FINITE ELEMENT METHODS WITH UPSTREAM DIFFERENCING FOR THE NAVIER-STOKES EQUATIONS

by

Dominique Pelletier

Ricardo Camarero

TABLE OF CONTENTS

1. INTRODUCTION

The Galerkin Finite Element Method (GFEM) has proved very successfull in the treatment of self adjoint boundary value problems. However, in non-self-adjoint cases, difficulties are encountered. Noteworthy examples occur in fluid mechanics with the Navier-Stokes equations and convective transport phenomena. Spurious oscillations are exhibited at high Reynolds number in the first case and high Péclêt number in the second. Until recently oscillations could only be removed by severe mesh refinement, which undermines the practical usefulness of the finite element method.

The oscillatory behaviour also results in finite difference schemes when central differences are used to approximate the convections terms. The use of upwind differencing has led to stable solutions. Observing that in one-dimensional problems, the GFEM leads to central differences, finite element researchers have turned their attention to the development of analogous schemes.

This paper presents an analysis of finite element schemes for advection-diffusion problems and a short review of upstreaming techniques in FEM. A simple scheme is then introduced for developing elements suitable for fluid dynamics problems.

This is applied to Navier-Stokes equations cast in the vorticity-stream function formulation using a linear triangular element.

Numerical results for the driven cavity problems are presented for Reynolds numbers ranging from 0 to 5000.

2. THE MODEL PROBLEM

Consider the two-dimensional flow of an incompressible fluid described by the Navier-Stokes equations written in terms of the vorticity ζ and the stream function ψ:

$$\nabla^T(u\zeta) = \nabla^T \, Re^{-1} \, \nabla\zeta \qquad (1)$$

$$\nabla^T\nabla\psi = -\zeta \qquad (2)$$

where

u is the velocity vector,

∇ is the gradient operator,

Re is the Reynolds number.

The discussion will be limited to Eq. (1), which is non-symmetric; Eq. (2) does not present any difficulties. Equation (1) is subject to the following set of boundary conditions:

$$\begin{aligned} \zeta &= f \quad \text{on} \quad \Gamma_1 \\ n^T \nabla\zeta &= g \quad \text{on} \quad \Gamma_2 \end{aligned} \tag{3}$$

f and g are known functions of the coordinates, n is the outward normal unit vector of Γ; and Γ_1, Γ_2 are two complementary subregions of the boundary of the domain of interest.

A weak form of the problem is given by

$$\int_V W \nabla u^T \zeta + \nabla W^T Re^{-1} \nabla\zeta \, dV = \int_S Re^{-1} Wg \, dS \tag{4}$$

Approximate solutions of (4) may be constructed by the finite element method in which one assumes:

$$\zeta = N_i \zeta_i$$

where

N_i is the shape function associated with node i

ζ_i is the approximation to ζ at node i

In the GFEM the weight functions W_i are set equal to the shape functions N_i.

3. THE NEED FOR UPSTREAMING

Consider the two limiting cases of Eq. (1):

$$\nabla^T \nabla\zeta = 0 \quad \text{for} \quad Re = 0 \tag{5}$$

and

$$\nabla^T u\zeta = 0 \quad \text{for} \quad Re = \infty \tag{6}$$

Equation (5) is elliptic while Eq. (6) is hyperbolic. Equation (5) is similar to Eq. (1) and does not present any difficulties.

Although the Navier-Stokes Eq. (1) will not be truly hyperbolic they can be expected to exhibit some characteristics of hyperbolic equations. This can be seen when written as follows.

$$\nabla^T u\zeta = \in \nabla^T \nabla\zeta \tag{7}$$

where $\in = Re^{-1}$ can be arbitrary small; the operator may be thought of as hyperbolic with an elliptic perturbation.

Mixed type equations are well known in transonic flow problems for which the changes of type are very sudden (i.e. through a shock wave). Switching between two discrete operators is current practice to cope with these changes of type. For the transport equation the change of type, although rapid, is smooth, continuous, and incomplete (the equation is never truly hyperbolic). We need a discrete approximation capable of reproducing these smooth changes throughout the domain of solution.

The hyperbolic nature of the equations will be best represented when the scheme uses information along the characteristic direction, which in this case is the streamline. The need to take into account the direction of the flow in numerical schemes for the present problem has already been recognised and discussed by Raithby [1,2]. In addition, it is proposed in this study, to do this is a gradual and controlled manner.

4. A SIMPLIFIED PROBLEM

The approach for developping such schemes is illustrated using the one-dimensional, linear, constant coefficient transport equation:

$$Pe \frac{d\zeta}{dx} = \frac{d^2\zeta}{dx^2} \tag{8}$$

where $Pe = \frac{UL}{K}$ is the Péclét number

U is the velocity

L is a reference length

K is the diffusivity

subject to the following boundary conditions

$$\zeta(0) = 0$$

$$\zeta(1) = 1$$

The exact solution of Eq. (8) is

$$\zeta(x) = \frac{1-e^{Pe \frac{x}{L}}}{1-e^{Pe}}$$

For small to moderate values of Pe, $\zeta(x)$ displays a solution which varies rather smoothly over the entire domain. However, as Pe increases much beyond unity (i.e. $Pe > 10$) the solution becomes one of a boundary layer type: $\zeta(x) = 0$ except in the region near $x = 1$ (fig. 1a).

A weak form of this problem is given by

$$\int_0^1 (W \, Pe \frac{d\zeta}{dx} + \frac{dW}{dx}\frac{d\zeta}{dX}) \, dx = 0$$

The usual technique of GFEM on a uniform mesh yields the following system of algebraic equations:

$$(-\frac{Pg}{2} - 1)\ \zeta i\text{-}1 + 2\zeta i + (\frac{Pg}{2} - 1)\ \zeta i\text{+}1 = 0 \qquad (9)$$

where $Pg = \frac{Uh}{K} = \frac{h}{L} Pe = \frac{Pe}{M}$ is the grid Péclét number.

M is the number of elements.

When applying the boundary conditions associated with Eq. (8), it is well known that the solution of Eq. (9) will exhibit significant and spurious oscillations for $Pg > 2$. This is readily seen from the analytical solution of Eq. (9):

$$\zeta_i = \frac{1 - \left[\dfrac{1+\frac{Pg}{2}}{1-\frac{Pg}{2}}\right]^i}{1 - \left[\dfrac{1+\frac{Pg}{2}}{1-\frac{Pg}{2}}\right]^M} \qquad i = 0,1,2,\ldots,M \qquad (10)$$

The very form of this equation suggest some sort of problem for $Pg \geqslant 2$. For $Pg = 2$, the limiting form of Eq. (10) gives:

$$\zeta_i = 0 \quad \text{for all} \quad i \neq M$$

$$\zeta_i = 1 \quad \text{for} \quad i = M$$

and for $Pg > 2$ it is obvious that node-to-node oscillations will occur. Furthermore, for high Pg the oscillations will depend strongly on the parity of M (fig. 1b, 1c) [3, 7]. The same results would have been obtained by applying central finite differences to Eq. (8).

One may legitimately wonder whether there exists a differential equation whose analytical solution identicaly exhibits the wiggles of the above discrete model. The answer is provided through a Hirt's analysis [4]: Substituting the Taylor series expansion of

$$\zeta i\pm1 = \zeta i \pm h\zeta' + \frac{h^2}{2!}\zeta'' \pm \frac{h^3}{3!}\zeta''' + 0(h^4)$$

in Eq. (9) and using Eq. (8) the third order derivative may be expressed as:

$$\zeta''' = Pe\ \zeta''$$

The discrete equation then becomes

$$Pe\frac{d\zeta}{dx} - (1 - \frac{Pg^2}{3!})\frac{d^2\zeta}{dx^2} = 0$$

The GFEM approximation actually introduces a negative artificial diffusivity which tends to promote oscillations for high enough Pg, and one in fact solves an equation of the form:

$$Pe\frac{d\zeta}{dx} + \gamma\frac{d^2\zeta}{dx^2} = 0$$

which admits an oscillatory solution with negative values of ζ. The GFEM, for high Pe, produces the exact solution to the wrong problem!

This is avoided in finite differences by using one-sided differences for the convection terms. This produces the following system of equations.

$$(-Pg - 1)\ \zeta_{i-1} + 2\zeta_i + (-1)\ \zeta_{i+1} = 0 \tag{11}$$

The exact solution is given by

$$\zeta_i = \frac{1 - (1+Pg)^i}{1 - (1+Pg)^M} \tag{12}$$

which admits no oscillations. For $Pg \gg 1$ we can use:

$$\zeta_i = Pg^{i-M}$$

which is closely approximated by

$$\zeta_i = 0 \quad \text{for} \quad i = 0,1,\ldots,M-1$$

$$\zeta_i = 1 \quad \text{for} \quad i = M$$

The "upwind" solution looks quite good exept near the outflow where the solution is <u>independent</u> of Pe. Hirt's analysis shows that the upwind solution produces an exact solution to the following O.D.E.:

$$Pe\,\frac{d\zeta}{dx} - \left(1 + \frac{Pg}{2}\right)\frac{d^2T}{dx^2} = 0$$

Hence whenever Pg is greater than 10 essentially all of the diffusion in the upwind scheme is artificial and the results are almost independent of the Péclét number. The effective Péclét number is given by

$$Pe_{eff} = \frac{Pe}{1 + \frac{Pg}{2}}$$

The independance of the solution comes from the fact that the mesh is too coarse to resolve the very fine outlet boundary layer. (Its thickness is of the order $\delta \simeq \frac{1}{Pe}$ [3]).

5. ONE-DIMENSIONAL UPWINDING ELEMENTS

A first attempt to overcome the oscillations was suggested by Zienkiewicz et al. [5] and established firmly by Chriestie et al. [6] in the context of one dimensional problems. They introduced the idea of using weight functions of a different order from that of the shape functions. For higher order schemes of the same family the reader is refered to [8].

Recently, Hughes [9] introduced the concept of an optimal one point integration procedure for the convection terms. It produces the following system of equations for the one-dimensional case:

$$[1 + \frac{Pg}{2}(1+\tilde{\xi})]T_{i-1} - 2[1 + \frac{Pg}{2}\tilde{\xi}]T_i + [1 - \frac{Pg}{2}(1-\tilde{\xi})]T_{i+1} = 0$$

The exact solution to this discrete system is given by:

$$T_i = A + B\left[\frac{1 + \frac{Pg}{2}(1+\tilde{\xi})}{1 - \frac{Pg}{2}(1-\tilde{\xi})}\right]^i$$

The constant A and B are determined from the boundary conditions.

Hirt's analysis yield the following differential equation

$$Pe\frac{dT}{dx} - [1 + \frac{Pg}{2}\tilde{\xi} - \frac{Pg^2}{3!}]\frac{d^2T}{dx^2} = 0$$

The artificial diffusion is given by

$$\nu_{art} = \frac{Pg}{2}\tilde{\xi} - \frac{Pg^2}{3!}$$

We now have the free parameter $\tilde{\xi}$ which can be used to control ν_{art}. The original differential equation is recovered if we set

$$\tilde{\xi} = \frac{Pg}{3} + O(h^2)$$

with an $O(h^2)$ accuracy. When using the entire Taylor series expansion, the original differential equation is identically recovered if

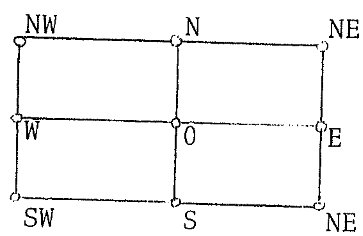
$$\tilde{\xi} = \coth\frac{Pg}{2} - \frac{2}{Pg} .$$

It must be pointed that exact nodal results are obtained on any mesh only for the one-dimensional, linear, constant coefficient differential equation.

Although the solution is nodaly exact, a very fine mesh must be used to accurately represent the outflow boundary layer. If the mesh is too coarse the boundary layer will lie entirely within the last element and the solution will not give any clue on its true thickness.

6. TWO-DIMENSIONAL UPWINDING ELEMENTS

Before discussing and describing any multi-dimensional schemes for convection dominated flows, it is very instructive to display the often quoted "central-difference nature" of the GFEM in modeling nonlinear advection in one of the simplest cases possible: the 4-noded bilinear element on a square mesh. Consider the following 4 elements patch:



The GFEM approximation to $U\frac{\partial T}{\partial x}$ at 0 is given by:

$$U\frac{dT}{dx}\bigg|_0 = \frac{\int_{-h}^{+h}\int_{-h}^{+h} N_0\, u\, \frac{\partial T}{\partial x}\, dxdy}{\int_{-h}^{+h}\int_{-h}^{+h} N_0\, dxdy}$$

Where

$$u = N_j\ U_j$$

$$T = N_j\ T_j$$

$$N_0 = \text{Shape function at } 0$$

The result is

$$u\frac{dT}{dx}\bigg|_0 = \frac{4}{9}\cdot\frac{6U_0+U_n+U_s}{8}\cdot\frac{(T_E-T_W)}{2h}+\frac{1}{9}\frac{U_0+U_n}{2}\frac{(T_{NE}-T_{NW})}{2h}+\frac{1}{9}\cdot\frac{U_0+U_S}{2}\frac{(T_{SE}-T_{SW})}{2h}$$

$$+\frac{1}{9}\cdot\frac{6U_E+U_{NE}+U_{SE}}{8}\cdot\frac{T_E-T_0}{h}+\frac{1}{36}\frac{U_E+U_{NE}}{2}\frac{(T_{NE}-T_N)}{h}+\frac{1}{36}\cdot\frac{U_E+U_{SE}}{2}\cdot\frac{(T_{SE}-T_S)}{h}$$

$$+\frac{1}{9}\cdot\frac{6U_W+U_{NW}+U_{SW}}{8}\cdot\frac{T_0-T_W}{h}+\frac{1}{36}\frac{U_W+U_{NW}}{2}\frac{(T_N-T_{NW})}{h}+\frac{1}{36}\frac{U_W+U_{SW}}{2}\frac{(T_S-T_{SW})}{h}$$

Inspection of this expression reveals the following:

1. GFEM approximation of advection involves very intense coupling and is much more complex than its finite difference counter part.

2. Although centered difference terms clearly dominate the approximation (2/3 of the total), the remaining 1/3 is equaly divided between upwind-and-downwind-type differences. Hence, the issue of the central difference nature of GFEM is more complex than one might expect from the previous one-dimensional analysis.

Heinrich et al. [10] generalised the scheme of [5,6] to the two-dimensional transport equation. A few month later Heinrich and Zienkiewicz [11] published a quadratic scheme. In these formulations the weight functions are set equal to the shape functions modified by higher-order terms. In both cases [10,11] various problems for the heat transport equation were solved successfully.

It is only with the work of Moult et al. [12] that an upstreaming technique is used to solve the full Navier-Stokes equations at high Reynolds numbers. The vorticity equation is approximated via a local potential approach. An iterative scheme is thus easily set up with the convection terms acting as a forcing function. The diagonal dominance of the iteration matrix expessed on a regular, triangular mesh is preserved by expressing the convection terms in a manner very similar to first order finite difference upstreaming techniques. The results presented for the driven cavity clearly show the well known cross-diffusion effects associated with these schemes. The authors achieved better results with higher order finite difference schemes.

At the same time Ikegawa [13] presented another scheme for the Navier-Stokes equations. The numerical approximation to the vorticity equation is obtained by the following simple integral:

$$\int_V \nabla^T U\zeta \, dv - \int_V \nabla^T Re^{-1} \nabla\zeta \, dv = 0 \qquad (13)$$

which yields, upon application of Gauss's divergence theorem:

$$\oint_S U^T n\zeta \, ds - \oint_S Re^{-1} \frac{\partial\zeta}{\partial n} \, ds = 0 \qquad (14)$$

Equation (9) may be regarded as a conservation form of vorticity. It is a special case of Eq. (4) in which the weight functions are chosen equal to one. It is also the basis of the well known finite volume technique. The first term of Eq. (14) is formulated in two different ways according to the sign of the normal velocity component V_n, as follows (fig. 2):

$$\oint_S U^T n\zeta \, ds = \oint_S v_n \, \zeta \, ds = \sum_{i=1}^{3} V_n(\zeta_k, \zeta_{ki}) \, d_i$$

where

$$(\zeta_k, \zeta_{ki}) = \zeta_k \quad \text{if} \quad V_n \geqslant 0$$

$$(\zeta_k, \zeta_{ki}) = \zeta_{ki} \quad \text{if} \quad V_n < 0$$

d_i is the length of the segment i and ζ_k is taken as the average value in the triangle. The vorticity and the stream-function equations are solved alternatively until convergence is reached. Results were reported for cavity flows with Reynolds numbers ranging from 500 to 1000.

7. A SIMPLE SCHEME

From the arguments of the previous sections, it is clear that the direction of the flow must be taken into account for the evaluation of the convection terms. It can be shown that the scheme introduced by Hughes [9] suffers from severe cross-wind diffusion [21]. Following the same line of thought we present a linear triangular element which overcomes these difficulties. The idea underlying the present scheme is a one dimensional analysis aligned on the streamlines passing through the element.

Equation (4) leads to the following system of nonlinear algebraic equation:

$$[B_2 + B_3]\ \{\zeta\} = \{G\} \tag{15a}$$

where

$$B_2(ij) = \int_V \nabla W_i^T\ Re^{-1}\ \nabla N_j\, dv = \text{a diffusion matrix} \tag{15b}$$

$$B_3(ij) = \int_V W_i\ \nabla^T u\ N_j\ dv = \text{a convection matrix} \tag{15c}$$

$$G(i) = \oint_S Re^{-1}\ W_i\ g\ ds = \text{Neumann boundary conditions} \tag{15d}$$

The elemental convection matrix is evaluated with a special one point Gauss integration formula:

$$B_3(ij) = W_i(\tau)\ \nabla^T\ u(0)\ N_j(\tau)\ J(0)\ V \tag{16}$$

where

- u is the velocity vector; it is known from the previous iteration.

- τ is a point located on the streamline passing through the isoparametric center 0 of the element. Its location controls the artificial viscosity, hence the degree of upwinding.

- J is the Jacobian determinant of the coordinate transformation.
- V is the volume of the element in that local system of coordinates. ($J*V$ is the true volume of the element).
- W_i are the weight functions and they are set equal to the shape function N_i.

Referring to fig. 3, the point τ is chosen as follows:

1. Determine the location of B and the arc length h of BO.
2. Calculate the element Reynolds number $\alpha = \|u\|*h*Re$ where u and h are non-dimensional velocity and length.
3. Compute the relative position $\tilde{\tau}$ from a one-dimensional analysis:

$$\tilde{\tau} = \coth \alpha - \frac{1}{\alpha}$$

4. The local coordinates of τ are given by:

$$\xi_\tau = \xi_0 + \tilde{\tau}(\xi_B - \xi_0)$$

$$\eta_\tau = \eta_0 + \tilde{\tau}(\eta_B - \eta_0)$$

and they are used to evaluate the convection matrix in Eq. (16).

8. APPLICATION

The scheme presented above was successfully applied to the driven cavity problem for Reynolds numbers varying from 0 to 5000. The results for moderate Reynolds numbers ($Re \leqslant 400$) compare very well with available numerical and experimental data.

Figures 4 - 15 illustrate the results obtained with the present method for various values of Reynolds numbers ranging from 0 to 5000. The unsteady state solution is shown as the evolution of vorticity and stream function at the center of the cavity. The steady state solution is shown as the map of isovorticity and streamlines. The computations were carried out for a 15×15 mesh.

It is noted that the present finite element solution is consistent with the physical characteristics of high Reynolds number flows; the presence of a nearly perfect circular core on the stream function contour map is typical of irrotational (inviscid) flow. This irrotationality is also found in a zone of zero vorticity on the vorticity contour plot. It should be noted that several published solutions do not respect this behaviour.

9. CONCLUSION

Several techniques to obtain finite elements suitable for solving the Navier-Stokes and transport equations at high Reynolds and Péclét numbers have been presented.

A general formulation is certainly that of Zienkiewicz et al. [5,6,8,10,11,18,22,23]. The latter two references present a very state of the art techniques.

The other effective methods by Moult et al. [12] and Ikegawa [13] are much more simple but are restricted to triangular elements. The former technique is an application of finite difference technique to a triangular mesh, while the later is very similar to the fluid-in-cell and finite-volume method [4]. It could be extended to quadrilateral elements.

The authors' method is very simple to implement into existing codes employing GFEM for convective transport phenomena. The nonsymmetric location of the integration point controls the degree of upwinding thus maximizing the accuracy of the solution. The formulation is valid for any type of element; higher-order elements are amenable to an analogue treatment. The method is similar to the scheme recently introduced by Hughes et al. [21] in which a controlled artificial diffusion is explicitely introduced in the equations which are then solved by the usual GFEM.

While of no physical nor practical use, laminar solutions at so high a Reynolds number open the way to the treatment of turbulent flows [24,25,26,27,28,29] which are of prime importance to practicing engineers, meteorologists and oceonographers.

A velocity-pressure [14,15,16,18,19,20,21,30] would allow the solution of a wider class of problems and boundary conditions (free surfaces, three-dimensional flows, pressure & stress boundary conditions.

REFERENCES

1. Raithby, G.D., "A Critical Evaluation of Upstream Differencing Applied to Problems Involving Fluid Flow", Comp. Meth. Appl. Mech. & Eng., Volume 9, 1976, pp. 75-103.

2. Raithby, G.D., "Skew Upstream Differencing Schemes for Problems Involving Fluid Flow", Comp. Meth. Appl. Mech. & Eng., Volume 9, 1976, pp. 153-164.

3. Gresho, P.M. and Lee, R.L., "Don't Suppress the Wiggles - They're Telling You Something!", In Finite Element Methods for Convection Dominated Flows. (T.J.R Hughes, editor) ASME annual meeting, December 2-7, 1979.

4. Roache P.J., "Computational Fluid Dynamics", Hermosa publishers, Albuquerque, New Mexico, 1976.

5. Zienkiewicz, O.C., Gallagher, R.H. and Hood, P., "Newtonian and Non-Newtonian Viscous Incompressible Flow , Temperature Induced Flow", 2nd Conf. on Mathematics of Finite Elements and Application, Brunel University, Brunel, 1975.

6. Chriestie, I., Griffith, D.F., Mitchell, A.R. and Zienkiewicz, O.C., "Finite Element Method for Second-Order Differential Equations with Significant First Derivative", Int. J. Num. Meth. Eng., Volume 10, 1976, pp. 1389-1396.

7. Lillington, J.N., Shepherd, I.M., "Central Difference Approximation to the Heat Transport Equation". Int. J. Num. Meth. Eng., Volume 12, 1978, pp. 1607-1617.

8. Chriestie, I., Mitchell, A.R., "Upwinding of High Order Galerkin Methods in Conduction-Convection Problems". Int. J. Num. Meth. Eng., Volume 12, 1978, pp. 1764-1771.

9. Hughes, T.J.R., "A Simple Scheme for Developing Upwind Finite Element", Int. J. Num. Meth. Eng., Volume 12, 1978, pp. 1359-1365.

10. Heinrich, J.C., Huyakorn, P.S., Zienkrewicz, O.C., Mitchell, A.R., "An Upwind Finite Element Scheme for Two-Dimensional Convective Transport Equation", Int. J. Num. Meth. Eng., Volume 11, 1977, pp. 131-143.

11. Heinrich, J.C., Zienkiewicz, O.C., "Quadratic Finite Element Schemes for Two-Dimensional Convective Transport Problems", Int. J. Num. Meth. Eng., Volume 11, 1977, pp. 1831-1844.

12. Moult, A., Burley, D., Rawson, H., "The Numerical Solution of Two-Dimensional Steady Flow Problems by the Finite Element Method", Int. J. Num. Meth. Eng., Volume 14, 1979 pp. 11-35.

13. Ikegawa, M., "A New Finite Element Technique for the Analysis of Steady Viscous Flow Problems", Int. J. Num. Meth. Eng., Volume 14, 1979, pp. 103-113.

14. Hughes, T.J.R., Liu, W.K., Brooks, A., "Finite Element Analysis of Incompressible Viscous Flows by the Penalty Function Formulation", Journal of Computational Physics, Volume 30, 1979, pp. 1-60.

15. Olson, M.D., Tuann, S.Y., "Primitive Variables Versus Stream Function Finite Element Solutions of the Navier-Stokes Equations", In Finite Elements in Fluids - Volume 3, John Wiley & Sons, 1978.

16. Hughes, T.J.R., Taylor, R.L., Levy, J.F., "High Reynolds Number, Seady, Incompressible Flows by a Finite Element Method", In Finite Elements in Fluids - Volume 3, John Wiley & Sons, 1978.

17. Tuann, S.Y., Olson, M.D., "A Review of Computing Methods for Recirculating Flows", J. Comp. Phys., Volume 29, Oct. 1978, pp. 1-19.

18. Zienkiewicz, O.C., Heinrich, J.C., "A Unified Treatment of Steady-State Shallow Water and Two-Dimensional Navier-Stokes Equations - Finite Element Method Penalty Functions Approach", Comp. Meth. Appl. Mech. Eng., Volume 17/18, 1979, pp. 673-698.

19. Huyakorn, P.S., Taylor, R.L., Lee, R.L., Ghescho, P.M., "A Comparison of Various Mixed Interpolation Finite Elements in the Velocity-Pressure Formulation of the Navier-Stokes Equations", Comp. and Fluids, Volume 6, 1978, pp. 25-35.

20. Bercovier, M., "A Family of Finite Elements with Penalisation for the Numerical Solution of Stokes and Navier-Stokes Equations". IFIP 77. (ed. B. Gilchrist), JFIP North Holland, 1977.

21. Hughes, T.J.R., Brooks, A., "A Multi-Dimensional Upwind Scheme with no Crosswind Diffusion", In Finite Element Methods for Convection Dominated Flows. (T.J.R. Hughes, editor), ASME annual meeting, Dec. 2-7, 1979.

22. Griffiths, D.F., Mitchell, A.R., "On Generating Upwind Finite Element Methods", In Finite Element Methods for Convection Dominated Flows. (T.J.R. Hughes, editor) ASME annual meeting, Dec. 2-7, 1979.

23. Heinrich, J., Zienkiewicz, O.C., "The Finite Element Method and "Up-winding" Techniques in the Numerical Solution of Convection Dominated Flow Problems". In Finite Element Methods for Convection Dominated Flows. (T.J.R. Hughes, editor) ASME annual meeting, Dec. 2-7, 1979.

24. Taylor, C., Thomas, C.E., Morgan, K., "Confined Turbulent Flow Utilising the Finite Element Method". In Finite Element Methods for Convection Dominated Flows. (T.J.R. Hughes, editor) ASME annual meeting, Dec. 2-7, 1979.

25. Baker, A.J., "Finite Element Analysis of Turbulent Flows", Proceeding First International Conference on Numerical Methods in Laminar and Turbulent Flow, Swansea, 1978, pp. 203-229.

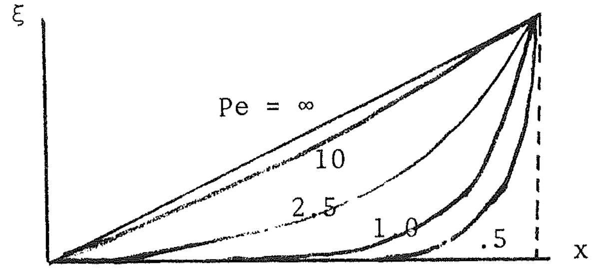
26. Taylor, C., Hughes, T.G., Morgan, K., "A Numerical Analysis of Turbulent Flow in Pipes", Comp. and Fluids, Volume 5, 1977, pp. 191-204.

27. Morgan, K., Hughes, T.G., Taylor, C., "A Numerical Model of Turbulent Shear Flow Behind a Prolate Spheroid", Applied Mathematical Modelling, Volume 2, 1978, pp. 271-274.

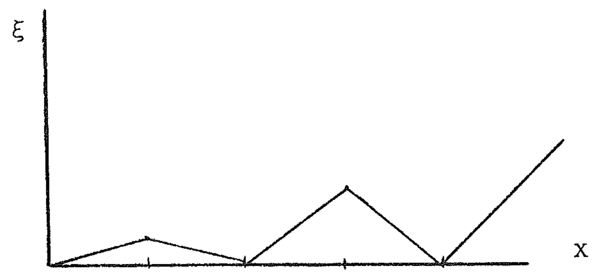
28. Taylor, C., Hughes, T.G., Morgan, K., "Finite Element Solution of One-Equation Models of TurbulentFlow", Journ. Comp. Physic, Volume 29, No. 2, 1978, pp. 163-172.

29. Morgan, K., Hughes, T.G., Taylor, C., "The Analysis of Turbulent Free Shear and Channel Flows by the Finite Element Method", Comp. Meth. in Appl. Mech. and Eng., Volume 19, 1979, pp. 117-125.

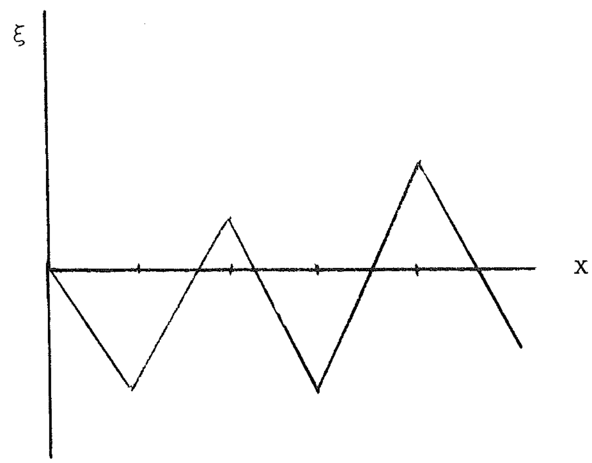
30. Taylor, C., Hood, P., "A Numerical Solution of the Navier-Stokes Equations Using the Finite Element Method", Computers and Fluids, Volume 1, No. 1, 1973, pp. 73-100.



1 a Analytical solution (Roache [4])



1 b GFEM with add number of elements



1 c GFEM with even number of elements

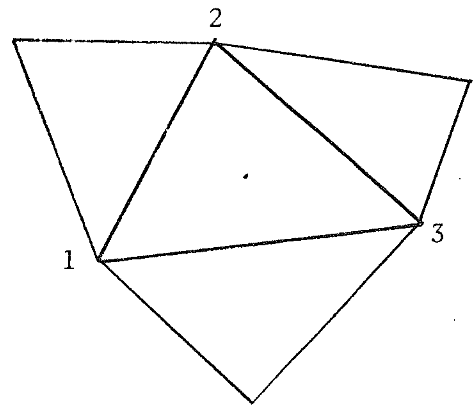


Fig. 2 The triangular element of
of Ikegawa [13]

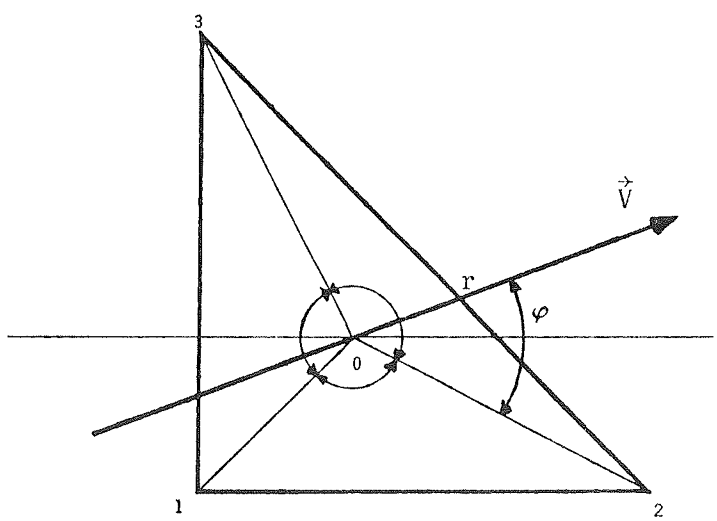


Fig. 3 Proposed up-winding element

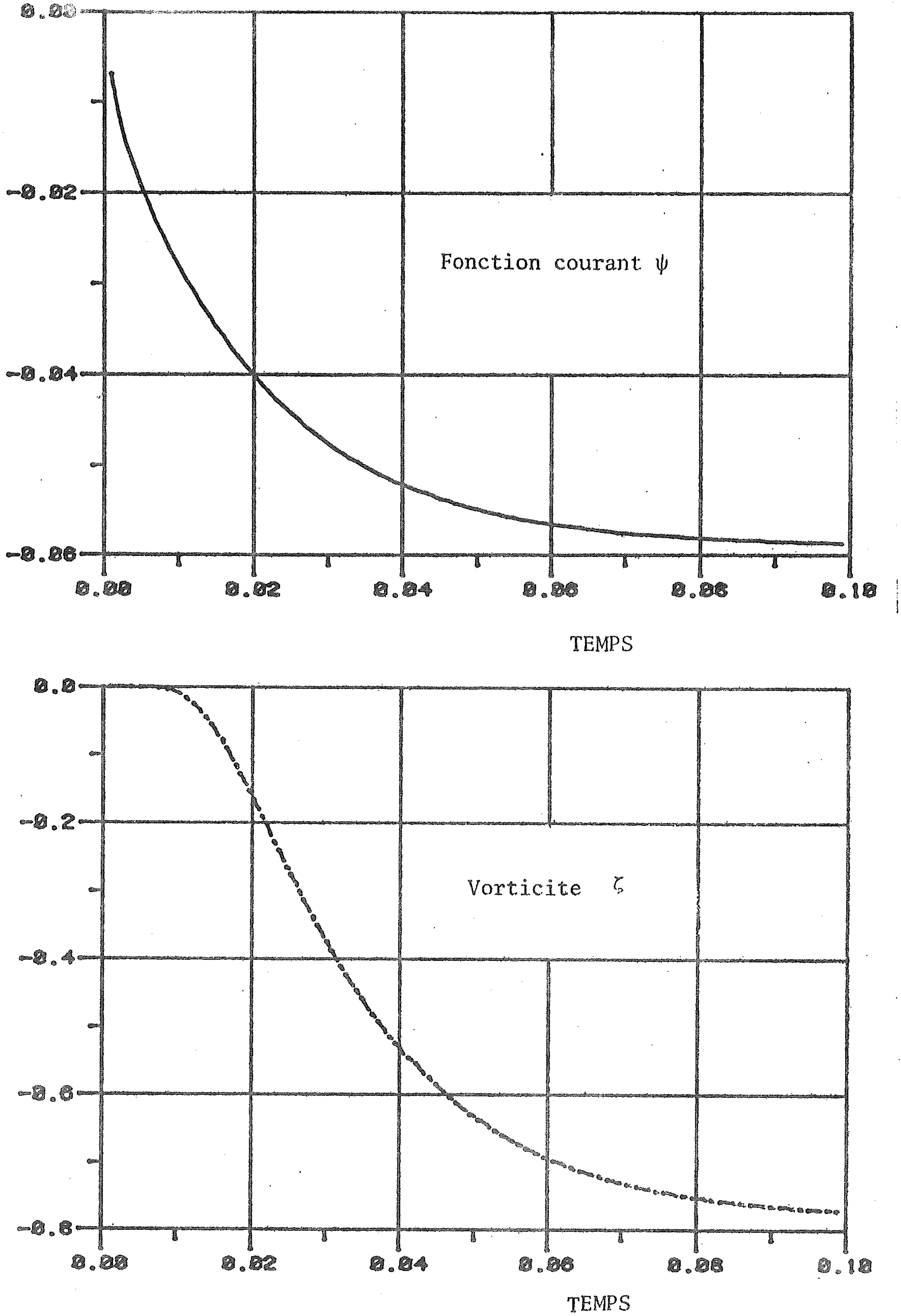
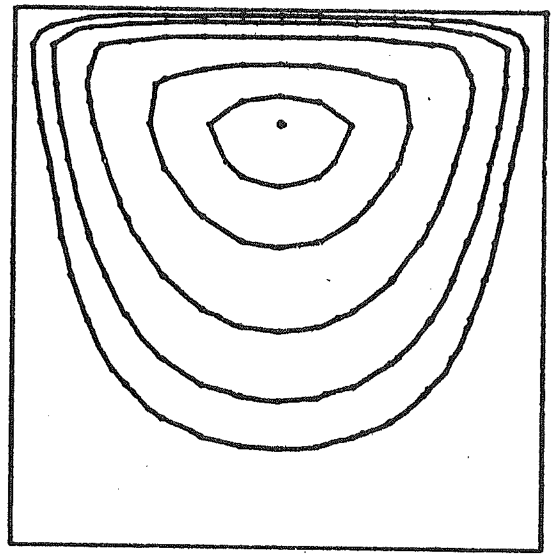
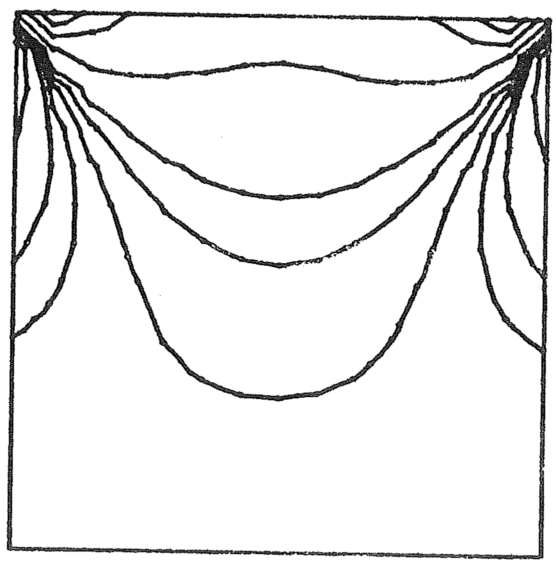


FIG. 4 HISTORIQUE DU POINT CENTRAL, SOLUTION PAR ELEMEMTS FINIS
POUR REYNOLDS DE 0 ET UN MAILLAGE DE 15X15



Lignes de courant



Lignes de vorticite

FIG. 5 LIGNES DE COURANT ET D ISO-VORTICITE A L ETAT STATIONNAIRE, SOLUTION PAR ELEMENTS FINIS POUR UN NOMBRE DE REYNOLDS DE 0.0 ET UN MAILLAGE DE 15X15

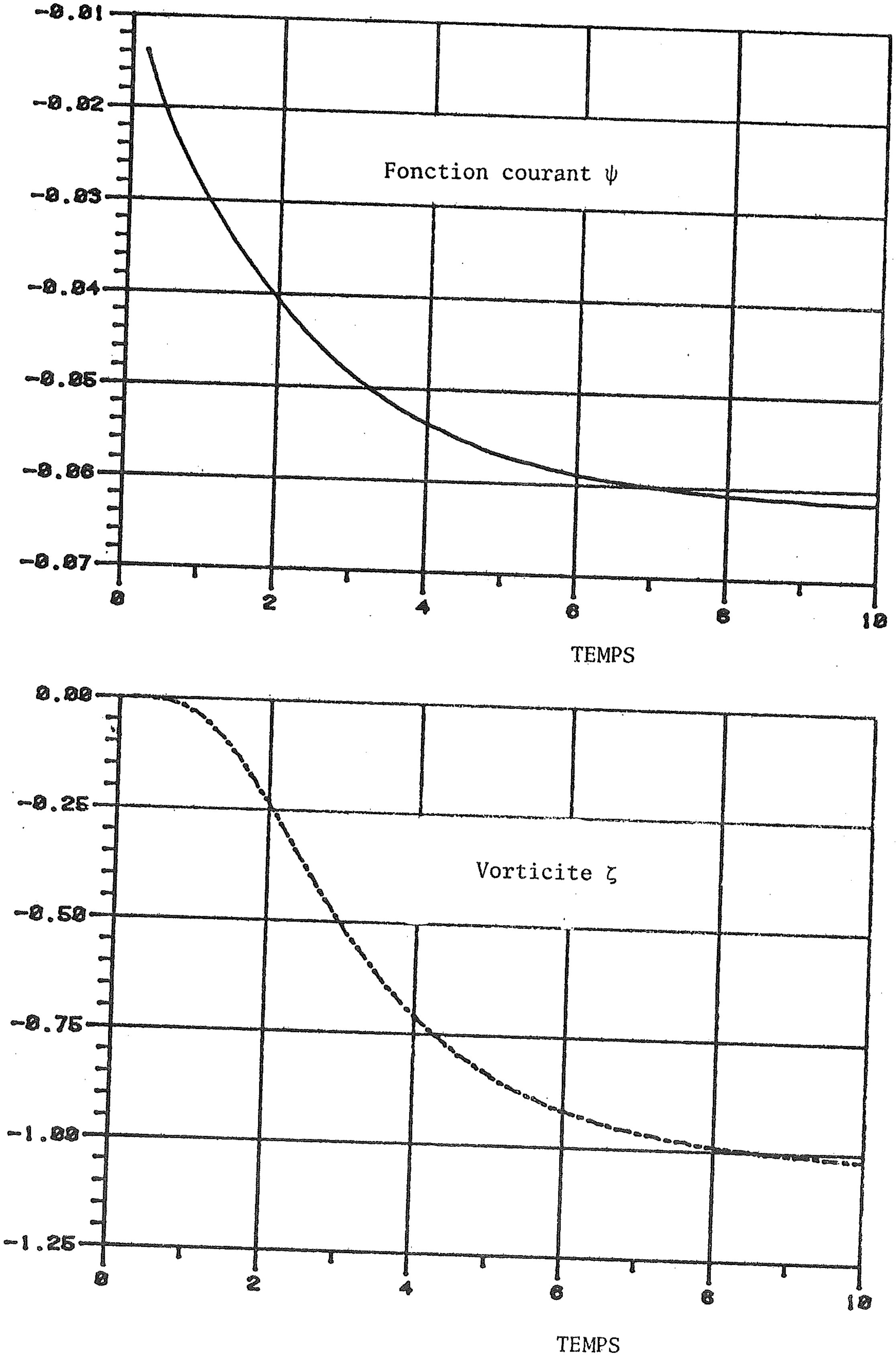
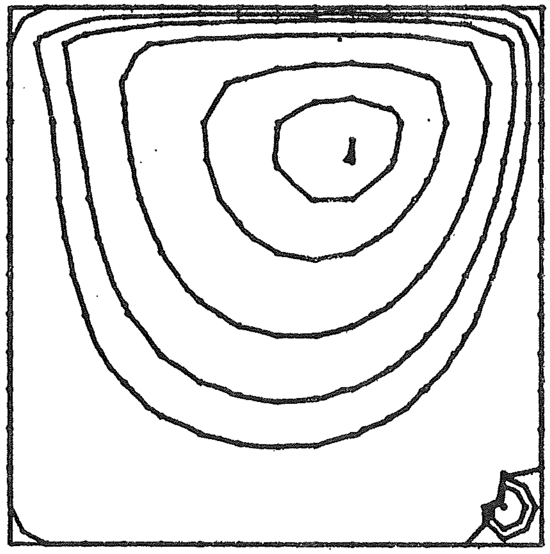
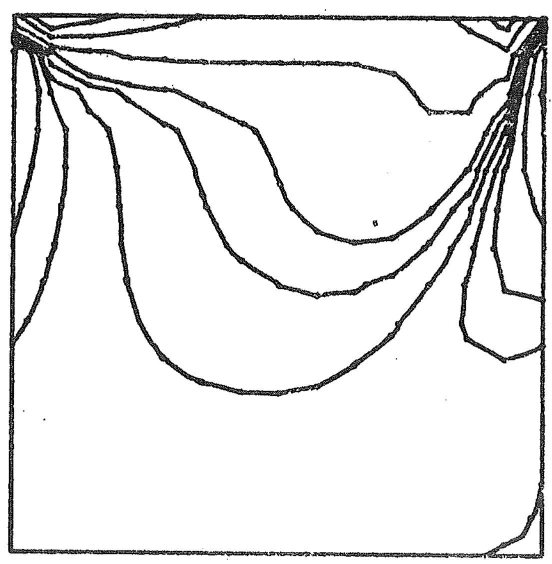


FIG. 6 HISTORIQUE DU POINT CENTRAL, SOLUTION PAR ELEMENTS FINIS POUR UN NOMBRE DE REYNOLDS DE 100 ET UN MAILLAGE DE 15X15

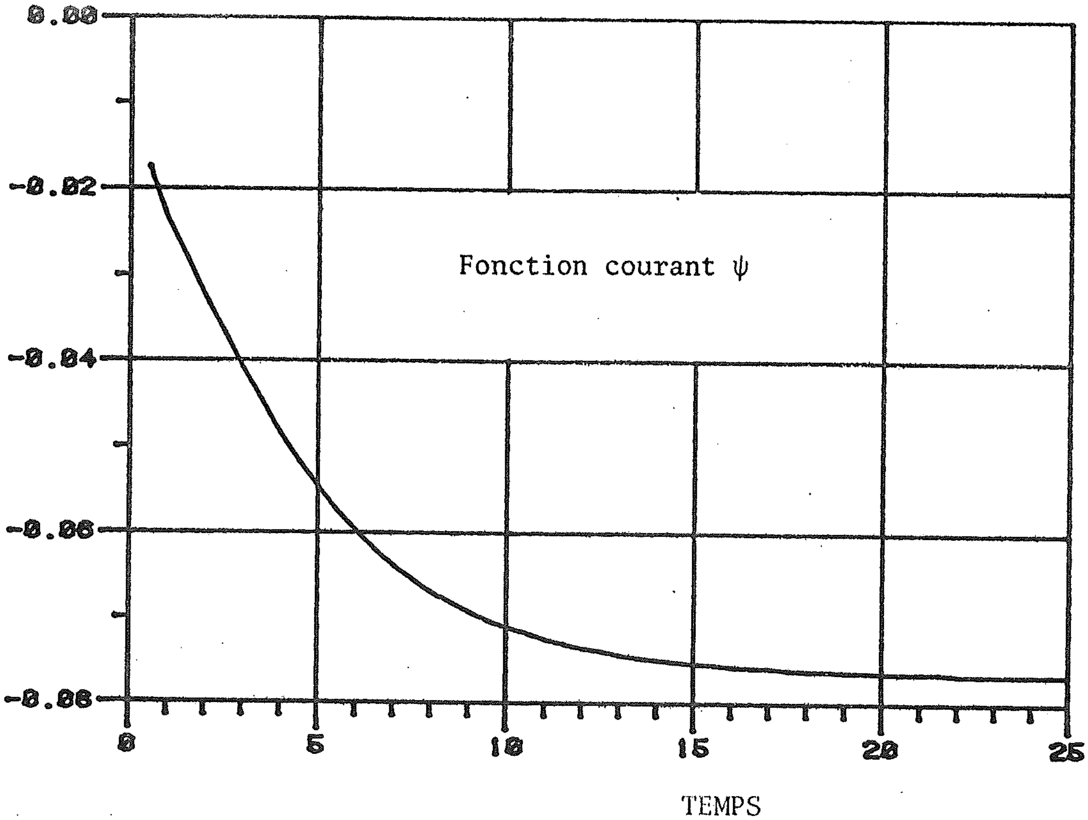


Lignes de courant



Lignes de vorticite

FIG. 7 LIGNES DE COURANT ET D ISO-VORTICITE A L ETAT STATIONNAIRE, SOLUTION PAR ELEMENTS FINIS POUR UN NOMBRE DE REYNOLDS DE 100 ET UN MAILLAGE DE 15X15



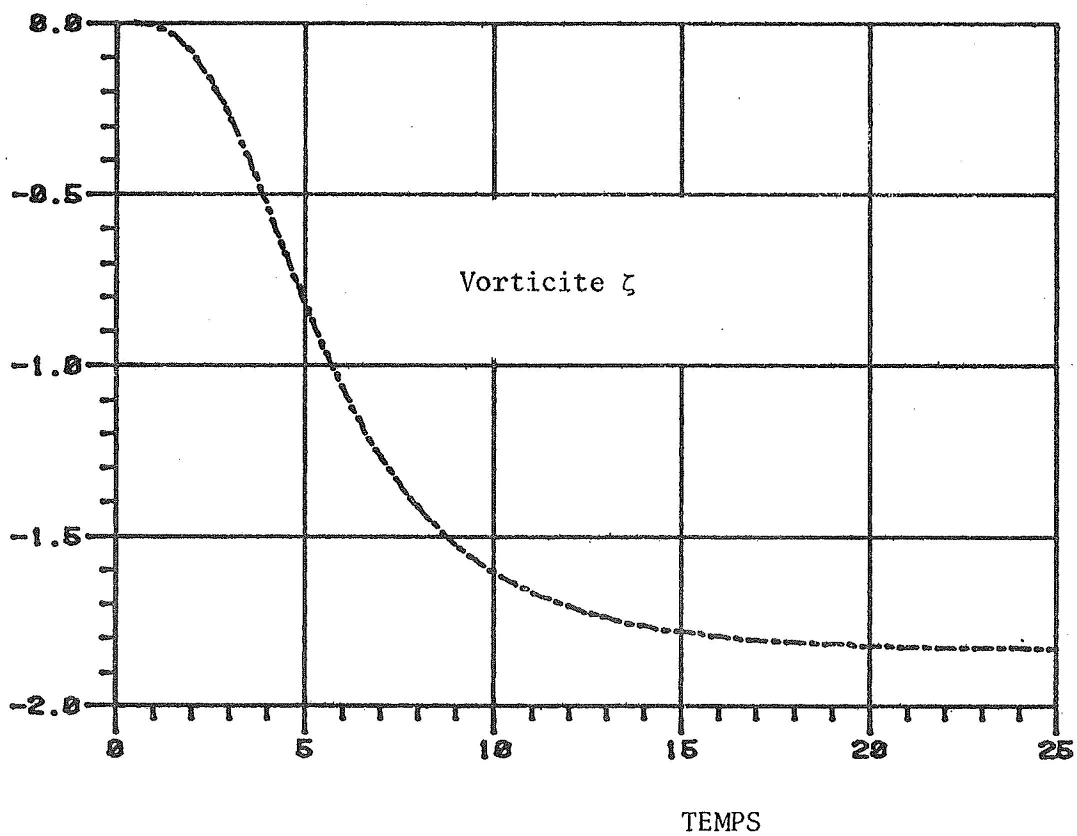
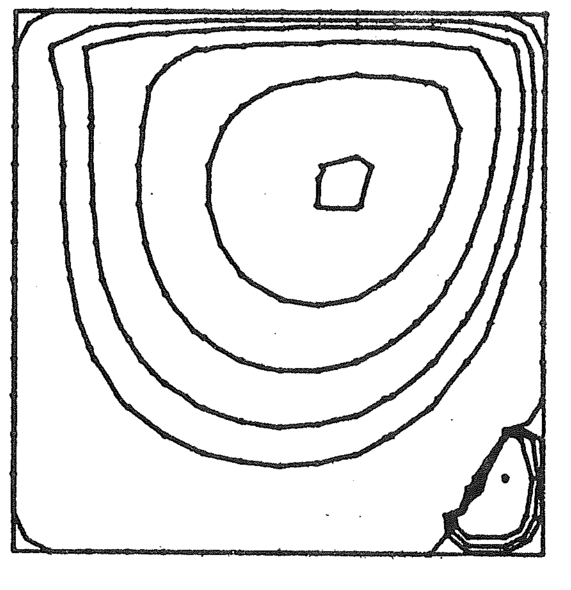
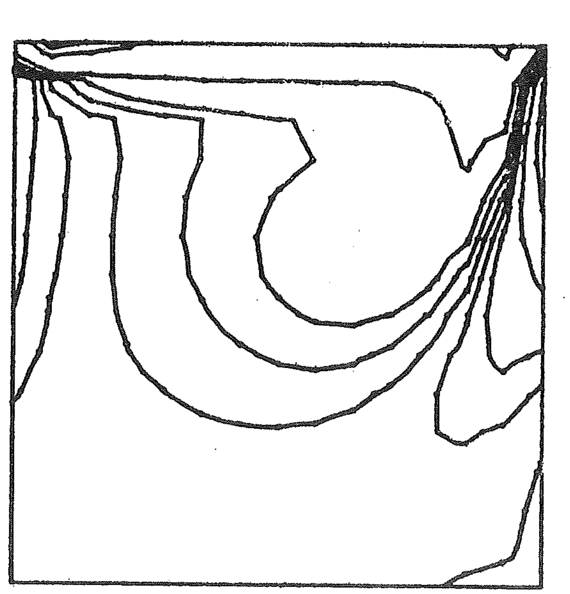


FIG. 8 HISTORIQUE DU POINT CENTRAL, SOLUTION PAR ELEMENTS FINIS POUR UN NOMBRE DE REYNOLDS DE 200 ET UN MAILLAGE DE 15X15



Lignes de courant



Lignes de vorticite

FIG. 9 LIGNES DE COURANT ET DE VORTICITE A L ETAT STATIONNAIRE, SOLUTION PAR ELEMENTS FINIS POUR UN NOMBRE DE REYNOLDS DE 200 ET UN MAILLAGE DE 15X15

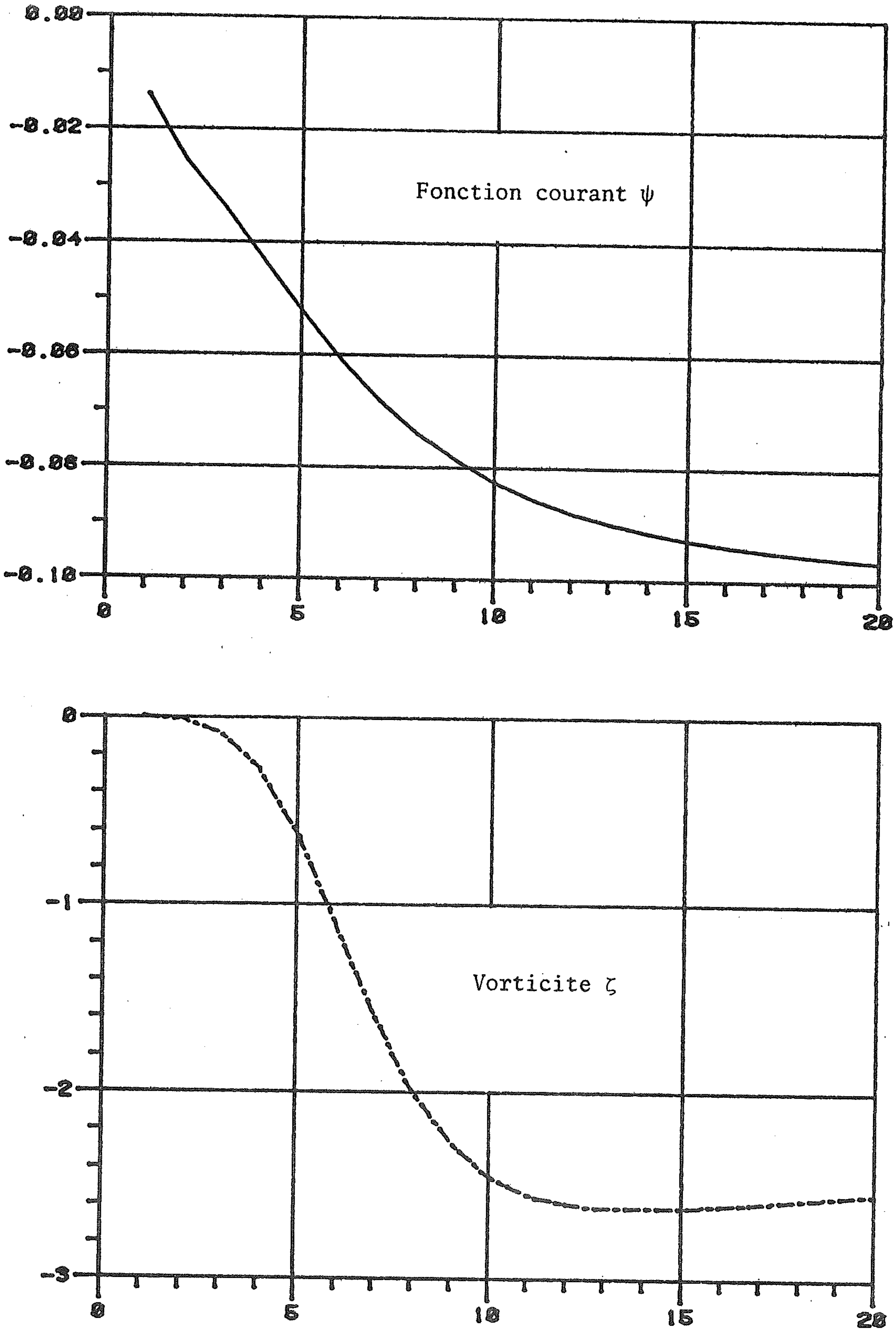
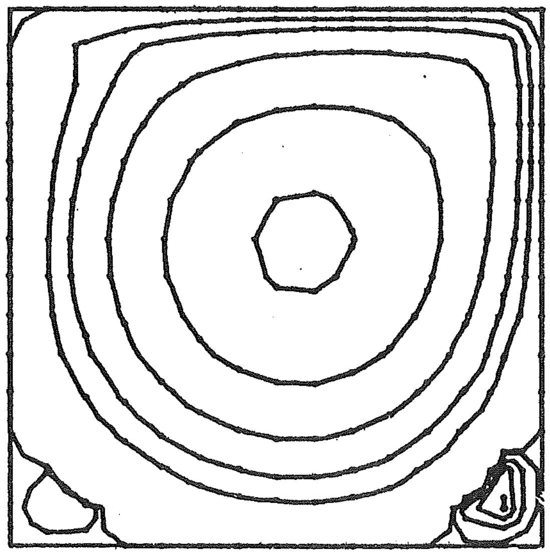
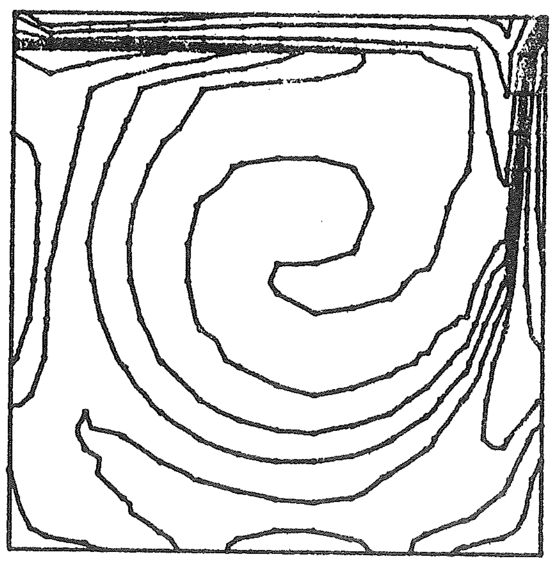


FIG. 10 HISTORIQUE DU POINT CENTRAL, SOLUTION PAR ELEMENTS FINIS POUR UN NOMBRE DE REYNOLDS DE 500 ET UN MAILLAGE DE 15X15



Lignes de courant



Lignes de vorticite

FIG. 11 LIGNES DE COURANT ET D ISO-VORTICITE A L ETAT STATIONNAIRE, SOLUTION PAR ELEMENTS FINIS POUR UN NOMBRE DE REYNOLDS DE 1000 ET UN MAILLAGE DE 15X15

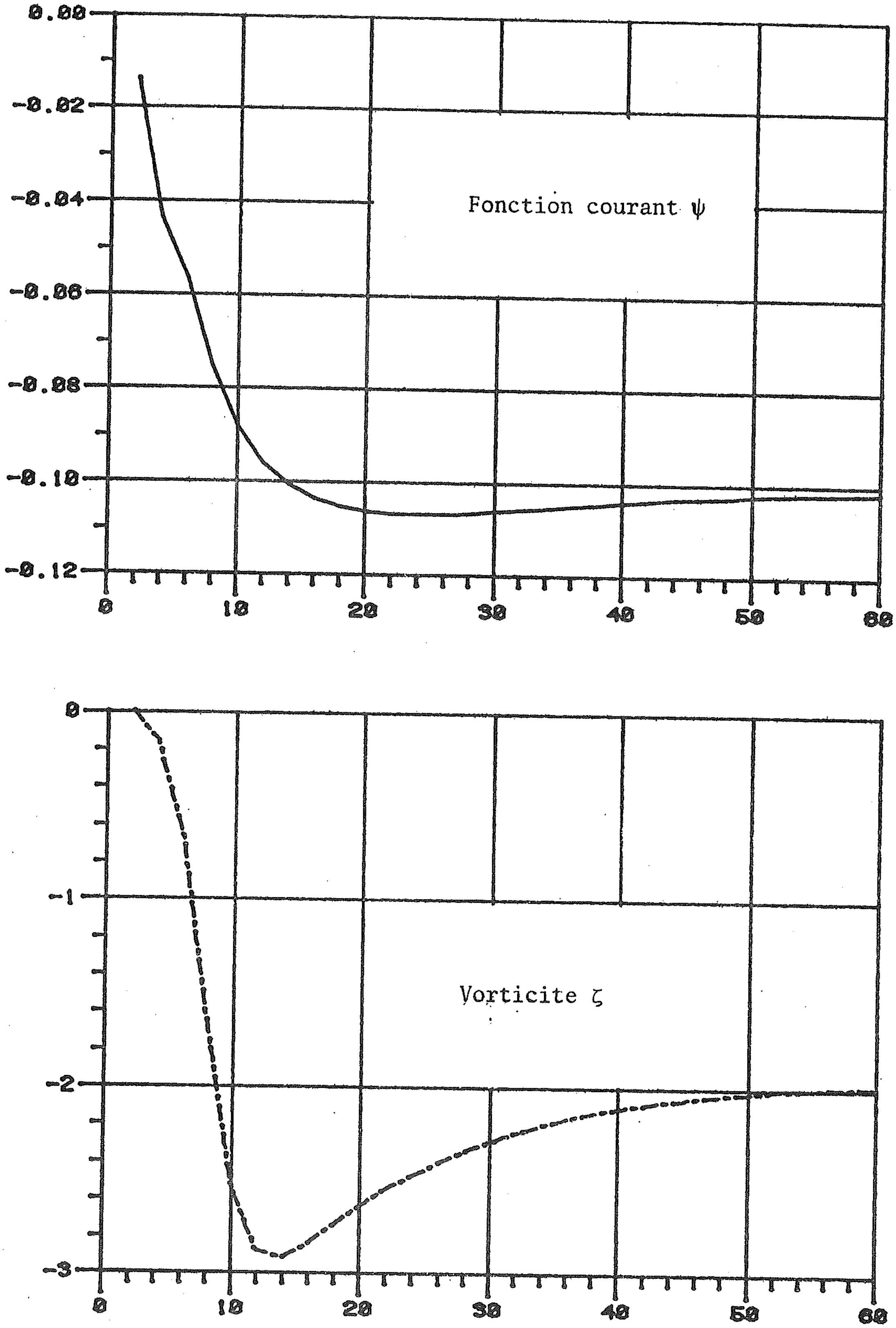
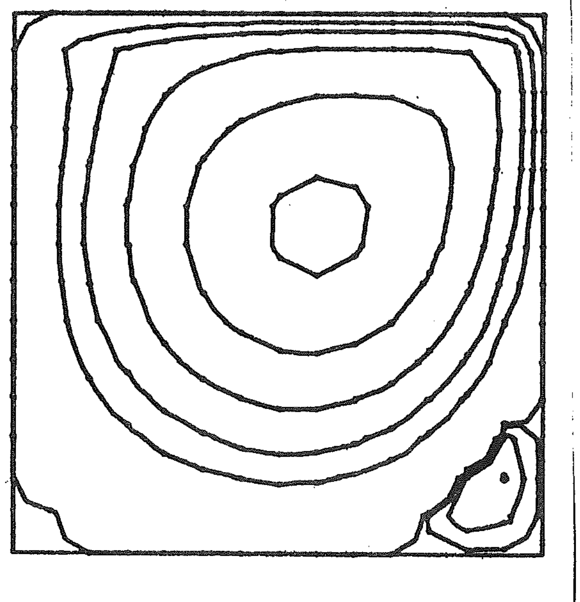
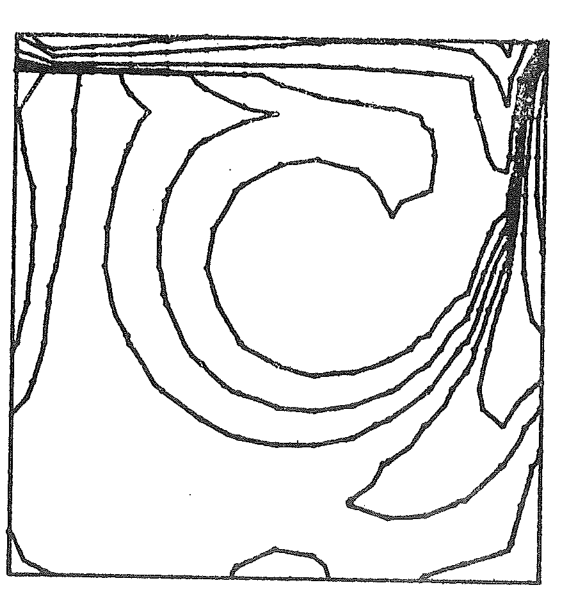


FIG. 12 HISTORIQUE DU POINT CENTRAL, SOLUTION PAR ELEMENTS FINIS POUR UN NOMBRE DE REYNOLDS DE 1000 et un MAILLAGE DE 15X15

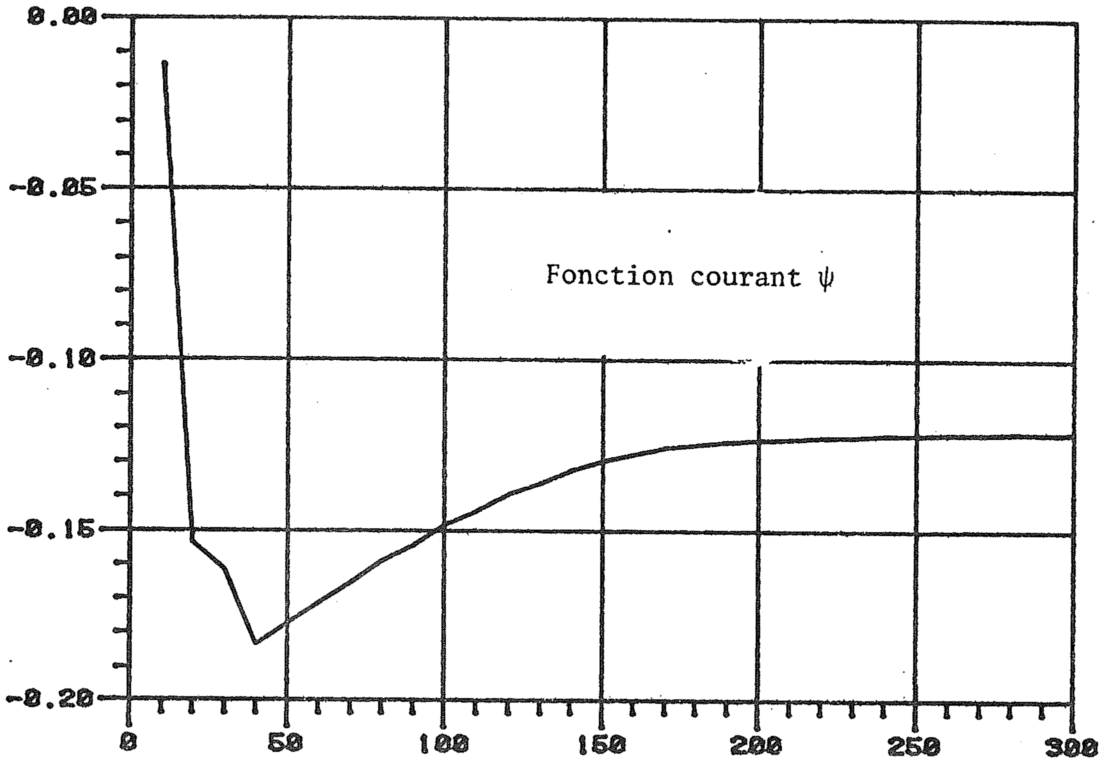


Lignes de courant



Lignes de vorticite

FIG. 13 LIGNES DE COURANT ET D ISO-VORTICITE A L ETAT STATIONNAIRE, SOLUTION PAR ELEMENTS FINIS POUR UN NOMBRE DE REYNOLDS DE 1000 et un MAILLAGE DE 15X15



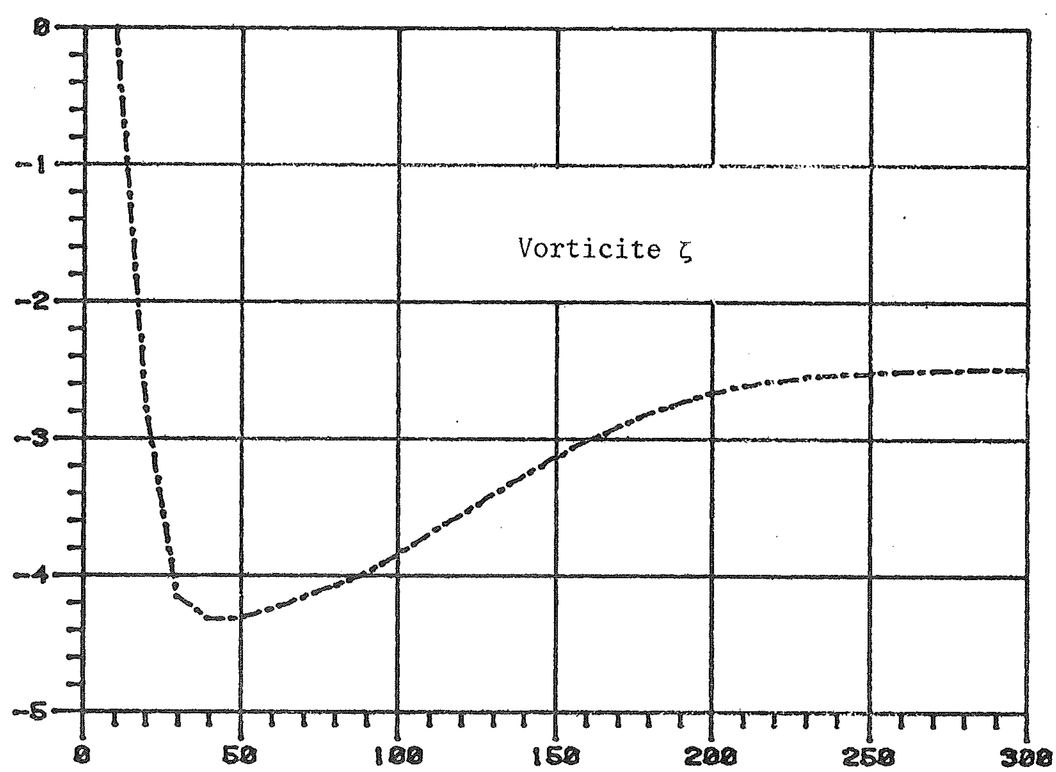
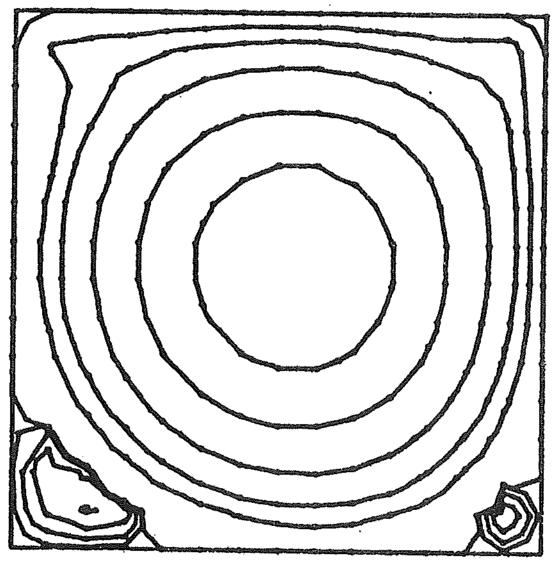
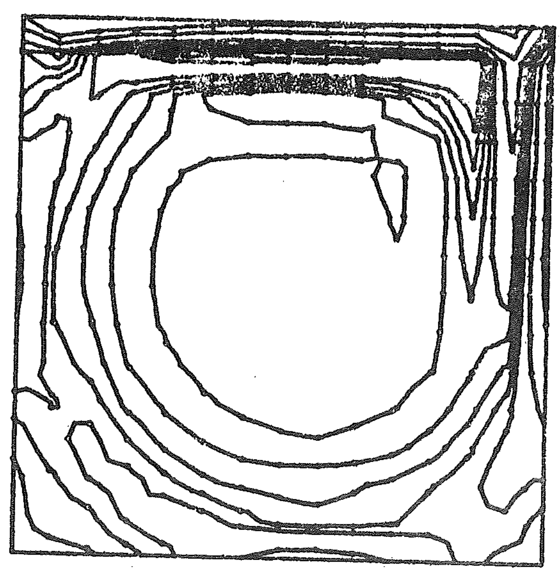


FIG. 14 HISTORIQUE DU POINT CENTRAL , SOLUTION PAR ELEMENTS FINIS POUR UN NOMBRE DE REYNOLDS DE 5000 ET UN MAILLAGE DE 15X15



Lignes de courant



Lignes de vorticite

FIG.15 LIGNES DE COURANT ET DE VORTICITE A L ETAT STATIONNAIRE, SOLUTION PAR ELEMENTS FINIS POUR UN NOMBRE DE REYNOLDS DE 5000 et un MAILLAGE DE 15X15

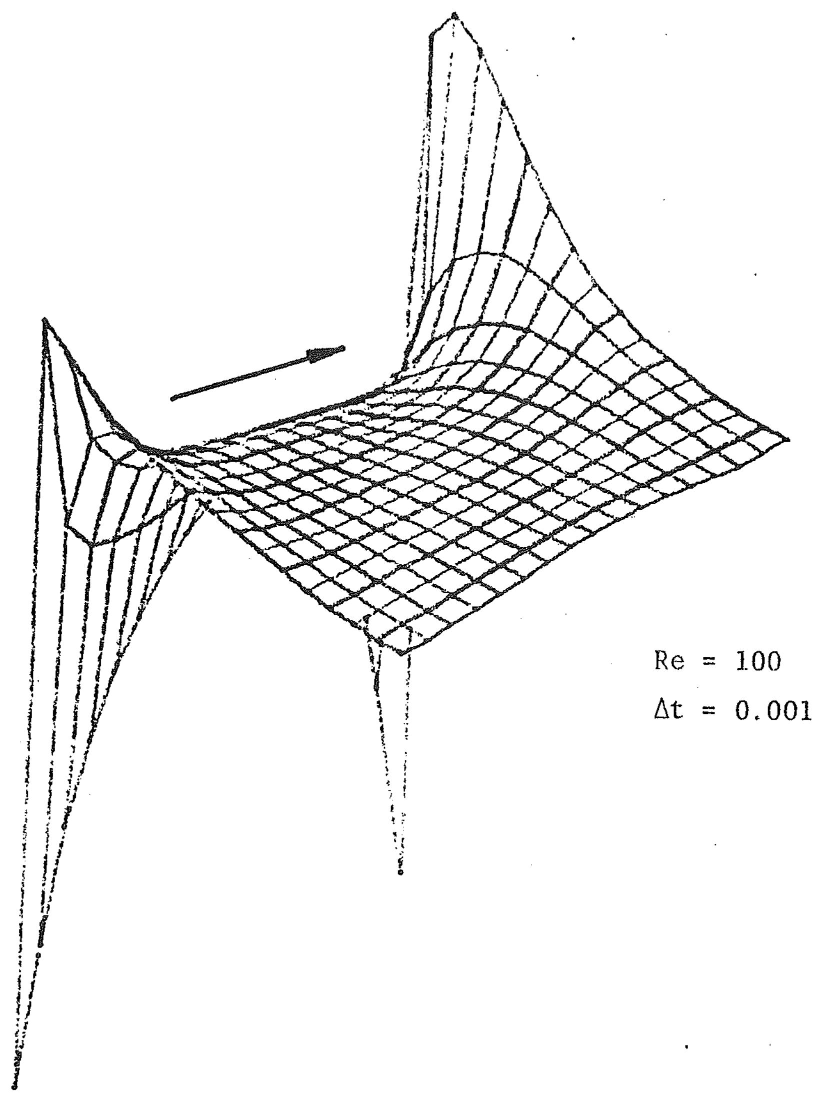


FIG. 16 REPRESENTATION TRIDIMENSIONNELLE DE LA SURFACE DE VORTICITE A L ETAT STATIONNAIRE

REMERCIEMENT

On remercie le Service de la recherche de l'Ecole Polytechnique et le Conseil de recherche en sciences naturelles et en génie pour l'appui financier accordé au cours de cette étude.