GRAVITATIONAL RADIATION: MAXWELL-HEAVISIDE FORMULATION

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Abstract: To fully describe gravitational energy flux by using an analog to the Maxwell-Heaviside equations for electrodynamics, the Liénard–Wiechert potentials and fields are derived for gravitation along with radiation patterns and corresponding Larmor formulas for total radiated power. Due to attraction of like gravitational charges (masses) as opposed to repulsion of like electrical charges, the mass-density and current-density terms pick up a negative sign along with a factor of the universal gravitational constant, G. This results in a sign change of the Poynting vector, indicating energy is gained by the field as opposed to energy being lost by the field, such is the case for electromagnetic radiation. The gravitational and cogravitational fields, analogous to the electric and magnetic fields, respectively, behave as described by Heaviside and Lorentz. Like an electric charge, a gravitic charge (mass) in uniform motion is found to produce a spherical field which contracts along the axis of motion as its velocity approaches the limiting speed of propagation. The speed of gravitational propagation is assumed to be equivalent to that of light, though this may not necessarily be the case. For an accelerated mass, the resulting gravitational radiation mirrors the dipole pattern produced by an accelerated electric charge, similarly oriented about the axis of acceleration yet contracting in the reverse direction at relativistic speeds. These results seek to further inquire on the nature of gravitational fields.

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Chapter I

INTRODUCTION

For over two thousand years, gravity has weighed heavy on the minds of ancient philosophers, mathematicians, and modern physicists. In 350 BC, Aristotle described how heavy objects tend toward their center. Natural philosophers continued exploring this concept and developed the theory of impetus (momentum), distinguished between impetus and force, and provided a description of acceleration. Translated from the Greek *baros* (*βάρος*) to the Latin *gravis* or *gravitas*, the understanding of "inner heaviness" led Galileo Galilei to propose that objects fall with uniform acceleration towards the earth at a distance proportional to the square of the elapsed time. About 90 years after Galileo's hypothesis, in 1687, Isaac Newton formulated his universal law of gravitation which states that the force of gravity is directly proportional to the product of the masses and inversely proportional to the square of the distance between them. In the beginning of the twentieth century, Albert Einstein published his works on special and general relativity which state that mass and energy are equivalent and that gravity and acceleration are indistinguishable, respectively. Yet, even a century later, the origin of gravity is still uncertain.

Comparatively, the theory of electrodynamics and its evolution to quantum electrodynamics has occurred in a much shorter time. Although the rigorous mathematical formulation of these concepts began around roughly the same time, the advancements from electrodynamics far exceed that for gravitation. The theory of electromagnetism grew from the work of William Gilbert and Robert Boyle in the 16th and 17th centuries and that of Henry Cavendish, Charles-Augustin de Coulomb, André-Marie Ampère, Hans Ørsted, Michael Faraday, and Gustav Kirchoff in the 18th and 19th centuries, culminating in James

Clerk Maxwell's equations of electrodynamics. The existence of the complete electromagnetic spectrum was predicted by Maxwell's equations, which describe light as a wave.

Following the wave description of light and the discovery of black-body radiation, Ludwig Boltzmann proposed discretized energy levels for physical systems using statistical mechanics. At the turn of the twentieth century, after roughly two decades of debate, Max Planck was motivated by Boltzmann's proposition and hypothesized the Planck relation and predicted the quantum mechanical nature of light. The De Broglie relation, proposed by Louis De Broglie in 1923, could describe particles as waves (and vice-versa) and led Erwin Schrödinger to develop his wave equation just three years later. Shortly after, Paul Dirac codified special relativity into quantum mechanics in the Dirac equation and created quantum field theory.

Building on the work of Dirac, Richard Feynman assumed there are no fields but rather "virtual photons" and formulated his iconic Feynman diagrams to explain the interaction of particles which could not be explained by the Schrödinger equation. Quantum field theory replaces the description of fields with *quanta*. Quantum electrodynamics was developed from the quantization of charge using Maxwell's equations and makes remarkably accurate predictions. Despite the achievements of Einstein, the theory of general relativity fails to quantize gravity or lead to a quantum theory of gravity.

Gravity as an Analogy to Electromagnetism

The necessity of localizing gravity and gravitational energy was emphasized by Oliver Heaviside in part one of his 1893 publication [1], in which he develops an analogy to Maxwell's equations for electromagnetism. Maxwell's equations were published in 1864 and were not written in their modern notation until 1884 when Heaviside reduced the set of twenty equations to four using his vector notation. As such, the equations of Maxwell and Heaviside shall be referred to as the Maxwell-Heaviside equations. Both Maxwell and Heaviside considered a gravitational analog to electromagnetism in their respective formalism. However, Maxwell found the field energy to be negative, which discouraged him from further investigation so he ultimately abandoned the analogy altogether [2]. Although Heaviside also obtained a negative field energy, he continued exploring the analogy and went on to propose gravitational waves.

Heaviside insisted that the nature of gravity cannot be fully explained without considering the localization of energy flux, which requires a complete description of both fields. He described how masses generate gravitational fields like charges generate electric fields and implies the existence of a gravitational analog to the magnetic field. From this analogy of the Maxwell-Heaviside equations, it may be possible to establish a classical description of gravitational radiation by following the development of radiation from an accelerated charge and evaluating the radiation patterns produced by an accelerated mass.

Chapter II

REVIEW OF LITERATURE

Beginning with Oliver Heaviside's proposition of a gravitational analog to electromagnetism, gravitational energy flux is discussed along with the resulting negative field energy and the suggestion of gravitational waves is investigated. Heaviside's gravitational analogy is further explored in Hills' book. Classical electromagnetic field theory is referenced and reproduced for gravitation from the textbook by Heald and Marion. References to the works of Oleg Jefimenko are acknowledged throughout.

Heaviside

In *A Gravitational and Electromagnetic Analogy*, Oliver Heaviside considers an analogy between gravitation and electromagnetism. Heaviside proposes gravitational waves by analogy and emphasizes the importance of localizing the flux of energy. He remarks that the analogy "serves, in fact, to further illustrate the mystery" of gravity [1]. These articles were republished by Oleg Jefimenko who updated the mathematics to modern notation [3].

A Gravitational and Electromagnetic Analogy, Part I

Heaviside begins by asserting that it is necessary to localize gravitational energy to accurately describe its flux, as had been done for electric and magnetic energies. He reasons that a generalized "moving force" can be described as the product of a mass-density and the forces intensity, where the intensity is related to a potential-gradient. Heaviside argues that, if the potential is defined by the distribution of matter, the speed of "gravitative influence" would be "infinitely great" [1]. He then describes the work done by the gravitational force as the flux of energy such that the "exhaustion of potential energy is, of course, the gain of kinetic energy, if no other forces have been in action" [1].

By making a comparison with electromagnetism, Heaviside characterizes the gravitational force as analogous to the electric force and affirms that a complete description of energy flux requires both forces. Thus, he identifies a cogravitational force analogous to the magnetic force, though he does not use the term "cogravitational" and instead simply denotes it as $\vec{\boldsymbol{h}}$. Heaviside extends the analogy by stating that a "simultaneous convergence of gravitational force" is like the "convective current of electrification" [1].

He returns to the discussion on work and flux with the magnetic analogy for gravitation, noting that the electromagnetic Poynting vector (also found by Heaviside) changes sign and becomes negative. He illustrates this phenomenon from the perspective of a moving charge, describing first the electric field as a spherical shell of electric charge and the magnetic field as rotating positively about the axis of motion such that the flux of energy emanates from the center of the shell outward in the radial direction. For a moving mass, the cogravitational field behaves just as the magnetic field, rotating positively about the axis of motion. However, since the direction of the gravitational field is radially reversed, the flux of energy is likewise reversed. This can be easily understood by recognizing that, in electromagnetism, like charges repel and, in gravitation, masses attract.

Newtonian gravity implies instantaneous action at a distance and Heaviside remarks how it "is as incredible now as it was in Newton's time that gravitative influence can be exerted without a medium" [1]. Turning to an alternative scenario, Heaviside considers a medium in which gravitational waves propagate at a finite speed and determines the vector curl equations for both fields. He reasons that the "circuital" $\vec{\boldsymbol{h}}$ equation suggests the curl equation for $\vec{\boldsymbol{g}}$ if a finite speed is assumed, noting this is implied by the analogy to electromagnetic waves. He comments that this description of gravity will appear as "instantaneous action, although expressed in terms of wave-propagation" [1].

Heaviside notes how the field lines for gravitation will contract laterally as the velocity of a mass increases to a substantial percentage of the finite propagation speed. He also extends his discussion to consider a medium which has a "translational motion of its own." Reflecting on his proposed analogy and commenting that finite propagation speed supersedes instantaneous action, Heaviside admits the nature of gravitation is yet to be explained.

Before concluding, Heaviside once more revisits the topic of work and its relation to potential and kinetic energies. He argues the best definition of potential energy is "energy that is not known to be kinetic," asserting the concept of potential energy is merely mathematics and "explains nothing" regarding physics [1]. Reiterating his initial claim, he insists that localization of energy is fundamental to describing the flux of energy. Heaviside's analogy between gravity and electromagnetism may provide a more complete characterization of gravitation or may "suggest something better."

A Gravitational and Electromagnetic Analogy, Part II

In the second part of this analogy, Heaviside responds to comments regarding "gravitational aberration," stating he has no evidence for such effects. To explain, he compares the intensity of the force of the Sun on the Earth when the Sun is considered at rest to that when the Sun is considered in motion. He emphasizes the phenomenon by allowing that the Sun move one in one-thousandth the speed of light, which suggests force intensity aberrations on the order of one in one-million. He considers a few other cases and concludes that, should these perturbations exist and be observable, the speed of gravity would be less than the speed of light. If they are too small to observe, this would suggest the speed of gravity is equal to the speed of light. Should they not exist at all, then the speed of gravity would be greater than that of light. Heaviside adds that gravitational waves may be described using other methods and that the "ether" problem calls for a solution.

Hills

The book authored by Brian Hills further examines Heaviside's analogy between gravitation and electromagnetism. *Gravito-electromagnetism & Mass Induction* is prefaced with a discussion of the gravitational analogy, referred to as gravitoelectromagnetism (GEM), and how the foundation for developing a quantum theory of gravity has been laid by electromagnetism. Despite the simplicity of the gravitational analogy, it is often considered as just an approximation of general relativity in the weak-field limit. Hills contends that GEM makes predictions that general relativity could not, such as describing mass induction and providing a theory of inertia, and attests that both theories could be an approximation of a yet more general theory of gravity. To this end, the author seeks to explore an alternative theory of gravity.

Referencing the recent work of José Heras, Hills provides a set of generalized Maxwell-Heaviside equations, purportedly valid for electromagnetism, gravitoelectromagnetism, as well as hydrodynamics. These equations were identified in a 2007 publication by Heras and demonstrate that the Maxwell-Heaviside equations can be derived from a continuity equation. Using this generalized set, the classical and neo-classical developments of electromagnetism are followed analogously to describe gravitational radiation and quantum gravity, respectively. The first chapter in the book is referenced for the generalized Maxwell-Heaviside equations and the generalized equations developed from them, such as the wave equations and the Jefimenko equations. Hills provides the following set of generalized Maxwell-Heaviside equations:

$$\nabla \cdot \vec{\boldsymbol{E}} = \alpha\rho\,, \tag{II.1}$$

$$\nabla \cdot \vec{\boldsymbol{B}} = 0\,, \tag{II.2}$$

$$\nabla \times \vec{\boldsymbol{E}} = -\gamma\frac{\partial\vec{\boldsymbol{B}}}{\partial t}\,, \tag{II.3}$$

$$\nabla \times \vec{\boldsymbol{B}} = \beta\vec{\boldsymbol{J}} + \left(\frac{\beta}{\alpha}\right)\frac{\partial\vec{\boldsymbol{E}}}{\partial t}\,, \tag{II.4}$$

where $\vec{\boldsymbol{E}}$ is a "conductive" field, $\vec{\boldsymbol{B}}$ is a "convective" field, ρ is the source density, and $\vec{\boldsymbol{J}} = \rho\vec{\boldsymbol{v}}$ is the source-current density. The coefficients (α, β, γ) for electromagnetism and gravitoelectromagnetism (GEM) are included in Table II.1 for SI units.

Table II.1. Coefficients for the Generalized Maxwell-Heaviside Equations in SI Units

Symbol	Electromagnetism, e	GEM, g (by analogy)
α	$1/\varepsilon_0$	$-4\pi G$
β	μ_0	$-4\pi G/c_g^2$
γ	1	1
$\vec{\boldsymbol{J}}$	$\rho_e \vec{\boldsymbol{v}}$	$\rho_g \vec{\boldsymbol{v}}$
$\vec{\boldsymbol{E}}$	$\vec{\boldsymbol{E}}_e$	$\vec{\boldsymbol{E}}_g$
$\vec{\boldsymbol{B}}$	$\vec{\boldsymbol{B}}_e$	$\vec{\boldsymbol{B}}_g$
ϕ	ϕ_e	ϕ_g
$\vec{\boldsymbol{A}}$	$\vec{\boldsymbol{A}}_e$	$\vec{\boldsymbol{A}}_g$
ε	ε_0	$-1/4\pi G$
μ	μ_0	$-4\pi G/c_g^2$
$\mu\varepsilon$	$1/c_e^2$	$1/c_g^2$

Note: ε is the permittivity, μ is the permeability, c_e is the speed of light, and c_g is the speed of gravity.

Heald and Marion

The *Classical Electromagnetic Radiation* textbook by Mark Heald and Jerry Marion became a crucial reference for developing gravitational radiation. Chapters 4, 5, and 8 are essentially replicated for gravitation, the latter most thoroughly, and Appendix A includes the equations for electromagnetism relevant for reference and comparison in CGS units. The Maxwell-Heaviside equations are presented in Chapter 4 along with the derivations of the scalar and vector potentials and their resulting wave equations in CGS units. This chapter also discusses energy flux and defines the Poynting vector and Poynting theorem.

Chapter 5 investigates the wave equations for electromagnetism in non-conducting and conducting media and analyzes complex wave solutions. Waves propagate information at a finite speed such that the measurement of information about some source at a distant point is observed at a delayed time, known as the *retarded time*. Due to this delayed transmission, the scalar and vector potentials should be evaluated at the retarded time, as suggested in 1858 by George Reimann [4], and are thus referred to as the *retarded potentials*. Likewise, the fields become the *retarded fields* evaluated at the retarded time.

In Chapter 8, the retarded potentials and fields are used to develop the Liénard–Wiechert potentials and fields. Using these fields in the Poynting vector, the magnitude and direction of energy flux can be determined along with the spatial dependence of the velocity and acceleration field components, where the acceleration components of the fields are shown to be responsible for radiation. The last four sections of Chapter 8 investigate the behavior of these fields and the radiation patterns produced by an accelerated charge in three special cases. The Larmor formulas for total power radiated are also determined.

For a charged particle moving at constant velocity, the electric field produced is spherical and contracts along the axis of motion at relatively high speeds. For an accelerated charge at low speeds, the charge produces a dipole radiation pattern about the axis of acceleration. As the acceleration continues and the speed increases, a similar contraction occurs along the axis of motion. For a charge confined to a circular orbit, the dipole pattern is oriented about the acceleration axis as before. Here, however, the direction of motion is perpendicular to the acceleration. Thus, contraction results in an asymmetrical radiation pattern.

Chapter III

METHODOLOGY

Following the standard procedure to describe electromagnetic radiation in classical electromagnetic field theory [4], a gravitational analog of the Maxwell-Heaviside equations is used to develop a description of gravitational radiation. Traditionally, a scalar and vector potential are defined from the Maxwell-Heaviside equations. These definitions are coupled in a statement of the electric field while the magnetic field is defined as the curl of the vector potential. After identifying the Lorentz gauge condition, wave equations can be determined for the potentials and the fields. Using the theory of advanced and retarded wave propagation, integral solutions to the wave equations can be found. Then, the Liénard–Wiechert potentials and fields can be derived to analytically evaluate the fields. Using the Poynting vector, the Larmor formula can be found for total power radiated by accelerated charges.

For gravitation, the Maxwell-Heaviside equations can be derived from four sets of assumptions or obtained using the generalized equations and corresponding set of coefficients in Table II.1 for GEM by analogy [5], which can be converted from SI units to CGS units by a simple redefinition. Analogous scalar and vector potentials will be defined to obtain equations for the gravitational fields. These equations should resemble those for electromagnetism. Wave equations will be determined for both the potentials and fields and likewise compared. Solutions to the wave equations for gravity will be found using retarded waves and the procedure of Liénard and Wiechert will be applied to derive analytic solutions for graphically evaluating the fields produced by masses. Analogously, the Lorentz force, the Poynting vector, and the Larmor formulas will be determined. Gravitational radiation produced by an accelerating mass will be plotted for a few special cases.

The analogy between gravitation and electromagnetism is strikingly similar, as is implied by the generalized Maxwell-Heaviside equations [5]. Another convincing comparison is that of Coulomb's force law with Newton's law of gravitation.

Coulomb's force law is given by:

$$\vec{\boldsymbol{F}} = k\frac{q_1 q_2}{R^2}\widehat{\boldsymbol{R}}, \tag{III.1}$$

and Newton's force law is given by:

$$\vec{\boldsymbol{F}} = -G\frac{m_1 m_2}{R^2}\widehat{\boldsymbol{R}}. \tag{III.2}$$

Such immediate similarity potentially suggests a more fundamental relationship of these force laws. Both forces vary as the product of the "charges" and vary inversely with the square of the distance between them. These forces are also proportional to some constant, be it either k or G. Since like charges repel whereas masses attract (there is no experimental evidence for negative mass), the forces have opposite signs, which is expected. The remarkable resemblance continues in the results of this analogy.

Chapter IV

FINDINGS

The results of the gravitational analogy to electromagnetism and analogical development of gravitational radiation are presented in this chapter.

Maxwell-Heaviside Equations for Gravitation

The Maxwell-Heaviside equations for gravitation are the following:

$$\nabla \cdot \vec{\boldsymbol{g}} = -4\pi G \rho_g \,, \tag{IV.1}$$

$$\nabla \cdot \vec{\boldsymbol{h}} = 0 \,, \tag{IV.2}$$

$$\nabla \times \vec{\boldsymbol{g}} = -\frac{1}{c_g}\frac{\partial \vec{\boldsymbol{h}}}{\partial t} \,, \tag{IV.3}$$

$$\nabla \times \vec{\boldsymbol{h}} = \frac{1}{c_g}\Big(-4\pi G \vec{\boldsymbol{J}}_g + \frac{\partial \vec{\boldsymbol{g}}}{\partial t}\Big) \,, \tag{IV.4}$$

where $\vec{\boldsymbol{g}}$ is the gravitational field, $\vec{\boldsymbol{h}}$ is the cogravitational field, ρ_g is the mass-density of the source, and $\vec{\boldsymbol{J}}_g = \rho_g \vec{\boldsymbol{v}}$ is the momentum-density of the source. These equations can be derived by making the following four assumptions (shown in Appendix B):

1. Newtonian gravitation, $\nabla \cdot \vec{\boldsymbol{g}} = -4\pi G \rho$
2. Mass conservation, $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{\boldsymbol{J}} = 0$
3. Gravitational wave solutions, $\nabla^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0$
4. Gravitomagnetic field has no sources, $\nabla \cdot \vec{\boldsymbol{h}} = 0$

or by using the coefficients given in Table II.1 for the generalized Maxwell-Heaviside equations. For completeness, it is standard to relate the secondary field to the curl of a vector potential, $\vec{\boldsymbol{A}}_g$, such that

$$\vec{\boldsymbol{h}} = \nabla \times \vec{\boldsymbol{A}}_g \tag{IV.5}$$

by the fact that the divergence of a curl vanishes identically, as given by

$$\nabla \cdot \vec{\boldsymbol{h}} = \nabla \cdot (\nabla \times \vec{\boldsymbol{A}}_g) = 0\,. \tag{IV.6}$$

It is also standard to define a scalar potential, ϕ_g, such that

$$\nabla \times \nabla \phi_g = 0\,, \tag{IV.7}$$

which is true for any continuously twice-differentiable scalar field. Thus, one can show

$$\vec{\boldsymbol{g}} = -\nabla \phi_g - \frac{1}{c_g}\frac{\partial \vec{\boldsymbol{A}}_g}{\partial t}\,. \tag{IV.8}$$

Comparing with electromagnetism (Appendix A), it is immediately obvious that the mass-density and momentum-density terms in the Maxwell-Heaviside equations pick up a negative sign along with a factor of the gravitational constant, G. These equations are identical to the generalized equations in SI units given in Hills, provided that $\vec{\boldsymbol{E}}_g = \vec{\boldsymbol{g}}$ and $\vec{\boldsymbol{B}}_g = \vec{\boldsymbol{h}}$ and that the generalized coefficient, β, in Table II.1 is redefined as

$$\beta = -\frac{4\pi G}{c_g} \tag{IV.9}$$

for CGS units. This conversion is justified by Equation V.100 in Appendix B and will be necessary when comparing results to Hills.

Note: The coefficient β is not to be confused with the scaled-velocity vector $\vec{\boldsymbol{\beta}}$ (or its magnitude) that will be used in the Liénard–Wiechert potentials and fields as well as the resulting fields and radiation.

Wave Equations for Gravitation

The wave equations found for the scalar and vector potentials are shown in Appendix C to be:

$$\nabla^2\phi_g - \frac{1}{c_g^2}\frac{\partial^2\phi_g}{\partial t} = 4\pi G\rho_g\,, \tag{IV.10}$$

$$\nabla^2\vec{\boldsymbol{A}}_g - \frac{1}{c_g^2}\frac{\partial^2\vec{\boldsymbol{A}}_g}{\partial t^2} = \frac{4\pi G}{c_g}\vec{\boldsymbol{J}}_g \tag{IV.11}$$

and the wave equations for the fields are:

$$\nabla^2\vec{\boldsymbol{g}} - \frac{1}{c_g^2}\frac{\partial^2\vec{\boldsymbol{g}}}{\partial t^2} = -4\pi G\left(\nabla\rho_g + \frac{1}{c_g^2}\frac{\partial\vec{\boldsymbol{J}}_g}{\partial t}\right), \tag{IV.12}$$

$$\nabla^2\vec{\boldsymbol{h}} - \frac{1}{c_g^2}\frac{\partial^2\vec{\boldsymbol{h}}}{\partial t^2} = \frac{4\pi G}{c_g}(\nabla\times\vec{\boldsymbol{J}}_g)\,. \tag{IV.13}$$

The same comparison can be made, where the terms associated with mass and momentum density involve the constant G and a negative sign. These equations are also identical to the wave equations in Hills, given the change in the coefficient β mentioned previously.

Retarded Potentials for Gravitation

Following the procedure in classical electromagnetic theory to develop the retarded solutions to the scalar and vector wave equations and using the bracket notation to indicate evaluation at the retarded time, we obtain (in Appendix D) the retarded scalar potential:

$$\phi_g(\vec{\boldsymbol{r}},t) = -G\int_V \frac{\left[\rho_g(\vec{\boldsymbol{r}}')\right]}{R}dV' \tag{IV.14}$$

and the retarded vector potential:

$$\vec{\boldsymbol{A}}_g(\vec{\boldsymbol{r}},t) = -\frac{G}{c_g}\int_V \frac{\left[\vec{\boldsymbol{J}}_g(\vec{\boldsymbol{r}}')\right]}{R}dV'\,. \tag{IV.15}$$

Again, the only difference between these equations and their electromagnetic counterparts is the change in the sign of the terms related to mass and the added factor of G. The retarded potentials agree with those provided in Hills, following the same comparison.

Retarded Fields for Gravitation

Using the retarded wave solutions to the scalar and vector potentials, it is shown that the gravitational field evaluated at the retarded time is given by:

$$\vec{\boldsymbol{g}}(\vec{\boldsymbol{r}},t) = -G\int_V \left(\frac{[\rho_g]\hat{\boldsymbol{e}}_R}{R^2} + \frac{\left[\frac{\partial \rho_g}{\partial t}\right]\hat{\boldsymbol{e}}_R}{c_g R} - \frac{\left[\frac{\partial \vec{\boldsymbol{J}}_g}{\partial t}\right]}{c_g^2 R} \right) dV' \, . \tag{IV.16}$$

This expression is analogous to the generalized Coulomb-Faraday Law for electromagnetism. The first and second terms relate to Coulomb's law (here, Newton's Law for mass) and represent the primary source of the gravitational field, mass. The third term is analogous to Faraday's law and represents the secondary source of the gravitational field, momentum. The cogravitational field is similarly evaluated as,

$$\vec{\boldsymbol{h}}(\vec{\boldsymbol{r}},t) = \frac{-G}{c_g}\int_V \left(\frac{[\vec{\boldsymbol{J}}_g] \times \hat{\boldsymbol{e}}_R}{R^2} + \frac{\left[\frac{\partial \vec{\boldsymbol{J}}_g}{\partial t}\right] \times \hat{\boldsymbol{e}}_R}{c_g R} \right) dV' \, . \tag{IV.17}$$

This expression is analogous to the generalized Biot-Savart Law for electromagnetism. The first term represents the primary source of the cogravitational field, momentum. The second term is the secondary source, mass. The second source is not immediately obvious as the Maxwell induction term does not appear explicitly. However, the partial-time derivative terms serve as retardation "proxies" for these sources [4]. From these generalized equations, it is clear that both fields are generated by both sources. Although it is common to express one field as being induced by the other, they are simultaneously generated by both sources when expressed as retarded fields.

These generalized retarded fields are shown in Appendix D to agree with the Jefimenko equations listed in Hills, while the notation used compliments the corresponding electromagnetic Jefimenko equations given in Heald and Marion (see Appendix A).

Liénard–Wiechert Potentials and Fields for Gravitation

Using the method of Alfred-Marie Liénard and Emil Wiechert, analytic solutions to the retarded potentials and fields are derived for gravitation that vary with the scaled velocity vector $\vec{\boldsymbol{\beta}}$. As before, the results (derived in Appendix E) differ from electromagnetism only by factor of G and a change in sign.

Liénard–Wiechert Potentials for Gravitation

It is shown that the retarded scalar potential evaluated at the retarded time, indicated by the bracket notation, is given by,

$$\phi_g(\vec{\boldsymbol{r}}, t) = -\frac{Gm}{\left[R(1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}})\right]} \,. \tag{IV.18}$$

And similarly, the retarded vector potential is shown to be,

$$\vec{\boldsymbol{A}}_g(\vec{\boldsymbol{r}}, t) = -\frac{Gm[\vec{\boldsymbol{\beta}}]}{\left[R(1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}})\right]} \tag{IV.19}$$

where the scaled velocity vector is defined as

$$\vec{\boldsymbol{\beta}} = \frac{\vec{\boldsymbol{v}}}{c_g} \,. \tag{IV.20}$$

Note:

$$\vec{\boldsymbol{A}}_g(\vec{\boldsymbol{r}}, t) = [\vec{\boldsymbol{\beta}}]\phi_g(\vec{\boldsymbol{r}}, t) \,. \tag{IV.21}$$

Liénard–Wiechert Fields for Gravitation

Matching notation with Heald and Marion, the Liénard–Wiechert gravitational field is shown to be:

$$\vec{\boldsymbol{g}} = -Gm\left[\frac{(\widehat{\boldsymbol{R}} - \vec{\boldsymbol{\beta}})(1-\beta^2)}{R^2(1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}})^3} + \frac{\widehat{\boldsymbol{R}} \times \left((\widehat{\boldsymbol{R}} - \vec{\boldsymbol{\beta}}) \times \dot{\vec{\boldsymbol{\beta}}}\right)}{c_g R(1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}})^3}\right] \tag{IV.22}$$

which can be separated into velocity and acceleration components as follows:

$$\vec{\boldsymbol{g}} = \vec{\boldsymbol{g}}_v + \vec{\boldsymbol{g}}_a\,, \tag{IV.23}$$

where

$$\vec{\boldsymbol{g}}_v = -Gm\left[\frac{(\widehat{\boldsymbol{R}} - \vec{\boldsymbol{\beta}})(1-\beta^2)}{R^2\big(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^3}\right], \tag{IV.24}$$

and

$$\vec{\boldsymbol{g}}_a = -Gm\left[\frac{\widehat{\boldsymbol{R}}\times\Big((\widehat{\boldsymbol{R}} - \vec{\boldsymbol{\beta}})\times\dot{\vec{\boldsymbol{\beta}}}\Big)}{c_g R\big(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^3}\right]. \tag{IV.25}$$

The cogravitational field is shown to be

$$\vec{\boldsymbol{h}} = -Gm\left[\frac{(\vec{\boldsymbol{\beta}}\times\widehat{\boldsymbol{R}})(1-\beta^2)}{R^2\big(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^3} + \frac{\Big(\dot{\vec{\boldsymbol{\beta}}}\cdot\widehat{\boldsymbol{R}}\Big)(\vec{\boldsymbol{\beta}}\times\widehat{\boldsymbol{R}})}{c_g R\big(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^3} + \frac{\dot{\vec{\boldsymbol{\beta}}}\times\widehat{\boldsymbol{R}}}{c_g R\big(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^2}\right] \tag{IV.26}$$

which can likewise be separated, such that:

$$\vec{\boldsymbol{h}} = \vec{\boldsymbol{h}}_v + \vec{\boldsymbol{h}}_a\,, \tag{IV.27}$$

where it is shown that,

$$\vec{\boldsymbol{h}}_v = \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{g}}_v\,, \tag{IV.28}$$

and

$$\vec{\boldsymbol{h}}_a = \widehat{\boldsymbol{R}}\times\vec{\boldsymbol{g}}_a\,. \tag{IV.29}$$

Lorentz Force for Gravitation

By analogy, the Lorentz force for gravitation is determined to be,

$$\vec{\boldsymbol{F}}_g = m\left(\vec{\boldsymbol{g}} + \frac{\vec{\boldsymbol{v}}}{c_g}\times\vec{\boldsymbol{h}}\right) \tag{IV.30}$$

which agrees with the generalized Lorentz force given in Hills. As shown in Appendix H, the first term can be easily identified as a linear Newtonian force and, by comparison, the second term can be recognized as a Coriolis force. Curiously, the fictitious centrifugal force is buried within $\vec{\boldsymbol{g}}$ since both accelerations act radially, and can be defined as an "effective" gravitational acceleration such that,

$$\vec{\boldsymbol{g}} = (a_f \pm a_c)\hat{\boldsymbol{g}} \tag{IV.31}$$

where a_f is the acceleration due to gravity measured by a fixed observer and a_c is the centrifugal acceleration due to the rotating frame. The relationship between these two accelerations is supported by the equivalence principle which says that acceleration and gravity are indistinguishable, hence why they are jointly an "effective" gravitational acceleration.

Poynting Vector for Gravitation

The Poynting vector for gravitation is shown to be

$$\vec{\boldsymbol{\mathcal{S}}} = -\frac{c_g}{4\pi G}\vec{\boldsymbol{g}} \times \vec{\boldsymbol{h}} \tag{IV.32}$$

which represents the energy per area per unit time flowing out the "sides" of the field. The Poynting theorem, given by Equation V.452 in Appendix F, can be written as,

$$\mathcal{P}_{field} = \frac{\partial \mathcal{W}}{\partial t} + \nabla \cdot \vec{\boldsymbol{\mathcal{S}}} = -\vec{\boldsymbol{g}} \cdot \vec{\boldsymbol{J}}\,. \tag{IV.33}$$

Although the form of the theorem is identical to its electromagnetic counterpart, the definitions of the energy density and energy flow, as seen in the Poynting vector, have a change in sign. This means that energy is flowing *into* the gravitational field.

Gravitational Field of a Mass in Uniform Motion

The gravitational field for a mass in uniform motion evaluated at present position and time is shown in Appendix G to be:

$$\vec{\boldsymbol{g}}(\vec{\boldsymbol{R}}_p, t) = \frac{-Gm(1-\beta^2)\hat{\boldsymbol{R}}_p}{R_p^2(1-\beta^2\sin^2\theta_p)^{\frac{3}{2}}}\,. \tag{IV.34}$$

These results agree with the equation originally derived by Oliver Heaviside in 1888 for a charge in uniform motion [4] and equally depict (Figure IV.1) the contraction of the field in the direction of motion. In the case of no acceleration, the cogravitational field is related to the gravitational field by,

$$\vec{\boldsymbol{h}}_v = \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{g}}_v \tag{IV.35}$$

which becomes,

$$\vec{\boldsymbol{h}}(\vec{\boldsymbol{R}}_p, t) = \frac{-Gm(1-\beta^2)}{R_p^2(1-\beta^2\sin^2\theta_p)^{\frac{3}{2}}}\vec{\boldsymbol{\beta}} \times \widehat{\boldsymbol{R}}_p \tag{IV.36}$$

for a mass in uniform motion.

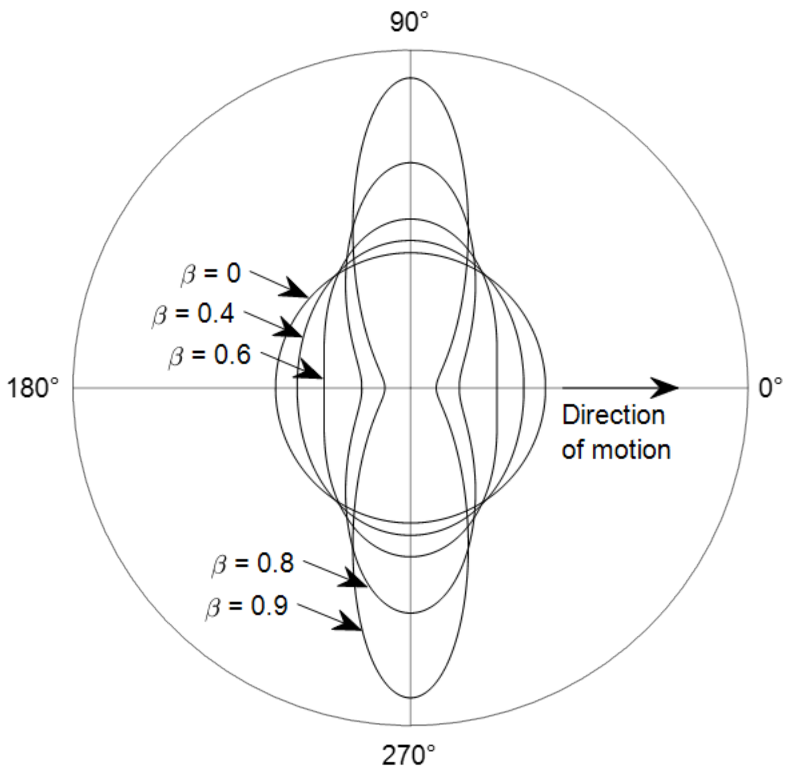

Figure IV.1. Polar Plot of Gravitational Field Magnitude for a Mass in Uniform Motion.

Gravitational Radiation

It is important to interpret the behavior of the velocity and acceleration components of the fields. The velocity term of the gravitational field has vectors parallel to the retarded position $\vec{\boldsymbol{R}}$ and to the velocity $\vec{\boldsymbol{\beta}}$. At a quick glance, the velocity component of the gravitational field reduces to the familiar inverse-square law for low velocities ($\beta << 1$). Thus,

$$\vec{\boldsymbol{g}}_v = -Gm\left[\frac{(\widehat{\boldsymbol{R}} - \vec{\boldsymbol{\beta}})(1-\beta^2)}{R^2\big(1 - \widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^3}\right] \tag{IV.37}$$

becomes,

$$\vec{\boldsymbol{g}}_v = -Gm\left[\frac{\widehat{\boldsymbol{R}}}{R^2}\right]. \tag{IV.38}$$

Although this interpretation isn't thorough, it demonstrates the inverse-square relation well enough. Since the velocity component of the cogravitational field is also proportionate to this inverse-square relationship, then

$$\vec{\boldsymbol{g}}_v, \vec{\boldsymbol{h}}_v \propto \frac{1}{[R^2]}. \tag{IV.39}$$

The acceleration term of the gravitational field is perpendicular to the position vector $\vec{\boldsymbol{R}}$ evaluated at the retarded position at the retarded time. This component varies as an inverse-first-power, given

$$\vec{\boldsymbol{g}}_a = -Gm\left[\frac{\widehat{\boldsymbol{R}}\times\Big((\widehat{\boldsymbol{R}} - \vec{\boldsymbol{\beta}})\times\dot{\vec{\boldsymbol{\beta}}}\Big)}{c_g R\big(1 - \widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^3}\right]. \tag{IV.40}$$

For $\beta << 1$, the above becomes

$$\vec{\boldsymbol{g}}_a = -Gm\left[\frac{\widehat{\boldsymbol{R}}\times\widehat{\boldsymbol{R}}\times\dot{\vec{\boldsymbol{\beta}}}}{c_g R}\right], \tag{IV.41}$$

and since the h-field is also proportionate:

$$\vec{\boldsymbol{g}}_a, \vec{\boldsymbol{h}}_a \propto \frac{1}{[R]}. \tag{IV.42}$$

So, in relation to distance, the velocity-component fields vary as:

$$\vec{\boldsymbol{g}}_v, \vec{\boldsymbol{h}}_v \propto \frac{1}{[R^2]}\,. \tag{IV.43}$$

And the acceleration-component fields vary as:

$$\vec{\boldsymbol{g}}_a, \vec{\boldsymbol{h}}_a \propto \frac{1}{[R]}\,. \tag{IV.44}$$

Recalling the Poynting vector for gravitation:

$$\vec{\boldsymbol{\mathcal{S}}} = -\frac{c_g}{4\pi G}\vec{\boldsymbol{g}} \times \vec{\boldsymbol{h}}\,. \tag{IV.45}$$

Consider the Poynting vector for the velocity components of both fields:

$$\vec{\boldsymbol{\mathcal{S}}}_{vv} = -\frac{c_g}{4\pi G}\vec{\boldsymbol{g}}_v \times \vec{\boldsymbol{h}}_v\,. \tag{IV.46}$$

Then, in relation to distance, the Poynting vector for this arrangement varies as:

$$\vec{\boldsymbol{\mathcal{S}}}_{vv} \propto \frac{1}{[R^4]}\,. \tag{IV.47}$$

Consider the Poynting vector for permutations of velocity and acceleration components,

$$\vec{\boldsymbol{\mathcal{S}}}_{va}, \vec{\boldsymbol{\mathcal{S}}}_{av} \propto \frac{1}{[R^3]} \tag{IV.48}$$

and for the acceleration components, such that:

$$\vec{\boldsymbol{\mathcal{S}}}_{aa} \propto \frac{1}{[R^2]}\,. \tag{IV.49}$$

This indicates that only the energy of the acceleration fields is capable of reaching infinity and is therefore responsible for gravitational radiation. Thus, only accelerated masses can radiate just as only accelerated charges can radiate. It is shown in Appendix G that an accelerated mass produces a dipole radiation pattern which is oriented about the axis of acceleration and which contracts along the axis of motion as described by Heaviside and Lorentz. However, contraction occurs in the reverse direction due to the change in sign. Thus, the results of gravitational radiation mirror those of electromagnetic radiation.

Gravitational Radiation from a Mass at Low Velocities

The Poynting vector is an energy flow vector with units of energy flux (energy per unit area per unit time). The power radiated by a mass is found by expressing the angular distribution of radiation as the power radiated in relation to an integrable "solid-angle" and multiplying the projection of the Poynting vector in the $\widehat{\boldsymbol{R}}$ direction by R^2 (i.e., by the area-per-unit-solid-angle at the radius R) [4]. Hence,

$$\frac{dP}{d\Omega} = (\vec{\mathcal{S}}_a \cdot \widehat{\boldsymbol{R}})R^2 = \frac{-Gm^2a^2\sin^2\theta}{4\pi c_g^3}\,. \tag{IV.50}$$

The total power radiated is found by integrating over the volume (sphere),

$$P = \int_{4\pi} \frac{dP}{d\Omega} d\Omega\,, \tag{IV.51}$$

so,

$$P = \frac{-2Gm^2a^2}{3c_g^3}\,. \tag{IV.52}$$

Equations IV.51 and IV.52 are known as the Larmor formulas and were derived by Joseph Larmor in 1897 [4]. The Larmor formulas for gravitation differ from electromagnetism in the usual way, having picked up a negative sign and a factor of G. The dipole pattern produced by the accelerated mass is shown in Figures IV.2 and IV.3 and clearly mirrors the radiation pattern produced by an accelerated charge at low velocities.

It is somewhat implied in the previous comparisons that e for electromagnetism turns in to $-Gm$ for gravitation. However, this is not the case as can be seen by comparing the Larmor formulas. In electromagnetism, the total power radiated is positive and e is squared. In the analogy of gravitation, the total power radiated is negative and, had e been substituted with $-Gm$, the negative sign and the factor of G would be squared as well, which would be incorrect.

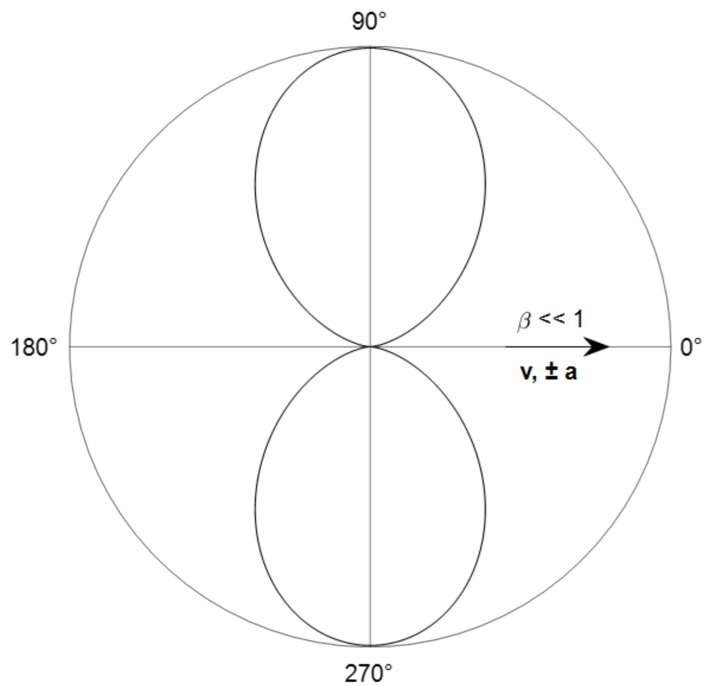

Figure IV.2. Radiation From an Accelerated Mass at Low Velocity (2D).

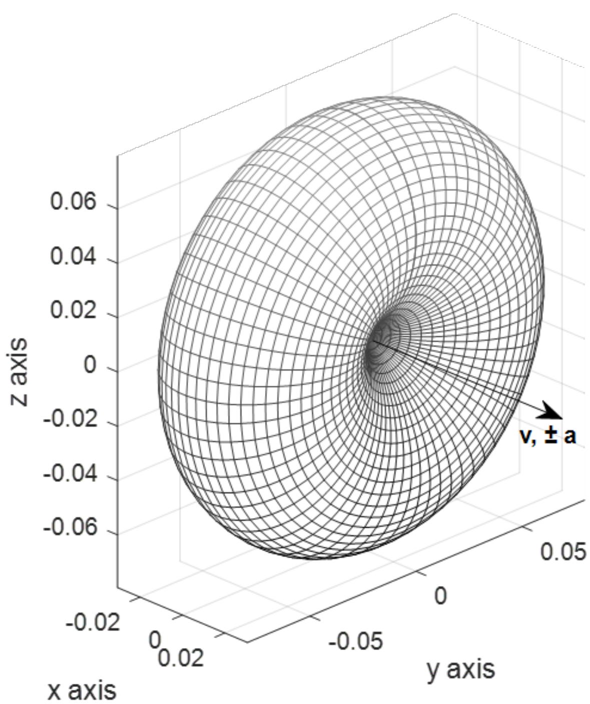

Figure IV.3. Radiation From an Accelerated Mass at Low Velocity (3D).

Gravitational Radiation from a Mass with Collinear Velocity and Acceleration

The power radiated per unit sold angle by a mass with collinear velocity and acceleration is shown to be:

$$\frac{dP}{d\Omega} = \frac{-Gm^2a^2\sin^2\theta}{4\pi c_g^3(1-\beta\cos\theta)^5}\,. \tag{IV.53}$$

The radiation produced by a mass under these conditions is depicted in Figure IV.4. The case of an accelerated mass moving at 60% the speed of gravity is shown in three-dimensions in Figures IV.5 and IV.6. Comparing the results of this scenario, the radiation pattern is a reflection of that from electromagnetism [4].

It is shown that the total power radiated is:

$$P = \frac{-2Gm^2a^2}{3c_g^3(1-\beta^2)^3}\,. \tag{IV.54}$$

And one can easily see that Equation IV.54 reduces to Equation IV.52 when $\beta << 1$.

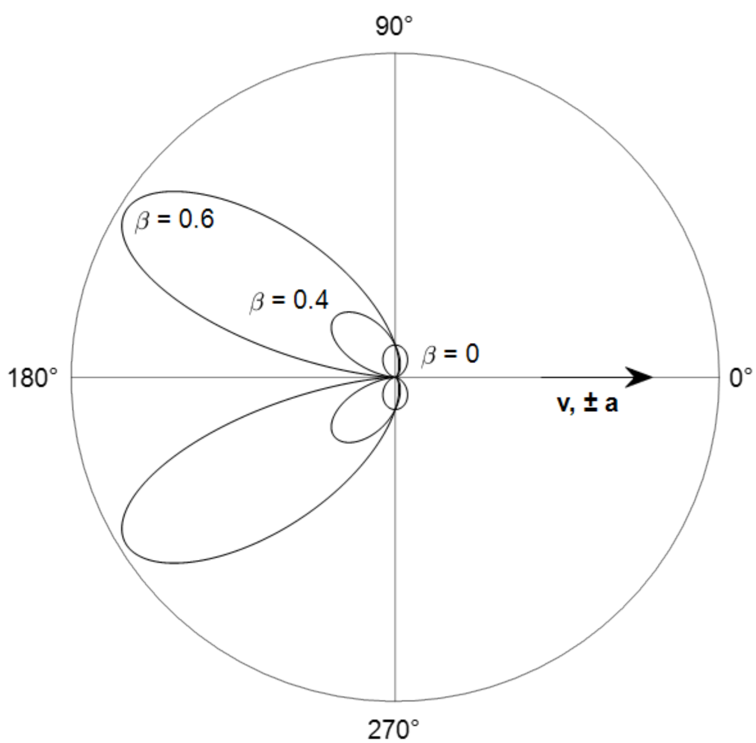

Figure IV.4. Radiation From a Mass with Collinear Velocity and Acceleration (2D).

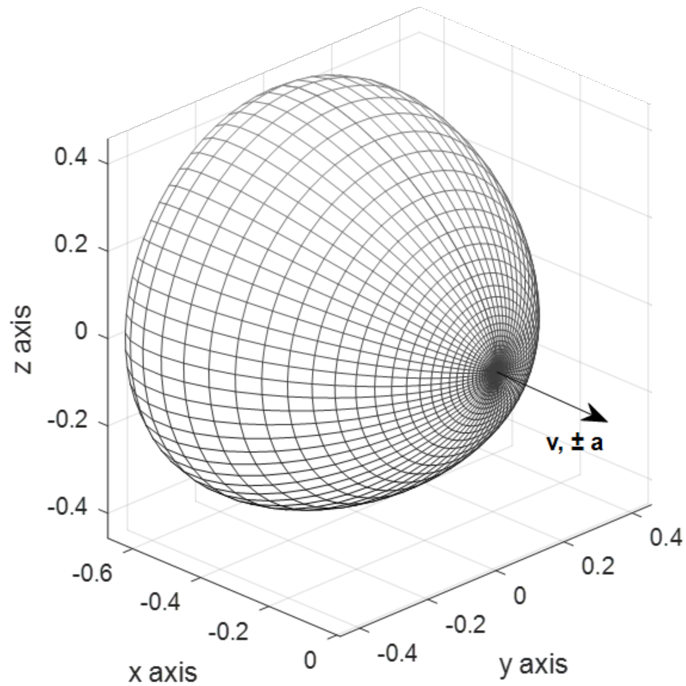

Figure IV.5. Radiation From a Mass with Collinear Velocity and Acceleration at Relativistic Speed - Front (3D).

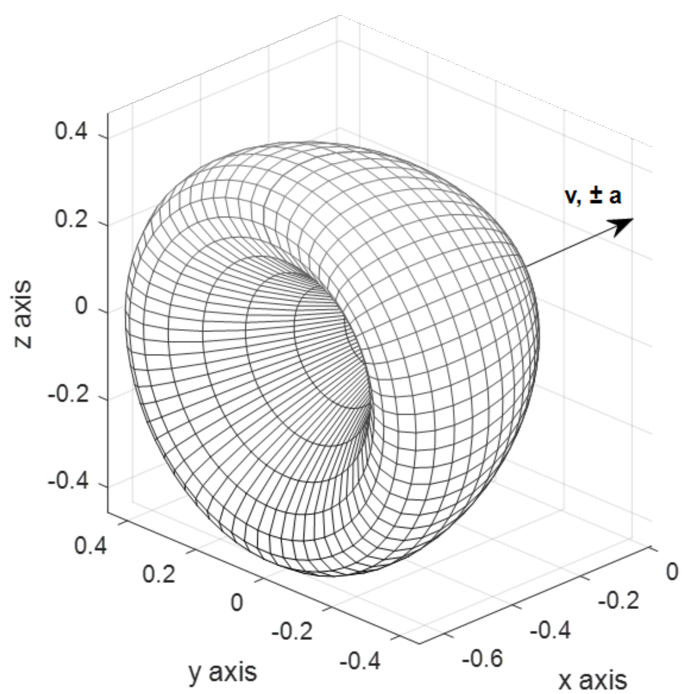

Figure IV.6. Radiation From a Mass with Collinear Velocity and Acceleration at Relativistic Speed - Back (3D).

Gravitational Radiation from a Mass in a Circular Orbit

For a mass confined to a circular orbit, the power radiated per unit solid angle is given by Equation V.532 in Appendix G,

$$\frac{dP}{d\Omega} = \frac{-Gm^2a^2\{(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\varphi\}}{4\pi c_g^3(1-\beta\cos\theta)^5} \,. \tag{IV.55}$$

The behavior of this scenario is depicted in Figure IV.6 where it can be seen that the dipole pattern is oriented about the acceleration axis while the contraction occurs in the direction of the velocity. Two cases are compared three-dimensionally in Figures IV.8 and IV.9. In the first figure, the velocity is much less than the speed of gravity. In the second, the mass is moving at 60% its limit and the contraction is quite clear. The total power radiated is:

$$P = \int_{4\pi} \frac{dP}{d\Omega} d\Omega = \frac{-2Gm^2a^2}{3c_g^3(1-\beta^2)^2} \,, \tag{IV.56}$$

which also reduces to Equation IV.52 when $\beta << 1$.

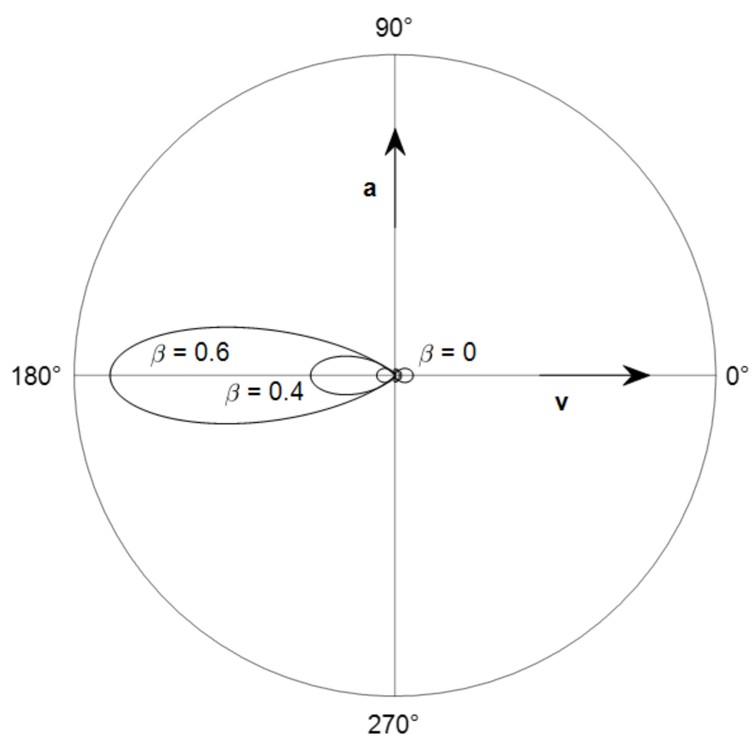

Figure IV.7. Radiation From a Mass in a Circular Orbit (2D).

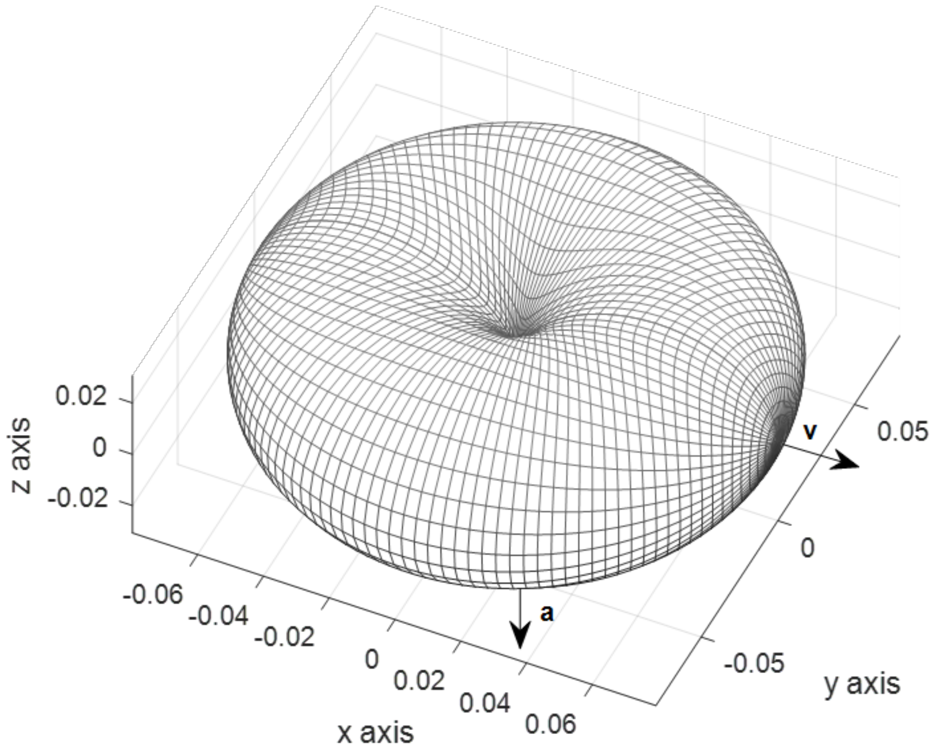

Figure IV.8. Radiation From a Mass in a Circular Orbit at Low Speed (3D).

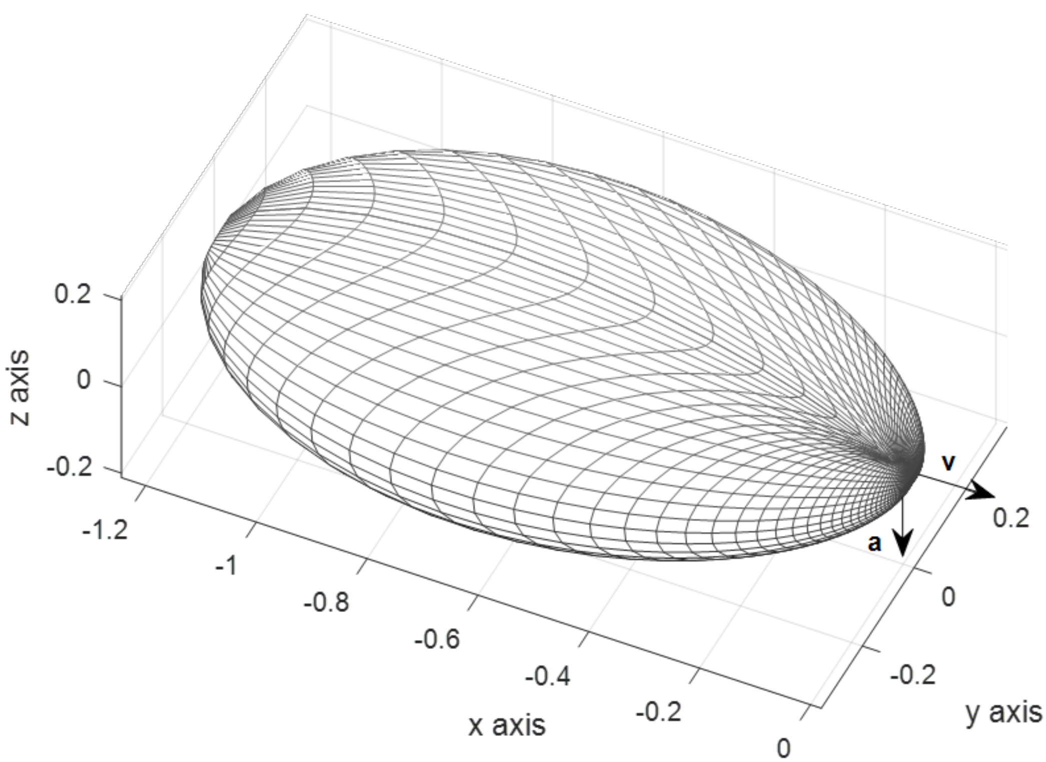

Figure IV.9. Radiation From a Mass in a Circular Orbit at Relativistic Speed (3D).

Chapter V

CONCLUSION

The Maxwell-Heaviside equations derived for gravitation and the resulting classical development of gravitational radiation reveals curious subtleties about the nature of gravity. As if in a mirror, this gravitational analogy reflects that of its electromagnetic counterpart.

Sources of Gravitational Fields

The first and most intuitive comparison is of the force between like masses and the force between like charges. The force is inward in the former and outward in the latter, as was seen when comparing Newton's law of gravitation with Coulomb's law. This is echoed in Gauss's law, where the source creates a field with negative divergence, which results in mass and momentum terms having opposite sign compared to charge and current in the Maxwell-Heaviside equations. These two physical constituents, mass and momentum, are the sources of gravitational and cogravitational fields. Like electric and magnetic fields, these two fields are intimately related.

Gravitational Fields Behave Relativistically

The relationship between the gravitational fields is revealed in their descriptions using retardation theory. In electromagnetism it is common to describe separate fields where the magnetic field is induced or "caused by" a moving electric field. However, when described using retarded wave solutions, the induction terms become "proxies for the charges and currents located elsewhere" [4]. In this description, the fields are generated *simultaneously*.

The concept of simultaneity is lost in relativity, however. Interestingly enough, the relativistic length contraction (in the direction of motion) at high velocities naturally arises in the results of the retarded Liénard–Wiechert potentials and fields. Following the usual procedure, it has been shown that the Liénard–Wiechert fields for gravitation likewise contract at relativistic velocities.

Negative Field Energy

After evaluating the energy flux for gravitation, the Poynting vector is found to be negative. This result agrees with those of Maxwell and Heaviside. To evaluate the implications of the negative field energy, first consider the effects of positive field energy for charges. In electromagnetism, the Poynting vector is positive which indicates that the field emits radiation and loses energy. For the case of an orbiting charge, this loss of energy decreases the radius of orbit. In gravitation, the Poynting vector is negative which indicates that the field gains energy. And for the case of an orbiting mass, this gain in energy would tend to increase the radius of orbit. This could possibly describe the observed expansion of orbiting bodies (i.e., the average orbit of the moon around the earth increases at a rate of approximately 3.8 cm per year) [6].

Radiation from Accelerated Masses

By evaluating the Poynting vector of the acceleration components of the gravitational fields, it is shown that these fields are solely responsible for radiating energy. Thus, an accelerated mass produces gravitational radiation just as an accelerated charge produces electromagnetic radiation. Similarly, an accelerated mass produces a dipole radiation pattern. The dipole pattern is oriented about the axis of acceleration and, as the velocity of the mass increases to its limiting speed, the radiation pattern contracts along the direction of motion as expected. However, rather than contracting in the forward direction like with an accelerated charge at high velocity, the radiation produced by an accelerated mass at high velocity contracts in the reverse direction.

Future Work

Further development of this analogy is necessary to develop testable results. Historically, it would be appropriate to calculate the precession of the perihelion of Mercury and compare the result to known measurements. Derivations for perihelion precession are proposed in Chapter 5 of Hills textbook though they were not the focus of this study. Measurements of precession are sensitive and the effects of gravitation are weak so accurate detection is difficult. The cogravitational field is suggested to be a cause of precession that results from a "frame-dragging" effect [5]. Research in to this effect, also known as Lense-Thirring precession, is underway with NASA's Juno mission at Jupiter.

This analogy motivates another area for future study. The Maxwell-Heaviside equations for electromagnetism were fundamental to describing electromagnetic waves, electromagnetic radiation, and the quantization of charge which led to quantum electrodynamics. Since gravitation can be formulated as a mirrored equivalent of the Maxwell-Heaviside equations, can be described as a pair of gravitational and cogravitational waves, together simply called a "gravitational" wave, and can also be shown to produce gravitational radiation, this would suggest a quantum field theory of gravity may be possible.

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APPENDICES

Appendix A: Classical Electromagnetism References

The following equations are referenced from Heald and Marion's textbook *Classical Electromagnetic Radiation* for comparison to the results of the gravitational analogy.

Maxwell-Heaviside Equations

The Maxwell-Heaviside equations for classical electromagnetism are:

$$\nabla \cdot \vec{\boldsymbol{E}} = 4\pi\rho_e\,, \tag{V.1}$$

$$\nabla \cdot \vec{\boldsymbol{B}} = 0\,, \tag{V.2}$$

$$\nabla \times \vec{\boldsymbol{E}} = -\frac{1}{c_e}\frac{\partial\vec{\boldsymbol{B}}}{\partial t}\,, \tag{V.3}$$

$$\nabla \times \vec{\boldsymbol{B}} = \frac{1}{c_e}\left(4\pi\vec{\boldsymbol{J}}_e + \frac{\partial\vec{\boldsymbol{E}}}{\partial t}\right) \tag{V.4}$$

where it is standard to relate the magnetic field to the curl of a vector potential, $\vec{\boldsymbol{A}}_e$, such that

$$\vec{\boldsymbol{B}} = \nabla \times \vec{\boldsymbol{A}}_e \tag{V.5}$$

by the fact that the divergence of a curl vanishes identically, as given by

$$\nabla \cdot \vec{\boldsymbol{B}} = \nabla \cdot (\nabla \times \vec{\boldsymbol{A}}_e) = 0 \tag{V.6}$$

and it is standard to define a scalar potential, ϕ_e, such that

$$\nabla \times \nabla\phi_e = 0 \tag{V.7}$$

which is true for any continuously twice-differentiable scalar field. Thus, one can show

$$\vec{\boldsymbol{E}} = -\nabla\phi_e - \frac{1}{c_e}\frac{\partial\vec{\boldsymbol{A}}_e}{\partial t}\,. \tag{V.8}$$

Wave Equations

The wave equations for the scalar and vector potentials are, respectively:

$$\nabla^2 \phi_e - \frac{1}{c_e^2}\frac{\partial^2 \phi_e}{\partial t} = -4\pi\rho_e \,, \tag{V.9}$$

$$\nabla^2 \vec{\boldsymbol{A}}_e - \frac{1}{c_e^2}\frac{\partial^2 \vec{\boldsymbol{A}}_e}{\partial t^2} = -\frac{4\pi}{c_e}\vec{\boldsymbol{J}}_e \tag{V.10}$$

and the wave equations for the electric and magnetic fields are, respectively:

$$\nabla^2 \vec{\boldsymbol{E}} - \frac{1}{c_e^2}\frac{\partial^2 \vec{\boldsymbol{E}}}{\partial t^2} = 4\pi\left(\nabla\rho_e + \frac{1}{c_e^2}\frac{\partial \vec{\boldsymbol{J}}_e}{\partial t}\right), \tag{V.11}$$

$$\nabla^2 \vec{\boldsymbol{B}} - \frac{1}{c_e^2}\frac{\partial^2 \vec{\boldsymbol{B}}}{\partial t^2} = -\frac{4\pi}{c_e}(\nabla \times \vec{\boldsymbol{J}}_e)\,. \tag{V.12}$$

Retarded Potentials and Fields

To simplify notation, define the source-density evaluated at the retarded time as:

$$\rho_e\left(\vec{\boldsymbol{r}}', t - \frac{R}{c_e}\right) \equiv \Big[\rho_e(\vec{\boldsymbol{r}}')\Big]\,, \tag{V.13}$$

such that the bracketed notation will represent the quantity evaluated at the retarded time. The distance, R, is defined as shown in Figure V.2 (Appendix D) and given in Equation V.208:

$$R = |\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|\,. \tag{V.14}$$

Now, the retarded scalar and vector potentials are given by:

$$\phi_e(\vec{\boldsymbol{r}}, t) = \int_V \frac{\Big[\rho_e(\vec{\boldsymbol{r}}')\Big]}{R} dV' \tag{V.15}$$

and

$$\vec{\boldsymbol{A}}_e(\vec{\boldsymbol{r}}, t) = \frac{1}{c_e}\int_V \frac{\Big[\vec{\boldsymbol{J}}_e(\vec{\boldsymbol{r}}')\Big]}{R} dV'\,. \tag{V.16}$$

The retarded electric field is given by:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}, t) = \int_V \left(\frac{[\rho_e]\hat{\boldsymbol{e}}_R}{R^2} + \frac{\left[\frac{\partial\rho_e}{\partial t}\right]\hat{\boldsymbol{e}}_R}{c_e R} - \frac{\left[\frac{\partial \vec{\boldsymbol{J}}_e}{\partial t}\right]}{c_e^2 R}\right) dV' \tag{V.17}$$

which is the generalized Coulomb-Faraday Law. The retarded magnetic field is given by:

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{r}}, t) = \frac{1}{c_e}\int_V \left(\frac{[\vec{\boldsymbol{J}}_e] \times \hat{\boldsymbol{e}}_R}{R^2} + \frac{\left[\frac{\partial \vec{\boldsymbol{J}}_e}{\partial t}\right] \times \hat{\boldsymbol{e}}_R}{c_e R}\right) dV' \tag{V.18}$$

which is the generalized Biot-Savart Law.

Liénard–Wiechert Potentials and Fields

The Liénard–Wiechert scalar and vector potentials, evaluated at the retarded time as denoted by the bracket notation, are given by:

$$\phi_e(\vec{\boldsymbol{r}},t) = \frac{e}{\left[R(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})\right]}\,, \tag{V.19}$$

$$\vec{\boldsymbol{A}}_e(\vec{\boldsymbol{r}},t) = \frac{e[\vec{\boldsymbol{\beta}}]}{\left[R(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})\right]}\,. \tag{V.20}$$

Note:

$$\vec{\boldsymbol{A}}_e(\vec{\boldsymbol{r}},t) = \vec{\boldsymbol{\beta}}\phi_e(\vec{\boldsymbol{r}},t)\,. \tag{V.21}$$

The Liénard–Wiechert electric field, evaluated at the retarded time, is given by:

$$\vec{\boldsymbol{E}} = e\left[\frac{(\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})(1-\beta^2)}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3} + \frac{\widehat{\boldsymbol{R}}\times\left((\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})\times\dot{\vec{\boldsymbol{\beta}}}\right)}{c_eR(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\right] \tag{V.22}$$

which can be separated in to velocity and acceleration components as,

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_v + \vec{\boldsymbol{E}}_a\,, \tag{V.23}$$

such that the velocity component of the electric field is,

$$\vec{\boldsymbol{E}}_v = e\left[\frac{(\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})(1-\beta^2)}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\right] \tag{V.24}$$

and the acceleration component of the electric field is,

$$\vec{\boldsymbol{E}}_a = e\left[\frac{\widehat{\boldsymbol{R}}\times\left((\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})\times\dot{\vec{\boldsymbol{\beta}}}\right)}{c_eR(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\right]. \tag{V.25}$$

The Liénard–Wiechert magnetic field, evaluated at the retarded time, is given by:

$$\vec{\boldsymbol{B}} = e\left[\frac{(\vec{\boldsymbol{\beta}}\times\widehat{\boldsymbol{R}})(1-\beta^2)}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3} + \frac{\left(\dot{\vec{\boldsymbol{\beta}}}\cdot\widehat{\boldsymbol{R}}\right)(\vec{\boldsymbol{\beta}}\times\widehat{\boldsymbol{R}})}{c_eR(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3} + \frac{\dot{\vec{\boldsymbol{\beta}}}\times\widehat{\boldsymbol{R}}}{c_eR(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^2}\right] \tag{V.26}$$

which can likewise be separated in to velocity and acceleration components, such that

$$\vec{\boldsymbol{B}} = \vec{\boldsymbol{B}}_v + \vec{\boldsymbol{B}}_a \tag{V.27}$$

where

$$\vec{\boldsymbol{B}}_v = \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}}_v \tag{V.28}$$

and

$$\vec{\boldsymbol{B}}_a = \widehat{\boldsymbol{R}}\times\vec{\boldsymbol{E}}_a\,. \tag{V.29}$$

Lorentz Force Law

The Lorentz force law for electromagnetism (in CGS units) is given by:

$$\vec{\boldsymbol{F}} = q\left(\vec{\boldsymbol{E}} + \frac{\vec{\boldsymbol{v}}}{c_e} \times \vec{\boldsymbol{B}}\right) \tag{V.30}$$

or, (in SI units):

$$\vec{\boldsymbol{F}} = q(\vec{\boldsymbol{E}} + \vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}})\,. \tag{V.31}$$

Poynting Theorem and Poynting Vector

The Poynting theorem states that the partial time derivative of the work-density added to the divergence of the Poynting vector and the dot product of the electric field in to the current-density is zero. This is essentially a statement of energy conservation.

$$\frac{\partial \mathcal{W}}{\partial t} + \nabla \cdot \vec{\boldsymbol{\mathcal{S}}} + \vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{J}} = 0\,. \tag{V.32}$$

Identifying the energy flux from the field being equal to the opposite of that for the particle, one can write that the power-density of the field is

$$\mathcal{P}_{field} = \frac{\partial \mathcal{W}}{\partial t} + \nabla \cdot \vec{\boldsymbol{\mathcal{S}}} = -\vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{J}}\,. \tag{V.33}$$

The Poynting vector, $\vec{\boldsymbol{\mathcal{S}}}$, describes the direction of energy flux for the field, and is given by

$$\vec{\boldsymbol{\mathcal{S}}} = \frac{c_e}{4\pi}\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}\,, \tag{V.34}$$

which is the energy per area per unit time flowing out of the "sides".

Fields Produced by a Charged Particle in Uniform Motion

For a charged particle in uniform motion, there is no acceleration, so the electric field evaluated at present position and time is:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{R}}_p, t) = \frac{e(1-\beta^2)\widehat{\boldsymbol{R}}_p}{R_p^2(1-\beta^2\sin^2\theta_p)^{\frac{3}{2}}}\,. \tag{V.35}$$

The magnetic field for a charge in uniform motion is:

$$\vec{\boldsymbol{B}}(\vec{\boldsymbol{R}}_p, t) = \frac{e(1-\beta^2)}{R_p^2(1-\beta^2\sin^2\theta_p)^{\frac{3}{2}}}\vec{\boldsymbol{\beta}} \times \widehat{\boldsymbol{R}}_p\,. \tag{V.36}$$

Radiation From an Accelerated Charged Particle at Low Velocities

For a charged particle being accelerated at low velocity, such that $\beta << 1$, the Poynting vector is given by:

$$\vec{\mathcal{S}}_a = \frac{e^2 a^2 sin^2\theta}{4\pi c_e^3 R^2}\,. \tag{V.37}$$

The power radiated per unit-solid-angle is given by:

$$\frac{dP}{d\Omega} = (\vec{\mathcal{S}}_a \cdot \widehat{\boldsymbol{R}})R^2 = \frac{e^2 a^2}{4\pi c_e^3}\sin^2\theta\,. \tag{V.38}$$

The total power radiated is found by integrating over the volume (sphere),

$$P = \int_{4\pi} \frac{dP}{d\Omega} d\Omega \,. \tag{V.39}$$

So that,

$$P = \frac{2e^2 a^2}{3c_e^3} \,. \tag{V.40}$$

Equations V.38 and V.40 are the Larmor formulas.

Radiation From a Charged Particle with Collinear Velocity and Acceleration

For a moving charged particle being accelerated in the same direction as its velocity, the power radiated per unit-solid-angle is given by:

$$\frac{dP}{d\Omega} = \frac{e^2 a^2 \sin^2\theta}{4\pi c_e^3 (1-\beta\cos\theta)^5} \,. \tag{V.41}$$

The total power radiated is:

$$P = \frac{2e^2 a^2}{3c_e^3 (1-\beta^2)^3} \,. \tag{V.42}$$

Radiation From a Charged Particle Confined to a Circular Orbit

For a charged particle confined to a circular orbit, the power radiated per solid-unit-angle is given by:

$$\frac{dP}{d\Omega} = \frac{e^2 a^2 \left\{(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\varphi\right\}}{4\pi c_e^3 (1-\beta\cos\theta)^5} \,. \tag{V.43}$$

The total power radiated is:

$$P = \frac{2e^2 a^2}{3c_e^3 (1-\beta^2)^2} \,. \tag{V.44}$$

Appendix B: Derivation of the Maxwell-Heaviside Equations for Gravitation

The Maxwell-Heaviside equations for gravitation can be easily derived by making four classical assumptions.

Assume Newtonian Gravitation

Begin by assuming (1) Newtonian gravitation. From Newton's law of universal gravitation where $\vec{\boldsymbol{F}}$ is the force, G is the universal gravitational constant, M is the parent mass, m is the child mass, and R is the distance of separation:

$$\vec{\boldsymbol{F}} = -G\frac{mM}{R^2}\widehat{\boldsymbol{R}} \tag{V.45}$$

where, by Newton's third law, the force on the child mass due to the parents mass is equal and opposite to the force on the parent mass due to the child mass,

$$\vec{\boldsymbol{F}}_{12} = -\vec{\boldsymbol{F}}_{21}\,. \tag{V.46}$$

One defines the gravitational field produced by the parent mass M as

$$\vec{\boldsymbol{g}}(\vec{\boldsymbol{r}}) = -G\frac{M}{R^2}\widehat{\boldsymbol{R}}\,. \tag{V.47}$$

From classical mechanics, the force experienced by an object in an inverse-square central force can be represented by:

$$\vec{\boldsymbol{F}} = -\frac{\lambda m}{R^2}\widehat{\boldsymbol{R}} \tag{V.48}$$

where

$$\lambda = GM\,, \tag{V.49}$$

such that one can use the above definition of the gravitational field to write the gravitational analog of Newton's second law:

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{g}}(\vec{\boldsymbol{r}})\,. \tag{V.50}$$

Gauss's Law for Gravitostatics from Newton's Universal Law of Gravitation

Begin with Equation V.47,

$$\vec{\boldsymbol{g}}(\vec{\boldsymbol{r}}) = -G\frac{M}{R^2}\widehat{\boldsymbol{R}}\,. \tag{V.51}$$

Expanding to an integral involving the mass density yields,

$$\vec{\boldsymbol{g}}(\vec{\boldsymbol{r}}) = -G\int \rho_g(\vec{\boldsymbol{r}}\,')\frac{\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,'|^3}d^3\vec{\boldsymbol{r}}\,'\,. \tag{V.52}$$

Taking the divergence of the above equation and knowing that

$$\nabla\cdot\left(\frac{\vec{\boldsymbol{r}}}{|\vec{\boldsymbol{r}}|^3}\right) = 4\pi\delta(\vec{\boldsymbol{r}})\,, \tag{V.53}$$

where δ is the Dirac delta function, one can show

$$\nabla \cdot \vec{\boldsymbol{g}}(\vec{\boldsymbol{r}}) = -4\pi G \int \rho_g(\vec{\boldsymbol{r}}\,')\delta(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}\,')d^3\vec{\boldsymbol{r}}\,' \,. \tag{V.54}$$

Thus, Gauss's law is given by,

$$\nabla \cdot \vec{\boldsymbol{g}}(\vec{\boldsymbol{r}}) = -4\pi G\rho_g \,. \tag{V.55}$$

*As an aside: Recall, gravitational fields are conservative (path-independent), thus we can say that

$$\vec{\boldsymbol{g}}(\vec{\boldsymbol{r}}) = -\nabla\phi(\vec{\boldsymbol{r}}) \,, \tag{V.56}$$

where $\phi(\vec{\boldsymbol{r}})$ is some potential field, such that

$$\phi(\vec{\boldsymbol{r}}) = -G\frac{M}{R} \,. \tag{V.57}$$

Plugging Equation V.56 into Gauss's law for gravitation yields

$$\nabla \cdot (-\nabla\phi) = -4\pi G\rho_g \tag{V.58}$$

so that,

$$\nabla^2\phi = 4\pi G\rho_g \,, \tag{V.59}$$

which is Poisson's equation for gravity and satisfies the wave equation for the scalar potential in the static condition.

Extending Gauss's Law for Gravitostatics to Time-dependence

Extending Gauss's law to time-dependence (instant propagation). *It's only an integral about space so it should hold!

$$\vec{\boldsymbol{g}}(\vec{\boldsymbol{r}}, t) = -G\int \frac{\rho_g(\vec{\boldsymbol{r}}, t)dV}{R^2}\hat{\boldsymbol{R}} \tag{V.60}$$

Thus, we have the time-dependent Gauss's law for gravitation:

$$\nabla \cdot \vec{\boldsymbol{g}}(\vec{\boldsymbol{r}}, t) = -4\pi G\rho_g(\vec{\boldsymbol{r}}, t) \,. \tag{V.61}$$

Conservative Gravitational Field

The nature of the gravitostatic field is conservative (as mentioned previously) and thus irrotational, such that

$$\nabla \times \vec{\boldsymbol{g}} = 0 \,. \tag{V.62}$$

The source of this field is mass.

Mass Conservation at Low Energies

(2) Assume that, for low energies, the law of conservation of mass can be expressed by the continuity equation

$$\frac{\partial \rho_g}{\partial t} + \nabla \cdot \vec{\boldsymbol{J}}_g = 0 \, . \tag{V.63}$$

To extend the static case of the irrotational field to the time-dependent case, take the partial time derivative of Gauss's law,

$$\frac{\partial}{\partial t}(\nabla \cdot \vec{\boldsymbol{g}}) = \frac{\partial}{\partial t}(-4\pi G \rho_g) \, , \tag{V.64}$$

$$\nabla \cdot \frac{\partial \vec{\boldsymbol{g}}}{\partial t} = -4\pi G \frac{\partial \rho_g}{\partial t} \, . \tag{V.65}$$

Using the mass-conservation equation,

$$\nabla \cdot \frac{\partial \vec{\boldsymbol{g}}}{\partial t} = -4\pi G(- \nabla \cdot \vec{\boldsymbol{J}}_g) = 4\pi G(\nabla \cdot \vec{\boldsymbol{J}}_g) \, . \tag{V.66}$$

Rearranging,

$$\nabla \cdot \left(\frac{\partial \vec{\boldsymbol{g}}}{\partial t} \right) - 4\pi G(\nabla \cdot \vec{\boldsymbol{J}}_g) = 0 \, , \tag{V.67}$$

$$\nabla \cdot \left(\frac{\partial \vec{\boldsymbol{g}}}{\partial t} - 4\pi G \vec{\boldsymbol{J}}_g \right) = 0 \tag{V.68}$$

or

$$\nabla \cdot \left(\frac{1}{4\pi G} \frac{\partial \vec{\boldsymbol{g}}}{\partial t} - \vec{\boldsymbol{J}}_g \right) = 0 \, . \tag{V.69}$$

Since the divergence of a curl vanishes identically, $\nabla \cdot (\nabla \times \vec{\boldsymbol{b}}) = 0$, the term in parentheses can be defined as the curl of an arbitrary vector such that,

$$\nabla \times \vec{\boldsymbol{b}} = \frac{1}{4\pi G} \frac{\partial \vec{\boldsymbol{g}}}{\partial t} - \vec{\boldsymbol{J}}_g \, . \tag{V.70}$$

Wave Solutions in Empty Space

(3) Assuming, as one would for electromagnetism, that waves propagate in empty space, then one expects wave solutions of the form:

$$\nabla^2 \vec{\boldsymbol{g}} - \frac{1}{c_g^2} \frac{\partial^2 \vec{\boldsymbol{g}}}{\partial t^2} = 0 \, , \tag{V.71}$$

$$\nabla^2 \vec{\boldsymbol{b}} - \frac{1}{c_g^2} \frac{\partial^2 \vec{\boldsymbol{b}}}{\partial t^2} = 0 \tag{V.72}$$

for gravitational waves where c_g is some finite propagation speed yet to be determined experimentally. Take the curl of $\nabla \times \vec{\boldsymbol{b}}$ and assume empty space (i.e. $\vec{\boldsymbol{J}}_g = 0$),

$$\begin{aligned} \nabla \times (\nabla \times \vec{\boldsymbol{b}}) &= \nabla \times \left(\frac{1}{4\pi G} \frac{\partial \vec{\boldsymbol{g}}}{\partial t} - \cancelto{0}{\vec{\boldsymbol{J}}_g} \right) \\ &= \frac{1}{4\pi G} \nabla \times \frac{\partial \vec{\boldsymbol{g}}}{\partial t} \, . \end{aligned} \tag{V.73}$$

Applying the vector identity $\nabla \times (\nabla \times \vec{\boldsymbol{b}}) = \nabla(\nabla \cdot \vec{\boldsymbol{b}}) - \nabla^2 \vec{\boldsymbol{b}}$,

$$\nabla(\nabla \cdot \vec{\boldsymbol{b}}) - \nabla^2 \vec{\boldsymbol{b}} = \frac{1}{4\pi G} \frac{\partial}{\partial t} (\nabla \times \vec{\boldsymbol{g}}) \,. \tag{V.74}$$

Using the wave equation, one obtains

$$\nabla(\nabla \cdot \vec{\boldsymbol{b}}) - \frac{1}{c_g^2} \frac{\partial^2 \vec{\boldsymbol{b}}}{\partial t^2} = \frac{1}{4\pi G} \frac{\partial}{\partial t} (\nabla \times \vec{\boldsymbol{g}}) \tag{V.75}$$

or

$$\nabla(\nabla \cdot \vec{\boldsymbol{b}}) = \frac{\partial}{\partial t} \left(\frac{1}{4\pi G} \nabla \times \vec{\boldsymbol{g}} + \frac{1}{c_g^2} \frac{\partial \vec{\boldsymbol{b}}}{\partial t} \right) . \tag{V.76}$$

Solenoidal Cogravitational Field

(4) Since there is no experimental evidence for a scalar source of the rotational field, $\vec{\boldsymbol{b}}$, we take $\nabla \cdot \vec{\boldsymbol{b}} = 0$, so

$$0 = \frac{\partial}{\partial t} \left(\frac{1}{4\pi G} \nabla \times \vec{\boldsymbol{g}} + \frac{1}{c_g^2} \frac{\partial \vec{\boldsymbol{b}}}{\partial t} \right) . \tag{V.77}$$

Then,

$$\frac{1}{4\pi G} \nabla \times \vec{\boldsymbol{g}} + \frac{1}{c_g^2} \frac{\partial \vec{\boldsymbol{b}}}{\partial t} = \vec{\boldsymbol{f}}(\vec{\boldsymbol{r}}) \,. \tag{V.78}$$

Since we wish to arrive at a set of universal fundamental equations valid for all time, the above must hold in the static case where $\nabla \times \vec{\boldsymbol{g}} = \vec{\boldsymbol{f}}(\vec{\boldsymbol{r}})$ and since $\nabla \times \vec{\boldsymbol{g}} = 0$ in the static case, we must take $\vec{\boldsymbol{f}}(\vec{\boldsymbol{r}}) = 0$, thus,

$$\frac{1}{4\pi G} \nabla \times \vec{\boldsymbol{g}} + \frac{1}{c_g^2} \frac{\partial \vec{\boldsymbol{b}}}{\partial t} = 0 \,. \tag{V.79}$$

Rearranging, Equation V.79 becomes

$$\nabla \times \vec{\boldsymbol{g}} = -\frac{4\pi G}{c_g^2} \frac{\partial \vec{\boldsymbol{b}}}{\partial t} \,. \tag{V.80}$$

Thus, we have the four vector equations:

$$\nabla \cdot \vec{\boldsymbol{g}} = -4\pi G \rho_g \,, \tag{V.81}$$

$$\nabla \cdot \vec{\boldsymbol{b}} = 0 \,, \tag{V.82}$$

$$\nabla \times \vec{\boldsymbol{g}} = -\frac{4\pi G}{c_g^2} \frac{\partial \vec{\boldsymbol{b}}}{\partial t} \,, \tag{V.83}$$

$$\nabla \times \vec{\boldsymbol{b}} = \frac{1}{4\pi G} \frac{\partial \vec{\boldsymbol{g}}}{\partial t} - \vec{\boldsymbol{J}}_g \,. \tag{V.84}$$

If we define a change of variable, such that

$$\vec{\boldsymbol{h}} = \mu_g \vec{\boldsymbol{b}} \tag{V.85}$$

where

$$\mu_g = \frac{4\pi G}{c_g^2} \tag{V.86}$$

one can show, multiplying Equation V.84 by μ_g, that

$$\nabla \times \vec{\boldsymbol{h}} = \frac{1}{c_g^2}\frac{\partial \vec{\boldsymbol{g}}}{\partial t} - \frac{4\pi G}{c_g^2}\vec{\boldsymbol{J}} \tag{V.87}$$

and trivially, Equation V.82 becomes,

$$\nabla \cdot \vec{\boldsymbol{h}} = 0\,. \tag{V.88}$$

Let us also define

$$\varepsilon_g = \frac{1}{4\pi G} \tag{V.89}$$

such that,

$$\mu_g \varepsilon_g = \frac{1}{c_g^2}\,. \tag{V.90}$$

Microscopic Maxwell-Heaviside Equations for Gravitation (SI Units)

After rearranging Equations V.81 and V.83 using Equations V.89 and V.90, it is possible to write them such that they resemble the microscopic Maxwell-Heaviside equations in SI units (with the expected sign change on mass and mass-flow densities for Gravitation):

$$\nabla \cdot \vec{\boldsymbol{g}} = -\frac{\rho_g}{\varepsilon_g}\,, \tag{V.91}$$

$$\nabla \cdot \vec{\boldsymbol{h}} = 0\,, \tag{V.92}$$

$$\nabla \times \vec{\boldsymbol{g}} = -\frac{\partial \vec{\boldsymbol{h}}}{\partial t}\,, \tag{V.93}$$

$$\nabla \times \vec{\boldsymbol{h}} = \mu_g\left(-\vec{\boldsymbol{J}} + \varepsilon_g \frac{\partial \vec{\boldsymbol{g}}}{\partial t}\right). \tag{V.94}$$

Macroscopic Maxwell-Heaviside Equations for Gravitation (SI Units)

To obtain the macroscopic form in SI units, one can simply use the definition,

$$\vec{\boldsymbol{d}} = \varepsilon_g \vec{\boldsymbol{g}} \tag{V.95}$$

to show that,

$$\nabla \cdot \vec{\boldsymbol{d}} = -\rho_g\,, \tag{V.96}$$

$$\nabla \cdot \vec{\boldsymbol{h}} = 0\,, \tag{V.97}$$

$$\nabla \times \vec{\boldsymbol{g}} = -\frac{\partial \vec{\boldsymbol{h}}}{\partial t}\,, \tag{V.98}$$

$$\nabla \times \vec{\boldsymbol{b}} = -\vec{\boldsymbol{J}} + \frac{\partial \vec{\boldsymbol{d}}}{\partial t}\,. \tag{V.99}$$

Microscopic Maxwell-Heaviside Equations for Gravitation (CGS Units)

To obtain the microscopic Maxwell-Heaviside equations in CGS units, one must simply define a change of variable such that:

$$\vec{\boldsymbol{h}}' = \frac{\vec{\boldsymbol{h}}}{c_g} \tag{V.100}$$

which implies μ_g becomes

$$\mu_g = \frac{4\pi G}{c_g}\,. \tag{V.101}$$

This can be easily identified by comparing the Lorentz force equation in both units. Recalling that we have defined the following coefficients for CGS units:

$$\varepsilon_g = \frac{1}{4\pi G} \quad \& \quad \mu_g = \frac{4\pi G}{c_g} \quad \& \quad \mu_g \varepsilon_g = \frac{1}{c_g}\,. \tag{V.102}$$

It is then easy to identify that Equations V.103 - V.106 become the same equations we started with (Equations IV.1 - IV.4):

$$\nabla \cdot \vec{\boldsymbol{g}} = -4\pi G \rho_g\,, \tag{V.103}$$

$$\nabla \cdot \vec{\boldsymbol{h}} = 0\,, \tag{V.104}$$

$$\nabla \times \vec{\boldsymbol{g}} = -\frac{1}{c_g}\frac{\partial \vec{\boldsymbol{h}}}{\partial t}\,, \tag{V.105}$$

$$\nabla \times \vec{\boldsymbol{h}} = \frac{1}{c_g}\bigg(-4\pi G \vec{\boldsymbol{J}}_g + \frac{\partial \vec{\boldsymbol{g}}}{\partial t}\bigg)\,. \tag{V.106}$$

In fact, it is possible to absorb the negative sign in to the coefficients as is done in Hills, although it is standard to keep these coefficients defined as positive quantities. For completeness, let

$$\varepsilon_g = -\frac{1}{4\pi G} \quad \& \quad \mu_g = -\frac{4\pi G}{c_g} \quad \& \quad \mu_g \varepsilon_g = \frac{1}{c_g}\,. \tag{V.107}$$

Thus, we have shown that these results agree with Hills for either set of units. Rewriting to identically match the Maxwell-Heaviside equations for electromagnetism in the CGS unit notation of Heald and Marion:

$$\nabla \cdot \vec{\boldsymbol{g}} = \frac{\rho_g}{\varepsilon_g}\,, \tag{V.108}$$

$$\nabla \cdot \vec{\boldsymbol{h}} = 0\,, \tag{V.109}$$

$$\nabla \times \vec{\boldsymbol{g}} = -\frac{1}{c_g}\frac{\partial \vec{\boldsymbol{h}}}{\partial t}\,, \tag{V.110}$$

$$\nabla \times \vec{\boldsymbol{h}} = \mu_g\bigg(\vec{\boldsymbol{J}} + \varepsilon_g \frac{\partial \vec{\boldsymbol{g}}}{\partial t}\bigg)\,. \tag{V.111}$$

Appendix C: Derivation of the Wave Equations for Gravitation

In summary, the wave equations for gravitation are as follows:

$$\nabla^2 \vec{\boldsymbol{A}}_g - \frac{1}{c_g^2}\frac{\partial^2 \vec{\boldsymbol{A}}_g}{\partial t^2} = \frac{4\pi G \vec{\boldsymbol{J}}_g}{c_g}\,, \tag{V.112}$$

$$\nabla^2 \phi_g - \frac{1}{c_g^2}\frac{\partial^2 \phi_g}{\partial t} = 4\pi G \rho_g\,, \tag{V.113}$$

$$\nabla^2 \vec{\boldsymbol{g}} - \frac{1}{c_g^2}\frac{\partial^2 \vec{\boldsymbol{g}}}{\partial t^2} = -4\pi G\Big(\nabla \rho_g + \frac{1}{c_g^2}\frac{\partial \vec{\boldsymbol{J}}_g}{\partial t}\Big)\,, \tag{V.114}$$

$$\nabla^2 \vec{\boldsymbol{h}} - \frac{1}{c_g^2}\frac{\partial^2 \vec{\boldsymbol{h}}}{\partial t^2} = \frac{4\pi G}{c_g}(\nabla \times \vec{\boldsymbol{J}}_g)\,, \tag{V.115}$$

which are identical to the wave equations in Hills, given that the generalized coefficient,

$$\beta = -\frac{4\pi G}{c_g} \tag{V.116}$$

for CGS units.

Scalar and Vector Wave Equations for Gravitation

Starting with the Maxwell-Heaviside equations for gravitation (Equations V.103 - V.106):

$$\nabla \cdot \vec{\boldsymbol{g}} = -4\pi G \rho_g\,, \tag{V.117}$$

$$\nabla \cdot \vec{\boldsymbol{h}} = 0\,, \tag{V.118}$$

$$\nabla \times \vec{\boldsymbol{g}} = -\frac{1}{c_g}\frac{\partial \vec{\boldsymbol{h}}}{\partial t}\,, \tag{V.119}$$

$$\nabla \times \vec{\boldsymbol{h}} = \frac{1}{c_g}\Big(-4\pi G \vec{\boldsymbol{J}}_g + \frac{\partial \vec{\boldsymbol{g}}}{\partial t}\Big)\,. \tag{V.120}$$

Define the vector potential, $\vec{\boldsymbol{A}}_g$, related to the cogravitational field, $\vec{\boldsymbol{h}}$, by:

$$\vec{\boldsymbol{h}} = \nabla \times \vec{\boldsymbol{A}}_g \tag{V.121}$$

such that,

$$\nabla \cdot \vec{\boldsymbol{h}} = \nabla \cdot (\nabla \times \vec{\boldsymbol{A}}_g) = 0\,. \tag{V.122}$$

Using Equation V.121 in Equation V.119 [V.105],

$$\nabla \times \vec{\boldsymbol{g}} = -\frac{1}{c_g}\frac{\partial}{\partial t}(\nabla \times \vec{\boldsymbol{A}}_g) = -\nabla \times \frac{1}{c_g}\frac{\partial \vec{\boldsymbol{A}}_g}{\partial t}\,. \tag{V.123}$$

Now,

$$\nabla \times \left(\vec{\boldsymbol{g}} + \frac{1}{c_g} \frac{\partial \vec{\boldsymbol{A}}_g}{\partial t} \right) = 0\,. \tag{V.124}$$

Defining the scalar potential, ϕ_g, as:

$$\nabla \times \nabla \phi_g = 0\,, \tag{V.125}$$

which is true for any continuously twice-differentiable scalar field, ϕ. So, let,

$$\vec{\boldsymbol{g}} + \frac{1}{c_g} \frac{\partial \vec{\boldsymbol{A}}_g}{\partial t} = -\nabla \phi_g\,. \tag{V.126}$$

Then,

$$\vec{\boldsymbol{g}} = -\nabla \phi_g - \frac{1}{c_g} \frac{\partial \vec{\boldsymbol{A}}_g}{\partial t}\,. \tag{V.127}$$

Now we use Equations V.121 and V.127 in Equation V.120 [V.106],

$$\nabla \times (\nabla \times \vec{\boldsymbol{A}}_g) = -\frac{4\pi G \vec{\boldsymbol{J}}_g}{c_g} + \frac{1}{c_g} \frac{\partial}{\partial t} \left(-\nabla \phi_g - \frac{1}{c_g} \frac{\partial \vec{\boldsymbol{A}}_g}{\partial t} \right). \tag{V.128}$$

Recall, for any vector, $\vec{\boldsymbol{A}}$,

$$\nabla \times (\nabla \times \vec{\boldsymbol{A}}) = \nabla(\nabla \cdot \vec{\boldsymbol{A}}) - \nabla^2 \vec{\boldsymbol{A}} \tag{V.129}$$

then,

$$\nabla(\nabla \cdot \vec{\boldsymbol{A}}_g) - \nabla^2 \vec{\boldsymbol{A}}_g = -\frac{4\pi G \vec{\boldsymbol{J}}_g}{c_g} - \nabla \frac{1}{c_g} \frac{\partial \phi_g}{\partial t} - \frac{1}{c_g^2} \frac{\partial^2 \vec{\boldsymbol{A}}_g}{\partial t^2}\,, \tag{V.130}$$

$$\nabla^2 \vec{\boldsymbol{A}}_g - \frac{1}{c_g^2} \frac{\partial^2 \vec{\boldsymbol{A}}_g}{\partial t^2} = \frac{4\pi G \vec{\boldsymbol{J}}_g}{c_g} + \nabla \left(\nabla \cdot \vec{\boldsymbol{A}}_g + \frac{1}{c_g} \frac{\partial \phi_g}{\partial t} \right). \tag{V.131}$$

Using the Lorenz Gauge Condition,

$$\nabla \cdot \vec{\boldsymbol{A}}_g = -\frac{1}{c_g} \frac{\partial \phi_g}{\partial t} \tag{V.132}$$

such that,

$$\nabla \cdot \vec{\boldsymbol{A}}_g + \frac{1}{c_g} \frac{\partial \phi_g}{\partial t} = 0\,. \tag{V.133}$$

Then, Equation V.131 becomes,

$$\nabla^2 \vec{\boldsymbol{A}}_g - \frac{1}{c_g^2} \frac{\partial^2 \vec{\boldsymbol{A}}_g}{\partial t^2} = \frac{4\pi G \vec{\boldsymbol{J}}_g}{c_g}\,. \tag{V.134}$$

Now, use Equation V.127 in Equation V.117 [V.103],

$$\nabla \cdot \left(-\nabla \phi_g - \frac{1}{c_g} \frac{\partial \vec{\boldsymbol{A}}_g}{\partial t} \right) = -4\pi G \rho_g\,, \tag{V.135}$$

$$-\nabla^2 \phi_g - \frac{1}{c_g} \frac{\partial}{\partial t} (\nabla \cdot \vec{\boldsymbol{A}}_g) = -4\pi G \rho_g\,. \tag{V.136}$$

Using Equation V.132,

$$-\nabla^2\phi_g - \frac{1}{c_g}\frac{\partial}{\partial t}\left(-\frac{1}{c_g}\frac{\partial \vec{\boldsymbol{A}}_g}{\partial t}\right) = -4\pi G\rho_g \tag{V.137}$$

and so,

$$\nabla^2\phi_g - \frac{1}{c_g^2}\frac{\partial^2\phi_g}{\partial t} = 4\pi G\rho_g\,. \tag{V.138}$$

Thus, the wave equations for the vector and scalar potentials are:

$$\nabla^2\vec{\boldsymbol{A}}_g - \frac{1}{c_g^2}\frac{\partial^2\vec{\boldsymbol{A}}_g}{\partial t^2} = \frac{4\pi G\vec{\boldsymbol{J}}_g}{c_g} \tag{V.139}$$

and

$$\nabla^2\phi_g - \frac{1}{c_g^2}\frac{\partial^2\phi_g}{\partial t} = 4\pi G\rho_g\,. \tag{V.140}$$

Gravitational Plane Waves in a Vacuum

Apply the general gauge transformation,

$$\phi_g = \phi_g' - \frac{1}{c_g}\frac{\partial f}{\partial t} \tag{V.141}$$

and

$$\vec{\boldsymbol{A}}_g = \vec{\boldsymbol{A}}_g' + \nabla f \tag{V.142}$$

to the scalar and vector potentials to show that Equation V.133 becomes:

$$\nabla\cdot(\vec{\boldsymbol{A}}_g - \nabla f) + \frac{1}{c_g}\frac{\partial}{\partial t}\left(\phi_g + \frac{1}{c_g}\frac{\partial f}{\partial t}\right) = 0\,, \tag{V.143}$$

$$\left(\nabla\cdot\vec{\boldsymbol{A}}_g + \frac{1}{c_g}\frac{\partial\phi_g}{\partial t}\right) - \nabla\cdot\nabla f + \frac{1}{c_g^2}\frac{\partial^2 f}{\partial t^2} = 0 \tag{V.144}$$

by the Lorentz gauge condition, such that,

$$-\nabla\cdot\nabla f + \frac{1}{c_g^2}\frac{\partial^2 f}{\partial t^2} = 0\,. \tag{V.145}$$

So,

$$\nabla^2 f - \frac{1}{c_g^2}\frac{\partial^2 f}{\partial t^2} = 0 \tag{V.146}$$

for any f that satisfies the homogenous wave equation. Now we can extend this to the $\vec{\boldsymbol{g}}$ and $\vec{\boldsymbol{h}}$ fields and their complex forms. Consider a linear isotropic media where there are no charges, so $\rho = 0$ and $\vec{\boldsymbol{J}} = 0$. The Maxwell-Heaviside equations for gravitation in a vacuum become:

$$\nabla\cdot\vec{\boldsymbol{g}} = 0\,, \tag{V.147}$$

$$\nabla\cdot\vec{\boldsymbol{h}} = 0\,, \tag{V.148}$$

$$\nabla \times \vec{\boldsymbol{g}} = -\frac{1}{c_g}\frac{\partial \vec{\boldsymbol{h}}}{\partial t}\,, \tag{V.149}$$

$$\nabla \times \vec{\boldsymbol{h}} = \frac{1}{c_g}\frac{\partial \vec{\boldsymbol{g}}}{\partial t}\,. \tag{V.150}$$

Taking the cross-product of Equation V.149,

$$\nabla \times (\nabla \times \vec{\boldsymbol{g}}) = \nabla \times \left(-\frac{1}{c_g}\frac{\partial \vec{\boldsymbol{h}}}{\partial t}\right) \tag{V.151}$$

which becomes,

$$\nabla(\nabla \cdot \vec{\boldsymbol{g}}) - \nabla^2 \vec{\boldsymbol{g}} = -\frac{1}{c_g}\frac{\partial}{\partial t}(\nabla \times \vec{\boldsymbol{h}})\,. \tag{V.152}$$

Applying Equation V.147 and V.150,

$$-\nabla^2 \vec{\boldsymbol{g}} = -\frac{1}{c_g}\frac{\partial}{\partial t}\left(\frac{1}{c_g}\frac{\partial \vec{\boldsymbol{g}}}{\partial t}\right). \tag{V.153}$$

Thus,

$$\nabla^2 \vec{\boldsymbol{g}} - \frac{1}{c_g^2}\frac{\partial^2 \vec{\boldsymbol{g}}}{\partial t^2} = 0\,. \tag{V.154}$$

One can easily show that the same operation on Equation V.150,

$$\nabla \times (\nabla \times \vec{\boldsymbol{h}}) = \nabla \times \left(\frac{1}{c_g}\frac{\partial \vec{\boldsymbol{g}}}{\partial t}\right), \tag{V.155}$$

leads to

$$\nabla^2 \vec{\boldsymbol{h}} - \frac{1}{c_g^2}\frac{\partial^2 \vec{\boldsymbol{h}}}{\partial t^2} = 0\,. \tag{V.156}$$

Rewriting the wave equations for $\vec{\boldsymbol{g}}$ and $\vec{\boldsymbol{h}}$ in a vacuum:

$$\nabla^2 \vec{\boldsymbol{g}} - \frac{1}{c_g^2}\frac{\partial^2 \vec{\boldsymbol{g}}}{\partial t^2} = 0\,, \tag{V.157}$$

$$\nabla^2 \vec{\boldsymbol{h}} - \frac{1}{c_g^2}\frac{\partial^2 \vec{\boldsymbol{h}}}{\partial t^2} = 0 \tag{V.158}$$

or

$$\left(\nabla^2 - \frac{1}{c_g^2}\frac{\partial^2}{\partial t^2}\right)\vec{\boldsymbol{g}} = 0\,, \tag{V.159}$$

$$\left(\nabla^2 - \frac{1}{c_g^2}\frac{\partial^2}{\partial t^2}\right)\vec{\boldsymbol{h}} = 0\,. \tag{V.160}$$

To determine a solution to the above wave equations and the gravitational Maxwell-Heaviside equations in a vacuum, evaluate complex plane waves of the form:

$$\vec{\mathfrak{g}}(\vec{\boldsymbol{r}}, t) = \vec{\mathfrak{g}}_o e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}} - \omega t)} \tag{V.161}$$

and

$$\vec{\mathfrak{h}}(\vec{r},t) = \vec{\mathfrak{h}}_o e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,, \tag{V.162}$$

where $\vec{k}$ is the wave vector in the direction of wave propagation, $\hat{k}$, and

$$k = \frac{2\pi}{\lambda}\,. \tag{V.163}$$

Define

$$\vec{g} = Re(\vec{\mathfrak{g}}) \tag{V.164}$$

and

$$\vec{h} = Re(\vec{\mathfrak{h}})\,. \tag{V.165}$$

One can show that,

$$\nabla \to i\vec{k}\,, \tag{V.166}$$

$$\frac{\partial}{\partial t} \to -i\omega\,, \tag{V.167}$$

$$\frac{\partial^2}{\partial t^2} \to -(i\omega)^2 = -\omega^2\,. \tag{V.168}$$

Now, Equation V.159 becomes,

$$\left[(i\vec{k})\cdot(i\vec{k}) - \frac{1}{c_g^2}(-i\omega)^2\right]\vec{\mathfrak{g}} = 0\,, \tag{V.169}$$

$$\left[-k^2 - \frac{1}{c_g^2}(-\omega)^2\right]\vec{\mathfrak{g}} = 0\,, \tag{V.170}$$

$$-k^2 + \frac{\omega^2}{c_g^2} = 0 \tag{V.171}$$

so,

$$\omega^2 = c_g^2 k^2 \tag{V.172}$$

or

$$c_g^2 = \frac{\omega^2}{k^2} \tag{V.173}$$

then,

$$c_g = \frac{\omega}{k} \tag{V.174}$$

is a solution of the wave equation. And just because it's a solution of the wave equation doesn't mean it satisfies the Maxwell-Heaviside equations! So let's see what is needed to satisfy them. In complex form, the Maxwell-Heaviside equations in a vacuum are:

$$i\vec{k}\cdot\vec{\mathfrak{g}} = 0\,, \tag{V.175}$$

$$i\vec{k}\cdot\vec{\mathfrak{h}} = 0\,, \tag{V.176}$$

$$i\vec{k}\times\vec{\mathfrak{g}} = -\frac{1}{c_g}(-i\omega)\vec{\mathfrak{h}}\,, \tag{V.177}$$

$$i\vec{\boldsymbol{k}} \times \vec{\mathfrak{h}} = \frac{1}{c_g}(-i\omega)\vec{\mathfrak{g}}\,. \tag{V.178}$$

Then,

$$\vec{\boldsymbol{k}} \cdot \vec{\mathfrak{g}}_o e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)} = 0\,, \tag{V.179}$$

$$\vec{\boldsymbol{k}} \cdot \vec{\mathfrak{h}}_o e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)} = 0\,, \tag{V.180}$$

$$\vec{\boldsymbol{k}} \times \vec{\mathfrak{g}}_o e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)} = \frac{\omega}{c_g}\vec{\mathfrak{h}}_o e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)}\,, \tag{V.181}$$

$$\vec{\boldsymbol{k}} \times \vec{\mathfrak{h}}_o e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)} = -\frac{\omega}{c_g}\vec{\mathfrak{g}}_o e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}-\omega t)}\,. \tag{V.182}$$

Now, we also have,

$$\vec{\boldsymbol{k}} \cdot \vec{\mathfrak{g}}_o = 0\,, \tag{V.183}$$

$$\vec{\boldsymbol{k}} \cdot \vec{\mathfrak{h}}_o = 0\,, \tag{V.184}$$

$$\vec{\boldsymbol{k}} \times \vec{\mathfrak{g}}_o = \frac{\omega}{c_g}\vec{\mathfrak{h}}_o\,, \tag{V.185}$$

$$\vec{\boldsymbol{k}} \times \vec{\mathfrak{h}}_o = -\frac{\omega}{c_g}\vec{\mathfrak{g}}_o\,. \tag{V.186}$$

We can see from the two dot-product equations that the wave vector, $\vec{\boldsymbol{k}}$, must be orthonormal to both $\vec{\mathfrak{g}}_o$ and $\vec{\mathfrak{h}}_o$. Using the cross-product relations,

$$k\hat{\boldsymbol{k}} \times \vec{\mathfrak{g}}_o = \frac{\omega}{c_g}\vec{\mathfrak{h}}_o\,. \tag{V.187}$$

From Equation V.174,

$$k = \frac{\omega}{c_g} \tag{V.188}$$

then,

$$\hat{\boldsymbol{k}} \times \vec{\mathfrak{g}}_o = \vec{\mathfrak{h}}_o\,. \tag{V.189}$$

Thus, $\vec{\mathfrak{g}}_o$ is orthonormal to $\vec{\mathfrak{h}}_o$:

$$\vec{\mathfrak{g}}_o \perp \vec{\mathfrak{h}}_o\,. \tag{V.190}$$

Also, we have,

$$|\hat{\boldsymbol{k}} \times \vec{\mathfrak{g}}_o| = |\vec{\mathfrak{h}}_o|\,. \tag{V.191}$$

Since the waves are orthonormal,

$$|\hat{\boldsymbol{k}}||\vec{\mathfrak{g}}_o| \sin 90^\circ = |\vec{\mathfrak{h}}_o| \tag{V.192}$$

then,

$$|\vec{\mathfrak{g}}_o| = |\vec{\mathfrak{h}}_o|\,. \tag{V.193}$$

We can do the same procedure with the last cross-product equations but it will only lead to the same result of the previous cross-product product relation. Thus, to be a solution to the planar wave equations and to satisfy the Maxwell-Heaviside equations for gravitation,

$$c_g = \frac{\omega}{k} \tag{V.194}$$

and

$$\vec{\boldsymbol{k}} \perp \vec{\mathfrak{g}}_o \perp \vec{\mathfrak{h}}_o \,, \tag{V.195}$$

which is analogous to the propagation of electromagnetic waves. Figure V.1 depicts the propagation of a gravitational wave.

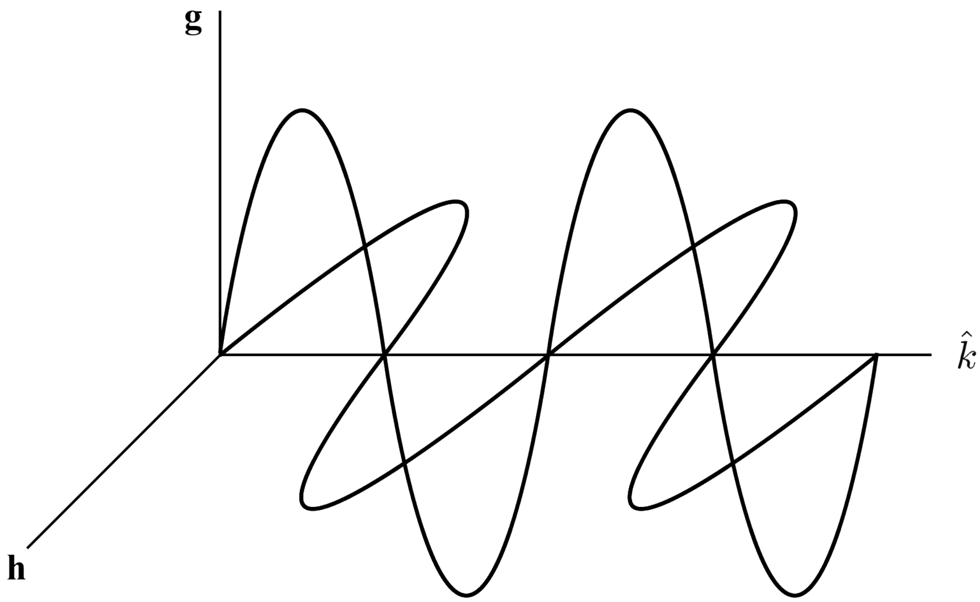

Figure V.1. A Gravitational Wave.

Gravitational and Cogravitational Wave Equations

Solving for the inhomogenous wave equations for the gravitational and cogravitational fields: Taking the curl of Equation V.105,

$$\nabla \times (\nabla \times \vec{\boldsymbol{g}}) = \nabla \times \left(-\frac{1}{c_g}\frac{\partial \vec{\boldsymbol{h}}}{\partial t} \right) = -\frac{1}{c_g}\frac{\partial}{\partial t}(\nabla \times \vec{\boldsymbol{h}}) \,, \tag{V.196}$$

and using Equation V.106,

$$\nabla \times (\nabla \times \vec{\boldsymbol{g}}) = -\frac{1}{c_g}\frac{\partial}{\partial t}\left[\frac{1}{c_g}\left(-4\pi G \vec{\boldsymbol{J}}_g + \frac{\partial \vec{\boldsymbol{g}}}{\partial t} \right) \right] . \tag{V.197}$$

Applying the vector identity:

$$\nabla \times (\nabla \times \vec{\boldsymbol{g}}) = \nabla(\nabla \cdot \vec{\boldsymbol{g}}) - \nabla^2 \vec{\boldsymbol{g}} \tag{V.198}$$

one obtains,

$$\nabla(\nabla \cdot \vec{\boldsymbol{g}}) - \nabla^2 \vec{\boldsymbol{g}} = \frac{4\pi G}{c_g^2}\frac{\partial \vec{\boldsymbol{J}}_g}{\partial t} - \frac{1}{c_g^2}\frac{\partial^2 \vec{\boldsymbol{g}}}{\partial t^2} \tag{V.199}$$

and from Equation V.103,

$$\nabla(-4\pi G\rho_g) - \nabla^2\vec{\boldsymbol{g}} = \frac{4\pi G}{c_g^2}\frac{\partial\vec{\boldsymbol{J}}_g}{\partial t} - \frac{1}{c_g^2}\frac{\partial^2\vec{\boldsymbol{g}}}{\partial t^2}\,. \tag{V.200}$$

Rearranging, we find,

$$\nabla^2\vec{\boldsymbol{g}} - \frac{1}{c_g^2}\frac{\partial^2\vec{\boldsymbol{g}}}{\partial t^2} = -4\pi G\nabla\rho_g - \frac{4\pi G}{c_g^2}\frac{\partial\vec{\boldsymbol{J}}_g}{\partial t}\,. \tag{V.201}$$

Finally, the wave equation for the gravitational field,

$$\nabla^2\vec{\boldsymbol{g}} - \frac{1}{c_g^2}\frac{\partial^2\vec{\boldsymbol{g}}}{\partial t^2} = -4\pi G\Big(\nabla\rho_g + \frac{1}{c_g^2}\frac{\partial\vec{\boldsymbol{J}}_g}{\partial t}\Big)\,. \tag{V.202}$$

The same procedure is implemented to obtain the wave equation for the cogravitational field, $\vec{\boldsymbol{h}}$. Taking the curl of Equation V.106,

$$\nabla\times(\nabla\times\vec{\boldsymbol{h}}) = \nabla\times\left[\frac{1}{c_g}\Big(-4\pi G\vec{\boldsymbol{J}}_g + \frac{\partial\vec{\boldsymbol{g}}}{\partial t}\Big)\right], \tag{V.203}$$

and applying the same vector identity as before,

$$\nabla(\nabla\cdot\vec{\boldsymbol{h}}) - \nabla^2\vec{\boldsymbol{h}} = \frac{-4\pi G}{c_g}(\nabla\times\vec{\boldsymbol{J}}_g)\frac{1}{c_g}\frac{\partial}{\partial t}(\nabla\times\vec{\boldsymbol{g}})\,. \tag{V.204}$$

From Equations V.104 and V.105, we obtain

$$\nabla(0) - \nabla^2\vec{\boldsymbol{h}} = \frac{-4\pi G}{c_g}(\nabla\times\vec{\boldsymbol{J}}_g) + \frac{1}{c_g}\frac{\partial}{\partial t}\Big(-\frac{1}{c_g}\frac{\partial\vec{\boldsymbol{h}}}{\partial t}\Big), \tag{V.205}$$

$$\nabla^2\vec{\boldsymbol{h}} = \frac{4\pi G}{c_g}(\nabla\times\vec{\boldsymbol{J}}_g) + \frac{1}{c_g^2}\frac{\partial^2\vec{\boldsymbol{h}}}{\partial t^2}\,. \tag{V.206}$$

And finally, the wave equation for the cogravitational field is,

$$\nabla^2\vec{\boldsymbol{h}} - \frac{1}{c_g^2}\frac{\partial^2\vec{\boldsymbol{h}}}{\partial t^2} = \frac{4\pi G}{c_g}(\nabla\times\vec{\boldsymbol{J}}_g)\,. \tag{V.207}$$

Appendix D: Derivation of the Retarded Potentials and Fields

Using the theory of retarded and advanced wave propagation (as done in Heald and Marion as well as Hills) it is possible to determine the retarded potentials and fields. Using Figure V.2 to represent the delay or retardation in the wave propagation, we can solve for the potentials and fields as a function of this retarded time as seen by the observer at point P.

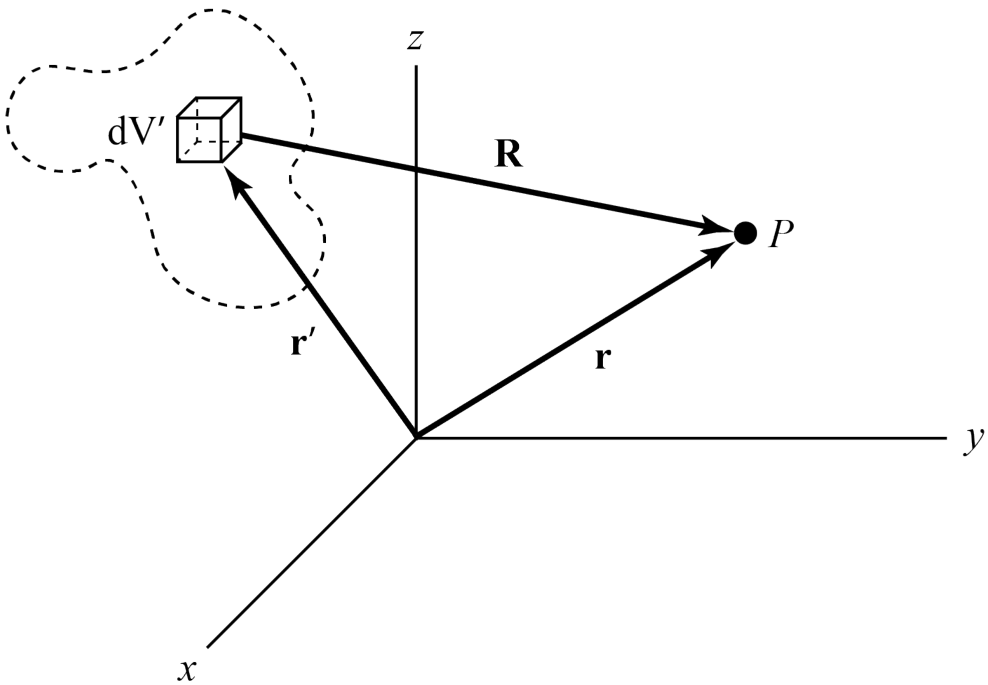

Figure V.2. Delay From Retardation.

Retarded Potentials for Gravitation

From Figure V.2,

$$\vec{\boldsymbol{R}} = \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}' \tag{V.208}$$

and

$$t_r = t' = t - \frac{R}{c_g} \,. \tag{V.209}$$

By analogy,

$$dm = \rho(\vec{\boldsymbol{r}}', t) dV' \equiv m(t) \tag{V.210}$$

and

$$d\phi_g(\vec{\boldsymbol{r}}, t) \equiv \phi_g(\vec{\boldsymbol{r}}, t) \,. \tag{V.211}$$

From Equation V.140,

$$\nabla^2 \phi_g - \frac{1}{c_g^2} \frac{\partial^2 \phi_g}{\partial t^2} = 4\pi G \rho_g \,, \tag{V.212}$$

we have,

$$\nabla^2 \phi_g(\vec{\boldsymbol{r}}, t) - \frac{1}{c_g^2} \frac{\partial^2 \phi_g(\vec{\boldsymbol{r}}, t)}{\partial t^2} = 4\pi G \rho_g(\vec{\boldsymbol{r}}, t) \tag{V.213}$$

where,

$$\rho_g(\vec{\boldsymbol{r}}, t) = m(t)\delta(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}') \,. \tag{V.214}$$

Let,

$$\vec{\boldsymbol{R}} = \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}' = \mathcal{X}\hat{\boldsymbol{e}}_x + \mathcal{Y}\hat{\boldsymbol{e}}_y + \mathcal{Z}\hat{\boldsymbol{e}}_z \tag{V.215}$$

and

$$\vec{\boldsymbol{r}} = x\hat{\boldsymbol{e}}_x + y\hat{\boldsymbol{e}}_y + z\hat{\boldsymbol{e}}_z \tag{V.216}$$

where

$$\mathcal{X} = x - x' \quad \& \quad \mathcal{Y} = y - y' \quad \& \quad \mathcal{Z} = z - z' \tag{V.217}$$

so,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \mathcal{X}}\frac{\partial \mathcal{X}}{\partial x} \tag{V.218}$$

then,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \mathcal{X}} \quad \& \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \mathcal{Y}} \quad \& \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial \mathcal{Z}} \tag{V.219}$$

therefore,

$$\nabla_r^2 = \nabla_R^2 \tag{V.220}$$

or

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial \mathcal{X}^2} + \frac{\partial^2}{\partial \mathcal{Y}^2} + \frac{\partial^2}{\partial \mathcal{Z}^2} \,. \tag{V.221}$$

Substituting, the wave equation becomes

$$\nabla_R^2 \phi_g(\vec{\boldsymbol{R}}, t) - \frac{1}{c_g^2}\frac{\partial^2 \phi_g(\vec{\boldsymbol{R}}, t)}{\partial t^2} = 4\pi G m(t)\delta(\vec{\boldsymbol{R}}) \,. \tag{V.222}$$

By symmetry and proper choice of origin,

$$\phi_g(\vec{\boldsymbol{R}}, t) = \phi_g(R, \not\theta, \not\varphi, t) = \phi_g(R, t) \tag{V.223}$$

becomes a function of R and t only, making the Laplacian is easier to solve! Thus, the wave equation becomes

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left[R^2 \frac{\partial}{\partial R}\phi_g(R, t)\right] - \frac{1}{c_g^2}\frac{\partial^2 \phi_g(R, t)}{\partial t^2} = 4\pi G m(t)\delta(\vec{\boldsymbol{R}}) \,. \tag{V.224}$$

Using the fact that

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial \phi_g}{\partial R}\right) = \frac{1}{R}\frac{\partial^2 (R\phi_g)}{\partial R^2} \tag{V.225}$$

then,

$$\frac{1}{R}\frac{\partial^2 (R\phi_g)}{\partial R^2} - \frac{1}{c_g^2}\frac{\partial^2 \phi_g}{\partial t^2} = 4\pi G m(t)\delta(\vec{\boldsymbol{R}}) \tag{V.226}$$

and for any $\vec{\boldsymbol{R}}$ except $\vec{\boldsymbol{R}} \equiv 0$,

$$\frac{1}{R}\frac{\partial^2 (R\phi_g)}{\partial R^2} - \frac{1}{c_g^2}\frac{\partial^2 \phi_g}{\partial t^2} = 0 \tag{V.227}$$

so,

$$\frac{\partial^2(R\phi_g)}{\partial R^2} - \frac{1}{c_g^2}\frac{\partial^2(R\phi_g)}{\partial t^2} = 0\,. \tag{V.228}$$

Let $u = R\phi_g$ such that,

$$\frac{\partial^2 u}{\partial R^2} - \frac{1}{c_g^2}\frac{\partial^2 u}{\partial t^2} = 0\,. \tag{V.229}$$

Let $\eta = t + \frac{R}{c_g}$ and $\zeta = t - \frac{R}{c_g}$ such that,

$$\frac{\partial}{\partial R} = \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial R} + \frac{\partial}{\partial \zeta}\frac{\partial \zeta}{\partial R} = \frac{1}{c_g}\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \zeta}\right) \tag{V.230}$$

and

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial \zeta}\frac{\partial \zeta}{\partial t} = \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \zeta}\,, \tag{V.231}$$

so now the wave equation for u becomes,

$$\left[\frac{1}{c_g^2}\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \zeta}\right)^2\right]u - \frac{1}{c_g^2}\left(\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \zeta}\right)^2 u = 0\,. \tag{V.232}$$

Expanding,

$$\frac{1}{-c_g^2}\left(\frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial \eta \partial \zeta} - \frac{\partial^2}{\partial \zeta \partial \eta} - \frac{\partial^2}{\partial \zeta^2}\right)u + \frac{1}{c_g^2}\left(\frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \eta \partial \zeta} + \frac{\partial^2}{\partial \zeta \partial \eta} + \frac{\partial^2}{\partial \zeta^2}\right)u = 0\,. \tag{V.233}$$

Assuming equal mixed partials (Clairaut's theorem),

$$\frac{\partial^2}{\partial \eta \partial \zeta} = \frac{\partial^2}{\partial \zeta \partial \eta} \tag{V.234}$$

then,

$$\frac{-4}{c_g^2}\frac{\partial^2 u}{\partial \eta \partial \zeta} = 0 \tag{V.235}$$

thus,

$$\frac{\partial^2 u}{\partial \eta \partial \zeta} = 0\,. \tag{V.236}$$

One might identify a function of only one variable:

$$\frac{\partial}{\partial \zeta}\left[\frac{\partial u}{\partial \eta}\right] = 0 \tag{V.237}$$

where

$$f(\eta) = \frac{\partial u}{\partial \eta} \tag{V.238}$$

such that,

$$\frac{\partial f(\eta)}{\partial \zeta} = 0\,. \tag{V.239}$$

Now we can write u as a function of two arbitrary functions (η,ζ):

$$u(\eta, \zeta) = \int^{\eta} f(\eta')d\eta' + g_1(\zeta) = g_2(\eta) + g_1(\zeta)\,. \tag{V.240}$$

Recall that, $u = R\phi_g$ & $\eta = t + \dfrac{R}{c_g}$ & $\zeta = t - \dfrac{R}{c_g}$, then,

$$u = R\phi_g = g_1\left(t - \frac{R}{c_g}\right) + g_2\left(t + \frac{R}{c_g}\right) \tag{V.241}$$

and so,

$$\phi_g(R,t) = \frac{1}{R}\left[g_1\left(t - \frac{R}{c_g}\right) + g_2\left(t + \frac{R}{c_g}\right)\right] \tag{V.242}$$

everywhere except $\vec{\boldsymbol{R}} = 0$ $\left(@\vec{\boldsymbol{R}} = 0, \phi_g \to \infty\right)$. Let

$$f_r(R,t) = g_1\left(t - \frac{R}{c_g}\right) \tag{V.243}$$

be the *retarded solution* and let

$$f_a(R,t) = g_2\left(t + \frac{R}{c_g}\right) \tag{V.244}$$

be the *advanced solution*. Then, the scalar potential becomes

$$\phi_g(R,t) = \frac{1}{R}(f_r + f_a)\,. \tag{V.245}$$

Given a form of ϕ_g, it is now possible to solve the left hand side (LHS) of the wave equation using integration over a volume and implementing the Divergence theorem, which states

$$\begin{aligned} \iiint_V \nabla^2\phi_g dV &= \iiint_V \nabla\cdot\nabla\phi_g dV \\ &= \oiint_S \nabla\phi_g\cdot\hat{\boldsymbol{n}} dS\,. \end{aligned} \tag{V.246}$$

Integrating the wave equation of ϕ_g (Equation V.213) over a volume yields

$$\iiint_V \left(\nabla^2\phi_g - \frac{1}{c_g^2}\frac{\partial^2\phi_g}{\partial t^2}\right)dV = \iiint_V 4\pi G m(t)\delta(\vec{\boldsymbol{R}})dV\,, \tag{V.247}$$

$$\iiint_V \nabla^2\phi_g dV - \frac{1}{c_g^2}\iiint_V \frac{\partial^2\phi_g}{\partial t^2}dV = 4\pi G m(t)\iiint_V \delta(\vec{\boldsymbol{R}})dV \tag{V.248}$$

and

$$\iiint_V \delta(\vec{\boldsymbol{R}})dV = 1 \tag{V.249}$$

then,

$$\iiint_V \nabla^2\phi_g dV - \frac{1}{c_g^2}\iiint_V \frac{\partial^2\phi_g}{\partial t^2}dV = 4\pi G m(t)\,. \tag{V.250}$$

Evaluating the first term on the LHS of Equation V.250 using the Divergence theorem:

$$\begin{aligned}
\iiint_V \nabla^2\phi_g dV &= \oiint_S \nabla\phi_g \cdot \widehat{\boldsymbol{R}} dS \\
&= \oiint_S \nabla\phi_g \cdot d\vec{\boldsymbol{a}} \\
&= \oiint_S \left[\frac{d\phi_g}{dR}\widehat{\boldsymbol{R}}\right] \cdot \widehat{\boldsymbol{R}} R^2 \sin\theta d\theta d\varphi \\
&= \oiint_S \frac{\partial\phi_g}{\partial R} R^2 \sin\theta d\theta d\varphi \\
&= \oiint_S \nabla\phi_g \cdot \widehat{\boldsymbol{R}} dS \\
\iiint_V \nabla^2\phi_g dV &= \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\left(\frac{\partial\phi_g}{\partial R}\right) R^2 \sin\theta d\theta d\varphi \,.
\end{aligned} \tag{V.251}$$

Continuing, we obtain

$$\begin{aligned}
\iiint_V \nabla^2\phi_g dV &= \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\left(\frac{\partial\phi_g}{\partial R}\right) R^2 \sin\theta d\theta d\varphi \\
&= 4\pi \frac{\partial\phi_g}{\partial R}\bigg|_{R=\epsilon} \epsilon^2 \\
&= 4\pi\epsilon^2 \Bigg\{ \frac{-f_r(t-\frac{R}{c_g})}{R^2} + \left(\frac{-1}{c_g}\right)\left[\frac{f_r'(t+\frac{R}{c_g})}{R}\right] + \frac{-f_a(t-\frac{R}{c_g})}{R^2} + \cdots \\
&\quad \cdots + \left(\frac{1}{c_g}\right)\left[\frac{f_a'(t+\frac{R}{c_g})}{R}\right]\Bigg\}\bigg|_{R=\epsilon} \\
&= 4\pi\epsilon^2 \Bigg\{ \frac{-f_r(t-\frac{\epsilon}{c_g})}{\epsilon^2} + \cancelto{0}{\left(\frac{-1}{c_g}\right)\left[\frac{f_r'(t+\frac{\epsilon}{c_g})}{\epsilon}\right]} + \frac{-f_a(t-\frac{\epsilon}{c_g})}{\epsilon^2} + \cdots \\
&\quad \cdots + \cancelto{0}{\left(\frac{1}{c_g}\right)\left[\frac{f_a'(t+\frac{\epsilon}{c_g})}{\epsilon}\right]}\Bigg\} \\
\iiint_V \nabla^2\phi_g dV &= 4\pi\big[-f_r(R=\epsilon,t) - f_a(R=\epsilon,t)\big] \,.
\end{aligned} \tag{V.252}$$

Letting $\epsilon \to 0$,

$$\iiint_V \nabla^2\phi_g dV = 4\pi\big[-f_r(t) - f_a(t)\big] \,. \tag{V.253}$$

Thus, for the first term,

$$\iiint_V \nabla^2\phi_g dV = -4\pi\big[f_r(t) + f_a(t)\big] \,. \tag{V.254}$$

Solving the second term on the LHS of Equation V.250,

$$\begin{aligned}
\iiint_V \frac{\partial^2 \phi_g}{\partial t^2} dV &= \iiint_V \left[\frac{f_r''(t-\frac{R}{c_g})}{R} + \frac{f_a''(t+\frac{R}{c_g})}{R}\right] dV \\
&= \int_{R=0}^{\epsilon}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{1}{R}\left[f_r''\left(t-\frac{R}{c_g}\right) + f_a''\left(t+\frac{R}{c_g}\right)\right] R^2 dR \sin\theta d\theta d\varphi \\
&= 4\pi \int_0^{\epsilon} \frac{1}{R}\left[f_r''\left(t-\frac{R}{c_g}\right) + f_a''\left(t+\frac{R}{c_g}\right)\right] R^2 dR \\
\iiint_V \frac{\partial^2 \phi_g}{\partial t^2} dV &= 4\pi \int_0^{\epsilon} \left[f_r''\left(t-\frac{R}{c_g}\right) + f_a''\left(t+\frac{R}{c_g}\right)\right] R dR \,.
\end{aligned} \tag{V.255}$$

For $\epsilon \to 0$, the above becomes,

$$\iiint_V \frac{\partial^2 \phi_g}{\partial t^2} dV = 4\pi \lim_{\epsilon\to 0} \int_0^{\epsilon} \left[f_r''\left(t-\frac{R}{c_g}\right) + f_a''\left(t+\frac{R}{c_g}\right)\right] R dR \,. \tag{V.256}$$

Thus,

$$\iiint_V \frac{\partial^2 \phi_g}{\partial t^2} dV = 0 \,. \tag{V.257}$$

Now, using Equation V.254 and V.257, Equation V.250 becomes,

$$-4\pi \left[f_r(t) + f_a(t)\right] - \frac{1}{c_g^2}(0) = 4\pi G m(t) \tag{V.258}$$

so,

$$m(t) = -\frac{1}{G}\left[f_r(t) + f_a(t)\right] . \tag{V.259}$$

Recalling,

$$\phi_g(R,t) = \frac{1}{R}\left[f_r\left(t-\frac{R}{c_g}\right) + f_r\left(t+\frac{R}{c_g}\right)\right] \tag{V.260}$$

and from Equation V.259,

$$m\left(t \pm \frac{R}{c_g}\right) = -\frac{1}{G}\left[f_r\left(t-\frac{R}{c_g}\right) + f_r\left(t+\frac{R}{c_g}\right)\right] \tag{V.261}$$

which, is really $m(t')$, however, the advanced solution is often discarded as it violates causality (though it is still a mathematical solution), so assuming

$$f_a(t') = 0 \,, \tag{V.262}$$

then the mass only depends on the retarded solution

$$m\left(t-\frac{R}{c_g}\right) = -\frac{1}{G} f_r\left(t-\frac{R}{c_g}\right) . \tag{V.263}$$

Rearranging,

$$f_r\left(t-\frac{R}{c_g}\right) = -Gm\left(t-\frac{R}{c_g}\right) \tag{V.264}$$

then,

$$\phi_g = \frac{f_r\left(t - \frac{R}{c_g}\right)}{R} = \frac{-Gm\left(t - \frac{R}{c_g}\right)}{R} \tag{V.265}$$

and

$$d\phi_g = \frac{-G\rho_g\left(\vec{\boldsymbol{r}}', t - \frac{R}{c_g}\right)}{R} dV' . \tag{V.266}$$

Finally,

$$\phi_g(\vec{\boldsymbol{r}}, t) = -G \iiint \frac{\rho_g\left(\vec{\boldsymbol{r}}', t - \frac{R}{c_g}\right)}{R} dV' \tag{V.267}$$

is the retarded potential of the scalar field ϕ_g. It has been shown that the retarded scalar potential, ϕ_g, has a solution of the form:

$$\phi_g(\vec{\boldsymbol{r}}, t) = -G \iiint \frac{\rho_g\left(\vec{\boldsymbol{r}}', t - \frac{R}{c_g}\right)}{R} dV' \tag{V.268}$$

and, using a similar method, one can show that the retarded vector potential has the form:

$$\vec{\boldsymbol{A}}_g(\vec{\boldsymbol{r}}, t) = -\frac{G}{c_g} \iiint \frac{\vec{\boldsymbol{J}}_g\left(\vec{\boldsymbol{r}}', t - \frac{R}{c_g}\right)}{R} dV' . \tag{V.269}$$

To simplify notation and to match with Heald and Marion, define the source-density evaluated at the retarded time as:

$$\rho_g(\vec{\boldsymbol{r}}', t - \frac{R}{c_g}) \equiv \left[\rho_g(\vec{\boldsymbol{r}}')\right] , \tag{V.270}$$

such that the bracketed notation will represent the quantity evaluated at the retarded time. Now, the retarded scalar and vector potentials are given by:

$$\phi_g(\vec{\boldsymbol{r}}, t) = -G \int_V \frac{\left[\rho_g(\vec{\boldsymbol{r}}')\right]}{R} dV' \tag{V.271}$$

$$\vec{\boldsymbol{A}}_g(\vec{\boldsymbol{r}}, t) = -\frac{G}{c_g} \int_V \frac{\left[\vec{\boldsymbol{J}}_g(\vec{\boldsymbol{r}}')\right]}{R} dV' , \tag{V.272}$$

which are identical to the retarded solutions for the potentials in Hills, given that the generalized coefficient

$$\beta = -\frac{4\pi G}{c_g} \tag{V.273}$$

for CGS units.

Retarded Fields for Gravitation

From Equations V.271 and V.272, again using bracket notation to indicate evaluation at the retarded time, $t' = t - {}^{R}/_{c_g}$,

$$\phi_g(\vec{\boldsymbol{r}}, t) = -G \int_V \frac{\left[\rho_g(\vec{\boldsymbol{r}}')\right]}{R} dV' , \tag{V.274}$$

$$\vec{A}_g(\vec{r},t) = \frac{-G}{c_g}\int_V \frac{\left[\vec{J}_g(\vec{r}')\right]}{R} dV' \,. \tag{V.275}$$

Recalling Equation V.127,

$$\vec{g} = -\nabla\phi_g - \frac{1}{c_g}\frac{\partial \vec{A}_g}{\partial t}\,, \tag{V.276}$$

where

$$\begin{aligned}\nabla\phi_g &= \nabla\left(\frac{-G}{R}[\rho_g]\right)\\ &= -G\left(\left(\frac{1}{R}\right)\nabla[\rho_g] + [\rho_g]\nabla\left(\frac{1}{R}\right)\right).\end{aligned} \tag{V.277}$$

The second term is easily found:

$$\nabla\left(\frac{1}{R}\right) = \frac{-\hat{\boldsymbol{e}}_R}{R^2} \tag{V.278}$$

but the first term, not so much.. (implicitly defined in time).. Following from Equation (8.25) in Heald and Marion,

$$\begin{aligned}\nabla\Big[\rho_g(\vec{r}',t')\Big] &= \left[\frac{\partial\rho_g}{\partial t}\right]\nabla\left(t - \frac{|\vec{r}-\vec{r}'|}{c_g}\right)\\ &= \frac{-1}{c_g}\left[\frac{\partial\rho_g}{\partial t}\right]\hat{\boldsymbol{e}}_R\,.\end{aligned} \tag{V.279}$$

Plugging all of the pieces into Equation V.276 [V.127] and integrating yields,

$$\begin{aligned}\vec{g}(\vec{r},t) = -&\int_V\left(-G\left(\frac{1}{R}\frac{-1}{c_g}\left[\frac{\partial\rho_g}{\partial t}\right]\hat{\boldsymbol{e}}_R + [\rho_g]\left(\frac{-\hat{\boldsymbol{e}}_R}{R^2}\right)\right)\right)dV'\cdots\\ &\cdots - \frac{1}{c_g}\left(\frac{-G}{c_g}\int_V\frac{\left[\frac{\partial\vec{J}}{\partial t}\right]}{R}dV'\right).\end{aligned} \tag{V.280}$$

Simplifying, we have

$$\vec{g}(\vec{r},t) = -G\int_V\left(\frac{[\rho_g]\hat{\boldsymbol{e}}_R}{R^2} + \frac{\left[\frac{\partial\rho_g}{\partial t}\right]\hat{\boldsymbol{e}}_R}{c_g R} - \frac{\left[\frac{\partial\vec{J}_g}{\partial t}\right]}{c_g^2 R}\right)dV'\,, \tag{V.281}$$

which is analogous to the generalized Coulomb-Faraday Law. And,

$$\vec{h} = \nabla\times\vec{A}_g \tag{V.282}$$

where,

$$\begin{aligned}\nabla\times\vec{A}_g &= \nabla\times\frac{[\vec{J}_g]}{R}\\ &= \frac{1}{R}\nabla\times[\vec{J}_g] - [\vec{J}_g]\times\nabla\left(\frac{1}{R}\right)\end{aligned} \tag{V.283}$$

and

$$\nabla \times [\vec{\boldsymbol{J}}_g] = -\left[\frac{\partial \vec{\boldsymbol{J}}_g}{\partial t}\right] \times \nabla\left(t - \frac{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}'|}{c_g}\right) = \frac{1}{c_g}\left[\frac{\partial \vec{\boldsymbol{J}}_g}{\partial t}\right] \times \hat{\boldsymbol{e}}_R\,. \tag{V.284}$$

So, plugging everything in, we find

$$\vec{\boldsymbol{h}} = \frac{-G}{c_g}\int_V \left(\frac{1}{R}\frac{1}{c_g}\left(\left[\frac{\partial \vec{\boldsymbol{J}}_g}{\partial t}\right] \times \hat{\boldsymbol{e}}_R\right) - [\vec{\boldsymbol{J}}_g] \times \frac{-\hat{\boldsymbol{e}}_R}{R^2}\right) dV'\,. \tag{V.285}$$

Apart from the brief and messy work, it was shown that the generalized gravitational field is:

$$\vec{\boldsymbol{g}}(\vec{\boldsymbol{r}}, t) = -G\int_V \left(\frac{[\rho_g]\hat{\boldsymbol{e}}_R}{R^2} + \frac{\left[\frac{\partial \rho_g}{\partial t}\right]\hat{\boldsymbol{e}}_R}{c_g R} - \frac{\left[\frac{\partial \vec{\boldsymbol{J}}_g}{\partial t}\right]}{c_g^2 R}\right) dV'\,. \tag{V.286}$$

This expression is analogous to the generalized Coulomb-Faraday Law for electromagnetism. The first and second terms can be traced to Coulomb's law (here, Newton's Law for mass) and represents the primary source of the gravitational field – mass. The third term is from Faraday's law and represents the secondary source of the gravitational field – momentum. The generalized cogravitational field is:

$$\vec{\boldsymbol{h}}(\vec{\boldsymbol{r}}, t) = \frac{-G}{c_g}\int_V \left(\frac{[\vec{\boldsymbol{J}}_g] \times \hat{\boldsymbol{e}}_R}{R^2} + \frac{\left[\frac{\partial \vec{\boldsymbol{J}}_g}{\partial t}\right] \times \hat{\boldsymbol{e}}_R}{c_g R}\right) dV'\,. \tag{V.287}$$

This expression is analogous to the generalized Biot-Savart Law for electromagnetism. The first term represents the primary source of the cogravitational field - momentum. The second term is the secondary source - mass. This is not immediately obvious as the source of Maxwell induction does not appear. However, the partial-time derivative terms serve as retardation "proxies" for these sources [4]. From these generalized equations, it is clear that both fields are generated by both sources. Although it is common to express one field as being induced by the other, they are simultaneously generated by both sources when expressed as retarded fields.

Appendix E: Derivation of the Liénard–Wiechert Potentials and Fields

Liénard–Wiechert Potentials for Gravitation

Consider Figure V.2 in the case of a moving point mass as shown in Figure V.3 below.

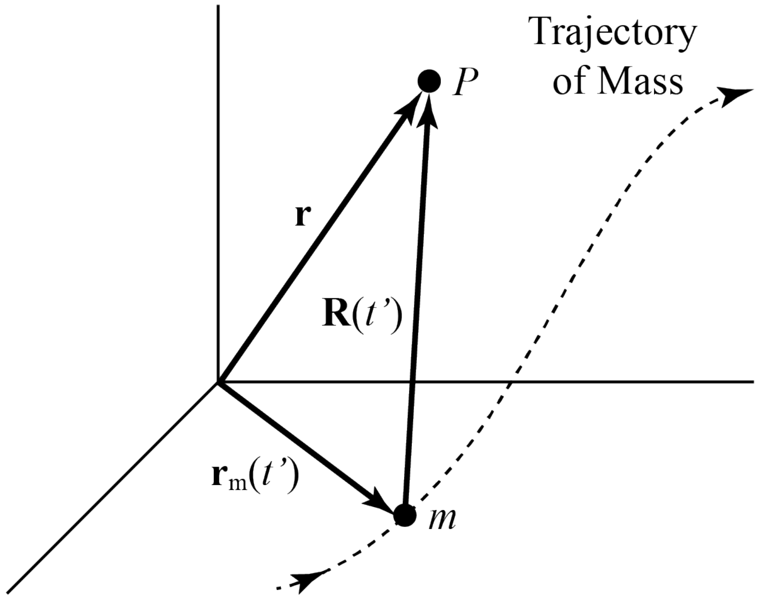

Figure V.3. Geometry of Moving Point Mass.

From Equation V.214, we know that we can use the Dirac delta expression to write,

$$\rho_g(\vec{\boldsymbol{r}}, t') = m\delta(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_m(t'))\,. \tag{V.288}$$

So it follows from $\vec{\boldsymbol{J}} = \rho\vec{\boldsymbol{v}}$ that

$$\vec{\boldsymbol{J}}_g(\vec{\boldsymbol{r}}, t') = m\vec{\boldsymbol{v}}_m(t')\delta(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_m(t'))\,, \tag{V.289}$$

and it has been determined that the retarded scalar potential is

$$\phi_g(\vec{\boldsymbol{r}}, t) = -G\iiint \frac{\rho_g\Big(\vec{\boldsymbol{r}}', t - \frac{R}{c_g}\Big)}{R} dV' \tag{V.290}$$

which becomes,

$$\begin{aligned}\phi_g(\vec{\boldsymbol{r}}, t) &= -G\iiint\int_{t'=-\infty}^{t'=\infty} \frac{\rho_g(\vec{\boldsymbol{r}}', t')}{R}\delta\bigg(t' - \Big(t - \frac{R}{c_g}\Big)\bigg) dV' dt' \\ &= -G\iiint\int_{t'=-\infty}^{t'=\infty} \frac{m\delta(\vec{\boldsymbol{r}}' - \vec{\boldsymbol{r}}_m(t'))\delta\Big(t' - \Big(t - \frac{R}{c_g}\Big)\Big)}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_m(t')|} dV' dt'\,.\end{aligned} \tag{V.291}$$

If the time integration is done first, the process will cycle backwards and repeat. So, integrating over space first,

$$\phi_g(\vec{\boldsymbol{r}}, t) = -G\int_{t'=-\infty}^{t'=\infty} \frac{m\delta\Big(t' - (t - \frac{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_m(t')|}{c_g})\Big)}{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_m(t')|} dt'\,. \tag{V.292}$$

Using a change of variable,

$$\zeta = t' - \left(t - \frac{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_m(t')|}{c_g}\right) \tag{V.293}$$

then,

$$d\zeta = dt' - dt + \frac{1}{c_g} d\big|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_m(t')\big| \,. \tag{V.294}$$

From page 264 in Heald & Marion, $dt = 0$ because t is the fixed time of observation, thus

$$d\zeta = dt' \left\{1 + \frac{1}{c_g}\frac{d}{dt'}|\vec{r} - \vec{\boldsymbol{r}}_m(t')|\right\} \tag{V.295}$$

or

$$d\zeta = dt' + \frac{1}{c_g}\frac{\partial R}{\partial t'} dt' \,. \tag{V.296}$$

The term $\frac{\partial R}{\partial t'}$ can obtained as follows:

$$\begin{aligned}\frac{\partial R}{\partial t'} &= \frac{\partial}{\partial t'}\sqrt{\left(x - x_m(t')\right)^2 + \left((y - y_m(t')\right)^2 + \left((z - z_m(t')\right)^2} \\ &= \frac{1}{\cancel{2}}\left\{\frac{\cancel{2}(x - x_m(t'))\left(\frac{-dx_m(t')}{dt'}\right)}{R} + \frac{\cancel{2}(y - y_m(t'))\left(\frac{-dy_m(t')}{dt'}\right)}{R} \cdots\right. \\ &\quad \left. \cdots + \frac{\cancel{2}(z - z_m(t'))\left(\frac{-dz_m(t')}{dt'}\right)}{R}\right\} \\ \frac{\partial R}{\partial t'} &= -\frac{1}{R}\left\{(x - x_m(t'))\frac{dx_m(t')}{dt'} + (y - y_m(t'))\frac{dy_m(t')}{dt'} + (z - z_m(t'))\frac{dz_m(t')}{dt'}\right\}.\end{aligned} \tag{V.297}$$

Recognizing the term in braces as the dot product between the position vector, $\vec{\boldsymbol{R}}$, and the mass velocity vector $\vec{\boldsymbol{v}}_m$, both of which depend on the retarded time t',

$$\begin{aligned}\frac{\partial R}{\partial t'} &= -\frac{\vec{\boldsymbol{R}}}{R}\cdot\vec{\boldsymbol{v}}_m(t') \\ &= -\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{v}}_m(t')\,.\end{aligned} \tag{V.298}$$

Now,

$$d\zeta = \left(1 - \frac{1}{c_g}\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{v}}_m(t')\right)dt'\,. \tag{V.299}$$

The retarded scalar potential in Equation V.290 [V.271] becomes

$$\phi_g(\vec{\boldsymbol{r}}, t) = -G\int_{\zeta=-\infty}^{\infty}\frac{m}{R}\frac{\delta(\zeta)d\zeta}{(1 - \frac{1}{c_g}\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{v}}_m(t'))}\,. \tag{V.300}$$

Let

$$\vec{\boldsymbol{\beta}}(t') = \frac{\vec{\boldsymbol{v}}_m}{c_g} \tag{V.301}$$

so,

$$\phi_g(\vec{\boldsymbol{r}}, t) = -G \int_{-\infty}^{\infty} \frac{m\delta(\zeta)d\zeta}{R(1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}})}\,, \tag{V.302}$$

$$\phi(\vec{\boldsymbol{r}}, t) = -\frac{Gm}{R(1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}})}\bigg|_{\zeta=0}\,. \tag{V.303}$$

Recall, $0 = t' - (t - {}^{R}\!/_{c_g})$. Thus,

$$\phi_g(\vec{\boldsymbol{r}}, t) = -\frac{Gm}{R(1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}})}\bigg|_{t'=t-\frac{R}{c_g}}\,. \tag{V.304}$$

And similarly, one can show for the retarded vector potential,

$$\vec{\boldsymbol{A}}_g(\vec{\boldsymbol{r}}, t) = -\frac{Gm\vec{\boldsymbol{\beta}}}{R(1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}})}\bigg|_{t'=t-\frac{R}{c_g}}\,. \tag{V.305}$$

Using the bracket notation to indicate evaluation at the retarded time, the Liénard–Wiechert scalar and vector potentials are shown to be,

$$\phi_g(\vec{\boldsymbol{r}}, t) = -\frac{Gm}{\left[R(1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}})\right]}\,. \tag{V.306}$$

And similarly,

$$\vec{\boldsymbol{A}}_g(\vec{\boldsymbol{r}}, t) = -\frac{Gm[\vec{\boldsymbol{\beta}}]}{\left[R(1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}})\right]}\,. \tag{V.307}$$

Note:

$$\vec{\boldsymbol{A}}_g(\vec{\boldsymbol{r}}, t) = [\vec{\boldsymbol{\beta}}]\phi_g(\vec{\boldsymbol{r}}, t)\,. \tag{V.308}$$

Liénard–Wiechert Fields for Gravitation

Now that the Liénard–Wiechert potentials are known for gravitation, it is possible to determine the fields. From Equations V.121 and V.127, we're able to expand the terms and solve for the fields but several derivative terms need to be found first, as will be seen. From Equation V.127,

$$\vec{\boldsymbol{g}} = -\nabla\phi_g - \frac{1}{c_g}\frac{\partial\vec{\boldsymbol{A}}_g}{\partial t}\,. \tag{V.309}$$

Plugging in Equation V.308,

$$\vec{\boldsymbol{g}} = -\nabla\phi_g - \frac{1}{c_g}\frac{\partial}{\partial t}(\vec{\boldsymbol{\beta}}\phi_g) \tag{V.310}$$

and expanding,

$$\vec{\boldsymbol{g}} = -\nabla\phi_g - \frac{1}{c_g}\left(\frac{\partial\vec{\boldsymbol{\beta}}}{\partial t}\right)\phi_g - \frac{1}{c_g}\left(\frac{\partial\phi_g}{\partial t}\right)\vec{\boldsymbol{\beta}} \tag{V.311}$$

more succinctly,

$$\vec{\boldsymbol{g}} = -\nabla\phi_g - \frac{\phi_g}{c_g}\left(\frac{\partial\vec{\boldsymbol{\beta}}}{\partial t}\right) - \frac{\vec{\boldsymbol{\beta}}}{c_g}\left(\frac{\partial\phi_g}{\partial t}\right). \tag{V.312}$$

And from Equation V.121,

$$\vec{\boldsymbol{h}} = \nabla \times \vec{\boldsymbol{A}}_g = \nabla \times (\vec{\boldsymbol{\beta}}\phi_g) \tag{V.313}$$

expanding,

$$\vec{\boldsymbol{h}} = \phi_g(\nabla \times \vec{\boldsymbol{\beta}}) - \vec{\boldsymbol{\beta}} \times \nabla\phi_g\,. \tag{V.314}$$

After expanding the $\vec{\boldsymbol{g}}$ and $\vec{\boldsymbol{h}}$ fields, it is apparent that the derivatives aforementioned are now needed. Since the retarded time, t', is implicitly defined,

$$t' = t - \frac{|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_m(t')|}{c_g}\,, \tag{V.315}$$

we need to solve the implicit partial time derivatives:

$$\begin{aligned} \frac{\partial t'}{\partial t} &= \frac{\partial}{\partial t}\left(t - \frac{R}{c_g}\right) \\ &= 1 - \frac{1}{c_g}\frac{\partial R}{\partial t'}\frac{\partial t'}{\partial t} \end{aligned} \tag{V.316}$$

and $\dfrac{\partial R}{\partial t'}$ was found previously to be (Equation V.298),

$$\frac{\partial R}{\partial t'} = -\widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{v}}_m(t')\,. \tag{V.317}$$

So plugging in Equation V.317 [V.298] in to Equation V.316 yields,

$$\frac{\partial t'}{\partial t} = 1 - \frac{1}{c_g}\big(- \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{v}}_m(t')\big)\frac{\partial t'}{\partial t}\,. \tag{V.318}$$

Rearranging to solve for $\dfrac{\partial t'}{\partial t}$,

$$\frac{\partial t'}{\partial t}\left[1 - \frac{\widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{v}}_m(t')}{c_g}\right] = 1 \tag{V.319}$$

$$\begin{aligned} \frac{\partial t'}{\partial t} &= \frac{1}{\left[1 - \dfrac{\widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{v}}_m(t')}{c_g}\right]} \\ &= \frac{1}{1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}} \end{aligned} \tag{V.320}$$

then,

$$\frac{\partial t'}{\partial t} = \frac{1}{1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}}\,. \tag{V.321}$$

Note: This term, which naturally arises in this procedure, relates directly to special relativity and will show up in the denominator of the fields! Now, computing the other time derivatives,

$$\begin{aligned}\frac{\partial R}{\partial t} &= \frac{1}{2R}\frac{\partial R^2}{\partial t} \\ &= \frac{1}{2R}\frac{\partial}{\partial t}(\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{R}}) \\ &= \frac{1}{\cancel{2}R}\left(\cancel{2}\vec{\boldsymbol{R}}\cdot\frac{\partial\vec{\boldsymbol{R}}}{\partial t}\right) \\ \frac{\partial R}{\partial t} &= \widehat{\boldsymbol{R}}\cdot\frac{\partial\vec{\boldsymbol{R}}}{\partial t}\end{aligned} \tag{V.322}$$

so,

$$\frac{\partial R}{\partial t} = \widehat{\boldsymbol{R}}\cdot\frac{\partial\vec{\boldsymbol{R}}}{\partial t} \tag{V.323}$$

where,

$$\frac{\partial\vec{\boldsymbol{R}}}{\partial t} = \frac{\partial\vec{\boldsymbol{R}}}{\partial t'}\frac{\partial t'}{\partial t} \tag{V.324}$$

then,

$$\begin{aligned}\frac{\partial\vec{\boldsymbol{R}}}{\partial t} &= \frac{\partial}{\partial t'}\left[\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}_m(t')\right]\frac{\partial t'}{\partial t} \\ &= -\vec{\boldsymbol{v}}_m(t')\frac{\partial t'}{\partial t} \\ \frac{\partial\vec{\boldsymbol{R}}}{\partial t} &= -\frac{\vec{\boldsymbol{v}}_m(t')}{1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}}\,.\end{aligned} \tag{V.325}$$

Using Equation V.325 in Equation V.323,

$$\frac{\partial R}{\partial t} = \widehat{\boldsymbol{R}}\cdot\left[-\frac{\vec{\boldsymbol{v}}_m(t')}{1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}}\right]. \tag{V.326}$$

Thus,

$$\frac{\partial R}{\partial t} = -\frac{\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{v}}_m(t')}{1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}} \tag{V.327}$$

and

$$\begin{aligned}\frac{\partial\vec{\boldsymbol{\beta}}}{\partial t} &= \frac{\partial\vec{\boldsymbol{\beta}}}{\partial t'}\frac{\partial t'}{\partial t} \\ &= \dot{\vec{\boldsymbol{\beta}}}\frac{\partial t'}{\partial t}\end{aligned} \tag{V.328}$$

so,

$$\frac{\partial\vec{\boldsymbol{\beta}}}{\partial t} = \frac{\dot{\vec{\boldsymbol{\beta}}}}{1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}}\,. \tag{V.329}$$

As well as $\frac{\partial \phi_g}{\partial t}$, thus, calculating,

$$\begin{aligned}\frac{\partial \phi_g}{\partial t} &= \frac{\partial}{\partial t}\left[\frac{-Gm}{R(1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\right] \\ &= -Gm\frac{\partial}{\partial t}\left(\frac{1}{R-\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}}\right) \\ &= -Gm\left[\frac{-1}{(R-\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^2}\frac{\partial}{\partial t}(R-\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})\right] \\ \frac{\partial \phi_g}{\partial t} &= \frac{Gm}{R^2(1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^2}\left[\frac{\partial R}{\partial t}-\frac{\partial}{\partial t}(\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})\right].\end{aligned} \tag{V.330}$$

Since $\frac{\partial R}{\partial t}$ is given in Equation V.327 we need to determine the second term.

$$\frac{\partial}{\partial t}(\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}) = \frac{\partial\vec{\boldsymbol{R}}}{\partial t}\cdot\vec{\boldsymbol{\beta}}+\vec{\boldsymbol{R}}\cdot\frac{\partial\vec{\boldsymbol{\beta}}}{\partial t} \tag{V.331}$$

Using Equation V.325 and Equation V.329,

$$\begin{aligned}\frac{\partial}{\partial t}(\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}) &= \left(\frac{-\vec{\boldsymbol{v}}_m(t')}{1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}}\right)\cdot\vec{\boldsymbol{\beta}}+\vec{\boldsymbol{R}}\cdot\left(\frac{\dot{\vec{\boldsymbol{\beta}}}}{1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}}\right) \\ &= \frac{1}{1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}}\left(\vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}-\vec{\boldsymbol{v}}_m(t')\cdot\vec{\boldsymbol{\beta}}\right).\end{aligned} \tag{V.332}$$

Plugging in Equation V.327 and Equation V.332 in to Equation V.330 yields,

$$\frac{\partial \phi_g}{\partial t} = \frac{Gm}{R^2(1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^2}\left(\left[\frac{-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{v}}_m(t')}{1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}}\right]-\left[\frac{1}{1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}}\left(\vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}-\vec{\boldsymbol{v}}_m(t')\cdot\vec{\boldsymbol{\beta}}\right)\right]\right). \tag{V.333}$$

Thus,

$$\frac{\partial \phi_g}{\partial t} = \frac{Gm}{R^2(1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[\vec{\boldsymbol{v}}_m(t')\cdot\vec{\boldsymbol{\beta}}-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{v}}_m(t')-\vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}\right]. \tag{V.334}$$

Now, computing the spatial partial derivatives,

$$\begin{aligned}\frac{\partial R}{\partial x_j} &= \frac{1}{2R}\frac{\partial R^2}{\partial x_j} \\ &= \frac{1}{2R}\frac{\partial}{\partial x_j}(\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{R}}) \\ &= \frac{1}{\cancel{2}R}\left(\cancel{2}\vec{\boldsymbol{R}}\cdot\frac{\partial\vec{\boldsymbol{R}}}{\partial x_j}\right) \\ &= \hat{\boldsymbol{R}}\cdot\frac{\partial\vec{\boldsymbol{R}}}{\partial x_j}\frac{\partial R}{\partial x_j} \\ \frac{\partial R}{\partial x_j} &= \hat{\boldsymbol{R}}\cdot\frac{\partial}{\partial x_j}\left[x_i\hat{\boldsymbol{e}}_i-\vec{\boldsymbol{r}}_m(t')\right]\end{aligned} \tag{V.335}$$

and evaluating the partial derivative:

$$\frac{\partial R}{\partial x_j} = \widehat{\boldsymbol{R}} \cdot \left[\delta_{ij}\hat{\boldsymbol{e}}_i - \dot{\vec{\boldsymbol{r}}}_m \frac{\partial t'}{\partial x_j}\right]. \tag{V.336}$$

Determining $\dfrac{\partial t'}{\partial x_j}$,

$$\begin{aligned}\frac{\partial t'}{\partial x_j} &= \frac{\partial}{\partial x_j}\left(t - \frac{R}{c_g}\right) \\ &= -\frac{1}{c_g}\frac{\partial R}{\partial x_j}.\end{aligned} \tag{V.337}$$

Now, Equation V.336 becomes

$$\begin{aligned}\frac{\partial R}{\partial x_j} &= \widehat{\boldsymbol{R}} \cdot \left[\delta_{ij}\hat{\boldsymbol{e}}_i - \dot{\vec{\boldsymbol{r}}}_m\left(-\frac{1}{c_g}\frac{\partial R}{\partial x_j}\right)\right] \\ &= \widehat{\boldsymbol{R}} \cdot \left(\hat{\boldsymbol{e}}_j + \frac{\vec{\boldsymbol{v}}_m}{c_g}\frac{\partial R}{\partial x_j}\right) \\ \frac{\partial R}{\partial x_j} &= \widehat{\boldsymbol{R}} \cdot \hat{\boldsymbol{e}}_j + \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}\frac{\partial R}{\partial x_j}.\end{aligned} \tag{V.338}$$

Rearranging to solve for $\dfrac{\partial R}{\partial x_j}$,

$$\frac{\partial R}{\partial x_j}(1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}) = \widehat{\boldsymbol{R}} \cdot \hat{\boldsymbol{e}}_j. \tag{V.339}$$

So, the previous equation becomes,

$$\frac{\partial R}{\partial x_j} = \frac{\widehat{\boldsymbol{R}} \cdot \hat{\boldsymbol{e}}_j}{1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}}. \tag{V.340}$$

And plugging Equation V.340 in Equation V.337 yields,

$$\begin{aligned}\frac{\partial t'}{\partial x_j} &= -\frac{1}{c_g}\left(\frac{\widehat{\boldsymbol{R}} \cdot \hat{\boldsymbol{e}}_j}{1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}}\right) \\ &= -\frac{\widehat{\boldsymbol{R}} \cdot \hat{\boldsymbol{e}}_j}{c_g(1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}})}.\end{aligned} \tag{V.341}$$

Now for the terms involving the del-operator (∇),

$$\begin{aligned}\nabla R &= \frac{\partial R}{\partial x_j}\hat{\boldsymbol{e}}_j \\ &= \left(\frac{\widehat{\boldsymbol{R}} \cdot \hat{\boldsymbol{e}}_j}{1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}}\right)\hat{\boldsymbol{e}}_j.\end{aligned} \tag{V.342}$$

So,

$$\nabla R = \frac{\widehat{\boldsymbol{R}}}{1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}} \tag{V.343}$$

and

$$\begin{aligned}
\nabla_i R_j &= \frac{\partial}{\partial x_i}(\vec{\boldsymbol{R}})_j \\
&= \frac{\partial}{\partial x_i}(\vec{\boldsymbol{r}}_j - \vec{\boldsymbol{r}}_{mj}) \\
&= \frac{\partial}{\partial x_i}(\vec{\boldsymbol{x}}_j - \vec{\boldsymbol{x}}_{mj}) \\
&= \delta_{ij} - \frac{\partial x_{mj}}{\partial x_i} \\
\nabla_i R_j &= \delta_{ij} - \frac{\partial x_{mj}}{\partial t'}\frac{\partial t'}{\partial x_i}\,.
\end{aligned} \tag{V.344}$$

Plugging in Equation V.341 to the above,

$$\begin{aligned}
\nabla_i R_j &= \delta_{ij} - v_{mj}(t')\left(\frac{-\widehat{\boldsymbol{R}}_i}{c_g(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\right) \\
&= \delta_{ij} + \frac{\widehat{\boldsymbol{R}}_i v_{mj}(t')}{c_g(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}
\end{aligned} \tag{V.345}$$

so,

$$\nabla_i R_j = \delta_{ij} + \frac{\widehat{\boldsymbol{R}}_i\beta_j}{1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}}\,. \tag{V.346}$$

Similarly,

$$\begin{aligned}
\nabla_i\beta_j &= \frac{\partial\beta_j}{\partial x_i} \\
&= \frac{\partial\beta_j}{\partial t'}\frac{\partial t'}{\partial x_i} \\
\nabla_i\beta_j &= \dot{\beta}_j\frac{\partial t'}{\partial x_i}
\end{aligned} \tag{V.347}$$

thus,

$$\begin{aligned}
\nabla_i\beta_j &= \dot{\beta}_j\left(-\frac{1}{c_g}\frac{\widehat{\boldsymbol{R}}_i}{(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\right) \\
&= \frac{-\widehat{\boldsymbol{R}}_i\dot{\beta}_j}{c_g(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\,.
\end{aligned} \tag{V.348}$$

We also need $\nabla\phi_g$,

$$\begin{aligned}
\nabla\phi_g &= \nabla\left[\frac{-Gm}{R-\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}}\right] \\
&= \frac{Gm}{(R-\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^2}\nabla(R-\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}) \\
\nabla\phi_g &= \frac{Gm}{(R-\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^2}\Big[\nabla R - \nabla(\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})\Big]
\end{aligned} \tag{V.349}$$

which, using the Einstein Summation Convention, becomes

$$\nabla_i\phi_g = \frac{Gm}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^2}\Big[\nabla_i R - R_j(\nabla_i\beta_j) - \beta_j(\nabla_i R_j)\Big]\,. \tag{V.350}$$

Partial Time Derivatives

Rewriting the previous partial time derivatives for succinctness,

$$\frac{\partial R}{\partial t'} = -\widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{v}}_m(t')\,, \tag{V.351}$$

$$\frac{\partial t'}{\partial t} = \frac{1}{1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}}\,, \tag{V.352}$$

$$\frac{\partial R}{\partial t} = -\frac{\widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{v}}_m(t')}{1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}}\,, \tag{V.353}$$

$$\frac{\partial \vec{\boldsymbol{R}}}{\partial t} = -\frac{\vec{\boldsymbol{v}}_m(t')}{1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}}\,, \tag{V.354}$$

$$\frac{\partial \vec{\boldsymbol{\beta}}}{\partial t} = \frac{\dot{\vec{\boldsymbol{\beta}}}}{1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}}\,, \tag{V.355}$$

$$\frac{\partial \phi_g}{\partial t} = \frac{Gm}{R^2\big(1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}\big)^3}\Big[\vec{\boldsymbol{v}}_m(t') \cdot \vec{\boldsymbol{\beta}} - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{v}}_m(t') - \vec{\boldsymbol{R}} \cdot \dot{\vec{\boldsymbol{\beta}}}\Big]\,. \tag{V.356}$$

Partial Spatial Derivatives

$$\frac{\partial R}{\partial x_j} = \frac{\widehat{\boldsymbol{R}} \cdot \hat{\boldsymbol{e}}_j}{1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}} \tag{V.357}$$

$$\frac{\partial t'}{\partial x_j} = -\frac{\widehat{\boldsymbol{R}} \cdot \hat{\boldsymbol{e}}_j}{c_g\big(1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}\big)} \tag{V.358}$$

$$\nabla_i R = \frac{\widehat{\boldsymbol{R}}_i}{1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}} \tag{V.359}$$

$$\nabla_i R_j = \delta_{ij} + \frac{\widehat{\boldsymbol{R}}_i \beta_j}{1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}} \tag{V.360}$$

$$\nabla_i \beta_j = \frac{-\widehat{\boldsymbol{R}}_i \dot{\beta}_j}{c_g\big(1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}\big)} \tag{V.361}$$

$$\nabla_i \phi_g = \frac{Gm}{R^2\big(1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}\big)^2}\Big[\nabla_i R - R_j\big(\nabla_i \beta_j\big) - \beta_j\big(\nabla_i R_j\big)\Big] \tag{V.362}$$

Gravitational Field

Now, it is possible to evaluate Equation V.312 [from V.127],

$$\vec{\boldsymbol{g}} = -\nabla\phi_g - \frac{\phi_g}{c_g}\left(\frac{\partial\vec{\boldsymbol{\beta}}}{\partial t}\right) - \frac{\vec{\boldsymbol{\beta}}}{c_g}\left(\frac{\partial\phi_g}{\partial t}\right) \tag{V.363}$$

by breaking it up into its three constituents. Firstly,

$$\begin{aligned}\nabla_i\phi_g &= \frac{Gm}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^2}\Big[\nabla_i R - R_j(\nabla_i\beta_j) - \beta_j(\nabla_i R_j)\Big]\\ &= \frac{Gm}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^2}\left[\frac{\widehat{\boldsymbol{R}}_i}{1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}} - R_j\left(\frac{-\widehat{\boldsymbol{R}}_i\dot{\beta}_j}{c_g(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\right) - \beta_j\left(\delta_{ij} + \frac{\widehat{\boldsymbol{R}}_i\beta_j}{1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}}\right)\right]\\ &= \frac{Gm}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[\widehat{\boldsymbol{R}}_i + \widehat{\boldsymbol{R}}_i\frac{R_j\dot{\beta}_j}{c_g} - \beta_i(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}) - \widehat{\boldsymbol{R}}_i\beta_j\beta_j\right]\\ \nabla_i\phi_g &= \frac{Gm}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[\widehat{\boldsymbol{R}}_i\left(1-\beta_j\beta_j + \frac{R_j\dot{\beta}_j}{c_g}\right) - \beta_i(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})\right]\end{aligned} \tag{V.364}$$

then,

$$\nabla\phi_g = \frac{Gm}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[\widehat{\boldsymbol{R}}\left(1-\beta^2 + \frac{\vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}}{c_g}\right) - \vec{\boldsymbol{\beta}}(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})\right]. \tag{V.365}$$

Expanding the second term of Equation V.363 [V.312],

$$\begin{aligned}\frac{\phi_g}{c_g}\frac{\partial\vec{\boldsymbol{\beta}}}{\partial t} &= \frac{1}{c_g}\left[\frac{-Gm}{R(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\right]\left(\frac{\dot{\vec{\boldsymbol{\beta}}}}{1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}}\right)\\ &= \frac{-Gm\dot{\vec{\boldsymbol{\beta}}}}{c_g R(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^2}.\end{aligned} \tag{V.366}$$

Lastly, the third term,

$$\frac{\vec{\boldsymbol{\beta}}}{c_g}\frac{\partial\phi_g}{\partial t} = \frac{1}{c_g}\vec{\boldsymbol{\beta}}\left(\frac{Gm}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[\vec{\boldsymbol{v}}_m(t')\cdot\vec{\boldsymbol{\beta}} - \widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{v}}_m(t') - \vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}\right]\right). \tag{V.367}$$

So,

$$\frac{\vec{\boldsymbol{\beta}}}{c_g}\frac{\partial\phi_g}{\partial t} = \frac{Gm\vec{\boldsymbol{\beta}}}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{\beta}} - \widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}} - \vec{\boldsymbol{R}}\cdot\frac{\dot{\vec{\boldsymbol{\beta}}}}{c_g}\right). \tag{V.368}$$

Now, plugging Equations V.365, v.366, and V.368 in to Equation V.363,

$$\begin{aligned}
\vec{\boldsymbol{g}} &= \frac{-Gm}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[\widehat{\boldsymbol{R}}\left(1-\beta^2+\frac{\vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}}{c_g}\right)-\vec{\boldsymbol{\beta}}(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})\right]\cdots \\
&\quad \cdots+\frac{-Gm\vec{\boldsymbol{\beta}}}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{\beta}}-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}-\vec{\boldsymbol{R}}\cdot\frac{\dot{\vec{\boldsymbol{\beta}}}}{c_g}\right]+\left[\frac{Gm\dot{\vec{\boldsymbol{\beta}}}}{c_gR(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^2}\right] \\
&= \frac{-Gm}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[\widehat{\boldsymbol{R}}\left(1-\beta_2+\frac{\vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}}{c_g}\right)-\vec{\boldsymbol{\beta}}+\cancel{\vec{\boldsymbol{\beta}}(\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}+\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{\beta}})\cdots\right. \\
&\quad \left.\cdots-\cancel{\vec{\boldsymbol{\beta}}(\widehat{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}})}-\vec{\boldsymbol{\beta}}\left(\frac{\vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}}{c_g}\right)\right]+\frac{Gm\dot{\vec{\boldsymbol{\beta}}}}{c_gR(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^2} \\
&= \frac{-Gm}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[\widehat{\boldsymbol{R}}\left(1-\beta^2+\frac{\vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}}{c_g}\right)-\vec{\boldsymbol{\beta}}\left(1-\beta^2+\frac{\vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}}{c_g}\right)\right]+\frac{Gm\dot{\vec{\boldsymbol{\beta}}}}{c_gR(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^2}\,.
\end{aligned} \tag{V.369}$$

Continuing,

$$\vec{\boldsymbol{g}} = \frac{-Gm}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[(\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})\left(1-\beta^2+\frac{\vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}}{c_g}\right)\right]+\frac{Gm\dot{\vec{\boldsymbol{\beta}}}}{c_gR(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^2}\,. \tag{V.370}$$

From here, it is possible to identify the velocity- and acceleration-dependent components as:

$$\vec{\boldsymbol{g}}_v = \frac{-Gm}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[(\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})(1-\beta^2)\right] \tag{V.371}$$

and

$$\vec{\boldsymbol{g}}_a = \frac{-Gm}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[(\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})\left(\frac{\vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}}{c_g}\right)\right]+\frac{Gm}{R(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^2}\frac{\dot{\vec{\boldsymbol{\beta}}}}{c_g}\,, \tag{V.372}$$

where

$$\vec{\boldsymbol{g}} = \vec{\boldsymbol{g}}_v+\vec{\boldsymbol{g}}_a\,. \tag{V.373}$$

It is also possible to write the acceleration-dependent term in a more compact manner. Expanding Equation V.372,

$$\begin{aligned}
\vec{\boldsymbol{g}}_a &= \frac{-Gm}{c_gR(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[(\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})\big(\widehat{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}\big)-\dot{\vec{\boldsymbol{\beta}}}(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})\right] \\
&= \frac{-Gm}{c_gR(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[(\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})\big(\widehat{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}\big)-\dot{\vec{\boldsymbol{\beta}}}(\widehat{\boldsymbol{R}}\cdot\widehat{\boldsymbol{R}}-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})\right] \\
\vec{\boldsymbol{g}}_a &= \frac{-Gm}{c_gR(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[(\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})\big(\widehat{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}\big)-\dot{\vec{\boldsymbol{\beta}}}\big[\widehat{\boldsymbol{R}}\cdot(\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})\big]\right].
\end{aligned} \tag{V.374}$$

Recognizing the BAC-CAB rule from vector calculus,

$$\vec{\boldsymbol{A}}\times(\vec{\boldsymbol{B}}\times\vec{\boldsymbol{C}}) = \vec{\boldsymbol{B}}(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{C}})-\vec{\boldsymbol{C}}(\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}})\,, \tag{V.375}$$

$$\vec{\boldsymbol{g}}_a = \frac{-Gm}{c_gR(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left\{\widehat{\boldsymbol{R}}\times\left[(\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})\times\dot{\vec{\boldsymbol{\beta}}}\right]\right\}. \tag{V.376}$$

So, finally,

$$\vec{\boldsymbol{g}} = \frac{-Gm}{(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left\{\frac{(\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})(1-\beta^2)}{R^2} + \frac{\widehat{\boldsymbol{R}}\times\left[(\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})\times\dot{\vec{\boldsymbol{\beta}}}\right]}{c_g R}\right\} \tag{V.377}$$

where, $\vec{\boldsymbol{g}} = \vec{\boldsymbol{g}}_v + \vec{\boldsymbol{g}}_a$. Now we have an expression for the Liénard–Wiechert (retarded) gravitational field, $\vec{\boldsymbol{g}}$,

$$\vec{\boldsymbol{g}} = \frac{-Gm}{(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left\{\frac{(\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})(1-\beta^2)}{R^2} + \frac{\widehat{\boldsymbol{R}}\times\left[(\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})\times\dot{\vec{\boldsymbol{\beta}}}\right]}{c_g R}\right\}, \tag{V.378}$$

which has velocity- and acceleration-dependent components

$$\vec{\boldsymbol{g}}_v = \frac{-Gm}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[(\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})(1-\beta^2)\right] \tag{V.379}$$

and

$$\vec{\boldsymbol{g}}_a = \frac{-Gm}{c_g R(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left\{\widehat{\boldsymbol{R}}\times\left[(\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})\times\dot{\vec{\boldsymbol{\beta}}}\right]\right\} \tag{V.380}$$

where, again,

$$\vec{\boldsymbol{g}} = \vec{\boldsymbol{g}}_v + \vec{\boldsymbol{g}}_a\,. \tag{V.381}$$

Cogravitational Field

Determining the expression for the cogravitational Liénard–Wiechert field through some "heroic algebra" [4]. From Equation V.314 [V.121]

$$\vec{\boldsymbol{h}} = \phi_g(\nabla\times\vec{\boldsymbol{\beta}}) - \vec{\boldsymbol{\beta}}\times\nabla\phi_g\,. \tag{V.382}$$

The scalar potential and the gradient are known, so solving the $\nabla\times\vec{\boldsymbol{\beta}}$ term,

$$\begin{aligned}\nabla\times\vec{\boldsymbol{\beta}} &= \begin{vmatrix}\hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}}\\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3}\\ \beta_1 & \beta_2 & \beta_3\end{vmatrix}\nabla\times\vec{\boldsymbol{\beta}}\\ &= \left(\frac{\partial\beta_3}{\partial x_2}-\frac{\partial\beta_2}{\partial x_3}\right)\hat{\boldsymbol{i}} + \left(\frac{\partial\beta_1}{\partial x_3}-\frac{\partial\beta_3}{\partial x_1}\right)\hat{\boldsymbol{j}} + \left(\frac{\partial\beta_2}{\partial x_1}-\frac{\partial\beta_1}{\partial x_2}\right)\hat{\boldsymbol{k}}\,,\end{aligned} \tag{V.383}$$

$$\begin{aligned}\nabla\times\vec{\boldsymbol{\beta}} =& \left(\frac{\partial\beta_3}{\partial t'}\frac{\partial t'}{\partial x_2}-\frac{\partial\beta_2}{\partial t'}\frac{\partial t'}{\partial x_3}\right)\hat{\boldsymbol{i}} + \left(\frac{\partial\beta_1}{\partial t'}\frac{\partial t'}{\partial x_3}-\frac{\partial\beta_3}{\partial t'}\frac{\partial t'}{\partial x_1}\right)\hat{\boldsymbol{j}}\cdots\\ &\cdots+\left(\frac{\partial\beta_2}{\partial t'}\frac{\partial t'}{\partial x_1}-\frac{\partial\beta_1}{\partial t'}\frac{\partial t'}{\partial x_2}\right)\hat{\boldsymbol{k}}\,.\end{aligned} \tag{V.384}$$

Recalling Equation V.358 [V.341],

$$\frac{\partial t'}{\partial x_i} = -\frac{\widehat{\boldsymbol{R}}_i}{c_g(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})} \tag{V.385}$$

and plugging in to Equation V.384,

$$
\begin{aligned}
\nabla \times \vec{\boldsymbol{\beta}} = \Bigg(\dot{\beta}_3\Bigg[-\frac{\widehat{\boldsymbol{R}}_2}{c_g(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\Bigg] - \dot{\beta}_2\Bigg[-\frac{\widehat{\boldsymbol{R}}_3}{c_g(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\Bigg]\Bigg)\hat{\boldsymbol{\imath}}\cdots \\
\cdots + \Bigg(\dot{\beta}_1\Bigg[-\frac{\widehat{\boldsymbol{R}}_3}{c_g(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\Bigg] - \dot{\beta}_3\Bigg[-\frac{\widehat{\boldsymbol{R}}_1}{c_g(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\Bigg]\Bigg)\hat{\boldsymbol{\jmath}}\cdots \\
\cdots + \Bigg(\dot{\beta}_2\Bigg[-\frac{\widehat{\boldsymbol{R}}_1}{c_g(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\Bigg] - \dot{\beta}_1\Bigg[-\frac{\widehat{\boldsymbol{R}}_2}{c_g(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\Bigg]\Bigg)\hat{\boldsymbol{k}} \\
= \frac{-1}{c_g(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\Big[(\dot{\beta}_3\widehat{\boldsymbol{R}}_2 - \dot{\beta}_2\widehat{\boldsymbol{R}}_3)\hat{\boldsymbol{\imath}} + (\dot{\beta}_1\widehat{\boldsymbol{R}}_3 - \dot{\beta}_3\widehat{\boldsymbol{R}}_1)\hat{\boldsymbol{\jmath}} + (\dot{\beta}_2\widehat{\boldsymbol{R}}_1 - \dot{\beta}_1\widehat{\boldsymbol{R}}_2)\hat{\boldsymbol{k}}\Big]
\end{aligned}
\tag{V.386}
$$

which, after rearranging, yields

$$
\nabla \times \vec{\boldsymbol{\beta}} = \frac{-1}{c_g(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\Big[(\widehat{\boldsymbol{R}}_2\dot{\beta}_3 - \widehat{\boldsymbol{R}}_3\dot{\beta}_2)\hat{\boldsymbol{\imath}} + (\widehat{\boldsymbol{R}}_3\dot{\beta}_1 - \widehat{\boldsymbol{R}}_{1\dot{\beta}_3})\hat{\boldsymbol{\jmath}} + (\widehat{\boldsymbol{R}}_1\dot{\beta}_2 - \widehat{\boldsymbol{R}}_2\dot{\beta}_1)\hat{\boldsymbol{k}}\Big]\,. \tag{V.387}
$$

Recognizing the cyclic nature of the cross product, as seen in:

$$
\nabla \times \vec{\boldsymbol{\beta}} = \Big(\frac{\partial}{\partial x_2}\beta_3 - \frac{\partial}{\partial x_3}\beta_2\Big)\hat{\boldsymbol{\imath}} + \Big(\frac{\partial}{\partial x_3}\beta_1 - \frac{\partial}{\partial x_1}\beta_3\Big)\hat{\boldsymbol{\jmath}} + \Big(\frac{\partial}{\partial x_1}\beta_2 - \frac{\partial}{\partial x_2}\beta_1\Big)\hat{\boldsymbol{k}}\,. \tag{V.388}
$$

Then, clearly,

$$
\begin{aligned}
\nabla \times \vec{\boldsymbol{\beta}} &= \frac{-1}{c_g(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\widehat{\boldsymbol{R}} \times \dot{\vec{\boldsymbol{\beta}}} \\
&= \frac{-\widehat{\boldsymbol{R}} \times \dot{\vec{\boldsymbol{\beta}}}}{c_g(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\,.
\end{aligned}
\tag{V.389}
$$

Plugging in Equations V.306, V.365, and V.389 into Equation V.382 yields,

$$
\begin{aligned}
\vec{\boldsymbol{h}} = \Bigg[\frac{-Gm}{R(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\Bigg]\Bigg[\frac{-\widehat{\boldsymbol{R}} \times \dot{\vec{\boldsymbol{\beta}}}}{c_g(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\Bigg] - \vec{\boldsymbol{\beta}} \times \Bigg(\frac{Gm}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\Bigg[\widehat{\boldsymbol{R}}(1-\beta^2\cdots \\
\cdots + \frac{\vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}}{c_g} - \vec{\boldsymbol{\beta}}(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})\Bigg]\Bigg) \\
= \widehat{\boldsymbol{R}} \times \Bigg[\frac{Gm\dot{\vec{\boldsymbol{\beta}}}}{c_gR(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^2}\Bigg] - \vec{\boldsymbol{\beta}} \times \Bigg[\frac{Gm\widehat{\boldsymbol{R}}}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}(1-\beta^2)\Bigg]\cdots \\
\cdots - \vec{\boldsymbol{\beta}} \times \Bigg[\frac{Gm\widehat{\boldsymbol{R}}}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\Bigg(\frac{\vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}}{c_g}\Bigg)\Bigg] + \vec{\boldsymbol{\beta}} \times \cancelto{0}{\Bigg[\frac{Gm(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\vec{\boldsymbol{\beta}}\Bigg]} \\
= \vec{\boldsymbol{\beta}} \times \Bigg[\frac{-Gm(1-\beta^2)}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\widehat{\boldsymbol{R}}\Bigg] + \widehat{\boldsymbol{R}} \times \Bigg[\frac{Gm\dot{\vec{\boldsymbol{\beta}}}}{c_gR(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^2}\Bigg]\cdots \\
\cdots + \widehat{\boldsymbol{R}} \times \Bigg[\frac{Gm\vec{\boldsymbol{\beta}}}{c_gR(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\Big(\widehat{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}\Big)\Bigg] \\
\vec{\boldsymbol{h}} = \vec{\boldsymbol{\beta}} \times \Bigg[\frac{-Gm(1-\beta^2)}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\widehat{\boldsymbol{R}}\Bigg] + \widehat{\boldsymbol{R}} \times \Bigg[\frac{Gm}{c_gR(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\Big[\Big(\widehat{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}\Big)\vec{\boldsymbol{\beta}} + \dot{\vec{\boldsymbol{\beta}}}(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})\Big]\Bigg]\,.
\end{aligned}
\tag{V.390}
$$

From here, it is possible to identify the velocity- and acceleration-dependent terms. Identifying the velocity and acceleration components of the cogravitational LW field.

$$\vec{\boldsymbol{h}}_v = \vec{\boldsymbol{\beta}} \times \left[\frac{-Gm(1-\beta^2)}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\widehat{\boldsymbol{R}}\right] \tag{V.391}$$

and

$$\vec{\boldsymbol{h}}_a = \widehat{\boldsymbol{R}} \times \left[\frac{-Gm}{c_g R(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[-\left(\widehat{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}\right)\vec{\boldsymbol{\beta}} - \dot{\vec{\boldsymbol{\beta}}}(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})\right]\right]. \tag{V.392}$$

Note: the negative signs added in Equation V.392 are there to match the $-Gm$ constant!

Where, as with the gravitational field,

$$\vec{\boldsymbol{h}} = \vec{\boldsymbol{h}}_v + \vec{\boldsymbol{h}}_a\,. \tag{V.393}$$

Recombining and rearranging to match Heald and Marion as was done for the gravitational field in Equation V.378,

$$\begin{aligned}\vec{\boldsymbol{h}} &= -Gm\left[\frac{(\vec{\boldsymbol{\beta}}\times\widehat{\boldsymbol{R}})(1-\beta^2)}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3} - \frac{\left(\widehat{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}\right)(\widehat{\boldsymbol{R}}\times\vec{\boldsymbol{\beta}})}{c_g R(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3} - \frac{\widehat{\boldsymbol{R}}\times\dot{\vec{\boldsymbol{\beta}}}}{c_g R(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^2}\right]\\ &= -Gm\left[\frac{(\vec{\boldsymbol{\beta}}\times\widehat{\boldsymbol{R}})(1-\beta^2)}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3} + \frac{\left(\dot{\vec{\boldsymbol{\beta}}}\cdot\widehat{\boldsymbol{R}}\right)(\vec{\boldsymbol{\beta}}\times\widehat{\boldsymbol{R}})}{c_g R(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3} + \frac{\dot{\vec{\boldsymbol{\beta}}}\times\widehat{\boldsymbol{R}}}{c_g R(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^2}\right],\end{aligned} \tag{V.394}$$

which matches the form of that for electromagnetism! It is possible for one to show that

$$\vec{\boldsymbol{h}}_v = \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{g}}_v \tag{V.395}$$

and

$$\vec{\boldsymbol{h}}_a = \widehat{\boldsymbol{R}} \times \vec{\boldsymbol{g}}_a\,. \tag{V.396}$$

So, checking the relations,

$$\begin{aligned}\vec{\boldsymbol{h}}_v &= \vec{\boldsymbol{\beta}} \times \left\{\frac{-Gm(1-\beta^2)}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}(\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})\right\}\\ &= \frac{-Gm(1-\beta^2)}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}(\vec{\boldsymbol{\beta}}\times\widehat{\boldsymbol{R}} - \cancelto{0}{\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{\beta}}}\;)\\ &= \frac{-Gm(1-\beta^2)}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\vec{\boldsymbol{\beta}}\times\widehat{\boldsymbol{R}}\\ \vec{\boldsymbol{h}}_v &= \vec{\boldsymbol{\beta}} \times \left[\frac{-Gm(1-\beta^2)}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\widehat{\boldsymbol{R}}\right].\end{aligned} \tag{V.397}$$

Thus, agreeing with Equation V.28,

$$\vec{\boldsymbol{h}}_v = \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{g}}_v\,. \tag{V.398}$$

And,

$$\begin{aligned}\vec{\boldsymbol{h}}_a &= \widehat{\boldsymbol{R}} \times \left\{ \frac{-Gm}{c_g R\big(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^3} \left[(\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})\Big(\widehat{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}\Big) - \dot{\vec{\boldsymbol{\beta}}}(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}) \right] \right\} \\ &= \frac{-Gm}{c_g R\big(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^3} \left[(\cancelto{0}{\widehat{\boldsymbol{R}}\times\widehat{\boldsymbol{R}}} - \widehat{\boldsymbol{R}}\times\vec{\boldsymbol{\beta}})\Big(\widehat{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}\Big) - \Big(\widehat{\boldsymbol{R}}\times\dot{\vec{\boldsymbol{\beta}}}\Big)\Big(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\Big) \right] \qquad \text{(V.399)} \\ \vec{\boldsymbol{h}}_a &= \widehat{\boldsymbol{R}} \times \left\{ \frac{-Gm}{c_g R\big(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^3} \left[-\Big(\widehat{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}\Big)\vec{\boldsymbol{\beta}} - \dot{\vec{\boldsymbol{\beta}}}(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}) \right] \right\}.\end{aligned}$$

Thus, agreeing with Equation V.29,

$$\vec{\boldsymbol{h}}_a = \widehat{\boldsymbol{R}} \times \vec{\boldsymbol{g}}_a\,. \qquad \text{(V.400)}$$

Now, matching with Heald and Marion and reapplying the bracket notation to indicate evaluation the retarded time,

$$\vec{\boldsymbol{g}} = -Gm\left[\frac{(\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})(1-\beta^2)}{R^2\big(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^3} + \frac{\widehat{\boldsymbol{R}}\times\Big((\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})\times\dot{\vec{\boldsymbol{\beta}}}\Big)}{c_g R\big(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^3} \right], \qquad \text{(V.401)}$$

where

$$\vec{\boldsymbol{g}} = \vec{\boldsymbol{g}}_v + \vec{\boldsymbol{g}}_a\,, \qquad \text{(V.402)}$$

$$\vec{\boldsymbol{g}}_v = -Gm\left[\frac{(\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})(1-\beta^2)}{R^2\big(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^3} \right], \qquad \text{(V.403)}$$

$$\vec{\boldsymbol{g}}_a = -Gm\left[\frac{\widehat{\boldsymbol{R}}\times\Big((\widehat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})\times\dot{\vec{\boldsymbol{\beta}}}\Big)}{c_g R\big(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^3} \right] \qquad \text{(V.404)}$$

and

$$\vec{\boldsymbol{h}} = -Gm\left[\frac{(\vec{\boldsymbol{\beta}}\times\widehat{\boldsymbol{R}})(1-\beta^2)}{R^2\big(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^3} + \frac{\Big(\dot{\vec{\boldsymbol{\beta}}}\cdot\widehat{\boldsymbol{R}}\Big)(\vec{\boldsymbol{\beta}}\times\widehat{\boldsymbol{R}})}{c_g R\big(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^3} + \frac{\dot{\vec{\boldsymbol{\beta}}}\times\widehat{\boldsymbol{R}}}{c_g R\big(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^2} \right], \qquad \text{(V.405)}$$

where

$$\vec{\boldsymbol{h}} = \vec{\boldsymbol{h}}_v + \vec{\boldsymbol{h}}_a\,, \qquad \text{(V.406)}$$

$$\vec{\boldsymbol{h}}_v = \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{g}}_v\,, \qquad \text{(V.407)}$$

$$\vec{\boldsymbol{h}}_a = \widehat{\boldsymbol{R}} \times \vec{\boldsymbol{g}}_a\,. \qquad \text{(V.408)}$$

Liénard–Wiechert Potentials for Gravitation Satisfy Lorentz Gauge

For completeness, it should also be worth showing that the Liénard–Wiechert potentials satisfy the Lorentz gauge condition. Recalling the Lorentz gauge condition from Equation V.133:

$$\nabla \cdot \vec{\boldsymbol{A}}_g + \frac{1}{c_g}\frac{\partial \phi_g}{\partial t} = 0\,, \tag{V.409}$$

and it was previously shown that $\vec{\boldsymbol{A}}_g = \vec{\boldsymbol{\beta}}\phi_g$, so

$$\nabla \cdot (\vec{\boldsymbol{\beta}}\phi_g) + \frac{1}{c_g}\frac{\partial \phi_g}{\partial t} = 0\,. \tag{V.410}$$

Expanding

$$\phi_g(\nabla \cdot \vec{\boldsymbol{\beta}}) + \vec{\boldsymbol{\beta}} \cdot \nabla\phi_g + \frac{1}{c_g}\frac{\partial \phi_g}{\partial t} = 0\,, \tag{V.411}$$

and rearranging, we obtain,

$$\vec{\boldsymbol{\beta}} \cdot \nabla\phi_g + \frac{1}{c_g}\frac{\partial \phi_g}{\partial t} + \phi_g(\nabla \cdot \vec{\boldsymbol{\beta}}) = 0\,. \tag{V.412}$$

Expressions for these terms have already been found so by using Equations V.306, V.356, V.361, and V.365 we get the following after dropping the bracket notation convention for simplicity,

$$\begin{aligned}
&\vec{\boldsymbol{\beta}} \cdot \left\{\frac{Gm}{R^2(1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[\hat{\boldsymbol{R}}\left(1-\beta^2+\frac{\vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}}{c_g}\right)-\vec{\boldsymbol{\beta}}(1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})\right]\right\}\cdots \\
&\cdots + \frac{1}{c_g}\left\{\frac{Gm}{R^2(1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[\vec{\boldsymbol{v}}_m(t')\cdot\vec{\boldsymbol{\beta}} - \hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{v}}_m(t') - \vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}\right]\right\}\cdots \\
&\cdots + \left[\frac{-Gm}{R(1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\right]\left[\frac{-\hat{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}}{c_g(1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\right] = 0\,.
\end{aligned} \tag{V.413}$$

Continuing...

$$\begin{aligned}
&\frac{Gm}{R^2(1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left\{\vec{\boldsymbol{\beta}}\cdot\left[\hat{\boldsymbol{R}}\left(1-\beta^2+\frac{\vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}}{c_g}\right)-\vec{\boldsymbol{\beta}}(1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})\right]\right\}\cdots \\
&\cdots + \frac{Gm}{R^2(1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{\beta}} - \hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}} - \frac{\vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}}{c_g}\right]\cdots \\
&\cdots + \frac{Gm\left(\hat{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}\right)}{c_gR(1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^2} = 0\,.
\end{aligned} \tag{V.414}$$

Pulling out common terms,

$$\begin{aligned}
&\frac{Gm}{R^2(1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left\{\cancel{\vec{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{R}}} + \cancel{\vec{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{R}}(-\beta^2)} + \vec{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{R}}\left(\frac{\vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}}{c_g}\right)\cdots\right. \\
&\cdots - \cancel{\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{\beta}}} - \cancel{\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{\beta}}(-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})} + \cancel{\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{\beta}}} - \cancel{\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}} - \frac{\vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}}{c_g}\cdots \\
&\left.\cdots + \frac{R\left(\hat{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}\right)(1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}{c_g}\right\} = 0\,.
\end{aligned} \tag{V.415}$$

Collecting and rearranging,

$$\frac{Gm}{R^2(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}\left[\cancel{\left(\frac{\vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}}{c_g}\right)(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})} - \cancel{\left(\frac{\vec{\boldsymbol{R}}\cdot\dot{\vec{\boldsymbol{\beta}}}}{c_g}\right)(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})}\right] = 0\,. \tag{V.416}$$

Therefore, the Lorentz gauge condition

$$\nabla\cdot\vec{\boldsymbol{A}}_g + \frac{1}{c_g}\frac{\partial\phi_g}{\partial t} = 0 \tag{V.417}$$

is satisfied using the retarded Liénard–Wiechert potentials for gravitation.

Appendix F: Derivation of the Poynting Vector for Gravitation

Recall, work is defined as,

$$W = Fd \tag{V.418}$$

or

$$dW = \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}\,, \tag{V.419}$$

and power is,

$$P = \frac{dW}{dt} \tag{V.420}$$

so,

$$dW = Pdt \tag{V.421}$$

then,

$$dW = \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} = Pdt\,. \tag{V.422}$$

Solving for power,

$$P = \vec{\boldsymbol{F}} \cdot \frac{d\vec{\boldsymbol{r}}}{dt} = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}\,. \tag{V.423}$$

Define work-volume, power-volume, and force-volume,

$$\mathcal{W} = \frac{W}{V}\,, \tag{V.424}$$

$$\mathcal{P} = \frac{P}{V}\,, \tag{V.425}$$

$$\mathcal{F} = \frac{F}{V}\,. \tag{V.426}$$

Thus,

$$\mathcal{P} = \vec{\boldsymbol{\mathcal{F}}} \cdot \vec{\boldsymbol{v}} = \frac{d\mathcal{W}}{dt}\,. \tag{V.427}$$

From Equation V.426 and Equation V.545 it follows that,

$$\vec{\boldsymbol{\mathcal{F}}} = \frac{m}{V}\left(\vec{\boldsymbol{g}} + \frac{\vec{\boldsymbol{v}}}{c_g} \times \vec{\boldsymbol{h}}\right) \tag{V.428}$$

so,

$$\vec{\boldsymbol{\mathcal{F}}} = \rho\vec{\boldsymbol{g}} + \frac{\vec{\boldsymbol{J}}}{c_g} \times \vec{\boldsymbol{h}}\,. \tag{V.429}$$

Now, plugging Equation V.429 into Equation V.427,

$$\begin{aligned} \mathcal{P} &= \left(\rho\vec{\boldsymbol{g}} + \frac{\vec{\boldsymbol{J}}}{c_g} \times \vec{\boldsymbol{h}}\right) \cdot \vec{\boldsymbol{v}} \\ &= \left(\rho\vec{\boldsymbol{g}} + \frac{\vec{\boldsymbol{J}}}{c_g} \times \vec{\boldsymbol{h}}\right) \cdot \frac{\vec{\boldsymbol{J}}}{\rho} \\ \mathcal{P} &= \vec{\boldsymbol{g}} \cdot \vec{\boldsymbol{J}}\,. \end{aligned} \tag{V.430}$$

Thus the energy gained per time by ρ and $\vec{\boldsymbol{J}}$ in dV is given by,

$$\mathcal{P}_{matter} = \vec{\boldsymbol{g}} \cdot \vec{\boldsymbol{J}}\,. \tag{V.431}$$

This energy comes from the gravitational field so the field strength decreases.

$$\mathcal{P}_{matter} = -\mathcal{P}_{field} \tag{V.432}$$

and so,

$$\mathcal{P}_{field} = -\vec{\boldsymbol{g}} \cdot \vec{\boldsymbol{J}} \tag{V.433}$$

as energy is lost from the field to matter. Recalling Equation V.106,

$$\nabla \times \vec{\boldsymbol{h}} = -\frac{4\pi G}{c_g}\vec{\boldsymbol{J}} + \frac{1}{c_g}\frac{\partial \vec{\boldsymbol{g}}}{\partial t}\,. \tag{V.434}$$

Solving for momentum density, $\vec{\boldsymbol{J}}$,

$$\vec{\boldsymbol{J}} = -\frac{c_g}{4\pi G}(\nabla \times \vec{\boldsymbol{h}}) + \frac{1}{4\pi G}\frac{\partial \vec{\boldsymbol{g}}}{\partial t}\,. \tag{V.435}$$

Now,

$$\mathcal{P}_{field} = -\vec{\boldsymbol{g}} \cdot \left[\frac{-c_g}{4\pi G}(\nabla \times \vec{\boldsymbol{h}}) + \frac{1}{4\pi G}\frac{\partial \vec{\boldsymbol{g}}}{\partial t}\right] \tag{V.436}$$

so,

$$\mathcal{P}_{field} = \frac{c_g}{4\pi G}\vec{\boldsymbol{g}} \cdot (\nabla \times \vec{\boldsymbol{h}}) - \frac{1}{4\pi G}\vec{\boldsymbol{g}} \cdot \frac{\partial \vec{\boldsymbol{g}}}{\partial t}\,. \tag{V.437}$$

Adding zero,

$$\mathcal{P}_{field} = -\frac{1}{4\pi G}\vec{\boldsymbol{g}} \cdot \frac{\partial \vec{\boldsymbol{g}}}{\partial t} - \frac{1}{4\pi G}\vec{\boldsymbol{h}} \cdot \frac{\partial \vec{\boldsymbol{h}}}{\partial t} + \frac{c_g}{4\pi G}\vec{\boldsymbol{g}} \cdot (\nabla \times \vec{\boldsymbol{h}}) + \frac{1}{4\pi G}\vec{\boldsymbol{h}} \cdot \frac{\partial \vec{\boldsymbol{h}}}{\partial t} \tag{V.438}$$

and using Equation V.105,

$$\nabla \times \vec{\boldsymbol{g}} = -\frac{1}{c_g}\frac{\partial \vec{\boldsymbol{h}}}{\partial t}\,, \tag{V.439}$$

which leads to

$$\frac{\partial \vec{\boldsymbol{h}}}{\partial t} = -c_g \nabla \times \vec{\boldsymbol{g}} \tag{V.440}$$

so,

$$\begin{aligned}\mathcal{P}_{field} &= -\frac{1}{4\pi G}\left(\vec{\boldsymbol{g}} \cdot \frac{\partial \vec{\boldsymbol{g}}}{\partial t} + \vec{\boldsymbol{h}} \cdot \frac{\partial \vec{\boldsymbol{h}}}{\partial t}\right) + \frac{c_g}{4\pi G}\vec{\boldsymbol{g}} \cdot (\nabla \times \vec{\boldsymbol{h}}) + \frac{1}{4\pi G}\vec{\boldsymbol{h}} \cdot (-c_g \nabla \times \vec{\boldsymbol{g}}) \\ &= -\frac{1}{4\pi G}\left(\vec{\boldsymbol{g}} \cdot \frac{\partial \vec{\boldsymbol{g}}}{\partial t} + \vec{\boldsymbol{h}} \cdot \frac{\partial \vec{\boldsymbol{h}}}{\partial t}\right) + \frac{c_g}{4\pi G}\vec{\boldsymbol{g}} \cdot (\nabla \times \vec{\boldsymbol{h}}) - \frac{c_g}{4\pi G}\vec{\boldsymbol{h}} \cdot (\nabla \times \vec{\boldsymbol{g}})\,.\end{aligned} \tag{V.441}$$

Using the vector identity:

$$-\nabla \cdot (\vec{\boldsymbol{g}} \times \vec{\boldsymbol{h}}) = \vec{\boldsymbol{g}} \cdot (\nabla \times \vec{\boldsymbol{h}}) - \vec{\boldsymbol{h}} \cdot (\nabla \times \vec{\boldsymbol{g}})\,, \tag{V.442}$$

$$\mathcal{P}_{field} = -\frac{1}{4\pi G}\left(\vec{\boldsymbol{g}} \cdot \frac{\partial \vec{\boldsymbol{g}}}{\partial t} + \vec{\boldsymbol{h}} \cdot \frac{\partial \vec{\boldsymbol{h}}}{\partial t}\right) - \frac{c_g}{4\pi G}\nabla \cdot (\vec{\boldsymbol{g}} \times \vec{\boldsymbol{h}})\,. \tag{V.443}$$

Consider the following,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{A}}\right) = \frac{1}{2}\frac{\partial\vec{\boldsymbol{A}}}{\partial t}\cdot\vec{\boldsymbol{A}} + \frac{1}{2}\vec{\boldsymbol{A}}\cdot\frac{\partial\vec{\boldsymbol{A}}}{\partial t} = \vec{\boldsymbol{A}}\cdot\frac{\partial\vec{\boldsymbol{A}}}{\partial t} \tag{V.444}$$

now,

$$\begin{aligned}\mathcal{P}_{field} &= -\frac{1}{4\pi G}\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{g}} + \frac{1}{2}\vec{\boldsymbol{h}}\cdot\vec{\boldsymbol{h}}\right) - \frac{c_g}{4\pi G}\nabla\cdot(\vec{\boldsymbol{g}}\times\vec{\boldsymbol{h}}) \\ &= \frac{\partial}{\partial t}\left[-\frac{1}{8\pi G}(\vec{\boldsymbol{g}}^2 + \vec{\boldsymbol{h}}^2)\right] + \nabla\cdot\left(-\frac{c_g}{4\pi G}\vec{\boldsymbol{g}}\times\vec{\boldsymbol{h}}\right).\end{aligned} \tag{V.445}$$

Define the work-energy per volume terms,

$$\mathcal{W}_g \equiv -\frac{1}{8\pi G}\vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{g}} \tag{V.446}$$

and

$$\mathcal{W}_h \equiv -\frac{1}{8\pi G}\vec{\boldsymbol{h}}\cdot\vec{\boldsymbol{h}} \tag{V.447}$$

such that,

$$\mathcal{W} = \mathcal{W}_g + \mathcal{W}_h = -\frac{1}{8\pi G}(\vec{\boldsymbol{g}}^2 + \vec{\boldsymbol{h}}^2) \tag{V.448}$$

and defining the Poynting vector,

$$\vec{\boldsymbol{\mathcal{S}}} = -\frac{c_g}{4\pi G}\vec{\boldsymbol{g}}\times\vec{\boldsymbol{h}} \tag{V.449}$$

which is the energy per area per unit time flowing out of the “sides”. Using Equations V.448 and V.449 in Equation V.445 yields,

$$\mathcal{P}_{field} = \frac{\partial\mathcal{W}}{\partial t} + \nabla\cdot\vec{\boldsymbol{\mathcal{S}}}\,. \tag{V.450}$$

Equating to Equation V.433, we can write,

$$-\vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{J}} = \frac{\partial\mathcal{W}}{\partial t} + \nabla\cdot\vec{\boldsymbol{\mathcal{S}}}\,. \tag{V.451}$$

Thus,

$$\begin{aligned}\mathcal{P}_{field} &= -\vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{J}} \\ &= \frac{\partial\mathcal{W}}{\partial t} + \nabla\cdot\vec{\boldsymbol{\mathcal{S}}}\end{aligned} \tag{V.452}$$

or, as a more common statement of mass-energy conservation, the Poynting theorem is given by:

$$\frac{\partial\mathcal{W}}{\partial t} + \nabla\cdot\vec{\boldsymbol{\mathcal{S}}} + \vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{J}} = 0\,. \tag{V.453}$$

Although the form of the relation is identical to its electromagnetic counterpart, the definitions of the energy density and energy flow (Poynting vector) have a change in sign! This means that energy is flowing *into* the gravitational field!

Momentum Density

Along with having defined work-energy density, force density, and power density, we can also define momentum density which we can relate to the Poynting vector $\vec{\boldsymbol{\mathcal{S}}}$ and the work-energy density. Recalling from Equation V.448,

$$\mathcal{W} = -\frac{1}{8\pi G}(\vec{\boldsymbol{g}}^2 + \vec{\boldsymbol{h}}^2) \tag{V.454}$$

and

$$\vec{\boldsymbol{\mathcal{S}}} = -\frac{c_g}{4\pi G}\vec{\boldsymbol{g}} \times \vec{\boldsymbol{h}}\,. \tag{V.455}$$

From EMII Lecture Notes (modified for gravitation):

$$\langle\vec{\boldsymbol{\mathcal{S}}}\rangle = -\frac{-c_g}{4\pi G}\frac{g_o^2}{2}\hat{\boldsymbol{k}} = -\frac{c_g g_o^2}{8\pi G}\hat{\boldsymbol{k}}\,. \tag{V.456}$$

If,

$$\langle\mathcal{W}\rangle = -\frac{1}{8\pi G}g_o^2 \tag{V.457}$$

then,

$$\vec{\boldsymbol{\mathcal{S}}} = c_g\langle\mathcal{W}\rangle\hat{\boldsymbol{k}}\,. \tag{V.458}$$

Recall that,

$$dW = \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} = F dx\,, \tag{V.459}$$

where

$$\vec{\boldsymbol{F}} = \frac{d\vec{\boldsymbol{p}}}{dt} \tag{V.460}$$

and

$$dW = \frac{dp}{dt}dx = dp\frac{dx}{dt}\,, \tag{V.461}$$

where

$$\vec{\boldsymbol{v}} = \frac{d\vec{\boldsymbol{x}}}{dt}\,. \tag{V.462}$$

So,

$$\frac{dW}{V} = \frac{dp}{V}v\,. \tag{V.463}$$

Defining the momentum density,

$$\overrightarrow{\boldsymbol{\mathcal{M}}} = \frac{d\vec{\boldsymbol{p}}}{V}\,. \tag{V.464}$$

Equation V.463 becomes,

$$\langle\mathcal{W}\rangle = \langle\mathcal{M}\rangle c_g\,. \tag{V.465}$$

Plugging Equation V.465 in to Equation V.458 yields,

$$\langle\vec{\boldsymbol{\mathcal{S}}}\rangle = c_g^2\langle\mathcal{M}\rangle\hat{\boldsymbol{k}}\,, \tag{V.466}$$

and solving for the momentum density,

$$\langle\overrightarrow{\boldsymbol{\mathcal{M}}}\rangle = \frac{\langle\vec{\boldsymbol{\mathcal{S}}}\rangle}{c_g^2}\,. \tag{V.467}$$

Now, plugging in the definition of the Poynting vector from Equation V.455,

$$\begin{aligned}\langle\vec{\boldsymbol{\mathcal{M}}}\rangle &= \left\langle\frac{-c_g}{4\pi G c_g^2}\vec{\boldsymbol{g}}\times\vec{\boldsymbol{h}}\right\rangle \\ &= \left\langle\frac{-1}{4\pi G c_g}\vec{\boldsymbol{g}}\times\vec{\boldsymbol{h}}\right\rangle .\end{aligned} \tag{V.468}$$

So, the momentum density is,

$$\vec{\boldsymbol{\mathcal{M}}} = \frac{-\vec{\boldsymbol{g}}\times\vec{\boldsymbol{h}}}{4\pi G c_g} \tag{V.469}$$

or

$$\vec{\boldsymbol{\mathcal{M}}} = \frac{\vec{\boldsymbol{h}}\times\vec{\boldsymbol{g}}}{4\pi G c_g} . \tag{V.470}$$

It might be worth noting for future reference that, in some textbooks, the momentum density is defined as $\boldsymbol{g}$ but, obviously, that variable is occupied by the gravitational field! Instead, the script M, $\boldsymbol{\mathcal{M}}$, is used since the work density is represented by $\boldsymbol{\mathcal{W}}$, the power density by $\boldsymbol{\mathcal{P}}$, and force density by $\boldsymbol{\mathcal{F}}$.

Appendix G: Derivation of Gravitational Fields and Radiation for Special Cases

Fields Produced by a Mass in Uniform Motion

Now, following Heald & Marion Chapter 8 - Section 5 - (8.5) Fields Produced By A Charged Particle in Uniform Motion. Given a particle (mass) in uniform motion (no acceleration), then the gravitational field becomes (simply the velocity term)

$$\vec{\boldsymbol{g}}(\vec{\boldsymbol{r}}, t) = \frac{-Gm(1-\beta^2)}{R^2(1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^3}(\hat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})\,. \tag{V.471}$$

Note: $\vec{\boldsymbol{\beta}} = \vec{v}/c_g$ is "unaffected by retardation." (pg 271). Since $\vec{\boldsymbol{g}}$ is expressed in terms of the retarded position, it will be useful to "transform" it to be in terms of a present position, say $\vec{\boldsymbol{R}}_p$ rather than $[\vec{\boldsymbol{R}}] \equiv \vec{\boldsymbol{R}}_r$. Referring to Figure V.4 for the relation between retarded time and present time,

$$\vec{\boldsymbol{R}}_r = \vec{\boldsymbol{R}}_p + (t - t_r)\vec{\boldsymbol{v}}\,, \tag{V.472}$$

where the time delay is given by

$$R_r = (t - t_r)c_g\,. \tag{V.473}$$

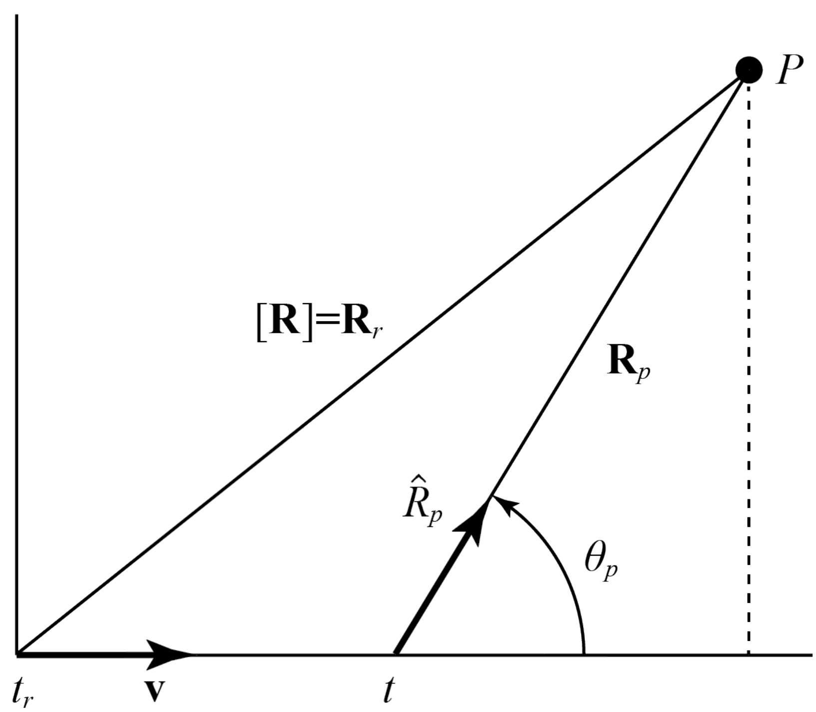

Figure V.4. Retarded Position Diagram.

So, rearranging Equation V.472,

$$\begin{aligned}\vec{\boldsymbol{R}}_p &= \vec{\boldsymbol{R}}_r - (t - t_r)\vec{\boldsymbol{v}} \\ &= \vec{\boldsymbol{R}}_r - (t - t_r)\vec{\boldsymbol{v}}\frac{c_g}{c_g} \\ &= \vec{\boldsymbol{R}}_r - \frac{R_r}{c_g}\vec{\boldsymbol{v}} \\ &= R_r\left(\widehat{\boldsymbol{R}}_r - \frac{\vec{\boldsymbol{v}}}{c_g}\right) \\ \vec{\boldsymbol{R}}_p &= R_r(\widehat{\boldsymbol{R}}_r - \vec{\boldsymbol{\beta}})\,. \end{aligned} \tag{V.474}$$

Modifying the velocity expression for $\vec{\boldsymbol{g}}$,

$$\vec{\boldsymbol{g}}(\vec{\boldsymbol{R}}_p, t) = \frac{-Gm(1-\beta^2)R_r(\widehat{\boldsymbol{R}}_r - \vec{\boldsymbol{\beta}})}{R_r^3(1 - \widehat{\boldsymbol{R}}_r \cdot \vec{\boldsymbol{\beta}})^3}\,, \tag{V.475}$$

which becomes

$$\vec{\boldsymbol{g}}(\vec{\boldsymbol{R}}_p, t) = \frac{-Gm(1-\beta^2)\vec{\boldsymbol{R}}_p}{R_r^3(1 - \widehat{\boldsymbol{R}}_r \cdot \vec{\boldsymbol{\beta}})^3}\,, \tag{V.476}$$

so it is then necessary to determine the denominator in terms of the present time and position. Consider that:

$$\begin{aligned} R_r^2(1 - \widehat{\boldsymbol{R}}_r \cdot \vec{\boldsymbol{\beta}})^2 &= (R_r - \vec{\boldsymbol{R}}_r \cdot \vec{\boldsymbol{\beta}})^2 \\ &= R_r^2 - 2R_r(\vec{\boldsymbol{R}}_r \cdot \vec{\boldsymbol{\beta}}) + (\vec{\boldsymbol{R}}_r \cdot \vec{\boldsymbol{\beta}})^2\,. \end{aligned} \tag{V.477}$$

Squaring Equation V.474 yields

$$\vec{\boldsymbol{R}}_p^2 = R_r^2(\widehat{\boldsymbol{R}} - \vec{\boldsymbol{\beta}})^2\,, \tag{V.478}$$

Expanding gives

$$\begin{aligned} R_p^2 &= R_r^2(\widehat{\boldsymbol{R}}_r^2 - \widehat{\boldsymbol{R}}_r \cdot \vec{\boldsymbol{\beta}} - \vec{\boldsymbol{\beta}} \cdot \widehat{\boldsymbol{R}}_r + \vec{\boldsymbol{\beta}}^2) \\ &= R_r^2\Big[1 - 2(\widehat{\boldsymbol{R}}_r \cdot \vec{\boldsymbol{\beta}}) + \beta^2\Big] \\ R_p^2 &= R_r^2 - 2R_r^2(\widehat{\boldsymbol{R}}_r \cdot \vec{\boldsymbol{\beta}}) + R_r^2\beta^2 \end{aligned} \tag{V.479}$$

or,

$$R_p^2 = R_r^2 - 2R_r(\vec{\boldsymbol{R}}_r \cdot \vec{\boldsymbol{\beta}}) + R_r^2\beta^2\,. \tag{V.480}$$

From Figure V.4, it can be seen that the perpendicular components of $\vec{\boldsymbol{R}}_r$ and $\vec{\boldsymbol{R}}_p$ (the dashed line) are equal. Thus,

$$|\vec{\boldsymbol{R}}_r \times \vec{\boldsymbol{\beta}}| = |\vec{\boldsymbol{R}}_p \times \vec{\boldsymbol{\beta}}|\,. \tag{V.481}$$

Applying the Pythagorean identity for two triangles by relating the two hypotenuses $\vec{\boldsymbol{R}}_r$ and $\vec{\boldsymbol{R}}_p$.

$$R_r^2\beta^2 - (\vec{\boldsymbol{R}}_r \cdot \vec{\boldsymbol{\beta}})^2 = R_p^2\beta^2 - (\vec{\boldsymbol{R}}_p \cdot \vec{\boldsymbol{\beta}})^2\,. \tag{V.482}$$

After rearranging Equation V.480 and plugging in to Equation V.477,

$$\begin{aligned} R_r^2(1-\widehat{\boldsymbol{R}}_r\cdot\vec{\boldsymbol{\beta}})^2 &= R_r^2 - 2R_r(\vec{\boldsymbol{R}}_r\cdot\vec{\boldsymbol{\beta}}) + (\vec{\boldsymbol{R}}_r\cdot\vec{\boldsymbol{\beta}})^2 \\ &= R_p^2 - R_r^2\beta^2 + (\vec{\boldsymbol{R}}_r\cdot\vec{\boldsymbol{\beta}})^2 \\ R_r^2(1-\widehat{\boldsymbol{R}}_r\cdot\vec{\boldsymbol{\beta}})^2 &= R_p^2 - \Big[R_r^2\beta^2 - (\vec{\boldsymbol{R}}_r\cdot\vec{\boldsymbol{\beta}})^2\Big]\,. \end{aligned} \tag{V.483}$$

Plugging in Equation V.482,

$$\begin{aligned} R_r^2(1-\widehat{\boldsymbol{R}}_r\cdot\vec{\boldsymbol{\beta}})^2 &= R_p^2 - \Big[R_p^2\beta^2 - (\vec{\boldsymbol{R}}_p\cdot\vec{\boldsymbol{\beta}})^2\Big] \\ &= R_p^2 - R_p^2\beta^2 + R_p^2(\widehat{\boldsymbol{R}}_p\cdot\vec{\boldsymbol{\beta}})^2 \\ &= R_p^2(1-\beta^2+\beta^2\cos^2\theta_p) \\ &= R_p^2\Big[1-\beta^2(1-\cos^2\theta_p)\Big] \\ R_r^2(1-\widehat{\boldsymbol{R}}_r\cdot\vec{\boldsymbol{\beta}})^2 &= R_p^2(1-\beta^2\sin^2\theta_p)\,. \end{aligned} \tag{V.484}$$

Now, using Equation V.484 in Equation V.476,

$$\begin{aligned} \vec{\boldsymbol{g}}(\vec{\boldsymbol{R}}_p,t) &= \frac{-Gm(1-\beta^2)\vec{\boldsymbol{R}}_p}{R_r^3(1-\widehat{\boldsymbol{R}}_r\cdot\vec{\boldsymbol{\beta}})^3} \\ &= \frac{-Gm(1-\beta^2)R_p\widehat{\boldsymbol{R}}_p}{R_r^3(1-\widehat{\boldsymbol{R}}_r\cdot\vec{\boldsymbol{\beta}})^3} \\ \vec{\boldsymbol{g}}(\vec{\boldsymbol{R}}_p,t) &= \frac{-Gm(1-\beta^2)\cancel{R_p}\widehat{\boldsymbol{R}}_p}{R_p^{\cancel{3}\,2}(1-\beta^2\sin^2\theta_p)^{\frac{3}{2}}}\,. \end{aligned} \tag{V.485}$$

So, the gravitational field for a mass in uniform motion evaluated at present position and time is:

$$\vec{\boldsymbol{g}}(\vec{\boldsymbol{R}}_p,t) = \frac{-Gm(1-\beta^2)\widehat{\boldsymbol{R}}_p}{R_p^2(1-\beta^2\sin^2\theta_p)^{\frac{3}{2}}}\,. \tag{V.486}$$

Note: $\left[\widehat{\boldsymbol{R}} = \dfrac{\vec{\boldsymbol{R}}_r}{R_r}\right]$. Using Equation V.474,

$$\begin{aligned} \vec{\boldsymbol{R}}_p &= R_r(\widehat{\boldsymbol{R}}_r - \vec{\boldsymbol{\beta}}) \\ &= \vec{\boldsymbol{R}}_r - R_r\vec{\boldsymbol{\beta}} \end{aligned} \tag{V.487}$$

solving for $\widehat{\boldsymbol{R}}_r$,

$$\vec{\boldsymbol{R}}_r = \vec{\boldsymbol{R}}_p + R_r\vec{\boldsymbol{\beta}} \tag{V.488}$$

so,

$$\begin{aligned} [\widehat{\boldsymbol{R}}_r] &= \frac{\vec{\boldsymbol{R}}_p + R_r\vec{\boldsymbol{\beta}}}{R_r} \\ &= \frac{R_p}{R_r}\widehat{\boldsymbol{R}}_p + \vec{\boldsymbol{\beta}}\,. \end{aligned} \tag{V.489}$$

We know in the case of no acceleration,

$$\vec{\boldsymbol{h}}_v = \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{g}}_v \tag{V.490}$$

there is no radiation! So,

$$\vec{\boldsymbol{h}}(\vec{\boldsymbol{R}}_p, t) = \frac{-Gm(1-\beta^2)}{R_p^2(1-\beta^2\sin^2\theta_p)^{\frac{3}{2}}}\vec{\boldsymbol{\beta}} \times \widehat{\boldsymbol{R}}_p \tag{V.491}$$

for a mass in uniform motion.

Radiation From an Accelerated Mass at Low Velocities

Following Section (8.6) of Heald and Marion, analogously: Given that a particle (mass) is being accelerated and whose speed is negligibly low where $\beta << 1$ and $\left(1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}\right)$ goes to 1. Then the g-field reduces to the acceleration component alone, thus,

$$\begin{aligned}
\vec{\boldsymbol{g}}_a &= -Gm\left\{\frac{\widehat{\boldsymbol{R}} \times \left[(\widehat{\boldsymbol{R}} - \cancelto{0}{\vec{\boldsymbol{\beta}}}\ ) \times \dot{\vec{\boldsymbol{\beta}}}\right]}{c_g R\,\cancelto{1}{(1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}})^3}}\right\} \\
&= \frac{-Gm}{c_g R}\left[\widehat{\boldsymbol{R}} \times \left(\widehat{\boldsymbol{R}} \times \dot{\vec{\boldsymbol{\beta}}}\right)\right] \\
&= \frac{-Gm}{c_g R^3}\left[\vec{\boldsymbol{R}} \times \left(\vec{\boldsymbol{R}} \times \dot{\vec{\boldsymbol{\beta}}}\right)\right] \\
&= \frac{-Gm}{c_g^2 R^3}\left[\vec{\boldsymbol{R}} \times (\vec{\boldsymbol{R}} \times \vec{\boldsymbol{a}})\right] \\
\vec{\boldsymbol{g}}_a &= \frac{-Gm}{c_g^2 R^3}\left[\vec{\boldsymbol{R}}(\vec{\boldsymbol{R}} \cdot \vec{\boldsymbol{a}}) - R^2\vec{\boldsymbol{a}}\right].
\end{aligned} \tag{V.492}$$

Note: $\dot{\vec{\boldsymbol{\beta}}}$ is the scaled acceleration such that

$$\dot{\vec{\boldsymbol{\beta}}} = \frac{\vec{\boldsymbol{a}}}{c_g}. \tag{V.493}$$

And we already know that

$$\vec{\boldsymbol{h}}_a = \widehat{\boldsymbol{R}} \times \vec{\boldsymbol{g}}_a \tag{V.494}$$

for an accelerated mass, so we can now determine the Poynting vector.

The Poynting vector is given by,

$$\vec{\mathcal{S}}_a = \frac{-c_g}{4\pi G}(\vec{\boldsymbol{g}}_a \times \vec{\boldsymbol{h}}_a) \tag{V.495}$$

so, calculating the cross-product,

$$\begin{aligned}
\vec{\boldsymbol{g}}_a \times \vec{\boldsymbol{h}}_a &= \vec{\boldsymbol{g}}_a \times (\widehat{\boldsymbol{R}} \times \vec{\boldsymbol{g}}_a) \\
&= \widehat{\boldsymbol{R}}(\vec{\boldsymbol{g}}_a \cdot \vec{\boldsymbol{g}}_a) - \vec{\boldsymbol{g}}_a(\vec{\boldsymbol{g}}_a \cdot \widehat{\boldsymbol{R}}) \\
\vec{\boldsymbol{g}}_a \times \vec{\boldsymbol{h}}_a &= g_a^2\widehat{\boldsymbol{R}} - \vec{\boldsymbol{g}}_a(\vec{\boldsymbol{g}}_a \cdot \widehat{\boldsymbol{R}}),
\end{aligned} \tag{V.496}$$

since $\vec{\boldsymbol{g}}_a$ is perpendicular to $\widehat{\boldsymbol{R}}$ the second term vanishes, thus,

$$\vec{\boldsymbol{g}}_a \times \vec{\boldsymbol{h}}_a = g_a^2\widehat{\boldsymbol{R}} \tag{V.497}$$

and the Poynting vector is

$$\vec{\mathcal{S}}_a = \frac{-c_g}{4\pi G} g_a^2 \widehat{\boldsymbol{R}}. \tag{V.498}$$

Squaring Equation V.492,

$$\begin{aligned}
g_a^2 &= \frac{G^2 m^2}{c_g^4 R^6}\left[\vec{\boldsymbol{R}}(\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{a}}) - R^2\vec{\boldsymbol{a}}\right]^2 \\
&= \frac{G^2 m^2}{c_g^4 R^6}\left[\vec{\boldsymbol{R}}^2(\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{a}})^2 + (R^2\vec{\boldsymbol{a}})^2 - 2R^2(\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{R}})(\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{a}})^2\right] \\
&= \frac{G^2 m^2}{c_g^4 R^6}\left[R^2(\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{a}})^2 + R^4 a^2 - 2R^2(\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{a}})^2\right] \\
&= \frac{G^2 m^2}{c_g^4 R^4}\left[R^2 a^2 - (\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{a}})^2\right] \qquad \text{(V.499)} \\
&= \frac{G^2 m^2}{c_g^4 R^4}(R^2 a^2 - R^2 a^2\cos^2\theta) \\
&= \frac{G^2 m^2 a^2}{c_g^4 R^2}(1 - \cos^2\theta) \\
g_a^2 &= \frac{G^2 m^2 a^2 \sin^2\theta}{c_g^4 R^2}.
\end{aligned}$$

Now, the Poynting vector is

$$\begin{aligned}
\vec{\mathcal{S}}_a &= \frac{-\cancel{c_g}}{4\pi \cancel{G}}\left(\frac{G^{\cancel{2}} m^2 a^2 \sin^2\theta}{c_g^{\cancel{4}\,3} R^2}\right)\widehat{\boldsymbol{R}} \\
&= \frac{-G m^2 a^2 \sin^2\theta}{4\pi c_g^3 R^2}\widehat{\boldsymbol{R}}.
\end{aligned} \tag{V.500}$$

The Poynting vector is an energy flow vector with units of energy flux (per unit area per unit time). The power radiated by the particle can be found by expressing the angular distribution of radiation as the power radiated in relation to an integrable "solid-angle" and multiplying the projection of the Poynting vector in the $\widehat{\boldsymbol{R}}$ direction by R^2 "i.e., by the area-per-unit-solid-angle at the radius R" [4]. Hence,

$$\frac{dP}{d\Omega} = (\vec{\mathcal{S}}_a \cdot \widehat{\boldsymbol{R}})R^2 = \frac{-G m^2 a^2 \sin^2\theta}{4\pi c_g^3}. \tag{V.501}$$

The total power radiated is found by integrating over the volume (sphere),

$$\begin{aligned}
P &= \int_{4\pi} \frac{dP}{d\Omega} d\Omega \\
&= \frac{-G m^2 a^2}{4\pi c_g^3}\int_0^{2\pi} d\varphi \int_0^{\pi} (\sin^2\theta)\cdot \sin\theta d\theta \\
&= \frac{-G m^2 a^2}{4\pi c_g^3}(2\pi)\left(\frac{4}{3}\right) \\
P &= \frac{-G m^2 a^2}{c_g^3}\left(\frac{2}{3}\right).
\end{aligned} \tag{V.502}$$

So,

$$P = \frac{-2Gm^2a^2}{3c_g^3}\,. \tag{V.503}$$

Radiation From a Mass With Collinear Velocity and Acceleration

Following Section (8.7) of Heald & Marion, analogously: The acceleration component of the fields carry radiation, and for collinear velocity and acceleration, $\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{a}} = 0$. Replacing the scaled acceleration vector $\dot{\vec{\boldsymbol{\beta}}}$ with the standard acceleration $\vec{\boldsymbol{a}}$ as defined in Equation V.493, Equation V.380 becomes,

$$\vec{\boldsymbol{g}}_a = \frac{-Gm}{c_g^2R^3\big(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^3}\Big[\vec{\boldsymbol{R}}\times(\vec{\boldsymbol{R}}\times\vec{\boldsymbol{a}})\Big]\,, \tag{V.504}$$

which is recognizably the same as that for an accelerated mass at low speeds, however, with one obvious difference: the denominator is now regulated by the speed proportion vector $\vec{\boldsymbol{\beta}} = \vec{v}/c_g$. This means it can have unrestricted values (from zero to one), and thus relativistically fast masses (which is improbable). And it has already been shown that (using the BAC-CAB rule) that the above gravitational acceleration becomes:

$$\vec{\boldsymbol{g}}_a = \frac{-Gm}{c_g^2R^3\big(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^3}\Big[\vec{\boldsymbol{R}}(\vec{\boldsymbol{R}}\cdot\vec{\boldsymbol{a}}) - R^2\vec{\boldsymbol{a}}\Big]\,, \tag{V.505}$$

which is equivalent to Equation V.492 with the only difference, again, being the relativistic dependency on $\vec{\boldsymbol{\beta}}$. Interpreting such scenarios accurately requires careful consideration that the radiation observed at time, t, was emitted at the retarded time, t'. To determine the power radiated by a particle (mass) in such relativistic scenarios and the shape of the distribution of radiation, consider again the projection $\vec{\mathcal{S}}_a\cdot\widehat{\boldsymbol{R}}$ as component of the Poynting vector in the outward direction (or inward, really just radially) elevated at the present time, t, which depends on the radiation at the retarded time, t'.

The differential energy emitted by the particle is radiated over the unit-solid-angle θ (as before) and is measured over the present time differential dt,

$$-dW(\theta) = (\vec{\mathcal{S}}_a\cdot\widehat{\boldsymbol{R}})R^2dt\,. \tag{V.506}$$

Thus, the amount of power radiated into a unit solid angle and crosses a surface at a distance R at a present time, t, is equal to the energy-per-unit-time emitted by the particle at the retarded time, t'.
Hence,

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{-dW(\theta)}{dt'}\\ &= (\vec{\mathcal{S}}_a\cdot\widehat{\boldsymbol{R}})R^2\frac{dt}{dt'}\,.\end{aligned} \tag{V.507}$$

And since the only change to the Poynting vector is the denominator, then,

$$\begin{aligned}\vec{\mathcal{S}}_a &= \frac{-c_g}{4\pi G}g_a^2\widehat{\boldsymbol{R}}\\ &= \frac{-Gm^2a^2\sin^2\theta\,\widehat{\boldsymbol{R}}}{4\pi c_g^3R^2\big(1-\widehat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^6}\,.\end{aligned} \tag{V.508}$$

Plugging Equation V.508 into Equation V.507,

$$\begin{aligned}\frac{dP}{d\Omega} &= \left[\frac{-Gm^2a^2\sin^2\theta\hat{\boldsymbol{R}}}{4\pi c_g^3R^2(1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^6}\cdot\hat{\boldsymbol{R}}\right]R^2\frac{dt}{dt'} \\ &= \frac{-Gm^2a^2\sin^2\theta}{4\pi c_g^3(1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}})^6}\frac{dt}{dt'}\,.\end{aligned} \tag{V.509}$$

From Equation V.352, and rearranging,

$$\begin{aligned}\frac{dt}{dt'} &= 1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}} \\ &= 1-\beta\cos\theta\,.\end{aligned} \tag{V.510}$$

So, plugging in Equation V.510 into Equation V.509 yields the following,

$$\frac{dP}{d\Omega} = \frac{-Gm^2a^2\sin^2\theta}{4\pi c_g^3(1-\beta\cos\theta)^6}(1-\beta\cos\theta)\,. \tag{V.511}$$

Thus, the power per unit sold angle radiated by a mass with collinear velocity and acceleration is:

$$\frac{dP}{d\Omega} = \frac{-Gm^2a^2\sin^2\theta}{4\pi c_g^3(1-\beta\cos\theta)^5}\,, \tag{V.512}$$

which one can easily see reduces to Equation V.501 when $\beta << 1$. One can also show that the total power radiated is:

$$P = \frac{-2Gm^2a^2}{3c_g^3(1-\beta^2)^3}\,. \tag{V.513}$$

Radiation From a Mass Confined to a Circular Orbit

Following Section (8.8) of Heald and Marion, analogously, In circular motion, the acceleration is perpendicular to the velocity, so the field component related to acceleration and radiation is unchanged. From Equation V.404, and using the standard acceleration notation,

$$\begin{aligned}\vec{\boldsymbol{g}}_a &= -Gm\left[\frac{\hat{\boldsymbol{R}}\times\Big((\hat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}})\times\vec{\boldsymbol{a}}\Big)}{c_g^2R\big(1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^3}\right] \\ &= \frac{-Gm\Big[\hat{\boldsymbol{R}}\times(\vec{\boldsymbol{b}}\times\vec{\boldsymbol{a}})\Big]}{c_g^2R\big(1-\hat{\boldsymbol{R}}\cdot\vec{\boldsymbol{\beta}}\big)^3}\,,\end{aligned} \tag{V.514}$$

where $\vec{\boldsymbol{b}} = \hat{\boldsymbol{R}}-\vec{\boldsymbol{\beta}}$.

So, to determine the power radiated per unit solid angle as before, we need to determine the following:

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{-dW(\theta)}{dt'} \\ &= (\vec{\mathcal{S}}_a\cdot\hat{\boldsymbol{R}})R^2\frac{dt}{dt'}\,,\end{aligned} \tag{V.515}$$

where

$$\vec{\mathcal{S}}_a = \frac{-c_g}{4\pi G} g_a^2 \widehat{\boldsymbol{R}} \tag{V.516}$$

so,

$$\frac{dP}{d\Omega} = \frac{-c_g}{4\pi G} g_a^2 R^2 (1 - \beta\cos\theta) \tag{V.517}$$

since, from Equation V.352,

$$\frac{dt}{dt'} = 1 - \beta\cos\theta\,. \tag{V.518}$$

Determining g_a^2 by squaring Equation V.514,

$$\begin{aligned} \vec{\boldsymbol{g}}_a^2 = g_a^2 &= \frac{G^2 m^2}{c_g^4 R^2 (1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}})^6} \Big[\widehat{\boldsymbol{R}} \times (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{a}})\Big]^2 \\ &= \frac{G^2 m^2}{c_g^4 R^2 (1 - \beta\cos\theta)^6} \Big[(\widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{a}})\vec{\boldsymbol{b}} - (\widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{b}})\vec{\boldsymbol{a}}\Big]^2 \\ g_a^2 &= \frac{G^2 m^2 \Big[(\widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{a}})^2 b^2 + (\widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{b}})^2 a^2 - 2(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}})(\widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{a}})(\widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{b}})\Big]}{c_g^4 R^2 (1 - \beta\cos\theta)^6}\,, \end{aligned} \tag{V.519}$$

where

$$\begin{aligned} \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} &= \vec{\boldsymbol{a}} \cdot (\widehat{\boldsymbol{R}} - \vec{\boldsymbol{\beta}}) \\ &= \vec{\boldsymbol{a}} \cdot \widehat{\boldsymbol{R}} - \cancelto{0 \text{ (orthogonal)}}{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{\beta}}} \\ &= \vec{\boldsymbol{a}} \cdot \widehat{\boldsymbol{R}} \\ &= \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{a}} \\ \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} &= a \sin\theta \cos\varphi \end{aligned} \tag{V.520}$$

and

$$\begin{aligned} \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{b}} &= \widehat{\boldsymbol{R}} \cdot (\widehat{\boldsymbol{R}} - \vec{\boldsymbol{\beta}}) \\ &= \widehat{\boldsymbol{R}} \cdot \widehat{\boldsymbol{R}} - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}} \\ &= 1 - \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}} \\ \widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{b}} &= 1 - \beta\cos\theta \end{aligned} \tag{V.521}$$

also,

$$\begin{aligned} b^2 &= (\vec{\boldsymbol{b}})^2 \\ &= (\widehat{\boldsymbol{R}} - \vec{\boldsymbol{\beta}})^2 \\ &= \widehat{\boldsymbol{R}}^2 + \vec{\boldsymbol{\beta}}^2 - 2(\widehat{\boldsymbol{R}} \cdot \vec{\boldsymbol{\beta}}) \\ b^2 &= 1 + \beta^2 - 2\beta\cos\theta\,. \end{aligned} \tag{V.522}$$

To fully justify the geometrical relationships used previously, consider the following:
If the orbit is confined to the $x_2 - x_3$ plane, then it is fairly easy to determine the relation between $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{R}}$. It does not depend on the new azimuthal angle φ and is easily related by the projection of $\vec{\boldsymbol{v}}$ on $\vec{\boldsymbol{R}}$. Thus,

$$\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{R}} = vR\cos\theta\,. \tag{V.523}$$

Note: The projection of $\vec{\boldsymbol{v}}$ on $\vec{\boldsymbol{R}}$ corresponds to the cosine of the angle θ multiplied by the magnitude (hypotenuse).

Equation V.523 can be rewritten as

$$\frac{\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{R}}}{c_g R} = \frac{v \cos\theta}{c_g} \,. \tag{V.524}$$

Or alternatively,

$$\vec{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{R}} = \beta \cos\theta \tag{V.525}$$

which we would expect!

One can also show that,

$$\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{R}} = aR \sin\theta \cos\varphi \,, \tag{V.526}$$

according to Heald and Marion (Equation 8.99) [4]. To get $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{R}}$ or $\vec{\boldsymbol{R}} \cdot \vec{\boldsymbol{v}}$ we simply take the projection of $\vec{\boldsymbol{v}}$ in the $\vec{\boldsymbol{R}}$ direction or $\vec{\boldsymbol{R}}$ in the $\vec{\boldsymbol{v}}$ direction. This corresponds to the cosine of the angle between them (θ) multiplied by the magnitudes (hypotenuse)

$$\therefore \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{R}} = |\vec{\boldsymbol{v}}||\vec{\boldsymbol{R}}| \cos\theta = vR\cos\theta \,. \tag{V.527}$$

To get$\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{R}}$ or $\vec{\boldsymbol{R}} \cdot \vec{\boldsymbol{a}}$ we simply deconstruct the projection such that it is the projection of $\vec{\boldsymbol{R}}$ in the θ direction projected on to $\vec{\boldsymbol{a}}$ in the φ direction. Since φ and θ are orthogonal, the projection multiplies like the sin in the cross-product. Thus,

$$\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{R}} = |\vec{\boldsymbol{a}}||\vec{\boldsymbol{R}}| \sin\theta \cos\varphi \,, \tag{V.528}$$

which can also be written as,

$$\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{R}} = |\vec{\boldsymbol{a}}||\vec{\boldsymbol{R}}| \sin\theta \cos\varphi = aR\sin\theta\cos\varphi \tag{V.529}$$

so,

$$\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{R}} = a \sin\theta \cos\varphi \,. \tag{V.530}$$

Now, plugging everything in to Equation V.517,

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{\cancel{-c_g}}{4\pi \cancel{G}} \left\{ \frac{G^{\cancel{2}} m^2 \left[(\hat{\boldsymbol{R}} \cdot \vec{\boldsymbol{a}})^2 b^2 + (\hat{\boldsymbol{R}} \cdot \vec{\boldsymbol{b}})^2 a^2 - 2(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}})(\hat{\boldsymbol{R}} \cdot \vec{\boldsymbol{a}})(\hat{\boldsymbol{R}} \cdot \vec{\boldsymbol{b}}) \right]}{c_g^{\overset{3}{\cancel{4}}} \cancel{R^2} (1-\beta\cos\theta)^{\overset{5}{\cancel{6}}} \cancel{R^{-2}} \cancel{(1-\beta\cos\theta)^{-1}}} \right\} \\
&= \frac{-Gm^2}{4\pi c_g^3 (1-\beta\cos\theta)^5} \{ (a\sin\theta\cos\varphi)^2 (1+\beta^2 - 2\beta\cos\theta) + (1-\beta\cos\theta)^2 a^2 \cdots \\
&\quad \cdots - 2(a\sin\theta\cos\varphi)^2 (1-\beta\cos\theta) \} \\
&= \frac{-Gm^2}{4\pi c_g^3 (1-\beta\cos\theta)^5} \{ \cancel{a^2 \sin^2\theta \cos^2\varphi} + \beta^2 a^2 \sin^2\theta\cos^2\varphi \cdots \\
&\quad \cdots - \cancel{2\beta\cos\theta a^2 \sin^2\theta\cos^2\varphi} + a^2 (1-\beta\cos\theta)^2 - \overset{1}{\cancel{2}}\, a^2 \sin^2\theta\cos^2\varphi \cdots \\
&\quad \cdots + \cancel{2\beta\cos\theta^2 \sin^2\theta\cos^2\varphi} \} \\
&= \frac{-Gm^2 a^2}{4\pi c_g^3 (1-\beta\cos\theta)^5} \{ (1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\varphi \} \\
\frac{dP}{d\Omega} &= \frac{-Gm^2 a^2 \{ (1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\varphi \}}{4\pi c_g^3 (1-\beta\cos\theta)^5} \,.
\end{aligned} \tag{V.531}$$

Thus, for a mass moving in a circular orbit with a relativistic velocity $\vec{\boldsymbol{\beta}} c_g = \vec{\boldsymbol{v}}$, the power radiated per unit solid angle is given by

$$\frac{dP}{d\Omega} = \frac{-Gm^2a^2\{(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\varphi\}}{4\pi c_g^3(1-\beta\cos\theta)^5}\,, \tag{V.532}$$

which likewise reduces to Equation V.501 at low speeds. One can show that the total power is:

$$P = \int_{4\pi} \frac{dP}{d\Omega} d\Omega = \frac{-2Gm^2a^2}{3c_g^3(1-\beta^2)^2}\,. \tag{V.533}$$

Also in considering the more likely scenario that the mass is moving such that $\beta << 1$, then $\dfrac{dP}{d\Omega}$ becomes,

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{-Gm^2a^2\{-\sin^2\theta\cos^2\varphi\}}{4\pi c_g^3(1-0)^5} \\ &= \frac{Gm^2a^2\sin^2\theta\cos^2\varphi}{4\pi c_g^3}\,. \end{aligned} \tag{V.534}$$

Appendix H: The Lorentz Force, Fictitious Forces, and the Zeeman Effect

The Lorentz force will be "derived" for electromagnetism and determined for gravitation by analogy. The results will be interpreted as they relate to the known forces and fictitious forces. A section regarding the Zeeman effect may be included to discuss the effect of specific reference frame rotation rates and the "visibility" of fictitious forces.

"Derivation" of Lorentz Force for Electromagnetism and Gravitation

Start with the force exerted by a current-carrying wire with current, I, cross-sectional area, A, length, l, and an induced magnetic field, $\vec{\boldsymbol{B}}$. From experiment:

$$\vec{\boldsymbol{F}} = I\,\vec{\boldsymbol{l}} \times \vec{\boldsymbol{B}}\,, \tag{V.535}$$

where

$$I = J \cdot A \tag{V.536}$$

and

$$\vec{\boldsymbol{J}} = \rho\vec{\boldsymbol{v}}\,. \tag{V.537}$$

Then,

$$\begin{aligned} d\vec{\boldsymbol{F}} &= Id\,\vec{\boldsymbol{l}} \times \vec{\boldsymbol{B}} \\ &= JAd\,\vec{\boldsymbol{l}}\,\hat{\boldsymbol{i}} \times \vec{\boldsymbol{B}} \\ &= \rho vAd\,\vec{\boldsymbol{l}}\,\hat{\boldsymbol{i}} \times \vec{\boldsymbol{B}} \\ d\vec{\boldsymbol{F}} &= \rho dV\,\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}\,. \end{aligned} \tag{V.538}$$

Recall that

$$\vec{\boldsymbol{F}} = q\vec{\boldsymbol{E}} \tag{V.539}$$

and

$$dq = \rho dV \tag{V.540}$$

so,

$$d\vec{\boldsymbol{F}} = dq\vec{\boldsymbol{E}}\,. \tag{V.541}$$

Thus,

$$\begin{aligned} d\vec{\boldsymbol{F}} &= dq\vec{\boldsymbol{E}} + \rho dV\,\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}} \\ &= dq\vec{\boldsymbol{E}} + dq\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}\,. \end{aligned} \tag{V.542}$$

Finally, in SI units, the Lorentz force is given by:

$$\vec{\boldsymbol{F}} = q(\vec{\boldsymbol{E}} + \vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}) \tag{V.543}$$

or, in CGS units,

$$\vec{\boldsymbol{F}} = q\left(\vec{\boldsymbol{E}} + \frac{\vec{\boldsymbol{v}}}{c} \times \vec{\boldsymbol{B}}\right). \tag{V.544}$$

Now, writing the "Lorentz Force" for gravitation in CGS units by analogy,

$$\vec{\boldsymbol{F}}_g = m\left(\vec{\boldsymbol{g}} + \frac{\vec{\boldsymbol{v}}}{c_g} \times \vec{\boldsymbol{h}}\right). \tag{V.545}$$

Fictitious Forces in Gravitation

If we decompose Equation V.545,

$$\vec{\boldsymbol{F}}_g = m\vec{\boldsymbol{g}} + \frac{m\vec{\boldsymbol{v}}}{c_g} \times \vec{\boldsymbol{h}}\,, \tag{V.546}$$

it is obvious that the first term is directly analogous to Newton's Second Law:

$$\vec{\boldsymbol{F}}_{\text{Newton}} = m\vec{\boldsymbol{a}}\,, \tag{V.547}$$

where $\vec{\boldsymbol{a}} = \vec{\boldsymbol{g}}$ as expected! But how does the second term relate in this analogy? At first it isn't entirely obvious, though after rearranging some terms, it becomes rather clear that the second term is the Coriolis force. The Coriolis is considered a "fictitious" force, as it results from a rotating reference frame, and is usually defined as,

$$\vec{\boldsymbol{F}}_{\text{Coriolis}} = -2m\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}\,. \tag{V.548}$$

Rearranging the Coriolis force,

$$\vec{\boldsymbol{F}}_{\text{Coriolis}} = m\vec{\boldsymbol{v}} \times 2\vec{\boldsymbol{\omega}}\,, \tag{V.549}$$

and comparing to the second term in Equation V.546 after shifting the factor of $1/c_g$,

$$\vec{\boldsymbol{F}}_{g2} = m\vec{\boldsymbol{v}} \times \frac{\vec{\boldsymbol{h}}}{c_g}\,, \tag{V.550}$$

it is possible to relate the angular velocity vector in the Coriolis force to the cogravitational force.

$$2\vec{\boldsymbol{\omega}} = \frac{\vec{\boldsymbol{h}}}{c_g} \tag{V.551}$$

so,

$$\vec{\boldsymbol{\omega}} = \frac{\vec{\boldsymbol{h}}}{2c_g}\,, \tag{V.552}$$

which is known as the Larmor frequency. The Coriolis force can then be written as

$$\begin{aligned}\vec{\boldsymbol{F}}_{\text{Coriolis}} &= m\vec{\boldsymbol{v}} \times 2\vec{\boldsymbol{\omega}} \\ &= m\vec{\boldsymbol{v}} \times 2\frac{\vec{\boldsymbol{h}}}{2c_g} \\ &= m\vec{\boldsymbol{v}} \times \frac{\vec{\boldsymbol{h}}}{c_g} \\ \vec{\boldsymbol{F}}_{\text{Coriolis}} &= \frac{m\vec{\boldsymbol{v}}}{c_g} \times \vec{\boldsymbol{h}}\,.\end{aligned} \tag{V.553}$$

Now, it has been shown that the second term in the "Lorentz force" for gravitation is the fictitious Coriolis force. Then, we can write

$$\vec{\boldsymbol{F}}_g = m\vec{\boldsymbol{g}} - 2m\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}\,, \tag{V.554}$$

$$\vec{\boldsymbol{F}_g} = \vec{\boldsymbol{F}}_{\text{Newton}} + \vec{\boldsymbol{F}}_{\text{Coriolis}}\,. \tag{V.555}$$

The occurrence of one fictitious force but not another - the centrifugal force - is rather curious. However, the centrifugal force and the Newtonian force are radial forces (with opposite directions). Considering this, it can be shown that the centrifugal force is buried within an "effective" gravitational force.

Method 1

When looking at relative motion using Newtonian dynamics, one writes the force in the fixed frame as related to the force in the rotating frame by:

$$\begin{aligned}\vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}}_f \\ &= m\Big(\vec{\boldsymbol{a}}_r + \vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{r}} + 2\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}_r + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})\Big)\,,\end{aligned} \tag{V.556}$$

where the force due to linear acceleration in the rotating frame is,

$$\vec{\boldsymbol{F}}_{\text{Newton}} = m\vec{\boldsymbol{a}}_r\,, \tag{V.557}$$

the Euler force due to angular acceleration in the rotating frame is,

$$\vec{\boldsymbol{F}}_{\text{Euler}} = -m\vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{r}}\,, \tag{V.558}$$

the fictitious Coriolis force due to rotation in the rotating frame is defined as,

$$\vec{\boldsymbol{F}}_{\text{Coriolis}} = -2m\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}\,. \tag{V.559}$$

The centrifugal force is also a "fictitious" force and is defined as,

$$\vec{\boldsymbol{F}}_{\text{Centrifugal}} = -m\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})\,. \tag{V.560}$$

Consider an inertial rotating reference frame ($\vec{\boldsymbol{\alpha}} = 0$).

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}_f = m\vec{\boldsymbol{a}}_r + 2m\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}_r + m\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})\,. \tag{V.561}$$

Rearranging,

$$\begin{aligned}m\vec{\boldsymbol{a}}_r &= m\vec{\boldsymbol{a}}_f - 2m\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}_r - m\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}) \\ &= m\vec{\boldsymbol{a}}_f + m\vec{\boldsymbol{v}}_r \times 2\vec{\boldsymbol{\omega}} - m\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}) \\ &= ma_f\hat{\boldsymbol{g}} + m\vec{\boldsymbol{v}}_r \times 2\vec{\boldsymbol{\omega}} - m\omega^2 r\hat{\boldsymbol{g}} \\ m\vec{\boldsymbol{a}}_r &= m(a_f - r\omega^2)\hat{\boldsymbol{g}} + m\vec{\boldsymbol{v}}_r \times 2\vec{\boldsymbol{\omega}}\,.\end{aligned} \tag{V.562}$$

Recall that the centripetal acceleration is defined as,

$$a_c = \frac{v^2}{r} \tag{V.563}$$

and

$$v = r\omega \tag{V.564}$$

so,

$$a_c = r\omega^2\,. \tag{V.565}$$

Then,

$$m\vec{\boldsymbol{a}}_r = m(a_f - a_c)\hat{\boldsymbol{g}} + m\vec{\boldsymbol{v}}_r \times 2\vec{\boldsymbol{\omega}}\,. \tag{V.566}$$

Since, as was claimed earlier, the Newtonian acceleration, a_f, and the centrifugal acceleration, a_c, occur on the same axis so it is possible to define an "effective" acceleration as related by the gravitational acceleration and the opposing centrifugal acceleration.

$$\vec{\boldsymbol{g}}_{\text{eff}} = (a_f - a_c)\hat{\boldsymbol{g}}\,, \tag{V.567}$$

and using the Larmor frequency from earlier, an "effective" cogravitational acceleration is defined as

$$\vec{\boldsymbol{h}}_{\text{eff}} = 2c_g\vec{\boldsymbol{\omega}}\,. \tag{V.568}$$

Thus,

$$m\vec{\boldsymbol{a}}_r = m\vec{\boldsymbol{g}}_{\text{eff}} + m\vec{\boldsymbol{v}}_r \times \frac{\vec{\boldsymbol{h}}_{\text{eff}}}{c_g}\,. \tag{V.569}$$

Rearranging to match with the Lorentz force notation in CGS units,

$$m\vec{\boldsymbol{a}}_r = m\left(\vec{\boldsymbol{g}}_{\text{eff}} + \frac{\vec{\boldsymbol{v}}_r}{c_g} \times \vec{\boldsymbol{h}}_{\text{eff}}\right). \tag{V.570}$$

So the Lorentz force for gravitation corresponds to linear acceleration in the rotating frame and is the sum of the effective gravitational force and the Coriolis cogravitational force. The effective gravitational acceleration is the gravitational acceleration $\vec{\boldsymbol{g}}$, as before, and the effective cogravitational acceleration is the cogravitational acceleration $\vec{\boldsymbol{h}}$. Then, finally,

$$\vec{\boldsymbol{F}}_g = m\left(\vec{\boldsymbol{g}} + \frac{\vec{\boldsymbol{v}}}{c_g} \times \vec{\boldsymbol{h}}\right). \tag{V.571}$$

Method 2

Starting with Equation V.554,

$$\vec{\boldsymbol{F}}_g = m\vec{\boldsymbol{g}} - 2m\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}\,. \tag{V.572}$$

Defining an effective gravitational acceleration as related by the Newtonian acceleration, a_f, and the opposing centrifugal acceleration, a_c,

$$\vec{\boldsymbol{g}}_{\text{eff}} = (a_f - a_c)\hat{\boldsymbol{g}}\,. \tag{V.573}$$

Plugging in for $\vec{\boldsymbol{g}}$,

$$\begin{aligned}\vec{\boldsymbol{F}}_g &= m(a_f - a_c)\hat{\boldsymbol{g}} - 2m\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}} \\ &= m\vec{\boldsymbol{a}}_f - m\vec{\boldsymbol{a}}_c - 2m\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}\,.\end{aligned} \tag{V.574}$$

Since the centrifugal acceleration can be expressed as,

$$\vec{\boldsymbol{a}}_c = \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})\,. \tag{V.575}$$

The force can now be expressed in a familiar way

$$\vec{\boldsymbol{F}}_g = m\vec{\boldsymbol{a}}_f - m\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}) - 2m\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}} \tag{V.576}$$

or

$$\vec{\boldsymbol{F}}_g = m\vec{\boldsymbol{a}}_f - 2m\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}} - m\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})\,, \tag{V.577}$$

where the first term is the gravitational force in the fixed frame, the second is the Coriolis force, and the third is the missing centrifugal force! Of course, the Euler force is missing because angular accelerations were not considered.

VITA

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