Coherent-Classical Estimation for Linear Quantum Systems ⋆

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Abstract

A coherent-classical estimation scheme for a class of linear quantum systems is considered. Here, the estimator is a mixed quantum-classical system that may or may not involve coherent feedback and yields a classical estimate of a variable for the quantum plant. We demonstrate that with no coherent feedback, such coherent-classical estimation provides no improvement over purely-classical estimation, when the quantum plant or the quantum part of the estimator (called the coherent controller) is a passive annihilation operator system. However, when both the quantum plant and the coherent controller are active systems (that cannot be described by annihilation operators only), coherent-classical estimation without coherent feedback can provide improvement over purely-classical estimation in certain cases. Furthermore, we show that with coherent feedback, it is possible to get better estimation accuracies with coherent-classical estimation, as compared to classical-only estimation, when either or both of the plant and the controller is not an annihilation operator only quantum system.

1 Introduction

Estimation and control problems for quantum systems have been of significant interest in recent years [3–5, 7, 9, 10, 14, 15, 24–28]. An important class of quantum systems are linear quantum systems [2, 3, 5, 7, 11, 13, 14, 17, 19, 21, 22, 24, 25], that describe quantum optical devices such as optical cavities [1, 21], linear quantum amplifiers [2], and finite bandwidth squeezers [2]. Much recent work has considered coherent feedback control for linear quantum systems, where the feedback controller itself is also a quantum system [6–11, 14, 17, 23]. A related coherent-classical estimation problem has been considered by the authors in Refs. [16, 18], where the estimator consists of a classical part, which produces the desired final estimate and a quantum part, which may also involve coherent feedback. Note that this is different from the problem considered in Ref. [12] which involves constructing a quantum observer. A quantum observer is a purely quantum system, that produces a quantum estimate of a variable for the quantum plant. On the other hand, a coherent-classical estimator is a mixed quantum-classical system, that produces a classical estimate of a variable for the quantum plant.

In this paper, we elaborate and build on the results of the conference papers [16, 18] to demonstrate two key theorems, propose three relevant conjectures, and illustrate our findings with several examples. We essentially show that a coherent-classical estimation scheme not involving any feedback, where either of the plant and the coherent controller is a physically realizable annihilation operator only system, it is not possible to get better estimates than the corresponding purely-classical estimation scheme. Although it is possible to get better estimates than purely-classical estimation with such a coherent-classical estimation scheme in certain cases, when both the plant and the controller are active systems, we observe that for the optimal choice of the homodyne detection angle, classical-only estimation is always superior. Moreover, we demonstrate that a coherent-classical estimation scheme with coherent feedback provides with higher estimation precision than classical-only estimation, if not both of the plant and the controller can be defined purely using annihilation operators. In addition, if there is any improvement with the coherent-classical estimation scheme with coherent feedback over purely-classical estimation for a given plant, we notice that for the optimal choice of the homodyne angle, classical-only estimation is always inferior. Furthermore, all these results are illustrated with pertinent examples.

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The paper is structured as follows. Section 2 introduces the class of linear quantum systems considered in this paper and discusses the notion of physical realizability for the same. Section 3 formulates the problem of the optimal purely-classical estimation of a quantum plant, that is a linear quantum system of the kind of interest here. In section 4, we formulate the optimal coherent-classical estimation problem, that does not involve any coherent feedback. We also present our first key theorem alongwith a formal proof. Section 5 considers examples involving dynamic optical squeezers to illustrate the theorem in the previous section and present a couple of relevant conjectures. Section 6 discusses the coherent-classical estimation scheme involving coherent feedback and lays down our second key theorem alongwith a formal proof. More examples using dynamic squeezers are presented in section 7 to support the theorem, before proposing another conjecture. Finally, section 8 concludes the paper with relevant remarks and possible directions for future work.

2 Linear Quantum Systems and Physical Realizability

The class of linear quantum systems we consider here are described by the quantum stochastic differential equations (QSDEs) [5, 7, 15, 16, 20]:

$$\begin{aligned} \begin{bmatrix} da(t) \\ da(t)^{\#} \end{bmatrix} &= F \begin{bmatrix} a(t) \\ a(t)^{\#} \end{bmatrix} dt + G \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^{\#} \end{bmatrix}; \\ \begin{bmatrix} d\mathcal{A}^{out}(t) \\ d\mathcal{A}^{out}(t)^{\#} \end{bmatrix} &= H \begin{bmatrix} a(t) \\ a(t)^{\#} \end{bmatrix} dt + K \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^{\#} \end{bmatrix}, \end{aligned} \tag{1}$$

where

$$\begin{aligned} F &= \Delta(F_1, F_2), \qquad G = \Delta(G_1, G_2), \\ H &= \Delta(H_1, H_2), \qquad K = \Delta(K_1, K_2). \end{aligned} \tag{2}$$

Here, $a(t) = [a_1(t) \dots a_n(t)]^T$ is a vector of annihilation operators. The adjoint of the operator a_i is called a creation operator, denoted by a_i^*. Also, the notation $\Delta(F_1, F_2)$ denotes the matrix $\begin{bmatrix} F_1 & F_2 \\ F_2^{\#} & F_1^{\#} \end{bmatrix}$. Here, F_1, $F_2 \in \mathbb{C}^{n\times n}$, G_1, $G_2 \in \mathbb{C}^{n\times m}$, H_1, $H_2 \in \mathbb{C}^{m\times n}$, and K_1, $K_2 \in \mathbb{C}^{m\times m}$. Moreover, $^{\#}$ denotes the adjoint of a vector of operators or the complex conjugate of a complex matrix. Furthermore, † denotes the adjoint transpose of a vector of operators or the complex conjugate transpose of a complex matrix. In addition, the vector $\mathcal{A} = [\mathcal{A}_1 \dots \mathcal{A}_m]^T$ represents a collection of external independent quantum field operators and the vector $\mathcal{A}^{out}$ represents the corresponding vector of output field operators.

Definition 2.1 *(See [7, 15, 16, 20]) A complex linear quantum system of the form (1), (2) is said to be physically realizable if there exists a complex commutation matrix $\Theta = \Theta^{\dagger}$, a complex Hamiltonian matrix $M = M^{\dagger}$, and a coupling matrix N such that*

$$\Theta = TJT^{\dagger}, \tag{3}$$

where $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$, $T = \Delta(T_1, T_2)$ is non-singular, M and N are of the form

$$M = \Delta(M_1, M_2), \qquad N = \Delta(N_1, N_2) \tag{4}$$

and

$$\begin{aligned} F &= -\iota\Theta M - \frac{1}{2}\Theta N^{\dagger} J N, \\ G &= -\Theta N^{\dagger} J, \\ H &= N, \\ K &= I. \end{aligned} \tag{5}$$

Here, the commutation matrix Θ satisfies the following commutation relation:

$$\begin{aligned} &\left[\begin{bmatrix} a \\ a^{\#} \end{bmatrix}, \begin{bmatrix} a \\ a^{\#} \end{bmatrix}^{\dagger} \right] \\ &= \begin{bmatrix} a \\ a^{\#} \end{bmatrix} \begin{bmatrix} a \\ a^{\#} \end{bmatrix}^{\dagger} - \left(\begin{bmatrix} a \\ a^{\#} \end{bmatrix}^{\#} \begin{bmatrix} a \\ a^{\#} \end{bmatrix}^{T} \right)^{T} \\ &= \Theta. \end{aligned} \tag{6}$$

Theorem 2.1 *(See [17, 20]) The linear quantum system (1), (2) is physically realizable if and only if there exists a complex matrix $\Theta = \Theta^{\dagger}$ such that Θ is of the form in (3), and*

$$\begin{aligned} F\Theta + \Theta F^{\dagger} + GJG^{\dagger} &= 0, \\ G &= -\Theta H^{\dagger} J, \\ K &= I. \end{aligned} \tag{7}$$

If the system (1) is physically realizable, then the matrices M and N define a complex open harmonic oscillator with a Hamiltonian operator

$$\mathbf{H} = \frac{1}{2} \begin{bmatrix} a^{\dagger} & a^T \end{bmatrix} M \begin{bmatrix} a \\ a^{\#} \end{bmatrix},$$

and a coupling operator

$$\mathbf{L} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} a \\ a^{\#} \end{bmatrix}.$$

2.1 Annihilation Operator Only Linear Quantum Systems

Annihilation operator only linear quantum systems are a special case of the above class of linear quantum systems, where the QSDEs (1) can be described purely in terms of the vector of annihilation operators [9, 10]. In this case, we consider Hamiltonian operators of the form $\mathbf{H} = a^\dagger M a$ and coupling vectors of the form $\mathbf{L} = Na$, where M is a Hermitian matrix and N is a complex matrix. The commutation relation (6), in this case, takes the form:

$$[a, a^\dagger] = \Theta, \tag{8}$$

where $\Theta > 0$. Also, the corresponding QSDEs are given by:

$$\begin{aligned} da &= Fadt + Gd\mathcal{A}; \\ d\mathcal{A}^{out} &= Hadt + Kd\mathcal{A}. \end{aligned} \tag{9}$$

Definition 2.2 *(See [9,17]) A linear quantum system of the form (9) is physically realizable if there exist complex matrices $\Theta > 0$, $M = M^\dagger$, N, such that the following is satisfied:*

$$\begin{aligned} &F = \Theta\left(-\iota M - \frac{1}{2}N^\dagger N\right), \\ &G = -\Theta N^\dagger, \\ &H = N, \\ &K = I. \end{aligned} \tag{10}$$

Theorem 2.2 *(See [9, 17]) An annihilation operator only linear quantum system of the form (9) is physically realizable if and only if there exists a complex matrix $\Theta > 0$ such that*

$$\begin{aligned} F\Theta + \Theta F^\dagger + GG^\dagger &= 0, \\ G &= -\Theta H^\dagger, \\ K &= I. \end{aligned} \tag{11}$$

2.2 Linear Quantum System from Quantum Optics

An example of a linear quantum system from quantum optics is a dynamic squeezer, that is an optical cavity with a non-linear optical element inside as shown in Fig. 1. Upon suitable linearizations and approximations, such an optical squeezer can be described by the following quantum stochastic differential equations [16]:

$$\begin{aligned} da &= -\frac{\gamma}{2}adt - \chi a^* dt - \sqrt{\kappa_1}d\mathcal{A}_1 - \sqrt{\kappa_2}d\mathcal{A}_2; \\ d\mathcal{A}_1^{out} &= \sqrt{\kappa_1}adt + d\mathcal{A}_1; \\ d\mathcal{A}_2^{out} &= \sqrt{\kappa_2}adt + d\mathcal{A}_2, \end{aligned} \tag{12}$$

where $\kappa_1, \kappa_2 > 0$, $\chi \in \mathbb{C}$, and a is a single annihilation operator of the cavity mode [1,2]. This leads to a linear quantum system of the form (1) as follows:

$$\begin{aligned} \begin{bmatrix} da(t) \\ da(t)^* \end{bmatrix} &= \begin{bmatrix} -\frac{\gamma}{2} & -\chi \\ -\chi^* & -\frac{\gamma}{2} \end{bmatrix} \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix} dt \\ &\quad - \sqrt{\kappa_1} \begin{bmatrix} d\mathcal{A}_1(t) \\ d\mathcal{A}_1(t)^{\#} \end{bmatrix} \\ &\quad - \sqrt{\kappa_2} \begin{bmatrix} d\mathcal{A}_2(t) \\ d\mathcal{A}_2(t)^{\#} \end{bmatrix}; \\ \begin{bmatrix} d\mathcal{A}_1^{out}(t) \\ d\mathcal{A}_1^{out}(t)^{\#} \end{bmatrix} &= \sqrt{\kappa_1} \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix} dt + \begin{bmatrix} d\mathcal{A}_1(t) \\ d\mathcal{A}_1(t)^{\#} \end{bmatrix}; \\ \begin{bmatrix} d\mathcal{A}_2^{out}(t) \\ d\mathcal{A}_2^{out}(t)^{\#} \end{bmatrix} &= \sqrt{\kappa_2} \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix} dt + \begin{bmatrix} d\mathcal{A}_2(t) \\ d\mathcal{A}_2(t)^{\#} \end{bmatrix}. \end{aligned} \tag{13}$$

The above quantum system requires $\gamma = \kappa_1 + \kappa_2$ in order for the system to be physically realizable.

Also, the above quantum optical system can be described purely in terms of the annihilation operator only if $\chi = 0$, i.e. there is no squeezing, in which case it reduces to a passive optical cavity. This leads to a linear quantum system of the form (9) as follows:

$$\begin{aligned} da &= -\frac{\gamma}{2}adt - \sqrt{\kappa_1}d\mathcal{A}_1 - \sqrt{\kappa_2}d\mathcal{A}_2; \\ d\mathcal{A}_1^{out} &= \sqrt{\kappa_1}adt + d\mathcal{A}_1; \\ d\mathcal{A}_2^{out} &= \sqrt{\kappa_2}adt + d\mathcal{A}_2, \end{aligned} \tag{14}$$

where again the system is physically realizable when we have $\gamma = \kappa_1 + \kappa_2$.

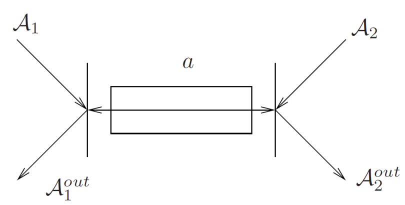

Fig. 1. Schematic diagram of a dynamic optical squeezer.

3 Purely-Classical Estimation

The schematic diagram of a purely-classical estimation scheme is provided in Fig. 2. We consider a quantum plant, which is a quantum system of the form (1), (2), defined as follows:

$$\begin{aligned}
\begin{bmatrix} da(t) \\ da(t)^{\#} \end{bmatrix} &= F \begin{bmatrix} a(t) \\ a(t)^{\#} \end{bmatrix} dt + G \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^{\#} \end{bmatrix}; \\
\begin{bmatrix} d\mathcal{Y}(t) \\ d\mathcal{Y}(t)^{\#} \end{bmatrix} &= H \begin{bmatrix} a(t) \\ a(t)^{\#} \end{bmatrix} dt + K \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^{\#} \end{bmatrix}; \\
z &= C \begin{bmatrix} a(t) \\ a(t)^{\#} \end{bmatrix}.
\end{aligned} \tag{15}$$

Here, z denotes a scalar operator on the underlying Hilbert space and represents the quantity to be estimated. Also, $\mathcal{Y}(t)$ represents the vector of output fields of the plant, and $\mathcal{A}(t)$ represents a vector of quantum noises acting on the plant.

In the case of a purely-classical estimator, a quadrature of each component of the vector $\mathcal{Y}(t)$ is measured using homodyne detection to produce a corresponding classical signal y_i:

$$\begin{aligned}
dy_1 &= \cos(\theta_1) d\mathcal{Y}_1 + \sin(\theta_1) d\mathcal{Y}_1^*; \\
&\vdots \\
dy_m &= \cos(\theta_m) d\mathcal{Y}_m + \sin(\theta_m) d\mathcal{Y}_m^*.
\end{aligned} \tag{16}$$

Here, the angles $\theta_1, \ldots, \theta_m$ determine the quadrature measured by each homodyne detector. The vector of classical signals $y = [y_1 \ldots y_m]^T$ is then used as the input to a classical estimator defined as follows:

$$\begin{aligned}
dx_e &= F_e x_e dt + G_e dy; \\
\hat{z} &= H_e x_e.
\end{aligned} \tag{17}$$

Here $\hat{z}$ is a scalar classical estimate of the quantity z. The estimation error corresponding to this estimate is

$$e = z - \hat{z}. \tag{18}$$

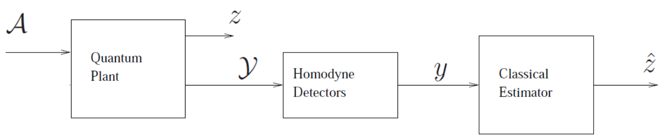

Fig. 2. Schematic diagram of purely-classical estimation.

Then, the optimal classical estimator is defined as the system (17) that minimizes the quantity

$$\bar{J}_c = \lim_{t \to \infty} \langle e^*(t) e(t) \rangle, \tag{19}$$

which is the mean-square error of the estimate. Here, $\langle \cdot \rangle$ denotes the quantum expectation over the joint quantum-classical system defined by (15), (16), (17).

The optimal classical estimator is given by the standard (complex) Kalman filter defined for the system (15), (16). This optimal classical estimator is obtained from the solution to the algebraic Riccati equation:

$$\begin{aligned}
&F_a \bar{P}_e + \bar{P}_e F_a^\dagger + G_a G_a^\dagger - (G_a K_a^\dagger + \bar{P}_e H_a^\dagger) L^\dagger \\
&\quad \times (L K_a K_a^\dagger L^\dagger)^{-1} L (G_a K_a^\dagger + \bar{P}_e H_a^\dagger)^\dagger = 0,
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
F_a &= F, \qquad G_a = G, \\
H_a &= H, \qquad K_a = K, \\
L &= \begin{bmatrix} L_1 & L_2 \end{bmatrix}, \\
L_1 &= \begin{bmatrix} \cos(\theta_1) & 0 & \ldots & 0 \\ 0 & \cos(\theta_2) & \ldots & 0 \\ & & \ddots & \\ & & & \cos(\theta_m) \end{bmatrix}, \\
L_2 &= \begin{bmatrix} \sin(\theta_1) & 0 & \ldots & 0 \\ 0 & \sin(\theta_2) & \ldots & 0 \\ & & \ddots & \\ & & & \sin(\theta_m) \end{bmatrix}.
\end{aligned} \tag{21}$$

Here we have assumed that the quantum noise $\mathcal{A}$ is purely canonical, i.e. $d\mathcal{A} d\mathcal{A}^\dagger = I dt$ and hence $K = I$.

Eq. (20) thus becomes:

$$F \bar{P}_e + \bar{P}_e F^\dagger + G G^\dagger - (G + \bar{P}_e H^\dagger) L^\dagger L (G + \bar{P}_e H^\dagger)^\dagger = 0. \tag{22}$$

The value of the cost (19) is given by

$$\bar{J}_c = C \bar{P}_e C^\dagger. \tag{23}$$

4 Coherent-Classical Estimation

The schematic diagram of the coherent-classical estimation scheme under consideration is provided in Fig. 3. In this case, the plant output $\mathcal{Y}(t)$ does not directly drive a bank of homodyne detectors as in (16). Rather, this output is fed into another quantum system called a coherent controller, which is defined as follows:

$$\begin{bmatrix} da_c(t) \\ da_c(t)^\# \end{bmatrix} = F_c \begin{bmatrix} a_c(t) \\ a_c(t)^\# \end{bmatrix} dt + G_c \begin{bmatrix} d\mathcal{Y}(t) \\ d\mathcal{Y}(t)^\# \end{bmatrix}; \\ \begin{bmatrix} d\tilde{\mathcal{Y}}(t) \\ d\tilde{\mathcal{Y}}(t)^\# \end{bmatrix} = H_c \begin{bmatrix} a_c(t) \\ a_c(t)^\# \end{bmatrix} dt + K_c \begin{bmatrix} d\mathcal{Y}(t) \\ d\mathcal{Y}(t)^\# \end{bmatrix}. \tag{24}$$

A quadrature of each component of the vector $\tilde{\mathcal{Y}}(t)$ is measured using homodyne detection to produce a corresponding classical signal $\tilde{y}_i$:

$$d\tilde{y}_1 = \cos(\tilde{\theta}_1)d\tilde{\mathcal{Y}}_1 + \sin(\tilde{\theta}_1)d\tilde{\mathcal{Y}}_1^*; \\ \vdots \\ d\tilde{y}_{\tilde{m}} = \cos(\tilde{\theta}_{\tilde{m}})d\tilde{\mathcal{Y}}_{\tilde{m}} + \sin(\tilde{\theta}_{\tilde{m}})d\tilde{\mathcal{Y}}_{\tilde{m}}^*. \tag{25}$$

Here, the angles $\tilde{\theta}_1, \ldots, \tilde{\theta}_{\tilde{m}}$ determine the quadrature measured by each homodyne detector. The vector of classical signals $\tilde{y} = [\tilde{y}_1 \ldots \tilde{y}_{\tilde{m}}]^T$ is then used as the input to a classical estimator defined as follows:

$$d\tilde{x}_e = \tilde{F}_e \tilde{x}_e dt + \tilde{G}_e d\tilde{y}; \\ \hat{z} = \tilde{H}_e \tilde{x}_e. \tag{26}$$

Here $\hat{z}$ is a scalar classical estimate of the quantity z. Corresponding to this estimate is the estimation error (18). Then, the optimal coherent-classical estimator is defined as the systems (24), (26) which together minimize the quantity (19).

We can now combine the quantum plant (15) and the coherent controller (24) to yield an augmented quantum linear system defined by the following QSDEs:

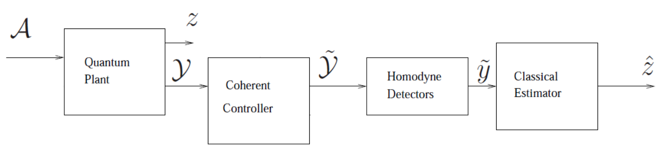

Fig. 3. Schematic diagram of coherent-classical estimation.

$$\begin{bmatrix} da \\ da^\# \\ da_c \\ da_c^\# \end{bmatrix} = \begin{bmatrix} F & 0 \\ G_c H & F_c \end{bmatrix} \begin{bmatrix} a \\ a^\# \\ a_c \\ a_c^\# \end{bmatrix} dt + \begin{bmatrix} G \\ G_c K \end{bmatrix} \begin{bmatrix} d\mathcal{A} \\ d\mathcal{A}^\# \end{bmatrix}; \\ \begin{bmatrix} d\tilde{\mathcal{Y}} \\ d\tilde{\mathcal{Y}}^\# \end{bmatrix} = \begin{bmatrix} K_c H & H_c \end{bmatrix} \begin{bmatrix} a \\ a^\# \\ a_c \\ a_c^\# \end{bmatrix} dt + K_c K \begin{bmatrix} d\mathcal{A} \\ d\mathcal{A}^\# \end{bmatrix}. \tag{27}$$

The optimal classical estimator is given by the standard (complex) Kalman filter defined for the system (27), (25). This optimal classical estimator is obtained from the solution $\tilde{P}_e$ to an algebraic Riccati equation of the form (20), where

$$F_a = \begin{bmatrix} F & 0 \\ G_c H & F_c \end{bmatrix}, \qquad G_a = \begin{bmatrix} G \\ G_c K \end{bmatrix}, \\ H_a = \begin{bmatrix} K_c H & H_c \end{bmatrix}, \qquad K_a = K_c K, \\ L = \begin{bmatrix} \tilde{L}_1 & \tilde{L}_2 \end{bmatrix}, \\ \tilde{L}_1 = \begin{bmatrix} \cos(\tilde{\theta}_1) & 0 & \ldots & 0 \\ 0 & \cos(\tilde{\theta}_2) & \ldots & 0 \\ & & \ddots & \\ & & & \cos(\tilde{\theta}_{\tilde{m}}) \end{bmatrix}, \\ \tilde{L}_2 = \begin{bmatrix} \sin(\tilde{\theta}_1) & 0 & \ldots & 0 \\ 0 & \sin(\tilde{\theta}_2) & \ldots & 0 \\ & & \ddots & \\ & & & \sin(\tilde{\theta}_{\tilde{m}}) \end{bmatrix}, \tag{28}$$

where since the quantum noise $\mathcal{A}$ is assumed to be purely canonical, i.e. $d\mathcal{A}d\mathcal{A}^\dagger = Idt$, we have $K_a = K_c K = I$, which requires $K_c = I$ too, as $K = I$.

Then, the corresponding optimal classical estimator (26) is defined by the equations:

$$\tilde{F}_e = F_a - \tilde{G}_e L H_a; \\ \tilde{G}_e = (G_a K_a^\dagger + \tilde{P}_e H_a^\dagger) L^\dagger (L K_a K_a^\dagger L^\dagger)^{-1}; \\ \tilde{H}_e = \begin{bmatrix} C & 0 \end{bmatrix}. \tag{29}$$

We write:

$$\tilde{P}_e = \begin{bmatrix} P_1 & P_2 \\ P_2^\dagger & P_3 \end{bmatrix}, \tag{30}$$

where P_1 is of the same dimension as $\bar{P}_e$.

Then, the value of the corresponding cost of the form (19) is then given by

$$\tilde{J}_c = \begin{bmatrix} C & 0 \end{bmatrix} \tilde{P}_e \begin{bmatrix} C^\dagger \\ 0 \end{bmatrix} = CP_1C^\dagger. \tag{31}$$

Also, we calculate:

$$\begin{aligned} G_aG_a^\dagger &= \begin{bmatrix} GG^\dagger & GG_c^\dagger \\ G_cG^\dagger & G_cG_c^\dagger \end{bmatrix}, \\ G_aK_a^\dagger + \tilde{P}_eH_a^\dagger &= \begin{bmatrix} G + P_1H^\dagger + P_2H_c^\dagger \\ G_c + P_2^\dagger H^\dagger + P_3H_c^\dagger \end{bmatrix}, \\ F_a\tilde{P}_e &= \begin{bmatrix} FP_1 & FP_2 \\ G_cHP_1 + F_cP_2^\dagger & G_cHP_2 + F_cP_3 \end{bmatrix}, \\ \tilde{P}_eF_a^\dagger &= \begin{bmatrix} P_1F^\dagger & P_1H^\dagger G_c^\dagger + P_2F_c^\dagger \\ P_2^\dagger F^\dagger & P_2^\dagger H^\dagger G_c^\dagger + P_3F_c^\dagger \end{bmatrix}. \end{aligned} \tag{32}$$

Thus, upon expanding the Riccati equation (20), we get the following set of equations:

$$\begin{aligned} &FP_1 + P_1F^\dagger + GG^\dagger - (G + P_1H^\dagger + P_2H_c^\dagger)L^\dagger \\ &\times L(G + P_1H^\dagger + P_2H_c^\dagger)^\dagger = 0, \\ &FP_2 + P_1H^\dagger G_c^\dagger + P_2F_c^\dagger + GG_c^\dagger - (G + P_1H^\dagger + P_2H_c^\dagger)L^\dagger \\ &\times L(G_c + P_2^\dagger H^\dagger + P_3H_c^\dagger)^\dagger = 0, \\ &G_cHP_2 + P_2^\dagger H^\dagger G_c^\dagger + F_cP_3 + P_3F_c^\dagger + G_cG_c^\dagger \\ &- (G_c + P_2^\dagger H^\dagger + P_3H_c^\dagger)L^\dagger L(G_c + P_2^\dagger H^\dagger + P_3H_c^\dagger)^\dagger = 0. \end{aligned} \tag{33}$$

Theorem 4.1 *Consider a coherent-classical estimation scheme defined by (9) ($\mathcal{A}^{out}$ being $\mathcal{Y}$), (24), (25) and (26), such that the plant is physically realizable, with the estimation error cost $\tilde{J}_c$ defined in (31). Also, consider the corresponding purely-classical estimation scheme defined by (9), (16) and (17), such that the plant is physically realizable, with the estimation error cost $\bar{J}_c$ defined in (23). Then,*

$$\tilde{J}_c = \bar{J}_c. \tag{34}$$

Proof We first consider the form of the system (15) under the assumption that the plant can be characterized purely by annihilation operators. This essentially implies that the plant is a passive quantum system. A quantum system (1), (2) is characterized by annihilation operators only when $F_2, G_2, H_2, K_2 = 0$.

Then, the equations for the annihilation operators in (15) take the form:

$$\begin{aligned} da &= F_1adt + G_1d\mathcal{A}; \\ d\mathcal{Y} &= H_1adt + K_1d\mathcal{A}. \end{aligned} \tag{35}$$

The corresponding equations for the creation operators are then:

$$\begin{aligned} da^* &= F_1^\# a^*dt + G_1^\# d\mathcal{A}^\#; \\ d\mathcal{Y}^\# &= H_1^\# a^*dt + K_1^\# d\mathcal{A}^\#, \end{aligned} \tag{36}$$

Hence, the plant is described by (15) where:

$$\begin{aligned} F &= \begin{bmatrix} F_1 & 0 \\ 0 & F_1^\# \end{bmatrix}, G = \begin{bmatrix} G_1 & 0 \\ 0 & G_1^\# \end{bmatrix}, \\ H &= \begin{bmatrix} H_1 & 0 \\ 0 & H_1^\# \end{bmatrix}, K = \begin{bmatrix} K_1 & 0 \\ 0 & K_1^\# \end{bmatrix}. \end{aligned} \tag{37}$$

Next, we use the assumption that the plant is physically realizable. Then, by applying Theorem 2.2 to (35), there exists a matrix $\Theta_1 > 0$, such that:

$$\begin{aligned} F_1\Theta_1 + \Theta_1F_1^\dagger + G_1G_1^\dagger &= 0, \\ G_1 &= -\Theta_1H_1^\dagger, \\ K_1 &= I. \end{aligned} \tag{38}$$

Hence,

$$\begin{aligned} F_1^\#\Theta_1^\# + \Theta_1^\# F_1^T + G_1^\# G_1^T &= 0, \\ G_1^\# &= -\Theta_1^\# H_1^T, \\ K_1^\# &= I. \end{aligned} \tag{39}$$

Combining (38) and (39), we get:

$$\begin{aligned} F\Theta + \Theta F^\dagger + GG^\dagger &= 0, \\ G &= -\Theta H^\dagger, \\ K &= I, \end{aligned} \tag{40}$$

where $\Theta = \begin{bmatrix} \Theta_1 & 0 \\ 0 & \Theta_1^\# \end{bmatrix} > 0$.

Substituting $\bar{P}_e = \Theta$ in the left-hand side of the Riccati equation (22), we get:

$$F\Theta + \Theta F^\dagger + GG^\dagger - (G + \Theta H^\dagger)L^\dagger L(G + \Theta H^\dagger)^\dagger,$$

which is clearly zero, owing to (40). Thus, Θ satisfies (22).

Also, it follows from the above that $\tilde{P}_e = \begin{bmatrix} \Theta & 0 \\ 0 & P_3 \end{bmatrix}$ satisfies (20) for the case of coherent-classical estimation, since (33) is satisfied, owing to (40), by substituting $P_1 = \Theta$ and $P_2 = 0$ to yield:

$$\begin{aligned} &F\Theta + \Theta F^\dagger + GG^\dagger - (G + \Theta H^\dagger)L^\dagger L(G + \Theta H^\dagger)^\dagger = 0, \\ &(G + \Theta H^\dagger)G_c^\dagger - (G + \Theta H^\dagger)L^\dagger L(G_c + P_3 H_c^\dagger)^\dagger = 0, \\ &F_c P_3 + P_3 F_c^\dagger + G_c G_c^\dagger - (G_c + P_3 H_c^\dagger)L^\dagger L(G_c + P_3 H_c^\dagger)^\dagger = 0, \end{aligned} \tag{41}$$

where $P_3 > 0$ is simply the error-covariance of the purely-classical estimation of the coherent controller alone.

Thus, we get $\bar{J}_c = \tilde{J}_c = C\Theta C^\dagger$. ∎

Remark 1 *We note that the Kalman gain of the purely-classical estimator is zero when $\bar{P}_e = \Theta$. This implies that the Kalman state estimate is independent of the measurement. This is consistent with Corollary 1 of Ref. [17], which states that for a physically realizable annihilation operator quantum system with only quantum noise inputs, any output field contains no information about the internal variables of the system.*

Remark 2 *Theorem 4.1 implies that a coherent-classical estimation of a physically realizable quantum plant, that can be described purely by annihilation operators, performs exactly identical to, and no better than, a purely-classical estimation of the plant. This is so because the output field of the quantum plant, as observed above, contains no information about the internal variables of the plant and, therefore, serves simply as a quantum white noise input for the coherent controller.*

5 Dynamic Squeezer Examples

In this section, we consider examples involving dynamic squeezers. First, we present an example to illustrate Theorem 4.1.

Let us consider the quantum plant to be described by the QSDEs:

$$\begin{aligned} \begin{bmatrix} da(t) \\ da(t)^* \end{bmatrix} &= \begin{bmatrix} -\frac{\gamma}{2} & -\chi \\ -\chi^* & -\frac{\gamma}{2} \end{bmatrix} \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix} dt - \sqrt{\kappa} \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \end{bmatrix}; \\ \begin{bmatrix} d\mathcal{Y}(t) \\ d\mathcal{Y}(t)^\# \end{bmatrix} &= \sqrt{\kappa} \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix} dt + \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \end{bmatrix}; \\ z &= \begin{bmatrix} 0.2 & -0.2 \end{bmatrix} \begin{bmatrix} a \\ a^* \end{bmatrix}. \end{aligned} \tag{42}$$

Here, we choose $\gamma = 4$, $\kappa = 4$ and $\chi = 0$. Note that this system is physically realizable, since $\gamma = \kappa$, and is annihilation operator only, since $\chi = 0$. In fact, this system corresponds to a passive optical cavity. The matrices corresponding to the system (15) are:

$$F = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, G = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

$$K = I, C = \begin{bmatrix} 0.2 & -0.2 \end{bmatrix}.$$

We then calculate the optimal classical-only state estimator and the error $\bar{J}_c$ of (23) for this system using the standard Kalman filter equations corresponding to homodyne detector angles varying from $\theta = 0^\circ$ to $\theta = 180^\circ$.

We next consider the case where a dynamic squeezer is used as the coherent controller in a coherent-classical estimation scheme. In this case, the coherent controller (24) is described by the QSDEs:

$$\begin{aligned} \begin{bmatrix} da(t) \\ da(t)^* \end{bmatrix} &= \begin{bmatrix} -\frac{\gamma}{2} & -\chi \\ -\chi^* & -\frac{\gamma}{2} \end{bmatrix} \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix} dt - \sqrt{\kappa} \begin{bmatrix} d\mathcal{Y}(t) \\ d\mathcal{Y}(t)^\# \end{bmatrix}; \\ \begin{bmatrix} d\tilde{\mathcal{Y}}(t) \\ d\tilde{\mathcal{Y}}(t)^\# \end{bmatrix} &= \sqrt{\kappa} \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix} dt + \begin{bmatrix} d\mathcal{Y}(t) \\ d\mathcal{Y}(t)^\# \end{bmatrix}. \end{aligned} \tag{43}$$

Here, we choose $\gamma = 16$, $\kappa = 16$ and $\chi = 2$, so that the system is physically realizable. The matrices corresponding to the system (24) are:

$$F_c = \begin{bmatrix} -8 & -2 \\ -2 & -8 \end{bmatrix}, G_c = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix},$$

$$H_c = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, K_c = I.$$

Then, the classical estimator for this case is calculated according to (27), (28), (20), (29) for the homodyne detector angle varying from $\theta = 0°$ to $\theta = 180°$. The resulting value of the cost $\tilde{J}_c$ in (31) alongwith the cost for the purely-classical estimator case is shown in Fig. 4.

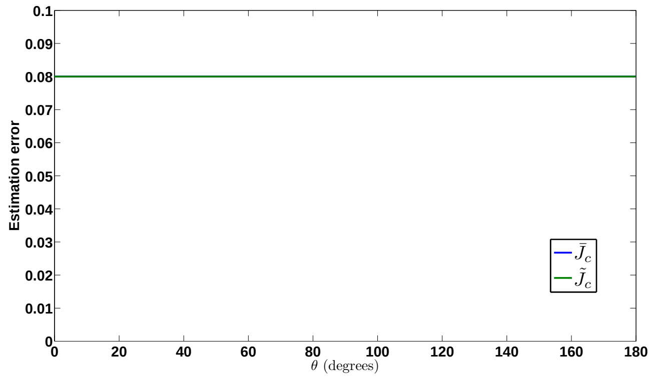

Fig. 4. Estimation error vs. homodyne detection angle θ in the case of an annihilation operator only plant.

It is clear from the figure that both the classical-only and coherent-classical estimators have the same estimation error cost for all homodyne angles. This illustrates Theorem 4.1.

Now, we shall consider an example where the controller is a physically realizable annihilation operator only system. However, the plant is physically realizable and has $\chi \neq 0$, i.e. it is not annihilation operator only.

In (42), we choose $\gamma = 4$, $\kappa = 4$ and $\chi = 0.5$. Note that this system is physically realizable, since $\gamma = \kappa$. Next, in (43), we choose $\gamma = 16$, $\kappa = 16$ and $\chi = 0$. Note that this system is also physically realizable, since $\gamma = \kappa$. Also, it is annihilation operator only, since $\chi = 0$. Fig. 5 then shows the comparison of the estimation error costs.

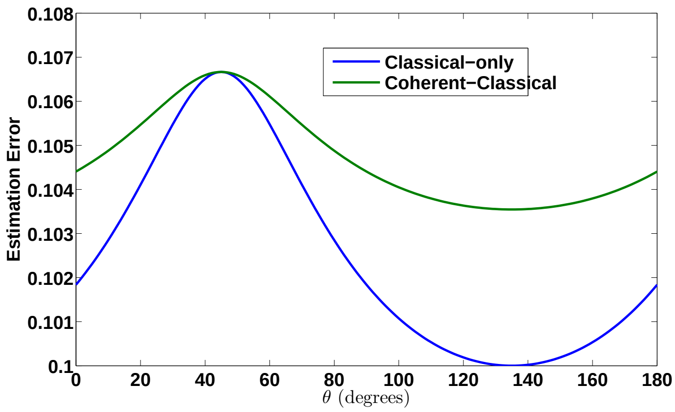

Fig. 5. Estimation error vs. homodyne detection angle θ in the case of an annihilation operator only controller.

From this figure, clearly the coherent-classical estimation error is larger than or equal to the purely-classical estimation error for all homodyne detector angles. In fact, we observed that the coherent-classical estimator can perform no better than the purely-classical estimator, when the coherent controller can be characterized by its annihilation operators only. This we present here as a conjecture.

Conjecture 5.1 *Consider a coherent-classical estimation scheme defined by (15), (24), (25) and (26), where the plant is physically realizable and the coherent controller is a physically realizable annihilation operator only system, with the estimation error cost $\tilde{J}_c$ defined in (31). Also, consider the corresponding purely-classical estimation scheme defined by (15), (16) and (17), such that the plant is physically realizable, with the estimation error cost $\bar{J}_c$ defined in (23). Then,*

$$\tilde{J}_c \geq \bar{J}_c. \tag{44}$$

Furthermore, we shall illustrate an example where both the plant and controller are physically realizable quantum systems with $\chi \neq 0$, i.e. none of them are annihilation operator only.

In (42), we choose $\gamma = 4$, $\kappa = 4$ and $\chi = 1$, and in (43), we choose $\gamma = 16$, $\kappa = 16$ and $\chi = 4$. Fig. 6 then shows the comparison of the estimation error costs.

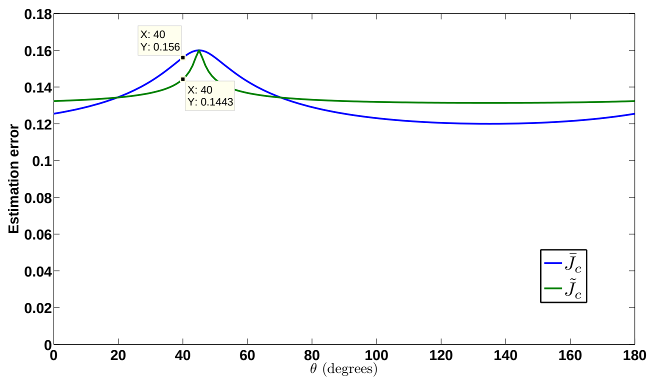

Fig. 6. Estimation error vs. homodyne detection angle θ in the case of squeezer plant and squeezer controller.

From this figure, we can see that the coherent-classical estimator can perform better than the purely-classical estimator, e.g. for a homodyne detector angle of $\theta = 40°$. It, however, appears that, for the best choice of homodyne detector angle, the classical-only estimator always outperforms the coherent-classical estimator. We present this too as a conjecture here.

Conjecture 5.2 *Consider a coherent-classical estimation scheme defined by (15), (24), (25) and (26) with an* *estimation error cost $\tilde{J}_c$ defined in (31). Also, consider the corresponding purely-classical estimation scheme defined by (15), (16) and (17) with an estimation error cost $\bar{J}_c$ defined in (23). Then, for the optimal choice of the homodyne angle given by θ_{opt}, we shall have:*

$$\tilde{J}_c(\theta_{opt}) \geq \bar{J}_c(\theta_{opt}). \tag{45}$$

6 Coherent-Classical Estimation with Feedback

In this section, we shall consider the case where there is quantum feedback from the coherent controller to the quantum plant [16]. For this purpose, the quantum plant is assumed to have an additional control input $\mathcal{U}$ as depicted in Fig. 7. The plant (15) then takes the following form:

$$\begin{aligned}
\begin{bmatrix} da(t) \\ da(t)^{\#} \end{bmatrix} &= F \begin{bmatrix} a(t) \\ a(t)^{\#} \end{bmatrix} dt + \begin{bmatrix} G_1 & G_2 \end{bmatrix} \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^{\#} \\ d\mathcal{U}(t) \\ d\mathcal{U}(t)^{\#} \end{bmatrix}; \\
\begin{bmatrix} d\mathcal{Y}(t) \\ d\mathcal{Y}(t)^{\#} \end{bmatrix} &= H \begin{bmatrix} a(t) \\ a(t)^{\#} \end{bmatrix} dt + \begin{bmatrix} K & 0 \end{bmatrix} \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^{\#} \\ d\mathcal{U}(t) \\ d\mathcal{U}(t)^{\#} \end{bmatrix}; \\
z &= C \begin{bmatrix} a(t) \\ a(t)^{\#} \end{bmatrix}.
\end{aligned} \tag{46}$$

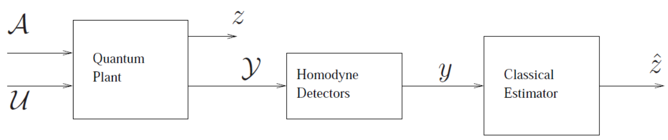

Fig. 7. Modified schematic diagram of purely-classical estimation.

The optimal purely-classical estimator is then obtained from the solution of an algebraic Riccati equation of the form (22), that yields:

$$\begin{aligned} F\bar{P}_e + \bar{P}_e F^{\dagger} + G_1 G_1^{\dagger} + G_2 G_2^{\dagger} - (G_1 + \bar{P}_e H^{\dagger}) \\ \times L^{\dagger} L (G_1 + \bar{P}_e H^{\dagger})^{\dagger} = 0, \end{aligned} \tag{47}$$

where we have assumed $K = I$ as before. The estimation error is then given by the cost (23).

The coherent controller here would have an additional output that is fed back to the control input of the quantum plant, as depicted in Fig. 8. The coherent controller in this case is defined as follows [16]:

$$\begin{aligned}
\begin{bmatrix} da_c(t) \\ da_c(t)^{\#} \end{bmatrix} &= F_c \begin{bmatrix} a_c(t) \\ a_c(t)^{\#} \end{bmatrix} dt + \begin{bmatrix} G_{c1} & G_{c2} \end{bmatrix} \begin{bmatrix} d\tilde{\mathcal{A}}(t) \\ d\tilde{\mathcal{A}}(t)^{\#} \\ d\mathcal{Y}(t) \\ d\mathcal{Y}(t)^{\#} \end{bmatrix}; \\
\begin{bmatrix} d\tilde{\mathcal{Y}}(t) \\ d\tilde{\mathcal{Y}}(t)^{\#} \\ d\mathcal{U}(t) \\ d\mathcal{U}(t)^{\#} \end{bmatrix} &= \begin{bmatrix} \tilde{H}_c \\ H_c \end{bmatrix} \begin{bmatrix} a_c(t) \\ a_c(t)^{\#} \end{bmatrix} dt + \begin{bmatrix} \tilde{K}_{c1} & \tilde{K}_{c2} \\ K_{c1} & K_{c2} \end{bmatrix} \begin{bmatrix} d\tilde{\mathcal{A}}(t) \\ d\tilde{\mathcal{A}}(t)^{\#} \\ d\mathcal{Y}(t) \\ d\mathcal{Y}(t)^{\#} \end{bmatrix}.
\end{aligned} \tag{48}$$

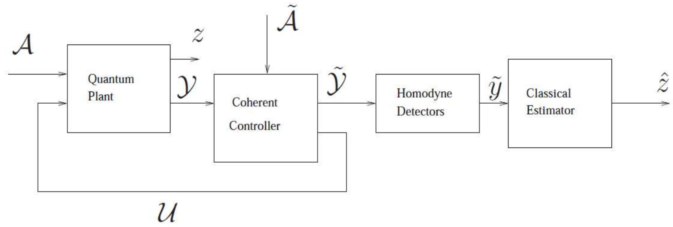

Fig. 8. Schematic diagram of coherent-classical estimation with coherent feedback.

The quantum plant (46) and the coherent controller (48) can be combined to yield an augmented quantum linear system [16]:

$$\begin{aligned}
\begin{bmatrix} da \\ da^{\#} \\ da_c \\ da_c^{\#} \end{bmatrix} &= \begin{bmatrix} F + G_2 K_{c2} H & G_2 H_c \\ G_{c2} H & F_c \end{bmatrix} \begin{bmatrix} a \\ a^{\#} \\ a_c \\ a_c^{\#} \end{bmatrix} dt \\
&\quad + \begin{bmatrix} G_1 + G_2 K_{c2} K & G_2 K_{c1} \\ G_{c2} K & G_{c1} \end{bmatrix} \begin{bmatrix} d\mathcal{A} \\ d\mathcal{A}^{\#} \\ d\tilde{\mathcal{A}} \\ d\tilde{\mathcal{A}}^{\#} \end{bmatrix}; \\
\begin{bmatrix} d\tilde{\mathcal{Y}} \\ d\tilde{\mathcal{Y}}^{\#} \end{bmatrix} &= \begin{bmatrix} \tilde{K}_{c2} H & \tilde{H}_c \end{bmatrix} \begin{bmatrix} a \\ a^{\#} \\ a_c \\ a_c^{\#} \end{bmatrix} dt + \begin{bmatrix} \tilde{K}_{c2} K & \tilde{K}_{c1} \end{bmatrix} \begin{bmatrix} d\mathcal{A} \\ d\mathcal{A}^{\#} \\ d\tilde{\mathcal{A}} \\ d\tilde{\mathcal{A}}^{\#} \end{bmatrix}.
\end{aligned} \tag{49}$$

The optimal coherent-classical estimator is then obtained from the solution $\tilde{P}_e$ (given by (30)) to an algebraic Riccati equation of the form (20), where

$$
\begin{aligned}
F_a &= \begin{bmatrix} F + G_2 K_{c2} H & G_2 H_c \\ G_{c2} H & F_c \end{bmatrix}, \\
G_a &= \begin{bmatrix} G_1 + G_2 K_{c2} K & G_2 K_{c1} \\ G_{c2} K & G_{c1} \end{bmatrix}, \qquad (50) \\
H_a &= \begin{bmatrix} \tilde{K}_{c2} H & \tilde{H}_c \end{bmatrix}, \qquad K_a = \begin{bmatrix} \tilde{K}_{c2} K & \tilde{K}_{c1} \end{bmatrix},
\end{aligned}
$$

and $\tilde{L}_1$, $\tilde{L}_2$ and L as in (28). Here, for the coherent controller to be physically realizable, we would have:

$$
\begin{bmatrix} \tilde{K}_{c1} & \tilde{K}_{c2} \\ K_{c1} & K_{c2} \end{bmatrix} = I,
$$

which implies $\tilde{K}_{c1} = K_{c2} = I$ and $K_{c1} = \tilde{K}_{c2} = 0$. The estimation error is then given by the cost (31).

Also, we calculate:

$$
\begin{aligned}
G_a G_a^\dagger &= \begin{bmatrix} (G_1+G_2)(G_1+G_2)^\dagger & (G_1+G_2)G_{c2}^\dagger \\ G_{c2}(G_1+G_2)^\dagger & G_{c1}G_{c1}^\dagger + G_{c2}G_{c2}^\dagger \end{bmatrix}, \\
G_a K_a^\dagger + \tilde{P}_e H_a^\dagger &= \begin{bmatrix} P_2 \tilde{H}_c^\dagger \\ G_{c1} + P_3 \tilde{H}_c^\dagger \end{bmatrix}, \\
F_a \tilde{P}_e &= \begin{bmatrix} F P_1 + G_2 H P_1 & F P_2 + G_2 H P_2 + G_2 H_c P_3 \\ G_{c2} H P_1 + F_c P_2^\dagger & G_{c2} H P_2 + F_c P_3 \end{bmatrix}, \\
\tilde{P}_e F_a^\dagger &= \begin{bmatrix} P_1 F^\dagger + P_1 H^\dagger G_2^\dagger + P_2 H_c^\dagger G_2^\dagger & P_1 H^\dagger G_{c2}^\dagger + P_2 F_c^\dagger \\ P_2^\dagger F^\dagger + P_2^\dagger H^\dagger G_2^\dagger + P_3 H_c^\dagger G_2^\dagger & P_2^\dagger H^\dagger G_{c2}^\dagger + P_3 F_c^\dagger \end{bmatrix}.
\end{aligned}
\qquad (51)
$$

Thus, upon expanding (20), we get the following set of equations:

$$
\begin{aligned}
& F P_1 + P_1 F^\dagger + G_1 G_1^\dagger + G_2 G_2^\dagger + G_2 (H P_1 + G_1^\dagger) \\
& + (P_1 H^\dagger + G_1) G_2^\dagger + G_2 H_c P_2^\dagger + P_2 H_c^\dagger G_2^\dagger \\
& - P_2 \tilde{H}_c^\dagger L^\dagger L \tilde{H}_c P_2^\dagger = 0, \\
& F P_2 + P_2 F_c^\dagger + (P_1 H^\dagger + G_1) G_{c2}^\dagger + G_2 (H_c P_3 + G_{c2}^\dagger) \qquad (52) \\
& + G_2 H P_2 - P_2 \tilde{H}_c^\dagger L^\dagger L (G_{c1} + P_3 \tilde{H}_c^\dagger)^\dagger = 0, \\
& F_3 P_3 + P_3 F_c^\dagger + G_{c1} G_{c1}^\dagger + G_{c2} G_{c2}^\dagger + G_{c2} H P_2 \\
& + P_2^\dagger H^\dagger G_{c2}^\dagger - (G_{c1} + P_3 \tilde{H}_c^\dagger) L^\dagger L (G_{c1} + P_3 \tilde{H}_c^\dagger)^\dagger = 0.
\end{aligned}
$$

Theorem 6.1 *Consider a coherent-classical estimation scheme defined by (46), (48), (25) and (26), such that both the plant and the coherent controller are physically realizable annihilation operator only systems, with the estimation error cost $\tilde{J}_c$ defined in (31). Also, consider the corresponding purely-classical estimation scheme defined by (46), (16) and (17), such that the plant is a physically realizable annihilation operator only system, with the estimation error cost $\bar{J}_c$ defined in (23). Then,*

$$
\tilde{J}_c = \bar{J}_c. \qquad (53)
$$

Proof The plant (46) may be augmented to account for an unused output $\bar{\mathcal{Y}}$ to recast the QSDE's in the desired form, that lends itself appropriately to the physical realizability treatment, as follows:

$$
\begin{aligned}
\begin{bmatrix} da(t) \\ da(t)^\# \end{bmatrix} &= F \begin{bmatrix} a(t) \\ a(t)^\# \end{bmatrix} dt + \begin{bmatrix} G_1 & G_2 \end{bmatrix} \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \\ d\mathcal{U}(t) \\ d\mathcal{U}(t)^\# \end{bmatrix}; \\
\begin{bmatrix} d\mathcal{Y}(t) \\ d\mathcal{Y}(t)^\# \\ d\bar{\mathcal{Y}}(t) \\ d\bar{\mathcal{Y}}(t)^\# \end{bmatrix} &= \begin{bmatrix} H \\ \bar{H} \end{bmatrix} \begin{bmatrix} a(t) \\ a(t)^\# \end{bmatrix} dt + \begin{bmatrix} K & 0 \\ 0 & \bar{K} \end{bmatrix} \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \\ d\mathcal{U}(t) \\ d\mathcal{U}(t)^\# \end{bmatrix}; \\
z &= C \begin{bmatrix} a(t) \\ a(t)^\# \end{bmatrix},
\end{aligned}
\qquad (54)
$$

where $\bar{K} = I$ too for the plant to be physically realizable. Additionally, using the same arguments as in the proof for Theorem (4.1), we must have:

$$
\begin{aligned}
F\Theta + \Theta F^\dagger + G_1 G_1^\dagger + G_2 G_2^\dagger &= 0, \\
G_1 &= -\Theta H^\dagger, \qquad (55) \\
G_2 &= -\Theta \bar{H}^\dagger,
\end{aligned}
$$

for the annihilation operator only plant with the commutation matrix $\Theta > 0$ to be physically realizable.

Similarly, if the coherent controller (48) is an annihilation operator only system, we must have the following for it to be physically realizable:

$$
\begin{aligned}
F_c \Theta_c + \Theta_c F_c^\dagger + G_{c1} G_{c1}^\dagger + G_{c2} G_{c2}^\dagger &= 0, \\
G_{c1} &= -\Theta_c \tilde{H}_c^\dagger, \qquad (56) \\
G_{c2} &= -\Theta_c H_c^\dagger,
\end{aligned}
$$

where $\Theta_c > 0$ is the controller's commutation matrix.

Clearly, $\bar{P}_e = \Theta$ satisfies the Riccati equation (47), owing to (55), for the purely-classical estimation case. Moreover, it follows from (55) and (56), that $\tilde{P}_e = \begin{bmatrix} \Theta & 0 \\ 0 & \Theta_c \end{bmatrix}$ satisfies (52), upon substituting $P_1 = \Theta$, $P_2 = 0$ and $P_3 = \Theta_c$, and, therefore, in turn satisfies (20) for the coherent classical estimation case considered here.

Thus, we get $\bar{J}_c = \tilde{J}_c = C\Theta C^\dagger$. ∎

Remark 3 *Theorem 6.1 implies that a coherent-classical estimation scheme involving coherent feedback, where both the plant and controller are physically realizable annihilation operator only quantum systems, performs exactly identical to, and no better than, a purely-classical estimation of the plant. Note that in addition to $P_2 = 0$, we need to have both $P_1 = \Theta$ and $P_3 = \Theta_c$, and not either of them, for the coherent-classical scheme to be equivalent to the classical-only scheme.*

Remark 4 *In the case of coherent-classical estimation not involving coherent feedback, we had observed that such a scheme cannot outperform purely-classical estimation, when either the plant or the controller is an annihilation operator only system. By contrast, a coherent-classical estimation scheme involving coherent feedback cannot perform better than classical-only estimation, only when both the plant and the controller can be purely described by annihilation operators. In other words, it is possible to obtain improvement with the coherent-classical estimation scheme involving coherent feedback over the purely-classical estimation scheme, when either or both of the plant and the controller has non-zero squeezing, as we shall demonstrate with examples next.*

7 More Dynamic Squeezer Examples

In this section, we present examples involving dynamic squeezers for the case of coherent-classical estimation involving coherent feedback. First, we consider an example to illustrate Theorem 6.1.

In this case, the quantum plant (42) takes the form:

$$\begin{aligned}
\begin{bmatrix} da(t) \\ da(t)^* \end{bmatrix} &= \begin{bmatrix} -\frac{\gamma}{2} & -\chi \\ -\chi^* & -\frac{\gamma}{2} \end{bmatrix} \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix} dt - \sqrt{\kappa_1} \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \end{bmatrix} \\
&\quad - \sqrt{\kappa_2} \begin{bmatrix} d\mathcal{U}(t) \\ d\mathcal{U}(t)^\# \end{bmatrix}; \\
\begin{bmatrix} d\mathcal{Y}(t) \\ d\mathcal{Y}(t)^\# \end{bmatrix} &= \sqrt{\kappa} \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix} dt + \begin{bmatrix} d\mathcal{A}(t) \\ d\mathcal{A}(t)^\# \end{bmatrix}; \\
z &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} a \\ a^* \end{bmatrix}.
\end{aligned} \tag{57}$$

Here, we choose $\gamma = 4$, $\kappa_1 = \kappa_2 = 2$ and $\chi = 0$. Note that this system is physically realizable, since $\gamma = \kappa_1 + \kappa_2$, and is annihilation operator only, since $\chi = 0$. The matrices corresponding to the system (46) are:

$$F = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, G_1 = \begin{bmatrix} -1.4142 & 0 \\ 0 & -1.4142 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} -1.4142 & 0 \\ 0 & -1.4142 \end{bmatrix}, H = \begin{bmatrix} 1.4142 & 0 \\ 0 & 1.4142 \end{bmatrix},$$

$$K = I, C = \begin{bmatrix} 0.7071 & -0.7071 \end{bmatrix}.$$

We then calculate the optimal classical-only state estimator and the error $\bar{J}_c$ of (23) for this system using the standard Kalman filter equations corresponding to homodyne detector angles varying from $\theta = 0°$ to $\theta = 180°$.

The coherent controller (43) in this case takes the form:

$$\begin{aligned}
\begin{bmatrix} da(t) \\ da(t)^* \end{bmatrix} &= \begin{bmatrix} -\frac{\gamma}{2} & -\chi \\ -\chi^* & -\frac{\gamma}{2} \end{bmatrix} \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix} dt - \sqrt{\kappa_1} \begin{bmatrix} d\tilde{\mathcal{A}}(t) \\ d\tilde{\mathcal{A}}(t)^\# \end{bmatrix} \\
&\quad - \sqrt{\kappa_2} \begin{bmatrix} d\mathcal{Y}(t) \\ d\mathcal{Y}(t)^\# \end{bmatrix}; \\
\begin{bmatrix} d\tilde{\mathcal{Y}}(t) \\ d\tilde{\mathcal{Y}}(t)^\# \end{bmatrix} &= \sqrt{\kappa_1} \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix} dt + \begin{bmatrix} d\tilde{\mathcal{A}}(t) \\ d\tilde{\mathcal{A}}(t)^\# \end{bmatrix}; \\
\begin{bmatrix} d\mathcal{U}(t) \\ d\mathcal{U}(t)^\# \end{bmatrix} &= \sqrt{\kappa_2} \begin{bmatrix} a(t) \\ a(t)^* \end{bmatrix} dt + \begin{bmatrix} d\mathcal{Y}(t) \\ d\mathcal{Y}(t)^\# \end{bmatrix}.
\end{aligned} \tag{58}$$

Here, we choose $\gamma = 16$, $\kappa_1 = \kappa_2 = 8$ and $\chi = 0$, so that it is a physically realizable annihilation operator only system. The matrices corresponding to the system (48) are:

$$F_c = \begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix}, G_{c1} = \begin{bmatrix} -2.8284 & 0 \\ 0 & -2.8284 \end{bmatrix},$$

$$G_{c2} = \begin{bmatrix} -2.8284 & 0 \\ 0 & -2.8284 \end{bmatrix}, \tilde{H}_c = \begin{bmatrix} 2.8284 & 0 \\ 0 & 2.8284 \end{bmatrix},$$

$$\tilde{K}_{c1} = I, \tilde{K}_{c2} = 0, H_c = \begin{bmatrix} 2.8284 & 0 \\ 0 & 2.8284 \end{bmatrix},$$

$$K_{c1} = 0, K_{c2} = I.$$

Then, the classical estimator for this case is calculated according to (49), (50), (20), (29) for the homodyne detector angle varying from $\theta = 0°$ to $\theta = 180°$. The resulting value of the cost $\tilde{J}_c$ in (31) alongwith the cost for the purely-classical estimator case is shown in Fig. 9.

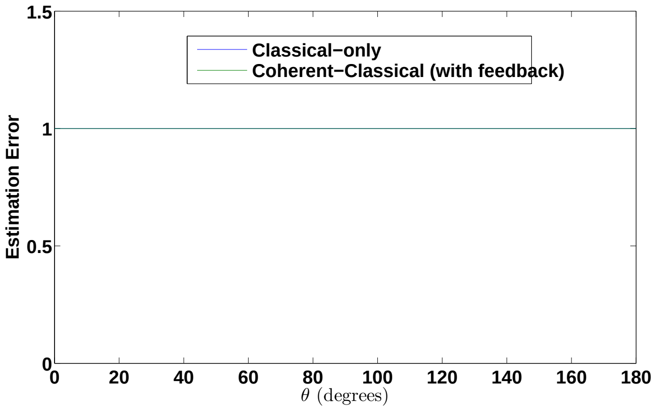

Fig. 9. Feedback: Estimation error vs. homodyne detection angle θ in the case of annihilation operator plant and controller systems.

It is clear from the figure that both the classical-only and coherent-classical estimators have the same estimation error cost for all homodyne angles. This illustrates Theorem 6.1.

We now want to demonstrate with examples that when either or both of the plant and the controller is an annihilation operator quantum system, the coherent-classical estimation scheme with coherent feedback can provide improvement over purely-classical estimation of the plant. We first consider an example where the controller is a physically realizable annihilation operator only system. However, the plant is physically realizable and has $\chi \neq 0$, i.e. it is not annihilation operator only.

In (57), we choose $\gamma = 4$, $\kappa_1 = \kappa_2 = 2$ and $\chi = 0.5$, and in (58), we choose $\gamma = 16$, $\kappa_1 = \kappa_2 = 8$ and $\chi = 0$. Fig. 10 then shows the comparison of the estimation error costs. Clearly, the coherent-classical estimation error is less than the purely-classical estimation error for all homodyne detector angles.

Furthermore, we consider the case where the plant is an annihilation operator only system, but the coherent controller is not, i.e. it has $\chi \neq 0$. In (57), we choose $\gamma = 4$, $\kappa_1 = \kappa_2 = 2$ and $\chi = 0$. Also, in (58), we choose $\gamma = 16$, $\kappa_1 = \kappa_2 = 8$ and $\chi = -0.5$. Fig. 11 then shows the comparison of the estimation error costs. Clearly, the coherent-classical estimation error is again less than the purely-classical estimation error for all homodyne detector angles.

Moreover, we illustrate an example where both the plant and the controller are physically realizable quantum systems with $\chi \neq 0$, i.e. none of them are annihilation operator only. In (57), we choose $\gamma = 4$, $\kappa_1 = \kappa_2 = 2$ and $\chi = 1$, and in (58), we choose $\gamma = 16$, $\kappa_1 = \kappa_2 = 8$ and $\chi = -0.5$. Fig. 12 then shows the comparison of the estimation error costs. Clearly, in this case too the coherent-classical estimation error is less than the purely-classical estimation error for all homodyne detector angles.

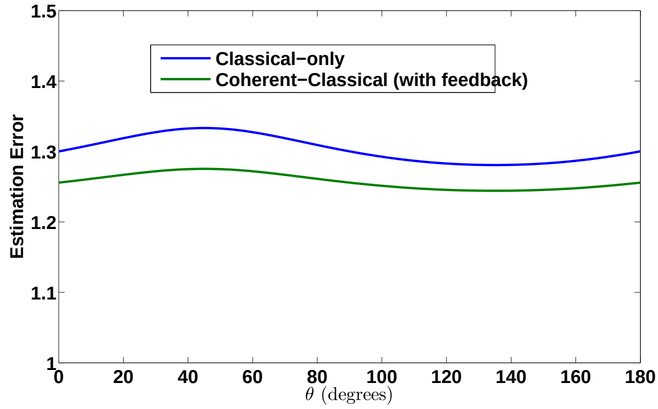

Fig. 10. Feedback: Estimation error vs. homodyne detection angle θ in the case of an annihilation operator only controller.

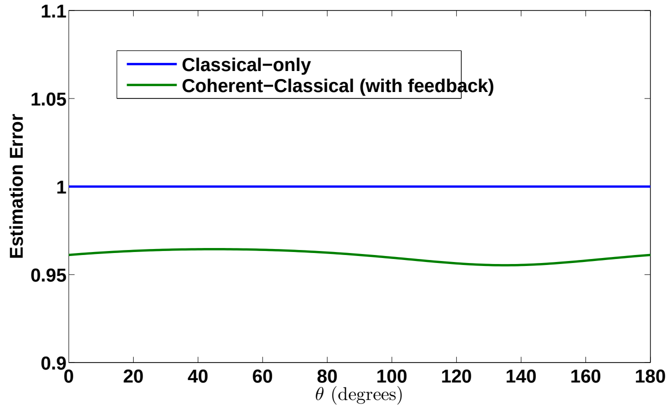

Fig. 11. Feedback: Estimation error vs. homodyne detection angle θ in the case of annihilation operator plant.

When both the plant and the controller are physically realizable quantum systems with $\chi \neq 0$, i.e. none of them are annihilation operator only, it is possible that the coherent-classical estimates are better than the corresponding purely-classical estimates for only certain homodyne angles. An example is shown in Fig. 13, where we used $\gamma = 4$, $\kappa_1 = \kappa_2 = 2$ and $\chi = 0.5$ in (57) and $\gamma = 16$, $\kappa_1 = \kappa_2 = 8$ and $\chi = 0.5$ in (58).

However, we observe that if there is any improvement observed with the coherent-classical estimation (with feedback) over purely-classical estimation, the coherent-classical scheme is always superior to classical-only estimation for the best choice of the homodyne detector

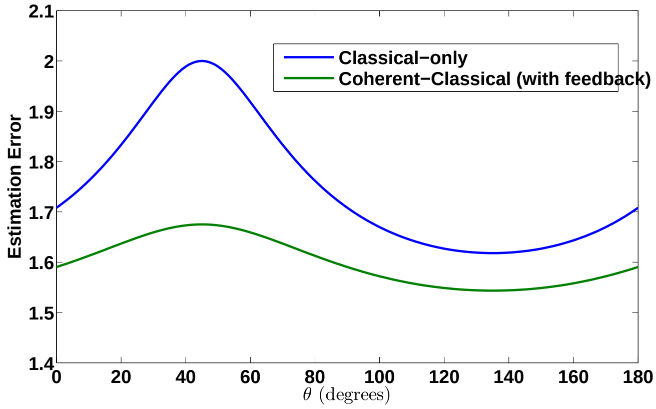

Fig. 12. Feedback: Estimation error vs. homodyne detection angle θ in the case of squeezer plant and squeezer controller.

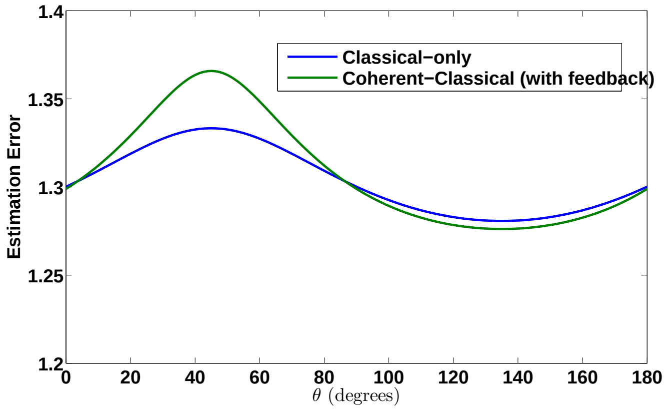

Fig. 13. Feedback: Estimation error vs. homodyne detection angle θ in the case of squeezer plant and squeezer controller, where it is possible to get better coherent-classical estimates than purely-classical estimates only for certain homodyne detector angles.

angle. This we propose as a conjecture here, that turns out to be just the opposite of conjecture 5.2. This is owing to the coherent feedback involved here as opposed to earlier.

Conjecture 7.1 *Consider a coherent-classical estimation scheme defined by (46), (48), (25) and (26) with an estimation error cost* $\tilde{J}_c$ *defined in (31). Also, consider the corresponding purely-classical estimation scheme defined by (46), (16) and (17) with an estimation error cost* $\bar{J}_c$ *defined in (23). Then, if there exists a homodyne angle* θ_i *for which* $\tilde{J}_c(\theta_i) \leq \bar{J}_c(\theta_i)$*, the following always holds for the optimal choice* θ_{opt} *of the homodyne angle:*

$$\tilde{J}_c(\theta_{opt}) \leq \bar{J}_c(\theta_{opt}). \tag{59}$$

8 Conclusion

In this paper, we have demonstrated that a coherent-classical estimation scheme without coherent feedback cannot provide better estimates than a classical-only estimation scheme, when either of the plant and the controller is assumed to be a physically realizable annihilation operator only quantum system. Otherwise, it is possible to get better estimation accuracy for certain homodyne detector angles; however, for the best choice of the homodyne angle, we observed that purely-classical estimation is always better than coherent-classical estimation (without feedback). On the other hand, a coherent-classical estimation scheme involving coherent feedback can be better than the corresponding purely-classical estimation scheme, only if not both the plant and the controller are assumed to be annihilation operator only systems. Moreover, in all cases where the coherent-classical estimation scheme with coherent feedback provides with improvement over purely-classical estimation, we observed that classical-only estimation is always worse than the coherent-classical estimation for the best choice of the homodyne angle, owing to the coherent feedback involved.

As part of future work, formal proofs may be derived for the conjectures proposed in this paper. Furthermore, a comparison study may be carried out between the two flavours of the coherent-classical estimation scheme, i.e. with and without coherent feedback, for the same nominal model of the plant to find possible interesting insights.

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