

A Direct Coupling Coherent Quantum Observer for a Qubit, including Observer Measurements

Ian R. Petersen and Elanor H. Huntington

Abstract—This paper proposes a direct coupling coherent quantum observer for a quantum plant which consists of a two level quantum system. The quantum observer, which is a quantum harmonic oscillator, includes homodyne detection measurements. It is shown that the observer can be designed so that it does not affect the quantum variable of interest in the quantum plant and that measured output converges in a given sense to the plant variable of interest. Also, the plant variable of interest-observer system can be described by a set of linear quantum stochastic differential equations. A minimum variance unbiased estimator form of the Kalman filter is derived for linear quantum systems and applied to the direct coupled coherent quantum observer.

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I. INTRODUCTION

A number of papers have recently considered the problem of constructing a coherent quantum observer for a quantum system; e.g., see [1]–[3]. In the coherent quantum observer problem, a quantum plant is coupled to a quantum observer which is also a quantum system. The quantum observer is constructed to be a physically realizable quantum system so that the system variables of the quantum observer converge in some suitable sense to the variables of interest for the quantum plant.

The papers [4]–[7] considered the problem of constructing a direct coupling quantum observer for a given quantum system. In particular, the paper [5] considered the case in which the quantum plant was a two level system and the quantum observer was a linear quantum harmonic oscillator. Also, the papers [8], [9] considered the problem of whether such direct coupling coherent observers could be experimentally implemented. In addition, the paper [10] considered with the direct coupling coherent observer of [4] could be experimentally implemented in an experiment which included homodyne detection measurements of the observer.

In this paper, we build on the results of [5] and [10] to consider the case in which the quantum plant is a two level system and the quantum observer is a linear quantum harmonic oscillator subject to measurements using homodyne detection. Similar convergence results for the quantum observer as obtained in [10] are obtained in this case. Also,

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as in [5], the plant observer system considering only the plant variable of interest is described by a set of linear quantum stochastic differential equations (QSDEs) in spite of the fact that finite level systems are normally described in terms of bilinear QSDEs; e.g., see [11]. However, in this case, measurements are available from the quantum observer. This means that we can apply a version of the Kalman filter to the linear plant observer QSDEs.

The paper develops a notion of a minimum variance unbiased estimator for a general set of linear QSDEs, building on the fact that the classical Kalman Filter can be regarded as the minimum variance unbiased estimator even in the case of non-Gaussian noises and initial conditions; e.g., see [12], [13]. The equations for this estimator are developed for the general case and then applied to the particular case of the plant observer system. This provides a numerically straightforward way of estimating the variable of interest for the qubit system when using homodyne detection measurements.

Some proofs and details have been omitted from this version of the paper due to page limitations. The full details and proofs can be found in the archive version of the paper [14].

II. DIRECT COUPLING COHERENT QUANTUM OBSERVER WITH OBSERVER MEASUREMENT

We first consider the dynamics of a single qubit spin system, which will correspond to the quantum plant; see also [11]. The quantum mechanical behavior of the system is described in terms of the system *observables* which are self-adjoint operators on the complex Hilbert space $\mathfrak{H}_p = \mathbb{C}^2$. The commutator of two scalar operators x and y in \mathfrak{H}_p is defined as $[x, y] = xy - yx$. Also, for a vector of operators x in \mathfrak{H}_p , the commutator of x and a scalar operator y in \mathfrak{H}_p is the vector of operators $[x, y] = xy - yx$, and the commutator of x and its adjoint x^\dagger is the matrix of operators

$$[x, x^\dagger] \triangleq xx^\dagger - (x^\# x^T)^T,$$

where $x^\# \triangleq (x_1^* x_2^* \cdots x_n^*)^T$ and $*$ denotes the operator adjoint. In the case of complex vectors (matrices) $*$ denotes the complex conjugate while † denotes the conjugate transpose.

The vector of system variables for the single qubit spin system under consideration is

$$x_p = (x_1, x_2, x_3)^T \triangleq (\sigma_1, \sigma_2, \sigma_3),$$

where σ_1, σ_2 and σ_3 are spin operators. Here, x_p is a self-adjoint vector of operators; i.e., $x_p = x_p^\#$. In particular $x_p(0)$

is represented by the Pauli matrices; i.e.,

$$\begin{aligned} \sigma_1(0) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2(0) = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \\ \sigma_3(0) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

Products of the spin operators satisfy

$$\sigma_i \sigma_j = \delta_{ij} + \mathbf{i} \sum_k \epsilon_{ijk} \sigma_k. \quad (1)$$

It then follows that the commutation relations for the spin operators are

$$[\sigma_i, \sigma_j] = 2\mathbf{i} \sum_k \epsilon_{ijk} \sigma_k, \quad (2)$$

where δ_{ij} is the Kronecker delta and ϵ_{ijk} denotes the Levi-Civita tensor. The dynamics of the system variables x are determined by the system Hamiltonian which is a self-adjoint operator on \mathfrak{H}_p . The Hamiltonian is chosen to be linear in x_p ; i.e.,

$$\mathcal{H}_p = r_p^T x_p(0)$$

where $r_p \in \mathbb{R}^3$. The plant model is then given by the differential equation

$$\begin{aligned} \dot{x}_p(t) &= -\mathbf{i}[x_p(t), \mathcal{H}_p]; \\ &= A_p x_p(t); \quad x_p(0) = x_{0p}; \\ z_p(t) &= C_p x_p(t) \end{aligned} \quad (3)$$

where z_p denotes the system variable to be estimated by the observer and $C_p \in \mathbb{R}^{1 \times 3}$; e.g., see [11]. Also, $A_p \in \mathbb{R}^{3 \times 3}$. In order to obtain an expression for the matrix A_p in terms of r_p , we define the linear mapping $\Theta : \mathbb{C}^3 \rightarrow \mathbb{C}^{3 \times 3}$ as

$$\Theta(\beta) = \begin{pmatrix} 0 & \beta_3 & -\beta_2 \\ -\beta_3 & 0 & \beta_1 \\ \beta_2 & -\beta_1 & 0 \end{pmatrix}. \quad (4)$$

Then, it was shown in [11] that

$$x_p(t)x_p(t)^T = I + \mathbf{i}\Theta(x_p(t)).$$

Similarly, the commutation relations for the spin operators are written as

$$[x_p(t), x_p(t)^T] = 2\mathbf{i}\Theta(x_p(t)). \quad (5)$$

Also, it was shown in [11] that

$$-\mathbf{i}[x_p(t), r_p^T x_p(t)] = -2\Theta(r_p)x_p(t) \quad (6)$$

and hence $A_p = -2\Theta(r_p)$.

Note that a quantum system of this form will be physically realizable which means that the commutation relation (5) will hold for all times $t \geq 0$.

We now describe the linear quantum system which will correspond to the quantum observer; see also [15]–[19]. This system is described by QSDEs of the form

$$\begin{aligned} dx_o &= A_o x_o dt + B_o dw; \quad x_o(0) = x_{0o}; \\ dy_o &= C_o x_o dt + dw; \\ z_o &= K y_o \end{aligned} \quad (7)$$

where $dw = \begin{bmatrix} dQ \\ dP \end{bmatrix}$ is a 2×1 vector of quantum noises expressed in quadrature form corresponding to the input field for the observer and dy_o is the corresponding output field; e.g., see [15], [17]. The observer output z_o will be a real scalar quantity obtained by applying homodyne detection to the observer output field. $A_o \in \mathbb{R}^{2 \times 2}$, $B_o \in \mathbb{R}^{2 \times 2}$, $C_o \in \mathbb{R}^{2 \times 2}$.

Also, $x_o = \begin{bmatrix} q_o \\ p_o \end{bmatrix}$ is a vector of self-adjoint system variables corresponding to the observer position and momentum operators; e.g., see [15]. We assume that the plant variables commute with the observer variables. The system dynamics (7) are determined by the observer system Hamiltonian and coupling operators which are operators on the underlying Hilbert space for the observer. For the quantum observer under consideration, this Hamiltonian is a self-adjoint operator given by the quadratic form: $\mathcal{H}_o = \frac{1}{2}x_o(0)^T R_o x_o(0)$, where R_o is a real symmetric matrix. Also, the coupling operator L is defined by a matrix $W_o \in \mathbb{R}^{2 \times 2}$ so that

$$\begin{bmatrix} L + L^* \\ \frac{L-L^*}{i} \end{bmatrix} = W_o x_o. \quad (8)$$

Then, the corresponding matrices A_o , B_o and C_o in (7) are given by

$$A_o = 2JR_o + \frac{1}{2}W_o^T J W_o, \quad B_o = J W_o^T J, \quad C_o = W_o \quad (9)$$

where

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix};$$

e.g., see [15], [17]. Furthermore, we will assume that the quantum observer is coupled to the quantum plant as shown in Figure 1. We define a coupling Hamiltonian which defines

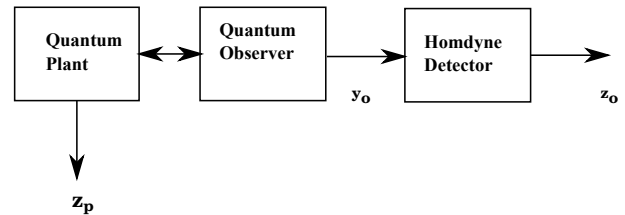


Fig. 1. Plant Observer System.

the coupling between the quantum plant and the quantum observer:

$$\mathcal{H}_c = x_p(0)^T R_c x_o(0).$$

The augmented quantum system consisting of the quantum plant and the quantum observer is then a quantum system described by the total Hamiltonian

$$\mathcal{H}_a = \mathcal{H}_p + \mathcal{H}_c + \mathcal{H}_o. \quad (10)$$

where the coupling operator L defined in (8).

Extending the approach used in [4], [5], we assume that $\mathcal{H}_p = 0$ and we can write

$$R_c = \alpha\beta^T, \quad (11)$$

$R_o = \omega_o I$, $W_o = \sqrt{\kappa} I$ where $\alpha \in \mathbb{R}^2$, $\beta \in \mathbb{R}^2$, $\omega_o > 0$ and $\kappa > 0$. In addition, we assume

$$\alpha = C_p^T. \quad (12)$$

Then, the total Hamiltonian (10) will be given by

$$\mathcal{H}_a = \alpha^T x_p(0) \beta^T x_o(0) + \frac{1}{2} x_o(0)^T R_o x_o(0)$$

since in this case the quantities $\alpha^T x_p(0)$ and $\beta^T x_o(0)$ are commuting scalar operators. Also, it follows that the augmented quantum system is described by the equations

$$\begin{aligned} dx_p(t) &= -2\Theta(\alpha)x_p(t)\beta^T x_o(t)dt; \quad x_p(0) = x_{0p}; \\ dx_o(t) &= -\frac{\kappa}{2}x_o dt + 2\omega_o Jx_o dt + 2J\beta\alpha^T x_p dt - \sqrt{\kappa}dw; \\ &\quad x_o(0) = x_{0o}; \\ dy_o &= \sqrt{\kappa}x_o dt + dw; \\ z_p(t) &= \alpha^T x_p(t); \\ z_o(t) &= Ky_o(t); \end{aligned} \quad (13)$$

e.g., see [11], [15], [17].

It follows from (13) that the quantity $z_p(t) = \alpha^T x_p(t)$ satisfies the differential equation

$$dz_p(t) = -2\alpha^T \Theta(\alpha)x_p(t)\beta^T x_o(t)dt = 0. \quad (14)$$

That is, the quantity $z_p(t)$ remains constant and is not affected by the coupling to the coherent quantum observer:

$$z_p(t) = z_p(0) \quad \forall t \geq 0.$$

Now using this result in (13), it follows that

$$dx_o(t) = -\frac{\kappa}{2}x_o dt + 2\omega_o Jx_o dt + 2J\beta z_p dt - \sqrt{\kappa}dw. \quad (15)$$

Combining equations (13), (14) and (15), we obtain the following reduced dimension QSDEs describing the augmented quantum plant variable of interest and the quantum observer:

$$\begin{aligned} dz_p(t) &= 0; \quad z_p(0) = \alpha^T x_{0p}; \\ dx_o(t) &= -\frac{\kappa}{2}x_o dt + 2\omega_o Jx_o dt + 2J\beta z_p dt - \sqrt{\kappa}dw; \\ &\quad x_o(0) = x_{0o}; \\ dy_o &= \sqrt{\kappa}x_o dt + dw. \end{aligned} \quad (16)$$

This is a set of linear QSDEs. Hence, we can analyze this system in a similar way to [10]. To analyse the system (16), we first calculate the steady state value of the quantum expectation of the observer variables as follows:

$$\langle \bar{x}_o \rangle = \frac{4}{\kappa^2 + 16\omega_o^2} \begin{bmatrix} \kappa & 4\omega_o \\ -4\omega_o & \kappa \end{bmatrix} J\beta z_p.$$

Then, we define the quantity

$$\tilde{x}_o = x_o - \langle \bar{x}_o \rangle = x_o - \frac{4}{\kappa^2 + 16\omega_o^2} \begin{bmatrix} \kappa & 4\omega_o \\ -4\omega_o & \kappa \end{bmatrix} J\beta z_p.$$

We can now re-write the equations (16) in terms of \tilde{x}_o as follows

$$\begin{aligned} d\tilde{x}_o &= \begin{bmatrix} -\frac{\kappa}{2} & 2\omega_o \\ -2\omega_o & -\frac{\kappa}{2} \end{bmatrix} \tilde{x}_o dt - \sqrt{\kappa}dw; \\ dy_o &= -2\sqrt{\kappa} \begin{bmatrix} -\frac{\kappa}{2} & 2\omega_o \\ -2\omega_o & -\frac{\kappa}{2} \end{bmatrix}^{-1} J\beta z_p dt + dw^{out} \end{aligned} \quad (17)$$

where

$$dw^{out} = \sqrt{\kappa}\tilde{x}_o dt + dw.$$

We now look at the transfer function of the system

$$\begin{aligned} \dot{\tilde{x}}_o &= \begin{bmatrix} -\frac{\kappa}{2} & 2\omega_o \\ -2\omega_o & -\frac{\kappa}{2} \end{bmatrix} \tilde{x}_o - \sqrt{\kappa}w; \\ w^{out} &= \sqrt{\kappa}\tilde{x}_o + w, \end{aligned} \quad (18)$$

which is given by

$$G(s) = -\kappa \begin{bmatrix} s + \frac{\kappa}{2} & -2\omega_o \\ 2\omega_o & s + \frac{\kappa}{2} \end{bmatrix}^{-1}.$$

It is straightforward to verify that this transfer function is such that

$$G(j\omega)G(j\omega)^\dagger = I$$

for all ω . That is $G(s)$ is all pass. Also, the matrix $\begin{bmatrix} -\frac{\kappa}{2} & 2\omega_o \\ -2\omega_o & -\frac{\kappa}{2} \end{bmatrix}$ is Hurwitz and hence, the system (18) will converge to a steady state in which dw^{out} represents a standard quantum white noise with zero mean and unit intensity. Hence, at steady state, the equation

$$dy_o = -2\sqrt{\kappa} \begin{bmatrix} -\frac{\kappa}{2} & 2\omega_o \\ -2\omega_o & -\frac{\kappa}{2} \end{bmatrix}^{-1} J\beta z_p dt + dw^{out} \quad (19)$$

shows that the output field converges to a constant value plus zero mean white quantum noise with unit intensity.

We now consider the construction of the vector K defining the observer output z_o . This vector determines the quadrature of the output field which is measured by the homodyne detector. We first re-write equation (19) as

$$dy_o = ez_p dt + dw^{out}$$

where

$$e = -2\sqrt{\kappa} \begin{bmatrix} -\frac{\kappa}{2} & 2\omega_o \\ -2\omega_o & -\frac{\kappa}{2} \end{bmatrix}^{-1} J\beta \quad (20)$$

is a vector in \mathbb{R}^2 . Then

$$dz_o = Kez_p dt + Kdw^{out}.$$

Hence, we choose K such that

$$Ke = 1 \quad (21)$$

and therefore

$$dz_o = z_p dt + dn$$

where

$$dn = Kdw^{out}$$

will be a white noise process at steady state with intensity $\|K\|^2$. Thus, to maximize the signal to noise ratio for our measurement, we wish to choose K to minimize $\|K\|^2$ subject to the constraint (21). If we choose

$$K = \frac{e^T}{\|e\|^2} \quad (22)$$

then (21) is satisfied and this value of K is the optimal K .

We now consider the special case of $\omega_o = 0$. In this case, we obtain

$$e = 2\sqrt{\kappa} \begin{bmatrix} \frac{2}{\kappa} & 0 \\ 0 & \frac{2}{\kappa} \end{bmatrix} J\beta = \frac{4}{\sqrt{\kappa}} J\beta.$$

Hence, as $\kappa \rightarrow 0$, $\|e\| \rightarrow \infty$ and therefore $\|K\| \rightarrow 0$. This means that we can make the noise level on our measurement arbitrarily small by choosing $\kappa > 0$ sufficiently small. However, as κ gets smaller, the system (18) gets closer to instability and hence, takes longer to converge to steady state.

III. KALMAN FILTER FOR THE PLANT OBSERVER SYSTEM

Since the QSDEs (16) describing the plant observer system are linear, it should be possible to apply Kalman filtering to this system in order to estimate z_p based on the available measurements. However, the QSDEs (16) are not physically realizable; e.g., see [15], [16]. Hence, the quantum Kalman filter such as discussed in [20], [21] formally does not apply; see also [22]. Although the QSDEs (16) could be made physically realizable by adding an extra fictitious quadrature variable to pair with z_p , using the technique described in [23], the issue would remain that z_p corresponds to a finite level quantum system and hence, its initial condition cannot be Gaussian. To overcome this issue, we will take another approach to Kalman filtering for quantum systems noting that in the classical case, the Kalman filter also has the property that it is optimal linear unbiased estimator for a linear stochastic system, even in the case of non-Gaussian noise and initial conditions; e.g., see [12], [13].

We first consider a general set of linear QSDSs:

$$\begin{aligned} dx(t) &= A(t)x(t)dt + B(t)dw; & x(t_0) &= x_0; \\ dy(t) &= C(t)x(t)dt + dw(t) \end{aligned} \quad (23)$$

where x is a $n \times 1$ vector of self adjoint operators on an underlying Hilbert space, dw is a $m \times 1$ vector of quantum noises expressed in quadrature form corresponding to the input field of the system and dy represents the corresponding output field; e.g., see [15]–[17]. Here m is assumed to be even. The measured output of the system $z(t)$ will be a real vector quantity of dimension $\frac{m}{2}$ obtained by applying homodyne detection to yield one quadrature of each of the output fields; i.e., we write

$$dz(t) = Ddy(t) = DC(t)x(t)dt + Ddw(t). \quad (24)$$

Also, $A(t) \in \mathbb{R}^{n \times n}$, $B(t) \in \mathbb{R}^{n \times m}$, $C(t) \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{\frac{m}{2} \times m}$. The augmented system (16) is a system of the form (23).

We will consider linear filters of the following form:

$$d\hat{x}(t) = F(t)\hat{x}(t)dt + G(t)dz(t); \quad \hat{x}(t_0) = \hat{x}_0; \quad (25)$$

where $\hat{x}(t) \in \mathbb{R}^n$ is a vector of estimates for x , $F(t) \in \mathbb{R}^{n \times n}$ and $G(t) \in \mathbb{R}^{n \times \frac{m}{2}}$. The filter (25) is said to be an *unbiased estimator* for the system (23) if

$$\langle x(t) \rangle = \mathbb{E} \{ \hat{x}(t) \} \quad \forall t \geq t_0;$$

e.g., see [12]. Here $\langle x(t) \rangle = \text{Tr}(\rho x(t))$ denotes the quantum expectation of $x(t)$ where ρ is the system density

operator; e.g. see [15]–[17]. It is straightforward to verify that if (25) is an unbiased estimator for the system (23) then

$$F(t) = A(t) - G(t)DC(t) \quad \forall t \geq t_0$$

and $\hat{x}_0 = \langle x_0 \rangle$; e.g., see [12]. Hence, an unbiased estimator for the system (23) will be a filter of the form

$$\begin{aligned} d\hat{x}(t) &= (A(t) - G(t)DC(t))\hat{x}(t)dt + G(t)dz(t); \\ \hat{x}(t_0) &= \langle x_0 \rangle. \end{aligned} \quad (26)$$

Corresponding to the system (16) and the filter (26) is the estimation error

$$e(t) = x(t) - \hat{x}(t)$$

which satisfies

$$de(t) = [A(t) - G(t)DC(t)]e(t)dt + [B(t) - G(t)D]dw(t).$$

The corresponding error variance is defined by

$$J = \langle e(T)^T e(T) \rangle = \text{Tr}[\Sigma(T)]$$

where

$$\Sigma(T) = \frac{1}{2} \langle e(t)e(t)^T + (e(T)e(T)^T)^T \rangle$$

is the error covariance matrix. It is straightforward to verify that the matrix $\Sigma(T)$ satisfies the following matrix differential equation:

$$\begin{aligned} \dot{\Sigma}(t) &= [A(t) - G(t)DC(t)]\Sigma(t) \\ &+ \Sigma(t)[A(t) - G(t)DC(t)]^T \\ &+ [B(t) - G(t)D][B(t) - G(t)D]^T \end{aligned}$$

where

$$\begin{aligned} \Sigma(t_0) &= \frac{1}{2} \langle e(t_0)e(t_0)^T + (e(t_0)e(t_0)^T)^T \rangle \\ &= \frac{1}{2} \langle (x(t_0) - \langle x_0 \rangle)(x(t_0) - \langle x_0 \rangle)^T \rangle \\ &+ \frac{1}{2} \langle ((x(t_0) - \langle x_0 \rangle)(x(t_0) - \langle x_0 \rangle)^T)^T \rangle \\ &= \Sigma_0; \end{aligned}$$

e.g., see [12]. The filter of the form (26) which minimizes the quantity J is the minimum variance unbiased estimator for the system (23). This filter is a version of the Kalman filter for the case of general QSDEs of the form (23).

Theorem 1: The minimum variance unbiased estimator for the system (23) is a filter of the form (26) where

$$G(t) = (\Sigma^*(t)C(t)^T D^T + B(t)D^T) (DD^T)^{-1}$$

and $\Sigma^*(t)$ is defined by the matrix differential equation

$$\begin{aligned} \dot{\Sigma}^*(t) &= [A(t) - B(t)D^T (DD^T)^{-1} DC(t)]\Sigma^*(t) \\ &+ \Sigma^*(t)[A(t) - B(t)D^T (DD^T)^{-1} DC(t)]^T \\ &- \Sigma^*(t)C(t)^T D^T (DD^T)^{-1} DC(t)\Sigma^*(t) \\ &+ B(t)B(t)^T - B(t)D^T (DD^T)^{-1} DB(t)^T; \\ \Sigma^*(t_0) &= \Sigma_0. \end{aligned}$$

Furthermore, the error covariance matrix for this estimator is given by $\Sigma(t) \equiv \Sigma^*(t)$.

We now construct the above Kalman filter for the system (16). We assume that the density operator for the quantum plant (3) is ρ_p . It follows that for $t_0 = 0$,

$$\langle z_p(t_0) \rangle = \sum_{i=1}^3 C_{pi} \text{Tr}(\rho_p \sigma_i) = \bar{z}_{p0}$$

and write

$$\langle x_o(t_0) \rangle = \bar{x}_{o0}.$$

Also using (1), we calculate

$$\begin{aligned} \langle (z_p(t_0) - \bar{z}_{p0})^2 \rangle &= \langle z_p(t_0)^2 \rangle - \bar{z}_{p0}^2 \\ &= \sum_{i=1}^3 C_{pi}^2 - \bar{z}_{p0}^2 \\ &= \sigma_{p0} \end{aligned}$$

and write

$$\begin{aligned} &\frac{1}{2} \langle (x_o(t_0) - \bar{x}_{o0})(x_o(t_0) - \bar{x}_{o0})^T \rangle \\ &+ \frac{1}{2} \langle ((x(t_0) - \bar{x}_{o0})(x(t_0) - \bar{x}_{o0})^T)^T \rangle \\ &= \Sigma_{o0}. \end{aligned}$$

Now the plant observer system (16) defines a set of QSDEs of the form (23), (24) where

$$\begin{aligned} A(t) &\equiv \begin{bmatrix} 0 & 0 \\ 2J\beta & -\frac{\kappa}{2}I + 2\omega_o J \end{bmatrix}; \\ B(t) &\equiv \begin{bmatrix} 0 \\ -\sqrt{\kappa}I \end{bmatrix}; \\ C(t) &\equiv \begin{bmatrix} 0 & \sqrt{\kappa}I \end{bmatrix}; \quad D = K; \\ \langle x_0 \rangle &= \begin{bmatrix} \bar{z}_{p0} \\ \bar{x}_{o0} \end{bmatrix}; \quad \Sigma_0 = \begin{bmatrix} \sigma_{p0} & 0 \\ 0 & \Sigma_{o0} \end{bmatrix}. \end{aligned}$$

Hence, the corresponding Kalman filter for the plant observer system (16) is defined by the equations

$$\begin{aligned} d \begin{bmatrix} \hat{z}_p \\ \hat{x}_o \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 2J\beta & -\frac{\kappa}{2}I + 2\omega_o J \end{bmatrix} \begin{bmatrix} \hat{z}_p \\ \hat{x}_o \end{bmatrix} dt \\ &+ G(t) \left(dz_o - \begin{bmatrix} 0 & \sqrt{\kappa}K \end{bmatrix} \begin{bmatrix} \hat{z}_p \\ \hat{x}_o \end{bmatrix} dt \right); \\ \begin{bmatrix} \hat{z}_p(0) \\ \hat{x}_o(0) \end{bmatrix} &= \begin{bmatrix} \bar{z}_{p0} \\ \bar{x}_{o0} \end{bmatrix}; \\ G(t) &= (\Sigma^*(t) + I) \begin{bmatrix} 0 \\ \sqrt{\kappa}I \end{bmatrix} K^T (KK^T)^{-1}; \\ \dot{\Sigma}^* &= \begin{bmatrix} 0 & 0 \\ 2J\beta & -\frac{\kappa}{2}I + 2\omega_o J + \kappa K^T (KK^T)^{-1} K \end{bmatrix} \Sigma^* \\ &+ \Sigma^* \begin{bmatrix} 0 & 0 \\ 2J\beta & -\frac{\kappa}{2}I + 2\omega_o J + \kappa K^T (KK^T)^{-1} K \end{bmatrix}^T \\ &- \Sigma^* \begin{bmatrix} 0 & 0 \\ 0 & \kappa K^T (KK^T)^{-1} K \end{bmatrix} \Sigma^* \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & \kappa I - \kappa K^T (KK^T)^{-1} K \end{bmatrix}; \\ \Sigma^*(0) &= \begin{bmatrix} \sigma_{p0} & 0 \\ 0 & \Sigma_{o0} \end{bmatrix}. \end{aligned}$$

REFERENCES

- [1] I. Vladimirov and I. R. Petersen, "Coherent quantum filtering for physically realizable linear quantum plants," in *Proceedings of the 2013 European Control Conference*, Zurich, Switzerland, July 2013.
- [2] Z. Miao, L. A. D. Espinosa, I. R. Petersen, V. Ugrynovskii, and M. R. James, "Coherent quantum observers for n-level quantum systems," in *Australian Control Conference*, Perth, Australia, November 2013.
- [3] Z. Miao, M. R. James, and I. R. Petersen, "Coherent observers for linear quantum stochastic systems," *Automatica*, vol. 71, pp. 264–271, 2016.
- [4] I. R. Petersen, "A direct coupling coherent quantum observer," in *Proceedings of the 2014 IEEE Multi-conference on Systems and Control*, Antibes, France, October 2014, also available arXiv 1408.0399.
- [5] —, "A direct coupling coherent quantum observer for a single qubit finite level quantum system," in *Proceedings of 2014 Australian Control Conference*, Canberra, Australia, November 2014, also arXiv 1409.2594.
- [6] —, "Time averaged consensus in a direct coupled distributed coherent quantum observer," in *Proceedings of the 2015 American Control Conference*, Chicago, IL, July 2015.
- [7] —, "Time averaged consensus in a direct coupled coherent quantum observer network for a single qubit finite level quantum system," in *Proceedings of the 10th ASIAN CONTROL CONFERENCE 2015*, Kota Kinabalu, Malaysia, May 2015.
- [8] I. R. Petersen and E. H. Huntington, "A possible implementation of a direct coupling coherent quantum observer," in *Proceedings of 2015 Australian Control Conference*, Gold Coast, Australia, November 2015.
- [9] —, "A reduced order direct coupling coherent quantum observer for a complex quantum plant," in *Proceedings of the European Control Conference 2016*, Aalborg, Denmark, June 2016.
- [10] I. R. Petersen and E. Huntington, "Implementation of a direct coupling coherent quantum observer including observer measurements," in *Proceedings of the 2016 American Control Conference*, Boston, MA, July 2016.
- [11] L. A. D. Espinosa, Z. Miao, I. R. Petersen, V. Ugrynovskii, and M. R. James, "Physical realizability and preservation of commutation and anticommution relations for n-level quantum systems," *SIAM Journal on Control and Optimization*, vol. 54, no. 2, pp. 632–661, 2016.
- [12] M. Athans and E. Tse, "A direct derivation of the optimal linear filter using the maximum principle," *IEEE Transactions on Automatic Control*, vol. AC-12, no. 6, pp. 690–698, 1967.
- [13] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*. Wiley, 1972.
- [14] I. R. Petersen and E. Huntington, "A direct coupling coherent quantum observer for a qubit, including observer measurements," 2017, arXiv.
- [15] M. R. James, H. I. Nurdin, and I. R. Petersen, " H^∞ control of linear quantum stochastic systems," *IEEE Transactions on Automatic Control*, vol. 53, no. 8, pp. 1787–1803, 2008.
- [16] I. R. Petersen, "Quantum linear systems theory," *Open Automation and Control Systems Journal*, vol. 8, pp. 67–93, 2016.
- [17] H. I. Nurdin, M. R. James, and A. C. Doherty, "Network synthesis of linear dynamical quantum stochastic systems," *SIAM Journal on Control and Optimization*, vol. 48, no. 4, pp. 2686–2718, 2009.
- [18] J. Gough and M. R. James, "The series product and its application to quantum feedforward and feedback networks," *IEEE Transactions on Automatic Control*, vol. 54, no. 11, pp. 2530–2544, 2009.
- [19] G. Zhang and M. James, "Direct and indirect couplings in coherent feedback control of linear quantum systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 7, pp. 1535–1550, 2011.
- [20] N. Yamamoto, "Robust observer for uncertain linear quantum systems," *Phys. Rev. A*, vol. 74, pp. 032 107–1 – 032 107–10, 2006.
- [21] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control*. Cambridge University Press, 2010.
- [22] L. Bouten, R. van Handel, and M. James, "An introduction to quantum filtering," *SIAM J. Control and Optimization*, vol. 46, no. 6, pp. 2199–2241, 2007.
- [23] S. Wang, H. I. Nurdin, G. Zhang, and M. R. James, "Quantum optical realization of classical linear stochastic systems," *Automatica*, vol. 49, no. 10, pp. 3090 – 3096, 2013.