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Gradient Estimates And The Fundamental Solution For Higher-Order Elliptic Systems With Lower-Order Terms

Michael J. Duffy Jr.
University of Arkansas, Fayetteville

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by

Michael J. Duffy Jr
Adelphi University
Bachelor of Science in Mathematics, 2012
Adelphi University
Master of Arts in Adolescent Education, 2013
University of Arkansas
Master of Science in Mathematics, 2017

July 2021
University of Arkansas

This dissertation is approved for recommendation to the Graduate Council.

Ariel Barton Ph.D.
Dissertation Director

Andrew Raich Ph.D.
Committee Member

Zachary Bradshaw Ph.D.
Committee Member

Abstract

Here we generalize the higher-order divergence-form elliptic differential equations studied by Barton in [4] by the inclusion of certain lower-order terms. The methods used here compare to those used in [4], with the addition of further Sobolev-type estimates to handle included lower-order terms. In section 3 we derive a Caccioppoli inequality in which we bound the L^2 norm of the m^{th} order gradient, in terms of the L^2 norm of the solution. In section 5 we adapt some of the ideas from [9] to derive L^p bounds on gradients of solutions as a substitute for a reverse Hölder inequality. Finally in section 4 we study the fundamental solution of the operator L. We prove existence and bounds first in the case that L is of sufficiently high order ($2m > d$), then in section 6.2 we extend these results to operators of lower order where $2m \leq d$.

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1 Introduction

The focus of this paper is the study of higher-order, divergence form elliptic operators L of order $2m$, in dimension d, with certain lower-order terms

$$(L\vec{u})_j = \sum_{k=1}^{N} \sum_{\substack{m-\frac{d}{2}<|\alpha|\leq m \\ m-\frac{d}{2}<|\beta|\leq m}} (-1)^{|\alpha|}\partial^\alpha (A^{j,k}_{\alpha,\beta}\partial^\beta u_k)$$

and in particular systems of the form given by

$$(L\vec{u})_j = \sum_{m-\frac{d}{2}<|\alpha|\leq m} (-1)^{|\alpha|}\partial^\alpha F_{j,\alpha}. \tag{1}$$

In the case of lower order operators, for instance where $2m = 2$ and $d \geq 3$, the assumption that $|\alpha| > m - \frac{d}{2}$ is automatically satisfied but it will be necessary here for certain technical reasons which will be outlined shortly.

First in section 3 we will use the methods of [4] to derive a Caccioppoli inequality similar to that of Campanato in [10] in which the L^2 norm of the m^{th} order gradient of the solution is bounded by a sum of L^2 norms of the lower-order gradient terms. Once we have this bound, we improve it to only include the L^2 norm of the solution via ideas from [4] and [19].

In section 5 we improve our Caccioppoli inequality to an L^p version for certain values of p close to 2. Here we use interpolation theory on the Banach spaces where our solutions lie. The ultimate goal here would be to prove some type of reverse Hölder inequality, but the presence of the lower order terms in our operator disallow us from using the Poincaré inequality, which is most commonly the main tool that is used to derive reverse Hölder inequalities. We are able to apply the interpolation theory result Šneĭberg's lemma and derive a reverse Hölder type inequality in terms of the norm of the solution space $Y^{m,p}(\Omega)$

as defined in section 2.

In section 4 we study the fundamental solution, and prove existence and L^2 bounds, as well as study properties of the dual operator L^*. In section 6.1 we focus on the high order case where $2m > d$, then in section 6.2 we extend our results from section 6.1 to the case where $2m \leq d$.

1.1 History

The theory of higher order partial differential equations is a relatively recent field of study when compared with its second order counterpart. Much of the recent history is discussed at great length by Barton and Mayboroda in the survey paper [7]. Many of the methods of dealing with second-order equations do not translate well into the higher-order setting.

In [16] Hofmann and Kim derive Green's function estimates for strongly elliptic systems of the form

$$(L\vec{u})_j = -\sum_{k=1}^{N} \sum_{|\alpha|=1} \sum_{|\beta|=1} \partial^\alpha (A^{\alpha,\beta}_{j,k}(x) \partial^\beta u_k)$$

in the case that $d \geq 3$. Here $A^{\alpha,\beta}(x)$ are $N \times N$ matrices that are strongly elliptic in the sense that there is some $\lambda > 0$ such that

$$\sum_{j,k=1}^{N} \sum_{|\alpha|=1} \sum_{|\beta|=1} A^{\alpha,\beta}_{j,k}(x) \xi^k_\beta \xi^j_\alpha \geq \lambda \sum_{j=1}^{N} \sum_{|\alpha|=1} |\xi^j_\alpha|^2$$

for all $x \in \mathbb{R}^d$ and $\xi^j_\alpha \in \mathbb{R}$.

This paper utilizes many of the methods discussed in [4] with the difference that the presence of lower-order terms causes technical differences. We are not able to obtain analogues to all of the results of [4], however we are able to obtain some desirable results.

In [4] and [1] the authors study solutions to operators of the form

$$L = (-1)^m \sum_{\substack{|\alpha|=m \\ |\beta|=m}} \partial^\alpha (a_{\alpha,\beta} \partial^\beta)$$

where L is a divergence form operator which only considers the m^{th} order gradient terms. Here the solutions have m^{th} order gradients in L^2 and coefficients are simply bounded and measurable. Many of the techniques used generalize those used in the second order case ($m = 1$) to great avail, however due to the nature of higher-order operators there are certain complications that arise.

In [11] the authors study local Hardy spaces associated with inhomogeneous higher-order elliptic operators with bounded, measurable, complex coefficients. In particular the operator L is of the form

$$L = \sum_{\substack{0\le|\alpha|\le m \\ 0\le|\beta|\le m}} (-1)^{|\alpha|} \partial^\alpha (a_{\alpha,\beta} \partial^\beta)$$

where the solutions lie in $W^{m,2}(\mathbb{R}^d)$ and the coefficients are bounded, measurable and complex valued. Two parabolic Caccioppoli inequalities are derived which involve the square of the m^{th} order gradient on the left-hand side, and sums of the squares of lower-order gradients on the right-hand side. One main difference between [11] and this paper is the assumption on the gradients of the solutions and coefficients which will be outlined in section 2.1.

The second order Caccioppoli inequality states that when L is a second-order bounded and elliptic operator, and $L\vec{u} = \text{div}\dot{\mathbf{F}}$ in $B(x_0, 2R)$ for some $\dot{\mathbf{F}} \in L^2(B(x_0, 2R))$, we have the inequality

$$\int_{B(x_0,R)} |\nabla \vec{u}|^2 \le \frac{C}{R^2} \int_{B(x_0,2R)} |\vec{u}|^2 + C \int_{B(x_0,2R)} |\dot{\mathbf{F}}|^2.$$

In [4] Barton proves a similar Caccioppoli inequality for the previously mentioned higher-order operator where $L\vec{u} = \text{div}_m \dot{\mathbf{F}}$, of the form

$$\int_{B(x_0,R)} |\nabla^m \vec{u}|^2 \le \frac{C}{R^2} \int_{B(x_0,2R)} |\vec{u}|^2 + C \int_{B(x_0,2R)} |\dot{\mathbf{F}}|^2.$$

We will generalize this result to our operator L in section 3.

Furthermore, Meyer's reverse Hölder inequality states that if L is a second order elliptic operator and $L\vec{u} = \text{div}\dot{\mathbf{F}}$ in some ball $B(x_0, 2R)$, then there exists a $p > 2$ that depends on L for which the following inequality holds when $\dot{\mathbf{F}} \in L^p(B(x_0, 2R))$

$$\left(\int_{B(x_0,R)} |\nabla \vec{u}|^p\right)^{\frac{1}{p}} \le \frac{C}{R^{d/2-d/p}} \left(\int_{B(x_0,2R)} |\nabla \vec{u}|^2\right)^{\frac{1}{2}} + C \left(\int_{B(x_0,2R)} |\dot{\mathbf{F}}|^p\right)^{\frac{1}{p}}.$$

In [4] Barton is able to achieve an analogous reverse Hölder inequality for an operator L which is elliptic and of order $2m$ where $L\vec{u} = \text{div}_m \dot{\mathbf{F}}$ in $B(x_0, R)$. In this situation for $0 < R$, there exists a $p^+ > 2$ which depends only on L so that for $0 < p \le 2 < q < p^+$ we have the inequality

$$\left(\int_{B(x_0,R)} |\nabla^m \vec{u}|^q\right)^{\frac{1}{q}} \le \frac{C}{R^{d/p-d/q}} \left(\int_{B(x_0,2R)} |\nabla^m \vec{u}|^p\right)^{\frac{1}{p}} + C \left(\int_{B(x_0,2R)} |\dot{\mathbf{F}}|^q\right)^{\frac{1}{q}}.$$

We are not able to derive an analogous result for our operator L, however we are able to derive several desirable bounds.

2 Definitions

We consider divergence-form elliptic systems of N partial differential equations of order $2m$ in d-dimensional Euclidean space $\mathbb{R}^d$. We employ the use of multiindices in $\mathbb{N}_0^d$. When $\gamma = (\gamma_1, ..., \gamma_d)$ is a multiindex, then

$$|\gamma| = \sum_{i=1}^{d} \gamma_i.$$

We also define $\alpha! = \alpha_1! \cdot \alpha_2! \cdots \alpha_d!$.

When δ is another multiindex in $\mathbb{N}^d$ we say that $\delta \leq \gamma$ if $\delta_i \leq \gamma_i$ for each $1 \leq i \leq d$. Futhermore, we say $\delta < \gamma$ if $\delta_i < \gamma_i$ for at least once such i.

We will employ the use of the Liebniz Rule for multiindices, i.e. that for all suitably differentiable functions u and v and a multiindex α, we have that

$$\partial^\alpha(uv) = \sum_{\gamma \leq \alpha} \frac{\alpha!}{\gamma!(\alpha-\gamma)!} \partial^\gamma u \partial^{\alpha-\gamma} v.$$

We will commonly use the notation $a_{\alpha,\gamma} := \frac{\alpha!}{\gamma!(\alpha-\gamma)!}$, and then use the fact that $a_{\alpha,0} = a_{\alpha,\alpha} = 1$.

We consider arrays $\dot{\mathbf{F}} = (F_{j,\gamma})$ indexed by integers $1 \leq j \leq N$ and multiindices γ for which $|\gamma| \leq k$ for some integer k. We define the inner product of two such arrays $\dot{\mathbf{F}}$ and $\dot{\mathbf{G}}$ by

$$\langle \dot{\mathbf{F}}, \dot{\mathbf{G}} \rangle = \sum_{j=1}^{N} \sum_{|\gamma| \leq k} \overline{F_{j,\gamma}} G_{j,\gamma}. \tag{2}$$

If $\dot{\mathbf{F}}$ and $\dot{\mathbf{G}}$ are two arrays of L^2 functions defined in a measurable subset Ω of $\mathbb{R}^d$, then the inner product is given by

$$\langle \dot{\mathbf{F}}, \dot{\mathbf{G}} \rangle_\Omega = \sum_{j=1}^{N} \sum_{|\gamma| \leq k} \int_\Omega \overline{F_{j,\gamma}} G_{j,\gamma}. \tag{3}$$

When $E \subset \mathbb{R}^d$ is a set of finite measure, we let $⨍_E f = \frac{1}{|E|} \int f$, where $|E|$ denotes the Lebesgue measure of E.

We denote by $L^p(\Omega)$ and $L^\infty(\Omega)$ the standard Lebesgue spaces with respect to

Lebesgue measure, with norms given by

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \right)^{1/p}$$

if $1 \le p < \infty$, and

$$\|u\|_{L^\infty(\Omega)} = \text{ess sup}_\Omega |u|.$$

We also define the space of local L^p functions $L^p_{\text{loc}}(\Omega)$ as the space of functions f for which $f \in L^p(K)$ for all compact sets $K \subset \Omega$. We define the inhomogeneous Sobolev spaces as

$$W^{k,p}(\Omega) = \{\vec{u} : \sum_{j=0}^{k} \|\nabla^j \vec{u}\|_{L^p(\Omega)} < \infty\}.$$

That is, the space of functions whose gradients up to order k are all in $L^p(\Omega)$. We then define the homogeneous Sobolev spaces as

$$\dot{W}^{k,p}(\Omega) = \{\vec{u} : \|\nabla^k \vec{u}\|_{L^p(\Omega)} < \infty\}.$$

That is, the space of functions whose k^{th}-order gradient is in $L^p(\Omega)$. In section 2.1 we define our solution space as the intersection of different Sobolev spaces both homogeneous and inhomogeneous. Similar to the definition of $L^p_{\text{loc}}(\Omega)$, we define the space of local Sobolev functions $W^{m,p}_{\text{loc}}(\Omega)$ as the space of functions f for which $f \in W^{m,p}(K)$ for all compact sets $K \subset \Omega$.

2.1 Elliptic Operators

Let $\mathbf{A} = (A^{j,k}_{\alpha,\beta})$ be an array of measurable real or complex coefficients defined on $\mathbb{R}^d$ indexed by integers j and k such that $1 \le j \le N$ and $1 \le k \le N$ and multiindices α and β

with $|\alpha| \leq m$ and $|\beta| \leq m$. If $\dot{\mathbf{F}}$ is an array, then

$$(\mathbf{A}\dot{\mathbf{F}})_{j,\alpha} = \sum_{k=1}^{N} \sum_{|\beta| \leq m} A_{\alpha,\beta}^{j,k} F_{k,\beta}$$

Recall from [12, Section 5.6] that for $1 \leq p < d$, the Sobolev conjugate of p is defined to be

$$p_* = \frac{dp}{d-p}.$$

Notice that

$$\frac{1}{p_*} = \frac{1}{p} - \frac{1}{d}. \tag{4}$$

We now mention a result which serves as motivation for the space where solutions to our systems will lie.

Theorem 2.1. *[12, Sec 5.6.1 Theorem 1 (Gagliardo-Nirenberg-Sobolev Inequality)] Assume $1 \leq p < d$. Then there is a constant C which depends only on p and d so that*

$$\|u\|_{L^{p_*}(\mathbb{R}^d)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^d)},$$

for all $u \in C_c^1(\mathbb{R}^d)$.

We will now generalize equation (4). Let i be an integer so that $m - \frac{d}{p} < i \leq m$. We then define $p_{m,d,i} = p_i$ so that

$$\frac{1}{p_i} = \frac{1}{p} - \frac{m-i}{d}. \tag{5}$$

In section 3 we will focus on the case where $p = 2$ and multiindices α such that $m - \frac{d}{2} < |\alpha| \leq m$. We will write $p_{|\alpha|} = p_\alpha$. Notice that when $|\alpha| = m$ we have that $2_\alpha = 2$, when $|\alpha| = m - 1$ then $2_\alpha = 2_*$ and so on. This definition for 2_α will help keep the notation throughout this paper relatively clean and help us to avoid any backwards summation.

We consider solutions $\vec{u} \in Y^{m,2}(B(x_0,r))$ where

$$Y^{m,2}(B(x_0,r)) := \{\vec{u} \in W^{m,2}(B(x_0,r)) : \partial^\alpha u \in L^{2_\alpha}(B(x_0,r)),\ \text{for each } m - \frac{d}{2} < |\alpha| \leq m\}.$$

We want to define the first integer $i \leq m$ for which 2_i exists. Based on equation (5) we see that the first finite value of 2_i will be when $i = m - \frac{d}{2} + 1$, if d is even, or when $i = m - \frac{d}{2} + \frac{1}{2}$ for odd values of d. It is by this fact that we define ω_2 so that $\omega_2 \geq 0$ as below.

$$\omega_2 = \begin{cases} m - \frac{d}{2} + 1 & \text{if } d \text{ is even, } d \leq 2m \\ m - \frac{d}{2} + \frac{1}{2} & \text{if } d \text{ is odd, } d < 2m \\ 0 & \text{if } d > 2m \end{cases} \tag{6}$$

For each $i \leq m$ we will use the notation

$$(\text{div}_i \dot{\mathbf{F}})_j = (-1)^i \sum_{|\alpha|=i} \partial^\alpha F_{j,\alpha}.$$

For each j such that $1 \leq j \leq N$, we define $F_i := \{F_{j,\beta} \text{ where } |\beta| = i\}$. We shall then write the system (1) as

$$L\vec{u} = \sum_{i=\omega_2}^{m} (-1)^i \text{div}_i F_i.$$

We endow the following norm on the space $Y^{m,2}(\Omega)$, and note that in section 5 we extend our results to allow for values of p in a range around 2.

$$\|u\|_{Y^{m,2}(\Omega)} := \sum_{i=\omega_2}^{m} \|\nabla^i u\|_{L^{2_i}(\Omega)} \tag{7}$$

Recall that many Sobolev type inequalities, such as theorem 2.1, bound $\|u\|_{L^{p_*}}$ in terms of something involving $\|\nabla u\|_{L^p}$ where we have introduced a gradient, and moved from p_* to p. It is for this reason that we define $Y^{m,2}$ as above. The inclusion of the lower order terms, and our assumption that $\nabla^m \vec{u} \in L^2(B(x_0, r))$, lead us to $\nabla^{m-1}\vec{u} \in L^{2_*}(B(x_0, r))$, and so on.

Whenever $m - \frac{d}{2} < |\alpha| \le m$ and $m - \frac{d}{2} < |\beta| \le m$, we define

$$2_{\alpha,\beta} := \frac{d}{2m - |\alpha| - |\beta|}.$$

We then require that our coefficients $A_{\alpha,\beta}$ satisfy the following, so that $\partial^\alpha \varphi A_{\alpha,\beta} \partial^\beta u \in L^1_{\text{loc}}(\Omega)$ for appropriate φ, u, and Ω:

$$A_{\alpha,\beta} \in L^{\frac{d}{2m-|\alpha|-|\beta|}}(\Omega) := L^{2_{\alpha,\beta}}(\Omega). \tag{8}$$

We then assume that there exists $\Lambda > 0$ so that

$$\Lambda = \max_{\substack{m-\frac{d}{2}<|\alpha|\le m \\ m-\frac{d}{2}<|\beta|\le m}} \|A_{\alpha,\beta}\|_{L^{2_{\alpha,\beta}}(\Omega)} < \infty, \qquad \sup_{\substack{|\alpha|\le m-\frac{d}{2} \\ |\beta|\le m-\frac{d}{2}}} \|A_{\alpha,\beta}\|_{L^\infty(\Omega)} = 0. \tag{9}$$

We arrive at this definition for $L^{2_{\alpha,\beta}}$ by considering a three way Hölder inequality in which $1 = \frac{1}{2_\alpha} + \frac{1}{2_\beta} + \frac{1}{2_{\alpha,\beta}}$. However, when $|\alpha| \le m - \frac{d}{2}$ or $|\beta| \le m - \frac{d}{2}$ we assume that $A_{\alpha,\beta} = 0$. This assumption is given for free to many lower-order operators such as second order in $\mathbb{R}^d$ when $d \ge 3$, but our methods limit us here to this assertion.

We will consider operators which satisfy the Gårding inequality

$$\operatorname{Re} \sum_{j,k=1}^{N} \sum_{|\alpha|\le m} \sum_{|\beta|\le m} \int_{\mathbb{R}^d} \overline{\partial^\alpha \varphi_j} A^{j,k}_{\alpha,\beta} \partial^\beta \varphi_k \ge \lambda \|\nabla^m \vec{\varphi}\|^2_{L^2(\mathbb{R}^d)} \tag{10}$$

for all $\vec{\varphi} \in Y^{m,2}(\mathbb{R}^d)$ and for some $\lambda > 0$ independent of $\vec{\varphi}$.

Remark 1. Notice that by the Gagliardo-Nirenberg-Sobolev inequality, and the definition of the $Y^{m,2}(\Omega)$ norm, we have the following inequalities for $\Omega = \mathbb{R}^d$.

$\|\varphi\|_{Y^{m,2}(\mathbb{R}^d)} \leq C\|\nabla^m\varphi\|_{L^2(\mathbb{R}^d)} \leq C\|\varphi\|_{Y^{m,2}(\mathbb{R}^d)}$ where C depends on d and m. Thus we have the relation $\frac{1}{C}\|\varphi\|_{Y^{m,2}(\mathbb{R}^d)} \leq \|\nabla^m\varphi\|_{L^2(\mathbb{R}^d)} \leq C\|\varphi\|_{Y^{m,2}(\mathbb{R}^d)}$ which gives us an equivalence between $\|\cdot\|_{Y^{m,2}(\mathbb{R}^d)}$ and $\|\nabla^m\cdot\|_{L^2(\mathbb{R}^d)} = \|\cdot\|_{\dot{W}^{m,2}(\mathbb{R}^d)}$.

We will also consider operators which satisfy the weak Gårding inequality

$$\text{(11)} \qquad \operatorname{Re}\sum_{j,k=1}^{N}\sum_{|\alpha|\leq m}\sum_{|\beta|\leq m}\int_{\mathbb{R}^d}\overline{\partial^\alpha\varphi_j}A^{j,k}_{\alpha,\beta}\partial^\beta\varphi_k \geq \lambda\|\nabla^m\vec{\varphi}\|^2_{L^2(\Omega)} - \delta\|\vec{\varphi}\|^2_{L^2(\Omega)}$$

where $\lambda > 0$ and $\delta > 0$ are real numbers, for all $\vec{\varphi}$ which are smooth and compactly supported. In [1], Auscher and Qafsaoui consider higher order elliptic systems under divergence form in which ellipticity is in the sense of the weak Gårding inequality (11) rather than (10).

In section 5, we generalize the space $Y^{m,2}(\Omega)$ and allow for varying values of $p \neq 2$, but still finite. Here we describe the spaces $Y^{m,p}(\Omega)$ and note several properties for the functions therein and relations based on our elliptic operator.

$$Y^{m,p}(\Omega) := \{\vec{u} \in W^{m,p}(\Omega) : \partial^\alpha\vec{u} \in L^{p_\alpha}(\Omega),\ \text{for each } m - \frac{d}{p} < |\alpha| \leq m\}.$$

We also define the norm on these spaces similarly as for $Y^{m,2}$ above with,

$$\text{(12)} \qquad \omega_p := \text{ the smallest natural number such that } p_\alpha \text{ exists, for } \alpha \text{ with } |\alpha| = \omega_p.$$

Notice that ω_p is also the smallest non-negative integer greater than $m - \frac{d}{p}$. We then define the norm on $Y^{m,p}(\Omega)$ to be

$$\|u\|_{Y^{m,p}(\Omega)} := \sum_{i=\omega_p}^{m}\|\nabla^i u\|_{L^{p_i}(\Omega)}.$$

Notice that the definitions of ω_2 and ω_p are equal when $p = 2$. Also note that by our definition above, we have that $f \in Y^{m,p}(\Omega)$ is defined modulo polynomials of degree up to $m - \frac{d}{p}$. This definition is derived from the values of p for which we are able to apply the Gagliardo-Nirenberg-Sobolev inequality.

We would also like to describe the the "negative" Sobolev space $Y^{-m,p}(\Omega)$. Recall that for $1 \leq p \leq \infty$ the space $W^{-m,p}(\Omega)$ is usually defined as the dual space of $W_0^{m,p'}(\Omega)$, it is for this reason that we define $Y^{-m,p}(\Omega) := (Y_0^{m,p'}(\Omega))^*$ where $\Omega \subseteq \mathbb{R}^d$.

We would like to consider function spaces $Y^{m,p}(\Omega)$ where we specify at least as many derivatives as we do in $Y^{m,2}(\Omega)$. For $f \in Y^{m,2}(\Omega)$, if d is even, then when $p = 2$, we have that $m - \frac{d}{2}$ is a whole number and we specify the $m - \frac{d}{2} + 1$ derivatives of f up to the m-th derivatives of f. This leads us to the inequality $m - \frac{d}{2} + 1 > m - \frac{d}{p}$. However when d is odd, we then specify the $m - \frac{d}{2} + \frac{1}{2}$ derivatives of f up to the m-th derivatives of f. This leads us to the inequality $m - \frac{d}{2} + \frac{1}{2} > m - \frac{d}{p}$, and thus regardless of the parity of d, we have that $p < \frac{2d}{d-1}$. We will also in section 5 need the inequality $p' < \frac{2d}{d-1}$, which leads us to the range $\frac{2d}{d+1} < p < \frac{2d}{d-1}$. Thus for p and q in the interval $(\frac{2d}{d+1}, \frac{2d}{d-1})$, we have that for $\vec{u} \in Y^{m,p}(\Omega)$ and $\vec{v} \in Y^{m,q}(\Omega)$ we specify the same number of top derivatives for the functions $\vec{u}$ and $\vec{v}$.

3 The Caccioppoli Inequality

The Caccioppoli inequality is a fundamental result when discussing elliptic partial differential equations. In [4] Barton derives a Caccioppoli inequality for a divergence-form elliptic operator of the form

$$(L\vec{u})_j = (-1)^m \sum_{k=1}^{N} \sum_{|\alpha|=m} \sum_{|\beta|=m} \partial^\alpha (A_{\alpha,\beta}^{j,k} \partial^\beta u_k)$$

in which the coefficients $A_{\alpha,\beta}^{j,k}$ are bounded and measurable, and $\vec{u} \in \dot{W}^{m,2}(\Omega)$. First, a preliminary lemma establishes a bound on $\|\nabla^m \vec{u}\|_{L^2(B(x_0,R))}$ in terms of the lower order gradient L^2 norms. Next, the dependence on the gradient terms is removed, resulting in a bound which depends only on an L^2 norm of the solution. We begin with a few preliminary lemmas that will help us to deal with bounding our various $L^{2\alpha}$ norms of the gradients in terms of the L^2 norm of the solution. We will also show that the product of certain functions lies in $Y^{m,2}(\Omega)$. After we have these lemmas, we will proceed by first bounding $\|\nabla^m \vec{u}\|_{L^2(B(x_0,R))}$ in terms of all lower order derivatives, then improve to the case of a bound only in terms of the solution.

Lemma 3.1. *[12, Sec 5.6.1 Theorem 2] Let U be a bounded open subset of $\mathbb{R}^d$, and suppose that ∂U is C^1. Assume $1 \leq p < d$, p_* is as in (4) and $u \in W^{1,p}(U)$. Then $u \in L^{p_*}(U)$, with the estimate $\|u\|_{L^{p_*}(U)} \leq C\|u\|_{W^{1,p}(U)}$, where the constant C depends only on p, d, and U.*

Corollary 3.2. *Let u, U, d, and p be as in Lemma 3.1. Let k be a positive integer such that $1 \leq k \leq m$ and p_k be as in (5). Then $\|u\|_{L^{p_{m-k}}(U)} \leq C(U)\|u\|_{W^{k,p}(U)}$.*

Proof. We will proceed by induction. For the base case we wish to show that $\|u\|_{L^{p_*}(U)} \leq C\|\nabla u\|_{L^p(U)} + C\|u\|_{L^p(U)} = \|u\|_{W^{1,p}(U)}$, which follows from Lemma 3.1. For the induction step, we assume that the statement holds for j derivatives, i.e. that $\|u\|_{L^{p_{m-j}}(U)} \leq C\|u\|_{W^{j,p}(U)}$.

Notice by lemma 3.1 we have $\|v\|_{L^{p_{m-(j+1)}}(U)} \leq C\|v\|_{W^{1,p_{m-j}}(U)}$ for any function v, thus

$$\|u\|_{L^{p_{m-(j+1)}}(U)} \leq C\|u\|_{L^{p_{m-j}}(U)} + C\|\nabla u\|_{L^{p_{m-j}}(U)}.$$

Then by the induction step,

$$\begin{aligned}\|u\|_{L^{p_{m-(j+1)}}(U)} &\leq C\|u\|_{L^{p_{m-j}}(U)} + C\|\nabla u\|_{W^{j,p}(U)} \\ &\leq C\|u\|_{W^{j,p}(U)} + C\|\nabla u\|_{W^{j,p}(U)} \leq C\|u\|_{W^{j+1,p}(U)}\end{aligned}$$

Where we have reiterated the above inequality on $\|u\|_{L^{p_{m-j}}(U)}$, collected extra terms in $C\|\nabla u\|_{W^{j,p}(U)}$, and we have our result. □

The next lemma will be used at the end of the proof of lemma 3.9 to bound our $L^{2\alpha}$ norms in terms of L^2 norms, and provide the right scale for our powers of R.

Lemma 3.3. *Let $u \in W^{k,2}(B(x_0, R))$, $d \geq 3$, and $\frac{1}{2} - \frac{k}{d} > 0$. Then*

$$\|u\|_{L^{2_{m-k}}(B(x_0,R))} \leq C \sum_{j=0}^{k} R^{j-k} \|\nabla^j u\|_{L^2(B(x_0,R))}.$$

Proof. Let $v(x) = u(x_0 + Rx)$ so that $\nabla^j v(x) = R^j(\nabla^j u)(x_0 + Rx)$, and thus $v \in W^{k,2}(B(0,1))$. Notice by a change of variables that $R^{-d/2_{m-k}} \|u\|_{L^{2_{m-k}}(B(x_0,R))} = \|v\|_{L^{2_{m-k}}(B(0,1))}$, and thus by the previous corollary, $R^{-d/2_{m-k}} \|u\|_{L^{2_{m-k}}(B(x_0,R))} \leq C \sum_{j=0}^{k} \left(\int_{B(0,1)} |\nabla^j v(x)|^2 dx \right)^{1/2}$. Then by using the definition of $v(x)$, and a change of variables,

$$\begin{aligned} R^{-d/2_{m-k}} \|u\|_{L^{2_{m-k}}(B(x_0,R))} &\leq C \sum_{j=0}^{k} \left(\int_{B(0,1)} |(\nabla^j u)(x_0 + Rx)|^2 R^{2j} dx \right)^{1/2} \\ &= C \sum_{j=0}^{k} \left(\int_{B(x_0,R)} |(\nabla^j u)(y)|^2 R^{2j-d} dy \right)^{1/2} \\ &= C \sum_{j=0}^{k} R^{j-\frac{d}{2}} \|\nabla^j u\|_{L^2(B(x_0,R))} \end{aligned}$$

Recall that $\frac{1}{2_{m-k}} = \frac{d-2k}{2d}$, and multiply both sides of the above inequality by $R^{d/2_{m-k}}$ to obtain the result. □

Our next goal will be to show that when we multiply a solution $\vec{u} \in Y^{m,2}(B(x_0, R))$ by a cutoff function $\chi \in C_c^\infty(\mathbb{R}^d)$, which has support in $B(x_0, R)$, then the product $\vec{u}\chi$ is in the space $Y^{m,2}(\mathbb{R}^d)$. Before we can prove this, we need the following results.

Lemma 3.4. *[12, Morrey's Inequality] Suppose that $1 \leq q \leq \infty$, that $k > d/q$, and that $\nabla^k v \in L^q(B(x_0, R))$ for some ball $B(x_0, R) \subset \mathbb{R}^d$. Then v is Hölder continuous in*

$B(x_0, R)$, and satisfies the local bound

$$\|v\|_{L^\infty(B(x_0,R))} \leq C(q,k) \sum_{i=0}^{k} R^{i-d/q} \|\nabla^i v\|_{L^q(B(x_0,R))}.$$

Furthermore, in the case that $d < p < \infty$, there is a constant $C(p,d)$ such that for $y \in B(x,R)$ we have the bound

$$|u(x) - u(y)| \leq CR^{1-\frac{d}{p}} \left(\int_{B(x,2R)} |\nabla u(z)|^p dz \right)^{1/p}$$

for all $u \in C^1(B(x,2R))$.

In the following lemma, we prepare ourselves to take advantage of the fact that for the operator L, we have that $A_{\alpha,\beta} = 0$ when $|\alpha| \leq m - \frac{d}{2}$ or when $|\beta| \leq m - \frac{d}{2}$. We are not able to normalize $\vec{u}$ with a polynomial to use the Poincaré inequality, but we will be able to use the next few results to derive a bound on the lower order gradient terms of $\vec{u}$.

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set, F a function, and $\vec{u} \in Y^{m,2}(\Omega)$. If $L\vec{u} = F$ in Ω, and P is a polynomial of degree at most $m - \frac{d}{2}$, then $L(\vec{u} - P) = F$ in Ω as well.*

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$. We then have the following.

$$\begin{aligned} \langle \varphi, L(\vec{u} - P) \rangle &= \sum_{|\alpha| \leq m} \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} \partial^\alpha \varphi A_{\alpha,\beta} \partial^\beta (\vec{u} - P) \\ &= \sum_{|\alpha| \leq m} \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} \partial^\alpha \varphi A_{\alpha,\beta} \partial^\beta \vec{u} \\ &= \langle \varphi, L\vec{u} \rangle = \langle \varphi, F \rangle \end{aligned}$$

Note that the above equality is true for terms involving $|\beta| \leq m - \frac{d}{2}$ since here $A_{\alpha,\beta} = 0$, and for terms involving $|\beta| > m - \frac{d}{2}$ since here we have $\partial^\beta(\vec{u} - P) = \partial^\beta \vec{u}$ (by the degree of P). Thus we have that $L(\vec{u} - P) = F$. □

We now use lemma 3.5 and choose such a polynomial P with certain properties to give us the following result.

Lemma 3.6. *Let $u \in Y^{m,2}(B(x_0,R))$, and let the dimension d of $\mathbb{R}^d$ be odd. Then there is a polynomial P of degree at most $m - \frac{d}{2} - \frac{1}{2}$ such that*
$\|\partial^\gamma(\vec{u} - P)\|_{L^\infty(B(x_0,R))} \leq CR^{m-\frac{d}{2}-|\gamma|}\|u\|_{Y^{m,2}(B(x_0,R))}$ for all γ such that $|\gamma| \leq m - \frac{d}{2} - \frac{1}{2}$.

Proof. We will proceed by reverse induction, beginning with the case where $|\gamma| = m - \frac{d}{2} - \frac{1}{2}$. First we choose a polynomial P of degree at most $m - \frac{d}{2}$ so that $\partial^\beta(\vec{u} - P)(x_0) = 0$, for all $0 \leq |\beta| \leq m - \frac{d}{2} - \frac{1}{2}$. Observe that $2_{m-\frac{d}{2}+\frac{1}{2}} = 2d$ so then by Morrey's inequality

$$|\nabla^{m-\frac{d}{2}-\frac{1}{2}}(\vec{u} - P)(x)| \leq CR^{\frac{1}{2}}\|\nabla^{m-\frac{d}{2}+\frac{1}{2}}(\vec{u} - P)\|_{L^{2d}(B(x_0,R))}.$$

Now by the definition of the $Y^{m,2}$ norm we have

$$|\nabla^{m-\frac{d}{2}-\frac{1}{2}}(\vec{u} - P)(x)| \leq CR^{\frac{1}{2}}\|\vec{u}\|_{Y^{m,2}(B(x_0,R))}.$$

For the $|\gamma| = m - \frac{d}{2} - \frac{3}{2}$ case, notice that by the above bound and because $\int_{B(x_0,R)} \partial^\gamma(\vec{u} - P) = 0$

$$|\partial^\gamma(\vec{u} - P)(x)| \leq CR \cdot \|\nabla\partial^\gamma(\vec{u} - P)\|_{L^\infty(B(x_0,R))} \leq CR^{\frac{3}{2}}\|\vec{u}\|_{Y^{m,2}(B(x_0,R))}.$$

Thus by how we have chosen P, for each $0 \leq |\gamma| \leq m - \frac{d}{2} - \frac{1}{2}$ we have a bound in the following form,

$$|\nabla^{|\gamma|}(\vec{u} - P)(x)| \leq CR \cdot \|\nabla^{|\gamma|+1}(\vec{u} - P)\|_{L^\infty(B(x_0,R))} \leq \cdots \leq CR^{m-\frac{d}{2}-|\gamma|}\|\vec{u}\|_{Y^{m,2}(B(x_0,R))}$$

which gives us our result. □

We now turn our attention to proving an analogue to lemma 3.6 in the case where d is even. One of the main differences between lemma 3.6 and lemma 3.7 is the presence of the space BMO (see [18] for further discussion on BMO). We use the standard argument in the following proof from [12, Section 5.8.1].

Lemma 3.7. *Let $u \in Y^{m,2}(B(x_0, R))$, $q < \infty$ be a real number, and the dimension of $\mathbb{R}^d$ be even. Then $\nabla^{m-\frac{d}{2}}\vec{u} \in BMO(B(x_0, R))$, and there exists a polynomial P of degree $m - \frac{d}{2}$ such that*

$$\fint_{B(x_0,R)} |\nabla^{m-\frac{d}{2}}(\vec{u} - P)|^q \leq C\|\vec{u}\|_{Y^{m,2}(B(x_0,R))}$$

where C depends on d and q. Furthermore,
$\|\partial^\gamma(\vec{u} - P)\|_{L^\infty(B(x_0,R))} \leq CR^{m-d/2-|\gamma|}\|u\|_{Y^{m,2}(B(x_0,R))}$ for all γ such that $|\gamma| \leq m - \frac{d}{2} - 1$.

Proof. Recall from (6) that if d is even, then $\omega_2 = m - \frac{d}{2} + 1$. Then from (5) we have that $2_{\omega_2} = d$. If we can show the bound

$$\fint_{B(y,r)} |\nabla^{m-\frac{d}{2}}(\vec{u} - P)|^d \leq C\|\nabla^{m-\frac{d}{2}+1}\vec{u}\|_{L^d(B(x_0,R))}$$

for all $B(y, r) \subset B(x_0, R)$ then $\nabla^{m-\frac{d}{2}}\vec{u} \in \text{BMO}(B(x_0, R))$ since by the definition of the $Y^{m,2}$ norm

$$\|\nabla^{m-\frac{d}{2}+1}\vec{u}\|_{L^d(B(x_0,R))} = \|\nabla^{m-\frac{d}{2}+1}\vec{u}\|_{L^{2_{m-\frac{d}{2}+1}}(B(x_0,R))} \leq \|\vec{u}\|_{Y^{m,2}(B(x_0,R))}.$$

Let P be a polynomial of degree $m - \frac{d}{2}$ so that $\fint_{B(y,r)} \nabla^{m-\frac{d}{2}}(\vec{u} - P) = 0$. Notice then by the Poincaré inequality that

$$\fint_{B(y,r)} |\nabla^{m-\frac{d}{2}}(\vec{u} - P)|^d \leq C \int_{B(y,r)} |\nabla^{m-\frac{d}{2}+1}\vec{u}|^d \leq C \int_{B(x_0,R)} |\nabla^{m-\frac{d}{2}+1}\vec{u}|^d.$$

Thus we have that $\nabla^{m-\frac{d}{2}}\vec{u} \in \text{BMO}(B(x_0, R))$. Now by the John-Nirenberg inequality [18], we can change the exponent d to any $q < \infty$. It is here that we pick up the dependence on q for C. To get the last part of the conclusion we now may apply the same argument from the proof of lemma 3.6 if we change the degree of P to be at most $m - \frac{d}{2} - 1$, and γ so that $0 \le |\gamma| \le m - \frac{d}{2} - 1$. Putting these facts together we finish the proof. $\square$

We are now ready to prove that the product $\vec{u}\chi$ is in the space $Y^{m,2}(\mathbb{R}^d)$.

Lemma 3.8. *Let $B(x_0, R) \subset \mathbb{R}^d$ be a ball, $\vec{u} \in Y^{m,2}(B(x_0, R))$, and $\chi \in C_c^\infty(\mathbb{R}^d)$ be a test function with supp$(\chi) \subset B(x_0, R)$. Then we have that $\vec{u}\chi \in Y^{m,2}(\mathbb{R}^d)$.*

Proof. Here we will be not be providing a bound on $\|\vec{u}\chi\|_{Y^{m,2}(\mathbb{R}^d)}$, but merely showing that the norm is finite, thus proving our result. We begin by using the definition of the $Y^{m,2}$-norm (recall the definition of ω_2 from equation (6)), and the Leibniz rule.

$$\|\vec{u}\chi\|_{Y^{m,2}(\mathbb{R}^d)} = \sum_{i=\omega_2}^{m} \|\nabla^i(\vec{u}\chi)\|_{L^{2_i}(\mathbb{R}^d)} \le C \sum_{i=\omega_2}^{m} \left(\int_{\mathbb{R}^d} \left(\sum_{k=0}^{i} |\nabla^{i-k}\chi| \cdot |\nabla^k \vec{u}| \right)^{2_i} \right)^{1/2_i}$$

Now we bound $|\nabla^{i-k}\chi|$ by $\sup_{k \le i \le m} |\nabla^{i-k}\chi| < \infty$ for each k.

$$\begin{aligned} \|\vec{u}\chi\|_{Y^{m,2}(\mathbb{R}^d)} &\le C \sum_{i=\omega_2}^{m} \left(\int_{B(x_0,R)} \left(\sum_{k=0}^{i} \sup_{k \le i \le m} |\nabla^{i-k}\chi| \cdot |\nabla^k \vec{u}| \right)^{2_i} \right)^{1/2_i} \\ &\le C \sum_{i=\omega_2}^{m} \sum_{k=0}^{i} C(\chi) \Big(\int_{B(x_0,R)} |\nabla^k \vec{u}|^{2_i} \Big)^{1/2_i} \end{aligned}$$

We now only need to bound the $(\int_{B(x_0,R)} |\nabla^k \vec{u}|^{2_i})^{1/2_i}$ terms. Notice that $2_i \le 2_k$ so for terms where $k > m - \frac{d}{2}$, we may use Hölder's inequality to bound these terms. For terms where $k \le m - \frac{d}{2}$ we apply lemmas 3.6 and 3.7 and we have our result. $\square$

Next, we will prove a preliminary Caccioppoli inequality in which we bound the L^2 norm of the m^{th} order gradient of the solution of $L\vec{u} = \sum_{i=\omega_2}^{m} (-1)^i \text{div}_i F_i$ in terms of the L^2 norms of the lower-order gradient terms. A key distinction between Lemma 3.9 and [4, Lemma 9] is the appearance of the L^{2_α} norms which appear on the right-hand side, which we must bound in terms of L^2 norms via lemma 3.3.

Lemma 3.9. *Let L be the operator of order $2m$ associated to the coefficients $\{A_{\alpha,\beta}\}_{|\alpha|,|\beta|\leq m}$, which satisfy* (9) *and the weak Gårding inequality* (11).

Let $x_0 \in \mathbb{R}^d$ and $R > 0$. Suppose that $\vec{u} \in Y^{m,2}(B(x_0, 2R))$, that $\dot{\mathbf{F}}$ is an array so that each $F_k \in L^{(2_k)'}(B(x_0, 2R))$ and that $L\vec{u} = \sum_{k=\omega_2}^{m} (-1)^k \operatorname{div}_k F_k$. Then we have that

(13)
$$\int_{B(x_0,R)} |\nabla^m \vec{u}|^2 \leq \sum_{k=0}^{m-1} \frac{C}{R^{2m-2k}} \int_{B(x_0,2R)} |\nabla^k \vec{u}|^2 + C\delta \int_{B(x_0,2R)} |\vec{u}|^2 + \sum_{k=\omega_2}^{m} \left(\int_{B(x_0,2R)} |F_k|^{(2_k)'} \right)^{\frac{2}{(2_k)'}}$$

where C is a constant depending on the dimension d, the order $2m$ of L, the number λ in (11)*, and Λ as outlined in equation* (9).

Proof. Let φ be a smooth, real valued test function with $0 \leq \varphi \leq 1$, supported in $B(x_0, 2R)$ and identically equal to 1 on $B(x_0, R)$. We require also that $|\nabla^k \varphi| \leq C_k R^{-k}$ for any integer $k \geq 0$.

Notice that $\vec{\Psi} = \varphi^{4m}\vec{u}$ is a function supported in $B(x_0, 2R)$ and also that by lemma 3.8, $\vec{\Psi} \in Y^{m,2}(B(x_0, 2R))$. By the definition of $L\vec{u}$, and the density of smooth functions in $Y^{m,2}(B(x_0, 2R))$, we have

$$\text{(14)} \quad \sum_{j=1}^{N} \sum_{\omega_2 \leq |\alpha| \leq m} \int_{B(x_0,2R)} \partial^\alpha (\varphi^{4m} \overline{u}_j) F_{j,\alpha} = \sum_{j,k=1}^{N} \sum_{|\alpha| \leq m} \sum_{|\beta| \leq m} \int_{B(x_0,2R)} \partial^\alpha (\varphi^{4m} \overline{u}_j) A^{j,k}_{\alpha,\beta} \partial^\beta u_k.$$

We first consider the left hand side, and note by the Leibniz rule, and separating the $\gamma = \alpha$ terms,

$$\begin{aligned}
\sum_{j=1}^{N} \sum_{\omega_2 \leq |\alpha| \leq m} \int_{B(x_0,2R)} \partial^\alpha (\varphi^{4m} \overline{u}_j) F_{j,\alpha} &= \sum_{j=1}^{N} \sum_{\omega_2 \leq |\alpha| \leq m} \int_{B(x_0,2R)} \sum_{\gamma \leq \alpha} a_{\alpha,\gamma} \partial^\gamma (\varphi^{2m} \overline{u}_j) \partial^{\alpha-\gamma} (\varphi^{2m}) F_{j,\alpha} \\
&= \sum_{j=1}^{N} \sum_{\omega_2 \leq |\alpha| \leq m} \int_{B(x_0,2R)} \partial^\alpha (\varphi^{2m} \overline{u}_j) \varphi^{2m} F_{j,\alpha} \\
&+ \sum_{j=1}^{N} \sum_{\omega_2 \leq |\alpha| \leq m} \sum_{\gamma < \alpha} \int_{B(x_0,2R)} a_{\alpha,\gamma} \partial^\gamma (\varphi^{2m} \overline{u}_j) \partial^{\alpha-\gamma} (\varphi^{2m}) F_{j,\alpha}.
\end{aligned}$$

Thus using Hölder's inequality, and properties of our φ, we get

$$|\sum_{j=1}^{N}\sum_{\omega_2\leq|\alpha|\leq m}\int_{B(x_0,2R)}\partial^\alpha(\varphi^{4m}\overline{u}_j)F_{j,\alpha}|\leq\sum_{k=\omega_2}^{m}\|\nabla^k(\varphi^{2m}\vec{u})\|_{L^{2_k}(B(x_0,2R))}\|F_k\|_{L^{(2_k)'}(B(x_0,2R))}$$
$$+\sum_{k=\omega_2}^{m}\sum_{i=0}^{k-1}\frac{C}{R^{k-i}}\|\nabla^i\vec{u}\|_{L^{2_k}(B(x_0,2R)\setminus B(x_0,R))}\|F_k\|_{L^{(2_k)'}(B(x_0,2R))}.$$

Now by Young's inequality we have

$$(15)\quad |\sum_{j=1}^{N}\sum_{\omega_2\leq|\alpha|\leq m}\int_{B(x_0,2R)}\partial^\alpha(\varphi^{4m}\overline{u}_j)F_{j,\alpha}|\leq C\sum_{k=0}^{m-1}\|\nabla^k(\varphi^{2m}\vec{u})\|^2_{L^{2_k}(B(x_0,2R))}$$
$$+\sum_{k=\omega_2}^{m}\sum_{i=0}^{k-1}\frac{C\|\nabla^i\vec{u}\|^2_{L^{2_k}(B(x_0,2R)\setminus B(x_0,R))}}{R^{2(k-i)}}+\sum_{k=\omega_2}^{m}C\|F_k\|^2_{L^{(2_k)'}(B(x_0,2R))}+\frac{\lambda}{4}\|\nabla^m(\varphi^{2m}\vec{u})\|^2_{L^2(B(x_0,2R))}$$

where λ is the number in our ellipticity condition (11).

We now consider the right hand side of (14). By the Leibniz rule, and again separating out the $\gamma=\alpha$ terms, we see the following.

$$\sum_{j,k=1}^{N}\sum_{|\alpha|\leq m}\sum_{|\beta|\leq m}\int_{B(x_0,2R)}\partial^\alpha(\varphi^{4m}\overline{u}_j)A^{j,k}_{\alpha,\beta}\partial^\beta u_k=\sum_{j,k=1}^{N}\sum_{|\alpha|\leq m}\sum_{|\beta|\leq m}\int_{B(x_0,2R)}\partial^\alpha(\varphi^{2m}\overline{u}_j)A^{j,k}_{\alpha,\beta}\varphi^{2m}\partial^\beta u_k$$
$$+\sum_{j,k=1}^{N}\sum_{1\leq|\alpha|\leq m}\sum_{|\beta|\leq m}\int_{B(x_0,2R)}\sum_{\gamma<\alpha}a_{\alpha,\gamma}\partial^{\alpha-\gamma}(\varphi^{2m})\partial^\gamma(\varphi^{2m}\overline{u}_j)A^{j,k}_{\alpha,\beta}\partial^\beta u_k$$

Now as in [4], we write

$$(16)\qquad \sum_{\gamma<\alpha}a_{\alpha,\gamma}\partial^{\alpha-\gamma}(\varphi^{2m})\partial^\gamma(\varphi^{2m}\overline{u}_j)=\sum_{\zeta<\alpha}\varphi^{2m}\Phi_{\alpha,\zeta}\partial^\zeta\overline{u}_j$$

for some functions $\Phi_{\alpha,\zeta}$ which are supported in $B(x_0,2R)\setminus B(x_0,R)$, and satisfy

$|\Phi_{\alpha,\zeta}| \leq CR^{|\zeta|-|\alpha|}$. Thus we have

(17)

$$\sum_{j,k=1}^{N} \sum_{|\alpha|\leq m} \sum_{|\beta|\leq m} \int_{B(x_0,2R)} \partial^\alpha(\varphi^{4m}\overline{u}_j) A^{j,k}_{\alpha,\beta} \partial^\beta u_k = \sum_{j,k=1}^{N} \sum_{|\alpha|\leq m} \sum_{|\beta|\leq m} \int_{B(x_0,2R)} \partial^\alpha(\varphi^{2m}\overline{u}_j) A^{j,k}_{\alpha,\beta} \varphi^{2m} \partial^\beta u_k$$
$$+ \sum_{j,k=1}^{N} \sum_{1\leq|\alpha|\leq m} \sum_{|\beta|\leq m} \int_{B(x_0,2R)} \sum_{\zeta<\alpha} \Phi_{\alpha,\zeta} \partial^\zeta \overline{u}_j A^{j,k}_{\alpha,\beta} \varphi^{2m} \partial^\beta u_k.$$

It is desirable to have our sum on the bottom of (17) in terms of $\partial^\beta(\varphi^{2m}u_k)$ rather than $\varphi^{2m}\partial^\beta u_k$, so after one more application of the Leibniz rule, and writing as in (16), we have for some functions $\Psi_{\beta,\xi}$ which are supported in $B(x_0, 2R) \setminus B(x_0, R)$, and satisfy $|\Psi_{\beta,\xi}| \leq CR^{|\xi|-|\beta|}$

(18)
$$\sum_{j,k=1}^{N} \sum_{|\alpha|\leq m} \sum_{|\beta|\leq m} \int_{B(x_0,2R)} \partial^\alpha(\varphi^{4m}\overline{u}_j) A^{j,k}_{\alpha,\beta} \partial^\beta u_k =$$
$$\sum_{j,k=1}^{N} \sum_{|\alpha|\leq m} \sum_{|\beta|\leq m} \int_{B(x_0,2R)} \partial^\alpha(\varphi^{2m}\overline{u}_j) A^{j,k}_{\alpha,\beta} \varphi^{2m} \partial^\beta u_k$$
$$+ \sum_{j,k=1}^{N} \sum_{1\leq|\alpha|\leq m} \sum_{|\beta|\leq m} \int_{B(x_0,2R)} \sum_{\zeta<\alpha} \Phi_{\alpha,\zeta} \partial^\zeta \overline{u}_j A^{j,k}_{\alpha,\beta} \partial^\beta(\varphi^{2m} u_k)$$
$$- \sum_{j,k=1}^{N} \sum_{1\leq|\alpha|\leq m} \sum_{1\leq|\beta|\leq m} \int_{B(x_0,2R)} \sum_{\zeta<\alpha} \Phi_{\alpha,\zeta} \partial^\zeta \overline{u}_j A^{j,k}_{\alpha,\beta} \sum_{\xi<\beta} \varphi^m \Psi_{\beta,\xi} \partial^\xi u_k.$$

Similar measures as taken above also give us,

(19)
$$\sum_{j,k=1}^{N} \sum_{|\alpha|\leq m} \sum_{|\beta|\leq m} \int_{B(x_0,2R)} \partial^\alpha(\varphi^{2m}\overline{u}_j) A^{j,k}_{\alpha,\beta} \partial^\beta(\varphi^{2m} u_k) =$$
$$\sum_{j,k=1}^{N} \sum_{|\alpha|\leq m} \sum_{1\leq|\beta|\leq m} \int_{B(x_0,2R)} \partial^\alpha(\varphi^{2m}\overline{u}_j) A^{j,k}_{\alpha,\beta} \sum_{\xi<\beta} \varphi^m \Psi_{\beta,\xi} \partial^\xi u_k$$
$$+ \sum_{j,k=1}^{N} \sum_{|\alpha|\leq m} \sum_{|\beta|\leq m} \int_{B(x_0,2R)} \partial^\alpha(\varphi^{2m}\overline{u}_j) A^{j,k}_{\alpha,\beta} \varphi^{2m} \partial^\beta u_k.$$

Thus combining (18) and (19) we see that

$$\sum_{j,k=1}^{N}\sum_{|\alpha|\le m}\sum_{|\beta|\le m}\int_{B(x_0,2R)}\partial^{\alpha}(\varphi^{2m}\overline{u}_j)A^{j,k}_{\alpha,\beta}\partial^{\beta}(\varphi^{2m}u_k)=$$
$$\sum_{j,k=1}^{N}\sum_{|\alpha|\le m}\sum_{|\beta|\le m}\int_{B(x_0,2R)}\partial^{\alpha}(\varphi^{4m}\overline{u}_j)A^{j,k}_{\alpha,\beta}\partial^{\beta}u_k$$
$$-\sum_{j,k=1}^{N}\sum_{1\le|\alpha|\le m}\sum_{|\beta|\le m}\int_{B(x_0,2R)}\sum_{\zeta<\alpha}\Phi_{\alpha,\zeta}\partial^{\zeta}\overline{u}_jA^{j,k}_{\alpha,\beta}\partial^{\beta}(\varphi^{2m}u_k)$$
$$+\sum_{j,k=1}^{N}\sum_{1\le|\alpha|\le m}\sum_{1\le|\beta|\le m}\int_{B(x_0,2R)}\sum_{\zeta<\alpha}\Phi_{\alpha,\zeta}\partial^{\zeta}\overline{u}_jA^{j,k}_{\alpha,\beta}\sum_{\xi<\beta}\varphi^{m}\Psi_{\beta,\xi}\partial^{\xi}u_k$$
$$+\sum_{j,k=1}^{N}\sum_{|\alpha|\le m}\sum_{1\le|\beta|\le m}\int_{B(x_0,2R)}\partial^{\alpha}(\varphi^{2m}\overline{u}_j)A^{j,k}_{\alpha,\beta}\sum_{\xi<\beta}\varphi^{m}\Psi_{\beta,\xi}\partial^{\xi}u_k. \tag{20}$$

We write this as I=II+III+IV+V. By (11), we have that

$$\lambda\|\nabla^m(\varphi^{2m}\vec{u})\|^2_{L^2(B(x_0,2R))}\le \operatorname{Re}\mathrm{I}+\delta\|\varphi^{2m}\vec{u}\|^2_{L^2(B(x_0,2R))}. \tag{21}$$

For II, we use (14) and (15) along with monotonicity, and reindexing the sum to get the following bound.

$$|\mathrm{II}|\le C\sum_{\omega_2\le|\alpha|\le m}\sum_{\zeta<\alpha}\|\partial^{\zeta}\vec{u}\|^2_{L^{2_\alpha}(B(x_0,2R))}R^{2(|\zeta|-|\alpha|)}$$
$$+\sum_{k=\omega_2}^{m}C\|F_k\|^2_{L^{(2_k)'}(B(x_0,2R))}+\frac{\lambda}{4}\|\nabla^m(\varphi^{2m}\vec{u})\|^2_{L^2(B(x_0,2R))} \tag{22}$$

The remainder of III, IV, and V use Hölder's inequality, Young's inequality, and (16).

$$|\mathrm{III}|\le C\sum_{\omega_2\le|\alpha|\le m}\sum_{\zeta<\alpha}\|\partial^{\zeta}\vec{u}\|^2_{L^{2_\alpha}(B(x_0,2R))}R^{2(|\zeta|-|\alpha|)}$$
$$+C\sum_{\omega_2\le|\beta|<m}\sum_{\xi\le\beta}\|\partial^{\xi}\vec{u}\|^2_{L^{2_\beta}(B(x_0,2R))}R^{2(|\xi|-|\beta|)}+\frac{\lambda}{4}\|\nabla^m(\varphi^{2m}\vec{u})\|^2_{L^2(B(x_0,2R))}. \tag{23}$$

By using our bounds on the Φ's, as well, we obtain,

$$(24)\quad |\mathrm{IV}| \le C \sum_{\omega_2 \le |\alpha| \le m} \sum_{\zeta < \alpha} \|\partial^\zeta \vec{u}\|^2_{L^{2_\alpha}(B(x_0,2R))} R^{2(|\zeta|-|\alpha|)} + C \sum_{\omega_2 \le |\beta| \le m} \sum_{\xi < \beta} \|\partial^\xi \vec{u}\|^2_{L^{2_\beta}(B(x_0,2R))} R^{2(|\xi|-|\beta|)}.$$

Splitting up V between our α derivatives, and our β derivatives, as well as using the bounds on the Ψ's, we have

$$(25)\quad |\mathrm{V}| \le C \sum_{\omega_2 \le |\alpha| < m} \sum_{\zeta \le \alpha} \|\partial^\zeta \vec{u}\|^2_{L^{2_\alpha}(B(x_0,2R))} R^{2(|\zeta|-|\alpha|)} + C \sum_{\omega_2 \le |\beta| \le m} \sum_{\xi < \beta} \|\partial^\xi \vec{u}\|^2_{L^{2_\beta}(B(x_0,2R))} R^{2(|\xi|-|\beta|)} + \frac{\lambda}{4} \|\nabla^m(\varphi^{2m}\vec{u})\|^2_{L^2(B(x_0,2R))}.$$

Now combine (21)-(25), change the order of summation, and reindex our sums to get

$$(26)\quad \frac{\lambda}{4} \|\nabla^m(\varphi^{2m}\vec{u})\|^2_{L^2(B(x_0,2R))} \le C \sum_{|\zeta| \le m-1} \sum_{\zeta \le \alpha} \sum_{\omega_2 \le |\alpha| \le m} \|\partial^\zeta \vec{u}\|^2_{L^{2_\alpha}(B(x_0,2R))} R^{2(|\zeta|-|\alpha|)} + \sum_{k=\omega_2}^{m} C \|F_k\|^2_{L^{(2_k)'}(B(x_0,2R))} + \delta \|\varphi^{2m}\vec{u}\|^2_{L^2(B(x_0,2R))}.$$

Notice that the right-hand side of (26) is in terms of L^{2_α} norms. By Lemma 3.3, we can write

$$(27)\quad \|\partial^\zeta u\|_{L^{2_\alpha}(B(x_0,R))} \le C \sum_{k=|\zeta|}^{m-|\alpha|+|\zeta|} \|\nabla^k u\|_{L^2(B(x_0,R))} R^{((k-|\zeta|)-(m-|\alpha|))}.$$

Thus by combining (26) and (27) we obtain the result. □

We may note at this point, that if we wanted to improve inequality (26) on the right-hand side by integrating over the annulus $B(x_0, R) \setminus B(x_0, r)$ (where r is any number such that $0 < r < R$) rather than the ball $B(x_0, 2R)$ our methods would allow us to do that at a cost of not being able to use Lemma 3.3. Thus we would not be able to end with L^2-norms on the right hand side, but rather L^{2_α}-norms instead.

We now wish to improve the bound (13) in terms of $\|u\|_{L^2}$ rather than in terms of all of the lower-order derivatives. The first step is to use the Vitali covering lemma so that the ball on the left-hand side is not half the radius of the ball on the right-hand side, but rather an arbitrarily smaller radius.

Lemma 3.10. *([13, Theorem 1.24]) Let $\mathcal{F}$ be any collection of nondegenerate closed balls in $\mathbb{R}^d$ with $\sup\{\operatorname{diam} B | B \in \mathcal{F}\} < \infty$. Then there exists a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that $\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{B\in\mathcal{G}} 5B$, where $5B$ is the concentric closed ball with radius 5 times the radius of B.*

In terms of what we need to improve our bound from (3.9), let $E = B(x_0, R) = \bigcup_{x\in B(x_0,R-\rho)} B(x,\rho)$, where $0 < \rho < R$. Then there exists a disjoint collection of balls, $\{B(x_j,\rho)\}_{j=1}^J$ with each $x_j \in B(x_0, R-\rho)$ such that $E \subset \bigcup_{j=1}^J B(x_j,\rho)$.

Corollary 3.11. *If $x \in B(x_0, R)$ then x is in at most $C(d)$ of the balls $B(x_j, 10\rho)$, where d is the dimension of $\mathbb{R}^d$.*

Proof. Let $x \in B(x_0, R)$, and define the set $I(x) = \{j : x \in B(x_j, 10\rho)\}$. If $j \in I(x)$ then $B(x_j,\rho) \subset B(x, 11\rho)$. Since the $B(x_j,\rho)$ are disjoint, we have $|I(x)| = \frac{C(d)}{\rho^d}|\bigcup_{j\in I(x)} B(x_j,\rho)| \leq \dfrac{C(d)}{\rho^d}|B(x,11\rho)| = C(d)$. □

We can use corollary 3.11 to improve the bound (13).

Corollary 3.12. *Let L, $\{A_{\alpha,\beta}\}_{|\alpha|,|\beta|\leq m}$, $\{F_k\}_{k=\omega_2}^m$, and $\vec{u}$ be as in lemma 3.9. Then for $0 < \rho < R$, $\vec{u}$ satisfies the inequality*

$$\begin{aligned}\int_{B(x_0,R)} |\nabla^m \vec{u}|^2 \leq \sum_{k=0}^{m-1} \frac{C}{\rho^{2m-2k}} \int_{B(x_0,R+10\rho)} |\nabla^k \vec{u}|^2 + C\delta \int_{B(x_0,R+10\rho)} |\vec{u}|^2 \\ + C\sum_{k=\omega_2}^{m} \left(\int_{B(x_0,R+10\rho)} |F_k|^{(2_k)'}\right)^{\frac{2}{(2_k)'}}\end{aligned} \tag{28}$$

Proof. The proof is short, and employs lemmas 3.10, 3.9, and 3.11. Let $I(x)$ and x_j be as in the proof of corollary 3.11, and $j \in I(x)$. We then have

$$\int_{B(x_0,R)} |\nabla^m \vec{u}|^2 \leq \sum_{j=1}^{J} \int_{B(x_j,5\rho)} |\nabla^m \vec{u}|^2$$
$$\leq \sum_{j=1}^{J} \sum_{k=0}^{m-1} \frac{C}{\rho^{2m-2k}} \int_{B(x_j,10\rho)} |\nabla^k \vec{u}|^2 + C\delta \int_{B(x_j,10\rho)} |\vec{u}|^2 + \sum_{k=\omega_2}^{m} \left(\int_{B(x_j,10\rho)} |F_k|^{(2_k)'} \right)^{\frac{2}{(2_k)'}}$$

where $0 < \rho < R$ comes from lemma 3.10. Now by lemma 3.11, we can eliminate the sum in j, and obtain the result. □

In the next theorem we will combine the above corollary with ideas from [4] and [19] to obtain a bound on $\|\nabla^m \vec{u}\|_{L^2(B(x_0,r))}$ in terms of only $\|\vec{u}\|_{L^2(B(x_0,R))}$.

Theorem 3.13. *Let $x_0 \in \mathbb{R}^d$, and let $R > 0$. Let $\vec{u} \in Y^{m,2}B(x_0,R)$ be a function that satisfies the inequality*

$$\int_{B(x_0,\rho)} |\nabla^m \vec{u}|^2 \leq \sum_{k=0}^{m-1} \frac{C_0}{(r-\rho)^{2m-2k}} \int_{B(x_0,r)} |\nabla^k \vec{u}|^2 + F \tag{29}$$

for all r, ρ with $0 < R/2 \leq \rho < r \leq R$ for some $F > 0$. Then $\vec{u}$ satisfies the inequality

$$\int_{B(x_0,\rho)} |\nabla^m \vec{u}|^2 \leq \frac{C}{(r-\rho)^{2m}} \int_{B(x_0,r)} |\vec{u}|^2 + CF \tag{30}$$

for some constant C depending only on m, the dimension d, and the constant C_0.

Proof. We prove this as in [4], except for the fact that we only establish the bound for a ball, rather than an annulus. We proceed by induction, showing that each term $\|\nabla^k \vec{u}\|_{L^2(B(x_0,r))}$ can be controlled by a sum involving gradients of order strictly less than k. Upon iterating this argument for gradients of order m, down to order 0, we effectively reduce our sum on the right-hand side of (29) to a single term. Thus if we can show this is true for all k in the range $0 \leq k \leq m$ then we will have the result. The argument is summarized in the following claim.

We claim that $\vec{u}$ satisfies the following. If $1 \le k \le m$ and $R/2 \le \eta \le \xi < R$, then

$$\int_{B(x_0,\eta)} |\nabla^k \vec{u}|^2 \le \sum_{i=0}^{k-1} \frac{C_k}{(\xi-\eta)^{2k-2i}} \int_{B(x_0,\xi)} |\nabla^i \vec{u}|^2 + R^{2m-2k} F$$

The fact that the claim is true for $k = m$ is formula (29), so we will work by induction and show that if the claim is true for some $k+1 \le m$, then it is indeed true for k as well. Choose an ascending sequence of balls $B_j = B(x_0, \rho_j)$ such that $\eta = \rho_0 < \rho_1 < \rho_2 < ... < \xi$ for some sequence $\{\rho_j\}_{j=1}^{\infty}$ to be chosen later on. Set $\delta_j = \rho_{j+1} - \rho_j > 0$ and define the ball $\widetilde{B}_j = B(x_0, \rho_j + \delta_j/2)$ so that we have the strict inclusion $B_j \subsetneq \widetilde{B}_j \subsetneq B_{j+1}$ for all j. Now choose a sequence of smooth cutoff functions $\varphi_j \in C_0^\infty(\widetilde{B}_j)$ such that $\varphi_j|_{B_j} = 1$. We will also need each φ_j to satisfy $\|\nabla \varphi_k\| \le \frac{C}{\delta_j}$, and also $\|\nabla^2 \varphi_k\| \le \frac{C}{\delta_j^2}$ for some constant C.

First by the monotonicity of integration and our choice of φ_j, we have for each $1 \le k \le m$ that

$$\int_{B_j} |\nabla^k \vec{u}|^2 \le \int_{\widetilde{B}_j} |\nabla(\varphi_j \nabla^{k-1} \vec{u})|^2.$$

Recall by Plancherel's theorem, if $f \in W^{2,2}(\mathbb{R}^d)$ then $\|\nabla f\|_{L^2(\mathbb{R}^d)} \le C \|\nabla^2 f\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)}$. Apply this to $f = \varphi_j \nabla^{k-1} \vec{u}$ and use the Leibniz rule so that we have the bound

$$\begin{aligned}\int_{B_j} |\nabla^k \vec{u}|^2 &\le C \left(\int_{\widetilde{B}_j} |\nabla^2(\varphi_j \nabla^{k-1} \vec{u})|^2 \right)^{1/2} \left(\int_{\widetilde{B}_j} |\varphi_j \nabla^{k-1} \vec{u}|^2 \right)^{1/2} \\ &\le C \left(\int_{\widetilde{B}_j} |\nabla^{k+1} \vec{u}|^2 + \frac{|\nabla^k \vec{u}|^2}{\delta_j^2} + \frac{|\nabla^{k-1} \vec{u}|^2}{\delta_j^4} \right)^{1/2} \left(\int_{\widetilde{B}_j} |\nabla^{k-1} \vec{u}|^2 \right)^{1/2}.\end{aligned}$$

Now we may apply the claim to the $|\nabla^{k+1} \vec{u}|^2$ term to get

$$\int_{B_j} |\nabla^k \vec{u}|^2 \le \left(\sum_{i=0}^{k} \frac{C_k}{\delta_j^{2k-2i+2}} \int_{B_{j+1}} |\nabla^i \vec{u}|^2 + C R^{2m-2k-2} F \right)^{1/2} \left(\int_{\widetilde{B}_j} |\nabla^{k-1} \vec{u}|^2 \right)^{1/2}.$$

Move the $\frac{C_k}{\delta_j^2}$ to the right most integral, and use Young's inequality so that

$$\int_{B_j} |\nabla^k \vec{u}|^2 \leq \frac{1}{2} \sum_{i=0}^{k} \frac{1}{\delta_j^{2k-2i}} \int_{B_{j+1}} |\nabla^i \vec{u}|^2 + \frac{1}{2} R^{2m-2k} F + \frac{C_k}{\delta_j^2} \int_{\widetilde{B}_j} |\nabla^{k-1} \vec{u}|^2.$$

By monotonicity since $\widetilde{B}_j \subset B_{j+1}$, and separating the k-term, we then see that

$$\int_{B_j} |\nabla^k \vec{u}|^2 \leq C_k \sum_{i=0}^{k-1} \frac{1}{\delta_j^{2k-2i}} \int_{B_{j+1}} |\nabla^i \vec{u}|^2 + \frac{1}{2} R^{2m-2k} F + \frac{1}{2} \int_{B_{j+1}} |\nabla^k \vec{u}|^2.$$

Since we have proven the above inequality for all $j > 0$, we can iterate on the above $\int_{B_{j+1}} |\nabla^k \vec{u}|^2$ term to gain the inequality,

$$\begin{aligned}\int_{B_0} |\nabla^k \vec{u}|^2 &\leq \sum_{j=0}^{\infty} 2^{-j} \left(C_k \sum_{i=0}^{k-1} \frac{1}{\delta_j^{2k-2i}} \int_{B_{j+1}} |\nabla^i \vec{u}|^2 + \frac{1}{2} R^{2m-2k} F \right) \\ &\leq C_k \sum_{i=0}^{k-1} \left(\sum_{j=0}^{\infty} \frac{2^{-j}}{\delta_j^{2k-2i}} \right) \int_{B_\infty} |\nabla^i \vec{u}|^2 + R^{2m-2k} F.\end{aligned}$$

Lastly we need to choose ρ_j appropriately so that the above sums converge. Choose $\rho_j = \eta + (\xi - \eta)(1 - \tau) \sum_{i=1}^{j} \tau^{i-1}$, where $0 < \tau < 1$, so that $\rho_0 = \eta$ and $\lim_{j \to \infty} \rho_j = \xi$ (using the convention that in the case $j = 0$ the empty sum is equal to 0). Thus we have the bound

$$\int_{B_0} |\nabla^k \vec{u}|^2 \leq C_{k,\tau} \sum_{i=0}^{k-1} \left(\sum_{j=1}^{\infty} \frac{1}{(2\tau^{2k-2i})^j (\xi - \eta)^{2k-2i}} \right) \int_{B_\infty} |\nabla^i \vec{u}|^2 + R^{2m-2k} F$$

when $\tau < 1$. Choose τ so that $2\tau^{2k} > 1$, so the sum converges in j and the claim holds. Now from the claim, we are able to bound each of the terms involving $|\nabla^k \vec{u}|^2$ by a single term, only involving $|\vec{u}|^2$, and thus we are done with the proof. $\square$

Now if we combine corollary 3.12 and theorem 3.13, we obtain the desired Caccioppoli inequality in which we bound $|\nabla^m \vec{u}|^2$ without the intermediate gradient terms, as stated in the following corollary.

Corollary 3.14. *(A higher order Caccioppoli inequality) Let* $x_0 \in \mathbb{R}^d$*, and* $0 < R/2 < r < R$*. Suppose* $\vec{u} \in Y^{m,2}(B(x_0, R))$ *and* $\dot{\mathbf{F}}$ *are as in lemma 3.9, so that* $L\vec{u} = \sum_{i=\omega_2}^{m} (-1)^i \text{div}_i F_i$*. Then* $\vec{u}$ *satisfies the inequality*

(31)
$$\int_{B(x_0,r)} |\nabla^m \vec{u}|^2 \leq \frac{C}{(R-r)^{2m}} \int_{B(x_0,R)} |\vec{u}|^2 + C \sum_{k=\omega_2}^{m} \left(\int_{B(x_0,R)} |F_k|^{(2_k)'} \right)^{\frac{2}{(2_k)'}} + C\delta \int_{B(x_0,R)} |\vec{u}|^2$$

where C *depends on* λ *and* Λ*, the dimension* d *and the order* $2m$ *of* L*.*

4 The Newton Potential

In this section we will construct the Newton potential, that is, the solution operator to the equation $L\vec{u} = \sum_{i=\omega_2}^{m} (-1)^i \text{div}_i F_i$. Recall the definition of ω_2 from equation (6). We will follow the method of [4], [15], and [16], where we assume that our solutions $\vec{u}$ lie in the space $Y^{m,2}(\mathbb{R}^d)$, and our array $\dot{\mathbf{F}}$ satisfies $F_\alpha \in L^{(2_\alpha)'}(\mathbb{R}^d)$ for $m - \frac{d}{2} < |\alpha| \leq m$. Notice that this requirement on $\dot{\mathbf{F}}$ is derived from the fact that for $\varphi \in Y^{m,2}(\mathbb{R}^d)$ and $m - \frac{d}{2} < |\alpha| \leq m$ we have $\partial^\alpha \varphi \in L^{2_\alpha}(\mathbb{R}^d)$.

Definition 4.1. Let $\Omega \subseteq \mathbb{R}^d$ be a connected open set. We say that an array of functions $\dot{\mathbf{F}}$ is in the space $\mathcal{F}_p(\Omega)$ if for each $1 \leq j \leq N$ and each integer i such that $\omega_p \leq i \leq m$ we have that $F_\alpha \in L^{(p_\alpha)'}(\Omega)$. We then write $\|\dot{\mathbf{F}}\|_{\mathcal{F}_p(\Omega)} := \sum_{i=\omega_p}^{m} \|F_i\|_{L^{(p_i)'}(\Omega)}$.

Notice that in section 3 we assumed that $\dot{\mathbf{F}} \in \mathcal{F}_2(B(x_0, 2R))$ so that we could derive a Caccioppoli inequality.

Remark 2. Every array $\dot{\mathbf{F}}$ that is in $\mathcal{F}_2(\Omega)$ gives rise to a bounded linear operator $T_{\dot{\mathbf{F}}}$ on $Y^{m,2}(\Omega)$, that is an element of $Y^{-m,2}(\Omega)$ given by $T_{\dot{\mathbf{F}}}(\vec{u}) = \sum_{j=1}^{N} \sum_{\omega_2 \leq |\alpha| \leq m} \int_\Omega \partial^\alpha u_j \overline{F_{j,\alpha}}$. Notice that $T_{\dot{\mathbf{F}}}(\vec{u}) \in \mathbb{C}$ with $|T_{\dot{\mathbf{F}}}(\vec{u})| \leq C\|\vec{u}\|_{Y^{m,2}(\Omega)}$. Conversely by the Hahn Banach theorem if $T \in Y^{-m,2}(\Omega)$ then there exists $\dot{\mathbf{F}} \in \mathcal{F}_2(\Omega)$ such that $T(\varphi) = T_{\dot{\mathbf{F}}}(\varphi)$ for all $\varphi \in Y^{m,2}(\Omega)$ and $\|T\|_{Y^{-m,2}(\Omega)} = \|\dot{\mathbf{F}}\|_{\mathcal{F}_2(\Omega)}$.

Now we will use the complex valued Lax-Milgram lemma to construct the Newton potential. Recall the complex Lax-Milgram lemma:

Theorem 4.1. *[3, Theorem 2.1] Let H_1 and H_2 be two Hilbert spaces, and let B be a bounded bilinear form on $H_1 \times H_2$ that is coercive in the sense that*

$$\sup_{w \in H_1 \setminus \{0\}} \frac{|B(w,v)|}{\|w\|_{H_1}} \geq \lambda \|v\|_{H_2}, \qquad \sup_{w \in H_2 \setminus \{0\}} \frac{|B(u,w)|}{\|w\|_{H_2}} \geq \lambda \|u\|_{H_1}$$

for every $u \in H_1$, and $v \in H_2$, for some fixed $\lambda > 0$. Then for every linear functional T defined on H_2 there is a unique $u_T \in H_1$ such that $B(v, u_T) = \overline{T(v)}$. Furthermore $\|u_T\|_{H_1} \leq \frac{1}{\lambda} \|T\|_{H_2'}$.

Let L be an operator of order $2m$ which satisfies the ellipticity conditions outlined in (9) and (10) from section 2.1. Let $B(\vec{u}, \vec{v})$ be the form given by

$$B(\vec{u}, \vec{v}) = \sum_{j,k=1}^{N} \sum_{|\alpha| \leq m} \sum_{|\beta| \leq m} \int_{\mathbb{R}^d} \overline{\partial^\alpha u_j} A_{\alpha,\beta}^{j,k} \partial^\beta v_k \tag{32}$$

where each $A_{\alpha,\beta} \in L^{2_{\alpha,\beta}}(\mathbb{R}^d)$. Notice that by the ellipticity assumptions and Hölder's inequality B is coercive and a bounded, bilinear operator on $Y^{m,2}(\mathbb{R}^d) \times Y^{m,2}(\mathbb{R}^d)$ with the bound

$$|B(\vec{u}, \vec{v})| \leq \Lambda \|\vec{u}\|_{Y^{m,2}(\mathbb{R}^d)} \|\vec{v}\|_{Y^{m,2}(\mathbb{R}^d)}. \tag{33}$$

Let T be a bounded linear operator on $Y^{m,2}(\mathbb{R}^d)$. In the case that $\dot{\mathbf{F}} = \{F_{j,\alpha}\ 1 \leq j \leq N, |\alpha| \leq m\}$ is an array of functions that lies in $\mathcal{F}_2(\mathbb{R}^d)$, then we may define such an operator T by $T_{\dot{\mathbf{F}}}(\vec{v}) = \sum_{j=1}^{N} \sum_{\omega_2 \leq |\alpha| \leq m} \int_{\mathbb{R}^d} \overline{F_{j,\alpha}} \partial^\alpha v_j$. Similarly to our bound on B, by Hölder's inequality we have that $T_{\dot{\mathbf{F}}}$ is a bounded linear operator on $Y^{m,2}(\mathbb{R}^d)$ with the bound

$$|T_{\dot{\mathbf{F}}}(\vec{v})| \leq \|\vec{v}\|_{Y^{m,2}(\mathbb{R}^d)}\|\dot{\mathbf{F}}\|_{\mathcal{F}_2(\mathbb{R}^d)}. \tag{34}$$

Let $u_T = L^{-1}T \in Y^{m,2}(\mathbb{R}^d)$ be the unique element given by the Lax-Milgram lemma, if $T = T_{\dot{\mathbf{F}}}$ we write $\vec{u}_T = \vec{\Pi}^L\dot{\mathbf{F}}$ so

$$\sum_{j,k=1}^{N}\sum_{|\alpha|\leq m}\sum_{|\beta|\leq m}\int_{\mathbb{R}^d}\overline{\partial^\alpha\varphi}A^{j,k}_{\alpha,\beta}\partial^\beta(\vec{\Pi}^L\dot{\mathbf{F}})_k = \sum_{j=1}^{N}\sum_{|\alpha|\leq m}\int_{\mathbb{R}^d}\overline{\partial^\alpha\varphi}F_{j,\alpha} \tag{35}$$

or

$$\sum_{j,k=1}^{N}\sum_{|\alpha|\leq m}\sum_{|\beta|\leq m}\int_{\mathbb{R}^d}\overline{\partial^\alpha\varphi}A^{j,k}_{\alpha,\beta}\partial^\beta(L^{-1}T)_k = \overline{T(\varphi)}. \tag{36}$$

for all $\varphi \in Y^{m,2}(\mathbb{R}^d)$.

Notice that by the uniqueness given by the Lax-Milgram lemma, the Newton potential $L^{-1}T$ or $\vec{\Pi}^L\dot{\mathbf{F}}$ is well defined as an element of $Y^{m,2}(\mathbb{R}^d)$ up to adding polynomials of order at most $m - \frac{d}{2}$, and we have that if $\Phi \in Y^{m,2}(\mathbb{R}^d)$ then by equation (35)

$$\Phi = \vec{\Pi}^L(\dot{\mathbf{F}}), \text{ when } F_{j,\alpha} = \sum_{k=1}^{N}\sum_{|\beta|\leq m}A^{j,k}_{\alpha,\beta}\partial^\beta\Phi_k. \tag{37}$$

Remark 3. Depending on the context, it may be convenient for us to switch between notation for $\vec{\Pi}^L$ and L^{-1}. In sections 4 and 6 we will describe our results in terms of $\vec{\Pi}^L$, and in section 5 we will phrase our results in terms of L^{-1}. Also note that in section 4 and 6 we will phrase results in terms of the spaces $\mathcal{F}_p$ and in section 5 we phrase our results in terms of $Y^{-m,p}$-spaces, which consists of linear operators associated to arrays in $\mathcal{F}_p$ (as described in remark 2).

Let us now introduce some new notation to aid in brevity, and write

$$\widetilde{\nabla}^m \vec{\Pi}^L = (\nabla^{\omega_2}\vec{\Pi}^L, \nabla^{\omega_2+1}\vec{\Pi}^L, ..., \nabla^m\vec{\Pi}^L) \tag{38}$$

where ω_2 is as in definition (6). Next we will describe the adjoint of the operator $\widetilde{\nabla}^m\vec{\Pi}^{L^*}$, in the following lemma.

Lemma 4.2. *The adjoint to the operator* $\widetilde{\nabla}^m\vec{\Pi}^L$ *is* $\widetilde{\nabla}^m\vec{\Pi}^{L^*}$*.*

Proof. We first recall the inner product mentioned in section 2 given by equation (3). Let each F_α be in the corresponding Lebesgue space, $L^{(2_\alpha)'}(\mathbb{R}^d)$ so that $T(\varphi) := \langle \dot{\mathbf{F}}, \widetilde{\nabla}^m\varphi\rangle$ is bounded on $Y^{m,2}(\mathbb{R}^d)$. Let $u = \Pi^L(\dot{\mathbf{F}}) \in Y^{m,2}(\mathbb{R}^d)$. By our definitions of $T(\cdot)$ and $B(\cdot,\cdot)$ above and of $\vec{\Pi}^L$ we have

$$\langle \dot{\mathbf{F}}, \widetilde{\nabla}^m\varphi\rangle = T(\varphi) = B(\vec{\Pi}^L\dot{\mathbf{F}}, \varphi) = \langle A\widetilde{\nabla}^m\vec{\Pi}^L\dot{\mathbf{F}}, \widetilde{\nabla}^m\varphi\rangle, \text{ for all } \varphi \in Y^{m,2}(\mathbb{R}^d).$$

Similarly, let $\vec{v} = \vec{\Pi}^{L^*}\dot{\mathbf{G}} \in \dot{Y}^{m,2}(\mathbb{R}^d)$ so $\langle \dot{\mathbf{G}}, \widetilde{\nabla}^m\varphi\rangle = \langle A^*\widetilde{\nabla}^m\vec{v}, \widetilde{\nabla}^m\varphi\rangle \in Y^{m,2}(\mathbb{R}^d)$. Notice first that $\langle \dot{\mathbf{F}}, \widetilde{\nabla}^m\vec{\Pi}^{L^*}\dot{\mathbf{G}}\rangle = \langle \dot{\mathbf{F}}, \widetilde{\nabla}^m\vec{v}\rangle = \langle A\widetilde{\nabla}^m\vec{\Pi}^L\dot{\mathbf{F}}, \widetilde{\nabla}^m\vec{v}\rangle = \langle A\widetilde{\nabla}^m\vec{\Pi}^L\dot{\mathbf{F}}, \widetilde{\nabla}^m\vec{\Pi}^{L^*}\dot{\mathbf{G}}\rangle$.

Next we observe,

$$\begin{aligned}\langle \dot{\mathbf{G}}, \widetilde{\nabla}^m\vec{\Pi}^L\dot{\mathbf{F}}\rangle = \langle \dot{\mathbf{G}}, \widetilde{\nabla}^m\vec{u}\rangle = \langle A^*\widetilde{\nabla}^m\vec{\Pi}^{L^*}\dot{\mathbf{G}}, \widetilde{\nabla}^m\vec{u}\rangle &= \langle A^*\widetilde{\nabla}^m\vec{\Pi}^{L^*}\dot{\mathbf{G}}, \widetilde{\nabla}^m\vec{\Pi}^L\dot{\mathbf{F}}\rangle \\ &= \overline{\langle \widetilde{\nabla}^m\vec{\Pi}^L\dot{\mathbf{F}}, A^*\widetilde{\nabla}^m\vec{\Pi}^{L^*}\dot{\mathbf{G}}\rangle} \\ &= \overline{\langle A\widetilde{\nabla}^m\vec{\Pi}^L\dot{\mathbf{F}}, \widetilde{\nabla}^m\vec{\Pi}^{L^*}\dot{\mathbf{G}}\rangle}.\end{aligned}$$

Where we have used the definition of our inner product to pick up the complex conjugate, then move over the A^* term. Thus we have $\langle \dot{\mathbf{G}}, \widetilde{\nabla}^m\vec{\Pi}^L\dot{\mathbf{F}}\rangle = \langle \widetilde{\nabla}^m\vec{\Pi}^{L^*}\dot{\mathbf{G}}, \dot{\mathbf{F}}\rangle$, and finally that $\langle (\widetilde{\nabla}^m\vec{\Pi}^L)^*\dot{\mathbf{G}}, \dot{\mathbf{F}}\rangle = \langle \widetilde{\nabla}^m\vec{\Pi}^{L^*}\dot{\mathbf{G}}, \dot{\mathbf{F}}\rangle$. □

In the case of the operator L given in [4], Barton is able to use a reverse Hölder inequality and a duality argument to show that the Newton potential is bounded on a range of L^p spaces where $|p-2| < \epsilon$ for some $\epsilon > 0$. As we do not have a reverse Hölder inequality available to us, we will next use a combination of the Lax-Milgram lemma, previous results, and Šneĭberg's lemma (lemma 4.3 below) to show that the Newton potential is bounded on a range of $\mathcal{F}_p(\mathbb{R}^d)$ spaces.

We now must explore a bit of interpolation theory in order to be able to apply Šneĭberg's lemma to the current problem. We first note that from [20], combining the facts that $\dot{W}^{m,p}(\Omega)$ forms a complex interpolation scale, and the map which sends an element of $\dot{W}^{m,p}(\Omega)$ to its unique representative in $Y^{m,p}(\Omega)$ is a retract [17, Lemma 7.11], we have that $Y^{m,p}(\mathbb{R}^d)$ forms a complex interpolation scale. Next we have from [8, Theorem 4.5.1] that the space $(Y^{m,p}(\mathbb{R}^d))^*$ also forms a complex interpolation scale.

As in [9, Lemma 3.4] the values of p around $p = 2$ for which L is invertible will depend on a range provided in Šneĭberg's lemma, as stated below.

Lemma 4.3. *(Šneĭberg's lemma [2, Theorem A.1]) Let $\overline{X} = (X_0, X_1)$ and $\overline{Z} = (Z_0, Z_1)$ be interpolation couples, and $T \in \mathcal{B}(\overline{X}, \overline{Z})$. Suppose that for some $\theta^* \in (0, 1)$ and some $\kappa > 0$, the lower bound $\|Tx\|_{Z_{\theta^*}} \geq \kappa \|x\|_{X_{\theta^*}}$ holds for all $x \in X_{\theta^*}$. Then the following are true.*

(i) Given $0 < \epsilon < 1/4$, the lower bound $\|Tx\|_{Z_\theta} \geq \epsilon\kappa \|x\|_{X_\theta}$ holds for all $x \in X_\theta$, provided that $|\theta - \theta^*| \leq \frac{\kappa(1-4\epsilon)\min\{\theta^*,1-\theta^*\}}{3\kappa+6M}$, where $M = \max_{j=0,1} \|T\|_{X_j \to Z_j}$.

(ii) If $T : X_{\theta^*} \longrightarrow Z_{\theta^*}$ is invertible, then the same is true for $T : X_\theta \longrightarrow Z_\theta$ if θ is as in (i). The inverse mappings agree on $X_\theta \cap X_{\theta^*}$ and their norms are bounded by $\frac{1}{\epsilon\kappa}$.

Now we are ready to show that the Newton potential is bounded on a range of $\mathcal{F}_p$ spaces.

Lemma 4.4. *Let L be an operator of order $2m$ that satisfies the ellipticity conditions* (9) *and* (10)*. Then there is a $\delta > 0$ such that $\vec{\mathbf{\Pi}}^L$ extends to an operator that is bounded from $\mathcal{F}_p(\mathbb{R}^d)$ to $Y^{m,p}(\mathbb{R}^d)$ where $2 - \delta < p < 2 + \delta$, and L^{-1} extends to an operator that is bounded $Y^{-m,p}(\mathbb{R}^d) \to Y^{m,p}(\mathbb{R}^d)$.*

Proof. First recall that the Lax-Milgram lemma provides us with the invertibility of L $Y^{m,p}(\mathbb{R}^d) \to Y^{m,p}(\mathbb{R}^d)$ when $p = 2$ by bound (33). Next by lemma 5.1 we have that L is bounded from $Y^{m,p}(\mathbb{R}^d)$ to $Y^{-m,p}(\mathbb{R}^d)$ when p is in the range $(\frac{2d}{d+1}, \frac{2d}{d-1})$. Let $\dot{\mathbf{F}} \in \mathcal{F}_2(\mathbb{R}^d) \cap \mathcal{F}_p(\mathbb{R}^d)$. Recall the Gårding inequality (10), and apply Šneĭberg's lemma 4.3.

We then have that L is bounded and invertible $Y^{-m,p}(\mathbb{R}^d) \to Y^{m,p}(\mathbb{R}^d)$ when $|p-2| < \delta$, where δ is as dictated by (i) from Šneĭberg's lemma. Combining this with the fact that $\vec{\Pi}^L \dot{\mathbf{F}} = L^{-1}(T_{\dot{\mathbf{F}}})$, invertibility of L and boundedness of $T_{\dot{\mathbf{F}}}$ we have our result. □

5 L^p Bounds On The Gradient

In the case of the operator L from [4], Barton employs techniques from the lower order case (for example the Laplacian $L = \Delta$) to derive a higher order version of Meyers's reverse Hölder inequality for the m^{th} order gradients of the solutions to $L\vec{u} = \operatorname{div}_m \dot{\mathbf{F}}$ in [4, Theorem 24]. One of the main tools used to accomplish this is the Poincaré inequality, which is of the form $\|u - u_\Omega\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}$, where $u_\Omega = \fint_\Omega u$. In order to use the Poincaré inequality in this context we must be able to normalize $\vec{u}$ by adding polynomials p of degree $m-1$ so that $\widetilde{u} = \vec{u} + p$ satisfies $\widetilde{u}_\Omega = 0$ and $L\widetilde{u} = \operatorname{div}_m \dot{F}$ (i.e, $\widetilde{u}$ is a solution to the same equation as $\vec{u}$, and $\widetilde{u} - \widetilde{u}_\Omega = \widetilde{u}$).

However in the case of our operator L as defined in section 1, we are unable to employ this technique (specifically we are unable to normalize solutions by adding polynomials of degree greater than $m - \frac{d}{2}$ and still obtain a solution to $L\vec{u} = \sum_{i=\omega_2}^{m} \operatorname{div}_i F_i$). As a substitute we will adapt the ideas of [9] to L in order to prove L^p bounds on solutions, where p is in a certain range as given by Šneĭberg's lemma.

In this section, we will explore how L behaves when paired with functions in certain $Y^{m,p}$ spaces when p is in the range $\frac{2d}{d+1} < p < \frac{2d}{d-1}$ (this range on p is precisely that from section 2.1 which guarantees us the right number of derivatives that we need for functions in our $Y^{m,p}(\Omega)$ spaces). We then prove a bound on $\vec{u}\chi$ (where $\vec{u}\chi$ is as in lemma 3.8) and use this bound to prove the invertibility of L in that range of values for p.

Lemma 5.1. *Let p be a real number such that $\frac{2d}{d+1} < p < \frac{2d}{d-1}$. Then $L : Y^{m,p}(\mathbb{R}^d) \to Y^{-m,p}(\mathbb{R}^d)$ is bounded.*

Proof. Let $\vec{u} \in Y^{m,p}(\mathbb{R}^d)$, and $\varphi \in Y^{m,p'}(\mathbb{R}^d)$. Recall that $\frac{1}{p_\beta} = \frac{1}{p} - \frac{m-|\beta|}{d}$ and

$\frac{1}{(p')_\alpha} = 1 - \frac{1}{p} - \frac{m-|\alpha|}{d}$, and that each coefficient $A_{\alpha,\beta}$ of L is in the space $L^r(\Omega)$ where $r = \frac{d}{2m-|\alpha|-|\beta|}$. Observe that $\frac{1}{(p')_\alpha} + \frac{1}{p_\beta} + \frac{1}{r} = 1$, and we can then apply Hölder's inequality to bound the following inner product.

$$\langle \varphi, L\vec{u} \rangle = \sum_{|\alpha|\leq m} \sum_{|\beta|\leq m} \int_{\mathbb{R}^d} \partial^\alpha \varphi A_{\alpha,\beta} \partial^\beta \vec{u}$$

Therefore, using the fact that $A_{\alpha,\beta} = 0$ for $|\alpha| \leq m - \frac{d}{2}$ or $|\beta| \leq m - \frac{d}{2}$ we have the following bound.

$$\begin{aligned}
|\langle \varphi, L\vec{u} \rangle| &\leq \sum_{\substack{\omega_p' \leq |\alpha| \leq m \\ \omega_p \leq |\beta| \leq m}} \int_{\mathbb{R}^d} |\partial^\alpha \varphi| \cdot |A_{\alpha,\beta}| \cdot |\partial^\beta \vec{u}| \\
&\leq \sum_{\substack{\omega_p' \leq |\alpha| \leq m \\ \omega_p \leq |\beta| \leq m}} \|\partial^\alpha \varphi\|_{L^{(p')_\alpha}(\mathbb{R}^d)} \|\partial^\beta \vec{u}\|_{L^{p_\beta}(\mathbb{R}^d)} \|A_{\alpha,\beta}\|_{L^r(\mathbb{R}^d)} \\
&\leq C \|\varphi\|_{Y^{m,p'}(\mathbb{R}^d)} \|\vec{u}\|_{Y^{m,p}(\mathbb{R}^d)}
\end{aligned}$$

where C depends on Λ from the coefficients $A_{\alpha,\beta}$, and the order m of the operator L. Note that we have only included terms where the coefficients $A_{\alpha,\beta}$ are non-zero. Observe that the right side of the inequality is finite by the assumptions on $\vec{u}$ and φ, and the proof is complete. □

Recall that in lemma 3.8 we showed that the product $\vec{u}\chi$ is in the space $Y^{m,2}$. We will now explore how we can write $L(\vec{u}\chi)$ when $\vec{u}$ is such that $L\vec{u} = 0$ in the weak sense.

Lemma 5.2. *Let $R > 0$ be a real number. Suppose that $\vec{u} \in Y^{m,2}(B(x_0, R))$ satisfies $L\vec{u} = 0$ in $B(x_0, R)$ in the weak sense. Then for all $\chi \in C_c^\infty(B(x_0, R))$, we have that $L(\chi\vec{u}) = \sum_{|\delta|\leq m} (-1)^{|\delta|} \partial^\delta F_\delta$ in the weak sense, where*

$$F_\delta = \sum_{|\beta|\leq m} \sum_{\gamma<\beta} A_{\delta,\beta} a_{\beta,\gamma} \partial^\gamma \vec{u} \partial^{\beta-\gamma} \chi - \sum_{|\beta|\leq m} \sum_{\alpha>\delta} A_{\alpha,\beta} a_{\alpha,\delta} \partial^\beta \vec{u} \partial^{\alpha-\delta} \chi.$$

Here the second sum is taken to be 0 *if* $|\delta| = m$.

Proof. We first outline our assumption that $L\vec{u} = 0$ in the weak sense. Using our definition of the inner product, we have that for all $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\sum_{|\alpha|\leq m} \sum_{|\beta|\leq m} \int_{B(x_0,R)} \partial^\alpha(\varphi\chi) A_{\alpha,\beta} \partial^\beta \vec{u} = \langle \varphi\chi, L\vec{u}\rangle_{B(x_0,R)} = 0.$$

We now compute $\langle \varphi, L(\vec{u}\chi)\rangle$, and use the above assumption, along with the Leibniz rule to derive F_δ as above.

$$\begin{aligned}
\langle \varphi, L(\vec{u}\chi)\rangle_{B(x_0,R)} &= \sum_{|\alpha|\leq m} \sum_{|\beta|\leq m} \int_{B(x_0,R)} \partial^\alpha \varphi A_{\alpha,\beta} \partial^\beta(\vec{u}\chi) \\
&= \langle \varphi\chi, L\vec{u}\rangle_{B(x_0,R)} + \sum_{|\alpha|\leq m} \sum_{|\beta|\leq m} \sum_{\gamma<\beta} \int_{B(x_0,R)} \partial^\alpha \varphi A_{\alpha,\beta} a_{\beta,\gamma} \partial^\gamma \vec{u} \partial^{\beta-\gamma}\chi \\
&\qquad - \sum_{|\alpha|\leq m} \sum_{|\beta|\leq m} \sum_{\delta<\alpha} \int_{B(x_0,R)} a_{\alpha,\delta} \partial^\delta \varphi \partial^{\alpha-\delta}\chi A_{\alpha,\beta} \partial^\beta \vec{u} \\
&= \sum_{|\alpha|\leq m} \sum_{|\beta|\leq m} \sum_{\gamma<\beta} \int_{B(x_0,R)} \partial^\alpha \varphi A_{\alpha,\beta} a_{\beta,\gamma} \partial^\gamma \vec{u} \partial^{\beta-\gamma}\chi \\
&\qquad - \sum_{|\alpha|\leq m} \sum_{|\beta|\leq m} \sum_{\delta<\alpha} \int_{B(x_0,R)} a_{\alpha,\delta} \partial^\delta \varphi \partial^{\alpha-\delta}\chi A_{\alpha,\beta} \partial^\beta \vec{u}
\end{aligned} \tag{39}$$

Our goal is to collect the terms in φ with the same number of derivatives. Now in the first sum of the last row above, replace α with δ, then switch the order of summation in the second sum to get

$$\begin{aligned}
\langle \varphi, L(\vec{u}\chi)\rangle_{B(x_0,R)} &= \sum_{|\delta|\leq m} \int_{B(x_0,R)} \partial^\delta \varphi \sum_{|\beta|\leq m} \sum_{\gamma<\beta} A_{\delta,\beta} a_{\beta,\gamma} \partial^\gamma \vec{u} \partial^{\beta-\gamma}\chi \\
&\qquad - \sum_{|\delta|\leq m-1} \int_{B(x_0,R)} \partial^\delta \varphi \sum_{|\beta|\leq m} \sum_{\alpha>\delta} a_{\alpha,\delta} A_{\alpha,\beta} \partial^\beta \vec{u} \partial^{\alpha-\delta}\chi
\end{aligned} \tag{40}$$

Writing F_δ as in the statement, we finish the proof. $\square$

Remark 4. Notice that in lemma 5.2 when $|\delta| = m$, equation (40) gives us that the second sum is equal to 0 and so

$$F_\delta = \sum_{|\beta|\leq m}\sum_{\gamma<\beta} A_{\delta,\beta} a_{\beta,\gamma}\partial^\gamma \vec{u}\partial^{\beta-\gamma}\chi.$$

Essentially lemma 5.2 shows us that when a solution of $L\vec{u} = 0$ is multiplied by a test function, we end up with a divergence form equation (with of course the presence of lower order terms). We will now use the inner product as written in lemma 5.2 to show that $L(\vec{u}\chi) \in Y^{-m,p}(\mathbb{R}^d)$, where p is in a range around 2.

Lemma 5.3. *Let $p, q \in (\frac{2d}{d+1}, \frac{2d}{d-1})$, $0 < R < \infty$ be a real number, $x_0 \in \mathbb{R}^d$, $\vec{u} \in Y^{m,2}(B(x_0, R)) \cap Y^{m,q}(B(x_0, R))$ be such that $L\vec{u} = 0$, and $\chi \in C_c^\infty(\mathbb{R}^d)$ be a test function such that $supp(\chi) \subset B(x_0, R)$. We extend $\vec{u}\chi$ by 0 outside of $B(x_0, R)$. Then $L(\vec{u}\chi) \in Y^{-m,p}(\mathbb{R}^d)$ and there is a normalization of $\vec{u}$ that satisfies the bound*

$$\|L(\vec{u}\chi)\|_{Y^{-m,p}(\mathbb{R}^d)} \leq CR^{d/p-d/q}\|\vec{u}\|_{Y^{m,q}(B(x_0,R))}. \tag{41}$$

Recall that if $m \geq \frac{d}{2}$ then elements of $Y^{m,2}(\Omega)$ are defined only up to adding polynomials of degree $\leq m - \frac{d}{2}$; the bound (41) is valid for an appropriate choice of polynomial.

Proof of lemma 5.3. Let $\varphi \in Y^{m,p'}(\mathbb{R}^d)$ where $\frac{1}{p'} = 1 - \frac{1}{p}$. Recall that $Y^{-m,p}(\mathbb{R}^d)$ is the dual space to $Y_0^{m,p'}(\mathbb{R}^d)$. So to show that $L(u\chi) \in Y^{-m,p}(\mathbb{R}^d)$, we need only bound $\langle\varphi, L(u\chi)\rangle_{\mathbb{R}^d}$ for all φ in $Y^{m,p'}(\mathbb{R}^d)$. From lemma 5.2 equation (39) we have

$$\langle\varphi, L(\vec{u}\chi)\rangle_{\mathbb{R}^d} =$$
$$\sum_{|\alpha|\leq m}\sum_{|\beta|\leq m}\sum_{\gamma<\beta}\int_{B(x_0,R)} \partial^\alpha\varphi A_{\alpha,\beta} a_{\beta,\gamma}\partial^\gamma\vec{u}\partial^{\beta-\gamma}\chi - \sum_{|\alpha|\leq m}\sum_{|\beta|\leq m}\sum_{\delta<\alpha}\int_{B(x_0,R)} a_{\alpha,\delta}\partial^\delta\varphi\partial^{\alpha-\delta}\chi A_{\alpha,\beta}\partial^\beta\vec{u}.$$

We will write this as $\langle\varphi, L(\vec{u}\chi)\rangle_{B(x_0,R)} = I - II$. Since $A_{\alpha,\beta} = 0$ if $|\alpha| \leq m - d/2$ or $|\beta| \leq m - d/2$, we can assume $|\alpha| > m - \frac{d}{2}$ and $|\beta| > m - d/2$.

We begin with the case where $|\gamma| > m - d/2$, as here we can bound the integral in I using a four-way Holder inequality if p is in an appropriate range, and each $\partial^{\beta-\gamma}\chi$ is in the correct Lebesgue space (call it L^x). Set $x = \frac{qpd}{qd-pd+qp(|\beta|-|\gamma|)}$ and then notice that we have

$$\frac{1}{(p')_\alpha} + \frac{1}{2_{\alpha,\beta}} + \frac{1}{q_\gamma} + \frac{1}{x} = 1.$$

Notice that $\frac{1}{x} = \frac{1}{p} - \frac{1}{q} + \frac{|\beta|-|\gamma|}{d}$. By our range on p and q, since $|\beta| \leq m$, and since $|\beta| > |\gamma| \geq m - \frac{d}{2} + \frac{1}{2}$, we have that

$$0 = \frac{d-1}{2d} - \frac{d+1}{2d} + \frac{1}{d} \leq \frac{1}{x} \leq \frac{d+1}{2d} - \frac{d-1}{2d} + \frac{m-(m-d/2+1/2)}{d} = \frac{1}{2d} + \frac{1}{2} < 1$$

and so $0 \leq \frac{1}{x} < 1$. We are now able to apply a 4-way Hölder inequality. Let Λ be as in (9) with $\Omega = \mathbb{R}^d$. Now using the fact that $\chi \in C_c^\infty(\mathbb{R}^d)$ we have that when $|\gamma| > m - \frac{d}{2}$,

$$\begin{aligned}
|I| &\leq C \sum_{\substack{m-\frac{d}{p'}<|\alpha|\leq m \\ m-\frac{d}{2}<|\beta|\leq m \\ m-\frac{d}{2}<|\gamma|<|\beta|}} \|\partial^\alpha \varphi\|_{L^{(p')_\alpha}(\mathbb{R}^d)} \|A_{\alpha,\beta}\|_{L^{2_{\alpha,\beta}}(\mathbb{R}^d)} \|\partial^\gamma \vec{u}\|_{L^{q_\gamma}(B(x_0,R))} \|\partial^{\beta-\gamma}\chi\|_{L^x(\mathbb{R}^d)} \\
&\leq C\Lambda \sum_{\substack{m-\frac{d}{p'}<|\alpha|\leq m \\ m-\frac{d}{2}<|\beta|\leq m \\ m-\frac{d}{2}<|\gamma|<|\beta|}} \|\partial^\alpha \varphi\|_{L^{(p')_\alpha}(\mathbb{R}^d)} |B(x_0,R)|^{\frac{1}{x}} \|\partial^\gamma \vec{u}\|_{L^{q_\gamma}(B(x_0,R))} \|\partial^{\beta-\gamma}\chi\|_{L^\infty(\mathbb{R}^d)} \\
&\leq C\Lambda \sum_{\substack{m-\frac{d}{p'}<|\alpha|\leq m \\ m-\frac{d}{2}<|\beta|\leq m \\ m-\frac{d}{2}<|\gamma|<|\beta|}} \|\partial^\alpha \varphi\|_{L^{(p')_\alpha}(\mathbb{R}^d)} R^{\frac{d}{x}} \|\partial^\gamma \vec{u}\|_{L^{q_\gamma}(B(x_0,R))} R^{-(|\beta|-|\gamma|)} \\
&\leq C R^{\frac{qd-pd}{qp}} \|\varphi\|_{Y^{m,p'}(\mathbb{R}^d)} \|\vec{u}\|_{Y^{m,q}(B(x_0,R))}.
\end{aligned} \tag{42}$$

Here we have used $\frac{d}{x} - (|\beta| - |\gamma|) = \frac{qd-pd}{qp}$, the definition of the $Y^{m,p}$ norms, and C depends on m and the ellipticity constant Λ. Similarly in II, in the case where $|\delta| > m - \frac{d}{2}$, we need each $\partial^{\alpha-\delta}\chi$ in the correct Lebesgue space (call it L^y). Set

$y = \frac{qpd}{qd-pd+qp(|\alpha|-|\delta|)}$ so we have that

$$\frac{1}{(p')_\delta} + \frac{1}{y} + \frac{2m-|\alpha|-|\beta|}{d} + \frac{1}{q_\beta} = 1.$$

This yields $\frac{1}{y} = \frac{1}{p} - \frac{1}{q} + \frac{|\alpha|-|\delta|}{d}$, and therefore by our range on p, and since $|\alpha| \leq m$ and $|\delta| \geq m - \frac{d}{2} + \frac{1}{2}$, we have that $0 \leq \frac{1}{y} < 1$.

We again are able to use Hölder's inequality to obtain the following bound on II when $|\delta| > m - \frac{d}{2}$:

$$\begin{aligned}
|II| &\leq C \sum_{\substack{m-\frac{d}{p'}<|\alpha|\leq m \\ m-\frac{d}{p'}<|\delta|<|\alpha| \\ m-\frac{d}{2}<|\beta|\leq m}} \|\partial^\delta \varphi\|_{L^{(p')_\delta}(\mathbb{R}^d)} \|\partial^{\alpha-\delta}\chi\|_{L^y(\mathbb{R}^d)} \|A_{\alpha,\beta}\|_{L^{2_{\alpha,\beta}}(\mathbb{R}^d)} \|\partial^\beta \vec{u}\|_{L^{q_\beta}(B(x_0,R))} \\
&\leq C\Lambda \sum_{\substack{m-\frac{d}{p'}<|\alpha|\leq m \\ m-\frac{d}{p'}<|\delta|<|\alpha| \\ m-\frac{d}{2}<|\beta|\leq m}} \|\partial^\delta \varphi\|_{L^{(p')_\delta}(\mathbb{R}^d)} |B(x_0,R)|^{\frac{1}{y}} \|\partial^{\alpha-\delta}\chi\|_{L^\infty(\mathbb{R}^d)} \|\partial^\beta \vec{u}\|_{L^{q_\beta}(B(x_0,R))} \\
&\leq C\Lambda \sum_{\substack{m-\frac{d}{p'}<|\alpha|\leq m \\ m-\frac{d}{p'}<|\delta|<|\alpha| \\ m-\frac{d}{2}<|\beta|\leq m}} \|\partial^\delta \varphi\|_{L^{(p')_\delta}(\mathbb{R}^d)} R^{\frac{d}{y}} R^{-(|\alpha|-|\delta|)} \|\partial^\beta \vec{u}\|_{L^{q_\beta}(B(x_0,R))} \\
&\leq C R^{\frac{qd-pd}{qp}} \|\varphi\|_{Y^{m,p'}(\mathbb{R}^d)} \|\vec{u}\|_{Y^{m,q}(B(x_0,R))},
\end{aligned} \tag{43}$$

where C depends on m and Λ.

In the case where $|\gamma| \leq m - \frac{d}{2}$ or $|\delta| \leq m - \frac{d}{2}$ we subtract off a polynomial P of degree at most $m - \frac{d}{2}$ from $\vec{u}$ so that $L(\vec{u} - P) = L\vec{u} = 0$ and $L(\vec{u}\chi - P) = L(\vec{u}\chi)$ and we may apply lemmas 3.6 and 3.7, (recall that $\vec{u} \in Y^{m,2}$ functions are defined up to polynomials of degree $m - \frac{d}{2}$) and we are done. $\square$

From lemma 5.3 we have a bound on $L(\vec{u}\chi)$. We are now in a position to derive a bound on $\|\vec{u}\|_{Y^{m,p}(B(x_0,\frac{R}{2}))}$ where p is in a range around 2 given by lemma 4.3.

Theorem 5.4. *Let $R > 0$ be a real number and $x_0 \in \mathbb{R}^d$. Then there exists a $\delta > 0$ that depends on L, so that for real numbers p and q in the range $2 - \delta < q < p < 2 + \delta$, if $\vec{u} \in Y^{m,2}(B(x_0, R)) \cap Y^{m,q}(B(x_0, R))$ and $L\vec{u} = 0$ in $B(x_0, R)$, then $\vec{u} \in Y^{m,p}(B(x_0, \frac{R}{2}))$ and $\|\vec{u}\|_{Y^{m,p}(B(x_0,\frac{R}{2}))} \leq CR^{d/p-d/q}\|\vec{u}\|_{Y^{m,q}(B(x_0,R))}$, where C depends on Λ, m, p, and d.*

Remark 5. Notice that theorem 5.4 is trivially true by Hölder's inequality in the case that $p \leq q$.

Proof of theorem 5.4. Let $\chi \in C_c^\infty(B(x_0, R))$ be a test function so that $\chi = 1$ in $B(x_0, \frac{R}{2})$, and $|\nabla^i \chi| \leq CR^{-i}$. By the discussion following theorem 4.1, we have that L is invertible $Y^{m,2}(\mathbb{R}^d) \to Y^{-m,2}(\mathbb{R}^d)$ via the Newton Potential. Since by lemma 3.8 we have that $\vec{u}\chi \in Y^{m,2}(\mathbb{R}^d)$ and since $L : Y^{m,2}(\mathbb{R}^d) \to Y^{-m,2}(\mathbb{R}^d)$ is invertible we have that $\vec{u}\chi = L^{-1}(L(\vec{u}\chi))$. By lemma 4.3 there is a $\delta > 0$ so that for p in the range $2 - \delta < p < 2 + \delta$ we have that L^{-1} is defined and bounded $Y^{-m,p}(\mathbb{R}^d) \to Y^{m,p}(\mathbb{R}^d)$. By lemma 5.3, $L(\vec{u}\chi) \in Y^{-m,p}(\mathbb{R}^d)$ so then $L^{-1}(L(\vec{u}\chi)) \in Y^{m,p}(\mathbb{R}^d)$. All of this together gives us that since $\vec{u}\chi \in Y^{m,2}(\mathbb{R}^d)$, we have that $\vec{u}\chi \in Y^{m,p}(\mathbb{R}^d)$. Then applying lemma 5.3 and since we have chosen χ so that $\chi = 1$ in $B(x_0, \frac{R}{2})$ we have the result. □

We can now combine the above results, with results from section 3 along with a Hölder argument to obtain a bound on $\|\vec{u}\|_{Y^{m,p}(B(x_0,\frac{R}{2}))}$ in terms of $\|\vec{u}\|_{L^p(B(x_0,\frac{R}{2}))}$.

Corollary 5.5. *Let $\vec{u} \in Y^{m,2}(B(x_0, \frac{R}{2}))$ such that $L\vec{u} = 0$ and p be a real number such that $2 < p < 2 + \delta$ where δ is as in theorem 5.4. Then we have the bound*

$$\|\vec{u}\|_{Y^{m,p}(B(x_0,\frac{R}{2}))} \leq CR^{-2m}\|\vec{u}\|_{L^p(B(x_0,R))}$$

where C depends on m, Λ, p, and d.

Proof. First from theorem 5.4 we have the bound $\|\vec{u}\|_{Y^{m,p}(B(x_0,\frac{R}{2})} \leq CR^{d/p-d/2}\|\vec{u}\|_{Y^{m,2}(B(x_0,R))}$. Now by lemma 3.3, note that every term of

$\|\vec{u}\|_{Y^{m,2}(\Omega)} = \sum_{i=\omega_2}^{m} \|\nabla^i \vec{u}\|_{L^{2_i}(\Omega)}$ is bounded by $C\sum_{j=0}^{m-i} R^{j-m+i}\|\nabla^{j+i}\vec{u}\|_{L^2(B(x_0,R))}$. Thus by rearranging the sum, we have the bound

$$\begin{aligned}\|\vec{u}\|_{Y^{m,p}(B(x_0,\frac{R}{2}))} &\leq CR^{d/p-d/2}\sum_{i=\omega_2}^{m}\sum_{j=0}^{m-i} R^{j-m+i}\|\nabla^{j+i}\vec{u}\|_{L^2(B(x_0,R))} \\ &= CR^{d/p-d/2}\sum_{j=\omega_2}^{m} R^{j-m}\|\nabla^{j}\vec{u}\|_{L^2(B(x_0,R))}.\end{aligned}$$

Then by the induction argument in theorem 3.13 we obtain the following bound

$$\|\vec{u}\|_{Y^{m,p}(B(x_0,\frac{R}{2}))} \leq CR^{d/p-d/2-2m}\|\vec{u}\|_{L^2(B(x_0,R))}.$$

Next let $s = \frac{p}{p-2}$, and since $2 < p$, by Hölder's inequality notice that $\|\vec{u}\|_{L^2(B(x_0,R))} \leq C_d R^{d/2s}\|\vec{u}\|_{L^p(B(x_0,R))}$. This gives us the bound $\|\vec{u}\|_{Y^{m,p}(B(x_0,\frac{R}{2}))} \leq CR^{d/p-d/2-2m+d/2s}\|\vec{u}\|_{L^p(B(x_0,R))} = CR^{-2m}\|\vec{u}\|_{L^p(B(x_0,R))}$. □

Despite not being able to fully normalize $\vec{u}$ with an appropriate polynomial, we are able to take advantage of the fact that we're able to normalize the lower order derivative parts where the derivative is sufficiently low. We have shown via Sobolev embedding where the resulting derivatives lie, and used properties of functions in those spaces along with properties of our $Y^{m,p}$ space, to obtain the bound in 5.5.

6 The Fundamental Solution

Here we will construct the fundamental solution as the kernel of the Newton potential from section 4. We first study the fundamental solution in the case that L is high enough order, then extend our results to the case that L is of lower order.

6.1 The Fundamental Solution For Operators Of High-Order

We now turn our attention to constructing the fundamental solution in the case that L is of sufficiently high order $2m > d$. In this case, by Morrey's inequality (lemma 3.4) any element of $Y^{m,2}(\mathbb{R}^d)$ is continuous. Similar to [4] where we wish to extend these results to an operator of arbitrarily higher order, we need to treat this case with some care. Later on in section 6.2 we will extend these results to operators of low order where $2m \le d$.

Recall that if $\dot{\mathbf{F}} \in \mathcal{F}_2(\mathbb{R}^d)$ the Newton potential of $\dot{\mathbf{F}}$ is an element of $Y^{m,2}(\mathbb{R}^d)$, and as such is defined up to adding polynomials of order $m - \frac{d}{2}$. We must choose a normalization so that the Newton potential is well-defined for any x in $\mathbb{R}^d$. Choose distinct points $h_1, h_2, ..., h_q \in \mathbb{R}^d$ in $\overline{B(0,1)}$ (so $|h_i| \le 1$ for all $1 \le i \le q$) where q is the number of multiindicies γ so that $|\gamma| \le m - \frac{d}{2}$. If the points $\{h_i\}_{i=1}^q$ are chosen appropriately (see [14] for a survey on polynomial interpolation in several variables) then for any numbers a_i there is a unique polynomial $P(x) = \sum_{|\gamma| \le m - \frac{d}{2}} p_\gamma x^\gamma$ of degree at most $m - \frac{d}{2}$ such that $P(h_i) = a_i$ for $1 \le i \le p$. Also there is some constant $H < \infty$ depending only on h_i such that $|p_\gamma| \le H \sup_i |a_i|$.

Now choose some point $z_0 \in \mathbb{R}^d$ and $r > 0$ and fix a normalization

$$\vec{\Pi}^L = \vec{\Pi}^L_{z_0,r} \text{ with the condition that } \vec{\Pi}^L_{z_0,r}\dot{\mathbf{F}}(z_0 + rh_i) = 0 \text{ for all } 1 \le i \le q. \tag{44}$$

Our goal is to now use duality to construct the fundamental solution as the kernel of some operator on $\mathcal{F}_2(\mathbb{R}^d)$. With this normalization, we now have that $\vec{\Pi}^L_{z_o,r}\dot{\mathbf{F}}(x)$ is well-defined for any $x \in \mathbb{R}^d$. Let $\vec{S}_x\dot{\mathbf{F}} = \vec{\Pi}^L_{z_0,r}\dot{\mathbf{F}}(x)$ so that $\vec{S}_x\dot{\mathbf{F}}$ is a well-defined linear operator on $Y^{m,2}(\mathbb{R}^d)$. We must now establish boundedness of $\vec{S}_x\dot{\mathbf{F}}$.

Lemma 6.1. *Let $r > 0$ and $z_0 \in \mathbb{R}^d$. Let $\vec{u} \in Y^{m,2}(\mathbb{R}^d)$ and normalize $\vec{u}$ such that*

$\vec{u}(z_0 + rh_i) = 0$ *for all* $1 \le i \le q$*. Then if* $x \in \mathbb{R}^d$ *and* $R = r + |x - z_0|$*, we have the bound*

$$|\vec{u}(x)| \le C\left(\frac{R}{r}\right)^{\omega_2 - 1} R^{m-\frac{d}{2}} \|\vec{u}\|_{Y^{m,2}(B(z_0,2R))}.$$

Proof. Recall that since $2m > d$ and $\nabla^m \vec{u} \in L^2(\mathbb{R}^d)$ (this comes from the fact that $\vec{u} \in Y^{m,2}(\mathbb{R}^d)$) we may apply Morrey's inequality (lemma 3.4) so we have that $\vec{u}$ is continuous and we also have the bound

$$|\vec{u}(x)| \le C\left(\sum_{i=0}^{m} R^{2i} \fint_{B(z_0,2R)} |\nabla^i \vec{u}|^2\right)^{1/2}.$$

Let $P(x)$ be a polynomial of order at most $\omega_2 - 1 \le m - \frac{d}{2}$ so that

$$\fint_{B(z_0,2R)} \partial^\gamma P(x)dx = \fint_{B(z_0,2R)} \partial^\gamma \vec{u}(x)dx$$

for all $|\gamma| \le m - \frac{d}{2}$. Then we have the following bound

$$|\vec{u}(x)| \le C\Bigg(\sum_{i=0}^{\omega_2-1} R^{2i} \fint_{B(z_0,2R)} |\nabla^i \vec{u} - \nabla^i P|^2 + \sum_{i=0}^{\omega_2-1} R^{2i} \fint_{B(z_0,2R)} |\nabla^i P|^2 + \sum_{i=\omega_2}^{m} R^{2i} \fint_{B(z_0,2R)} |\nabla^i \vec{u}|^2\Bigg)^{1/2}.$$

Since the degree of P is at most $m - \frac{d}{2}$, we may apply the Poincaré inequality in the first sum so that

$$R^{2i} \fint_{B(z_0,2R)} |\nabla^i \vec{u} - \nabla^i P|^2 \le R^{2\omega_2} \fint_{B(z_0,2R)} |\nabla^{\omega_2} \vec{u}|^2.$$

We can then improve this using Hölder's inequality with

$$R^{\omega_2}\left(\fint_{B(z_0,2R)} |\nabla^{\omega_2} \vec{u}|^2\right)^{1/2} \le C_d R^{\omega_2 - \frac{d}{2\omega_2}}\left(\int_{B(z_0,2R)} |\nabla^{\omega_2} \vec{u}|^{2\omega_2}\right)^{1/2\omega_2}$$
$$\le C_d R^{\omega_2 - \frac{d}{2\omega_2}} \|\vec{u}\|_{Y^{m,2}(B(z_0,2R))} = CR^{m-\frac{d}{2}} \|\vec{u}\|_{Y^{m,2}(B(z_0,2R))}$$

Where we have used that $\frac{1}{2_{\omega_2}} = \frac{1}{2} - \frac{m-\omega_2}{d}$ so then $\omega_2 - \frac{d}{2_{\omega_2}} = m - \frac{d}{2}$. Applying this bound we have then that

$$(45)\quad |\vec{u}(x)| \leq CR^{m-\frac{d}{2}} \|\vec{u}\|_{Y^{m,2}(B(z_0,2R))} + \left(C \sum_{i=0}^{\omega_2 - 1} R^{2i} \fint_{B(z_0,2R)} |\nabla^i P|^2 + \sum_{i=\omega_2}^{m} R^{2i} \fint_{B(z_0,2R)} |\nabla^i \vec{u}|^2 \right)^{1/2}.$$

Next we can bound the last sum in (45). We have by Hölder's inequality, and the definition of the norm on $Y^{m,2}(B(z_0, 2R))$,

$$(46)\quad \sum_{i=\omega_2}^{m} R^i \left(\fint_{B(z_0,2R)} |\nabla^i \vec{u}|^2 \right)^{1/2} \leq \sum_{i=\omega_2}^{m} R^i \left(\fint_{B(z_0,2R)} |\nabla^i \vec{u}|^{2_i} \right)^{1/2_i}$$
$$\leq \sum_{i=\omega_2}^{m} R^i R^{\frac{-d}{2_i}} \left(\int_{B(z_0,2R)} |\nabla^i \vec{u}|^{2_i} \right)^{1/2_i} \leq \sum_{i=\omega_2}^{m} R^i R^{\frac{-d}{2_i}} \|\vec{u}\|_{Y^{m,2}(B(z_0,2R))}$$
$$\leq CR^{m-\frac{d}{2}} \|\vec{u}\|_{Y^{m,2}(B(z_0,2R))}.$$

Where we have used that $\frac{1}{2_i} = \frac{1}{2} - \frac{m-i}{d}$ so $R^{i-\frac{d}{2_i}} = R^{i-\frac{d}{2}+m-i} = R^{m-\frac{d}{2}}$. We can now improve the estimate in (45) to obtain the following bound on $|\vec{u}(x)|$.

$$(47)\quad |\vec{u}(x)| \leq CR^{m-\frac{d}{2}} \|\vec{u}\|_{Y^{m,2}(B(z_0,2R))} + \left(C \sum_{i=0}^{\omega_2 - 1} R^{2i} \fint_{B(z_0,2R)} |\nabla^i P|^2 \right)^{1/2}.$$

We now bound the sum in (47) in terms of $\vec{u}$. By Morrey's inequality, and the bound provided by the Poincaré inequality, if $1 \leq i \leq q$ then

$$|P(z_0 + rh_i)| = |P(z_0 + rh_i) - \vec{u}(z_0 + rh_i)| \leq CR^{m-\frac{d}{2}} \|\nabla^{\omega_2} \vec{u}\|_{L^{2\omega_2}(B(z_0,2R))}$$
$$\leq CR^{m-\frac{d}{2}} \|\vec{u}\|_{Y^{m,2}(B(z_0,2R))}.$$

Let $P(x) = Q((x - z_0)/r)$ so that $Q(h_i) = P(z_0 + rh_i)$ and $Q(x) = \sum_{|\gamma| \leq \omega_2 - 1} q_\gamma x^\gamma$ for some q_γ

where $|q_\gamma| \le C R^{m-\frac{d}{2}} \|\vec{u}\|_{Y^{m,2}(B(z_0,2R))}$. Now we have the following for P

$$\partial^\delta P(x) = \sum_{\gamma \ge \delta} r^{-|\gamma|} q_\gamma \frac{\gamma!}{(\gamma-\delta)!} (x - z_0)^{\gamma-\delta}.$$

If $x \in B(z_0, 2R)$ then

$$|\nabla^i P| \le C(R/r)^{\omega_2 - 1} R^{-i} \sup_\gamma |q_\gamma| \le C(R/r)^{\omega_2-1} R^{m-i-\frac{d}{2}} \|\vec{u}\|_{Y^{m,2}(B(z_0,2R))}. \tag{48}$$

Combining (47) and (48) we have our result. □

We can now apply a duality argument to the operator $\vec{S}_x \dot{\mathbf{F}}$, but before we do so, we will need one more technical lemma to identify vector fields that arise as m^{th} order gradients. The following lemma is from [4, Lemma 41] which generalizes the classical result that irrotational vector fields may be written as gradients.

Lemma 6.2. *Let $(f_\alpha)_{|\alpha|=m}$ be a set of functions in $L^1_{loc}(\Omega)$, where Ω is a simply connected domain. Suppose that whenever $\alpha + \vec{e}_a = \beta + \vec{e}_b$, we have that $\langle \partial_b \varphi, f_\beta \rangle_\Omega = \langle \partial_a \varphi, f_\alpha \rangle_\Omega$ (in the sense of $L^2(\Omega)$) for all smooth and compactly supported functions φ on Ω. Then there is some function $f \in \dot{W}^{m,1}_{loc}(\Omega)$ such that $f_\alpha = \partial^\alpha f$ for all α.*

We can now apply lemma 6.1 to $\vec{S}_x \dot{\mathbf{F}}$ to obtain the following theorem. In theorem 6.4 we will extend theorem 6.3 to operators of order $2m \le d$.

Theorem 6.3. *Let L be an operator of order $2m > d$ whose coefficients satisfy the ellipticity conditions (9) and (10). Then for each $z_0 \in \mathbb{R}^d$ and $r > 0$ there exist functions $E^L_{j,k,z_0,r}(x,y)$ with the following properties. For every $x \in \mathbb{R}^d$, and $\dot{\mathbf{F}} \in \mathcal{F}_2(\mathbb{R}^d)$, we have that $(\partial^\beta_y E^L_{j,k,z_0,r}(x,\cdot))_{\omega_2 \le |\beta| \le m} \in (\mathcal{F}_2(\mathbb{R}^d))^*$ and for all $1 \le j \le N$*

$$(\Pi^L_{z_0,r} \dot{\mathbf{F}})_j(x) = \sum_{k=1}^N \sum_{\omega_2 \le |\beta| \le m} \int_{\mathbb{R}^d} \partial^\beta_y E^L_{j,k,z_0,r}(x,y) F_{k,\beta}(y)\,dy. \tag{49}$$

When $\omega_2 \leq |\gamma| < m$, and $\dot{\mathbf{F}} \in \mathcal{F}_2(\mathbb{R}^d)$ has compact support, and for each i with $\omega_2 \leq i \leq m$ satisfies $F_i \in L^{q_i}_{loc}(\mathbb{R}^d)$ for some $q_i > d/((m-i)+(m-|\gamma|))$, $q_i \geq 1$, we have for a.e. $x \in \mathbb{R}^d$

$$\partial_x^\gamma \vec{\Pi}^L_j \dot{\mathbf{F}}(x) = \sum_{k=1}^N \sum_{\omega_2 \leq |\beta| \leq m} \int_{\mathbb{R}^d} \partial_x^\gamma \partial_y^\beta E^L_{j,k}(x,y) F_{k,\beta}(y)dy. \tag{50}$$

When $|\gamma| = m$ equation (50) is still true for $\dot{\mathbf{F}} \in \mathcal{F}_2(\mathbb{R}^d)$ when additionally $x \notin \operatorname{supp}(\dot{\mathbf{F}})$ and $\operatorname{supp}(\dot{\mathbf{F}}) \subsetneq \mathbb{R}^d$.

Also for any fixed j and k with $1 \leq k \leq N$ and $1 \leq j \leq N$, and multiindices ζ and ξ with $\omega_2 \leq |\zeta| \leq m$ and $\omega_2 \leq |\xi| \leq m$ we have the following symmetry property

$$\partial_x^\zeta \partial_y^\xi E^L_{j,k,z_0,r}(x,y) = \overline{\partial_y^\xi \partial_x^\zeta E^{L^*}_{k,j,z_0,r}(y,x)}. \tag{51}$$

For these indices we also have that $\partial_x^\zeta \partial_y^\xi E^L_{j,k,z_0,r}(x,y)$ doesn't depend on z_0 and r.

We also have that for fixed $x_0, y_0 \in \mathbb{R}^d$ so that $|x_0 - z_0| = |y_0 - z_0| = |x_0 - y_0| = 8r$, and for $0 \leq i \leq m$ and $0 \leq l \leq m$ we have the bound

$$\int_{B(x_0,r)} \int_{B(y_0,r)} |\nabla_x^i \nabla_y^l E^L_{j,k,z_0,r}(x,y)|^2 dydx \leq C r^{4m-2i-2l}. \tag{52}$$

In the case that both $i \geq \omega_2$ and $l \geq \omega_2$, then for any $\rho > 0$ and points $x_0, y_0 \in \mathbb{R}^d$ with $|x_0 - y_0| = 8\rho$ we have the bound

$$\int_{B(x_0,\rho)} \int_{B(y_0,\rho)} |\nabla_x^i \nabla_y^l E^L_{j,k,z_0,r}(x,y)|^2 dydx \leq C \rho^{4m-2i-2l}. \tag{53}$$

Proof. By lemma 6.1 with $\vec{u}(x) = \vec{S}_x \dot{\mathbf{F}}$, and along with the fact that $\vec{\Pi}^L$ is bounded on $\mathcal{F}_2(\mathbb{R}^d)$ we have for $R = |x - z_0| + r$,

$$|\vec{S}_x \dot{\mathbf{F}}| \leq C R^{m-\frac{d}{2}} \left(\frac{R}{r}\right)^{\omega_2 - 1} \|\dot{\mathbf{F}}\|_{\mathcal{F}_2(\mathbb{R}^d)}.$$

Now there is some array $\mathbf{E^L} \in (\mathcal{F}_2(\mathbb{R}^d))^*$ such that

$$(\vec{\Pi}^L_{z_0,r}\dot{\mathbf{F}})_j(x) = (\vec{S}_x\dot{\mathbf{F}})_j = \sum_{k=1}^{N}\sum_{\omega_2\le|\beta|\le m}\int_{\mathbb{R}^d} E^L_{j,k,\beta,z_0,r}(x,y)F_{k,\beta}(y)dy.$$

Also we have the following bound

$$\|E^L_{j,k,\beta,z_0,r}(x,\cdot)\|_{(\mathcal{F}_2(\mathbb{R}^d))^*} \le CR^{m-\frac{d}{2}}\left(\frac{R}{r}\right)^{\omega_2-1}. \tag{54}$$

Note that by the duality of L^p spaces for p in the range $1 < p < \infty$ we have that $\dot{\mathbf{G}} \in (\mathcal{F}_2(\mathbb{R}^d))^*$ if and only if $G_\beta \in L^{2_\beta}(\mathbb{R}^d)$ for all β with $\omega_2 \le |\beta| \le m$. Now we wish to show that there is some function $E^L_{j,k,z_0,r}(x,y)$ such that $E^L_{j,k,\beta,z_0,r}(x,y) = \partial_y^\beta E^L_{j,k,z_0,r}(x,y)$. Suppose that $|\alpha| = |\beta| = \omega_2$, $\alpha + \vec{e}_a = \beta + \vec{e}_b$, and $\varphi, \Psi \in C_c^\infty(\mathbb{R}^d)$ are smooth functions with compact support. Fix j and k. Then by the definition of the Newton potential $\vec{\Pi}^L_{z_0,r}$ we have

$$\sum_{i,l=1}^{N}\sum_{\substack{|\zeta|\le m\\|\xi|\le m}}\int_{\mathbb{R}^d}\partial^\zeta\Psi_l A^{l,i}_{\zeta,\xi}\partial^\xi\Pi^L_{i,z_0,r}(\partial_b\varphi\dot{e}_{\beta,k}) = \int_{\mathbb{R}^d}\partial^\beta\Psi_k\partial_b\varphi$$

and again by the definition of the Newton Potential

$$\sum_{i,l=1}^{N}\sum_{\substack{|\zeta|\le m\\|\xi|\le m}}\int_{\mathbb{R}^d}\partial^\zeta\Psi_l A^{l,i}_{\zeta,\xi}\partial^\xi\Pi^L_{i,z_0,r}(\partial_a\varphi\dot{e}_{\alpha,k}) = \int_{\mathbb{R}^d}\partial^\alpha\Psi_k\partial_a\varphi.$$

Then using integration by parts we have that

$$\int_{\mathbb{R}^d}\partial^\beta\Psi_k\partial_b\varphi = \int_{\mathbb{R}^d}-\partial^\beta\partial_b\Psi_k\varphi \text{ and } \int_{\mathbb{R}^d}\partial^\alpha\Psi_k\partial_a\varphi = \int_{\mathbb{R}^d}-\partial^\alpha\partial_a\Psi_k\varphi.$$

Since $\alpha + \vec{e}_a = \beta + \vec{e}_b$, both sets of the above equations are equal. So with the bilinear form B as in (32), we have shown $B(\Psi, \vec{\Pi}^L_{z_0,r}(\partial_a\varphi\dot{e}_{\alpha,k})) = B(\Psi, \vec{\Pi}^L_{z_0,r}(\partial_b\varphi\dot{e}_{\beta,k}))$. By the coercivity of B, we have $\Pi^L_{z_0,r}(\partial_a\varphi\dot{e}_{\alpha,k})_j = \Pi^L_{z_0,r}(\partial_b\varphi\dot{e}_{\beta,k})_j$ as $Y^{m,2}$ functions, and by our normalization

(44) they are equal pointwise. Combined with the definitions of $E^L_{j,k,\beta,z_0,r}$, and $E^L_{j,k,\alpha,z_0,r}$ we have

$$\int_{\mathbb{R}^d} E^L_{j,k,\beta,z_0,r}\partial_b\varphi dy = \Pi^L_{z_0,r}(\partial_b\varphi\dot{e}_{\beta,k})(x)_j = \Pi^L_{z_0,r}(\partial_a\varphi\dot{e}_{\alpha,k})(x)_j = \int_{\mathbb{R}^d} E^L_{j,k,\alpha,z_0,r}(x,y)\partial_a\varphi dy$$

for all a, b, α, β with $|\alpha| = |\beta|$ and $\alpha + \vec{e}_a = \beta + \vec{e}_b$. Thus we can apply lemma 6.2 so there is some $E^L_{j,k,z_0,r}(x,y)$ such that $E^L_{j,k,\alpha,z_0,r}(x,y) = \partial^\alpha_y E^L_{j,k,z_0,r}(x,y)$ for all $|\alpha| = \omega_2$.

We now consider the case of $\omega_2 < |\alpha| \leq m$. Let $\gamma < \alpha$ so that $|\gamma| = \omega_2$. It suffices to show that $E^L_{j,k,\alpha,z_0,r}(x,y) = \partial^{\alpha-\gamma}E_{j,k,\gamma,z_0,r(x,y)}$ in the weak sense. That is we need to show

$$\int_{\mathbb{R}^d} \varphi E^L_{j,k,\alpha,z_0,r}(x,y)dy = \int_{\mathbb{R}^d} (-1)^{|\alpha-\gamma|}\partial^{\alpha-\gamma}\varphi E^L_{j,k,\gamma,z_0,r}(x,y)dy \tag{55}$$

for all $\varphi \in C^\infty_0(\mathbb{R}^d)$. We begin with the left-hand side of (55). By the definition of $E^L_{j,k,\alpha,z_0,r}$, we have

$$\int_{\mathbb{R}^d} \varphi E^L_{j,k,\alpha,z_0,r}(x,y)dy = \Pi^L_{z_0,r}(\varphi\dot{e}_{\alpha,k})(x)_j.$$

By the definition of the Newton potential we have that if $\Psi \in Y^{m,2}(\mathbb{R}^d)$ then

$$\sum_{i,l=1}^N \sum_{\substack{|\zeta|\leq m\\|\xi|\leq m}} \int_{\mathbb{R}^d} \partial^\zeta\Psi_l A^{l,i}_{\zeta,\xi}\partial^\xi\Pi^L_i(\varphi\dot{e}_{\alpha,k}) = \int_{\mathbb{R}^d} \partial^\alpha\Psi_k\varphi.$$

For the right-hand side of (55) we have

$$\int_{\mathbb{R}^d} (-1)^{|\alpha-\gamma|}\partial^{\alpha-\gamma}\varphi E^L_{j,k,\gamma,z_0,r}(x,y)dy = \Pi^L_{z_0,r}((-1)^{|\alpha-\gamma|}\partial^{\alpha-\gamma}\varphi\dot{e}_{\gamma,k})(x)_j.$$

Now by the definition of the Newton potential for $\vec{\Pi}^L_{z_0,r}((-1)^{|\alpha-\gamma|}\partial^{\alpha-\gamma}\varphi\dot{e}_{\gamma,k})$ and integration by parts,

$$\sum_{i,l=1}^{N}\sum_{\substack{|\zeta|\le m\\|\xi|\le m}}\int_{\mathbb{R}^d}\partial^\zeta\Psi_i A^{i,l}_{\zeta,\xi}\partial^\xi\Pi^L_l((-1)^{|\alpha-\gamma|}\partial^{\alpha-\gamma}\varphi\dot{e}_{\gamma,k}) = \int_{\mathbb{R}^d}(-1)^{|\alpha-\gamma|}\partial^\gamma\Psi_k\partial^{\alpha-\gamma}\varphi$$

$$= \int_{\mathbb{R}^d}\partial^\alpha\Psi_k\varphi.$$

Thus $\vec{\Pi}^L_{z_0,r}(\varphi\dot{e}_{\alpha,k}) = \vec{\Pi}^L_{z_0,r}((-1)^{|\alpha-\gamma|}\partial^{\alpha-\gamma}\varphi\dot{e}_{\gamma,k})$ as $Y^{m,2}(\mathbb{R}^d)$ functions. By our normalization of $\vec{\Pi}^L_{z_0,r}\dot{\mathbf{F}}(z_0+rh_i)=0$ they are equal pointwise as well. This establishes equation (55). Thus there exists a function $E^L_{j,k,z_0,r}$ such that $E^L_{j,k,\beta,z_0,r} = \partial^\beta_y E^L_{j,k,z_0,r}$ for any β with $\omega_2 \le |\beta| \le m$, and we have established equation (49).

Notice that $E^L_{j,k,z_0,r}(x,y)$ is defined up to adding polynomials in y of degree $m-\frac{d}{2}$. Similar to earlier we can fix a normalization so that $E^L_{z_0,r}(x,z_0+rh_i)=0$ for all $x\in\mathbb{R}^d$ and $1\le i\le q$. Also since $\vec{\Pi}^L_{z_0,r}\dot{\mathbf{F}}(z_0+rh_i)=0$, by the discussion preceding lemma 6.1 we see that $\partial^\beta_y E^L_{z_0,r}(z_0+rh_i,y)=0$ for all $m-\frac{d}{2}<|\beta|\le m$ and $0\le i\le q$. Thus we have that $E^L_{z_0,r}(z_0+rh_i,y)$ is a polynomial in y of order ω_2-1 and since it is zero for $y=z_0+rh_i$ we also have $E^L_{z_0,r}(z_0+rh_i,y)=0$ for all $y\in\mathbb{R}^d$ and $0\le i\le q$. We can also apply lemma 6.1 and the equation (54) to obtain the bound

$$|E^L_{z_0,r}(x,y)| \le Cr^{2m-d}\left(1+\frac{|x-z_0|}{r}\right)^{m-\frac{d}{2}+\omega_2-1}\left(1+\frac{|y-z_0|}{r}\right)^{m-\frac{d}{2}+\omega_2-1}. \tag{56}$$

Now we will begin our proof of equations (51) and (52). Let η be a nonnegative, smooth cutoff function so that $\int_{B(0,1)}\eta=1$, and $\eta\equiv 0$ outside of $B(0,1)$. We will use a standard mollifier argument (see [12, Appendix C.5]) to prove the symmetry property of $E^L_{z_0,r}(x,y)$. Let $\epsilon>0$ be a real number, and denote $\eta_\epsilon=\epsilon^{-d}\eta(\frac{x}{\epsilon})$. Let $*_x$ denote convolution in the x variable, and $*_y$ convolution in the y variable. Let $\epsilon,\delta>0$ be real numbers, ζ,ξ be multiindices, so that $\omega_2 \leq |\zeta| \leq m$, and $\omega_2 \leq |\xi| \leq m$, and lastly let

$$E^L_{j,k,\zeta,\xi,\delta,\epsilon}(x,y) := \partial_x^\zeta \partial_y^\xi (\eta_\delta *_x E^L_{j,k,z_0,r} *_y \eta_\epsilon)(x,y). \tag{57}$$

Notice that for each ζ and ξ, we have $E^L_{j,k,\zeta,\xi,\delta,\epsilon}(x,y) = ((\partial^\zeta \eta_\delta) *_x E^L_{j,k,\xi,z_0,r} *_y \eta_\epsilon)(x,y)$. Choose some $\dot{\mathbf{F}} \in \mathcal{F}_2(\mathbb{R}^d)$. Notice that since $\dot{\mathbf{F}} \in \mathcal{F}_2(\mathbb{R}^d)$ we have $F_{k,\xi} \in (L^{2\xi}(\mathbb{R}^d))^*$. Multiply $E^L_{j,k,\zeta,\xi,\delta,\epsilon}(x,y)$ by $F_{k,\xi}(y)$ and integrate over $\mathbb{R}^d$. Notice then by equations (49) and (57), when $|\xi| \geq \omega_2$ we have

$$\begin{aligned} \int_{\mathbb{R}^d} E^L_{j,k,\zeta,\xi,\delta,\epsilon}(x,y) F_{k,\xi}(y) dy &= (\partial^\zeta \eta_\delta) *_x \int_{\mathbb{R}^d} \partial_y^\xi E^L_{j,k,z_0,r}(x,y)(\eta_\epsilon *_y F_{k,\xi})(y) dy \\ &= \eta_\delta *_x \partial^\zeta \Pi_j^L (\eta_\epsilon *_y F \dot{e}_{k,\xi}). \end{aligned} \tag{58}$$

By the definition of $\vec{\Pi}^L$, we have that $F \to \eta_\delta *_x \partial^\zeta \Pi_j^L (\eta_\epsilon *_y F\dot{e}_{k,\xi})(x)$ is bounded $\mathcal{F}_2(\mathbb{R}^d) \to \mathbb{C}$, in this case with that bound also depending on δ. Thus $E^L_{j,k,\zeta,\xi,\delta,\epsilon}$ is the kernel of this operator and if $|\zeta| \geq \omega_2$ and $|\xi| \geq \omega_2$, it does not depend on the previous normalization in z_0 and r. We can also apply lemma 4.2 to obtain

$$E^L_{j,k,\zeta,\xi,\delta,\epsilon}(x,y) = \overline{E^{L^*}_{k,j,\xi,\zeta,\epsilon,\delta}(y,x)}. \tag{59}$$

Next we will prove a similar symmetry property for $E^L_{j,k,z_0,r}(x,y)$ and use that to prove equation (52). First, let ζ have $|\zeta| = \omega_2$ so that from equation (59) and (57) we have

$$\partial_x^\zeta E^L_{j,k,0,\xi,\delta,\epsilon}(x,y) = \partial_x^\zeta \overline{E^{L^*}_{k,j,\xi,0,\epsilon,\delta}(y,x)}.$$

Again this is by the construction of $E^L_{j,k,z_0,r}$ from equation (49), and so $E^L_{j,k,0,\xi,\delta,\epsilon}(x,y)$ and $\overline{E^{L^*}_{k,j,\xi,0,\epsilon,\delta}(y,x)}$ differ by a polynomial in x of degree $\omega_2 - 1$. Also by the definition of z_0, r

and h_i, for $1 \le i \le q$,

$$E^L_{j,k,0,\xi,\delta,\epsilon}(z_0 + rh_i, y) = 0 = \overline{E^{L^*}_{k,j,\xi,0,\epsilon,\delta}(y, z_0 + rh_i)}$$

which implies

$$E^L_{j,k,0,\xi,\delta,\epsilon}(x, y) = \overline{E^{L^*}_{k,j,\xi,0,\epsilon,\delta}(y, x)}.$$

Now let ξ be a multiindex with $|\xi| = \omega_2$, and then by a similar argument to above we obtain

$$E^L_{j,k,0,0,\delta,\epsilon}(x, y) = \overline{E^{L^*}_{k,j,0,0,\epsilon,\delta}(y, x)}.$$

Recall that $\mathbf{E}^L$ is continuous, and so as we take limits as $\delta \to 0$ and $\epsilon \to 0$ and obtain

$$E^L_{j,k,z_0,r}(x, y) = \overline{E^{L^*}_{k,j,z_0,r}(y, x)}. \tag{60}$$

We can now use equation (60), theorem 3.13, and equation (56) to bound derivatives of $E^L_{j,k,z_0,r}$ in $L^2(\mathbb{R}^d)$ to prove equation (52). Let $\varphi \in Y^{m,2}(\mathbb{R}^d)$, and recall by equation (37) that $\varphi_j(x) = \Pi^L_j(\sum_{i,k=1}^N \sum_{\substack{|\beta| \le m \\ |\alpha| \le m}} A^{i,k}_{\alpha,\beta} \partial^\beta \varphi_k \dot{e}_{i,\alpha})(x)$. Combined with our equation (49) we have

$$\varphi_j(x) = \sum_{k=1}^N \sum_{\omega_2 \le |\beta| \le m} \int_{\mathbb{R}^d} \partial^\beta_y E^L_{j,k,z_0,r}(x, y) \sum_{l=1}^N \sum_{|\zeta| \le m} A^{k,l}_{\beta,\zeta} \partial^\zeta \varphi_l(y) dy \tag{61}$$

as $Y^{m,2}(\mathbb{R}^d)$ functions. Furthermore if φ is normalized by the condition $\vec{\varphi}(z_0 + rh_i) = 0$ for $1 \le i \le q$ then (61) is true pointwise for all x. So then for the function $\vec{u}(y)$ defined by $u_k(y) = E^L_{j,k,z_0,r}(x, y)$ we have that $L^*\vec{u} = 0$ in $\mathbb{R}^d \setminus \{x\} \setminus \overline{B(z_0, r)}$. Now choose some points x_0 and y_0 so that $|x_0 - y_0| = |y_0 - z_0| = |x_0 - z_0| = 8r$. By theorem 3.13 and the definition of $E^L_{j,k,\zeta,\xi,\delta,\epsilon}(x,y)$ we have

$$
\begin{aligned}
\int_{B(y_0,r)} |E^L_{j,k,\zeta,\xi,\delta,\epsilon}(x,y)|^2 dy &= \int_{B(y_0,r)} |\eta_\delta *_y (\partial_y^\xi(\partial^\zeta \eta_\epsilon *_x E^L_{j,k,z_0,r})(x,y))|^2 dy \\
&\leq \int_{B(y_0,2r)} |(\partial_y^\xi(\partial^\zeta \eta_\epsilon *_x E^L_{j,k,z_0,r})(x,y))|^2 dy \\
&\leq Cr^{-2|\xi|} \int_{B(y_0,4r)} |(\partial^\zeta \eta_\epsilon *_x E^L_{j,k,z_0,r})(x,y)|^2 dy
\end{aligned} \tag{62}
$$

Now once we bound the last part of (62) in terms of x, we will arrive at equation (52). Now use theorem 3.13 again in x, along with (60) and (56) to get

$$
\begin{aligned}
\int_{B(x_0,r)} \int_{B(y_0,r)} |E^L_{j,k,\zeta,\xi,\delta,\epsilon}(x,y)|^2 dydx &\leq Cr^{-2|\xi|} \int_{B(y_0,4r)} \int_{B(x_0,r)} |(\partial^\zeta \eta_\epsilon *_x E^L_{j,k,z_0,r})(x,y)|^2 dxdy \\
&= Cr^{-2|\xi|} \int_{B(y_0,4r)} \int_{B(x_0,r)} |(\eta_\epsilon *_x \partial_x^\zeta E^{L^*}_{k,j,z_0,r})(y,x)|^2 dxdy \\
&\leq Cr^{-2|\xi|} \int_{B(y_0,4r)} \int_{B(x_0,2r)} |\partial_x^\zeta E^{L^*}_{k,j,z_0,r}(y,x)|^2 dxdy \\
&\leq Cr^{-2|\xi|-2|\zeta|} \int_{B(y_0,4r)} \int_{B(x_0,4r)} |E^{L^*}_{k,j,z_0,r}(y,x)|^2 dxdy \\
&\leq Cr^{4m-2|\xi|-2|\zeta|}
\end{aligned} \tag{63}
$$

Thus we have that $E^L_{\zeta,\xi,\delta,\epsilon}(x,y) \in L^2(B(x_0,r)) \times L^2(B(y_0,r))$ independent of δ and ϵ. Taking the limits as $\delta \to 0$ and $\epsilon \to 0$ we obtain a weakly convergent subsequence. By the definition of weak limit and weak derivative, the limit must be $\partial_x^\zeta \partial_y^\xi \mathbf{E}^L_{z_0,r}(x,y)$ which provides us with equations (52) and (51). From equation (52) we see that we have a bound on $E^L_{j,k,z_0,r}$ when we take derivatives in both x and y.

Recall that we have already established boundedness of $\vec{\Pi}^L$ for $2-\delta < p < 2+\delta$ in lemma 4.4. Now in terms of equation (50), we must show that the integrand on the right-hand side of equation (50) is in $L^1(B(x_0,R) \times \mathbb{R}^d)$. Notice by splitting up the integral we have the following bound

(64)

$$\sum_{k=1}^{N}\sum_{\omega_2\le|\beta|\le m}\int_{B(x_0,R)}\int_{\mathbb{R}^d}\left|\partial_x^\gamma\partial_y^\beta E^L_{j,k}(x,y)F_{k,\beta}(y)\right|dydx$$
$$\le C\sum_{i=\omega_2}^{m}\int_{B(x_0,R)}\int_{B(x_0,2R)}|\nabla_x^{|\gamma|}\nabla_y^i E^L(x,y)|\cdot|F_i|dydx$$
$$+C\sum_{n=1}^{\infty}\sum_{i=\omega_2}^{m}\int_{B(x_0,R)}\int_{B(x_0,2^{n+1}R)\setminus B(x_0,2^nR)}|\nabla_x^{|\gamma|}\nabla_y^i E^L(x,y)|\cdot|F_i|dydx$$

We will write equation (64) as $I+II$. First for I, we will bound

$$\int_{B(x_0,R)}\int_{B(x_0,2R)}|\nabla_x^{m-s}\nabla_y^{m-t}E^L(x,y)|^{q'}dydx$$

where $s<d/2$, $t<d/2$, $q'<d/(d-(s+t))$ and $q'\le 2$. We cover $B(x_0,R)$ by cubes Q_k and use a dyadic decomposition. We then let Q_0 be a cube of sidelength $2R$ and $B(x_0,R)\subset Q_0$ so that

$$\int_{B(x_0,R)}\int_{B(x_0,2R)}|\nabla_x^{m-s}\nabla_y^{m-t}E^L(x,y)|^{q'}dydx\le C\int_{Q_0}\int_{2Q_0}|\nabla_x^{m-s}\nabla_y^{m-t}E^L(x,y)|^{q'}dydx \tag{65}$$

Now let G_a be a grid of dyadic subcubes of Q_0 of sidelength $2^{1-a}R$. If $y\in B(x_0,R)$ let $Q_a(y)$ be the cube that has $y\in Q_a(y)\in G_a$. If $Q\in G_{a+1}$, let $P(Q)$ be the dyadic parent of the cube Q, that is the unique cube with $Q\subset P(Q)\in G_a$. Then we have the following

(66)

$$\int_{Q_0}\int_{2Q_0}|\nabla_x^{m-s}\nabla_y^{m-t}E^L(x,y)|^{q'}dydx$$
$$=\int_{Q_0}\sum_{a=0}^{\infty}\int_{2Q_a(y)\setminus 2Q_{a+1}(y)}|\nabla_x^{m-s}\nabla_y^{m-t}E^L(x,y)|^{q'}dydx$$
$$=\sum_{a=0}^{\infty}\sum_{Q\in G_{a+1}}\int_Q\int_{2P(Q)\setminus 2Q}|\nabla_x^{m-s}\nabla_y^{m-t}E^L(x,y)|^{q'}dydx.$$

By Hölder's inequality since $q' \leq 2$, and by equation (53)

$$\int_Q \int_{2P(Q)\setminus 2Q} |\nabla_x^{m-s}\nabla_y^{m-t} E^L(x,y)|^{q'} dydx \leq C\ell(Q)^{d(2-q')} \left(\int_Q \int_{2P(Q)\setminus 2Q} |\nabla_x^{m-s}\nabla_y^{m-t} E^L(x,y)|^2 dydx \right)^{q'/2} \leq C\ell(Q)^{d(2-q')+sq'+tq'}. \tag{67}$$

Notice that there are 2^{ad} cubes in each G_a, then by equations (66) and (67) we have

$$\int_{Q_0} \int_{2Q_0} |\nabla_x^{m-s}\nabla_y^{m-t} E^L(x,y)|^{q'} dydx \leq CR^{d(2-q')+q'(s+t)} \sum_{a=0}^{\infty} 2^{-ad+a(d-t-s)q'}. \tag{68}$$

By how we have chosen q', we get $-d+(d-t-s)q' < 0$ and the above series converges and gives us the bound

$$\int_{B(x_0,R)} \int_{B(x_0,2R)} |\nabla_x^{m-s}\nabla_y^{m-t} E^L(x,y)|^{q'} dydx \leq CR^{2d-q'(d-s-t)}. \tag{69}$$

For each i so $\omega_2 \leq i \leq m$, let $q_i > 0$ be such that $\frac{1}{q_i} + \frac{1}{(q_i)'} = 1$, then if $q_i > d/((m-i)+(m-|\gamma|))$, then $(q_i)' < d/(d-(m-i)-(m-|\gamma|))$. From equations (64) and (69), we have by Hölder's inequality since $F_i \in L^{q_i}(B(x_0,2R))$

$$I \leq C \sum_{i=\omega_2}^{m} R^{d/q_i} \|F_i\|_{L^{q_i}(B(x_0,2R))} \left(R^{2d-(q_i)'(d-(m-|\gamma|)-(m-i))} \right)^{1/(q_i)'} = C \sum_{i=\omega_2}^{m} \|F_i\|_{L^{q_i}(B(x_0,2R))} R^{d-d/q_i+2m-|\gamma|-i} < \infty. \tag{70}$$

Now for the second part of equation (64) we have to bound II. Define $A_n(x_0,R)$ to be the annulus $A_n(x_0,R) = B(x_0,2^{n+1}R) \setminus B(x_0,2^nR)$ and notice that since $\dot{\mathbf{F}}$ has compact support, there exists some $M \geq 1$ such that

$$II = C\sum_{n=1}^{M}\sum_{i=\omega_2}^{m}\int_{B(x_0,R)}\int_{A_n(x_0,R)}|\nabla_x^{|\gamma|}\nabla_y^i E^L(x,y)|\cdot|F_i|dydx. \tag{71}$$

Fix i and n, we have for equation (71) by Hölder's inequality

$$\begin{aligned}\int_{B(x_0,R)}\int_{A_n(x_0,R)}&|\nabla_x^{|\gamma|}\nabla_y^i E^L(x,y)|\cdot|F_i|dydx\\ &\leq \int_{B(x_0,R)}\left(\int_{A_n(x_0,R)}|F_i|^{(2_i)'}\right)^{1/(2_i)'}\left(\int_{A_n(x_0,R)}|\nabla_x^{|\gamma|}\nabla_y^i E^L(x,y)|^{2_i}dy\right)^{1/2_i}dx\\ &= \|F_i\|_{L^{(2_i)'}(A_n(x_0,R))}\int_{B(x_0,R)}\left(\int_{A_n(x_0,R)}|\nabla_x^{|\gamma|}\nabla_y^i E^L(x,y)|^{2_i}dy\right)^{1/2_i}dx.\end{aligned} \tag{72}$$

Notice by the discussion preceeding equation (62) we have that the function $\vec{v}(y) := \partial_x^\gamma E^L(x,y)$ is a solution to $L^*\vec{u} = 0$ in $A_n(x_0,R)$. Then by the Gagliardo-Nirenberg-Sobolev inequality, and the Caccioppoli inequality (theorem 3.13) we have

$$\begin{aligned}\left(\int_{A_n(x_0,R)}|\nabla^i\vec{v}|^{2_i}\right)^{\frac{1}{2_i}} &\leq C\sum_{k=0}^{m-i}R^{m-k-i}\left(\int_{A_n(B(x_0,R)}|\nabla^{k+i}\vec{v}|^2\right)^{1/2}\\ &\leq \frac{1}{R^m}\left(\int_{\widetilde{A_n}(B(x_0,R))}|\vec{v}|^2\right)^{1/2}\end{aligned} \tag{73}$$

where $\widetilde{A_n}(B(x_0,R))$ is the enlarged annulus $B(x_0,2^{n+2}R)\setminus B(x_0,(3/4)2^nR)$. Again recall that $\vec{v}$ is defined up to adding polynomials of degree at most $\omega_2 - 1$, so we may use the Poincaré inequality to give us the bound

$$\left(\int_{A_n(x_0,R)}|\nabla^i\vec{v}|^{2_i}\right)^{\frac{1}{2_i}} \leq R^{-m+\omega_2}\left(\int_{\widetilde{A_n}(B(x_0,R))}|\nabla^{\omega_2}\vec{v}|^2\right)^{1/2}. \tag{74}$$

Now by combining equations (72) and (74) we have

(75)
$$\int_{B(x_0,R)}\int_{A_n(x_0,R)}|\nabla_x^{|\gamma|}\nabla_y^i E^L(x,y)|\cdot|F_i|dydx$$
$$\leq R^{-m+\omega_2}\|F_i\|_{L^{(2_i)'}(A_n(x_0,R))}\int_{B(x_0,R)}\left(\int_{\widetilde{A_n}(B(x_0,R))}|\nabla_y^{\omega_2}\nabla_x^{|\gamma|}E^L(x,y)|^2dy\right)^{1/2}dx.$$

Next use Hölder's inequality, and the bound from equation (53) to get

(76)
$$\begin{aligned}
&\int_{B(x_0,R)}\int_{A_n(x_0,R)}|\nabla_x^{|\gamma|}\nabla_y^i E^L(x,y)|\cdot|F_i|dydx\\
&\leq CR^{-m+\omega_2+d}\|F_i\|_{L^{(2_i)'}(A_n(x_0,R))}\left(\int_{B(x_0,R)}\int_{\widetilde{A_n}(B(x_0,R))}|\nabla_y^{\omega_2}\nabla_x^{|\gamma|}E^L(x,y)|^2dydx\right)^{1/2}\\
&\leq CR^{-m+\omega_2+d}\|F_i\|_{L^{(2_i)'}(A_n(x_0,R))}\left(\int_{\widetilde{A_n}(B(x_0,R))}\int_{B(x_0,2^{n-1}R)}|\nabla_x^{|\gamma|}\nabla_y^{\omega_2}E^L(x,y)|^2dxdy\right)^{1/2}\\
&\leq CR^{-m+\omega_2+d}\|F_i\|_{L^{(2_i)'}(A_n(x_0,R))}(2^nR)^{2m-\omega_2-|\gamma|}\\
&= C2^{n(2m-\omega_2-|\gamma|)}R^{m+d-|\gamma|}\|F_i\|_{L^{(2_i)'}(A_n(x_0,R))}.
\end{aligned}$$

Now by combining equations (71) and (76) we have

$$II\leq C\sum_{n=1}^{M}\sum_{i=\omega_2}^{m}2^{n(2m-\omega_2-|\gamma|)}R^{m+d-|\gamma|}\|F_i\|_{L^{(2_i)'}(A_n(x_0,R))}<\infty. \tag{77}$$

Then by combining equations (70) and (77) we have that the integrand from the right-hand side of equation (50) is in $L^1(B(x_0,R)\times\mathbb{R}^d)$. Thus by Fubini's theorem, we are able to bring the derivative into equation (49) and get equation (50). This completes the proof. □

6.2 The Fundamental Solution For Lower-Order Operators

In this section we will extend the results of theorem 6.3 to operators of order $2m\leq d$. We will proceed as in [4], [5] and [6]. For an operator L of order $2m\leq d$ we will use the poly-Laplacian Δ^M where M is chosen to be large enough so that $\Delta^M L \Delta^M$ is an operator of order high enough so that we are able to apply results from section 6.1.

Theorem 6.4. *Let L be an operator of order $2m \leq d$ that satisfies the ellipticity conditions (8) and (10). Let $r > 0$, and q and s be integers so that $q < d/2$ and $s < d/2$. Then there exists some array of functions $E^L_{j,k}(x, y)$ that satisfies the following properties. First we have that $E^L_{j,k}(x, y)$ satisfies the following symmetry property where $|\zeta| = m - s$ and $|\xi| = m - q$*

$$\partial_x^\zeta \partial_y^\xi E^L_{j,k}(x, y) = \overline{\partial_x^\zeta \partial_y^\xi E^{L^*}_{k,j}(y, x)}. \tag{78}$$

Furthermore we have the $L^2_{loc} \times L^2_{loc}$ bound for all $x_0, y_0 \in \mathbb{R}^d$ with $|x_0 - y_0| = 8r$

$$\int_{B(x_0,r)} \int_{B(y_0,r)} |\nabla_x^{m-s} \nabla_y^{m-q} E^L_{j,k}(x, y)|^2 \leq C r^{2q+2s}. \tag{79}$$

Next we have that when $\omega_2 \leq |\gamma| < m$ and $\dot{\mathbf{F}} \in \mathcal{F}_2(\mathbb{R}^d)$ has compact support, and for each i with $\omega_2 \leq i \leq m$ satisfies $F_i \in L^{q_i}_{loc}(\mathbb{R}^d)$ for some $q_i > d/((m - i) + (m - |\gamma|))$, $q_i \geq 1$, we have for a.e. $x \in \mathbb{R}^d$

$$\partial_x^\gamma \vec{\Pi}^L_j \dot{\mathbf{F}}(x) = \sum_{k=1}^{N} \sum_{\omega_2 \leq |\beta| \leq m} \int_{\mathbb{R}^d} \partial_x^\gamma \partial_y^\beta E^L_{j,k}(x, y) F_{k,\beta}(y) dy. \tag{80}$$

When $|\gamma| = m$ equation (80) is still true for $\dot{\mathbf{F}} \in \mathcal{F}_2(\mathbb{R}^d)$ when additionally $x \notin \operatorname{supp}(\dot{\mathbf{F}})$ and $\operatorname{supp}(\dot{\mathbf{F}}) \subsetneq \mathbb{R}^d$.

Proof. We will begin by constructing an operator $\widetilde{L}$ of order high enough so that we can apply theorem 6.3. Let M be large enough so that $\widetilde{m} = m + 2M > \frac{d}{2}$, then define the operator $\widetilde{L}$ by $\widetilde{L} = \Delta^M L \Delta^M$ in the weak sense, where $\Delta^M = \sum_{|\alpha|=M} \frac{M!}{\alpha!} \partial^{2\alpha}$ is the poly-Laplacian. That is for $\vec{u} \in Y^{\widetilde{m},2}(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$ we have

$$\langle \vec{\varphi}, \widetilde{L} \vec{u} \rangle_\Omega = \langle \Delta^M \vec{\varphi}, L \Delta^M \vec{u} \rangle_\Omega \tag{81}$$

in the sense of the $L^2(\Omega)$ inner product. Notice that $\widetilde{L}$ is a bounded elliptic operator of order $2\widetilde{m} > d$ and so by theorem 6.3 we have that a fundamental solution $E^{\widetilde{L}}_{j,k}(x,y)$ exists, and equations (49), (51), and (52) hold in terms of $\widetilde{L}$ and $\widetilde{m}$. Notice that since $\vec{\varphi}$ is smooth, we may write $\Delta^M \vec{\varphi} = \sum_{|\zeta|=2M} a_\zeta \partial^\zeta \vec{\varphi}$ where a_ζ are all constants depending on m and ζ. Now write

$$E^L_{j,k}(x,y) = \sum_{|\zeta|=2M} \sum_{|\xi|=2M} a_\zeta a_\xi \partial^\zeta_x \partial^\xi_y E^{\widetilde{L}}_{j,k}(x,y) \tag{82}$$

and we will show that $E^L_{j,k}(x,y)$ is the fundamental solution for the operator L. Notice that $E^{\widetilde{L}}_{j,k}(x,y)$ must satisfy the symmetry property (51) based on our choice of M. Observe that by equation (51), for $|\zeta| = m - s$ and $|\xi| = m - q$,

$$\begin{aligned}
\partial^\zeta_x \partial^\xi_y E^L_{j,k}(x,y) &= \partial^\zeta_x \partial^\xi_y \sum_{|\alpha|=2M} \sum_{|\beta|=2M} a_\alpha a_\beta \partial^\alpha_x \partial^\beta_y E^{\widetilde{L}}_{j,k}(x,y) \\
&= \partial^\zeta_x \partial^\xi_y \sum_{|\beta|=2M} \sum_{|\alpha|=2M} \overline{a_\beta a_\alpha \partial^\beta_y \partial^\alpha_x E^{\widetilde{L}_*}_{k,j}(y,x)} \\
&= \overline{\partial^\zeta_x \partial^\xi_y E^{L^*}_{k,j}(y,x)}
\end{aligned}$$

and so (78) holds. Next notice that by equation (52)

$$\begin{aligned}
\int_{B(x_0,r)} \int_{B(y_0,r)} |\nabla^{m-s}_x \nabla^{m-q}_y E^L_{j,k}(x,y)|^2 dy dx &\leq C \int_{B(x_0,r)} \int_{B(y_0,r)} |\nabla^{m-s+2M}_x \nabla^{m-q+2M}_y E^{\widetilde{L}}_{j,k}(x,y)|^2 dy dx \\
&\leq C r^{4\widetilde{m}-2(m-s)-2(m-q)-8M} = C r^{2q+2s}
\end{aligned}$$

which is equation (79). Let $\dot{\mathbf{F}} \in \mathcal{F}_p(\mathbb{R}^d)$ where p is in the range $2 - \delta < 2 < 2 + \delta$ as in lemma 4.4. Notice that $F_\beta = 0$ if $|\beta| < \omega_2$. We now wish to relate the Newton potential $\vec{\Pi}^L \dot{\mathbf{F}}$, to the Newton potential $\vec{\Pi}^{\widetilde{L}} \dot{\widetilde{\mathbf{F}}}$, where $\dot{\widetilde{\mathbf{F}}}$ is constructed so that we may extend theorem 6.3. For each $\widetilde{\beta}$ where $|\widetilde{\beta}| \leq \widetilde{m}$, define

$$\widetilde{F}_{k,\widetilde{\beta}} = \sum_{\substack{|\xi|=2M \\ \xi<\widetilde{\beta}}} a_\xi F_{k,\widetilde{\beta}-\xi} \tag{83}$$

Notice that $\widetilde{F}_{k,\widetilde{\beta}} = 0$ for $|\widetilde{\beta}| < \omega_2 + 2M$. Similarly to equation (58), multiply $\partial_x^\gamma \partial_y^\beta E_{j,k}^L(x,y)$ by $F_{k,\beta}$ and sum in β for $0 \leq |\beta| \leq m$ and define the operator t_j by

$$t_j(x) := \sum_{k=1}^{N} \sum_{|\beta|\leq m} \int_{\mathbb{R}^d} \partial_x^\gamma \partial_y^\beta E_{j,k}^L(x,y) F_{k,\beta}(y) dy. \tag{84}$$

Then by equations (82), (83) and (84)

$$\begin{aligned} t_j(x) &= \sum_{k=1}^{N} \sum_{|\beta|\leq m} \sum_{|\zeta|=2M} \sum_{|\xi|=2M} \int_{\mathbb{R}^d} \partial_x^{\gamma+\zeta} \partial_y^{\beta+\xi} a_\zeta a_\xi E_{j,k}^{\widetilde{L}}(x,y) F_{k,\beta} dy \\ &= \sum_{|\zeta|=2M} a_\zeta \sum_{k=1}^{N} \sum_{|\widetilde{\beta}|\leq m} \int_{\mathbb{R}^d} \partial_x^{\gamma+\zeta} \partial_y^{\widetilde{\beta}} E_{j,k}^{\widetilde{L}}(x,y) \widetilde{F}_{k,\widetilde{\beta}} dy \end{aligned} \tag{85}$$

By equation (49) we have

$$t_j(x) = \sum_{|\zeta|=2M} a_\zeta \partial_x^{\gamma+\zeta} \vec{\Pi}_j^{\widetilde{L}} \dot{\widetilde{\mathbf{F}}}(x) = \partial_x^\gamma \Delta^M \vec{\Pi}_j^{\widetilde{L}} \dot{\widetilde{\mathbf{F}}}(x). \tag{86}$$

Now in order to finish proving equation (80), we need to show that $\Delta^M \vec{\mathbf{\Pi}}^{\widetilde{L}} \dot{\widetilde{\mathbf{F}}} = \vec{\mathbf{\Pi}}^L \dot{\mathbf{F}}$. In order to distinguish between the varying level of gradients, we will need to utilize the notation from equation (38), again where the inner product is in terms of the L^2 inner product. Choose $\vec{\varphi} \in Y^{m,2}(\mathbb{R}^d)$, then there is some function $\vec{\widetilde{\varphi}} \in Y^{\widetilde{m},2}(\mathbb{R}^d)$ so that $\vec{\varphi} = \Delta^M \vec{\widetilde{\varphi}}$. Then we have

$$\langle \widetilde{\nabla}^m \vec{\varphi}, A \widetilde{\nabla}^m (\Delta^M \vec{\mathbf{\Pi}}^{\widetilde{L}} \dot{\widetilde{\mathbf{F}}}) \rangle_{\mathbb{R}^d} = \langle \vec{\varphi}, L(\Delta^M \vec{\mathbf{\Pi}}^{\widetilde{L}} \dot{\widetilde{\mathbf{F}}}) \rangle_{\mathbb{R}^d} = \langle \Delta^M \vec{\widetilde{\varphi}}, L(\Delta^M \vec{\mathbf{\Pi}}^{\widetilde{L}} \dot{\widetilde{\mathbf{F}}}) \rangle_{\mathbb{R}^d} \tag{87}$$

Recall that $\widetilde{L} = \Delta^M L \Delta^M$, so then using the definition of the Newton potential we have

$$\langle \Delta^M \widetilde{\vec{\varphi}}, L(\Delta^M \vec{\mathbf{\Pi}}^{\widetilde{L}} \dot{\widetilde{\mathbf{F}}}) \rangle_{\mathbb{R}^d} = \langle \widetilde{\vec{\varphi}}, \widetilde{L}(\vec{\mathbf{\Pi}}^{\widetilde{L}} \dot{\widetilde{\mathbf{F}}}) \rangle_{\mathbb{R}^d} = \langle \widetilde{\nabla}^{\widetilde{m}} \widetilde{\vec{\varphi}}, \dot{\widetilde{\mathbf{F}}} \rangle_{\mathbb{R}^d}.$$

Recall from equation (83) we have that

$$\langle \widetilde{\nabla}^{\widetilde{m}} \widetilde{\vec{\varphi}}, \dot{\widetilde{\mathbf{F}}} \rangle_{\mathbb{R}^d} = \sum_{k=1}^{N} \sum_{|\widetilde{\beta}| \le \widetilde{m}} \langle \partial^{\widetilde{\beta}} \widetilde{\varphi}_k, \widetilde{F}_{k,\widetilde{\beta}} \rangle_{\mathbb{R}^d} = \sum_{k=1}^{N} \sum_{|\widetilde{\beta}| \le \widetilde{m}} \sum_{\substack{|\xi|=2M \\ \xi < \widetilde{\beta}}} \langle \partial^{\widetilde{\beta}} \widetilde{\varphi}_k, a_\xi F_{k,\widetilde{\beta}-\xi} \rangle_{\mathbb{R}^d}.$$

Then by the definition of $\widetilde{\beta}$ and rearranging the sums, and definition of $\widetilde{\vec{\varphi}}$ we have

(88)
$$\begin{aligned}
\langle \widetilde{\nabla}^{\widetilde{m}} \widetilde{\vec{\varphi}}, \dot{\widetilde{\mathbf{F}}} \rangle_{\mathbb{R}^d} &= \sum_{k=1}^{N} \sum_{|\beta| \le m} \sum_{|\xi|=2M} \langle a_\xi \partial^{\beta+\xi} \widetilde{\varphi}_k, F_{k,\beta} \rangle_{\mathbb{R}^d} \\
&= \sum_{k=1}^{N} \sum_{|\beta| \le m} \langle \partial^{\beta} \Delta^M \widetilde{\varphi}_k, F_{k,\beta} \rangle_{\mathbb{R}^d} \\
&= \sum_{k=1}^{N} \sum_{|\beta| \le m} \langle \partial^{\beta} \varphi_k, F_{k,\beta} \rangle_{\mathbb{R}^d} \\
&= \langle \widetilde{\nabla}^m \vec{\varphi}, \dot{\mathbf{F}} \rangle_{\mathbb{R}^d}.
\end{aligned}$$

Now combining equation (87) and equation (88) we are left with

$$\langle \widetilde{\nabla}^m \vec{\varphi}, A \widetilde{\nabla}^m (\Delta^M \vec{\mathbf{\Pi}}^{\widetilde{L}} \dot{\widetilde{\mathbf{F}}}) \rangle_{\mathbb{R}^d} = \langle \widetilde{\nabla}^m \vec{\varphi}, \dot{\mathbf{F}} \rangle_{\mathbb{R}^d}$$

which by the uniqueness of the Newton potential gives us that $\Delta^M \vec{\mathbf{\Pi}}^{\widetilde{L}} \dot{\widetilde{\mathbf{F}}} = \vec{\mathbf{\Pi}}^L \dot{\mathbf{F}}$ and we have established equation (80). $\square$

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