \dots, M\}$ if $\boldsymbol{y}_p(Mq) \in \widetilde{\Delta}_n$, for any q.
- P2 If $\boldsymbol{y}_p(Mq) = \mathbf{e}_i$ for any i, then $\boldsymbol{G}_p(\boldsymbol{y}_p(Mq)) = \mathbf{e}_i$, i.e. the n vertices of Δ_n are fixed points of $\boldsymbol{G}_p$.
- P3 The function $\boldsymbol{G}_p(\boldsymbol{y}_p(Mq)) : \Delta_n \to \Delta_n$ is continuous.
- P4 There exists at least one fixed point for $\boldsymbol{G}_p$ in $\text{int}(\Delta_n)$.

Theorem 7. Suppose that Assumption 4 holds. Suppose further that for some q_1, there holds $\boldsymbol{x}(M(q_1-1)+p) = \boldsymbol{y}^*_p$, where $\boldsymbol{y}^*_p \in \text{int}(\Delta_n)$ is a fixed point of $\boldsymbol{G}_p$. Then, for all $s = M(q-1)+j \geq M(q_1-1)+p$, $q \geq q_1$, there holds

$$\boldsymbol{x}(s) = \boldsymbol{y}^*_j, \text{ for any } j \in \{1, 2, \dots, M\} \tag{27}$$

where $\boldsymbol{y}^*_j \in \text{int}(\Delta_n)$ is a fixed point of $\boldsymbol{G}_j$, and $\boldsymbol{y}^*_j = \boldsymbol{y}^*_p$ for $j = p$.

The proofs for the above two results are not included here due to space limitations, and can be found in the extended arXiv version of the paper (Ye et al., 2017).

4. SIMULATIONS

In this section, simulations are provided which corroborate and illustrate the statements of Lemma 2, Lemma 6, Theorem 3 and Theorem 7. The simulated social network has 8 individuals, with three possible sets of interactions described by three different irreducible relative interaction matrices, $\boldsymbol{C}_1$, $\boldsymbol{C}_2$ and $\boldsymbol{C}_3$. Their precise forms (i.e. the values of the entries of $\boldsymbol{C}_1$, $\boldsymbol{C}_2$ and $\boldsymbol{C}_3$) are omitted due to space limitations and can be found in the extended arXiv version of the paper (Ye et al., 2017).

Figure 1 shows the evolution of the individual social power (self-weight $x_i(s)$) over a sequence of issues for the periodically switching relative interaction matrices $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$. The initial condition $\boldsymbol{x}(s=1)$ was generated randomly. For the same two relative interaction matrices, Fig. 2 shows the evolution for a second randomly generated initial condition $\boldsymbol{x}(s=1)$ different from the first figure. Figure 3 shows the evolution of $\boldsymbol{x}(s)$ for a social network that periodically switches between $\boldsymbol{C}_1$, $\boldsymbol{C}_2$ and $\boldsymbol{C}_3$.

Figures 1 and 2 illustrate that Theorem 3 holds. In other words, $\boldsymbol{x}(s)$ has a periodic solution. Notice from Figures 1 and 2 that even for different initial conditions, $\boldsymbol{x}(s)$ asymptotically reaches the same periodic solution. This supports our conjecture that $\bar{\boldsymbol{F}}$ has a unique fixed point and that the fixed point is attractive for all $\boldsymbol{x}(s) \in \text{int}(\Delta_n)$. Our goal is to verify this in future work. Figure 3 illustrates the results developed in subsection 3.3 on generalising to multiple periodically switching relative interaction matrices.

5. CONCLUSION

In this paper, the DeGroot-Friedkin model has been used to analyse a social network with dynamically changing net-

[2] Note that any given $s \in \mathcal{S}$ can be uniquely expressed by a given fixed positive integer M, a positive integer q, and positive $p \in \{1, 2, \dots, M\}$, as shown.

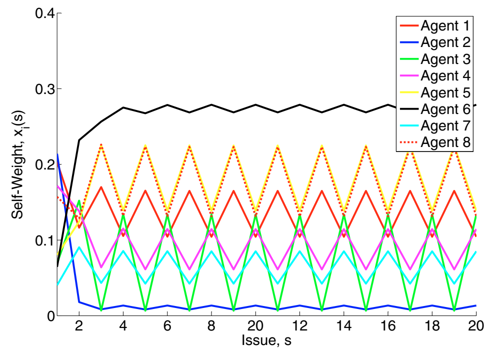

Fig. 1. Evolution of an individual's self-weight for $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$.

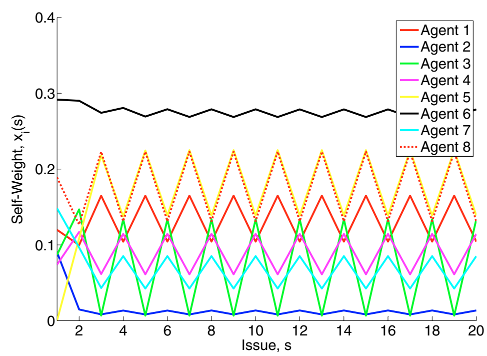

Fig. 2. Evolution of an individual's self-weight for $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$, different initial conditions.

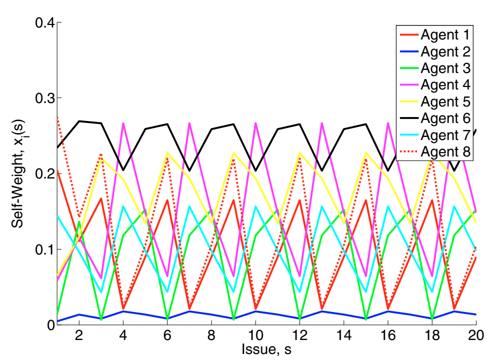

Fig. 3. Evolution of an individual's self-weight for $\boldsymbol{C}_1$, $\boldsymbol{C}_2$ and $\boldsymbol{C}_3$.

work topology described by relative interaction matrices which are allowed to change between any two consecutive issues discussed by the social network. In particular, we have developed results on the evolution of an individual's social power (or self-weight). We have shown that when the relative interaction matrices change periodically, then an individual's self-weight admits at least one periodic solution, where the individual's self-weight is always strictly positive and less than 1. Future work will focus on obtaining a convergence result for the periodic switching scenario. Beyond this, we aim to generalise the DeGroot-Friedkin model to allow for arbitrary switching between irreducible relative interaction matrices.

REFERENCES

Altafini, C. (2013). Consensus problems on networks with antagonistic interactions. *IEEE Transactions on Automatic Control*, 58(4), 935–946.

Altafini, C. and Lini, G. (2015). Predictable dynamics of opinion forming for networks with antagonistic interactions. *IEEE Transactions on Automatic Control*, 60(2), 342–357.

Cao, Y., Yu, W., Ren, W., and Chen, G. (2013). An overview of recent progress in the study of distributed multi-agent coordination. *IEEE Transactions on Industrial Informatics*, 9(1), 427–438.

Chen, X., Liu, J., Belabbas, M.A., Xu, Z., and Başar, T. (2017). Distributed evaluation and convergence of self-appraisals in social networks. *IEEE Transactions on Automatic Control*, 62(1), 291–304.

Chen, X., Liu, J., Xu, Z., and Başar, T. (2015). Distributed evaluation and convergence of self-appraisals in social networks. In *54th IEEE Conference on Decision and Control (CDC)*, 2895–2900.

DeGroot, M.H. (1974). Reaching a consensus. *Journal of the American Statistical Association*, 69(345), 118–121.

Etesami, S.R. and Başar, T. (2015). Game-theoretic analysis of the Hegselmann-Krause model for opinion dynamics in finite dimensions. *IEEE Transactions on Automatic Control*, 60(7), 1886–1897.

Friedkin, N.E. and Johnsen, E.C. (1990). Social influence and opinions. *Journal of Mathematical Sociology*, 15(3-4), 193–206.

Godsil, C.D., Royle, G., and Godsil, C. (2001). *Algebraic Graph Theory*, volume 207. Springer New York.

Hegselmann, R. and Krause, U. (2002). Opinion dynamics and bounded confidence models, analysis, and simulation. *Journal of Artificial Societies and Social Simulation*, 5(3).

Jadbabaie, A., Lin, J., and Morse, A.S. (2003). Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6), 988–1001.

Jia, P., MirTabatabaei, A., Friedkin, N.E., and Bullo, F. (2015). Opinion dynamics and the evolution of social power in influence networks. *SIAM Review*, 57(3), 367–397.

Knorn, S., Chen, Z., and Middleton, R. (2016). Overview: Collective control of multi-agent systems. *IEEE Transactions on Control of Network Systems*, 3(4), 334–347.

Shi, G. and Johansson, K.H. (2013). The role of persistent graphs in the agreement seeking of social networks. *IEEE Journal on Selected Areas in Communications*, 31(9), 595–606.

Xia, W., Liu, J., Johansson, K.H., and Başar, T. (2016). Convergence rate of the modified DeGroot-Friedkin model with doubly stochastic relative interaction matrices. In *American Control Conference (ACC)*, 1054–1059.

Xu, Z., Liu, J., and Başar, T. (2015). On a modified DeGroot-Friedkin model of opinion dynamics. In *American Control Conference (ACC)*, 1047–1052.

Ye, M., Liu, J., Anderson, B.D.O., Yu, C., and Başar, T. (2017). On the analysis of the DeGroot-Friedkin model with dynamic relative interaction matrices. URL `https://arxiv.org/abs/1703.04901`. ArXiv:1703.04901 [cs.SI].