Formation shape control with distance and area constraints

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Abstract

This paper discusses a formation control problem in which a target formation is defined with both distance and *signed* area constraints. The control objective is to drive spatially distributed agents to reach a unique target rigid formation shape (up to rotation and translation) with desired inter-agent distances. We define a new potential function by incorporating both distance terms and signed area terms and derive the formation system as a gradient system from the potential function. We start with a triangle formation system with detailed analysis on the equilibrium and convergence property with respect to a weighting gain parameter. For an equilateral triangle example, analytic solutions describing agents' trajectories are also given. We then examine the four-agent double-triangle formation and provide conditions to guarantee that both triangles converge to the desired side distances and signed areas.

1. Introduction

This paper deals with an aspect of formation shape control that has been largely untreated in the literature to this point. To explain this aspect, and to state our contribution, we first recall the two broad approaches to shape control, viz., displacement-based and distance-based approaches [1].

Assuming point agents, and for convenience an ambient $\mathbb{R}^2$, displacement-based shape control is concerned with securing a formation where the relative positions of a number of pairs of agents assume certain prescribed values. More precisely, if there are n agents, located at positions $p_1, p_2, \ldots, p_n$, and there is a set of prescribed relative positions involving certain pairs of the agents, say q^*_{ij} for certain pairs ij, we aim to move the agents so that $q_{ij} = p_i - p_j$ approaches q^*_{ij} as $t \to \infty$. (Of course, it is assumed that the problem is well-posed, i.e. there exists a formation with the prescribed relative positions). Define a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ associated with the formation, with the vertex set $\mathcal{V}$ defined by the agents, and the edge set $\mathcal{E}$ defined by the agent pairs appearing in the list of relative positions. If the undirected graph obtained by neglecting the orientations of the edges is connected, then there is a distributed linear control law leading to a steady state formation which is unique up to displacement. The angular orientation of the formation is determined by the data, as is the signed area of a triangle formed by any three agents in the formation. The displacement-based approach has been discussed in e.g. [2] and [3]. A more recent paper [4] discussed a distributed displacement-based control law which allows each agent to measure relative positions with its own independent local coordinate frame, which is contrary to most existing works on displacement-based control where a global coordinate system should be known to all agents.

In contrast, in distance-based shape control, a set of distances between certain agent pairs is prescribed. Again, there must be a formation for which the distances are achievable. An undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is defined in an obvious manner, and if a certain condition known as rigidity is satisfied by the graph [5, 6], then there are a finite number of noncongruent formation orbits (and sometimes a unique orbit) which will achieve the distances, where a formation orbit is a set of formations differing simply by one or more of translation, angular rotation, or reflection. In implementing a distance-based shape control, all the agents do not need to align their local coordinate bases with a global coordinate frame.

Some attempts have been made to solve problems that are in some sense intermediate between these two classes. For example, as discussed in [7], one can consider the distance-based control approach and superimpose a requirement that a particular angular orientation be achieved, and indeed one or possibly two relative positions alone can be controlled, along with the distances associated with enough edges to guarantee rigidity, to achieve the objective. We note that the formation control strategy involving orientation constraints in [7] still has not fully solved the formation reflection issue, unless the initial formation shape is assumed to be sufficiently close to the target one. In [8], this issue has been tackled by considering the equilibrium set including the freedom of translation and rotation but excluding reflection, namely the special Euclidean group. However, the global convergence has not been discussed.

In this paper, we do not seek to control the angular orientation, but we do seek to control the reflection variable. More precisely, in the case of a triangular formation, we seek to control the sign of the area of the triangle, as well as the lengths of its sides. Of course, the general aim is of broader applicability than just a single triangle, and we illustrate its applicability with a four agent formation also in the paper. For a four agent formation with five edge lengths specified, we can specify two triangle signed errors to pin down a desired formation that is unique up to translation and angular rotation, and we aim to provide a control law to converge to it. A more relevant paper is [9], which considered a similar distance-based formation control problem by including both distance and planar (or volume) restrictions in a performance index function. The planar (or volume) constraint is defined by the relative angular information in certain selected edges, which helped to exclude symmetric counterparts of a target formation if a gradient formation control system derived from the performance function is applied. However, the paper [9] established only a local convergence result based on convergence results of gradient systems, and a detailed analysis on the equilibrium set and its properties is still lacking.

Aside from the scientific interest in doing this, one might well ask when one might have a preference for a formation with one signed area over the other. In general, such a preference might arise if the agents are in some way heterogeneous. Suppose for example that three agents are fixed-wing unmanned aerial vehicles (UAVs), undertaking surveillance. It may be that each agent is equipped with a camera with limited angular field of view whose look-direction is a priori fixed relative to the direction of forward motion of the UAV; for example, one of the three UAVs might look left, another straight ahead and the third look right. It would be possible then to envisage that the three agents are supposed to achieve velocity consensus, with the front-most agent looking forwards, the left-most of the three agents looking right and the right-most of the three agents looking left, with the three cones associated with the look directions having a common area of intersection. Then the sign of the triangle becomes critical in securing this last property. If the second and third agents were to be interchanged in their positions but otherwise the interagent distances were the same (thus were there to be a change of the sign of the triangle area), then the three agents would no longer be able to simultaneously see a target. Heterogeneity of the area under surveillance as opposed to heterogeneity of the UAVs could also give rise to a preference. Suppose a coastline is being searched by a triangular formation of three UAVs to locate a missing person, who may be on land or sea. A UAV over land may well need to fly lower than an identical UAV over water, given the likely need for different resolution in imaging. A UAV directly over the land-water boundary may fly at an intermediate height. Interchange of the left and right UAV of a triangle could then place the higher UAV over land, and the lower UAV over water, which would be unhelpful.

The control laws we shall propose are gradient descent laws, and as such are modifications of those used for ordinary distance-based control achieved through the addition of an extra term or terms in the associated performance index. The modifications for both a three agent problem and a four agent problem are such that the performance index after modification is a sum of terms each of which depends on a subset of agents corresponding to a clique (i.e., complete subgraph) of the formation graph. Interestingly, the inclusion of the fourth summand is consistent with work of [10], which we now explain. This reference considers a number of different formation control tasks, among them consensus and distance-based shape control, in $\mathbb{R}^2$, with an arbitrary graph, and with n agents located at time t at $p_1(t), p_2(t), \dots, p_n(t)$. The formation task is specified using a performance index $V(p_1, p_2, \dots, p_n)$ whose global minimum corresponds to achievement of the formation objective. The question is asked: if a gradient descent law is to be used, so that the i-th agent moves according to $\dot{p}_i = -\frac{\partial V}{\partial p_i}$, what performance indices will give rise to a *distributed* control law, i.e. one for which the motion of agent i can be obtained independently of the position of all agents save those which are its neighbors. The paper demonstrates that a distributed gradient-based controller results if and only if the performance index is a sum of functions of those subsets of agents which are cliques. In relation to a triangle, the cliques are formed by the agent subsets $\{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}$. The paper thus legitimizes the idea of choosing a performance index V of the form $V(p_1, p_2, p_3) = V_{12}(p_1, p_2) + V_{23}(p_2, p_3) + V_{31}(p_1, p_3) + V_{123}(p_1, p_2, p_3)$. In relation to a two-triangle formation with four agents, with edge 23 common to the triangles, a distributed control law should result from a performance index of the form $V(p_1, p_2, p_3, p_4) = V_{12}(p_1, p_2) + V_{23}(p_2, p_3) + V_{31}(p_1, p_3) + V_{24}(p_2, p_4) + V_{34}(p_3, p_4) + V_{123}(p_1, p_2, p_3) + V_{234}(p_2, p_3, p_4)$.

The second general comment we make is that the final formation will be specified only up to angular rotation and translation if and only if the summands in the performance index expression are themselves translation and rotation independent, e.g. because they depend just on Euclidean distances within the clique. Such an observation draws on the work of Vasile et.al. [11]; however this reference does not so straightforwardly allow the treatment of the particular terms we will use for $V_{123}(p_1, p_2, p_3)$ and $V_{234}(p_2, p_3, p_4)$.

The remaining parts of this paper are organized as follows. We start with a triangular formation in Section 2, derive the control law from a modified index function involving both distance errors and signed area, and present a detailed analysis on the equilibrium properties and convergence of the formation control system. The following section examines in some detail an equilateral triangle, This is because it is possible to analytically obtain the trajectories followed by the agents from certain initial conditions. For an extension, in Section 4 we treat a four agent formation comprising two triangles with five interagent distances specified, and show that it is possible to choose the performance index so that from all but a thin set of initial conditions, one will asymptotically obtain a formation where both triangles have the desired orientation and the distances are correct. The final Section 5 offers concluding remarks and directions for future work.

2. Securing a triangle shape and signed area

In this section, we set out in a detailed way the problem of formation shape control with signed area for a triangular (three agent) formation.

In the usual approach to shape control (with no account taken of signed area, see e.g. [12, 13]), three desired lengths for the triangle sides and satisfying the standard triangle inequality are specified. Then a performance index computable from the agent positions (which correspond to the triangle corners) is established which is zero if and only if the actual distances correspond to the desired values, and is otherwise nonzero. With the aim of adjusting the agent positions to secure the desired inter-agent distances, the agents are moved along steepest descent directions computed using the performance index. In general each agent's motion is a sum of two components; each component is a vector directed towards or away from a neighbor agent, according as the moving agent is too far from or too close to the neighbor agent.

There are some well known properties of this law, and indeed many like it [12, 13, 14]. Here are some:

1. If the initial positions of the agents are distinct and not collinear, convergence occurs to a triangle with the correct distances (the centroid and orientation being irrelevant)
2. If the initial positions of the agents are distinct and *not collinear*, the same is true of all subsequent positions. Hence the area of the contained triangle never goes to zero.
3. If the initial positions of the agents are distinct and *collinear*, the same is true of all subsequent positions; convergence occurs but the lengths are not those desired–indeed, the desired lengths satisfy the triangle inequality, while the three lengths for collinear agents cannot.
4. The centroid of the triangle remains fixed during the motions.

Given a formation with the desired lengths and fixed centroid, there is a manifold of such formations formed by rotation. But in addition, there is a mirror image manifold: a triangle with corners defined by 2-vectors p_1, p_2, p_3 is congruent to one defined by $-p_1, -p_2, -p_3$ and one triangle cannot be smoothly transformed to the other without encountering a collinearity of the three 'corners' at some intermediate point of the deformation process. There are evidently two branches in the set of desired formations. It is possible to think about these two branches in a quite systematic way, by distinguishing them on the basis of the sign of the enclosed area.

Indeed, as is well known, it is possible to define a signed area for a triangle, denoted by Z, as

$$Z = \frac{1}{2}\det\begin{bmatrix} 1 & 1 & 1 \\ p_1 & p_2 & p_3 \end{bmatrix} \tag{1}$$

where 'det' denotes *determinant*. The quantity Z is positive or negative according as the ordering of p_1, p_2, p_3 around the boundary of the triangle is counterclockwise, or clockwise. This observation suggests that if we wish to achieve a particular formation shape with a prescribed cyclic ordering of the triangle vertices, i.e. a prescribed sign for the area of the triangle, we should incorporate a function reflecting that area into the performance index.

Let us now study how we can control both distances and sign of the triangle area. First, recall that with no account taken of the sign of the area, and three prescribed distances $d^*_{12}, d^*_{23}, d^*_{13}$, a commonly used index, see e.g. [12], has been

$$V(p_1, p_2, p_3) = \frac{1}{4}((\|p_1 - p_2\|^2 - d^{*2}_{12})^2 + (\|p_2 - p_3\|^2 - d^{*2}_{23})^2 + (\|p_3 - p_1\|^2 - d^{*2}_{13})^2) \tag{2}$$

Now let Z^* denote the area (including sign) of the desired triangle. The magnitude of Z^* is of course determined from the d^*_{ij} but the sign is not. Then an adjustment to the index reflecting the area is evidently available as

$$V(p_1, p_2, p_3) = \frac{1}{4}((\|p_1 - p_2\|^2 - d^{*2}_{12})^2 + (\|p_2 - p_3\|^2 - d^{*2}_{23})^2 + (\|p_3 - p_1\|^2 - d^{*2}_{13})^2) + \frac{1}{2}(Z - Z^*)^2 \tag{3}$$

Note that all four summands dimensionally involve distance raised to the fourth power. Separately, we note that the index has the general properties of invariance to displacement and angular rotation. However, it is not invariant to reflection.

2.1. Triangle motion under the gradient descent law

In this subsection, we obtain equations of motion related to use of the index (3), and we derive several properties of the motion.

For notation simplicity we denote the distance error term e_{ij} for edge (i, j) associated with agents i and j as $e_{ij} = \|p_i - p_j\|^2 - d^{*2}_{ij}$. The gradient descent law obtained from a performance index V is of the form $\dot{p}_i = -\frac{\partial V}{\partial p_i}$ which in the case of the common index of (2) becomes

$$\begin{aligned}\dot{p}_1 &= -e_{12}(p_1 - p_2) - e_{13}(p_1 - p_3)\\ \dot{p}_2 &= -e_{12}(p_2 - p_1) - e_{23}(p_2 - p_3)\\ \dot{p}_3 &= -e_{13}(p_3 - p_1) - e_{23}(p_3 - p_2)\end{aligned} \tag{4}$$

It is not much harder to verify that the law obtained from (3) is, with

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{5}$$

given by

$$\begin{aligned}\dot{p}_1 &= -e_{12}(p_1 - p_2) - e_{13}(p_1 - p_3) - (Z - Z^*)J(p_2 - p_3)\\ \dot{p}_2 &= -e_{12}(p_2 - p_1) - e_{23}(p_2 - p_3) - (Z - Z^*)J(p_3 - p_1)\\ \dot{p}_3 &= -e_{13}(p_3 - p_1) - e_{23}(p_3 - p_2) - (Z - Z^*)J(p_1 - p_2)\end{aligned} \tag{6}$$

We remark that, as is well known, to implement the law (4) each agent needs to be able to measure the relative position of its neighbors, but all agents can have their own coordinate bases, i.e. there is no requirement for a global coordinate basis to be available for each agent (see e.g., [12, 13]). Now we show the same is true for (6). Considering the last term in the first equation of (6) for example, observe that $p_2 - p_3 = (p_2 - p_1) - (p_3 - p_1)$ and from (1) there holds that

$$\begin{aligned} Z &= \frac{1}{2}\det\begin{bmatrix} 1 & 0 & 0 \\ p_1 & p_2 - p_1 & p_3 - p_1 \end{bmatrix}\\ &= \frac{1}{2}\det\begin{bmatrix} p_2 - p_1 & p_3 - p_1 \end{bmatrix}\end{aligned} \tag{7}$$

Evidently, the additional term $(Z - Z^*)J(p_2 - p_3)$ in the law again involves relative positions. We now show the function $f(p) := (Z - Z^*)J(p_2 - p_3)$ is an *SE(2)*-invariant function (see e.g., [11] for its definition), where *SE(2)* denotes the special Euclidean group in $\mathbb{R}^2$. For any rotation matrix $R \in SO(2)$ and a real vector $w \in \mathbb{R}^2$, there holds

$$Z(Rp_1 + w, Rp_2 + w, Rp_3 + w) = \frac{1}{2}\det[R(p_2 - p_1) \quad R(p_3 - p_1)] = \frac{1}{2}\det(R)\det[p_2 - p_1 \quad p_3 - p_1] = Z(p_1, p_2, p_3),$$

and

$$J((Rp_2 + w) - (Rp_3 + w)) = JR(p_2 - p_3) = RJ(p_2 - p_3),$$

where we have used the fact that $\det(R) = 1$ and $JR = RJ$ because $R, J \in SO(2)$ and $SO(2)$ is a commutative group under matrix multiplication. Thus, the additional term is *SE(2)*-invariant. The same analysis also applies to the second and the third equation of (6). Therefore, all agents with control (6) do not require access to a global coordinate basis. [1]

Our interest is in studying how the solutions of these equations evolve. As a preliminary observation, observe that, *with or without* the inclusion of the area term, there holds

$$\sum_{i=1}^{3} \dot{p}_i = 0 \tag{8}$$

which implies that the centroid of the triangle remains fixed. Two other observations are:

1. $p_1 = p_2 = p_3$ is an equilibrium point of both sets of equations. It is trivial to see that any incremental motion away from this equilibrium point reduces V, implying that the equilibrium point defines a maximum of V and accordingly is an unstable equilibrium point for a gradient descent algorithm.
2. If three agents are collinear and at most two are collocated, the trajectory of the system with no area term in the performance index retains collinearity of the agents, but this is not the case for the system involving an area term, as we now check in some detail.

To this end, we will now prove the following lemma.

[1] Note that this property implies the *SE(N)* invariance (i.e., translational and rotational invariance) [11] of the proposed formation controller, which enables a convenient implementation of the formation control law without coordinate frame alignment for all of the agents.

Lemma 1. *Consider the motion of three agents according to Equations (6), and suppose that at some point in time the three agents are collinear, with agent 2 between agents 1 and 3, or with agent 2 and agent 3 collocated. Suppose that Z^* is positive (negative). Then the components of motion of the agents at right angles to the line will be such that after an incremental motion the resulting triangle will have p_1, p_2 and p_3 occurring counterclockwise (clockwise), i.e. Z is positive (negative).*

Proof. Without loss of generality, suppose that the agents lie on the x-axis, with agent 1 the left most agent, and suppose that $Z^* > 0$. Suppose initially that no two of the three agents are collocated. Thus $x_1 < x_2 < x_3$. Then we can show that the y-components of $\dot{p}_1$ and $\dot{p}_3$ are positive and the y-component of $\dot{p}_2$ is negative. This is easily checked. In obvious notation, we have immediately from (6)

$$\begin{aligned} \dot{y}_1 &= -(0 - Z^*)(x_3 - x_2) > 0 \\ \dot{y}_2 &= -(0 - Z^*)(x_1 - x_3) < 0 \\ \dot{y}_3 &= -(0 - Z^*)(x_2 - x_1) > 0 \end{aligned} \tag{9}$$

Next suppose that agents 2 and 3 are collocated, so that $x_2 = x_3$. Then it is easily seen that $\dot{y}_1 = 0, \dot{y}_2 < 0, \dot{y}_3 > 0$ and collocation is again broken. In both cases, when the collocation is broken there arises $Z > 0$, as claimed. □

Of course, equivalent results to those of the lemma occur with other orderings of the agents.

There is an immediate corollary to Lemma 1. Observe that if on a trajectory, the three agents define a triangle of area Z with the same sign as Z^*, they will never become collinear. This is evident from Lemma 1, which shows that movement from a collinear position is always in the direction which causes Z and Z^* to have the same sign. We summarize:

Corollary 1. *Consider the motion of three agents according to Equations (6), and suppose at some time the area of the triangle formed by the three agents, Z, has the same sign as the desired final area, Z^*. Then at subsequent times, the agents will never become collinear.*

2.2. Defining the equilibrium points

We now aim to study the remaining equilibrium points arising from the algorithm, especially the stable ones which result from local or global minima of V. It is well understood that the set of equilibria breaks into orbits, with an orbit being obtained through translation and rotation of an equilibrium point. Each orbit is evidently defined by the three inter-agent distances and sign of the enclosed area. We shall show that (up to rotation and translation) there is only one stable equilibrium point (i.e., more precisely there is one orbit of equilibrium points) for which the associated Z and the prescribed Z^* have the same sign, namely the equilibrium point at which all distances (and therefore also the area) equal the desired values. To analyse other cases, we shall investigate a generalization of the previous index. The generalization is obtained by varying the weighting of the area error relative to the distance errors, and with K a positive constant is given by

$$V(p_1, p_2, p_3) = \frac{1}{4}((\|p_1 - p_2\|^2 - d_{12}^{*2})^2 + (\|p_2 - p_3\|^2 - d_{23}^{*2})^2 + (\|p_3 - p_1\|^2 - d_{13}^{*2})^2) + \frac{1}{2}K(Z - Z^*)^2 \tag{10}$$

The gradient descent law from the modified potential (10) is

$$\begin{aligned} \dot{p}_1 &= -e_{12}(p_1 - p_2) - e_{13}(p_1 - p_3) - K(Z - Z^*)J(p_2 - p_3) \\ \dot{p}_2 &= -e_{12}(p_2 - p_1) - e_{23}(p_2 - p_3) - K(Z - Z^*)J(p_3 - p_1) \\ \dot{p}_3 &= -e_{13}(p_3 - p_1) - e_{23}(p_3 - p_2) - K(Z - Z^*)J(p_1 - p_2) \end{aligned} \tag{11}$$

Then at an equilibrium, there holds

$$\begin{aligned} -e_{12}(p_1 - p_2) - e_{31}(p_1 - p_3) - K(Z - Z^*)J(p_2 - p_3) &= 0 \\ -e_{23}(p_2 - p_3) - e_{12}(p_2 - p_1) - K(Z - Z^*)J(p_3 - p_1) &= 0 \\ -e_{31}(p_3 - p_1) - e_{23}(p_3 - p_2) - K(Z - Z^*)J(p_1 - p_2) &= 0 \end{aligned} \tag{12}$$

We will also use the observation of the following lemma in our calculations.

Lemma 2. *For a triangle with corners defined by vectors p_1, p_2, p_3, the triangle area Z is given by*

$$Z = -\frac{1}{2}(p_2 - p_3)^\top J(p_1 - p_2) = -\frac{1}{2}(p_2 - p_3)^\top J(p_1 - p_3) \tag{13}$$

Proof. Set $p_i = [x_i\ y_i]^\top$. Then observe that, expanding by the determinant by the first row, there holds

$$2Z = \det\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = [x_2y_3 - y_2x_3 - x_1y_3 + y_1x_3 + x_1y_2 - y_1x_2] \tag{14}$$

It is easily checked that $-(p_2 - p_3)^\top J(p_1 - p_2)$ and $-(p_2 - p_3)^\top J(p_1 - p_3)$ evaluate to the same result. □

The following result deals with the case when Z and Z^* have the same sign:

Theorem 1. *With notation as given above, consider the above set of equations (12) in which $Z^* > 0, Z > 0, K > 0$. Then there necessarily holds $e_{12} = e_{23} = e_{31} = 0$ and $Z = Z^*$.*

Proof. Our proof will be by contradiction. Suppose then there does exist an equilibrium with $Z > 0$, but without one at least of the distances or Z assuming correct values. We will use the consequence of the equilibrium equations (12) and divide the possibilities into three cases: $Z = Z^*, Z < Z^*$ and $Z > Z^*$. Taking these equations multiply successively by $-(p_2 - p_3)^\top J, -(p_3 - p_1)^\top J$ and $-(p_1 - p_2)^\top J$, and use the above Lemma 2. There results

$$\begin{aligned} -2(e_{12} + e_{31})Z - K(Z - Z^*)\|p_2 - p_3\|^2 &= 0 \\ -2(e_{23} + e_{12})Z - K(Z - Z^*)\|p_3 - p_1\|^2 &= 0 \\ -2(e_{31} + e_{23})Z - K(Z - Z^*)\|p_1 - p_2\|^2 &= 0 \end{aligned} \tag{15}$$

Case 1: $Z = Z^*$. In this case, the equilibrium equations become identical to those applying when shape control is achieved using solely distances as the basis for the cost function. It is well known (see e.g. [15, 14]) that the only (noncollinear) equilibria are those where all distances are correct. Alternatively, we can see from (15) that $e_{12} + e_{23} = 0,\ e_{23} + e_{31} = 0,\ e_{31} + e_{12} = 0$, whence all three e_{ij} are zero.

Case 2: $Z < Z^*$. In this instance, there may be three, two or one of the associated equilibrium distances $d_{ij} = \|p_i - p_j\|$ which are less than the corresponding d^*_{ij}. Suppose firstly there are two or three such distances; without loss of generality, suppose $e_{12} < 0,\ e_{31} < 0$. Then the various sign constraints imply a contradiction of the first equation of (15). Next suppose there is just one distance d_{ij} with $d_{ij} < d^*_{ij}$, without loss of generality d_{12}, thus $e_{12} < 0,\ e_{23} \geq 0,\ e_{31} \geq 0$. Next, observe that from (12), when multiplying the equations successively by $(p_2 - p_3)^\top,\ (p_3 - p_1)^\top, (\ p_1 - p_2)^\top$, there results

$$\begin{aligned} -e_{12}(p_2 - p_3)^\top(p_1 - p_2) + e_{31}(p_2 - p_3)^\top(p_3 - p_1) &= 0 \\ -e_{23}(p_3 - p_1)^\top(p_2 - p_3) + e_{12}(p_3 - p_1)^\top(p_1 - p_2) &= 0 \\ -e_{31}(p_1 - p_2)^\top(p_3 - p_1) + e_{23}(p_1 - p_2)^\top(p_2 - p_3) &= 0 \end{aligned} \tag{16}$$

Now the sign of each of the inner products in these equations is determined by the cosine of the angle between the relevant vectors. The vectors themselves are the relative position vectors associated with the sides of the triangular formation; and taking account of directions, and noting that a triangle can have at most one obtuse angle, we see that at most one of the signs can be positive. Hence one at least of the first two equations has inner products which are both negative. Suppose without loss of generality, it is the first equation. Then the sign condition on the inner products coupled with the assumption that $e_{12} < 0,\ e_{31} > 0$ yields a contradiction. Hence there cannot be an equilibrium satisfying the condition of Case 2.

A very similar argument also shows there cannot be an equilibrium satisfying the condition of Case 3, i.e. $Z > Z^*$, and the theorem is then proved. □

Next, we will investigate equilibria when the associated Z is of opposite sign to Z^*. Our next result shows that if K is chosen sufficiently large, there is no stable equilibrium with Z, Z^* of opposite signs.

Theorem 2. *With notation as given above, consider the above set of equations (12) in which $Z^* > 0, Z < 0, K > 0$. Then for K sufficiently large, there exists no solution of the equations. A sufficient condition on the magnitude of K for there to be no solution is*

$$K > \frac{d^{*2}_{12} + d^{*2}_{23} + d^{*2}_{13}}{\sqrt{3}Z^*} \tag{17}$$

Proof. Our aim is to show that the equations (15) cannot be satisfied if K is sufficiently large. From these equations, by addition, there follows

$$\begin{aligned} &4(e_{12} + e_{23} + e_{31})Z \\ &\quad = -K(Z - Z^*)(\|p_1 - p_2\|^2 + \|p_2 - p_3\|^2 + \|p_3 - p_1\|^2) \end{aligned}$$

Observe that in this equation, the theorem hypothesis implies $Z - Z^*$ is negative, K is positive, and Z is negative, and therefore also $(e_{12} + e_{23} + e_{31})$ is negative. Now we appeal to Weitzenbock's inequality [16], which states that for any triangle

$$\|p_1 - p_2\|^2 + \|p_2 - p_3\|^2 + \|p_3 - p_1\|^2 \geq 4\sqrt{3}|Z| \tag{18}$$

with equality holding for an equilateral triangle. Together with the preceding equation, this implies that

$$|e_{12} + e_{23} + e_{31}| \geq \sqrt{3}K(|Z| + Z^*) \tag{19}$$

But recall that $(e_{12} + e_{23} + e_{31})$ is negative. The fact $e_{ij} = d_{ij}^2 - d_{ij}^{*2}$ means that the sum is bounded below by the negative sum of the desired interagent squared distances, thus

$$0 \geq e_{12} + e_{23} + e_{31} \geq -d_{12}^{*2} - d_{23}^{*2} - d_{31}^{*2}$$

or

$$|e_{12} + e_{23} + e_{31}| \leq d_{12}^{*2} + d_{23}^{*2} + d_{31}^{*2} \tag{20}$$

Hence at the equilibrium with incorrect sign of triangle area, there holds

$$d_{12}^{*2} + d_{23}^{*2} + d_{13}^{*2} \geq \sqrt{3}K(|Z| + Z^*) > \sqrt{3}KZ^* \tag{21}$$

Obviously for sufficiently large K, in fact for K satisfying (17), this inequality cannot be satisfied. □

We make several observations. First, note that the inequality (17) is only a sufficient condition for there to be no equilibrium point with Z and Z^* of opposite sign. Thus it may be that a smaller value of K would suffice. In the case of an *equilateral* triangle, one can actually show that the bound from this inequality turns out to be exact. Second, observe using Weitzenbock's inequality that the right hand side of (17) is lower bounded by 4. Third, observe that there exist triangles with fixed perimeter and arbitrarily small area, implying the lower bound in K applying to such triangles can be arbitrarily large.

As a partial converse to the above result, we can argue that for K positive but sufficiently small, there is necessarily an equilibrium point for which Z and Z^* have opposite signs.

Theorem 3. *With notation as given above, consider the above set of equations (15) in which $Z^* > 0, Z < 0, K > 0$. Then for K sufficiently small, there exists a solution to the equations and this corresponds to a minimum of the performance index.*

Proof. Suppose $K = 0$; then there is an equilibrium at which correct distances are maintained but the signed triangle area Z is the negative of Z^*. Now suppose that K is allowed to be nonzero. Let $A = A(d_{12}, d_{23}, d_{31})$ be the function expressing the unsigned area of a triangle in terms of the lengths of its sides. Thus A is always a positive (and smooth) function of its arguments, provided they satisfy the triangle inequality. Now observe that when Z^* is positive and Z is negative, there holds $Z^* = A(d_{12}^*, d_{23}^*, d_{31}^*)$ and $Z = -A(d_{12}, d_{23}, d_{31})$ and the equilibrium equation (15) can be rewritten as:

$$\begin{aligned}
2(d_{12}^2 - d_{12}^{*2} + d_{31}^2 - d_{31}^{*2})A(d_{12}, d_{23}, d_{31}) - K(-A(d_{12}, d_{23}, d_{31}) - A(d_{12}^*, d_{23}^*, d_{31}^*))d_{23}^2 &= 0 \\
2(d_{23}^2 - d_{23}^{*2} + d_{12}^2 - d_{12}^{*2})A(d_{12}, d_{23}, d_{31}) - K(-A(d_{12}, d_{23}, d_{31}) - A(d_{12}^*, d_{23}^*, d_{31}^*))d_{31}^2 &= 0 \\
2(d_{31}^2 - d_{31}^{*2} + d_{23}^2 - d_{23}^{*2})A(d_{12}, d_{23}, d_{31}) - K(-A(d_{12}, d_{23}, d_{31}) - A(d_{12}^*, d_{23}^*, d_{31}^*))d_{12}^2 &= 0
\end{aligned} \tag{22}$$

Write this equation set in shorthand form as $F(d_{12}, d_{23}, d_{31}, K) = 0$. Then the equilibrium corresponding to $K = 0$ with negative signed area corresponds to having $F(d_{12}^*, d_{23}^*, d_{31}^*, 0) = 0$. Now by the implicit function theorem, if K is sufficiently close to zero, there will exist values of d_{12}, d_{23}, d_{31} satisfying $F(d_{12}, d_{23}, d_{31}, K) = 0$ if and only if the 3×3 Jacobian matrix

$$\frac{\partial F(d_{12}, d_{23}, d_{31}, K)}{\partial(d_{12}, d_{23}, d_{31})}$$

evaluated at $d_{12}^*, d_{23}^*, d_{31}^*, 0$ is nonsingular. Moreover, the satisfying values d_{12}, d_{23}, d_{31} will depend continuously on K. At the particular point, the Jacobian is actually easy to evaluate and is

$$\frac{\partial F(d_{12}, d_{23}, d_{31}, K)}{\partial(d_{12}, d_{23}, d_{31})}\Big|_{d_{12}^*, d_{23}^*, d_{31}^*, 0} = 4A(d_{12}^*, d_{23}^*, d_{31}^*)\begin{bmatrix} d_{12}^* & 0 & d_{31}^* \\ d_{12}^* & d_{23}^* & 0 \\ 0 & d_{23}^* & d_{31}^* \end{bmatrix} \tag{23}$$

which is evidently nonsingular.

To show that such an equilibrium corresponds to a minimum of the performance index, note that when $K = 0$ the associated equilibrium is a minimum, and the Hessian computed with the distances is positive definite. This Hessian clearly depends continuously on the values of its arguments and hence will be positive definite at equilibrium points obtained with sufficiently small K. This completes the proof. □

If the weighting K in the performance index is zero, we know that besides the two equilibrium points minimizing the performance index, corresponding to two triangles with correct distances and oppositely signed areas, there are saddle point equilibria (more strictly equilibrium orbits) "between" the two minimizing equilibria, at which the p_i are collinear. If K is small and nonzero, because there are two equilibria (strictly speaking equilibrium orbits) known to be minima, there will be boundaries of the regions of attraction for each equilibrium which include points other than at infinity, and the set or sets of such boundary points itself forms an invariant set on which there will be an equilibrium. It will necessarily be a saddle however. For nonzero K, the set is obviously not straightforward to characterize.

Also, if K is taken sufficiently large that there is no equilibrium with incorrectly signed area, and if the initial condition for a trajectory is such that the area for that initial condition is incorrectly signed, it is clear that at some point in the motion, the area must pass through zero, i.e. the three agents will be collinear. Of course, they do not remain collinear.

2.3. *Infinite weighting on the area error*

In this subsection, and for the sake of completeness, we consider the effect of letting the gain K go to infinity. Equivalently, we consider a performance index

$$V(p_1, p_2, p_3) = \frac{1}{2}(Z - Z^*)^2 \tag{24}$$

A steepest descent algorithm yields

$$\begin{aligned} \dot{p}_1 &= -(Z - Z^*)J(p_2 - p_3) \\ \dot{p}_2 &= -(Z - Z^*)J(p_3 - p_1) \\ \dot{p}_1 &= -(Z - Z^*)J(p_1 - p_2) \end{aligned} \tag{25}$$

and this leads to

$$\dot{V} = -(Z - Z^*)^2[\|p_1 - p_2\|^2 + \|p_2 - p_3\|^2 + \|p_3 - p_1\|^2] \tag{26}$$

It is easily seen that the only equilibrium points apart from those where $Z = Z^*$ correspond to $p_1 = p_2 = p_3$, i.e. all points are collocated. At such a point $Z = 0$ and $V = Z^{*2}$, while at all points in the neighborhood of such a point, V is obviously decreasing (or equivalently, $\dot{V} < 0$), since both $(Z - Z^*)^2$ and $[\|p_1 - p_2\|^2 + \|p_2 - p_3\|^2 + \|p_3 - p_1\|^2]$ are nonzero. Therefore, there can be no trajectory reaching such a point. Consequently, a trajectory starting from any position other than one where the three agents are collocated will converge to one where $Z = Z^*$, i.e. a correct area, while the final formation shape (i.e., edge lengths) is unspecified.

Further, since V is monotonic decreasing, once Z has acquired the same sign as Z^*, it will never change its sign.

One can view the effect of using a large as opposed to infinite K as ensuring first that the correct area is obtained, and after that as a secondary objective, correct edge lengths are obtained.

3. Example: an equilateral triangle formation

In this section, we shall consider a specific example of triangle control and establish

1. Existence of a correct stable equilbrium
2. Existence of an incorrect stable equilibrium
3. Existence of an unstable equilibrium

The example is deliberately constructed so as to allow calculation.

The desired equilibrium is defined by an equilateral triangle in which the vertices 1,2,3 are arranged in a counterclockwise fashion, and for which the side length is 2. Notice that the signed area of such a triangle, call it Z^*, is $\sqrt{3}$.

As an initial condition, we shall also assume an equilateral triangle, with a side length a_0 and with vertices 1,2,3 arranged in a clockwise fashion. Hence the signed area is initially $Z_0 = -\frac{\sqrt{3}}{4}a_0^2$. Take as the coordinates of the initial point

$$p_1(0) = \begin{bmatrix} 0 \\ \frac{\sqrt{3}}{2}a_0 \end{bmatrix} \quad p_2(0) = \begin{bmatrix} \frac{1}{2}a_0 \\ 0 \end{bmatrix} \quad p_3(0) = \begin{bmatrix} -\frac{1}{2}a_0 \\ 0 \end{bmatrix} \tag{27}$$

From simple symmetry considerations, we should expect the trajectory to evolve and retain the equilateral property. To explore this, and supposing for the moment that the weighting K is taken as 1, observe that the agent positions evolve according to equation (6). Note that if the triangle is equilateral at any time, with side length $a(t)$, then there holds (if the vertices 1,2,3 occur in a clockwise fashion)

$$Z(t) = -\frac{\sqrt{3}}{4}a^2(t) \tag{28}$$

Lemma 3. *While the agent positions defining the triangle remain finite, the triangle motion is such that it remains equilateral, with side length $a(t)$ obeying the equation*

$$\dot{a} = -\frac{5}{2}a^3 + 6a \tag{29}$$

Proof. Observe that the definitions

$$\bar{p}_1(t) = \begin{bmatrix} 0 \\ \frac{\sqrt{3}}{2}a(t) \end{bmatrix}; \quad \bar{p}_2(t) = \begin{bmatrix} \frac{1}{2}a(t) \\ 0 \end{bmatrix}; \quad \bar{p}_3(t) = \begin{bmatrix} -\frac{1}{2}a(t) \\ 0 \end{bmatrix} \tag{30}$$

define an equilateral triangle with side length $a(t)$ and one can verify that

$$-(\|\bar{p}_1 - \bar{p}_2\|^2 - 4)(\bar{p}_1 - \bar{p}_2) - (\|\bar{p}_1 - \bar{p}_3\|^2 - 4)(\bar{p}_1 - \bar{p}_3) - (-\frac{\sqrt{3}}{4}a^2(t) - \sqrt{3})J(p_2 - p_3) \tag{31}$$

$$= -(a^2(t) - 4)\begin{bmatrix} -\frac{a(t)}{2} \\ \frac{\sqrt{3}}{2}a(t) \end{bmatrix} - (a^2(t) - 4)\begin{bmatrix} \frac{a(t)}{2} \\ \frac{\sqrt{3}}{2}a(t) \end{bmatrix} - (-\frac{\sqrt{3}}{4}a^2(t) - \sqrt{3})\begin{bmatrix} 0 \\ -a(t) \end{bmatrix} \tag{32}$$

Using the differential equation (29), we now see that $\bar{p}_1(t))$ satisfies the same differential equation as $p_1(t)$ provided also $Z(t)$ satisfies the equation given above. Hence the motion of the agent 1 and (by a further easy calculation) the motions of agent 2 and 3 are given by the formulas for the $\bar{p}_i$, and then the lemma conclusion is immediate. ☐

Observe using the differential equation for $a(t)$ that with $a(0) > 0$, $a(t)$ remains bounded. Thus the hypothesis of the lemma statement requiring the explicit assumption that $a(t)$ remains finite can actually be dropped.

Now let us consider in more detail how the equilateral triangle evolves. For this purpose, we need to understand the equilibrium points of the differential equation for $a(t)$. These are the nonnegative solutions of the polynomial equation

$$-\frac{5}{2}a^3 + 6a = 0 \tag{33}$$

One is zero, and the other is $a = 2\sqrt{\frac{3}{5}}$. These steady state solutions of (29) are immediately seen (using linearization) to be unstable and asymptotically stable, respectively. However, we need to be more careful in the consideration of the solution at 0. In an arbitrary neighborhood of the point $p_1 = p_2 = p_3 = 0$, which corresponds to the case $a = 0$, there are points in which the three points p_1, p_2, p_3 occur clockwise, and the differential equation accordingly changes to reflect the fact that $Z(t)$ is now $\frac{\sqrt{3}}{4}a^2(t)$ rather than its negative. For an equilateral triangle with side length $a(t)$, the equation replacing is now

$$-\dot{a} = \frac{5}{2}a^3 - 10a \tag{34}$$

The steady state solutions are now $a = 0$, which is again unstable, and $a = 2$, which is asymptotically stable and corresponds to the desired formation.

To summarize, there is a stable equilibrium with an incorrectly oriented triangle and with an incorrect side length (an 'incorrect' equilibrium), there is an unstable equilibrium with all points coinciding, and there is a correct (and asymptotically stable) equilibrium. A small perturbation away from the unstable equilibrium will result in the system moving to one of the two stable equilibria.

In order to demonstrate the above points, we will show some simulation results. In Fig. 1, three agents 1, 2 and 3 start with a large equilateral triangle which is incorrectly oriented ($Z < -\sqrt{3}$). Then, the triangle formation shrinks due to the control law, but converges to an equilateral triangle with the side length $2\sqrt{\frac{3}{5}}$ while retaining the incorrectly signed area ($Z < 0$). This obviously is not the desired formation. On the other hand, in Fig. 2, the three agents start with the same size of equilateral triangular formation but correctly oriented ($Z > \sqrt{3}$). Then they converge to the desired formation (i.e., equilateral triangular with $Z = \sqrt{3}$) as shown in Fig. 2. These figures clearly show the two types of stable equilibria.

We can reasonably expect similar results to hold with initial conditions other than equilateral and indeed for other desired triangles. Also, as noted in the previous section, given the existence of two stable equilibria as here, it follows at once that there will necessarily be points which are both finite and on the boundary of the region of convergence of each point. We now consider further the equilateral triangle example with adjustable weighting given to the area error term in the performance index.

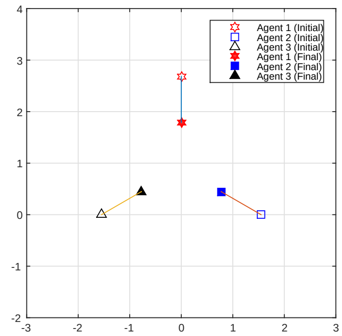


Figure 1: Convergence to an incorrect stable equilibrium

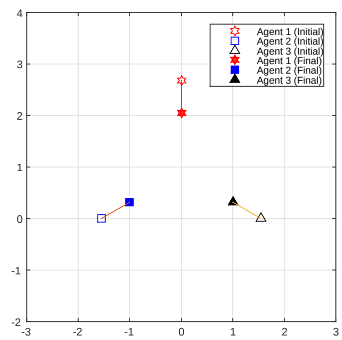


Figure 2: Convergence to a correct stable equilibrium

3.1. Equilateral triangle with adjusted weighting

The law we have just examined is one obtained by using a performance index of the form

$$V(p_1, p_2, p_3) = \frac{1}{4}\{[||p_1 - p_2||^2 - 2^2]^2 + [||p_2 - p_3||^2 - 2^2]^2 + [||p_3 - p_1||^2 - 2^2]^2 + \frac{1}{2}[Z - \sqrt{3}]^2 \tag{35}$$

where $Z(t)$ denotes the signed error of the triangle formed by $p_1(t)$, $p_2(t)$ and $p_3(t)$.

We now adjust the weighting given to the area mismatch, by introducing a positive constant K; the associated performance index is

$$V(p_1, p_2, p_3) = \frac{1}{4}\{[||p_1 - p_2||^2 - 2^2]^2 + [||p_2 - p_3||^2 - 2^2]^2 + [||p_3 - p_1||^2 - 2^2]^2 + \frac{1}{2}K[Z(t) - \sqrt{3}]^2 \tag{36}$$

It is not hard to check that if we start with an equilateral triangle, an equilateral triangle is still retained along the motion; the evolution of the side length however depends on K, as illustrated in Fig. 3. The final triangle (for $K = 2$) is smaller than that of Fig. 1 (for $K = 1$). The sign of the area is of course incorrect.

It is now straightforward to verify that in the clockwise agent case, for $K > 4$, there are no nonnegative real solutions other than $a = 0$ of the steady state equation (there are two imaginary solutions). This means that there is no longer an incorrect equilibrium point with agents arranged in a clockwise pattern. Furthermore, the equilibrium at $a = 0$ becomes an asymptotically stable equilibrium, since the linearized equation is $\dot{a} = (8 - 2K)a$. For $K < 4$, there is a nonnegative real solution however, corresponding to an incorrect stable equilibrium. Note that our results for general triangles identified a lower bound, see (17), on K to secure a unique stable equilibrium point, and we have in effect verified that for an equilateral triangle the bound is tight.

On the other hand, the steady state equation for the counterclockwise arrangement case has nonnegative real solutions at $a = 0, a = 2$ for all positive K. These are straightforwardly verified to be unstable and asymptotically stable respectively. Thus the effect of choosing K larger than 4 is to eliminate the possibility of any undesired stable equilibrium, at least of equilateral structure.

If one starts with an initial formation that is an equilateral triangle with a negative area, and $K > 4$, the formation will converge so that the triangle area shrinks to zero, i.e. all points are collocated, and then noise or round-off error will cause the system to converge towards an equilateral triangle with the correct side and area. in effect, the trajectories pass through a saddle, corresponding to $a = 0$.

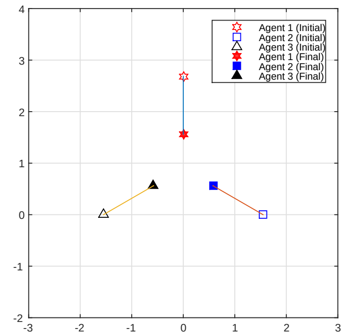


Figure 3: Preservation of equilaterality with $K = 2$

Fig. 4 illustrates this process with $K = 5$ for the same initial state as in Fig. 1.

4. Formations comprising two triangles

In this section, we will treat the case of a four agent formation with five edges, as a starting point of extensions to more general formations. In this case, there are precisely two triangles, with a common edge. To fix ideas, suppose that the two triangles are formed by agents $1, 2, 3$ and $2, 3, 4$, with a common edge $(2, 3)$. Taking into account the possible separate clockwise/anticlockwise orientations of the two triangles, we note that there are in all four possible formations (centroid and angular orientation being irrelevant in this classification). Two of these have agents 1 and 4 on the same side of edge $(2, 3)$, and are congruent but differ through the signs of the areas, one being obtainable from the other by replacing the four position vectors by their negatives (one case is shown in Fig. 5(a)). The other two have agents 1 and 4 on opposite sides of edge $(2, 3)$ (one case is shown in Fig. 5(b)). Again, these two are congruent, and one can be obtained from the other by replacing the four position vectors by their negatives.

We shall show that it is possible to formulate a performance index through the inclusion of terms reflecting the signed area of each triangle to achieve a particular one of the four formations. Suppose that the desired distances are $d^*_{12}, d^*_{23}, d^*_{31}, d^*_{34}$ and d^*_{42}. Let the corresponding desired areas of triangle 123 and triangle 234 be respectively Z^*_A, Z^*_B. With no real loss of generality, take these as positive, so the counterclockwise orderings of the vertices of the desired triangles are 123 and 234 respectively (see an illustration in Fig. 5(a)). The performance index we use is:

$$V(p_1, p_2, p_3, p_4) = \frac{1}{4}(e^2_{12} + e^2_{23} + e^2_{13} + e^2_{24}) + \frac{1}{2}K\left((Z_A - Z^*_A)^2 + (Z_B - Z^*_B)^2\right) \tag{37}$$

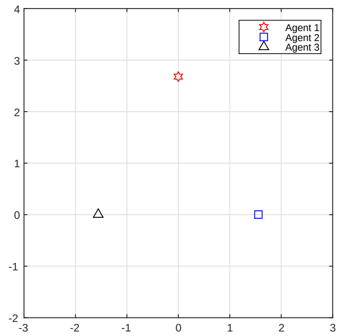
There would be no problem in having different weighting for $(Z_A - Z^*_A)^2$ and $(Z_B - Z^*_B)^2$ but no theoretical advantage appears to emerge.

For the case of a formation comprising a single triangle, we showed that use of a sufficiently large weighting K for the squared area error meant that there was no equilibrium with the associated triangle having an area with incorrect sign. For the case of a two-triangle formation, the situation is slightly different; we shall show that if there is an equilibrium with a triangle having the wrong sign, the associated Hessian matrix has a negative eigenvalue, i.e. the equilibrium cannot be a minimum. Accordingly, for almost all initial conditions, an equilibrium with the correct signs of the triangle areas will be attained, and we can further show that indeed the distances are correct for such an equilibrium. To provide the necessary tools then, we first obtain the equilibrium equations and the Hessian.

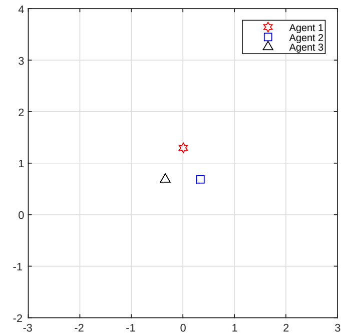
4.1. Equilibrium equations and Hessian of the performance index

The equilibrium equations are simply obtained by setting $\nabla V = 0$ with V defined in (37). The calculations are little different to those applying with a single triangle. The end result is as follows:

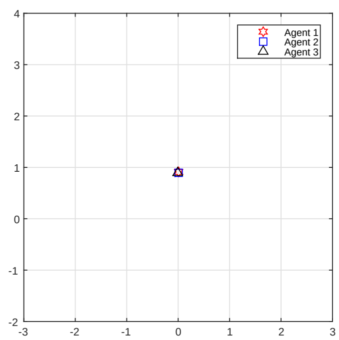
$$\begin{aligned}
-e_{12}(p_1 - p_2) - e_{13}(p_1 - p_3) - K(Z_A - Z^*_A)J(p_2 - p_3) &\overset{(1)}{=} 0\\
-e_{12}(p_2 - p_1) - e_{23}(p_2 - p_3) - e_{24}(p_2 - p_4) - K(Z_A - Z^*_A)J(p_3 - p_1) - K(Z_B - Z^*_B)J(p_3 - p_4) &\overset{(2)}{=} 0\\
-e_{13}(p_3 - p_1) - e_{23}(p_3 - p_2) - e_{34}(p_3 - p_4) - K(Z_A - Z^*_A)J(p_1 - p_2) - K(Z_B - Z^*_B)J(p_4 - p_2) &\overset{(3)}{=} 0\\
-e_{24}(p_4 - p_2) - e_{34}(p_4 - p_3) - K(Z_B - Z^*_B)J(p_2 - p_3) &\overset{(4)}{=} 0
\end{aligned} \tag{38}$$



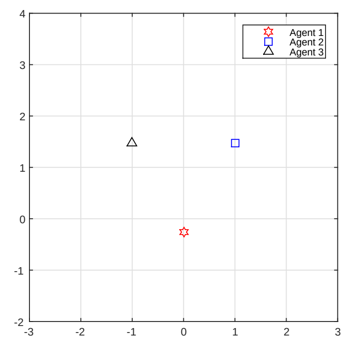
(a) $t = 0$ (Initial)



(b) $t = 0.1$



(c) $t = 0.2$



(d) $t = 0.3$

Figure 4: Convergence by adjusted weighting

Write the performance index V as a sum $V_1 + V_2$, where V_1 contains the distance error terms and V_2 the area error terms. The Hessian of V_1 has been computed for an almost identical case in [17], and is given by

$$\nabla^2 V_1 = 2R^\top R + E(p) \otimes I_2 \tag{39}$$

where R is the 5×8 rigidity matrix (essentially the Jacobian of the mapping from agent positions to squares of edge lengths, with the edge ordering having immaterial effect on the Hessian), and E is the matrix

$$E = \begin{bmatrix} e_{12} + e_{13} & -e_{12} & -e_{13} & 0 \\ -e_{12} & e_{12} + e_{23} + e_{24} & -e_{23} & -e_{24} \\ -e_{13} & -e_{23} & e_{13} + e_{23} + e_{34} & -e_{34} \\ 0 & -e_{24} & -e_{34} & e_{24} + e_{34} \end{bmatrix}$$

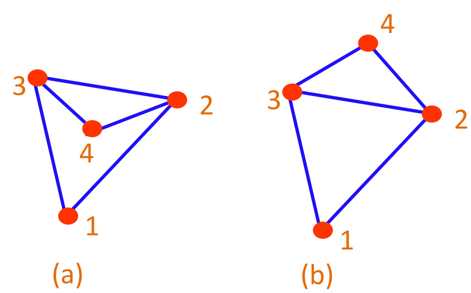


Figure 5: Examples of formations comprising two triangles

A tedious calculation also delivers

$$\nabla^2 V_2 = \frac{1}{4}K\left\{Y_AY_A^\top + 2(Z_A - Z_A^*)\begin{bmatrix} 0 & J & -J & 0 \\ -J & 0 & J & 0 \\ J & -J & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + Y_BY_B^\top + 2(Z_B - Z_B^*)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & J & -J \\ 0 & -J & 0 & J \\ 0 & J & -J & 0 \end{bmatrix}\right\} \tag{40}$$

with

$$Y_A = \begin{bmatrix} J(p_2 - p_3) \\ J(-p_1 + p_3) \\ J(p_1 - p_2) \\ 0 \end{bmatrix} \quad Y_B = \begin{bmatrix} 0 \\ J(p_3 - p_4) \\ J(-p_2 + p_4) \\ J(p_2 - p_3) \end{bmatrix} \tag{41}$$

Of course, $\nabla^2 V = \nabla^2 V_1 + \nabla^2 V_2$.

4.2. Large area weighting implies no equilibria with incorrect triangle signs

In this subsection, we will establish the following result.

Theorem 4. *Consider the four agent control problem with specified and achievable interagent distances $d_{12}^*, d_{23}^*, d_{31}^*, d_{24}^*, d_{34}^*$ and let the associated signed triangle areas for agents 123 and 234 be $Z_A^* > 0, Z_B^* > 0$. Suppose that a control law is established as a gradient descent law using the performance index* (37)*. Then for sufficiently large K, there can be no stable equilibrium in which the signs of the triangle areas are incorrect.*

Proof. We shall first study (Case 1) the effect of large K when at an equilibrium, at least one of the pairs Z_A, Z_A^* or Z_B, Z_B^* have differing signs. Then we shall consider the situation where one of Z_A, Z_B is zero at an equilibrium (Case 2). Our aim is to establish that there can be no such stable equilibrium.

Case 1: Suppose then without loss of generality that $Z_A < 0, Z_A^* > 0$ at an equilibrium. Multiplying the first of equation of (38) on the left by $[p_2 - p_3]^\top J$ yields

$$-(e_{12} + e_{13}) = K\left(1 + \frac{Z_A^*}{|Z_A|}\right)\|p_2 - p_3\|^2 \tag{42}$$

It is immediate that $e_{12} + e_{13} < 0$ independently of K, so that

$$d_{12}^2 + d_{13}^2 < d_{12}^{*2} + d_{13}^{*2} \tag{43}$$

independent of K. Hence at any equilibrium with the signs as assumed, the separation of the three agents located at p_1, p_2 and p_3 can be bounded independently of the value of K. Consequently the value of $|Z_A|$ can also be bounded independently of K.

Next, let w_1 be a unit length 2-vector orthogonal to $J(p_2 - p_3)$ and let w_2 be a second linearly independent vector of unit length. Let ϵ be an arbitrary small positive constant. Consider the vector $w = [w_1^\top \ \epsilon w_2^\top \ 0 \ 0]^\top$, and the quadratic form $w^\top \nabla^2 V_2 w$. It is easy to check that

$$w^\top \nabla^2 V_2 w = \frac{1}{4}K[4(Z_A - Z_A^*)\epsilon w_1^\top J w_2 + O(\epsilon^2)] \tag{44}$$

Notice that, under Case 1, $Z_A^* > 0$ and $Z_A < 0$ and is bounded independently of K. Hence $(Z_A - Z_A^*)$ is negative, bounded, and bounded away from zero with both bounds independent of K. Evidently then, there is a choice of w_2 and ϵ which ensures the quantity $w^\top \nabla^2 V_2 w$ is negative. Because of the K multiplier, it follows that $w^\top \nabla^2 V w < 0$ for the same w and sufficiently large K, implying the Hessian has at least one negative eigenvalue. Hence there cannot be a stable equilibrium with the stated sign constraints of Case 1 on Z_A, Z_A^*.

Case 2: We must also consider (and exclude) the possibility that at an equilibrium, $Z_A = 0, Z_A^* > 0$, by showing that if such an equilibrium exists, then it cannot be stable. The argument is effectively the same as applying for the case $Z_A < 0$. Note that (42) yields from the assumption that $Z_A = 0$ that $p_2 = p_3$ (which will imply also that $Z_B = 0$). The first block entry of Y_A is zero, and vectors w_1, w_2 can be constructed as previously, save that w_1 can now be arbitrary. □

4.3. Uniqueness of equilibrium with correct triangle area signs

In this subsection, we shall assume that at an equilibrium, the signs of the triangle areas are correct. We aim to show that there is a unique equilibrium as a consequence, i.e. the desired formation is attained. Note that we do not assume for this purpose that K is necessarily large.

Theorem 5. *Adopt the same hypothesis as for Theorem 4, and suppose further that at an equilibrium the signs of Z_A, Z_B are both positive. Then the equilibrium is necessarily the desired equilibrium, i.e. $V(p_1, p_2, p_3, p_4)_{(37)} = 0$.*

Proof. Our argument will be by contradiction. Suppose that at an equilibrium consistent with the sign constraints, there holds $Z_A < Z_A^*$. Then one at least of d_{12}, d_{23}, d_{31} must be too small. If either of d_{12}, d_{31} is too small, there will be an incremental motion of agent 1 to increase that value (or values) which is (or are) too small, leaving the other unaltered if necessary. The relevant squared distance error summands in $V_{(37)}$ will decrease as will the relevant squared area error summand, so that $V_{(37)}$ will decrease. So the situation could not correspond to an equilibrium. Hence we are left to consider the case when d_{12}, d_{13} are too big (or assume correct values) and d_{23} is too small.

Now we need to consider how such an equilibrium fits in with a consideration of Z_B. First, if $Z_B > Z_B^*$, an analogous argument to the above shows one cannot be at an equilibrium unless d_{24}, d_{34} are too small (or assume correct values) and d_{23} is too large. However, the conclusion regarding d_{23} is in contradiction to the conclusion arising from assuming $Z_A < Z_A^*$.

If $Z_B < Z_B^*$, then an analogous argument to that involving Z_A implies that we cannot be at an equilibrium unless d_{24}, d_{34} are too large and d_{23} is too small. But in this case, infinitesimal increase of the separation of agents 2 and 3 with maintenance of their distances from agents 1 and 4 will move d_{23} and both of Z_A, Z_B nearer to their correct values, and thus there cannot be an equilibrium of this type.

Last, suppose we are at an equilibrium in which $Z_A < Z_A^*$, d_{23} is too small, and $Z_B = Z_B^*$. In this case, one at least of d_{24} and d_{34} must be too large. Consider an incremental movement of agents in which the separation of agents 2 and 3 is increased, and agent 4 moves so as to reduce whichever of d_{24}, d_{34} is too large, while maintaining the area $Z_B = Z_B^*$. Such an incremental movement reduces the performance index, and thus again establishes that there can be no equilibrium of the nominated type.

In summary, if $Z_A < Z_A^*$, there can be no equilibrium. Effectively the same argument establishes that there can be no equilibrium with $Z_A > Z_A^*$, with the same conclusions for Z_B, Z_B^*. Hence at any equilibrium, both $Z_A = Z_A^*$ and $Z_B = Z_B^*$.

Now consider (38). Noting that the agent triples 1,2,3 and 2,3,4 cannot be collinear if the triangle areas are correct, we see immediately from the first equation that $e_{12} = e_{13} = 0$ and from the last equation that $e_{24} = e_{34} = 0$. Either of the remaining equations then yields $e_{23} = 0$. Hence the equilibrium is correct. □

4.4. *Four agents forming a complete graph*

It is possible to consider a formation comprising 4 agents and 6 edges. Such a formation is unique, up to rotation, translation, and reflection, with the associated graph a complete graph. The same formation can clearly be specified up to simply rotation and translation, but not up to reflection by using 5 lengths and two signed triangle areas. This means that the methods of this section can be used to cause a complete four-agent formation, with specified signed area, to be attained from all but a thin set of initial conditions.

4.5. *Examples*

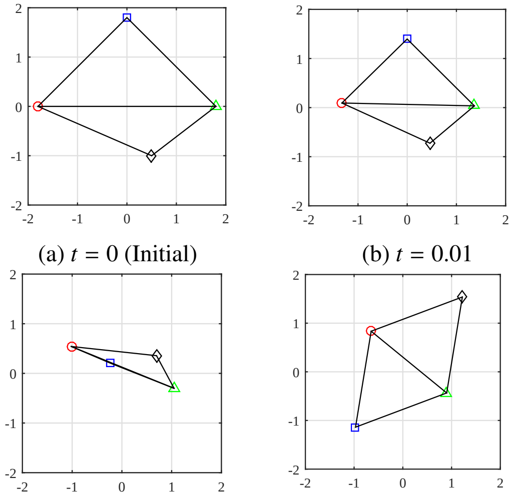
One example of a four agent formation with five edges is shown in Fig. 6, where the performance index (37) is employed with the gain $K = 3$. Agents 1, 2, 3 and 4 are represented by a square, triangle, circle and diamond, respectively. The goal formation comprises two equilateral triangles with the area constraints $Z_A^* = -Z_B^* = \sqrt{3}$. At $t = 0$, triangles 123 and 234 start with the wrong orientation $Z_A < 0$ and $Z_B > 0$, respectively as shown in Fig. 6 (a). At $t = 0.01$, both agents 1 (square) and 4 (diamond) are seen to be approaching the correct sides of the edge 23 as shown Fig. 6 (b), and both of them almost reach the correct sides (i.e., $Z_A \geq 0$ and $Z_B < 0$) at $t = 0.02$ as shown in Fig. 6 (c). At $t = 0.1$, the agents are almost in the steady state and achieve the goal formation. In fact, it is confirmed that $\|p_1 - p_2\| = \|p_1 - p_3\| = \|p_2 - p_3\| = \|p_2 - p_4\| = \|p_3 - p_4\| = 2$ holds with $Z_A = -Z_B = \sqrt{3}$ in Fig. 6 (d).

When the area gain is chosen as $K = 0.3$ and they start with the same initial positions, the agents converge to an incorrect formation in which the desired area signs are not attained.

5. Conclusions

In this paper we have discussed the formation control problem for rigid formation shapes with both distance and area constraints. For a rigid target formation, the signed area is incorporated in the performance index function to address the formation reflection issue. By taking the triangle formation as an example, we analyze the equilibrium property and provide conditions with a weighting parameter to guarantee the uniqueness of the desired equilibrium point and a global convergence of the formation to correct side lengths and signed area. The results are then extended to a four-agent formation shape comprising two triangles, which could be further generalized to more complex formations.

We remark some future directions based on this work. The first and rather obvious potential extension would be to frameworks comprising interconnected triangles. We anticipate that the techniques and results established in the recent paper [18] on distance-based global stabilization of triangulated rigid formations would also deliver promising results for this problem. Furthermore, the ideas of controlling 2-D formations with signed area constraints defined in (7) could also be generalized to the control of 3-D formations with signed *volume* constraints [19], in which signed volume constraints in the 3-D case could be defined by triple products, each involving three relative position vectors that define a *tetrahedron*. Apart from these, one



(a) $t = 0$ (Initial) (b) $t = 0.01$

(c) $t = 0.02$ (d) $t = 0.1$

Figure 6: Convergence of a four-agent formation with five edges and area constraints

further interesting direction is to consider a second-order agent model which involves a velocity consensus term so that other formation motions (e.g. flocking) of a rigid formation shape with distances and area constraints can be generated. Another extension would be the development of formation controllers with further constraints, e.g. collision avoidance, unidirectional sensing/measurements, with a general theory on the formation convergence analysis.

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References

[1] K.-K. Oh, M.-C. Park, and H.-S. Ahn, "A survey of multi-agent formation control," *Automatica*, vol. 53, pp. 424–440, 2015.

[2] W. Ren and R. W. Beard, *Distributed consensus in multi-vehicle cooperative control*. Springer, 2008.

[3] F. Xiao, L. Wang, J. Chen, and Y. Gao, "Finite-time formation control for multi-agent systems," *Automatica*, vol. 45, no. 11, pp. 2605–2611, 2009.

[4] M. Aranda, G. López-Nicolás, C. Sagüés, and M. M. Zavlanos, "Distributed formation stabilization using relative position measurements in local coordinates," *IEEE Transactions on Automatic Control*, vol. 61, no. 12, pp. 3925–3935, 2016.

[5] B. Hendrickson, "Conditions for unique graph realizations," *SIAM Journal on Computing*, vol. 21, no. 1, pp. 65–84, 1992.

[6] B. D. O. Anderson, C. Yu, B. Fidan, and J. M. Hendrickx, "Rigid graph control architectures for autonomous formations," *IEEE Control Systems Magazine*, vol. 28, no. 6, pp. 48–63, 2008.

[7] Z. Sun, M.-C. Park, B. D. O. Anderson, and H.-S. Ahn, "Distributed stabilization control of rigid formations with prescribed orientation," *Automatica*, vol. 78, no. 4, pp. 250–257, 2017.

[8] K. Sakurama, "Distributed control of networked multi-agent systems for formation with freedom of special euclidean group," in *Proc. of the 2016 IEEE Conference on Decision and Control (CDC)*, IEEE, 2016.

[9] E. Ferreira-Vazquez, J. Flores-Godoy, E. Hernandez-Martinez, and G. Fernandez-Anaya, "Adaptive control of distance-based spatial formations with planar and volume restrictions," in *Proc. of the 2016 IEEE Conference on Control Applications (CCA)*, pp. 905–910, IEEE, 2016.

[10] K. Sakurama, S.-i. Azuma, and T. Sugie, "Distributed controllers for multi-agent coordination via gradient-flow approach," *IEEE Transactions on Automatic Control*, vol. 60, pp. 1471–1485, June 2015.

[11] C. Vasile, M. Schwager, and C. Belta, "Translational and rotational invariance in networked dynamical systems," *IEEE Transactions on Control of Network Systems*, p. in press, 2017.

[12] L. Krick, M. E. Broucke, and B. A. Francis, "Stabilisation of infinitesimally rigid formations of multi-robot networks," *International Journal of Control*, vol. 82, no. 3, pp. 423–429, 2009.

[13] Z. Sun, S. Mou, B. D. O. Anderson, and M. Cao, "Exponential stability for formation control systems with generalized controllers: a unified approach," *Systems and Control Letters*, vol. 93, pp. 50–57, 2016.

[14] Z. Sun, U. Helmke, and B. D. O. Anderson, "Rigid formation shape control in general dimensions: an invariance principle and open problems," in *Proc. of the IEEE 54th Annual Conference on Decision and Control (CDC)*, pp. 6095–6100, December 2016.

[15] M. Cao, A. S. Morse, C. Yu, B. D. O. Anderson, and S. Dasgupta, "Controlling a triangular formation of mobile autonomous agents," in *Proc. of 45th IEEE Conference on Decision and Control (CDC)*, pp. 3603–3608, IEEE, 2007.

[16] R. Weitzenböck, "Über eine Ungleichung in der Dreiecksgeometrie," *Mathematische Zeitschrift*, vol. 5, pp. 137–146, 1919.

[17] T. Summers, C. Yu, B. D. O. Anderson, and S. Dasgupta, "Formation shape control: global asymptotic stability of a four-agent formation," in *Proc. of IEEE 48th Annual Conference on Decision and Control (CDC)*, pp. 8334–8339, December 2009.

[18] X. Chen, M.-A. Belabbas, and T. Basar, "Global stabilization of triangulated formations," *SIAM Journal on Control and Optimization*, vol. 55, no. 1, pp. 172–199, 2017.

[19] J. M. Chappell, S. P. Drake, C. L. Seidel, L. J. Gunn, A. Iqbal, A. Allison, and D. Abbott, "Geometric algebra for electrical and electronic engineers," *Proceedings of the IEEE*, vol. 102, no. 9, pp. 1340–1363, 2014.