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On some methods of calculating the integrals of trigonometric rational functions

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Abstract

The paper presents original methods of calculating integrals of selected trigonometric rational functions.

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1. Introduction

The aim of this work is to present an "original" methods of determining the integrals of the form

$$\int \frac{p\sin^2 x + q\sin x\cos x + r\cos^2 x}{\left(a\sin x + b\cos x\right)^n}\,\mathrm{d}x, \tag{1.1}$$

where $p, q, r, a, b \in \mathbb{R}$, $n \in \mathbb{N}$. Presented methods are useful for manual as well as machine symbolic calculations.

[*]Marcin Jochlik is a first-year undergraduate student.
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In the era of omnipotent and, above all, commonly available symbolic calculations, including integration, this article may seem archaic. But the reason for creating this paper is neither complicated nor artificial. The article was initiated during classes in mathematical analysis in the second semester of undergraduate studies in mathematics, which were led by the third author in the March of this year. The way from a simple task – a special case of the integral described by (1.1) – to creative and gripping generalizations turned out to be easy and very fast. It resulted in the presented article that is an effect of pure creative passion.

Let us emphasize that from the very beginning we were looking for alternative sources of the presented methods and computational techniques [1–5]. Only in [5] a solution was found, rather a run-of-the-mill solution, for a certain special case of integral (1.1). Besides, we did not come across any at least promising traces of similar or comparable methods. Therefore, we can confidently say that the proving methods and technical tricks presented in the paper are original. It is worth pointing out that the obtained formulae, e.g. (5.10) and (5.12), may be used in both numerical and symbolic applications.

Notation. We will denote by $\alpha \times (k) \pm \beta \times (l)$, over all numbered identities (k), (l) in this paper, the following operation: identity (k) is multiplied by α, identity (l) is multiplied by β and then the obtained identities are summed (substracted, respectively).

First, we describe the method for the simple case of $n = 1$ basing on a 3-step reduction in computation.

2. The first step of the method

We reduce the numerator in (1.1) to

$$p\sin^2 x + q\sin x\cos x + r\cos^2 x = \alpha f(x) + \beta g(x) + \gamma h(x), \tag{2.1}$$

where

$$f(x) = \big(M(x)\big)^2, \quad g(x) = M(x)M'(x), \quad h(x) = \sin x\cos x,$$

and $M(x)$ denotes denominator in (1.1) in the case $n = 1$, i.e.

$$M(x) := a\sin x + b\cos x. \tag{2.2}$$

By solving the appropriate system of equations (created by comparing the coefficients at $\cos^2 x$, $\sin^2 x$ and $\sin x\cos x$)

$$\begin{cases} \alpha b^2 + \beta ab = r, \\ \alpha a^2 - \beta ab = p, \\ (a^2 - b^2)\beta + 2ab\alpha + \gamma = q, \end{cases}$$

we get

$$\alpha = \frac{p + r}{a^2 + b^2}, \quad \beta = \frac{-b^2 p + a^2 r}{ab(a^2 + b^2)}, \quad \gamma = q - \frac{b}{a}p - \frac{a}{b}r.$$

Then integral (1.1) takes the form

$$\int \frac{p\sin^2 x + q\sin x\cos x + r\cos^2 x}{a\sin x + b\cos x}\,\mathrm{d}x \\ = \alpha\int M(x)\,\mathrm{d}x + \beta\int M'(x)\,\mathrm{d}x + \gamma\int \frac{\sin x\cos x}{a\sin x + b\cos x}\,\mathrm{d}x.$$

We only need to calculate the integral

$$\int \frac{\sin x\cos x}{a\sin x + b\cos x}\,\mathrm{d}x.$$

3. The second step of the method

Integrating by parts in two ways, we find

$$\int \frac{\sin x\cos x}{a\sin x + b\cos x}\,\mathrm{d}x = \int (\sin x)'\frac{\sin x}{a\sin x + b\cos x}\,\mathrm{d}x \\ = \frac{\sin^2 x}{a\sin x + b\cos x} - \int \frac{b\sin x}{(a\sin x + b\cos x)^2}\,\mathrm{d}x \tag{3.1}$$

and

$$\int \frac{\sin x\cos x}{a\sin x + b\cos x}\,\mathrm{d}x = \int (-\cos x)'\frac{\cos x}{a\sin x + b\cos x}\,\mathrm{d}x \\ = \frac{-\cos^2 x}{a\sin x + b\cos x} - \int \frac{a\cos x}{(a\sin x + b\cos x)^2}\,\mathrm{d}x. \tag{3.2}$$

Moreover, by $\frac{a^2}{a^2+b^2} \times (3.1) + \frac{b^2}{a^2+b^2} \times (3.2)$, we obtain

$$\int \frac{\sin x\cos x}{a\sin x + b\cos x}\,\mathrm{d}x \\ = \frac{1}{a^2+b^2}\cdot\frac{a^2\sin^2 x - b^2\cos^2 x}{a\sin x + b\cos x} - \frac{ab}{a^2+b^2}\int \frac{a\sin x + b\cos x}{(a\sin x + b\cos x)^2}\,\mathrm{d}x \\ = \frac{1}{a^2+b^2}(a\sin x - b\cos x) - \frac{ab}{a^2+b^2}\int \frac{\mathrm{d}x}{a\sin x + b\cos x}. \tag{3.3}$$

Moreover, from (3.1) and (3.2), we get

$$\int \frac{\sin x\cos x}{a\sin x + b\cos x}\,\mathrm{d}x = \frac{v\sin^2 x - u\cos^2 x}{a\sin x + b\cos x} - \int \frac{au\cos x + bv\sin x}{(a\sin x + b\cos x)^2}\,\mathrm{d}x \tag{3.4}$$

whenever $u, v \in \mathbb{R}$, $u + v = 1$.

4. The third step of the method (supplementary reminder)

We still have to determine the integral $\int \frac{\mathrm{d}x}{a\sin x + b\cos x}$. We calculate it as follows

$$\int \frac{\mathrm{d}x}{a\sin x + b\cos x} = \int \frac{\mathrm{d}x}{\sqrt{a^2+b^2}\sin(x+\varphi)} = \left| \begin{array}{l} \text{where} \\ \cos\varphi = \frac{a}{\sqrt{a^2+b^2}} \\ \sin\varphi = \frac{b}{\sqrt{a^2+b^2}} \end{array} \right|$$
$$= \frac{1}{\sqrt{a^2+b^2}} \int \frac{\mathrm{d}x}{2\cos^2\frac{x+\varphi}{2}\tan\frac{x+\varphi}{2}} = \frac{1}{\sqrt{a^2+b^2}} \ln\left|\tan\frac{x+\varphi}{2}\right| + C,$$

where, after applying the identity

$$\tan\frac{x+\varphi}{2} = \frac{\cos\frac{\varphi}{2}\sin\frac{x}{2} + \sin\frac{\varphi}{2}\cos\frac{x}{2}}{\cos\frac{\varphi}{2}\cos\frac{x}{2} - \sin\frac{\varphi}{2}\sin\frac{x}{2}} = \frac{2\cos^2\frac{\varphi}{2}\sin\frac{x}{2} + 2\cos\frac{\varphi}{2}\sin\frac{\varphi}{2}\cos\frac{x}{2}}{2\cos^2\frac{\varphi}{2}\cos\frac{x}{2} - 2\cos\frac{\varphi}{2}\sin\frac{\varphi}{2}\sin\frac{x}{2}}$$
$$= \frac{(1+\cos\varphi)\sin\frac{x}{2} + \sin\varphi\cos\frac{x}{2}}{(1+\cos\varphi)\cos\frac{x}{2} - \sin\varphi\sin\frac{x}{2}} = \frac{\left(a+\sqrt{a^2+b^2}\right)\sin\frac{x}{2} + b\cos\frac{x}{2}}{\left(a+\sqrt{a^2+b^2}\right)\cos\frac{x}{2} - b\sin\frac{x}{2}},$$

we get

$$\int \frac{\mathrm{d}x}{a\sin x + b\cos x} = \frac{1}{\sqrt{a^2+b^2}} \ln\left|\frac{\left(a+\sqrt{a^2+b^2}\right)\sin\frac{x}{2} + b\cos\frac{x}{2}}{\left(a+\sqrt{a^2+b^2}\right)\cos\frac{x}{2} - b\sin\frac{x}{2}}\right| + C.$$

Another method of calculating the discussed integral, without using the half-angle formula, is presented below

$$\int \frac{\mathrm{d}x}{a\sin x + b\cos x} = \int \frac{a\sin x - b\cos x}{a^2\sin^2 x - b^2\cos^2 x}\,\mathrm{d}x$$
$$= \int \frac{a\sin x}{a^2 - (a^2+b^2)\cos^2 x}\,\mathrm{d}x + \int \frac{b\cos x}{b^2 - (a^2+b^2)\sin^2 x}\,\mathrm{d}x$$
$$= \frac{1}{2}\int\left(\frac{\sin x}{a - \sqrt{a^2+b^2}\cos x} + \frac{\sin x}{a + \sqrt{a^2+b^2}\cos x}\right)\mathrm{d}x$$
$$+ \frac{1}{2}\int\left(\frac{\cos x}{b - \sqrt{a^2+b^2}\sin x} + \frac{\cos x}{b + \sqrt{a^2+b^2}\sin x}\right)\mathrm{d}x$$
$$= \frac{1}{2\sqrt{a^2+b^2}}\left(\ln\left|\frac{a - \sqrt{a^2+b^2}\cos x}{a + \sqrt{a^2+b^2}\cos x}\right| + \ln\left|\frac{b + \sqrt{a^2+b^2}\sin x}{b - \sqrt{a^2+b^2}\sin x}\right|\right) + C.$$

At the end of this section, we present the conventional method of calculating $\int \frac{\mathrm{d}x}{a\sin x + b\cos x}$ using the Weierstrass substitution. We consider it in a more general case with an additional constant in the denominator.

$$
\begin{aligned}
&\int \frac{\mathrm{d}x}{a\sin x + b\cos x + c} \\
&\quad = \int \frac{\mathrm{d}x}{a\big(2\sin\frac{x}{2}\cos\frac{x}{2}\big) + b\big(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\big) + c\big(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\big)} \\
&\quad = \int \frac{\mathrm{d}x}{\big((c-b)\tan^2\frac{x}{2} + 2a\tan\frac{x}{2} + b + c\big)\cos^2\frac{x}{2}} = \left| \begin{matrix} \text{substitution} \\ t = \tan\frac{x}{2} \end{matrix} \right| \\
&\quad = \int \frac{2\,\mathrm{d}t}{(c-b)t^2 + 2at + b + c} = \left| \begin{matrix} \text{we only give} \\ \text{the final result} \end{matrix} \right| \\
&\quad = \begin{cases}
\dfrac{1}{\sqrt{a^2+b^2-c^2}} \ln \dfrac{\sqrt{a^2+b^2-c^2} - a - (c-b)\tan\frac{x}{2}}{\sqrt{a^2+b^2-c^2} + a + (c-b)\tan\frac{x}{2}}, & \text{when } a^2+b^2 > c^2, \\
\dfrac{2}{\sqrt{c^2-a^2-b^2}} \arctan \dfrac{a + (c-b)\tan\frac{x}{2}}{\sqrt{c^2-a^2-b^2}}, & \text{when } a^2+b^2 < c^2, \\
-\dfrac{2}{a + (c-b)\tan\frac{x}{2}}, & \text{when } a^2+b^2 = c^2.
\end{cases}
\end{aligned}
$$

5. A generalization due to the power of the denominator

In the case of the integrals

$$\int \frac{\sin x \cos x}{(a\sin x + b\cos x)^k}\,\mathrm{d}x, \quad k \in \mathbb{N},\ k \geq 2, \tag{5.1}$$

our attempts of the finding of a generalization of formula (3.3) did not provide desired results. Following the discussed methods, we generated only

$$\int \frac{b\sin x}{(a\sin x + b\cos x)^3}\,\mathrm{d}x = \frac{\sin^2 x}{2(a\sin x + b\cos x)^2} + C, \tag{5.2}$$

$$\int \frac{a\cos x}{(a\sin x + b\cos x)^3}\,\mathrm{d}x = \frac{-\cos^2 x}{2(a\sin x + b\cos x)^2} + C. \tag{5.3}$$

Hence, by $\frac{a}{b} \times (5.2) + \frac{b}{a} \times (5.3)$ we get

$$\int \frac{\mathrm{d}x}{(a\sin x + b\cos x)^2} = \frac{\frac{a}{b}\sin^2 x - \frac{b}{a}\cos^2 x}{2(a\sin x + b\cos x)^2} + C = \frac{1}{2ab} \cdot \frac{a\sin x - b\cos x}{a\sin x + b\cos x} + C \tag{5.4}$$

and generally

$$\int \frac{au\cos x + bv\sin x}{(a\sin x + b\cos x)^3}\,\mathrm{d}x = \frac{v\sin^2 x - u\cos^2 x}{2(a\sin x + b\cos x)^2} + C \tag{5.5}$$

for any $u, v \in \mathbb{R}$. So, we got certain analogues of formulae (3.3) and (3.4). Formulae (5.2) and (5.3) can be easily verified directly and they prompted us to calculate the following derivatives (and it was a bull's-eye)

$$\left(\frac{\cos^2 x}{(a\sin x + b\cos x)^k}\right)' = \frac{-2a\cos x - (k-2)\cos^2 x(a\cos x - b\sin x)}{(a\sin x + b\cos x)^{k+1}},$$
$$\left(\frac{\sin^2 x}{(a\sin x + b\cos x)^k}\right)' = \frac{2b\sin x - (k-2)\sin^2 x\big(a\cos x - b\sin x\big)}{(a\sin x + b\cos x)^{k+1}}.$$

After integrating the above identities, we get

$$\frac{\cos^2 x}{(a\sin x + b\cos x)^k}$$
$$= -\int \frac{2a\cos x}{(a\sin x + b\cos x)^{k+1}}\,\mathrm{d}x - (k-2)\int \cos^2 x\left(\frac{-\frac{1}{k}}{(a\sin x + b\cos x)^k}\right)'\mathrm{d}x$$

(integrating by parts)

$$= -\int \frac{2a\cos x}{(a\sin x + b\cos x)^{k+1}}\,\mathrm{d}x + \frac{k-2}{k}\cdot\frac{\cos^2 x}{(a\sin x + b\cos x)^k}$$
$$+ \frac{2(k-2)}{k}\int \frac{\sin x\cos x}{(a\sin x + b\cos x)^k}\,\mathrm{d}x,$$

which implies

$$(k-2)\int \frac{\sin x\cos x}{(a\sin x + b\cos x)^k}\,\mathrm{d}x$$
$$= \frac{\cos^2 x}{(a\sin x + b\cos x)^k} + k\int \frac{a\cos x}{(a\sin x + b\cos x)^{k+1}}\,\mathrm{d}x \tag{5.6}$$

for $k \in \mathbb{N},\ k \geq 3$. Similarly

$$\frac{\sin^2 x}{(a\sin x + b\cos x)^k}$$
$$= \int \frac{2b\sin x}{(a\sin x + b\cos x)^{k+1}}\,\mathrm{d}x - (k-2)\int \sin^2 x\left(\frac{-\frac{1}{k}}{(a\sin x + b\cos x)^k}\right)'\mathrm{d}x$$
$$= \int \frac{2b\sin x}{(a\sin x + b\cos x)^{k+1}}\,\mathrm{d}x + \frac{k-2}{k}\cdot\frac{\sin^2 x}{(a\sin x + b\cos x)^k}$$
$$- \frac{2(k-2)}{k}\int \frac{\sin x\cos x}{(a\sin x + b\cos x)^k}\,\mathrm{d}x,$$

which implies

$$(k-2)\int \frac{\sin x\cos x}{(a\sin x + b\cos x)^k}\,\mathrm{d}x$$

$$= \frac{-\sin^2 x}{(a\sin x + b\cos x)^k} + k\int \frac{b\sin x}{(a\sin x + b\cos x)^{k+1}}\,\mathrm{d}x \tag{5.7}$$

for $k \in \mathbb{N}$, $k \geq 3$. Additionally, by $\frac{b^2}{a^2+b^2} \times (5.6) + \frac{a^2}{a^2+b^2} \times (5.7)$, we obtain

$$(k-2)\int \frac{\sin x\cos x}{(a\sin x + b\cos x)^k}\,\mathrm{d}x = \frac{1}{a^2+b^2}\cdot\frac{b\cos x - a\sin x}{(a\sin x + b\cos x)^{k-1}} + \frac{abk}{a^2+b^2}\int \frac{1}{(a\sin x + b\cos x)^k}\,\mathrm{d}x \tag{5.8}$$

for $k \in \mathbb{N}$, $k \geq 3$. Moreover, by $u \times (5.6) + v \times (5.7)$ we get

$$(k-2)\int \frac{\sin x\cos x}{(a\sin x + b\cos x)^k}dx = \frac{u\cos^2 x - v\sin^2 x}{(a\sin x + b\cos x)^k} + k\int \frac{au\cos x + bv\sin x}{(a\sin x + b\cos x)^{k+1}}dx \tag{5.9}$$

whenever $u, v \in \mathbb{R}$, $u+v = 1$, and $k \in \mathbb{N}$, $k \geq 3$. For $k = 1$, from (5.8) and (5.9), we obtain (3.3) and (3.4), respectively. Furthermore, for any $k \in \mathbb{N}$, $k \geq 3$, formulae (5.8) and (5.9) are generalizations of formulae (3.3) and (3.4), respectively, for any $k \in \mathbb{N}$, $k \geq 3$. Let us recall that the case $k = 2$ is not covered by these formulae and it is described by identities (5.4) and (5.5) - we can obtain them also from (5.8) and (5.9) after substitution $k = 2$. In this way, we also obtain a solution to problem (5.1), previously unsuccessfully investigated with the method from Section 3.

Corollary 5.1. *Suppose $ab > 0$ and let $x_0 \in \left(-\frac{\pi}{2}, 0\right)$ be such that $\tan x_0 = -\frac{b}{a}$. Then for each $\varphi \in \left(0, \frac{\pi}{2}\right)$, there is $x(\varphi) \in (x_0, 0)$ that satisfies the condition*

$$\int\limits_{x(\varphi)}^{0} \frac{\sin x\cos x}{(a\sin x + b\cos x)^k}\,\mathrm{d}x = -\int\limits_{0}^{\varphi} \frac{\sin x\cos x}{(a\sin x + b\cos x)^k}\,\mathrm{d}x.$$

Hence, based on formula (5.8), we get the formulae

$$\int\limits_{x(\varphi)}^{\varphi} \frac{\mathrm{d}x}{(a\sin x + b\cos x)^k} = \frac{1}{k}\cdot\frac{1}{ab}\cdot\left.\frac{a\sin x - b\cos x}{(a\sin x + b\cos x)^{k-1}}\right|_{x(\varphi)}^{\varphi},$$

$$\int\limits_{x(\varphi)}^{\varphi} \frac{au\cos x + bv\sin x}{(a\sin x + b\cos x)^{k+1}}\,\mathrm{d}x = -\frac{1}{k}\cdot\left.\frac{u\cos^2 x - v\sin^2 x}{(a\sin x + b\cos x)^k}\right|_{x(\varphi)}^{\varphi},$$

whenever $u, v \in \mathbb{R}$, $u + v = 1$, and $k \in \mathbb{N}$, $k \geq 3$.

Proof. Note that

$$\frac{\sin x\cos x}{(a\sin x + b\cos x)^k} < 0$$

in the interval $(x_0, 0)$ and

$$\int\limits_{x_0}^{0} \frac{\sin x \cos x}{(a\sin x + b\cos x)^k}\,\mathrm{d}x = -\infty.$$

The function

$$(x_0, 0] \ni x \mapsto \int\limits_{x}^{0} \frac{\sin\tau\cos\tau}{(a\sin\tau + b\cos\tau)^k}\,\mathrm{d}\tau$$

is continuous. It remains to use the Darboux property. □

Remark 5.2. In according to identity (5.8), we propose to derive a recurrent identity for the integrals

$$I_k = \int \frac{\mathrm{d}x}{(a\sin x + b\cos x)^k}, \quad k \in \mathbb{N}.$$

So, we have

$$\begin{aligned}
(a^2+b^2)I_k &= \int \frac{(a\sin x + b\cos x)^2 + (a\cos x - b\sin x)^2}{(a\sin x + b\cos x)^k}\,\mathrm{d}x \\
&= I_{k-2} + \int (a\cos x - b\sin x)\cdot\left(\frac{-\frac{1}{k-1}}{(a\sin x + b\cos x)^{k-1}}\right)'\mathrm{d}x \\
&= I_{k-2} + \frac{1}{k-1}\cdot\frac{b\sin x - a\cos x}{(a\sin x + b\cos x)^{k-1}} - \frac{1}{k-1}I_{k-2} \\
&= \frac{k-2}{k-1}I_{k-2} + \frac{1}{k-1}\cdot\frac{b\sin x - a\cos x}{(a\sin x + b\cos x)^{k-1}}.
\end{aligned} \tag{5.10}$$

Hence, for example, by (5.4) we obtain

$$3(a^2+b^2)I_4 = \frac{1}{ab}\cdot\frac{a\sin x - b\cos x}{a\sin x + b\cos x} + \frac{b\sin x - a\cos x}{(a\sin x + b\cos x)^3} + C.$$

Remark 5.3. In according to identities (5.8) and (5.4), it is worth pointing out that

$$\begin{aligned}
&\int \frac{\sin x\cos x}{(a\sin x + b\cos x)^2}\,\mathrm{d}x \\
&\quad = \frac{a^2-b^2}{(a^2+b^2)^2}\ln|a\sin x + b\cos| \\
&\qquad + \frac{b}{a^2+b^2}\cdot\frac{\cos x}{a\sin x + b\cos x} + \frac{2abx}{(a^2+b^2)^2} + C,
\end{aligned} \tag{5.11}$$

where the calculations were done using the following decomposition in an ingenious way

$$\int \frac{\sin x\cos x}{(a\sin x + b\cos x)^2}\,\mathrm{d}x = \int \frac{\tan x}{(a\tan x + b)^2(\tan^2 x + 1)}\,\mathrm{d}(\tan x)$$

(where $u = \tan x$)

$$= \int \frac{u}{(au+b)^2(u^2+1)}\,\mathrm{d}u = \int \left(\frac{\alpha}{au+b} + \frac{\beta}{(au+b)^2} + \frac{\gamma u + \delta}{u^2+1} \right) \mathrm{d}u$$

(after an observation of the obtained integrals)

$$= A \ln |a \sin x + b \cos x| + B \frac{\cos x}{a \sin x + b \cos x} + Dx + C$$

(we have only 3 unknown constants A, B, D), which, after differentiation, easily implies formula (5.11). Therefore, from (5.4) and (5.11) results that a simple functional identity, as for example formula (5.8), between the integrals

$$\int \frac{\mathrm{d}x}{(a \sin x + b \cos)^2} \quad \text{and} \quad \int \frac{\sin x \cos x}{(a \sin x + b \cos)^2}\,\mathrm{d}x$$

does not exist. But there exists such a connection between the discussed integral

$$\int \frac{\sin x \cos x}{(a \sin x + b \cos x)^2}\,\mathrm{d}x$$

and the other surprising integral

$$\int \frac{a \cos x + b \sin x}{a \sin x + b \cos x}\,\mathrm{d}x.$$

Based on the identity

$$\begin{aligned}(b^2 + a^2) \sin x \cos x + ab &= (b^2 + a^2) \sin x \cos x + ab(\sin^2 x + \cos^2 x) \\ &= b \sin x (b \cos x + a \sin x) + a \cos x (a \sin x + b \cos x) \\ &= (a \sin x + b \cos x)(a \cos x + b \sin x)\end{aligned}$$

we get

$$\begin{aligned}&\int \frac{\sin x \cos x}{(a \sin x + b \cos x)^2}\,\mathrm{d}x \\ &\quad = \frac{1}{a^2+b^2} \int \frac{(b^2+a^2) \sin x \cos x + ab}{(a \sin x + b \cos x)^2}\,\mathrm{d}x - \frac{ab}{a^2+b^2} \int \frac{\mathrm{d}x}{(a \sin x + b \cos x)^2} \\ &\quad = \frac{1}{a^2+b^2} \int \frac{a \cos x + b \sin x}{a \sin x + b \cos x}\,\mathrm{d}x - \frac{ab}{a^2+b^2} \int \frac{\mathrm{d}x}{(a \sin x + b \cos x)^2} \\ &\quad \overset{(5.4)}{=} \frac{1}{a^2+b^2} \int \frac{a \cos x + b \sin x}{a \sin x + b \cos x}\,\mathrm{d}x - \frac{1}{2(a^2+b^2)} \cdot \frac{a \sin x - b \cos x}{a \sin x + b \cos x}.\end{aligned}$$

Remark 5.4. Using formulae (2.1), (5.7) and (5.9) we obtain a generalization of the identities presented in Section 2

$$\int \frac{p \sin^2 x + q \sin x \cos x + r \cos^2 x}{(a \sin x + b \cos x)^n}\,\mathrm{d}x$$

$$\overset{n\geq 2}{=} \alpha I_{n-2} + \beta \int \frac{M'(x)}{M^n(x)}\,\mathrm{d}x + \gamma \int \frac{\sin x \cos x}{M^n(x)}\,\mathrm{d}x$$
$$\overset{n\geq 3}{=} \alpha I_{n-2} - \frac{\beta}{n-1} \cdot \frac{1}{M^{n-1}(x)} + \frac{\gamma(b\cos x - a\sin x)}{(n-2)(a^2+b^2)M^{n-1}(x)} + \frac{abn\gamma}{(n-2)(a^2+b^2)} I_n$$
$$= \left(\alpha + \frac{abn\gamma}{(n-1)(a^2+b^2)^2}\right) I_{n-2} + \left(-\frac{\beta}{n-1} + \frac{\gamma}{(n-2)(a^2+b^2)}(b\cos x - a\sin x) + \frac{abn\gamma}{(n-2)(n-1)(a^2+b^2)^2}(b\sin x - a\cos x)\right)\frac{1}{M^{n-1}(x)}, \tag{5.12}$$

where $M(x)$ is defined in (2.2) and I_n is discussed in Remark 5.2.

References

[1] I. N. Bronstein, H. Mühlig, G. Musiol, K. A. Semendjajew: *Taschenbuch der Mathematik*, Verlag Harri Deutsch GmbH, 2001.

[2] I. S. Gradshteyn, I. M. Ryzhik: *Table of Integrals, Series, and Products*, ed. by A. Jeffrey, D. Zwillinger, New York: Academic Press, 2000, DOI: https://doi.org/10.1016/B978-0-12-294757-5.X5000-4.

[3] A. P. Prudnikov, J. A. Brychkov, O. I. Marichev: *Integrals and Series, vol. 1: Elementary Functions*, New York: Gordon & Breach, 1986.

[4] A. F. Timofeev: *Integration of functions*, Russian, Moscow: OGIZ, 1948.

[5] I. A. Vinogradova, S. N. Olekhnik, V. A. Sadovnichiy: *Problems and exercises in Mathematical Analysis*, Russian, Moscow: Moscow University Press, 1999.