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Septembre 1995

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BASIC THEOREMS ON SLIDING MODE CONTROLLERS

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Abstract.

Basic results are presented that play an essential role in the analysis and design of sliding mode controllers. They include necessary and sufficient conditions for a controller to be on a sliding mode regime, characterization properties of the ensuing feedback system dynamics, and sensitivity properties concerning the influence of parameter and external perturbations over this dynamics. Although perhaps excessively short and necessarily incomplete, this presentation offers the advantage to be carried out under a unified setting and should provide a suitable introduction to this mathematically difficult subject.

1. Introduction

In these last few years sliding mode controllers have been recognized as a useful addition to the already formidable arsenal of techniques that are relevant to the control of both linear and nonlinear plants(see, for example, Utkin 1992, DeCarlo Zak and Matthews 1988, and Buhler 1986). By inducing a feedback behavior that can potentially be made independent of plant parameter variations and external perturbations, this usefulness appears to be particularly promising in applications where the controlled system is required to have a high tracking accuracy and a high speed of response and where, at the same time, fast parameter variations and high level nonstationary perturbations may be present.

It follows that one of the most important reasons for the adoption of sliding mode controllers is that they offer a particularly robust performance with respect to certain classes of perturbations and parameter variations (see, for example, El-Ghezawi et al. 1883, DeCarlo et al. 1988, Utkin 1991, Slotine 1992). A second important reason is that they play a significant role in the performance improvement of existing controllers such as adaptive and model following controllers (Balestrino et al. 1984). A third reason is that they provide a natural vehicle for the analysis and design of discrete controllers (in particular multivariable relay controllers as illustrated in Buhler 1986 and DeSantis 1992) as well as for the improvement of gain tuning procedures for more standard controllers (DeSantis 1988, 1989). Finally, yet an additional important reason is that ideas and techniques relevant to sliding mode controllers are also relevant an provide analogous benefits in the realm of state estimation and parameter identification problems (Edwards and Spurgeon 1994).

A sliding mode controller is a «variable structure controller» operating in a «sliding mode regime». Intuitively, a variable structure controller is a controller of which the structure depends on the belonging of the state of the plant to certain regions in the state space; operation in a sliding mode regime is an operation where the state of the plant is constrained to evolve (slide) on the surface defined by the intersections of these regions (sliding surface). As suggested by these characterization, the basic idea governing the analysis and design of these controllers unfolds as follows. First, the desired behavior of the system to be controlled is translated in terms of the convergence of the state to an appropriately selected sliding surface. Second, the control strategy is designed so as to guarantee that this convergence takes place. Third, care is taken to ensure that the behavior of the feedback system remains satisfactory in correspondence to the (realistic) situation where plant parameter variations external perturbations and non-modelled dynamics are present, and where the envisioned control strategy is only implemented within a certain approximation and the state constraints are only satisfied within a certain tolerance.

In what follows we present the elements of the theory of sliding mode controllers that are essential to carry out these steps. These elements consist of a formal definition of sliding mode controllers and five basic theorems. These theorems concern the following aspects: necessary and sufficient conditions for a controller to be of the sliding mode type, necessary and sufficient conditions for the system dynamics to be invariant to the presence of perturbations, characterization of state sensitivity to the presence of unmodelled behavior, stability of the dynamics under a sliding mode regime, and special sliding mode characterization properties in the case of linear plants. Two additional theorems illustrate the application of sliding mode controllers to the design of discrete controllers. While for brevity of exposition we will not discuss detailed applications attention is called to the fact that examples of convincing engineering applications of these theorems abound, and we encourage the reader to look for them in the cited literature (some of the applications developed at the Ecole Polytechnique may be found in Serfass 1987, DeSantis 1992, Hammada and El Ferik 1993, Krau 1994).

2. Formal Definition of a Sliding Mode Controller

Given a (time invariant, linear in the control) dynamic system,

$$\dot{x} = f(x) + B(x)u(x,t),$$

with x and u n- and m-dimensional state and control vectors, a **sliding mode controller** is a controller with the following three properties:

1) **it has a variable structure**: the control is described by $u(x,t) = u(x,t)^*$, where

$$u_i(x,t)^* = u_i^+(x) \text{ for } s_i(x) > 0 \quad i=1...m,$$

$$u_i(x,t)^* = u_i^-(x) \text{ for } s_i(x) < 0 \quad i=1...m,$$

where: u_i^+, u_i^- and s_i are pre-assigned continuous functions;

2) **its structure is continuously switching**: no trajectory, $x(t)$ $t\in[0, \infty)$, entirely generated by one of these 2^m potential control strategies is contained in the set

$$S:= \{x(t)|x(t) \text{ such that } s_i(x(t)) = 0 \text{ for } i=1,..m\};$$

3) **its action on the system may be viewed as the limit of a sequence of control actions** which, from a physical point of view, are easier to implement than the action provided by the controller described by 1) and 2). More formally: there exists an open set Ω in R^n, $\Omega\cap S\neq 0$, such that: given any $\varepsilon>0$, there exists $\Delta(\varepsilon)>0$, $\delta(\varepsilon)>0$ such that: for any $x_0 \varepsilon \Omega$, such that $|s(x_0)|<\delta$, the trajectory of

$$\dot{x} = f(x) + B(x)u(x,t), \; x(t_o) = x_o$$

with u(x,t) such that

$$u_i = u_i^* \quad \text{if } |s_i(x,t)| > \Delta$$

$$\min(u^+_i, u^-_i) \leq u_i \leq \max(u^+_i, u^-_i) \text{ otherwise,}$$

has the property that

$$|s(x)|<\varepsilon.$$

Note that vector function $s(x):=[s_1(x) \ \ldots \ s_m(x)]'$ is usually referred to as the **sliding mode function.**

Remark 1. Properties 1) and 2) make a sliding mode controller somewhat "ideal" and difficult to implement in practice. However, property 3), stipulates that this ideal behavior must be attainable within any desired degree of tolerance in terms of a more realistic, physically implementable controller.

3. On Necessary and Sufficient Conditions for Existence

The following theorem states that, for controller to be in a sliding mode , there must exist an **equivalent controller** that does not necessarily have a variable structure and which must satisfy a certain property with respect to the original variable structure controller.

Theorem 1:(A Necessary Condition for the Existence of a Sliding Mode Controller).

Given a dynamic system,

$$\dot{x} = f(x) + B(x)u(x,t),$$

a **necessary condition** for a variable structure controller defined by

$$u_i(x,t)^* = u_i^+(x) \text{ for } s_i(x) > 0 \quad i=1...p,$$

$$u_i(x,t)^* = u_i^-(x) \text{ for } s_i(x) < 0 \quad i=1...p,$$

to be a **sliding mode controller** is that an equivalent control u_{equ}, solution of

$$G(x)[f(x) + B(x)u_{equ}]=0,$$

$$G(x):=grad_x(s)$$

exists and satisfies the inequality

$$in\{u^+_i, u^-_i\} \leq u_{equi} \leq max\{u^+_i, u^-_i\}. \qquad \blacksquare$$

The next theorem stipulates that to check whether a variable structure controller is indeed a sliding mode controller, the task of verifying the validity of property 3) required by its definition may be replaced by the considerably simpler task of verifying that a certain Lyapunov-type function exists.

Theorem 2:(A Sufficient Condition for a Controller to be of the Sliding Mode Type).

Given a dynamic system,

$$\dot{x} = f(x) + B(x)u(x,t),$$

a **sufficient condition** for a variable structure controller described by

$$u_i(x,t)^* = u_i^+(x) \text{ for } s_i(x) > 0 \quad i=1...m,$$

$$u_i(x,t)^* = u_i^-(x) \text{ for } s_i(x) < 0 \quad i=1...m,$$

to be a **sliding mode controller** is that there exist a scalar function $v(s,x,t)$, continuously differentiable with respect to each of its arguments, such that in a certain open set Ω contained in R^n, $\Omega \cap S \neq 0$, with $S := \{x | x \text{ such that } s_i(x) = 0 \text{ for } i=1,..m\}$, the following properties hold:

i. $v(s,x,t)$ is positive definite with respect to s; $v(0,x,t)=0$;

ii. for a sufficiently small $R > 0$ and for all $x \in \Omega$ and any t:

$$\inf_{|s(x)| = R} v(s,x,t) = h_R \qquad \sup_{|s(x)| = R} v(s,x,t) = H_R$$

with $h_R < H_R$ only dependent on R;

iii. a total time derivative of v is negative everywhere this function is defined and

$$\sup_{|s(x)| = R} \dot{v}(s,x,t) = -m_R.$$

with $m_R > 0$ only dependent on R. ■

It should be noted that while theorems 1 and 2 imply that the sliding mode vector function eventually converges to zero in spite of the eventual presence of parameter or external perturbations this property does not necessarily imply that the dynamics of the system is unaffected by such a presence. However, as established by the following theorem, this implication holds if these perturbations satisfy some appropriate "matching conditions".

Theorem 3 (Dynamics Invariance when the Matching Conditions are Satisfied).

A necessary and sufficient condition for the dynamics of a system

$$\dot{x} = f(x) + B(x)u(x,t) + D(x)\xi,$$

under sliding mode control to be **invariant with respect to the perturbation**, ξ, is that the range of $D(x)$ be contained in the range of $B(x)$. ■

In the same circle of ideas as above, the fact that the value of the sliding mode vector function is close to zero, $|s(x(t))|<\varepsilon$ for arbitrarily selected ε, does not necessarily imply that the trajectory of the system state is close to the state trajectory obtained when the sliding function is equal to zero $(s(x(t)=0)$. However, as established by the following theorem, this is the case under appropriate hypotheses.

Theorem 4 (State Trajectory Sensitivity to Sliding Mode Controller Approximation).

Suppose that:

i) Given any $\varepsilon_1>0$, there exists $\delta_1(\varepsilon_1)>0$ such that on an interval [0, T] any solution of

$$\dot{x} = f(x) + B(x)u(x,t) \quad , \; x(t_o) = x_o$$

with u(x,t) such that

$$u_i = u_i^* \quad \text{if } |s_i(x)| > \delta_1$$

$$\min(u^+_i, u^-_i) \leq u_i \leq \max(u^+_i, u^-_i) \text{ otherwise,}$$

has the property that $|s(x)|<\varepsilon_1$;

ii) partial derivatives of $B(x)[G(x)B(x)]^{-1}$ with respect to all arguments exist and are bounded in any bounded domain;

iii) there exist positive quantities M and N such that for any solution, x(t), of the system in i) one has

$$|f(x) + B(x)u(x,t)| < M + N|x|;$$

iv) the function

$$f(x^*) - B(x^*)[G(x^*)B(x^*)]^{-1}G(x^*)f(x^*),$$

with x^* solution of (the system under equivalent control)

$$\dot{x} = f(x) - B(x)[G(x)B(x)]^{-1}G(x)f(x), \quad x^*(t_o) = x_0^*$$

is Lipschitz continuous;

Then, given any $\varepsilon_2>0$, there exists a $\delta_2(\varepsilon_2)>0$ such that

$$|x(0)-x^*(0)|<\delta_2$$

implies

$$|x(t)-x^*(t)|<\varepsilon_2, \qquad t\varepsilon[0,T]. \qquad \blacksquare$$

The following theorem 5 specializes the ensemble of the above results to the case where the plant is linear and time invariant. In particular: the characterization of an equivalent model to describe plant dynamics under sliding mode; a procedure to design the sliding mode function so that the dynamics of the plant under sliding mode has a certain behavior; the explicit characterization of a family of control laws that allow the sliding mode to be attained.

Theorem 5 (Special Properties in the Case of a Linear Plant).

Given a linear scalar-input system,

$$\dot{x} = Ax + Bu(x,t) + D\xi,$$

consider the sliding mode scalar function $s(x)=Cx$. Then:

i) under sliding mode, **a necessary and sufficient condition for the stability of the system dynamics** is that the system obtained by using the equivalent controller

$$\dot{x} = [I - B(CB)^{-1}C]Ax$$

is asymptotically stable (its non zero eigenvalues have negative real part);

ii) if the dynamics associated with the equivalent controller is asymptotically stable, then **the trajectory characterized by $|s(x)|<\varepsilon$ is as close to that produced by the equivalent controller**

as desired provided that ε is small enough and the initial conditions of the two controllers are sufficiently close;

iii) if A,B is a controllable pair then by appropriately choosing C **the eigenvalues of the system under sliding mode may be assigned arbitrarily;**

iv) a **sufficient condition** for a controller of the family

$$u = -\Sigma\Phi_i x_i - \delta SIGN(s(x))$$

with $s(x):=Cx$, and

$$\Phi_i=\alpha_i \text{ if } x_i s(x) < 0$$

$$\Phi_i=\beta_i \text{ if } x_i s(x) > 0$$

$$\delta > |[CB]^{-1}CD\xi|$$

to be a sliding mode controller is that

$$\alpha_i \leq \min_t \langle C,a_i\rangle/\langle C,B\rangle$$

$$\max_t \langle C,a_i\rangle/\langle C,B\rangle \leq \beta_i$$

where: a_i is the i-th column of A;

v) **a necessary and sufficient condition** for the system dynamics under sliding mode **to be invariant with respect to the perturbation,** ξ, is that the range of D be contained in the range of B.. ■

Remark 2. A detailed illustration of the application of theorem 5 to the design of a sliding mode controller for a position servo with an elastic link may be found in Serfass and DeSantis 1987.

4. Implementation of a Continuous Controller Via a Discrete Controller

Controllers are usually designed by assuming the control action to be arbitrarily selectable within a certain continuous range of values. However, on occasion, practical considerations may make it convenient or necessary to implement the controller by imposing that the control action be restricted to have one out of a discrete set of values. We must then deal with the problem of modifying an already designed (continuous) controller into a controller which, while subject to the discrete set of values constraint, does nevertheless produce an effect equivalent to that of the designed controller. The following theorems suggest that adoption of an appropriate sliding mode controller may provide a natural solution to this problem.

Theorem 6. (Discrete Implementation of a Continuous Nonlinear Controller)

Under the assumption that $B_0(x,t)$ is of a full rank, $B_0(x,t)B_0^+(x,t)$ is independent of x and t, and the inverse of $B_0(x,t)'B(x,t)$ exists;

then: the **dynamics** of the system

.

$\dot{x}(t) = f(x,t) + B(x,t)u(t) + p(t), \quad x(0) = x_0$

where $x(t) \in R^n$ represents the plant state, $u(t) \in R^m$ is the control, $p(t) \in R^n$ a disturbance; $f(x,t)$ and $B(x,t)$, appropriately dimensioned real valued vector and matrix functions, are given by

$$f(x,t) := f_0(x,t) + \delta f(x,t)$$

$$B(x,t) := B_0(x,t) + \delta B(x,t)$$

where $f_0(x,t)$ and B_0 characterize the "nominal" behavior of the plant, and $\delta f(x,t)$ $\delta B(x,t)$ describe the influence of parameter variations;

submitted to a discrete control satisfying

$u_i^*(t) \in \{u_{i1}, \ldots, u_{iNi}\}$, $i = 1, \ldots, m$

$SGN\{u_i^*(t) - u_{Di}(x,t) + \mu_i(t)\} := - SGN\{[B_0'\sigma(t)]_i\}$

$\mu(t) := B_0^+\{\delta f(x,t) + \delta B(x,t)u(t) + p(t)\}$

$\sigma(t) := B_0(x,t)B_0^+(x,t)S(t)$

$B_0^+(x,t) :=$ pseudo-inverse of $B_0(x,t)$,

$$S(t) := \int_0^t \{\dot{x}(t) - f_0(x,t) - B_0(x,t)u_D(x,t)\}dt$$

has the following properties:

i) the state trajectory of the system is described by

$\dot{x}(t) = f_0(x,t) + B_0(x,t)u_D(x,t)$

$+ [I - B[B_0^+B]^{-1}B_0^+]\{- B_0u_D(x,t) + \delta f(x,t) + p(t)\}$

ii) **if** the columns of $\delta f(x,t)$, $\delta B(x,t)$ and $p(t)$ are a linear combination of the columns of $B_0(x,t)$, **then**

$$\dot{x}(t) = f_0(x,t) + B_0 u_D(x,t),$$

iii) **if** the columns of $\delta f(x,t)$, $\delta B(x,t)$ and $p(t)$ are orthogonal to the columns of $B_0(x,t)$, **then**

$$\dot{x}(t) = f(x,t) + B(x,t)u_D(x,t) + p(t),$$

iv) **if** only the columns of $\delta B(x,t)$ are a linear combination of the columns of $B_0(x,t)$, **then**

$$\dot{x}(t) = f_0(x,t) + B_0 u_D(x,t) + v(t),$$

where

$$v(t) := [I- [B_0 B_0^{+}]]\{\delta f(x,t) + p(t)\}. \; .$$ ■

In the case of a linear plant, the above result may be simplified as follows.

Theorem 7: (Discrete Implementation of a Continuous Linear Controller).

Under the assumption that matrix B_0 is of a full rank, and that the inverse of $B_0{}'B$ exists;

then: **the dynamics of the system described by**

$$\dot{x}(t) = Ax(t) + Bu(t) + p(t), \quad x(0) = x_0$$

where $x(t) \varepsilon R^n$ represents the plant state, $u(t) \varepsilon R^m$ is the control, $p(t) \varepsilon R^n$ a disturbance; A and B, appropriately dimensioned real matrices, are given by

$$A := A_0 + \delta A$$

$$B := B_0 + \delta B$$

where A_0 and B_0 characterize the "nominal" behavior of the plant, and δA, δB describe the influence of parameter variations;

submitted to a discrete control satisfying

$u_i^*(t) \; \varepsilon \; \{u_{i1}, \; ..., \; u_{iNi}\}, \; i= 1, \; .. \; , \; m$

$SGN\{u_i^*(t)-u_{Di}(x,t)+\mu_i(t)\} := - \; SGN\{[B_0'\sigma(t)]_i\}$

$\mu(t) \; := B_0^+\{\delta Ax(t) \; + \; \delta Bu(t) + \; p(t)\}$

$\sigma(t) := B_0(x,t)B_0^+(x,t)S(t)$

$B_0^+(x,t)$:= pseudo-inverse of $B_0(x,t)$,

$$S(t) \; := \; \int_0^t \{\dot{x}(t) \; - \; A_0x(t) \; - \; B_0u_D(x,t)\}dt$$

has the following properties:

i) the state trajectory of the system is described by

$\dot{x}(t) \; = \; A_0x(t) \; + \; B_0u_D(x,t)$

$+ \; [I- \; B[B_0^+B]^{-1}B_0^+]\{- \; B_0u_D(x,t) \; + \; \delta Ax(t) \; + \; p(t)\}$

ii) **if** $Rank([\delta A|\delta B|p(t)|B_0]=Rank(B_0)$ **then**

$\dot{x}(t) \; = \; A_0x(t) \; + \; B_0u_D(x,t)$,

iii) **if**

$$[I- [B_0 B_0^+]]\delta = \delta \quad \text{for } \delta \varepsilon \text{SPAN}\{\delta A | \delta B | p(t)\}$$

then

$$\dot{x}(t) = Ax(t) + Bu_D(x,t) + p(t),$$

iv) **if** $\text{Rank}([\delta B | B_0]) = \text{Rank}(B_0)$

then

$$\dot{x}(t) = A_0 x(t) + B_0 u_D(x,t) + v(t),$$

where

$$v(t) := [I- [B_0 B_0^+]]\{\delta A x(t) + p(t)\}. \; . \qquad \blacksquare$$

Remark 3. An application of theorem 6 to the design of a bang bang controller for an overhead Eucledian crane may be found in DeSantis and Krau 1994. DeSantis and DeSantis 1993 apply theorem 7 to design a bang bang controller for a flexible beam.

Conclusions

Using the concept of an equivalent control together with the nominal model of the plant, we can use theorems 1, 2, 5, 6 and 7 to design a controller so as to attain a prescribed sliding mode regime and a desired state trajectory behavior. However, unaccounted for non-idealities, in both the physical plant and in the components of the controller, often prevent the sliding mode conditions to be satisfied exactly and the state trajectory to conditions to be satisfied exactly and the state trajectory to correspond to that predicted by the theory. We must then be careful to verify that, in spite of the influence of these non-idealities, the value of the sliding mode function remain small and that the trajectory produced by the real controller remain close to that produced by the ideal controller under nominal operating conditions.

In the case of special non-idealities (memoryless approximations of an ideal switching function, perturbations satisfying the matching or mis-matching conditions, and similar), theorem 3 helps us establish the extent within which this is indeed the case. However, this theorem is not of a great help in predicting what happens in the case of other more general non-idealities such as time delays, hysteresis and neglected actuator dynamics. It becomes then useful, in such cases, to explore the influence of these elements over the actual behavior of the system by means of a direct experimentation, or of extended simulations or by mean of both approaches.

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