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On the hereditary character of certain spectral properties and some applications

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Abstract

In this paper we study the behavior of certain spectral properties of an operator T on a proper closed and T-invariant subspace $W \subseteq X$ such that $T^n(X) \subseteq W$, for some $n \geq 1$, where $T \in L(X)$ and X is an infinite-dimensional complex Banach space. We prove that for these subspaces a large number of spectral properties are transmitted from T to its restriction on W and vice-versa. As consequence of our results, we give conditions for which semi-Fredholm spectral properties, as well as Weyl type theorems, are equivalent for two given operators. Additionally, we give conditions under which an operator acting on a subspace can be extended on the entire space preserving the Weyl type theorems. In particular, we give some applications of these results for integral operators acting on certain functions spaces.

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1. Introduction

In 1909, H. Weyl [28] studied the spectra of compact perturbations for a Hermitian operator and showed that a point belong to spectra of all compact perturbations of the operator if and only if this point is not an isolated point of finite multiplicity in the spectrum of the operator. L. Coburn [15] was one of the first to make a systematic investigation about this result and introduced in abstract form the Weyl's theorem for operators acting on a Banach space. Later, W. Rakočević [23] introduce a stronger property, called a-Weyl's theorem. Berkani and Koliha [11] introduced generalized versions for the Weyl's theorems by using some spectra of a new theory of semi B-Fredholm operators given in [9]. After them many authors have introduced and studied a large number of spectral properties associated to an operator by using spectra derived from either Fredholm operators theory or B-Fredholm operators theory. Today all these results are known as Weyl type theorems or Weyl type properties and over the last years there has been a considerable interest to study these properties in operator theory. On the other hand B. Barnes [5](resp. [6]) studied the relationship between some properties of an operator and its extensions (resp. restrictions) on certain superspaces (resp. subspaces) and showed that some Fredholm properties (resp. closed range and generalized inverses) are transmitted from the operator to its extensions (resp. restrictions). In this paper, using the framework dealt by Barnes [6] (which extends the context treated by Berkani [7]), we study the behavior of certain spectral properties of an operator T on a proper closed and T-invariant subspace $W \subseteq X$ such that $T^n(X) \subseteq W$, for some $n \geq 1$, where T is a bounded linear operator acting on an infinite-dimensional complex Banach space X. We prove that for these subspaces (which generalize the case $R(T^n)$ closed for some $n \geq 0$, [16], [18]) a large number of spectral properties are transmitted from T to its restriction on W and vice-versa. As consequence of our results, we give sufficient conditions for which semi-Fredholm spectral properties, as well as Weyl type theorems, are equivalent for two given operators. Additionally, we give conditions under which an operator acting on a subspace can be extended on the entire space preserving the Weyl type theorems. In particular, we give some applications of these results for integral operators acting on certain functions spaces.

2. Preliminaries

Throughout this paper $L(X)$ denotes the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X. The classes of operators studied in the classical Fredholm theory generate several spectra associated with an operator $T \in L(X)$. The *Fredholm spectrum* is defined by

$$\sigma_{\mathrm{f}}(T) = \{\lambda \in \mathbf{C} : \lambda I - T \text{ is not Fredholm}\},$$

and the *upper semi-Fredholm spectrum* is defined by

$$\sigma_{\mathrm{uf}}(T) = \{\lambda \in \mathbf{C} : \lambda I - T \text{ is not upper semi-Fredholm}\}.$$

The *Browder spectrum* and the *Weyl spectrum* are defined, respectively, by

$$\sigma_{\mathrm{b}}(T) = \{\lambda \in \mathbf{C} : \lambda I - T \text{ is not Browder}\},$$

and

$$\sigma_{\mathrm{w}}(T) = \{\lambda \in \mathbf{C} : \lambda I - T \text{ is not Weyl}\}.$$

Since every Browder operator is Weyl, $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$. Analogously, the *upper semi-Browder spectrum* and the *upper semi-Weyl spectrum* are defined by

$$\sigma_{\mathrm{ub}}(T) = \{\lambda \in \mathbf{C} : \lambda I - T \text{ is not upper semi-Browder}\},$$

and

$$\sigma_{\mathrm{uw}}(T) = \{\lambda \in \mathbf{C} : \lambda I - T \text{ is not upper semi-Weyl}\}.$$

For further information on Fredholm operators theory, we refer to [1] and [20].

According [7] and [9], T_n denotes the restriction of $T \in L(X)$ on the subspace $R(T^n) = T^n(X)$. Also, $T \in L(X)$ is said to be *B-Fredholm* (resp. *upper semi B-Fredholm*, *lower semi B-Fredholm*, *semi B-Fredholm*, *B-Browder*, *upper semi B-Browder*, *lower semi B-Browder*), if for some integer $n \geq 0$ the range $R(T^n)$ is closed and T_n, viewed as an operator from the space $R(T^n)$ into itself, is a Fredholm (resp. upper semi-Fredholm, lower semi-Fredholm, semi-Fredholm, Browder, upper semi-Browder, lower semi-Browder). If T_n is a semi-Fredholm operator, it follows from [9, Proposition 2.1] that also T_m is semi-Fredholm for every $m \geq n$, and $\operatorname{ind} T_m = \operatorname{ind} T_n$. This enables us to define the *index* of a semi B-Fredholm operator T as the index of the semi-Fredholm operator T_n. Thus, $T \in L(X)$ is said to be a *B-Weyl operator* if T is a B-Fredholm operator having index 0. $T \in L(X)$ is said to be *upper semi B-Weyl* (resp. *lower semi B-Weyl*) if T is upper semi B-Fredholm (resp. lower semi B-Fredholm) with index $\operatorname{ind} T \leq 0$ (resp. $\operatorname{ind} T \geq 0$). Note that if T is B-Fredholm and T^* denotes the dual of T, then also T^* is B-Fredholm with $\operatorname{ind} T^* = -\operatorname{ind} T$.

The spectra related with semi B-Fredholm operators are defined as follows. The *B-Browder spectrum* is defined by

$$\sigma_{\mathrm{bb}}(T) = \{\lambda \in \mathbf{C} : \lambda I - T \text{ is not B-Browder}\},$$

while the *B-Weyl spectrum* is defined by

$$\sigma_{\mathrm{bw}}(T) = \{\lambda \in \mathbf{C} : \lambda I - T \text{ is not B-Weyl}\}.$$

Another class of operators related with semi B-Fredholm operators is the quasi-Fredholm operators defined in the sequel. Previously, we consider the following set.

$$\Delta(T) = \{n \in \mathbf{N} : m \geq n, m \in \mathbf{N} \Rightarrow T^n(X) \cap N(T) \subseteq T^m(X) \cap N(T)\}.$$

The *degree of stable iteration* is defined as $\operatorname{dis}(T) = \inf \Delta(T)$ if $\Delta(T) \neq \emptyset$, while $\operatorname{dis}(T) = \infty$ if $\Delta(T) = \emptyset$.

Definition 2.1. *$T \in L(X)$ is said to be* quasi-Fredholm of degree d, *if there exists $d \in \mathbf{N}$ such that:*

(a) *$\operatorname{dis}(T) = d$,*
(b) *$T^n(X)$ is a closed subspace of X for each $n \geq d$,*
(c) *$T(X) + N(T^d)$ is a closed subspace of X.*

It should be noted that by [9, Proposition 2.5], every semi B-Fredholm operator is quasi-Fredholm. For further information on quasi-Fredholm operators, we refer to [2], [3], [8] and [9].

An operator $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_0 \in \mathbf{C}$ (abbreviated, SVEP at λ_0)[17], if for every open disc $\mathbf{D}_{\lambda_0} \subseteq \mathbf{C}$ centered at λ_0 the only analytic function $f : \mathbf{D}_{\lambda_0} \to X$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0 \quad \text{for all } \lambda \in \mathbf{D}_{\lambda_0},$$

is the function $f \equiv 0$ on $\mathbf{D}_{\lambda_0}$. The operator T is said to have SVEP if T has the SVEP at every point $\lambda \in \mathbf{C}$. Evidently, $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) = \mathbf{C} \setminus \sigma(T)$. Also, the single valued extension property is inherited by restrictions on invariant closed subspaces. Moreover, from the identity theorem for analytic functions it is easily seen that T has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of the spectrum. Note that (see [1, Theorem 3.8])

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{2.1}$$

and dually

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda. \tag{2.2}$$

Recall that $T \in L(X)$ is said to be *bounded below* if T is injective and has closed range. Denote by $\sigma_{\mathrm{ap}}(T)$ the classical *approximate point spectrum* defined by

$$\sigma_{\mathrm{ap}}(T) = \{\lambda \in \mathbf{C} : \lambda I - T \text{ is not bounded below}\}.$$

Note that if $\sigma_{\mathrm{su}}(T)$ denotes the *surjectivity spectrum*

$$\sigma_{\mathrm{su}}(T) = \{\lambda \in \mathbf{C} : \lambda I - T \text{ is not onto}\},$$

then $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{su}}(T^*)$, $\sigma_{\mathrm{su}}(T) = \sigma_{\mathrm{ap}}(T^*)$ and $\sigma(T) = \sigma_{\mathrm{ap}}(T) \cup \sigma_{\mathrm{su}}(T)$.

It is easily seen from definition of localized SVEP, that

$$\lambda \notin \operatorname{acc} \sigma_{\mathrm{ap}}(T) \Rightarrow T \text{ has SVEP at } \lambda, \tag{2.3}$$

and

$$\lambda \notin \operatorname{acc} \sigma_{\mathrm{su}}(T) \Rightarrow T^* \text{ has SVEP at } \lambda, \tag{2.4}$$

where $\operatorname{acc} K$ means the set of all accumulation points of a subset $K \subseteq \mathbf{C}$.

Remark 2.2. The implications (2.1),(2.2),(2.3) and (2.4) are actually equivalences, if $T \in L(X)$ is semi-Fredholm (see [1, Chapter 3]). More generally, if $T \in L(X)$ is quasi-Fredholm (see [2]). On the other hand $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T) \cup \operatorname{acc} \sigma(T)$, $\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{uw}}(T) \cup \operatorname{acc} \sigma_{ap}(T)$ and $\sigma(T) = \sigma_{\mathrm{ap}}(T) \cup \Xi(T)$, where $\Xi(T)$ denote the set $\{\lambda \in \mathbf{C} : T \text{ does not have SVEP at } \lambda\}$ (see [1, Chapter 3]).

The following definition summarizes a large number of properties called Weyl type theorems.

Definition 2.3. *An operator $T \in L(X)$ is said to satisfy property:*

(i) (w), *if* $\sigma_{ap}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$ *([24]);*

(ii) (aw), *if* $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi^a_{00}(T)$ *([12]);*

(iii) (b), *if* $\sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{uw}}(T) = p_{00}(T)$ *([10],[13]);*

(iv) (ab), *if* $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = p^a_{00}(T)$ *([12]);*

(v) (z) *if* $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi^a_{00}(T)$ *([26]);*

(vi) (az), *if* $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) = p^a_{00}(T)$ *([26]);*

(vii) (h), *if* $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi_{00}(T)$ *([25],[27]);*

(viii) (ah), *if* $\sigma(T) \setminus \sigma_{\mathrm{uw}}(T) = p_{00}(T)$ *([25],[27]);*

Also, T is said to satisfy:

(ix) Browder's theorem, if $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$ *([19]);*

(x) a-Browder's theorem, if $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$ *([24]);*

(xi) generalized Browder's theorem, if $\sigma_{\mathrm{bw}}(T) = \sigma_{\mathrm{bb}}(T)$ *([19]);*

(xii) Weyl's theorem, if $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T)$ *([15]);*

(xiii) a-Weyl's theorem, if $\sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{uw}}(T) = \pi^a_{00}(T)$ *([23]);*

where

$$\begin{aligned} p_{00}(T) &= \sigma(T) \setminus \sigma_{\mathrm{b}}(T), \\ p^a_{00}(T) &= \sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{ub}}(T), \\ \pi_{00}(T) &= \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\}, \\ \pi^a_{00}(T) &= \{\lambda \in \operatorname{iso} \sigma_{\mathrm{ap}}(T) : 0 < \alpha(\lambda I - T) < \infty\}, \end{aligned}$$

and $\operatorname{iso} K$ *denote the set of all isolated points of a subset* $K \subseteq \mathbf{C}$.

As in Barnes [6], in the sequel of this paper we always assume that W is a proper closed subspace of a Banach space X. Also, we denote

$$\mathcal{P}(X, W) = \{T \in L(X) : T(W) \subseteq W \text{ and for some integer } n \geq 1, T^n(X) \subseteq W\}.$$

For each $T \in \mathcal{P}(X, W)$, T_W denote the restriction of T on the subspace T-invariant W of X. Observe that $0 \in \sigma_{\text{su}}(T)$ for all $T \in \mathcal{P}(X, W)$. Because, $T \in \mathcal{P}(X, W)$ and T onto implies that $X = T^n(X) \subseteq W$, for some $n \geq 1$, contradicting our assumption that W is a proper subspace of a X. Later we shall see that $\sigma_{\text{su}}(T)$ and $\sigma_{\text{su}}(T_W)$ may differ only in 0.

3. Relations between the parameters of T and T_W

In this section we give some fundamental facts, by citing several previous results which will be used in the proof of the main results of this paper.

Lemma 3.1. *(see [14]) If $T \in \mathcal{P}(X, W)$, then for all $\lambda \neq 0$:*

(i) $N((\lambda I - T_W)^m) = N((\lambda I - T)^m)$, *for any* m,

(ii) $R((\lambda I - T_W)^m) = R((\lambda I - T)^m) \cap W$, *for any* m,

(iii) $\alpha(\lambda I - T_W) = \alpha(\lambda I - T)$,

(iv) $p(\lambda I - T_W) = p(\lambda I - T)$,

(v) $\beta(\lambda I - T_W) = \beta(\lambda I - T)$.

Moreover, we have the following equivalences.

Lemma 3.2. *(see [14]) If $T \in \mathcal{P}(X, W)$, then:*

(i) $p(T) < \infty$ *if and only if* $p(T_W) < \infty$,

(ii) $q(T) < \infty$ *if and only if* $q(T_W) < \infty$.

Theorem 3.3. *(see [14]) If $T \in \mathcal{P}(X, W)$ and $p(T) = \infty$, or $q(T) = \infty$, then the following equalities are true:*

(i) $\sigma_{\text{su}}(T) = \sigma_{\text{su}}(T_W)$;

(ii) $\sigma_{\text{ap}}(T) = \sigma_{\text{ap}}(T_W)$;

(iii) $\sigma(T) = \sigma(T_W)$;

(iv) $\sigma_{\text{w}}(T) = \sigma_{\text{w}}(T_W)$;

(v) $\sigma_{\text{uw}}(T) = \sigma_{\text{uw}}(T_W)$;

(vi) $\sigma_{\text{b}}(T) = \sigma_{\text{b}}(T_W)$;

(vii) $\sigma_{\text{ub}}(T) = \sigma_{\text{ub}}(T_W)$;

(viii) $\sigma_{\text{f}}(T) = \sigma_{\text{f}}(T_W)$;

(ix) $\sigma_{\text{uf}}(T) = \sigma_{\text{uf}}(T_W)$.

Remark 3.4. Recall that for $T \in L(X)$, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ precisely when λ is a pole of the resolvent of T (see [20, Prop. 50.2]). Also, is well known that if λ is a pole of the resolvent of T, then $\lambda \in \operatorname{iso}\sigma(T)$. Evidently, if $\lambda \in \operatorname{iso}\sigma(T)$ then $\lambda \in \partial\sigma(T)$. Thus, for $T \in \mathcal{P}(X, W)$, if $0 \notin \operatorname{iso}\sigma(T)$ (resp. $0 \notin \partial\sigma(T)$, $0 \in \Xi(T)$, $0 \in \Xi(T^*)$) then $p(T) = \infty$ or $q(T) = \infty$. Therefore, the conclusions of Theorem 3.3 remain true if the hypothesis $p(T) = \infty$ or $q(T) = \infty$ is replaced by one of the following hypothesis: $0 \notin \operatorname{iso}\sigma(T)$, $0 \notin \partial\sigma(T)$, $0 \in \Xi(T)$ or $0 \in \Xi(T^*)$. On the other hand, according to Lemma 3.2 we can change the hypothesis $p(T) = \infty$ or $q(T) = \infty$ by $p(T_W) = \infty$ or $q(T_W) = \infty$ in the Theorem 3.3. Hence, the conclusions of Theorem 3.3 remain true if the hypothesis $p(T) = \infty$ or $q(T) = \infty$ is replaced by one of the following hypothesis: $0 \notin \operatorname{iso}\sigma(T_W)$, $0 \notin \partial\sigma(T_W)$, $0 \in \Xi(T_W)$ or $0 \in \Xi((T_W)^*)$.

4. Weyl type theorems for T and T_W

In this section we present the main results of this paper. We show that, for all $T \in \mathcal{P}(X, W)$, Weyl type theorems studied in section one are transmitted from T to its restriction T_W and vice-versa.

By Lemma 3.1, Theorem 3.3 and Remark 3.4, we obtain the following consequences.

Lemma 4.1. *If $T \in \mathcal{P}(X, W)$ and $q(T) = \infty$, or $p(T) = \infty$, then the following relations are true:*

(i) $p_{00}(T) = p_{00}(T_W)$;

(ii) $p_{00}^a(T) = p_{00}^a(T_W)$.

Proof. It follows from Theorem 3.3.

■

Lemma 4.2. *If $T \in \mathcal{P}(X,W)$ and $0 \notin \text{iso}\,\sigma(T)$, then the following equalities are true:*

(i) $p_{00}(T) = p_{00}(T_W)$;

(ii) $p_{00}^a(T) = p_{00}^a(T_W)$;

(iii) $\pi_{00}(T) = \pi_{00}(T_W)$;

(iv) $\pi_{00}^a(T) = \pi_{00}^a(T_W)$.

Proof. (i) and (ii) follows from Theorem 3.3 and Remark 3.4.

(iii) Suppose that $\lambda \in \pi_{00}(T)$, then $\lambda \in \text{iso}\,\sigma(T)$ and $0 < \alpha(\lambda I - T) < \infty$. Assuming that 0 is not an isolated point of $\sigma(T)$, then $\lambda \neq 0$. Thus, from Lemma 3.1, Theorem 3.3 and Remark 3.4, follows that $0 < \alpha(\lambda I - T_W) = \alpha(\lambda I - T) < \infty$ and $\lambda \in \text{iso}\,\sigma(T) = \text{iso}\,\sigma(T_W)$. Hence $\lambda \in \pi_{00}(T_W)$, and we have the inclusion $\pi_{00}(T) \subseteq \pi_{00}(T_W)$. Similarly, by the same argument above, we can prove the inclusion $\pi_{00}(T_W) \subseteq \pi_{00}(T)$.

(iv) The proof is analogous to that of part (iii).

■

Now, we are ready to prove the main results.

Theorem 4.3. *If $T \in \mathcal{P}(X,W)$ and $q(T) = \infty$, or $p(T) = \infty$, then (iii) (resp.,(iv), (vi), (viii), (ix), (x), (xi)) in Definition 2.3 holds for T if and only if (iii) (resp.,(iv), (vi), (viii), (ix), (x), (xi)) in Definition 2.3 holds for T_W.*

Proof. It follows by Theorem 3.3 and Lemma 4.1. For (xi), observe the equivalence between Browder's theorem and generalized Browder's theorem proved in [3].

■

From the Theorem 4.3 and Lemma 3.2, the following corollary is obtained.

Corollary 4.4. *Let T be an operator in $\mathcal{P}(X, W)$ such that its restriction T_W verifies one of the following conditions: $q(T_W) = \infty$ or $p(T_W) = \infty$. Then (iii) (resp.,(iv), (vi), (viii), (ix), (x), (xi)) in Definition 2.3 holds for T if and only if (iii) (resp.,(iv), (vi), (viii), (ix), (x), (xi)) in Definition 2.3 holds for T_W.*

∎

Theorem 4.3, may be extended assuming weaker hypotheses as follows.

Theorem 4.5. *If $T \in \mathcal{P}(X, W)$ verifies one of the following conditions: (i) $0 \notin \text{iso}\,\sigma(T)$, (ii) $0 \notin \partial\sigma(T)$, (iii) $0 \in \Xi(T)$ or (iv) $0 \in \Xi(T^*)$. Then (i) (resp.,(ii)-(xiii)) in Definition 2.3 holds for T if and only if (i)(resp.,(ii)-(xiii)) in Definition 2.3 holds for T_W.*

Proof. It follows by Theorem 3.3, Remark 3.4 and Lemma 4.2. Again, for (xi), consider the equivalence between Browder's theorem and generalized Browder's theorem proved in [3].

∎

From the Theorem 4.5 and Remark 3.4, we get the following corollary.

Corollary 4.6. *Let T be an operator in $\mathcal{P}(X, W)$ such that its restriction T_W verifies one of the following conditions: (i) $0 \notin \text{iso}\,\sigma(T_W)$, (ii) $0 \notin \partial\sigma(T_W)$, (iii) $0 \in \Xi(T_W)$ or (iv) $0 \in \Xi(T_W^*)$. Then (i) (resp.,(ii)-(xiii)) in Definition 2.3 holds for T if and only if (i)(resp.,(ii)-(xiii)) in Definition 2.3 holds for T_W.*

∎

5. Applications

In this section we give sufficient conditions for which semi-Fredholm spectral properties, as well as Weyl type theorems, are equivalent for two given operators. Also, we give conditions under which an operator acting on a subspace can be extended on the entire space preserving the Weyl type theorems. In particular, we give applications of these results for integral operators acting on certain functions spaces.

As an immediate consequence of Theorem 3.3, we obtain sufficient conditions for which semi-Fredholm spectral properties are equivalent for two given operators.

Corollary 5.1. *Suppose that T, $S \in \mathcal{P}(X, W)$ and T, S coincide on W. If ones of the following conditions is valid*

(i) $0 \notin \operatorname{iso}\sigma(T) \cup \operatorname{iso}\sigma(S)$ *(resp.* $0 \notin \operatorname{iso}\sigma(T_W) \cup \operatorname{iso}\sigma(S_W)$*),*

(i) $0 \notin \partial\sigma(T) \cup \partial\sigma(S)$ *(resp.* $0 \notin \partial\sigma(T_W) \cup \partial\sigma(S_W)$*),*

(iii) $0 \in \Xi(T) \cap \Xi(S)$ *(resp.* $0 \in \Xi(T_W) \cap \Xi(S_W)$*),*

(iv) $0 \in \Xi(T^*) \cap \Xi(S^*)$ *(resp.* $0 \in \Xi((T_W)^*) \cap \Xi((S_W)^*)$*),*

then the following equalities are true:

(i) $\sigma_{\mathrm{su}}(T) = \sigma_{\mathrm{su}}(S)$;

(ii) $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{ap}}(S)$;

(iii) $\sigma(T) = \sigma(S)$;

(iv) $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{w}}(S)$;

(v) $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{uw}}(S)$;

(vi) $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{b}}(S)$;

(vii) $\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{ub}}(S)$;

(viii) $\sigma_{\mathrm{f}}(T) = \sigma_{\mathrm{f}}(S)$;

(ix) $\sigma_{\mathrm{uf}}(T) = \sigma_{\mathrm{uf}}(S)$.

Proof. Observe that if T, S coincide on W, then $T_W = S_W$. ■

Similarly, as the above theorem, from Theorem 4.5 and Corollary 4.6, we obtain sufficient conditions for which Weyl type theorems are equivalent for two given operators.

Corollary 5.2. *Suppose that T, $S \in \mathcal{P}(X, W)$ and T, S coincide on W. If ones of the following conditions is valid*

(i) $0 \notin \operatorname{iso}\sigma(T) \cup \operatorname{iso}\sigma(S)$ *(resp.* $0 \notin \operatorname{iso}\sigma(T_W) \cup \operatorname{iso}\sigma(S_W)$*),*

(i) $0 \notin \partial\sigma(T) \cup \partial\sigma(S)$ *(resp.* $0 \notin \partial\sigma(T_W) \cup \partial\sigma(S_W)$*),*

(iii) $0 \in \Xi(T) \cap \Xi(S)$ *(resp.* $0 \in \Xi(T_W) \cap \Xi(S_W)$*),*

(iv) $0 \in \Xi(T^*) \cap \Xi(S^*)$ *(resp.* $0 \in \Xi((T_W)^*) \cap \Xi((S_W)^*)$*),*

then (i) (resp.,(ii)-(xiii)) in Definition 2.3 holds for T *if and only if (i)(resp.,(ii)-(xiii)) in Definition 2.3 holds for* S*.*

∎

Proof. If T, S coincide on W, then $T_W = S_W$.

∎

The following theorem ensures that bounded operators acting on complemented subspaces can always be extended on the entire space preserving the Weyl type properties.

Theorem 5.3. *Let* W *be a complemented subspace of* X *and* $T \in L(W)$*. If ones of the following conditions is valid: (i)* $0 \notin \operatorname{iso}\sigma(T)$*, (ii)* $0 \notin \partial\sigma(T)$*, (iii)* $0 \in \Xi(T)$ *or (iv)* $0 \in \Xi(T^*)$*. Then* T *has an extension* $\overline{T} \in L(X)$ *such that (i) (resp.,(ii)-(xiii)) in Definition 2.3 holds for* $\overline{T}$ *if and only if (i)(resp.,(ii)-(xiii)) in Definition 2.3 holds for* T*.*

Proof. Since W is a complemented subspace of X, then there exists a bounded projection $P \in L(X)$ such that $P(X) = W$. Thus $\overline{T} = TP$ defines an operator in $\mathcal{P}(X, W)$ such that $T = \overline{T}_W$. From this, by Corollary 4.6, we obtain the equalities (i), (ii) and (iii).

∎

As a particular case of the above theorem, we obtain the following corollary.

Corollary 5.4. *Let* W *be a closed proper subspace of a Hilbert space* H *and* $T \in L(W)$*. If ones of the following conditions is valid: (i)* $0 \notin \operatorname{iso}\sigma(T)$*, (ii)* $0 \notin \partial\sigma(T)$*, (iii)* $0 \in \Xi(T)$ *or (iv)* $0 \in \Xi(T^*)$*. Then* T *has an extension* $\overline{T} \in L(H)$ *such that (i) (resp.,(ii)-(xiii)) in Definition 2.3 holds for* $\overline{T}$ *if and only if (i)(resp.,(ii)-(xiii)) in Definition 2.3 holds for* T*.*

∎

Proof. Follows immediately from Theorem 5.3, because every closed subspaces of a Hilbert space is complemented.

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For a locally compact Hausdorff space Ω, let $BC(\Omega)$ denote the space of all functions $f : \Omega \to \mathbf{C}$ such that f is bounded and continuous on

Ω, considered with the norm, $\|f\|_u = \sup\{|f(\omega)| : \omega \in \Omega\}$. If μ is a regular Borel measure on Ω, arise some interesting Banach spaces treated in the theory of integral operators acting on functions spaces. The following examples summarize some applications of our results, when $\Omega = G$ and G is a locally compact Hausdorff topological group, G unimodular and equipped with a Haar measure (see [21, 10.1 and 12.4]).

Example 5.5. *For $f \in L^1(G)$, we consider the operator convolution T given by $T(g) = f * g$, where $g \in L^p(G)$ (operation on G as multiplication). By the convolution results given in [21, pp 297-298], if $f \in L^1(G) \cap L^2(G)$ then*

(1) $T(L^2(G)) \subset L^2(G) \cap BC(G)$;

(2) $T^2(L^1(G)) \subset L^1(G) \cap BC(G)$.

Thus, the results of this paper (Theorem 3.3, Theorem 4.3, Corollary 4.4, Theorem 4.5 and Corollary 4.6), allows us reduce the study of semi-Fredholm spectral properties and Weyl type theorems of T to the subspace $W = L^2(G) \cap BC(G)$ in (1) *(resp. $W = L^1(G) \cap BC(G)$ in* (2)*)*

Example 5.6. *For $f \in L^1(G) \cap L^2(G)$, we consider the integral operator T determined by the kernel $\varphi(x)f(xt^{-1}\psi(t))$, where $\varphi, \psi \in L^\infty(G)$. By the results given in [22, 2.19 (iii), (iv)], for every $r \in [2, s]$, $2 \leq s < \infty$, T has an extension T_r such that $T_r(L^r(G)) \subset L^\infty(G)$. Thus, from the results of this paper we conclude that semi-Fredholm spectral properties and Weyl type theorems for T_r and T are the same, for all $r \in [2, s]$.*

For $n \geq 1$, $BC^n(0, \infty)$ is the set of all functions $f : (0, \infty) \to \mathbf{C}$ such that for $0 \leq k \leq n$, $f^{(k)}$ (the k^{th} derivative of f; $f^{(0)} = f$) exists and $f^{(k)} \in BC(0, \infty)$. The norm on $BC^n(0, \infty)$ is given by $\|f\| = \sum_{k=0}^{n} \|f^{(k)}\|_u$.

Example 5.7. *Suppose that $g \in BC^{n-1}(0, \infty) \cap L^1(0, \infty)$, $n \geq 1$, and $k \in BC^n(0, \infty)$ (here $BC^0(0, \infty) = BC(0, \infty)$). Set*

$$T(f)(x) = k(x) \int_0^x g(t)f(t)dt, \qquad f \in BC(0, \infty), x > 0.$$

If $f \in BC^m(0, \infty)$ for some m, $0 \leq m < n$, then it is easily verified that $T(f) \in BC^{m+1}(0, \infty)$. Therefore $T^n(BC(0, \infty)) \subset BC^n(0, \infty)$, so the study of semi-Fredholm spectral properties and Weyl type theorems of T can be reduced to the subspace $W = BC^n(0, \infty)$.

Let $BV[0,1]$ be the space of functions of bounded variation on $[0,1]$. Suppose that $u \in BV[0,1]$ and consider T the multiplication operator induced by the symbol u. That is $T(f) = u \cdot f$, for all $f \in BV[0,1]$. In the same style of the previous examples, we have the following result for these class of operators.

Example 5.8. *If $Z_u = \{t \in [0,1] : u(t) = 0\}$ is an infinite set and $X_{Z_u} = \{f \in BV[0,1] : f(t) = 0, \forall t \in Z_u\} = \emptyset$. By [4, Proposition 6], X_{Z_u} is a proper closed T-invariant subspace of $BV[0,1]$ such that $T(BV[0,1]) \subset X_{Z_u}$. Therefore, we can concluded that the study of semi-Fredholm spectral properties and Weyl type theorems of T can be reduced to the subspace $W = X_{Z_u}$.*

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