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Passivity-Based Control of Sampled-Data Systems on Lie Groups with Linear Outputs ⋆

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Abstract: We present a method of stabilizing a sampled-data system that evolves on a matrix Lie group using passivity. The continuous-time plant is assumed passive with known storage function, and its passivity is preserved under sampling by redefining the output of the discretized plant and keeping the storage function. We show that driftlessness is a necessary condition for a sampled-data system on a matrix Lie group to be zero-state observable. The closed-loop sampled-data system is stabilized by any strictly passive controller, and we present a synthesis procedure for a strictly positive real LTI controller. The closed-loop system is shown to be asymptotically stable. This stabilization method is applied to asymptotic tracking of piecewise constant references.

Keywords: nonlinear control, discrete-time systems, stabilization, Lie groups, passivity

1. INTRODUCTION

Systems on matrix Lie groups are common in engineering applications. Rotational dynamics, such as those of UAVs, evolve on $\mathsf{SO}(3)$ or $\mathsf{SO}(2)$; if translational dynamics are also considered, they evolve on $\mathsf{SE}(3)$ (Roza and Maggiore, 2012) or $\mathsf{SE}(2)$ (Justh and Krishnaprasad, 2004), respectively. Mobile ground robots can be modelled on $\mathsf{SE}(2)$, and quantum systems on $\mathsf{SU}(n)$ (Altafini and Ticozzi, 2012). The Kuramoto oscillator evolves on the circle (Dörfler and Bullo, 2014), $\mathsf{SO}(2)$.

We refer the reader to (Sachkov, 2009) for a recent treatment of control theory on Lie groups. Control theory on Lie groups differs from classical control in that the state space is not a Euclidean vector space. Such systems are usually controlled using coordinate charts on the Lie group to represent the system dynamics in local coordinates as systems in $\mathbb{R}^n$. This effects artificial singularities that arise from the choice of local coordinates as opposed to being intrinsic to the system's dynamics.

Control on Lie groups has also been treated using global coordinates, for example, motion tracking in $\mathsf{SE}(3)$ (Park and Kim, 2014), the control of UAV (Forbes, 2013) and spacecraft (Egeland and Godhavn, 1994) orientation on $\mathsf{SO}(3)$, and the synchronization of networks of rigid bodies on $\mathsf{SE}(3)$ (Igarashi et al., 2009). The latter two works take an input-output approach, specifically the notion of passivity, for their control design and analysis.

The sampled-data setup, i.e., a continuous-time plant and a discrete-time controller, is ubiquitous in applied control systems. In practice, controllers are often designed in continuous-time and it is assumed that the sampling period is small enough that the sampled-data behaviour will adequately match the theoretical continuous-time behaviour. But, in general, such setups are not guaranteed to even be stable (Nešić and Teel, 2004). Lyapunov-like sufficient conditions for the preservation of closed-loop stability of nonlinear systems under sampling were identified in (Karafyllis and Kravaris, 2009). In addition, the standard notion of discrete-time passivity is not preserved under sampling and this directly affects the sampled-data design (Monaco et al., 2011). Passivity analysis under sampling has been studied in (Costa-Castelló and Fossas, 2006; Nešić et al., 1999; Stramigioli et al., 2005), but remains a challenging unsolved problem. In the LTI case it is known (Hoagg et al., 2004; De La Sen, 2002) that passivity is preserved under step-invariant discretization, but the two systems generally have different storage functions (Costa-Castelló and Fossas, 2006).

Analytic solutions to the sampled discrete-time dynamics of nonlinear systems do not exist, in general. Right invariant systems on matrix Lie groups are an exception, in that their sampled discrete-time time dynamics have closed-form solutions (Elliott, 2009). The closely related class of bilinear systems has received some attention in the discrete-time (Elliott, 2009) and sampled-data settings (Sontag, 1986; Omran et al., 2014).

We present a method of controller design for the stabilization, and consequently setpoint tracking, of passive continuous-time plants on matrix Lie groups in a sampled-data setting. The proposed analysis and design is done entirely in global coordinates. Our contributions are: 1) a method of preserving passivity for systems on matrix Lie groups, under sampling based on the results from (Costa-Castelló and Fossas, 2006); 2) stability analysis and control design for these systems; 3) proof that driftlessness is a

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necessary condition for systems on matrix Lie groups to be zero-state observable; 4) a controller synthesis procedure.

1.1 Notation and Terminology

Let $\mathsf{GL}(n)$ denote the real general linear group, i.e., the group of real, invertible, $n \times n$ matrices. If G is a matrix Lie group [1], $\mathfrak{g}$ is its associated Lie algebra. Given a matrix $M \in \mathbb{R}^{n\times n}$, trace$(M)$ denotes its trace; $M \succ 0$ ($M \succeq 0$) denotes that M is positive (semi) definite.

If $A \in \mathbb{R}^{m\times n}$, then $A^\top$ denotes its transpose and vec(A) is its vectorization, i.e., $\mathrm{vec}(A) := [A_{11} \cdots A_{m1}\ A_{21} \cdots A_{mn}]^\top \in \mathbb{R}^{mn}$. Given $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, let $A \otimes B \in \mathbb{R}^{mp\times nq}$ denote their Kronecker product. If $x \in \mathbb{R}^n$, then $\|x\|$ denotes its Euclidean norm.

A positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ is proper, or radially unbounded, if there exists a class $\mathcal{K}_\infty$ function α such that, for all $x \in \mathbb{R}^n$, $\alpha(\|x\|) \le V(x)$.

2. CLASS OF SYSTEMS

We consider systems on Lie groups with linear outputs:

$$\dot{X}(t) = \left(A + \sum_{i=1}^{m} B_i u_i(t)\right) X(t) \quad (1a)$$

$$y(t) = C\mathrm{vec}(X(t)) + Du(t), \quad (1b)$$

where the input $u := [u_1 \cdots u_m]^\top \in \mathbb{R}^m$, the state $X \in \mathsf{G}$, and $A, B_i \in \mathfrak{g}$ for $i \in \{1, \ldots, m\}$. Study of this class of systems is motivated by works such as (Igarashi et al., 2009) and (Forbes, 2013). Restraining the output to be linear allows for closed-form expressions of some of our results, but these results generalize in a straightforward manner to systems with nonlinear outputs.

Our controller design and stability analysis depend on the concept of passivity. In Section 4, we assume that (1) is zero-state observable (Byrnes et al., 1991) and show this class of system is zero-state observable only if it is driftless, i.e., $A = 0$. However, the discussion and results theretofore do not require driftlessness. We also show that zero-state observability is preserved under sampling.

Assumption 1. (Passivity). System (1) is passive.

The class of system (1a) is called the class of right invariant systems on matrix Lie groups [2]. The solution to (1a) for piecewise constant inputs is (Elliott, 2009, Equation 1.30)

$$X(t) = \exp\left((t - t_k)\left(A + \sum_{i=1}^{m} B_i u_i(t_k)\right)\right) X(t_k), \quad (2)$$

for $t \in [t_k, t_{k+1})$, where $0 < t_1 < \cdots < t_k < t_{k+1}$ and $u(t)$ is constant on each interval $[t_k, t_{k+1})$.

We are interested in the sampled-data plant configuration illustrated in Figure 1, where H and S are, respectively, ideal hold and sample operators.

Like LTI systems, system (1a) has a closed-form exact solution under sampling with period T.

[1] $\mathsf{G} \subset \mathsf{GL}(n)$ is a matrix Lie group if it is closed in $\mathsf{GL}(n)$.

[2] Any right invariant system can be rewritten as the left invariant system $\dot{Y} = -Y(A + \sum_{i=1}^{m} B_i u_i)$, where $Y := X^{-1}$.

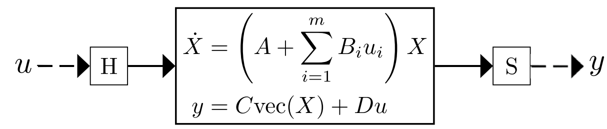

Fig. 1. Sampled-data plant.

Proposition 2.1. (Step-Invariant Transform). Given the continuous-time system (1a), the discrete-time system

$$X^+ = \exp\left(T\left(A + \sum_{i=1}^{m} B_i u_i\right)\right) X, \quad (3)$$

which is the sampled system illustrated in Figure 1, has the same step response at the sampling instants.

Proof Set $t_k := kT$, $T > 0$, in (2), then $X(t) = \exp\left(T\left(A + \sum_{i=1}^{m} B_i u_i(kT)\right)\right) X(kT),\ t \in [kT, (k+1)T)$. Sampling with period T yields (3). □

Note that (3) is an exact discretization for any u that is constant between the sampling instants.

3. CONTROL OBJECTIVE

We address the problem of stabilizing the identity of (3). This is itself an important problem, and easily extends to constant reference tracking, as discussed in Section 6.

3.1 Loss of Passivity under Sampling

The passivity of (1) does not guarantee the passivity of (3) with output (1b) and the same storage function.

Example 3.1. Consider the system on $\mathsf{SO}(3)$:

$$\dot{X} = u^\times X, \qquad y = \frac{1}{2}(X - X^\top)^\vee, \quad (4)$$

where $^\times : \mathbb{R}^3 \to \mathfrak{so}(3)$ is the standard map from vectors to skew symmetric matrices, and $^\vee$ is its inverse. We use this notation for compactness, but (4) can easily be expressed in the form of (1).

System (4) is passive with storage function $V = \frac{1}{2}\,\mathrm{trace}(I - X)$ in (Igarashi et al., 2009). Its step-invariant transform is $X^+ = \exp(Tu^\times)X$. Then $\Delta V = \frac{1}{2}\left(\mathrm{trace}(X) - \mathrm{trace}\left(\exp(Tu^\times)X\right)\right)$. If $X = I$, then $y = 0$ and $\Delta V = \frac{3}{2} - \frac{1}{2}\,\mathrm{trace}(\exp(Tu^\times))$. It can be shown that $\mathrm{trace}(\exp(Tu^\times)) = 1 + 2\cos(T\|u\|)$, so if $u[k] \neq 0$, we have $\Delta V > 0 = u^\top y$. Thus, passivity is lost under sampling with this choice of storage function and output. △

Storage functions often have intuitive physical interpretations, e.g., energy. This is useful in passivity-based control because physical intuition may guide the search for storage functions. Thus, it is desirable to be able to use the storage function of the continuous-time system for the sampled-data system. Preserving passivity under sampling with the same storage function may be achieved by redefining the output of the sampled-data system (Costa-Castelló and Fossas, 2006). By the Fundamental Theorem of Calculus, $\Delta V[k] = \int_{kT}^{(k+1)T} \dot{V}(t)\mathrm{d}t = \int_{kT}^{(k+1)T} \left(y(t)^\top u(t) + \beta(X(t), u(t))\right)\mathrm{d}t$, where $\beta : \mathsf{G} \times \mathbb{R}^m \to \mathbb{R}$ is continuous and negative semidefinite, and exists by the assumption that (1) is passive. Using $u(t) = u(kT) = u[k]$ for $t \in [kT, (k+1)T)$,

$\Delta V[k] = \int_{kT}^{(k+1)T} y(t)^\top \mathrm{d}t \quad u[k] + \beta^\star$, where $\beta^\star := \int_{kT}^{(k+1)T} \beta(X(t), u[k])\mathrm{d}t \leq 0$. By the definition of passivity, the sampled-data system is passive with the same storage function if the output is redefined as

$$y^\star[k] := \frac{1}{T}\int_{kT}^{(k+1)T} y(t)\mathrm{d}t. \tag{5}$$

Note that the factor of $1/T$ is not necessary for passivity, but it endows (5) with the interpretation of being the time-average of the output y over one sampling interval.

Proposition 3.2. The passive output (5) at time step k depends only on values that are available at time $t = kT$.

Proof Using the change of variable $t \mapsto t - kT$ we have

$$\begin{aligned} y^\star[k] &= \frac{1}{T}\int_0^T y(t+kT)\mathrm{d}t \\ &= \frac{1}{T}\int_0^T (C\mathrm{vec}(X(t+kT)) + Du(t+kT))\mathrm{d}t \\ &= \frac{1}{T}C\int_0^T \mathrm{vec}(X(t+kT))\mathrm{d}t + TDu(kT), \end{aligned}$$

where we have used $u(t + kT) = u(kT)$ for $t \in [0, T)$. Since the integral of a matrix is computed elementwise and vec is smooth, the order of the integral and vec can be changed. Using (3), we have that $\int_0^T X(t+kT)\mathrm{d}t$ equals $\int_0^T \exp\left(t\left(A + \sum_{i=1}^m B_i u_i(kT)\right)\right)\mathrm{d}tX(kT)$. Thus,

$$y^\star[k] = Du(kT) + \frac{C}{T}\mathrm{vec}\left(\int_0^T \exp\left(t\left(A + \sum_{i=1}^m B_i u_i(kT)\right)\right)\mathrm{d}tX(kT)\right).$$

□

Corollary 3.3. The passive output (5) is linear in $X[k]$.

Proof The X term in the expression for $y^\star$ can be written

$$\frac{C}{T}\left(I \otimes \int_0^T \exp\left(t\left(A + \sum_{i=1}^m B_i u_i(kT)\right)\right)\mathrm{d}t\right)\mathrm{vec}(X(kT)).$$

□

Defining $\bar{C} : \mathbb{R}^m \to \mathbb{R}^{mn \times mn}$, $u \mapsto \frac{1}{T}C\left(I \otimes \int_0^T \exp\left(t\left(A + \sum_{i=1}^m B_i u_i(kT)\right)\right)\mathrm{d}t\right)$, we can write the sampled system (3) with output $y^\star$ as

$$\begin{aligned} X^+ &= \exp\left(T\left(A + \sum_{i=1}^m B_i u_i\right)\right)X \\ y^\star &= \bar{C}(u)\mathrm{vec}(X) + Du. \end{aligned} \tag{6}$$

The sampled-data system (6) is of a form similar to the underlying continuous-time system (1), but the vector field is in G instead of $\mathfrak{g}$ and the output is not affine in u.

The integral term in the expression for $y^\star$ always has an analytic solution. If $H \in \mathbb{R}^{n\times n}$ is invertible, then $\int_0^T \exp(tH)\mathrm{d}t = H^{-1}(e^{TH} - I)$. If H is singular, then its Jordan form comprises invertible blocks and strictly upper triangular blocks. The integral-exponential term for the latter is straightforward to compute. Thus, $y^\star$ can always be computed analytically, as the only other computations involved are matrix multiplications.

4. CONTROL DESIGN

Our approach to control design uses the well-known fact that the negative feedback interconnection of passive systems is passive. Our design leverages existing stability results for interconnected passive systems that provide sufficient conditions for closed-loop asymptotic stability.

4.1 Proposed Method

We will prove that the closed-loop sampled-data system is stabilized by any strictly passive controller, but present a synthesis procedure for a special class of strictly passive LTI controllers, as existing passivity results for LTI systems more readily admit a systematic synthesis procedure.

Since passivity is a characterization of the input-output behaviour of a dynamical system, in our analysis, we must appeal to a notion of observability, so that driving the output to 0 has implications for the behaviour of the states.

Definition 4.1. ((Byrnes et al., 1991, Definition 3.1)). A dynamical system on a Lie group, (1a) or (3), with output $y \in \mathbb{R}^m$, is **locally zero-state observable (LZSO)**, if there exists a neighbourhood U of I such that if $X \in U$, $u = 0$, and $y = 0$, then $X = I$. ◊

Remark 4.2. We use the term "zero-state", which refers to the states being zero, because it is common in the existing literature for systems on vector spaces.

Lemma 4.3. ((Hill and Moylan, 1976, Lemma 1)). If a system $\dot{x} = f(x) + g(x)u$, $y = h(x) + J(x)u$ is passive with storage function V and LZSO, then V is positive definite.

Lemma 4.4. If (1) is passive with storage function V and LZSO, then V is positive definite.

Proof The dynamics of (1) can be rewritten as $\mathrm{vec}(\dot{X}) = (I \otimes A)\mathrm{vec}(X) + [(I \otimes B_1)\mathrm{vec}(X) \ \cdots \ (I \otimes B_m)\mathrm{vec}(X)]\, u$, which is in the form required by Lemma 4.3. □

Local zero-state observability is crucial for passivity analysis, but it has various implications for systems (1) and (6).

Proposition 4.5. If (1) is LZSO, then it is driftless.

Proof If $u(t) = u[k] = 0$ and $y = y^\star = 0$, implying $X = I$, then $\dot{X} = A$ and $X^+ = \exp(TA)$. But since $X = I$ identically, the systems must be at equilibrium, which is true if and only if $A = 0$. □

Because of this result, we make the following standing assumption, which we impose on all following results.

Assumption 2. (Driftlessness). $A = 0$ in (1) and (6).

The next result shows that zero-state observability is preserved under sampling for any sampling period.

Proposition 4.6. Let (6) be the discrete-time system effected by sampling the continuous-time system (1) with $A = 0$, so $u(t) = H(u[k])$ and $X[k] = S(X(t))$. Then (6) is LZSO if and only if (1) is LZSO.

Proof Assume (1) is LZSO on U_1. If $X[k] \in U_1$, $y^\star[k] = 0$, and $u[k] = 0$, then $C\mathrm{vec}(X[k]) = 0$. Since $u(t) = H(u[k])$, we have $u(t) = 0$ if and only if $u[k] = 0$. Thus $y(t) = C\mathrm{vec}(X(t)) = 0$. Since $A = 0$, if $u = 0$, then $\dot{X} = 0$ and $X^+ = X[k]$, thus $X(t) = X[k]$, $t \in [kT, (k+1)T)$. The local zero-state observability of (1) implies $X(t) = X[k] = I$.

Next consider the case where (6) is LZSO on neighbourhood U_2 containing I. If $X(t) \in U_2$, $y(t) = 0$, and $u(t) = 0$, then $C\text{vec}(X(t)) = 0$. The latter two equalities imply $y^\star[k] = C\text{vec}(X[k]) = 0$. The local zero-state observability of (6) implies $X[k] = X(t) = I$. □

In our main result, we consider a strictly passive controller of the form

$$x_c^+ = f(x_c) + g(x_c)u_c, \qquad y_c = h(x_c) + J(x_c)u_c, \qquad (7)$$

where $f : \mathbb{R}^{n_c} \to \mathbb{R}^{n_c}$, $g : \mathbb{R}^{n_c} \to \mathbb{R}^{n_c \times m_c}$, $h : \mathbb{R}^{n_c} \to \mathbb{R}^{m_c}$, $J : \mathbb{R}^{n_c} \to \mathbb{R}^{m_c \times m_c}$, $f(0) = 0$, $h(0) = 0$, and $g(x_c)$ is full rank for all $x_c \in \mathbb{R}^{n_c}$.

Theorem 4.7. If the plant (1) is passive, driftless, and LZSO, then $(X, x_c) = (I, 0)$ is asymptotically stable for the negative feedback connection of the sampled plant (6) and the discrete-time strictly passive controller (7), i.e., $u_c = y^\star$ and $u = -y_c$.

Proof By hypothesis, the plant and controller are passive. Therefore, their respective storage functions satisfy $\Delta V_P \le u^\top y^\star$ and $\Delta V_C \le u_c^\top y_c$. Define the function $V := V_P + V_C$, which is positive definite, since, by Lemma 4.4, it is the sum of positive definite functions. Since (7) is strictly passive, $\Delta V_C \le u_c^\top y_c - \psi(x_c)$ for some positive definite function ψ. Then

$$\Delta V = \Delta V_P + \Delta V_C \le u^\top y^\star + u_c^\top y_c - \psi(x_c) = -\psi(x_c).$$

Thus, $\Delta V = 0$ implies $\psi(x_c) = 0$, which by positive definiteness implies $x_c = 0$. From (7), $x_c = 0$ identically implies $0 = f(0) + g(0)u_c$. Since $f(0) = 0$ and $g(0)$ is full rank, we have $u_c = y^\star = 0$. This implies $u = y_c = h(0) + J(0)u_c = 0$. By Proposition 4.6, $y^\star = 0$ implies $X = I$.

Let $c > 0$ be sufficiently small such that $\mathcal{D} := \{(X, x_c) \in \mathsf{G} \times \mathbb{R}^{n_c} : V \le c\}$ is compact. Let $\mathcal{R} := \{(X, x_c) \in \mathcal{D} : \Delta V = 0\}$. From the previous discussion, we have $\mathcal{R} = \{(I, 0)\}$. Since $\mathcal{R}$ is a singleton comprising an equilibrium, it is itself the largest invariant set contained therein. By the invariance principle, $(X, x_c) \to (I, 0)$ as $k \to \infty$. □

Corollary 4.8. If V is radially unbounded and the set U from the definition of local zero-state observability equals G, then (6) is globally asymptotically stable.

Proof This follows from the invariance principle argument in the proof of Theorem 4.7 and (Haddad and Chellaboina, 2008, Theorem 13.5). □

The proposed system topology is illustrated in Figure 2.

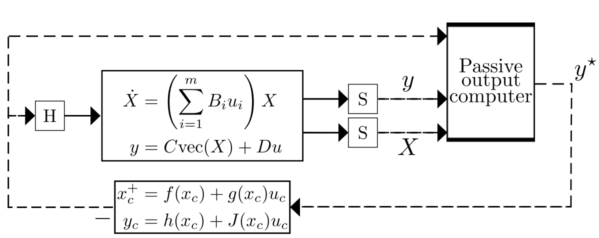

Fig. 2. Closed-loop sampled-data system.

4.2 Controller Synthesis

Theorem 4.7 holds for any strictly passive controller, but its proof is not constructive. For synthesis, we appeal to a special class of strictly passive LTI systems. Consider a discrete-time LTI system with minimal realization:

$$x_c^+ = A_c x_c + B_c u_c, \qquad y_c = C_c x_c + D_c u_c, \qquad (8)$$

where $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times m_c}$, $C_c \in \mathbb{R}^{m_c \times n_c}$, and $D_c \in \mathbb{R}^{m_c \times m_c}$.

System (8) is **strictly positive real (SPR)** if and only if it satisfies the Kalman-Yakubovich-Popov (KYP) Lemma.

Lemma 4.9. (Discrete-Time KYP Lemma). (Hitz and Anderson, 1969), (Haddad and Chellaboina, 2008, Theorem 13.28). System (8) is SPR if and only if there exist matrices $P \in \mathbb{R}^{n \times n}$, $L \in \mathbb{R}^{q \times n}$, $W \in \mathbb{R}^{q \times m}$ with $P = P^\top \succ 0$, and a scalar $0 < \rho < 1$ such that

$$\rho P = A_c^\top P A_c + L^\top L \qquad (9a)$$

$$0 = A_c^\top P B_c - C_c^\top + L^\top W \qquad (9b)$$

$$0 = D_c + D_c^\top - B_c^\top P B_c - W^\top W. \qquad (9c)$$

There is an analogous definition for continuous-time SPR systems; it is a standard result that such systems are strictly passive. Although it too is likely a standard result, we state this result for discrete-time. We omit the relatively straightforward proof due to lack of space.

Proposition 4.10. If (8) is SPR, then it is strictly passive.

The controller can be designed using emulation, as the bilinear transform preserves positive realness (Hoagg et al., 2004), or using direct design, as discussed below.

The KYP lemma provides necessary and sufficient conditions for an LTI system to be SPR, but does not itself provide a method for synthesizing such systems. We present one method for designing a discrete-time SPR controller.

1) Set $A_c := 0$ and choose invertible diagonal B_c; 2) Choose $0 < \rho < 1$ and diagonal $L \in \mathbb{R}^{n_c \times n_c}$, $L \succ 0$, then set $P := L^2/\rho$, which is also diagonal; 3) Letting λ be the minimum eigenvalue of $-PB_c^2$, set $D_c := |\lambda| I$; 4) Set $W := \sqrt{2D_c - PB_c^2}$; 5) Set $C_c := WL$.

We now outline our justification for the proposed SPR controller synthesis procedure.

For $A_c = 0$ and $L \succ 0$ diagonal, (9a) reduces to the equation in step 2. Thus $P \succ 0$ exists and is diagonal for any $0 < \rho < 1$.

If B_c and D_c are diagonal, then (9c) reduces to $W^\top W = 2D_c - PB_c^2$, which is diagonal, thus its eigenvalues are its diagonal elements. If $D_c = |\lambda| I$, as defined in step 3, then $|\lambda| > 0$ bounds the eigenvalues of $W^\top W$ from below. Since $2D_c - PB_c^2 \succ 0$ is diagonal, its square root is computed elementwise. Thus $W = \sqrt{2D_c - PB_c^2} \succ 0$ is diagonal.

For $A_c = 0$, (9b) reduces to the equation in step 5. Since $W \succ 0$ and $L \succ 0$ are diagonal, $C_c \succ 0$ is diagonal. Since B_c and C_c are invertible, (A_c, B_c) is controllable and (A_c, C_c) is observable, thus the KYP applies. A_c, B_c, C_c, D_c, ρ satisfy the KYP by construction.

Remark 4.11. This procedure ensures asymptotic stability of the closed-loop system, but provides no information about the dynamic response of the closed-loop system.

Note that the foregoing procedure is presented only to provide a systematic method to design SPR controllers.

Suitable controllers could also be designed, for example, using frequency-domain design.

5. SIMULATIONS

To illustrate our results, we compare the performance of a discrete-time SPR (DSPR) controller to a continuous-time SPR controller that has been discretized using the bilinear transform. The plant is (4) with initial condition

$$R(0) = R[0] = \begin{bmatrix} 0.5 & 1/\sqrt{2} & -0.5 \\ 0.5 & -1/\sqrt{2} & -0.5 \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}.$$

Note that system (4) is LZSO. If $u = 0$ and $X = I$, then $y = 0$. The topologies of the DSPR- and SPR-controlled systems are illustrated in Figures 2 and 3, respectively.

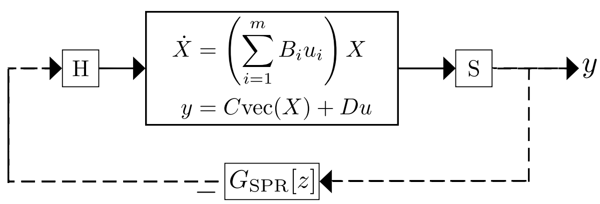

Fig. 3. Closed-loop SPR-controlled system.

The DSPR controller is designed using the algorithm in Section 4.2 with $B = L = I$ and $\rho = 0.5$:

$$G_{\mathrm{DSPR}}[z] = I \otimes \frac{2z + \sqrt{2}}{z}. \quad (10)$$

The SPR controller is the lead-lag compensator

$$G_{\mathrm{SPR}}(s) = I \otimes \frac{s + 1000}{s + 100} \frac{s + 0.005}{s + 0.05}, \quad (11)$$

which satisfies the continuous-time KYP lemma (Khalil, 2002, Lemma 6.3). This controller was tuned to generate control signals similar in magnitude to those of (10) and discretized using the bilinear transform.

We plot the trajectories of (4) in roll-pitch-yaw coordinates. If $R = I$, then roll, pitch, and yaw are 0. If we sample relatively quickly, for example, with $T = 0.01$, then both topologies achieve asymptotic stability. We simulate using $T = 0.2$. Figure 5 shows that the DSPR controller in the proposed topology achieves asymptotic stability, whereas Figure 4 shows that the naïve sampled-data topology using the discretized SPR controller does not. Even though (11) is SPR, the sampled plant is not passive from u to y, as shown in Example 3.1, so Theorem 4.7 does not apply.

6. STEP TRACKING

Stabilizing the identity extends to reference tracking by letting X_{ref} be the reference value for the states, and defining the error $E := XX_{\mathrm{ref}}^{-1}$, stabilizing the point $E = I$ implies $X \to X_{\mathrm{ref}}$. The error E has dynamics $\dot{E} = \dot{X}X_{\mathrm{ref}}^{-1} = (\sum_{i=1}^{m} B_i u_i) X X_{\mathrm{ref}}^{-1} = (\sum_{i=1}^{m} B_i u_i) E$, which are the same as the plant's, with X replaced by E. Thus, the results in Section 4 easily extend to setpoint tracking. The reference tracking topology is illustrated in Figure 6. If $X_{\mathrm{ref}} = I$, then the topology reduces to that in Figure 2.

Example 6.1. Consider the same plant and initial conditions used to demonstrate our passivity-based controller

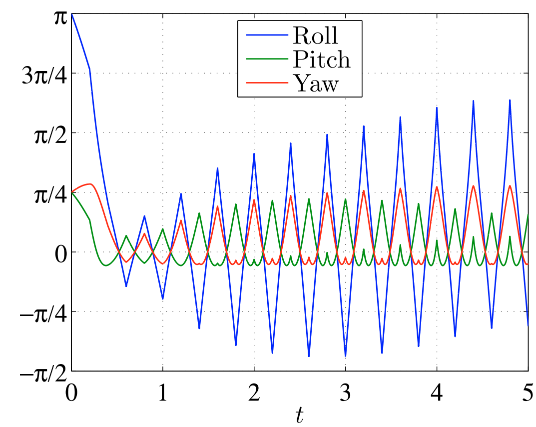

Fig. 4. Local trajectories of (4) using $G_{\mathrm{SPR}}[z]$, $T = 0.2$.

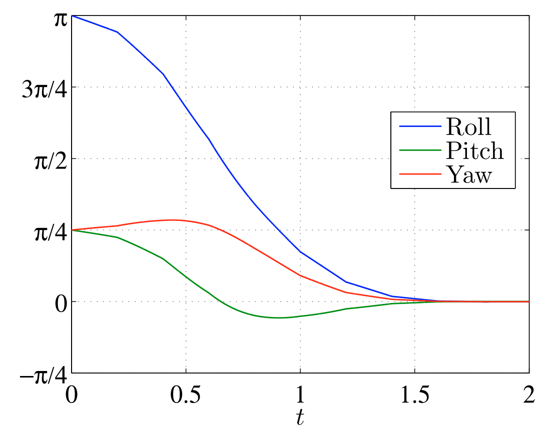

Fig. 5. Local trajectories of (4) using $G_{\mathrm{DSPR}}[z]$, $T = 0.2$.

in Section 5, with controller (10). The topology is similar to that illustrated in Figure 2, except $E := XX_{\mathrm{ref}}^{-1}$, instead of X, is fed to the *Passive output computer* block; this is possible because X and X_{ref} are known in discrete-time.

The sampling period is $T = 0.2$. The reference is defined in roll-pitch-yaw coordinates $\xi \in \mathbb{R}^3$, which in turn define $X_{\mathrm{ref}}(t) \in \mathsf{SO}(3)$: $\xi(t) = \left[-\frac{\pi}{4}\ \frac{\pi}{3}\ \frac{\pi}{4}\right]$, $t \in [0, 3]$, $\xi(t) = \left[\frac{\pi}{4}\ -\frac{\pi}{3}\ -\frac{\pi}{4}\right]$, $t \in (3, 6]$. Figure 7 shows that asymptotic tracking is achieved. △

7. CONCLUSIONS

We proposed a method of stabilizing sampled-data systems on matrix Lie groups using passivity-based control design. We proved that under certain assumptions that the resultant closed-loop system is asymptotically stable. The proposed method was compared to an emulation-based control design and topology in simulation, which suggested that the proposed method is more robust to high sampling periods. The proposed method was also shown to be applicable to constant reference tracking.

Future research includes revising our DSPR synthesis procedure to optimize dynamical performance, developing a nonlinear strictly passive controller synthesis procedure based on the KYP-like conditions in (Navarro-López, 2007), and identifying KYP-like conditions for systems on matrix Lie groups to help identify storage functions.

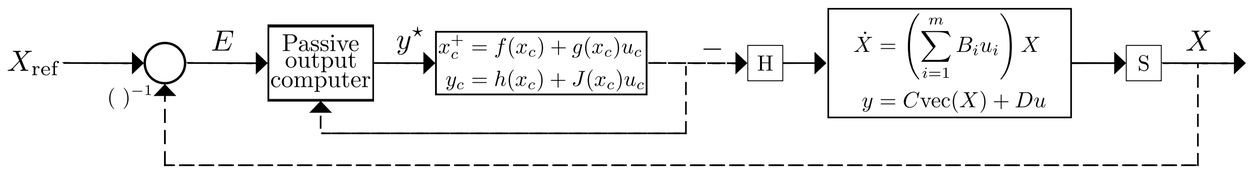

Fig. 6. Passivity-based reference tracking topology.

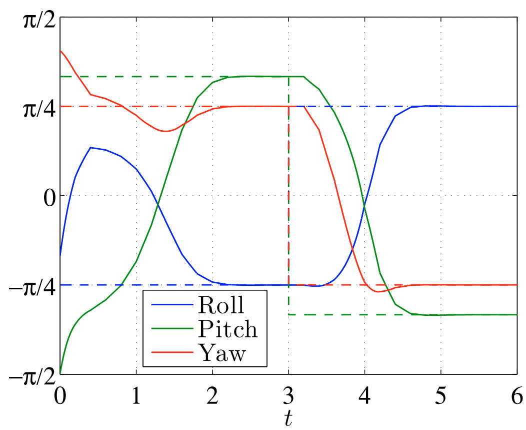

Fig. 7. Series of step responses of (4) controlled by $G_{\text{SPR}}[z]$ in local coordinates. The local coordinates of the plant and reference states are represented by the solid and dotted lines, respectively.

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