Thomassen's 5-Choosability Theorem Extends to Many Faces

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

We prove in this thesis that planar graphs can be L-colored, where L is a list-assignment in which every vertex has a 5-list except for a collection of arbitrarily large faces which have 3-lists, as long as those faces are at least a constant distance apart. Such a result is analogous to Thomassen's 5-choosability proof where arbitrarily many faces, rather than just one face, are permitted to have 3-lists. This result can also be thought of as a stronger form of a conjecture of Albertson which was solved in 2012 and asked whether a planar graph can be 5-list-colored even if it contains distant precolored vertices. Our result has useful applications in proving that drawings with arbitrarily large pairwise far-apart crossing structures are 5-choosable under certain conditions, and we prove one such result at the end of this thesis.

Acknowledgments

First and foremost, I would like to thank my supervisor, Bruce Richter. It is difficult for me to put into words the debt of gratitude that I owe Bruce. The project for this thesis ended up being considerably longer than either of us had anticipated, and there were certainly moments over the course of the last several years when it looked like it had hit some roadblocks that simply could not be overcome. Throughout all of this, as the unexpected technical obstacles continued to pile up and the thesis got longer and longer, Bruce was completely unwavering in his belief that I was capable of seeing this thesis through to the end. He always believed in me, even in those moments when I did not believe in me. Without this unwavering commitment from my supervisor to the belief that this project was doable and that my approach was the right one, this document would simply not have been possible.

The two of us spent many hours carefully discussing the numerous and often unexpected highly technical issues that came up seemingly on every page of work. As anyone who has ever written a doctoral thesis in mathematics knows all too well, you essentially have to resign yourself to the fact that it will need to be written twice, since the length and complexity of the document are such that there is really no way to know in advance what the right way to organize and structure the results and their proofs is. Bruce's careful close reading and his extensive and highly detailed comments, questions, sanity checks, and suggestions were all essential to the final version, and I am indebted to him for the amount of work he put in.

Secondly, I would like to thank my partner, Julienne Forrester. As above, it is difficult for me to put into words the full scope of the support I received from Julienne. Julienne and I met in November of 2015, two months after I had begun my PhD program at the University of Waterloo. This means that she has been with me almost since the beginning of the extremely long road from my first day at Waterloo to the end of my PhD. She has had a front-row seat to pretty much the entire thing and supported me at every step: The late nights I spent working on assignments for my many courses, or marking papers as a TA, the stress of preparing for my comprehensive exams, and then later preparing for my thesis proposal, and of course, she supported me throughout the long grind of me writing this thesis, as my stack of notebooks with sketches of ideas (of which some worked and many others did not) got taller and taller. She was always there for me, with infinite patience and infinite support, even though I'm confident that the stress of the work, and the fact that I was constantly lost in thought, made me insufferable (or at least very annoying to live with) at many points over the last 6 years. Like Bruce, she never wavered in the belief that I was capable of getting this done, even in those moments when I did not.

Even under normal circumstances, I would have relied on Julienne always being there for me, but the circumstances under which this thesis was written were very abnormal. I completed about three quarters of the work on this thesis in the period between March 2020 and July 2021, which meant that I did most of the work during the pandemic. Writing a dissertation is often a very isolating experience even without a once-in-a-century public health emergency which forces everyone to physically stay far away from each other, but however isolating the experience is under normal circumstances, it is is triply so when the world comes to a grinding halt and you have to work from home for over 500 days in a row. Without Julienne by my side, I cannot imagine how I would have dealt with the combined stressors of the pandemic and the ever-growing scope of this project. This PhD would simply not have been possible without her.

I would like to thank my parents, who have always been so supportive of me in my very long journey through my time as a student (a 4-year BSc, a 2-year MSc, and a 6-year PhD together make 12 years as a student, for those keeping score). They have always instilled in me the value of learning and intellectual curiosity, and I am so fortunate to have been raised by them. I also suspect that their humoring of my unrealistic early-childhood aspirations, like trying to build an airplane in our backyard, was an important part of raising me with the stubbornness that I needed in order to complete this PhD.

I would like to thank Dori, Chris E, Chris P, Alex, Wade, and Gjergji. I have very vivid memories from my time at Caltech of the late nights we spent working on math or talking about math. I remember us ordering pizza while we all sat around a blackboard in Sloane and worked through problem sets late into the night. I remember how we all crammed together in the jam room or in the Ricketts library to stare at a whiteboard until one of us had an epiphany that we needed to solve a problem for an assignment that we probably shouldn't have started at the last minute every week. It was such a pleasure doing math alongside all of you.

I would like to thank all of the grad students here in the C&O Department of the University of Waterloo who I spent so much time working with in the early phase of our PhD program while we were all in the same boat with our courses and preparation for comprehensive exams. I would also like to thank all of the professors here who taught me. When I arrived at Waterloo, I found a fantastic, highly research-intense environment full of intellectually curious people who thrive on tackling difficult problems. The commitment to research and problem-solving in the C&O Department here is absolutely infectious, and when I think back on how much I have learned from all of you during my time here, I am absolutely floored.

Dedication

This thesis is dedicated to the mathematicians in my family who came before me: My grandfather David Borwein and David's sons (my uncles) Jonathan and Peter Borwein. I had the extraordinary privilege of David being present at the oral defence of this thesis. Jonathan and Peter did not live to see me complete my PhD, as they passed away in 2016 and 2020 respectively. David passed away at 97, several weeks after the oral defence. This thesis is dedicated to their memories.

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This index consists of symbolic terminology and definitions which are specific to this thesis. The pages referenced indicate where each piece of terminology is first defined.

Chapter 0

Introduction

0.1 Embeddings and Drawings

An *embedding* of a graph G into the plane is an assignment of the vertices of G to points of $\mathbb{R}^2$ and edges of G to arcs in $\mathbb{R}^2$ homeomorphic to $[0, 1]$, where the following conditions are satisfied.

1) For any edge $e = xy \in E(G)$, the endpoints of the arc associated to e are the points of $\mathbb{R}^2$ associated to x and *y; AND*

2) For any edge $e = xy \in E(G)$ and $v \in V(G)$ with $v \notin \{x, y\}$, the arc associated to e does not contain the point of in $\mathbb{R}^2$ associated to *v; AND*

3) For any two distinct edges $e_1, e_2 \in E(G)$, there is no point in $\mathbb{R}^2$ which is both an interior point of the arc associated to e_1 and an interior point of the arc associated to e_2.

An embedding satisfying conditions 1)-3) is called a *planar embedding* or a *planar graph*. An assignment of the vertices and edges of a graph G to points and arcs of $\mathbb{R}^2$ respectively for which conditions 1) and 2) are satisfied, but for which condition 3) is dropped, is called a *drawing of G*, or just a *drawing*. Given a drawing G and two edges $e_1, e_2 \in E(G)$, a point of $\mathbb{R}^2$ which is both an interior point of e_1 and an interior point of e_2 is called a *crossing point* of G.

Throughout this thesis, unless otherwise specified, a graph is always understood to be a fixed drawing (or embedding, if the graph is planar) in $\mathbb{R}^2$. In some cases (in particular, in the last chapter), we deal with drawings on the sphere $\mathbb{S}^2$ rather than in $\mathbb{R}^2$. We define embeddings and drawings on the sphere analogously to the above, where $\mathbb{R}^2$ is replaced by $\mathbb{S}^2$ in the definition above.

If we want to talk about a graph G as an abstract collection of vertices and edges, without reference to sets of points and arcs in the plane or the sphere, then we call G an *abstract graph*. All graphs in this thesis are simple (that is, free of loops or repeated edges). We also do not deal with directed graphs in this thesis. All graphs in this thesis are undirected.

We adopt that standard convention that all drawings G satisfy the property that, if e_1 and e_2 are edges which cross in G, then e_1 and e_2 share no endpoint (since if e_1 and e_2 share an endpoint, then the crossing can be undone by rerouting the two edges and then deforming the arcs in a sufficiently small open neighborhood around the crossing point).

Definition 0.1.1. For any $U \subseteq \mathbb{R}^2$ and drawing G in $\mathbb{R}^2$, we have the following terminology.

1) $\text{Cl}(U)$ denotes the closure of U and $\partial(U)$ denotes the boundary of U.

2) A *subgraph* of G is a drawing obtained by deleting some of the vertices of G, removing some of the arcs from $E(G)$, and, in particular, for each deleted vertex v, removing from $E(G)$ every arc with v as an endpoint. We write $H \subseteq G$ to indicate that H is a subgraph of G.

3) We write $G \subseteq U$ to mean that the vertices and edges of G, regarded as sets of points of $\mathbb{R}^2$, are all contained in U.

4) Given two subgraphs $H_1, H_2 \subseteq G$, the graph $H_1 \cup H_2$ is the drawing consisting of all the vertices in $V(H_1) \cup V(H_2)$ and all the edges in $E(H_1) \cup E(H_2)$.

5) The notation $G \setminus H$ refers to the drawing obtained from G by deleting the points of $\mathbb{R}^2$ corresponding to the vertices of H and deleting the interiors of the arcs of $\mathbb{R}^2$ corresponding to the edges of G with at least one endpoint in H.

We have analogous definitions in the case where $U \subseteq \mathbb{S}^2$ and G is a drawing in $\mathbb{S}^2$.

Given a planar graph G, the deletion of G partitions $\mathbb{R}^2$ into a collection of disjoint, open path-connected components called the *faces* of G. Our main objects of study are the subgraphs of G bounding the faces of G.

Definition 0.1.2. Given a planar graph G and a subgraph H of G, we introduce the following terminology.

1) H is called a *facial subgraph* of G if there exists a connected component U of $\mathbb{R}^2 \setminus G$ such that $H = \partial(U)$.

2) H is called a *cyclic facial subgraph* (or, more simply, a *facial cycle*) if H is both a facial subgraph of G and a cycle.

Definition 0.1.2 does not require H to be connected. Indeed, if G is not connected, then at least one facial subgraph of G is not connected. We also use the following standard notation.

Definition 0.1.3. Given a planar graph G and a cycle C in G, we let $\text{Int}_G(C)$ denote the subgraph of G consisting of all the edges and vertices in the closure of the unique bounded simply connected component of $\mathbb{R}^2 \setminus C$, and we let $\text{Ext}_G(C)$ denote the subgraph $G \setminus (\text{Int}_G(C) \setminus C)$ of G. We now make the following definition: An expression of G as a union of the form $G = G_0 \cup G_1$, where G_0, G_1 is a pair of subgraphs of G, is called the *natural C-partition* of G if there exists an $i \in \{0, 1\}$ such that $G_i = \text{Int}_G(C)$ and $G_{1-i} = \text{Ext}_G(C)$.

We also introduce the following standard notation.

Definition 0.1.4. For any graph G, vertex set $X \subseteq V(G)$, integer $j \geq 0$, and real number $r \geq 0$, we have the following.

1) We set $D_j(X, G) := \{v \in V(G) : d(v, X) = j\}$.

2) We set $B_r(X, G) := \{v \in V(G) : d(v, X) \leq r\}$.

3) For any subgraph H of G, we usually just write $D_j(H, G)$ to mean $D_j(V(H), G)$, and likewise, we usually write $B_r(H, G)$ to mean $B_r(V(H), G)$.

If the underlying graph G is clear from the context, then we drop the second coordinate from the above notation in order to avoid clutter.

Definition 0.1.5. Given a graph G, a subgraph H of G, a subgraph P of G, and an integer $k \geq 0$, we call P a *k-chord* of H if $|E(P)| = k$ and P is of the following form.

1) $P := v_1 \cdots v_k v_1$ is a cycle with $v_1 \in V(H)$ and $v_2, \cdots, v_k \notin V(H)$; ***OR***

2) $P := v_1 \cdots v_{k+1}$, and P is a path with distinct endpoints, where $v_1, v_{k+1} \in V(H)$ and $v_2, \cdots, v_k \notin V(H)$.

Given a $k \geq 1$ and a k-chord P of H, P is called a *proper* k-chord of H if P is not a cycle, i.e P intersects H on two distinct vertices. Note that, for any $1 \leq k \leq 2$, any k-chord of H is a proper k-chord of H, since G has no loops or duplicated edges. A 1-chord of H is simply referred to as a *chord* of H. In some cases, we are interested in analyzing k-chords of H in G where the precise value of k is not important. We thus introduce the following definition. We call P a *generalized chord* of H if there exists an integer $k \geq 1$ such that P is a k-chord of H. We call P a *proper generalized chord* of H if there exists an integer $k \geq 1$ such that P is a proper k-chord of H.

Given a planar graph G, a cyclic facial subgraph C of G, and a proper generalized Q of C, there is a natural way to talk about one or the other "side" of Q in G. That is, analogous to Definition 0.1.3, there is a natural topological way to partition G into two sides of Q, which is made precise below.

Definition 0.1.6. Let G be a planar graph, let C be a cyclic facial subgraph of G, and let Q be a generalized chord of C. The unique *natural* (C, Q)*-partition* of G is an expression of G as a union of the form $G = G_0 \cup G_1$, where G_0, G_1 is a pair of subgraphs of G such that the following hold.

1) $G_0 \cap G_1 = Q$*; AND*
2) For each $i \in \{0, 1\}$, there is a unique open simply connected region U of $\mathbb{R}^2 \setminus (C \cup Q)$ such that G_i consists of all the edges and vertices of G in the closed region Cl(U).

If the facial cycle C is clear from the context then we usually just refer to $\{G_0, G_1\}$ as the *natural* Q*-partition* of G. Note that this is consistent with Definition 0.1.3 in the sense that, if Q is not a proper generalized chord of C (i.e Q is a cycle) then the natural Q-partition of G is the same as the natural (C, Q)-partition of G.

Throughout this thesis, we frequently analyze paths, and use the following standard notation related to paths.

Definition 0.1.7. Given a graph G, a path P in G, and a pair of vertices $x, y \in V(P)$, we let xPy denote the subpath of P with endpoints x, y. If we have a specified ordering of P as $P := x_1 \cdots x_k$ for some integer $k \geq 1$, then, for each $i \in \{1, \cdots, k\}$, we write Px_i to mean the subpath $x_1 \cdots x_i$ of P, and we write x_iP to mean the subpath $x_i \cdots x_k$ of P. Furthermore, for any path $P := x_1 \cdots x_k$, we let $\mathring{P}$ denote the path $x_2 \cdots x_{k-1}$. We also adopt the convention that, for any cycle $C \subseteq G$, we have $\mathring{C} = C$.

Definition 0.1.8. Given a graph G and a pair of subsets $X, Y \subseteq V(G)$, a path $P \subseteq G$ is called a (X, Y)*-path* if $P := x_1 \cdots x_k$, where $x_1 \in X$, $x_k \in Y$, and $V(P)$ is otherwise disjoint to $X \cup Y$.

0.2 Colorings and List-Colorings

Given a graph G, a *list-assignment* for G is a family of sets $\{L(v) : v \in V(G)\}$ indexed by the vertices of G, such that $L(v)$ is a finite subset of $\mathbb{N}$ for each $v \in V(G)$. The elements of $L(v)$ are called *colors*.

Definition 0.2.1. Let G be a graph and let L be a list-assignment for G. Let H be a subgraph of G. A function $\phi : V(H) \to \bigcup_{v \in V(H)} L(v)$ is called an L*-coloring of* H if $\phi(v) \in L(v)$ for each $v \in V(H)$, and, for each pair of vertices $x, y \in V(H)$ such that $xy \in E(H)$, we have $\phi(x) \neq \phi(y)$. Given a set $S \subseteq V(G)$ and a function $\phi : S \to \bigcup_{v \in S} L(v)$, we call ϕ an L*-coloring of* S if $\phi(v) \in L(v)$ for each $v \in S$ and ϕ is an L-coloring of the induced graph $G[S]$.

Definition 0.2.2. Let $k \geq 1$ be an integer. A graph G is called k*-choosable* if, for every list-assignment L for G such that $|L(v)| \geq k$ for all $v \in V(G)$, G is L-colorable.

In 1994, Thomassen demonstrated in [17] that all planar graphs are 5-choosable, settling a problem that had been posed in the 1970's.

Theorem 0.2.3. *Let G be a planar graph with facial cycle C. Let xy be an edge of G with $x, y \in V(C)$. Let L be a list assignment for $V(G)$ such that the following hold.*

1) xy is L-colorable; AND

2) $|L(v)| \geq 3$ for $v \in V(C) \setminus \{x, y\}$; AND

3) $|L(v)| \geq 5$ for $v \in V(G) \setminus V(C)$.

Then G is L-colorable.

This theorem also has the following useful corollary.

Corollary 0.2.4. *Let G be a planar graph with outer cycle C, where $|V(C)| \leq 4$, and let L be a list-assignment for $V(G)$ where each vertex of $G \setminus C$ has a list of size at least five and $V(C)$ is L-colorable. Then G is L-colorable.*

The proof technique for Theorem 0.2.3 can be applied to other settings, in order to prove the 5-choosability of some more general classes of graphs. One such result, which is proven in [6], is as follows.

Theorem 0.2.5. *Let G be a graph drawn in the plane so that all crossings in G are pairwise of distance at least 15 apart. Then G is 5-choosable.*

In the statement of Theorem 0.2.5, the distance between two crossings in a drawing G refers to the graph-theoretic distance between the two pairs of edges which are incident to the respective crossings.

One of the results we rely on in order to prove the results of this thesis is the following useful theorem about two precolored cycles, which follows from Theorem 5.2.9 in [8].

Theorem 0.2.6. *There exists an integer $\gamma \geq 1$ such that the following holds: Let G be a planar graph and let C_1, C_2 be cyclic facial subgraphs of G such that $d(C_1, C_2) \geq \gamma$ and, for each $i = 1, 2$, we have $3 \leq |V(C_i)| \leq 4$. Let L be a 5-list-assignment for G and let ϕ be a proper L-coloring of $V(C_1 \cup C_2)$. Then ϕ extends to an L-coloring of G.*

0.3 Results of this Thesis

This thesis consists of two results. The first result of this thesis is the following, which is a generalization of Theorem 0.2.3 to a planar graph with a collection of specified faces which are pairwise at least a constant distance from each other.

Theorem 0.3.1. *There exists a constant α such that the following holds: Let G be a planar graph and let $F_1, \cdots, F_m$ be a collection of facial subgraphs of G such that $d(F_i, F_j) \geq \alpha$ for each $1 \leq i < j \leq m$. Let $x_1y_1, \cdots, x_my_m$ be a collection of edges in G, where $x_iy_i \in E(F_i)$ for each $i = 1, \cdots, m$. Let L be a list-assignment for G such that the following hold.*

1) For each $v \in V(G) \setminus (\bigcup_{i=1}^m V(C_i))$, $|L(v)| \geq 5$; AND

2) For each $i = 1, \cdots, m$, x_iy_i is L-colorable, and, for each $v \in V(F_i) \setminus \{x_i, y_i\}$, $|L(v)| \geq 3$.

Then G is L-colorable. In particular, letting γ be as in Theorem 0.2.6, the choice $\alpha = 48749 + 3\gamma$ suffices.

In the process of proving Theorem 0.3.1, we also prove many stand-alone intermediate results along the way which have useful applications in other contexts. Theorem 0.3.1 is a strengthening of the following result, which is proven in [7] and positively resolved a conjecture of Albertson.

Theorem 0.3.2. *There exists a constant* d *such that the following holds: Let* G *be a planar graph and let* $S \subseteq V(G)$*, where the vertices of* S *are pairwise of distance at least* d *apart. Let* L *be a list-assignment for* $V(G)$ *such that every vertex of* S *has a list of size one and every vertex of* $G \setminus S$ *has a list of size least five. Then* G *is* L*-colorable.*

In [13], Postle and Thomas proved a very nice result which is also a strengthening of Theorem 0.3.2.

Theorem 0.3.3. *Given a planar graph* G*, a subgraph* H*, and a list-assignment* L *of* G*, where all the vertices of* $G \setminus H$ *have lists of size at least five, either* G *is* L*-colorable or there is a subgraph of* G *of size* $O(|V(H)|)$ *which is not* L*-colorable.*

Theorem 0.3.3 immediately implies a weaker version of Theorem 0.3.1 where the lower bound on the pairwise-distance between the faces of $\{F_1, \cdots, F_m\}$ is linearly dependent on the quantity $\max\{|V(F_i)| : i = 1, \cdots, m\}$. Our result brings the required lower bound on this pairwise-distance down to a constant. Postle and Thomas also have an independent proof of Theorem 0.3.1, with a different distance constant, which consists of a sequence of papers for which publication is ongoing at the time of writing. The last paper in the sequence, which is [12], appeared several weeks after the oral defence of this thesis.

Theorem 0.3.1 gives a positive answer to a conjecture posed at the very end of [8] and also gives a positive answer to a list-coloring version of the following conjecture from [16] for ordinary colorings, albeit with a different distance constant.

Conjecture 0.3.4. *Let* G *be a planar graph and* $W \subseteq V(G)$ *such that* $G[W]$ *is bipartite and any two components of* $G[W]$ *have distance at least 100 from each other. Can any coloring of* $G[W]$ *such that each component is 2-colored be extended to a 5-coloring of* G*?*

A positive answer to Conjecture 0.3.4 in the special case where each component of W is a lone vertex was provided by Albertson in [1]. Note that Conjecture 0.3.4 generalizes the setting of pairwise far-apart precolored vertices to that of precolored far-apart 2-colored bipartite components. That is, Theorem 0.3.2 provides a positive answer to a list-coloring version of Conjecture 0.3.4 in the case where each component of W is a lone vertex, albeit with a different distance constant.

The proof of Theorem 0.3.1 consists of Chapters 1-13, which is the majority of the thesis. In Chapter 14, we prove an analogue to Theorem 0.2.5 for more general crossing structures. The lone result of Chapter 14 does not rely on the details of the proof of Theorem 0.3.1. That is, for the purposes of Chapter 14, Theorem 0.3.1 is just a black box, so Chapter 14 can be read independently of Chapters 1-13. Chapter 14 consists of the following result.

Theorem 0.3.5. *There exists a constant* α' *such that the following holds: Let* G *be a drawing on the sphere of a graph and let* $C_1, \cdots, C_m$ *be a collection of cycles in* G *such that* $d(C_i, C_j) \geq \alpha'$ *for each* $1 \leq i < j \leq m$*. Suppose that, for each* $1 \leq i \leq m$*, there is a connected component* U_i *of* $\mathbb{S}^2 \setminus C_i$ *such that the following hold.*

1) *For each crossing point* x *of* G*, there is an* $i \in \{1, \cdots, m\}$ *such that* $x \in U_i$ *; AND*
2) *For each* $v \in V(C_i)$*,* $V(G) \cap U_i = \varnothing$ *and there is at most one chord of* C_i *in* $\mathrm{Cl}(U_i)$ *which is incident to* v*.*

Then G *is 5-choosable. In particular, letting* γ *be as in Theorem 0.2.6, the choice* $\alpha' = 48751 + 3\gamma$ *suffices.*

0.4 Notational Conventions and Formatting of Proofs in this Thesis

Definitions and symbols introduced in this thesis can be found in the index or symbol index respectively. Whenever notation and definitions are introduced in this thesis, subscripts, superscripts, and coordinates that denote underlying structures are frequently suppressed in later uses of these definitions in order to avoid clutter, as long as the suppressed symbols are clear from the context. For example, instead of writing $\text{Int}_G(C)$ to denote the interior of a cycle C in a planar embedding G, we usually just write $\text{Int}(C)$. The suppressed subscripts, superscripts, and coordinates typically denote ambient graphs and/or list-assignments which are clear from the context.

All proofs in this thesis are structured in the following way. Any proof environment which ends with a white box is the proof of a statement which is contained in one of the environments *Theorem, Lemma, Proposition*, or *Observation*. The statement of any intermediate result within a white-box proof environment is contained in the *Claim* environment, and the proof of this intermediate result is contained within a proof environment which ends with a black box. If the proof within this black-box environment itself requires an intermediate statement, then this statement is contained within the *Subclaim* environment. The proof of the subclaim is contained in a nested environment that ends with a black box as well, but both the subclaim and its proof are indented so as to easier distinguish them from the proof environment of the claim in which they are nested. There is no further nesting within this thesis (i.e proofs of subclaims do not contain any nested proof environments). The template below shows an example of this.

Proposition 0.4.1. *This is a proposition.*

Proof. In order to prove Proposition 0.4.1 we first prove the following intermediate result.

Claim 0.4.2. *This is a claim within the proof of Proposition 0.4.1.*

Proof: In order to prove Claim 0.4.2, we need the following fact.

> **Subclaim 0.4.3.** *This is a subclaim within the proof of Claim 0.4.2.*
>
> Proof: This is the proof of Subclaim 0.4.3. ■

Now we leave the indented environment and continue with the proof of Claim 0.4.2. This completes the proof of Claim 0.4.2. ■

This completes the proof of Proposition 0.4.1. □

0.5 Layout and Organization of this Thesis

Because the proof of Theorem 0.3.1 is very long, Chapters 1-13 are, to the greatest extent possible, organized in a modular way. More precisely, most of the chapters between 1 and 13 each consist of a lone main result, where this lone main result is an intermediate result in the proof of Theorem 0.3.1. Each of these chapters has the additional property that no subsequent chapter relies on the inner workings of the proof of this lone result. That is, for any $1 \leq k \leq 13$ and any chapter k which consists only of a lone result, we only make reference to Chapter k either to invoke the main result of Chapter k (which is, for the purposes of subsequent chapters, a black box) or to use a definition or a piece of notation which was introduced in Chapter k.

Chapter	Lone Main Result
3	Theorem 3.0.2
4	Theorem 4.0.1
5	Theorem 5.1.6
6	Theorem 6.0.9
7	Theorem 7.0.1

Chapter	Lone Main Result
8	Theorem 8.0.4
9	Theorem 9.0.1
10	Theorem 10.0.7
11	Theorem 2.1.7
13	Theorem 13.0.1

Table 0.5.1: Chapters and their main results

Table 0.5.1 shows all chapters with this property. The purpose of this modular organization is to avoid forcing the reader to read all the details of one chapter before moving onto the next. Because the proof of Theorem 0.3.1 is long and technical and contains many intermediate results, it is organized in a way that allows the reader to take note of intermediate results, skip the details of their proofs, read ahead to later chapters to see where and how these intermediate results are applied, and then return to the earlier chapters to read through the details of the proofs of these intermediate results at their leisure. In particular, if a section is part of a chapter which consists of a lone main result and its proof, and that section itself consists only of a lone intermediate result, then that intermediate result is not referenced in subsequent chapters, since it is only used to obtain the main result of the chapter it belongs to. The reader thus has a considerable amount of freedom in deciding which details to skip or read later.

The modular organization of the proof of Theorem 0.3.1 outlined above, is, to the greatest extent possible, replicated at the level of individual chapters. That is, there are many sections of this thesis which each consist solely of a lone intermediate result and its proof, and possibly several corollaries to this result, where the only subsequent references to this section are for the purpose of invoking this result or its corollaries, which are otherwise black boxes. The table below lists, in order, all sections which consist of a lone result.

Chapter	Section	Lone Intermediate Result
1	1.6	Theorem 1.6.1
2	2.2	Theorem 2.2.4
2	2.3	Theorem 2.3.2
3	3.1	Lemma 3.1.1
3	3.3	Lemma 3.3.1
4	4.1	Theorem 4.1.3
4	4.2	Lemma 4.2.1
4	4.4	Theorem 4.4.1
5	5.2	Lemma 5.2.1
5	5.3	Lemma 5.3.1
6	6.3	Theorem 6.3.2
7	7.1	Theorem 7.1.1
8	8.2	Lemma 8.2.1
8	8.3	Lemma 8.3.3
8	8.4	Lemma 8.4.1
8	8.5	Lemma 8.5.1

Chapter	Section	Lone Intermediate Result
9	9.1	Lemma 9.1.1
9	9.2	Lemma 9.2.4
10	10.1	Lemma 10.1.1
10	10.3	Lemma 10.3.2
10	10.4	Lemma 10.4.2
10	10.7	Lemma 10.7.1
10	10.8	Lemma 10.8.1
11	11.1	Theorem 11.1.1
11	11.2	Theorem 11.2.3
11	11.3	Lemma 11.3.2
12	12.2	Theorem 12.2.10
12	12.4	Theorem 12.4.1
12	12.5	Theorem 12.3.3
13	13.1	Lemma 13.1.1
13	13.3	Lemma 13.3.1

Table 0.5.2: Sections and their main results

Chapter 1

Charts, Path Colorings, and Tessellations

In this chapter, we introduce the basic tools, definitions, and language used to prove Theorem 0.3.1. We prove some basic topological and coloring facts which we use over the course of this thesis, and we also give a brief overview of the proof of Theorem 0.3.1.

1.1 Charts and Colorings

We begin with the following definition, which is our main object of study.

Definition 1.1.1. Let $k, \alpha \geq 1$ be integers. A tuple $\mathcal{T} = (G, \mathcal{C}, L)$ is an (α, k)-*chart* if G is a planar graph with list-assignment L, $\mathcal{C}$ is a family of facial subgraphs of G, and the following conditions are satisfied.

1) For any distinct facial subgraphs $H_1, H_2 \in \mathcal{C}$, $d(H_1, H_2) \geq \alpha$*; AND*
2) $|L(v)| \geq 5$ for all $v \in V(G) \setminus \left(\bigcup_{H \in \mathcal{C}} V(H)\right)$*; AND*
3) For each $H \in \mathcal{C}$, there exists a connected subgraph $\mathbf{P}_{\mathcal{T}}(H)$ of H satisfying the following.
    i) $|E(\mathbf{P}_{\mathcal{T}}(H))| \leq k$ and $\mathbf{P}_{\mathcal{T}}(H)$ is induced in H*; AND*
    ii) $|L(v)| \geq 3$ for all $v \in V(H) \setminus V(\mathbf{P}_{\mathcal{T}}(H))$*; AND*
    iii) $V(\mathbf{P}_{\mathcal{T}}(H))$ is L-colorable and $|L(v)| = 1$ for all $v \in V(\mathbf{P}_{\mathcal{T}}(H))$.

A tuple $\mathcal{T}$ is a *chart* if there exist integers $k, \alpha \geq 1$ such that $\mathcal{T}$ is an (α, k)-chart. A chart $\mathcal{T} = (G, \mathcal{C}, L)$ is *colorable* if G is L-colorable.

For any chart $\mathcal{T} = (G, \mathcal{C}, L)$ and any $H \in \mathcal{C}$, we let $\mathbf{P}_{\mathcal{T}}(H)$ denote the uniquely specified subgraph of H satisfying i)-iii) of 3) of Definition 1.1.1. We call $\mathbf{P}_{\mathcal{T}}(H)$ the *precolored subgraph* of H.

Given a chart $\mathcal{T} = (G, \mathcal{C}, L)$, the elements of $\mathcal{C}$ are called the *rings* of the chart. This terminology, together with the notation $\mathcal{C}$, is suggestive of the fact that our primary interest is the case where the rings of the chart are cyclic facial subgraphs of G, but in general, the definition of a chart does not require the elements of $\mathcal{C}$ to be cyclic facial subgraphs of G, or even connected subgraphs of G.

In most applications of the terminology above, we are dealing with the case where each $H \in \mathcal{C}$ is a facial cycle of G, and thus $\mathbf{P}_{\mathcal{T}}(H)$ is either a path or a cycle. If either the underlying chart or the ring containing the precolored subgraph, or both, are clear from the context, we usually drop either the H or the $\mathcal{T}$ or both from the notation $\mathbf{P}_{\mathcal{T}}(H)$ to avoid clutter, i.e we just write $\mathbf{P}$. The bold-font $\mathbf{P}$ is only ever used to refer to these paths in charts so there is no danger of confusing them with other paths. For our purposes, it is essential to distinguish the special case in which the entirety of an element of $\mathcal{C}$ is precolored, so we introduce the following terminology.

Definition 1.1.2. Let $\mathcal{T} = (G, \mathcal{C}, L)$ be a chart and let $H \in \mathcal{C}$. We say that H is a *closed* $\mathcal{T}$-ring if $\mathbf{P}_{\mathcal{T}}(H) = H$. Otherwise, we say that C is an *open* $\mathcal{T}$-ring.

We now restate Theorem 0.3.1 in the language of charts.

Theorem 1.1.3. *Every* $(48749 + 3\gamma, 1)$*-chart is colorable, where* γ *is as in Theorem 0.2.6.*

Since we are dealing with graphs as topological objects, we adopt the following standard definitions.

Definition 1.1.4. Let G be a planar embedding, and let A, B, H be subgraphs of G. We say that H *disconnects* A from B if the following hold.

1) Each of $V(A \setminus H)$ and $V(B \setminus H)$ is nonempty*; AND*
2) Every (A, B)-path in G contains a vertex of H.

We say that H *separates* A from B if H satisfies the following stronger properties.

1) $A \cap B \subseteq H$ and each of $E(A) \setminus E(H)$ and $E(B) \setminus E(H)$ are nonempty*; AND*
2) For any edges $e_1 \in E(A)$ and $e_2 \in E(B)$, any points $x \in e_1, y \in e_2$ with $x, y \notin V(H)$, and any arc $P \subseteq \mathbb{R}^2$ with endpoints x, y, P has nonempty intersection with H, where H, e_1, e_2 are regarded as subsets of $\mathbb{R}^2$.

In some cases, to avoid clutter, we abuse this notation in the following way: Given a planar embedding G, a subgraph H of G, an integer $k \geq 1$ and a k-chord P of H, we write "P separates a from b" to mean that the deletion of $H \cup P$ leaves a, b in two connected components of $\mathbb{R}^2 \setminus (H \cup P)$.

Given a planar graph G, a *separating cycle* in G is a cycle C in G such that both $\text{Int}(C) \setminus C$ and $\text{Ext}(C) \setminus C$ have nonempty intersection with $V(G)$. Of particular importance to us over the course of the proof of Theorem 1.1.3 are planar graphs which do not have separating cycles of length 3 or 4.

Definition 1.1.5. Given a planar graph G, we call G *short-separation-free* if G does not contain any separating cycle of length 3 or 4. Likewise, given a chart $\mathcal{T} = (G, \mathcal{C}, L)$, we call $\mathcal{T}$ a *short-separation-free chart* if G is a short-separation-free graph.

We now introduce some additional notation related to list-assignments for graphs. We very frequently analyze the situation where we begin with a partial L-coloring ϕ of a subgraph of a graph G, and then delete some or all of the vertices of $\text{dom}(\phi)$ and remove the colors of the deleted vertices from the lists of their neighbors in $G \setminus \text{dom}(\phi)$. We thus make the following definition.

Definition 1.1.6. *Let G be a graph, let ϕ be a partial L-coloring of G, and let $S \subseteq V(G)$. We define a list-assignment L^S_ϕ for $G \setminus (\text{dom}(\phi) \setminus S)$ as follows.*

$$L^S_\phi(v) := \begin{cases} \{\phi(v)\} \text{ if } v \in \text{dom}(\phi) \cap S \\ L(v) \setminus \{\phi(w) : w \in N(v) \cap (\text{dom}(\phi) \setminus S)\} \text{ if } v \in V(G) \setminus \text{dom}(\phi) \end{cases}$$

If $S = \varnothing$, then $L^\varnothing_\phi$ is a list-assignment for $G \setminus \text{dom}(\phi)$ in which the colors of the vertices in $\text{dom}(\phi)$ have been deleted from the lists of their neighbors in $G \setminus \text{dom}(\phi)$. The situation where $S = \varnothing$ arises so frequently that, in this case, we simply drop the superscript and let L_ϕ denote the list-assignment $L^\varnothing_\phi$ for $G \setminus \text{dom}(\phi)$. In some cases, we specify a subgraph H of G rather than a vertex-set S. In this case, to avoid clutter, we write L_ϕ^H to mean $L_\phi^{V(H)}$. We now introduce the following natural way to combine two partial colorings of a graph.

Definition 1.1.7. Given two subsets $S_1, S_2 \subseteq V(G)$ and, for each $i \in \{1,2\}$, an L-coloring ϕ_i of S_i, we have the following: If $\phi_1(v) = \phi_2(v)$ for each $v \in S_1 \cap S_2$, and $\phi_1(x) \neq \phi_2(y)$ for each $xy \in E(G)$ with $x \in S_1$ and $y \in S_2$, then there is a well-defined proper L-coloring ϕ of $S_1 \cup S_2$ which is compatible with ϕ_1, ϕ_2, where

$$\phi(v) := \begin{cases} \phi_1(v) \text{ if } v \in S_1 \\ \phi_2(v) \text{ if } v \in S_2 \end{cases}$$

We denote this coloring by $\phi_1 \cup \phi_2$.

Definition 1.1.8. Given a graph G, a list-assignment L for $V(G)$, a subgraph H of G, and a partial L-coloring ϕ of G, we let $\Phi_{G,L}(\phi, H)$ denote the set of extensions of ϕ to L-colorings of dom$(\phi) \cup V(H)$. If ϕ is the empty coloring (i.e dom$(\phi) = \varnothing$) then $\Phi_{G,L}(\phi, H)$ is just the set of L-colorings of H, which we denote by $\Phi_{G,L}(H)$.

In some cases, it is more convenient to specify a vertex set rather than a subgraph of G. Given a vertex set $S \subseteq V(G)$, we define $\Phi_{G,L}(\psi, S)$ analogously. That is, $\Phi_{G,L}(\psi, S)$ is the set of extensions of ψ to L-colorings of dom$(\psi) \cup S$.

In certain cases we want to analyze the possible extensions of a partial coloring obtained by extending the domain of the partial coloring by a lone vertex, particularly when we are considering colorings of a subpath of a path, and want to extend this coloring to the next vertex in the path. We thus introduce the following compact notation.

Definition 1.1.9. Let G be a graph and let L be a list-assignment for $V(G)$. Let ψ be a partial L-coloring of G. For each $c \in L_\psi(v)$, the notation $\psi\langle v : c\rangle$ denotes the extension of ψ to an L-coloring of dom$(\phi) \cup \{v\}$ obtained by coloring v with c.

Given a family of partial L-colorings of G, it is useful to keep track of the colors given to a particular vertex by the elements of this family.

Definition 1.1.10. If $\mathcal{F}$ is a family of partial L-colorings of G, then we define $\text{Col}_{G,L}(v, \mathcal{F}) := \{\phi(v) : \phi \in \mathcal{F} \text{ and } v \in \text{dom}(\phi)\}$.

In view of Theorem 0.2.3 and Definition 0.1.2, we introduce the following terminology, which we use repeatedly throughout this thesis.

Definition 1.1.11. Given a graph G, a list-assignment L for G, and a facial subgraph H of G, we call H a *Thomassen facial subgraph* of G with respect to L if H is a facial subgraph of G and there is an edge $xy \in E(H)$ such that xy is L-colorable, and, for all $v \in V(H) \setminus \{x, y\}$, $|L(v)| \geq 3$.

We also introduce the following natural notation, which we use frequently both in this chapter and subsequent chapters.

Definition 1.1.12. Given a graph G and a cycle C in G, we let $\text{Int}^+(C)$ denote the graph $G[\text{Int}_G(C)]$, and, likewise, we let $\text{Ext}^+_G(C)$ denote the graph $G[\text{Ext}_G(C)]$. That is, $\text{Int}^+_G(C)$ consists of $\text{Int}_G(C)$ together with any chords of C which lie in $\text{Ext}_G(C)$, and $\text{Ext}^+_G(C)$ is defined analogously. Furthermore, we let $\text{Int}^-_G(C)$ denote the graph $\text{Int}_G(C) \setminus E(C)$ and let let $\text{Ext}^-_G(C)$ denote the graph $\text{Ext}_G(C) \setminus E(C)$.

1.2 Coloring and Deleting Paths

Given a graph G, a list-assignment L for G, and a path P in G, we sometimes want to find an L-coloring ψ of P such that G does not have too many vertices of distance one from P which have L_ψ-lists of size less than three. Since we frequently perform analyses of this form throughout this thesis, we introduce the following compact notation.

Definition 1.2.1. Let G be a graph with list-assignment L. Given an integer $n \geq 1$, a path $P := p_1 \cdots p_n$, and a partial L-coloring ϕ of G with $V(P) \subseteq \text{dom}(\phi)$, we denote the L-coloring $\phi|_P$ of P as $(\phi(p_1), \phi(p_2), \cdots, \phi(p_n))$.

We introduce several more very natural pieces of terminology.

Definition 1.2.2. Let G be a graph and let $P \subseteq G$ be a path.

1) We say that P is a *quasi-shortest path* if P is an induced path in G with the additional property that, for any $v \in D_1(P)$ and any two vertices $w, w' \in V(P) \cap N(v)$, the path wPw' has length at most two.

2) If P is a quasi-shortest path, then, given a $v \in V(\mathring{Q})$, we say that v is a *P-gap* if there is no vertex of $D_1(P)$ such that $G[N(v) \cap V(P)]$ is a subpath of P of length two with midpoint v.

3) Given a list-assignment L, we introduce the following sets of L-colorings of P.

    i) We let $\text{Avoid}_{G,L}(P)$ be the set of L-colorings ϕ of $V(P)$ such that, for every $v \in D_1(P)$, either $|L(v)| < 5$ or $|L_\phi(v)| \geq 3$.

    ii) We let $\text{Avoid}^\dagger_{G,L}(P)$ be the set of L-colorings ϕ of $V(P)$ such that, for some $v^\dagger \in D_1(P)$, we have either $|L(v^\dagger)| < 5$ or $|L_\phi(v^\dagger)| \geq 2$, and furthermore, for every $v \in D_1(P) \setminus \{v^\dagger\}$, either $|L(v)| < 5$ or $|L_\phi(v)| \geq 3$.

We now prove a simple results about coloring and deleting paths in short-separation free planar graphs.

Proposition 1.2.3. *Let G be a short-separation-free planar graph, where $|V(G)| > 5$, and let L be a list-assignment for G. Let $P := p_1 \cdots p_k$ be a quasi-shortest path, let P' be a terminal subpath of P. Suppose further that every vertex of $P \setminus P'$ has an L-list of size at least five. Then for any $\phi \in$* $\text{Avoid}_{G,L}(P')$, *ϕ extends to an element of* $\text{Avoid}_{G,L}(P)$.

Proof. Let $P' := p_1 \cdots p_\ell$ for some $1 \leq \ell \leq r$. For each $j = \ell, \cdots, r$, let B_j be the set of extensions of ϕ to an element of $\text{Avoid}_{G,L}(p_1 P p_j)$. Note that $\phi \in B_\ell$. We claim that $B_j \neq \varnothing$ for all $j = \ell, \cdots, r$. We show this by induction on j. This holds if $j = \ell$ since $\phi \in B_\ell$. If $r = \ell$, then we are done, so let $j \in \{\ell, \cdots, r-1\}$ and let $\psi \in B_j$.

Since P is an induced subpath of G, we have $|L_\psi(p_{j+1})| \geq 4$. If p_j is a P-gap, then any extension of ψ to an L-coloring of $V(p_1 P p_{j+1})$ lies in B_{j+1}, so we are done in that case, so now suppose that p_j is not a P-gap. Since $|V(G)| > 5$ and G is short-separation-free, G is $K_{2,3}$-free, so there is a unique vertex w such that $G[N(w) \cap V(P)] = p_{j-1}p_jp_{j+1}$. If $|L(w)| < 5$, then, as above, any any extension of ψ to an L-coloring of $V(p_1 P p_{j+1})$ lies in B_{j+1}, so we are done in that case, so now suppose that $|L(w)| \geq 5$. Thus, we have $|L_\psi(w)| \geq 3$, as $N(w) \cap \text{dom}(\psi) = \{p_{j-1}, p_j\}$.

Since $|L_\psi(p_{j+1})| \geq 4$, there is an extension of ψ to an L-coloring ψ^* of $p_1 P p_{j+1}$ such that $|L_{\psi^*}(w)| \geq 3$, and thus $\psi^* \in B_{j+1}$. We conclude that $B_{j+1} \neq \varnothing$, as desired. $\square$

Now we have the following.

Proposition 1.2.4. *Let G be a short-separation-free planar graph, where $|V(G)| > 5$, and let L be a list-assignment for G. Let P be a quasi-shortest path in G with $|E(P)| = 4$. Let L be a list-assignment for G such that each endpoint of P has a list of size at least one and each internal vertex of P has a list of size at least five. Then* $\text{Avoid}^\dagger_{G,L}(P) \neq \varnothing$.

Proof. Let $P := p_1p_2p_3p_4p_5$. Let ψ be an L-coloring of $\{p_1, p_5\}$. Let $\text{Sub}^{\pentagon}(P) := \{w \in D_1(P) : |L(w)| \geq 5\}$. Since P is a quasi-shortest path and $|L_\psi(p_3)| \geq 5$, there is a $d \in L_\psi(p_3)$ such that either no vertex of $\text{Sub}^{\pentagon}(P)$ adjacent to all of p_1, p_2, p_3, or there is a unique $w \in \text{Sub}^{\pentagon}(P)$ such that $|L_\psi(w_2) \setminus \{d\}| \geq 4$.

Let ψ' be an extension of ψ to an L-coloring of $\{p_1, p_3, p_5\}$ obtained by coloring p_3 with d. Since P is a quasi-shortest path, we have $|L_{\psi'}(p_2)| \geq 3$ and $|L_{\psi'}(p_4)| \geq 3$, and thus, there is an extension of ψ' to an L-coloring ψ^* of $V(P)$ such that either no vertex of $\text{Sub}^{\pentagon}(P)$ is adjacent to all three of $p_2p_3p_4$, or there is a $w_3 \in \text{Sub}^{\pentagon}(P)$ such that $G[N(w_3) \cap V(P)] = p_2p_3p_4$ and $|L_{\psi^*}(w_3)| \geq 3$. If there is a $w \in \text{Sub}^{\pentagon}(P)$ with $|L_{\psi^*}(w)| < 3$, then $|L_{\psi^*}(w)| = 2$ and w is the unique vertex of G adjacent to all three of p_3, p_4, p_5, so $\psi^* \in \text{Avoid}^\dagger(P)$ and we are done. $\square$

We have an analogous result for paths of length six.

Proposition 1.2.5. *Let G be a short-separation-free planar graph, where $|V(G)| > 5$, and let L be a list-assignment for G. Let P be a quasi-shortest path in G with $|E(P)| = 6$, and suppose that the midpoint of P is a P-gap. Let L be a list-assignment for G such that each endpoint of P has a list of size at least one and each internal vertex of P has a list of size at least five. Then* $\text{Avoid}^\dagger_{G,L}(P) \neq \varnothing$.

Proof. Let $P := p_1p_2p_3p_4p_5p_6p_7$. Let ψ be an L-coloring of $\{p_1, p_7\}$. Let $\text{Sub}^{\pentagon}(P) := \{w \in D_1(P) : |L(w)| \geq 5\}$. Since P is a quasi-shortest path, there exist colors $d_3 \in L_\psi(p_3)$ and $d_5 \in L_\psi(p_5)$ such that both of the following hold.

1) Either no vertex of $\text{Sub}^{\pentagon}(P)$ is adjacent to all three of p_1, p_2, p_3 or, letting w_2 be the unique vertex of $\text{Sub}^{\pentagon}(P)$ such that $G[N(w_2) \cap V(P)] = p_1p_2p_3$, we have $|L_\psi(w_2) \setminus \{d_3\}| \geq 4$*; AND*
2) Either no vertex of $\text{Sub}^{\pentagon}(P)$ is adjacent to all three of p_5, p_6, p_7 or, letting w_6 be the unique vertex of $\text{Sub}^{\pentagon}(P)$ such that $G[N(w_6) \cap V(P)] = p_5p_6p_7$, we have $|L_\psi(w_6) \setminus \{d_5\}| \geq 4$.

Since P is an induced path, ψ extends to an L-coloring ψ' of $\{p_1, p_3, p_5, p_7\}$ such that $\psi'(p_3) = d_3$ and $\psi'(p_5) = d_5$. We have $|L_{\psi'}(p_2)| \geq 3$ and $|L_{\psi'}(p_4)| \geq 3$, and since P is a quasi-shortest path, it follows that ψ' extends to an L-coloring ψ'' of $P - p_6$ such that either no vertex of $\text{Sub}^{\pentagon}(P)$ is adjacent to all three of p_2, p_3, p_4, or there is a $w_3 \in \text{Sub}^{\pentagon}(P)$ with $G[N(w_3) \cap V(P)] = p_2p_3p_4$ and $|L_{\psi''}(w_3)| \geq 3$.

In any case, we have $|L_{\psi''}(p_6)| \geq 3$, and ψ'' extends to an L-coloring ψ^* of $V(P)$. If there is no vertex w_5 of $\text{Sub}^{\pentagon}(P)$ with $G[N(w_5) \cap V(P)] = p_4p_5p_6$, then we have $\psi^* \in \text{Avoid}(P)$, as p_4 is a P-gap vertex. In that case, we are done since $\text{Avoid}(P) \subseteq \text{Avoid}^\dagger(P)$. If such a vertex w_5 exists, then w_5 is unique and $|L_{\psi^*}(w_5)| \geq 2$. In that case, since p_4 is a P-gap, we again have $\psi^* \in \text{Avoid}^\dagger(P)$, so we are done. $\square$

Now we have the following.

Proposition 1.2.6. *Let G be a short-separation-free planar graph with $|V(G)| > 5$ and let $P := p_1p_2 \cdots p_k$ be a path in G which is a shortest path between its endpoints. Suppose that there exists a $j \in \{2, \cdots, k-3\}$ such that no vertex of $p_jp_{j+1}p_{j+2}$ is a P-gap. Then there exists a $w \in D_1(P)$ such that $G[N(w) \cap V(P)] = p_{j+1}p_{j+2}p_{j+3}$ and such that, letting P^* be the path obtained from P by replacing p_{j+2} with w, the vertex p_{j+1} is a P^*-gap.*

Proof. Since no vertex of $p_jp_{j+1}p_{j+2}$ is a P-gap and P is a shortest path between its endpoints, it follows that, for each $r = 0, 1, 2$, there is a $w_r \in D_1(P)$ such that $G[N(w_r) \cap V(P)] = p_{(j+r)-1}p_{j+r}p_{j+r+1}$.

Note that P^* is also a shortest path between its endpoints. Suppose toward a contradiction that p_{j+1} is not a P^*-gap. Thus, there exists a $q \in D_1(P^*)$ such that $G[N(q) \cap V(P^*)] = p_jp_{j+1}w_2$. If $q = w_1$, then $w_1w_2 \in E(G)$ and G contains a copy of K_4 on the vertices $\{w_1, w_2, p_{j+1}, p_{j+2}\}$, contradicting the fact that G is short-separation-free.

If $q = w_0$, then $w_0w_2 \in E(G)$, contradicting the fact that P is a shortest path between its endpoints. Thus, we have $q \neq w_0, w_2$, and G contains a $K_{2,3}$ with bipartition $\{w_0, w_1, q\}$, $\{p_j, p_{j+1}\}$, contradicting the fact that G is short-separation-free. $\square$

Likewise, we have the following

Proposition 1.2.7. *Let G be a short-separation-free planar graph and let P be a shortest path between its endpoints, where $|V(P)| > 6$. Suppose that there exists a subpath Q of $\mathring{P}$ with $|E(Q)| = 4$ such that no vertex of Q is a P-gap. Then there exists a $w \in D_1(P)$ such that $G[N(w) \cap V(P)] = \mathring{Q}$ and such that, letting P^* be the path obtained by from P by replacing the midpoint of $\mathring{Q}$ with w, P^* is a quasi-shortest path and each endpoint of $\mathring{Q}$ is a P^*-gap.*

Proof. Let $Q := p_j \cdots p_{j+4}$ for some $j \in \{2, \cdots, k-5\}$. The result just follows by applying Proposition 1.2.6 to each of the two paths $p_jp_{j+1}p_{j+2}$ and $p_{j+2}p_{j+3}p_{j+4}$ $\square$

We use the results above to color and delete paths between facial subgraphs of short-separation-free graphs. We also use the following useful notation.

Definition 1.2.8. Let G be a short-separation-free planar graph. Given a cycle $A \subseteq G$ be a cycle and an edge $e = wz$ of G with $w \in D_3(A)$ and $z \in D_2(A)$, we let $\text{Bar}_A(wz)$ be the set of vertices $v \in V(G) \setminus \{w, z\}$ such that $v \in N(w) \cap N(z)$ and v has a neighbor in $D_1(A)$.

We conclude this section with the following simple observation.

Observation 1.2.9. *Let G be a short-separation-free planar graph with $|V(G)| > 5$, let $A \subseteq G$ be a cycle, and let $w \in D_3(A)$, where $N(w) \not\subseteq D_2(A)$. Then there exists a $z \in N(w) \cap D_2(A)$ such that $|\text{Bar}_A(wz)| \leq 1$.*

Proof. We first note the following simple observation.

Claim 1.2.10. *Each connected component of $G[N(w) \cap D_2(A)]$ is an induced path.*

<u>Proof:</u> Since G is short-separation-free and $|V(G)| > 5$, G is $K_{2,3}$-free, so each vertex of $N(w)$ has degree at most two in the graph $G[N(w)]$. If there is a connected component of $G[N(w) \cap D_2(A)]$ in which every vertex has degree two, then G contains a wheel with central vertex w, where every other vertex of the wheel lies in $N(w) \cap D_2(A)$. Since G is short-separation-free, it then follows that $N(w) \subseteq D_2(A)$, contradicting our assumption. Since every vertex of $N(w)$ has degree at most two in $G[N(w)]$, it follows that every connected component of $G[N(w) \cap D_2(A)]$ is an induced path. $\blacksquare$

For each $z \in N(w) \cap D_2(A)$ and $u \in \text{Bar}_A(wz)$, we have $u \in D_2(A)$, as u has a neighbor of distance one from A and a neighbor of distance three from A. In particular, for any connected component P of $G[N(w) \cap D_2(A)]$ and any endpoint p of P, we have $|\text{Bar}_A(wp)| \leq 1$. $\square$

1.3 An Overview of the Proof of Theorem 1.1.3

We now provide a brief overview of the proof of Theorem 1.1.3, whose remaning details are worked out in chapters 2-13 of this thesis. We first introduce the following terminology. One of the key ingredients in the proof of Theorem 1.1.3 is the reduction to a particular subclass of charts which are easier to study.

Definition 1.3.1. Let $\mathcal{T} = (G, \mathcal{C}, L)$ be a chart.

1) We call $\mathcal{T}$ *near-triangulated* if, for every facial subgraph H of G, with $H \notin \mathcal{C}$, H is a triangle.

2) We call $\mathcal{T}$ a *tessellation* if it is near-triangulated and short-separation-free.

3) Given integers $k, \alpha \geq 1$, we call $\mathcal{T}$ an (α, k)-*tessellation* if it is both a tessellation and an (α, k)-chart.

In Chapters 2-11, we show that, for some $\beta \geq 1$, all $(\beta, 1)$-tessellations are colorable. More precisely, we prove something stronger by defining a structure called a *mosaic* and showing that all mosaics are colorable. This result, which is the main step in the proof of Theorem 1.1.3, is stated in Theorem 2.1.7, and the entirety of Chapters 2-11 consists of the proof of Theorem 2.1.7.

In Chapters 12 and 13, we complete the proof of Theorem 1.1.3 by showing that Theorem 2.1.7 implies Theorem 1.1.3. That is, we show that, since all mosaics are colorable, there exists an $\alpha \geq 1$ such that $(\alpha, 1)$-charts are colorable.

The key to the proof of Theorem 2.1.7 is to choose the right definition of mosaics, i.e to choose the right induction hypothesis. We show that Theorem 2.1.7 holds by a minimal counterexample argument. In Chapter 2, we gather some basic structural properties of minimal counterexamples, and we also analyze k-chords of the rings of a minimal counterexample for small values of k. The primary purpose of Chapter 2 is to show that, given a minimal counterexample $\mathcal{T} = (G, \mathcal{C}, L)$ (for some suitable definition of minimal counterexample, which is made precise later) and a $C \in \mathcal{C}$, for sufficiently small values of k, there is no k-chord of C which separates the faces of $\mathcal{C} \setminus \{C\}$. This means that, given a k-chord of C in G, there is a "small" side of the k-chord in G in which all vertices outside of C have L-lists of size at least five. This fact is essential to our construction of a smaller counterexample from a minimal counterexample.

In Chapters 3 and 4, we continue to investigate k-chords of the open rings of a minimal counterexample, where $k \leq 5$, and show that a minimal counterexample has a very regular structure near each open ring. In Chapters 5 and 6, we show how to carefully color and delete vertices on and near each open ring of a minimal counterexample. In Chapters 7 and 8, we perform an analysis for closed rings analogous to the results for open rings proven in Chapters 3 and 4. In Chapters 9 and 10, we show how to carefully color and delete vertices on and near each closed ring of a minimal counterexample, i.e we prove a result for closed rings which is analogous to the result for open rings in Chapter 6. Finally, in Chapter 11, we use the work of Chapters 2-10 to produce a smaller counterexample by carefully coloring and deleting a path between two rings of a minimal counterexample. Over the course of this proof, we repeatedly rely on the following simple standard fact.

Theorem 1.3.2. *Let G be a planar graph and let F, F' be facial subgraphs of G (possibly $F = F'$). Let $T \subseteq G$ be a subgraph of G satisfying the following properties.*

1) T has nonempty intersection with each of F, F', and $T \cup F \cup F'$ is connected; AND

2) Either $F = F'$ or $F \cap F' \subseteq T$; AND

3) For every $v \in V(T)$, every facial subgraph of G containing v, except possibly F, F', is a triangle.

Then there is a facial subgraph F^* of $G \setminus T$ such that $V(F^*) = D_1(T) \cup V((F \cup F') \setminus T)$ and $E((F \cup F') \setminus T) \subseteq E(F^*)$.

Given a graph G with a list-assignment L, and a subgraph H of G, we frequently deal with situations where we construct a partial L-coloring ϕ of H such that, for any extension ψ of ϕ to $G \setminus (H \setminus \text{dom}(\phi))$, ψ extends to the rest of $V(H)$. This is very useful in instances where we have a set of vertices that we want to delete, and it is desirable to color as few vertices in the deletion set as possible. We thus introduce the following definition, which we use all throughout the remainder of this thesis.

Definition 1.3.3. Let G be a graph with a list-assignment L. Given a subset $Z \subseteq V(G)$ and a partial L-coloring ϕ of $V(G)$, we say that Z is (L, ϕ)*-inert in* G if every extension of ϕ to an L-coloring of $G \setminus (Z \setminus \text{dom}(\phi))$ extends to an L-coloring of all of G.

If the ambient graph G is clear from the context, then we just say that Z is (L, ϕ)-inert. If ϕ is the empty coloring, then we just say that Z is L-inert in G. Note that, for a partial L-coloring ϕ of G, if $Z \subseteq V(G) \setminus \text{dom}(\phi)$, then Z is (L, ϕ)-inert in G if and only if Z is L_ϕ-inert in $G \setminus \text{dom}(\phi)$.

In situations where we color and delete vertices on or near a specified facial subgraph C of a given graph, where the vertices of C have lists of size three, it is useful to be able to delete some vertices without coloring them, which we do if any coloring of the remaining graph extends to the uncolored vertices. Given a chart $\mathcal{T} = (G, \mathcal{C}, L)$, the two cases where this usually arises are the case where some specified subgraph of G is separated from all the vertices of G with lists of size less than five by a short cycle, and the case where, for some $C \in \mathcal{C}$, some specified subgraph of G is separated from all the vertices of $G \setminus C$ with lists of size less than five by a generalized chord of C with short length. In the remaining sections of Chapter 1, we gather some results which provide inertness conditions for a vertex set S of a specified planar graph G with a list assignment L, where, for some facial cycle C of G, S is separated from all the vertices of $G \setminus C$ with lists of size less than five by a 2- or 3-chord of C.

The proof of Theorem 1.1.3 outlined in the paragraphs above is mostly self-contained. However, in addition to Theorem 0.2.3 and Theorem 0.2.6, we rely on two additional useful results which we state below. The first of these results is proven in [9].

Theorem 1.3.4. *Let G be a planar graph, let F be a facial subgraph of G, and let $v, w \in V(F)$. Let L be a list-assignment for $V(G)$ where $|L(v)| \geq 2$, $|L(w)| \geq 2$, and furthermore, for each $x \in V(F) \setminus \{v, w\}$, $|L(x)| \geq 3$, and, for each $x \in V(G \setminus F)$, $|L(x)| \geq 5$. Then G is L-colorable.*

In addition to Theorem 1.3.4, we use a simple but very useful theorem from [2] that characterizes the obstructions to extending a precoloring of a short cycle in a planar graph.

Theorem 1.3.5. *Let G be a short-separation-free graph with facial cycle C, and let L be a list-assignment for G where $|L(v)| \geq 1$ for each $v \in V(C)$ and $|L(v)| \geq 5$ for all $v \in V(G \setminus C)$. Suppose that $|V(C)| \leq 6$ and $V(C)$ is L-colorable, but G is not L-colorable. Then $5 \leq |V(C)| \leq 6$, and the following hold.*

1) If $|V(C)| = 5$, then C is induced in G and $G \setminus C$ consists of a lone vertex which is adjacent to all five vertices of C; AND

2) If $|V(C)| = 6$, then $G \setminus C$ consists of at most three vertices, each of which has at least three neighbors in $G \setminus C$. Furthermore, if C is induced in G, then $G \setminus C$ is one of the following.

i) A lone vertex adjacent to all six vertices of C; OR

ii) An edge x_1x_2 such that, for each $i = 1, 2$, the graph $G[N(x_i) \cap V(C)]$ is a path of length three; OR

iii) A triangle $x_1x_2x_3$ such that $G[N(x_i) \cap V(C)]$ is a path of length two for each $i = 1, 2$.

Proof. This is just an immediate consequence of Theorem 7 of [2], since, in each of the three configurations a), b), c) listed, the obstruction is the entirety of $G \setminus C$, as G is short-separation-free. $\square$

Theorem 1.3.5 has the following immediate corollary.

Corollary 1.3.6. *Let H be a short-separation-free planar graph with facial cycle $C := p_1p_2p_3p_4p_5$. Let L be a list-assignment for H where $|L(p_i)| \geq 1$ for each $1 \leq i \leq 5$, C is L-colorable, and each vertex of $H \setminus C$ has an L-list of size at least five. Let $S \subseteq V(C)$ and let ϕ be an L-coloring of S. If either $|V(H \setminus C)| > 1$, or there is a vertex w of $H \setminus C$ adjacent to all five vertices of C, where $|L(w) \cap \{\phi(p) : p \in S\}| < |S|$, then $V(H \setminus C)$ is L_ϕ-inert in H.*

We use the standard terminology for the structure specified in Corollary 1.3.6.

Definition 1.3.7. A *wheel* is a graph H with a vertex $p \in V(H)$ such that $H - p$ is a chordless cycle and p is adjacent to all of the vertices of $V(H) \setminus \{p\}$.

In the remaining four sections of this chapter, we gather some preliminary facts we need about the list-coloring situations which occur very frequently in the subsequent chapters.

1.4 Extending Colorings of 2-Paths: Broken Wheels

Throughout this thesis, we frequently analyze planar graphs with a list-assignment in which a specified facial subgraph has a precolored path of length two. In Sections 1.4 and 1.5, we gather some facts which we use all throughout the remainder of the proof of Theorem 1.1.3.

Definition 1.4.1. Let H be a graph and let $P := p_1p_2p_3$ be a specified subpath of H of length two. Let L be a list-assignment for $V(H)$.

1) For each color $c \in L(p_2)$ and $d \in L(p_3)$, we set $\mathcal{Z}^P_{H,L}(\bullet, c, d) \subseteq L(p_1)$ to be the set of colors $f \in L(p_1)$ such that there is a proper L-coloring of H using f, c, d on the respective vertices p_1, p_2, p_3.
2) For any $f \in L(p_1)$ and $c \in L(p_2)$, we define the subset $\mathcal{Z}^P_{H,L}(f, c, \bullet)$ of $L(p_3)$ analogously to 1), and, for any $f \in L(p_1)$ and $d \in L(p_3)$, we define the subset $\mathcal{Z}^P_{H,L}(f, \bullet, d)$ of $L(p_2)$ analogously to 1).

In practice, the notation above is always used in the context where H is a planar graph and P is a subpath of a specified facial cycle of H, since we are interested in precolorings of paths of length two of a specified subpath of a facial cycle which extend to the entire graph. The use of the notation above always requires us to specify an ordering of the vertices of given 2-path. That is, whenever we write $\mathcal{Z}^P_{H,L}(\cdot, \cdot, \cdot)$, where two of the coordinates are colors of two of the vertices of P and one is a bullet denoting the remaining uncolored vertex of P, we have specified beforehand which vertices the first, second, and third coordinates correspond to. By Theorem 0.2.3, we immediately have the following:

Observation 1.4.2. *Let G be a planar graph with facial cycle C, and let $P := p_1p_2p_3$ be a subpath of C of length two. Let L be a list-assignment for $V(G)$ where each vertex of $C \setminus P$ has a list of size at least three, and each vertex of $G \setminus C$ has an L-list of size at least five. If ϕ is an L-coloring of p_1p_2 and $|L(p_3) \setminus \{\phi(p_2)\}| \geq 2$, then ϕ extends to an L-coloring of G.*

One of the structures analyzed frequently throughout the remaining chapters is the broken wheel, which is defined as follows.

Definition 1.4.3. A *broken wheel* is a graph H with a vertex $p \in V(H)$ such that $H - p$ is a path $q_1 \cdots q_r$ with $r \geq 2$, where $N(p) = \{q_1, \cdots, q_r\}$. The subpath p_1qp_r of H is called the *principal path* of H. The vertex q is called the *principal vertex* of H.

Note that, if $|V(H)| < 4$, then the above definition does not uniquely specify the principal path, although in practice, whenever we deal with broken wheels in this thesis, we specify the principal path beforehand so that there is no ambiguity.

In this section, we state and prove several useful facts about broken wheels which we use frequently throughout the remainder of this thesis. In Section 1.5, we consider the general case of a 2-path of a facial cycle in an arbitrary planar graph which is not necessarily a broken wheel. In Theorem 1.5.3, we show that in a certain sense, the broken wheel is the only nontrivial case. The first of the facts which make up Section 1.4, which is stated below, is trivial and is stated without proof.

Proposition 1.4.4. *Let H be a broken wheel with principal path $P := p_1p_2p_3$, and let L be a list-assignment for H such that $|L(u)| \geq 3$ for all $u \in V(H \setminus P)$. Let $H - p_2 = p_1u_1 \cdots u_tp_3$ for some $t \geq 0$. Then the following hold.*

1) *If $t \geq 1$ and ϕ is an L-coloring of P which does not extend to H, then $\phi(p_1) \in L(u_1)$, $\phi(p_3) \in L(u_t)$, and, for each $i = 1, \cdots, t$, we have $|L(u_i)| = 3$ and $\phi(p_2) \in L(u_i)$; AND*
2) *If ψ is an L-coloring of p_1p_2 with $|L(p_3) \setminus \{\psi(p_2)\}| \geq 2$, and $|\mathcal{Z}_H(\psi(p_1), \psi(p_2), \bullet)| = 1$, then $\psi(p_1) \in L(u_1)$, $|L(p_3) \setminus \{\psi(p_2)\}| = 2$, and, for each $i = 1, \cdots, t$, we have $|L(u_i)| = 3$ and $\psi(p_2) \in L(u_i)$.*

We now have the following facts.

Proposition 1.4.5. *Let H be a broken wheel with principal path $P := p_1p_2p_3$, and let L be a list-assignment for H such that $|L(p_1)| \geq 1$ and $|L(v)| \geq 3$ for all $v \in V(H) \setminus \{p_1, p_2\}$. Let $c \in L(p_1)$, and suppose further that $|L(p_2) \setminus \{c\}| \geq 3$. Then the following hold.*

1) *There exists a color $d \in L(p_2) \setminus \{c\}$ such that $|\mathcal{Z}_H(c, d, \bullet)| \geq 2$; AND*
2) *If $d_1, d_2, d_3 \in L(p_2) \setminus \{c\}$ are distinct colors and there is a lone color $e \in L(p_3)$ such that $\mathcal{Z}_H(c, d_1, \bullet) = \mathcal{Z}_H(c, d_2, \bullet) = \{e\}$, then $\mathcal{Z}_H(c, d_3, \bullet) \supseteq L(p_3) \setminus \{e\}$.*

Proof. Fact 1 is trivial if H is a triangle, so suppose now that H is not a triangle. Let $H \setminus \{p_2\}$ be the path $p_1x_1 \cdots x_tp_3$, where $t \geq 1$. If $c \in L(x_1)$, then, since $|L(x_1)| \geq 3$ and $|L(p_2) \setminus \{c\}| \geq 3$, there is a color $d \in L(p_2) \setminus \{c\}$ with $|L(x_1) \setminus \{c\}| \geq 3$. In that case, for any color $e \in L(p_3) \setminus \{d\}$, the coloring (c, d, e) of the principal path $p_1p_2p_3$ extends to an L-coloring of H, so $|\mathcal{Z}_H(c, d, \bullet)| \geq 2$, since $\mathcal{Z}_H(c, d, \bullet) = L(p_3) \setminus \{d\}$. Thus, if $c \in L(x_1)$, then we are done. Now suppose that $c \notin L(x_1)$. In that case, for any color $d \in L(p_2) \setminus \{c\}$ and any color $e \in L(p_3) \setminus \{d\}$, the coloring (c, d, e) of the principal path $p_1p_2p_3$ extends to an L-coloring of H, so $|\mathcal{Z}_H(c, d, \bullet)| \geq 2$ for any $d \in L(p_2) \setminus \{c\}$. So, again, we are done.

Now we prove Fact 2. By removing colors from p_2 if necessary, we may suppose that $L(p_2) \setminus \{c\} = \{d_1, d_2, d_3\}$. Suppose that $\mathcal{Z}_H(c, d_1, \bullet) = \mathcal{Z}_H(c, d_2, \bullet) = \{e\}$. In that case, we have $d_1, d_2 \in \bigcap_{u \in V(H) \setminus \{p_1, p_2\}} L(u)$ by Proposition 1.4.4. Thus, H is not a triangle, or else we have $d_1 \in \mathcal{Z}_H(c, d_2, \bullet)$ and $d_2 \in \mathcal{Z}_H(c, d_1, \bullet)$, contradicting our assumption. Thus, let $H \setminus \{p_2\} = p_1v_1 \cdots v_tp_3$ for some $t \geq 1$. Now, since $\mathcal{Z}_H(c, d_1, \bullet) = \mathcal{Z}_H(c, d_2, \bullet) = \{e\}$, we have $L(x_t) = L(p_3) = \{d_1, d_2, e\}$ by Proposition 1.4.4. Thus, if $d_3 \neq e$, then $\mathcal{Z}_H(c, d_3, \bullet) = L(p_3)$, so we are done in that case. Now suppose that $d_3 = e$. By Fact 1, we have $|\mathcal{Z}_H(c, e, \bullet)| \geq 2$. Since $e \notin \mathcal{Z}_H(c, e, \bullet)$, we have $\mathcal{Z}_H(c, e, \bullet) = \{d_1, d_2\} = L(p_3) \setminus \{e\}$, so, again, we are done. This completes the proof of Fact 2. $\square$

Proposition 1.4.5 has the following immediate corollary.

Corollary 1.4.6. *Let H be a broken wheel with principal path $P := p_1p_2p_3$, and let L be a list-assignment for H such that $|L(p_1)| \geq 1$, and $|L(v)| \geq 3$ for all $v \in V(H) \setminus \{p_1, p_2\}$. Let $c \in L(p_1)$, and suppose further that $|L(p_2) \setminus \{c\}| \geq 3$. Then the following hold.*

1) *There exists a pair of distinct colors $d, d' \in L(p_3)$ such that $\mathcal{Z}_H(c, \bullet, d) \cap \mathcal{Z}_H(c, \bullet, d') \neq \varnothing$; AND*

2) *There exists a pair of distinct colors $f, f' \in L(p_2) \setminus \{c\}$ such that $\mathcal{Z}_H(c, f, \bullet) \cap \mathcal{Z}_H(c, f', \bullet) \neq \varnothing$.*

The remainder of Section 1.4 consists of two useful propositions.

Proposition 1.4.7. *Let H be a broken wheel with principal path $p_1p_2p_3$ and let L be a list-assignment for H where $|L(v)| \geq 3$ for all $v \in V(H) \setminus \{p_1, p_2\}$. Then we have the following two facts.*

1) *Let $c \in L(p_1)$ and suppose there exist distinct colors $c_1, c_2 \in L(p_2)\setminus\{c\}$ such that $|\mathcal{Z}_H(c, c_1, \bullet)| = |\mathcal{Z}_H(c, c_2, \bullet)| = 1$ and $\mathcal{Z}_H(c, c_1, \bullet) \neq \mathcal{Z}_H(c, c_2, \bullet)$. Then $c_1, c_2 \in L(p_3)$, $\mathcal{Z}_H(c, c_1, \bullet) = c_2$, and $\mathcal{Z}_H(c, c_2, \bullet) = c_1$; AND*

2) *Suppose there exist two L-colorings ϕ, ϕ' of p_1p_2 such that $\phi(p_1) \neq \phi'(p_1)$, where $\mathcal{Z}_H(\phi(p_1), \phi(p_2), \bullet) = \mathcal{Z}_H(\phi'(p_1), \phi'(p_2), \bullet)$ and $|\mathcal{Z}_H(\phi(p_1), \phi(p_2), \bullet)| = |\mathcal{Z}_H(\phi'(p_1), \phi'(p_2), \bullet)| = 1$. Then $\phi(p_1) = \phi'(p_2)$ and $\phi(p_2) = \phi'(p_1)$.*

Proof. We first prove 1). Let H be a vertex-minimal counterexample to 1) and let L, c_1, c_2 be as in the statement of 1). It is trivial to check that a triangle satisfies 1), so H is not a triangle. Thus, let $H \setminus \{p_2\} = p_1u_1 \cdots u_tp_3$ for some $t \geq 1$. Let $H' := H \setminus \{p_3\}$. Then H' is a broken wheel with principal path $p_1p_2u_t$. We claim now that $|\mathcal{Z}_{H'}(c, c_1, \bullet)| = |\mathcal{Z}_{H'}(c, c_2, \bullet)| = 1$. Suppose not, and suppose without loss of generality that $|\mathcal{Z}_{H'}(c, c_1, \bullet)| > 1$. Let $d, d' \in \mathcal{Z}_{H'}(c, c_1, \bullet)$. Then $L(p_3) \setminus \{c_1, d\} \subseteq \mathcal{Z}_H(c, c_1, \bullet)$, and $L(p_3) \setminus \{c_1, d'\} \subseteq \mathcal{Z}_H(c, c_1, \bullet)$, so we have $|\mathcal{Z}_H(c, c_1, \bullet)| > 1$, contradicting our assumption.

Thus, we indeed have $|\mathcal{Z}_{H'}(c, c_1, \bullet)| = |\mathcal{Z}_{H'}(c, c_2, \bullet)| = 1$. We claim now that $\mathcal{Z}_{H'}(c, c_1, \bullet) \neq \mathcal{Z}_{H'}(c, c_2, \bullet)$. Suppose that $\mathcal{Z}_{H'}(c, c_1, \bullet) = \mathcal{Z}_{H'}(c, c_2, \bullet)$ and let d be the common color of both sets. Since $d \notin \{c_1, c_2\}$, we then have $c_1 \in \mathcal{Z}_H(c, c_2, \bullet)$ and $c_2 \in \mathcal{Z}_H(c, c_1, \bullet)$, as each of the colorings (c, c_1, c_2) and (c, c_2, c_1) of $p_1p_2p_3$ leaves d for u_t. But then, since $|\mathcal{Z}_H(c, c_1, \bullet)| = |\mathcal{Z}_H(c, c_2, \bullet)| = 1$. we have $\mathcal{Z}_H(c, c_1, \bullet) = \{c_2\}$, and $\mathcal{Z}_H(c, c_2, \bullet) = \{c_1\}$, contradicting our assumption that H is a counterexample.

We conclude that $|\mathcal{Z}_{H'}(c, c_1, \bullet)| = |\mathcal{Z}_{H'}(c, c_2, \bullet)| = 1$ and $\mathcal{Z}_{H'}(c, c_1, \bullet) \neq \mathcal{Z}_{H'}(c, c_2, \bullet)$. Since $|\mathcal{Z}_H(c, c_1, \bullet)| = |\mathcal{Z}_H(c, c_2, \bullet)| = 1$, it follows from 2) of Proposition 1.4.4 that $c_1, c_2 \in L(u_t)$, and, by the minimality of H, we have $\mathcal{Z}_{H'}(c, c_1, \bullet) = \{c_2\}$ and $\mathcal{Z}_{H'}(c, c_2, \bullet) = \{c_1\}$. But then, letting $d \in L(p_3) \setminus \{c_1, c_2\}$, we have $d \in \mathcal{Z}_H(c, c_1, \bullet) \cap \mathcal{Z}_H(c, c_2, \bullet)$, contradicting our assumption that $\mathcal{Z}_H(c, c_1, \bullet) \neq \mathcal{Z}_H(c, c_2, \bullet)$. This proves 1).

Now we prove 2). Suppose that 2) does not hold and let H be a vertex-minimal counterexample to 2). Let (r, r') and (s, s') be two L-colorings of p_1p_2, where these colorings satisfy the conditions of 2) but either $r \neq s'$ or $r' \neq s$. Since r, s are distinct, we we have $\{r, r'\} \neq \{s, s'\}$. Let c be the lone color of $L(p_3)$ such that $\mathcal{Z}_H(r, r', \bullet) = \mathcal{Z}_H(s, s', \bullet) = \{c\}$. As above, it is trivial to check that a triangle satisfies 2), so H is not a triangle. Now we have the following:

Claim 1.4.8. $r' \neq s'$.

Proof: Suppose there is a color d such that $r' = s' = d$. Thus, we have $r, s \in L(p_1) \setminus \{d\}$. Since $|L(p_3)| \geq 3$, let $d' \in L(p_3) \setminus \{c, d\}$. Then, by Observation 1.4.2, since $r \neq s$, the L-coloring (d, d') of p_2p_3 extends to an L-coloring of H using one of r, s on p_1, contradicting the fact that $d' \notin \mathcal{Z}_H(r, d, \bullet) \cup \mathcal{Z}_H(s, d, \bullet)$. ■

Since H is not a triangle, let $H - p_2 = p_1 u_1 \cdots u_t p_3$ for some $t \geq 1$. Since $|\mathcal{Z}_H(r, r', \bullet)| = |\mathcal{Z}_H(s, s', \bullet)| = 1$, it follows from Proposition 1.4.4 that, for all $j = 1, \cdots, t$, $|L(u_j)| = 3$ and $\{r', s'\} \subseteq L(u_j)$. Furthermore, $\{r, s\} \subseteq L(u_1)$, $|L(p_3)| = 3$, and $\{r', s'\} \subseteq L(p_3)$. By assumption, we have $|\{r, s\}| = 2$, and, by Claim 1.4.8, we have $|\{r', s'\}| = 2$. Since $|L(u_1)| = 3$, we have $\{r, s\} \cap \{r', s'\} \neq \varnothing$, so suppose without loss of generality that $r \in \{r', s'\}$. Since $r \neq r'$, we have $r = s'$, and our two L-colorings of $p_1 p_2$ are (r, r') and (s, r).

Claim 1.4.9. $t > 1$.

Proof: Suppose that $t = 1$. Since $r = s'$, we have $r \in L(p_3) \setminus \{c\}$, and the L-coloring (r, r', r) of $p_1 p_2 p_3$ leaves a color for u_1, so $r \in \mathcal{Z}_H(r, r', \bullet)$, contradicting the fact that $\mathcal{Z}_H(r, r', \bullet) = \{c\}$. ■

Let $H^* := H - \{p_3, u_t\}$. Since $t > 1$, H^* is a broken wheel with principal path $p_1 p_2 u_{t-1}$. By Theorem 0.2.3, each of $\mathcal{Z}_{H^*}(r, r', \bullet)$ and $\mathcal{Z}_{H^*}(r, r', \bullet)$ is nonempty. If $|\mathcal{Z}_{H^*}(r, r', \bullet) \cup \mathcal{Z}_{H^*}(s, s', \bullet)| = 1$, then it follows from the minimality of H that $\{r, r'\} = \{s, s'\}$, contradicting our assumption. Thus, there exist distinct colors $f, g \in L(u_{t-1})$ such that $f \in \mathcal{Z}_{H^*}(r, r', \bullet)$ and $g \in \mathcal{Z}_{H^*}(s, s', \bullet)$.

Let $H^\dagger$ be the broken wheel with principal path $u_{t-1} p_2 p_3$, where $H^\dagger - p_2 = u_{t-1} u_t p_3$. Since each of $\mathcal{Z}_{H^\dagger}(f, r', \bullet)$ and $\mathcal{Z}_{H^\dagger}(g, s', \bullet)$ is nonempty, it follows that $\mathcal{Z}_{H^\dagger}(f, r', \bullet) = \mathcal{Z}_{H^\dagger}(g, s', \bullet) = \{c\}$. Since $\{r', s'\} = L(p_3) \setminus \{c\}$, it follows that $\{s'\} = L(u_t) \setminus \{f, r'\}$ and $\{r'\} = L(u_t) \setminus \{g, s'\}$. But then, since $f \neq r'$ and $g \neq s'$, it follows that $\{f, g\}$ and $\{r', s'\}$ are two disjoint sets of size two, each of which lies in $L(u_t)$, contradicting the fact that $|L(u_t)| = 3$. This completes the proof of Proposition 1.4.7. □

The last fact we prove in Section 1.4 is the following result which we use in the special case where we have a broken wheel with a vertex outside the principal path with a list of size two.

Proposition 1.4.10. *Let H be a broken wheel with principal path $P := p_1 p_2 p_3$. Let $u \in V(H \setminus P)$ and L be a list-assignment for such that the following hold.*

1) $|L(u)| \geq 2$ *and, for each* $v \in V(H \setminus P) \setminus \{u\}$, $|L(v)| \geq 3$*; AND*
2) $|L(p_1)| \geq 1$ *and* $|L(p_3)| \geq 2$*; AND*
3) $|L(p_2)| \geq 5$.

Then H is L-colorable.

Proof. Let $H - p_2 = p_1 v_1 \cdots v_t p_3$ for some $t \geq 1$, where $u \in \{v_1, \cdots, v_t\}$. Let c be the lone color of $L(p_1)$. By removing some colors from some of the lists if necessary, we suppose that $|L(u)| = |L(p_3)| = 2$ and we also suppose that either $u = v_1$ or $|L(v_1) \setminus \{c\}| = 2$. Suppose toward a contradiction that H is not L-colorable. Let $d \in L(p_2) \setminus \{c\}$. Since the L-coloring (c, d) of $p_1 p_2$ does not extend to L-color H, it follows that d is in at least two of the lists $L(u)$, $L(p_3)$, $L(v_1) \setminus \{c\}$. This is true for each $d \in L(p_2) \setminus \{c\}$, so we have $u \neq v_1$ and $|L(v_1) \setminus \{c\}| = 2$. But then there are at most three colors that lie in at least two of the lists among $L(u), L(p_3), L(v_1) \setminus \{c\}$. Since $|L(p_1) \setminus \{c\}| \geq 4$, we have a contradiction. □

1.5 Extending Colorings of 2-Paths: The General Case

In many instances in the subsequent chapters, we partition a planar graph into two sides which intersect on a generalized chord of a specified cycle in this graph, color one side of this cycle and then show that this coloring extends to the other side. In particular, the two situations described in the proposition below occur very frequently. Note that the proposition below deals with paths of arbitrary length, not just 2-paths.

Proposition 1.5.1. *Let G be a planar graph with facial cycle C and let P be a subpath of C. Let L be a list-assignment for G such that, for each $v \in V(G \setminus C)$, $|L(v)| \geq 5$, and, for each $v \in V(C \setminus P)$, $|L(v)| \geq 3$. Then the following hold.*

1) *If there exists an L-coloring ϕ of $V(P)$ which does not extend to an L-coloring of G, then either there is a chord of C with an endpoint in P, or there is a vertex $v \in V(G \setminus C)$ with $|L_\phi(v)| \leq 2$. In particular, v has at least three neighbors in P; AND*
2) *If ϕ is an L-coloring of the endpoints of P which does not extend to L-color G, then there exists a vertex of $\mathring{P}$ with an L_ϕ-list of size less than three. In particular, if every vertex of $\mathring{P}$ has an L-list of size at least five, then any L-coloring of the endpoints of P extends to an L-coloring of G.*

Proof. For convenience, we suppose, by applying an appropriate stereographic projection, that C is the outer face of G. Since ϕ does not extend to an L-coloring of G, it follows from Theorem 0.2.3 that $|V(P)| \geq 3$, so $V(\mathring{P}) \neq \varnothing$. Let p, p' be the endpoints of P and let $C \setminus \mathring{P} = pu_1 \cdots u_t p'$ for some $t \geq 0$ (if $t = 0$ then $V(C) = V(P)$). Let F be the outer face of $G \setminus P$. By our assumption, every vertex of $F \setminus C$ has an L_ϕ-list of size at least three, and since there is no chord of C with an endpoint in P, each internal vertex of the path $C \setminus P$ has an L_ϕ-list of size at least three as well. If $t = 0$ then $G \setminus P$ is L_ϕ-colorable by Theorem 0.2.3. Likewise, if $t = 1$, then $|L_\phi(u_1)| \geq 1$, since $p_2u_1 \notin E(G)$, and thus, again applying Theorem 0.2.3, $G \setminus P$ is L_ϕ-colorable. Finally, if $t \geq 2$, then, since there is no chord of C with an endpoint in P, we have $|L_\phi(u_1)| \geq 2$ and $|L_\phi(u_t)| \geq 2$. Thus, by Theorem 1.3.4, $G \setminus P$ is L_ϕ-colorable. Thus, in any case, ϕ extends to an L-coloring of G, contradicting our assumption. This proves 1).

Now we prove 2). Suppose toward a contradiction that 2) does not hold and let G be a vertex-minimal counterexample to the proposition. Let $P := p_1 \cdots p_k$ for some $k \geq 1$. By assumption, there is an L-coloring ϕ of $\{p_1, p_k\}$ which does not extend to L-color G, and every vertex of $\mathring{P}$ has an L_ϕ-list of size at least three. By Theorem 0.2.3, we have $|E(P)| > 1$. Let $C \setminus \mathring{P} = p_1u_1 \cdots u_t p_k$ for some $t \geq 0$. If $t = 0$, then it again follows from Theorem 0.2.3 that ϕ extends to an L-coloring of G, contradicting our assumption, so $t > 0$.

Claim 1.5.2. *Any chord of C has an endpoint in $\mathring{P}$.*

Proof: Suppose toward a contradiction that there is a chord xy of C with $x, y \notin V(\mathring{P})$. Let $G = G_0 \cup G_1$ be the canonical xy-partition of G, where $P \subseteq G_0$. Note that, for each $i = 0, 1$, $|V(G_i)| < |V(G)|$. By the minimality of G, ϕ extends to an L-coloring ψ of G_0, and, by Theorem 0.2.3, G_1 is L_ψ^{xy}-colorable, so ϕ extends to an L-coloring of G, contradicting our assumption. ■

Let F be the outer face of $G \setminus \{p_1, p_k\}$. By Claim 1.5.2, there is no chord of C with one endpoint in $\{p_1, p_k\}$ and the other endpoint in $V(C) \setminus \{p_2, \cdots, p_{k-1}\}$, so each vertex of $\{u_2, \cdots, u_{t-1}\}$ has an L_ϕ-list of size at least three. Furthermore, every vertex of $\{p_2, \cdots, p_{k-1}\}$ also has an L_ϕ-list of size at least three. By assumption, all the vertices of $\mathring{P}$ have L_ϕ-lists of size at least three, so every vertex of $F \setminus \{u_1, u_t\}$ has an L_ϕ-list of size at least three.

If $t = 1$, then, since $|L_\phi(u_1)| \geq 1$, it follows from Theorem 1.5.2 that ϕ extends to an L-coloring of G, contradicting our assumption. If $t > 1$ then $|L_\phi(u_1)| \geq 2$ and $|L_\phi(u_t)| \geq 2$, and it follows from Theorem 1.3.4 that ϕ extends to an L-coloring of G, contradicting our assumption. □

A useful consequence of Proposition 1.5.1 is the following.

Theorem 1.5.3. *Let G be a short-separation-free planar graph with facial cycle C and let $P := p_1p_2p_3$ be a subpath of C of length two. Let L be a list-assignment for G such that, for each $v \in V(G \setminus C)$, $|L(v)| \geq 5$, and, for each $v \in V(C \setminus P)$, $|L(v)| \geq 3$. Suppose further that every chord of C has p_2 as an endpoint. Then either G is a broken wheel with principal path P, or there is at most one L-coloring of $V(P)$ which does not extend to an L-coloring of G.*

Proof. Let G be a vertex minimal counterexample to the theorem. For convenience, we suppose, by applying an appropriate stereographic projection, that C is the outer face of G. Let $P := p_1p_2p_3$ be a subpath of C and L be a list-assignment for G such that the specified conditions are satisfied. Let ϕ, ϕ' be two distinct L-colorings of P, neither of which extends to an L-coloring of G. Since neither ϕ nor ϕ' extends to an L-coloring of G, it follows from Corollary 0.2.4 that $|V(C)| \geq 5$. Let $C = p_3p_2p_1u_1 \cdots u_t$ for some $t \geq 2$.

Claim 1.5.4. *C is an induced subgraph of G.*

Proof: By assumption, any chord of C has p_2 as an endpoint, so suppose toward a contradiction that there is a chord p_2u_j of C for some $j \in \{1, \cdots, t\}$, and let $G = G^* \cup G^{**}$ be the natural p_2u_j-partition of G, where $p_1 \in V(G^*)$ and $p_3 \in V(G^{**})$. Let $P^* := p_1p_2u_j$ and $P^{**} := u_jp_2p_3$. Let $C^* := p_1u_1 \cdots p_jp_2$ and $C^{**} := p_3u_t \cdots u_jp_2$. Since every chord of C in G has p_2 as an endpoint, every chord of C^* in G^* has p_2 as an endpoint.

If G^* is a broken wheel with principal path P^*, and G^{**} is a broken wheel with principal path P^{**}, then G is a broken wheel with principal path P, contradicting our assumption, so suppose without loss of generality that G^* is not a broken wheel with principal path P^*. Since G is short-separation-free, G^* is not a triangle, and $u_j \neq u_1$.

By Theorem 0.2.3, there are colors $r, r' \in L(u_j)$ such that the colorings $(\phi(p_3), \phi(p_2), r)$ and $(\phi'(p_3), \phi'(p_2), r')$ of P^{**} extend to an L-coloring of G. If either of the colorings $(\phi(p_1), \phi(p_2), r)$, $(\phi'(p_1), \phi'(p_2), r')$ of P^* extends to an L-coloring of G^*, then one of ϕ, ϕ' extends to an L-coloring of G, contradicting our assumption. Thus, since G^* is not a broken wheel with principal path P^*, and every chord of C^* in G^* has p_2 as an endpoint, it follows from the minimality of G that the colorings $(\phi(p_1), \phi(p_2), r)$ and $(\phi'(p_1), \phi'(p_2), r')$ of P^* are not distinct, so ϕ, ϕ' use the same color on p_1 and the same color on p_2, and $r = r'$. Since $\phi \neq \phi'$, it follows that ϕ, ϕ' differ precisely on p_3. Let $a = \phi(p_1) = \phi'(p_1)$ and $b = \phi(p_2) = \phi'(p_2)$.

By Theorem 0.2.3, there is an extension of the coloring (a, b) of p_1p_2 to an L-coloring ψ of G^*. Since the colors of $\{b, \phi(p_3), \phi'(p_3)\}$ are all distinct, it follows from Observation 1.4.2 that there is an extension of the coloring $(\psi(u_j), b)$ of u_jp_2 to an L-coloring of G^{**} using one of $\phi(p_3), \phi'(p_3)$ on p_3. But then one of ϕ, ϕ' extends to an L-coloring of G, contradicting our assumption. ■

Since there is no chord of C in G it follows from 1) of Proposition 1.5.1 that $G \setminus C$ contains a vertex v^* adjacent to all three vertices of P, and, since neither ϕ nor ϕ' extends to an L-coloring of G, we have $|L_\phi(v^*)| = |L_{\phi'}(v^*)| = 2$. Let $L_\phi(v^*) = \{r, s\}$ and $L_{\phi'}(v^*) = \{r', s'\}$.

Since G is short-separation-free, $G - p_2$ has outer cycle $p_1v^*p_3u_t \cdots u_1$, and there is no chord of $p_1v^*p_3u_t \cdots u_1$ which is not incident to v^*, or else there is a chord of C in G which is not incident to p_2. Thus, if $G - p_2$ is not a broken wheel with principal path $p_1v^*p_3$, then it follows from the minimality of G that one of the two colorings $(\phi(p_1), r, \phi(p_3))$, $(\phi(p_1), s, \phi(p_3))$ extends to an L-coloring of $G - p_2$. If that holds, then ϕ extends to an L-coloring of G, contradicting our assumption. We conclude that v^* is adjacent to each of $u_1, \cdots, u_t$, and G is a wheel with central vertex v^*. Since neither ϕ nor ϕ' extends to an L-coloring of G, we have the following by Proposition 1.4.4.

i) $\phi(p_1), \phi'(p_1) \in L(u_1)$ and $\phi(p_3), \phi'(p_3) \in L(u_t)$; *AND*

ii) For each $i = 1, \cdots, t$, $|L(u_i)| = 3$, $\{r, s\} \subseteq L(u_i)$ and $\{r', s'\} \subseteq L(u_i)$

Now consider the following cases.

Case 1: ϕ, ϕ' use the same color on p_1 and the same color on p_3

In this case, let $a = \phi(p_1) = \phi'(p_1)$ and $b = \phi(p_3) = \phi'(p_3)$. Since $L(v^*) = \{\phi(p_1), \phi(p_2), \phi(p_3), r, s\} = \{\phi'(p_1), \phi'(p_2), \phi'(p_3), r', s'\}$, we have $\{\phi(p_2), r, s\} = \{\phi'(p_2), r', s'\}$. Since ϕ, ϕ' are distinct colorings of P, we have $\phi(p_2) \neq \phi'(p_2)$, so $|\{r, s, r', s'\}| \geq 3$. But then, by ii), $L(u_1)$ consists of three colors which are not a, b, contradicting i).

Case 2: ϕ, ϕ' differ on at least one of p_1, p_3

In this case, suppose without loss of generality that $\phi(p_1) \neq \phi'(p_1)$. Let $\phi(p_1) = a$ and $\phi'(p_1) = b$. By i), we have $\{a, b\} \subseteq L(u_1)$. By ii), $\{r, s\} \subseteq L(u_1)$ and $|L(u_1)| = 3$. Since $\{r, s\} = L_\phi(v^*)$, it follows that $b \in \{r, s\}$. Likewise, since $L_{\phi'}(v^*) = \{r', s'\}$ and $\{r', s'\} \subseteq L(u_1)$, we have $a \in \{r', s'\}$. Suppose without loss of generality that $a = r'$ and $b = r$. Since $|L(u_1)| = 3$, it follows that $s = s'$, and, by ii), we have $L(u_i) = \{a, b, s\}$ for each $i = 1, \cdots, t$. Since $\{r, s\} = L_\phi(v^*)$ and $\{r', s'\} = L_{\phi'}(v^*)$, and $\phi(p_3), \phi'(p_3) \in L(u_t)$, it follows that $\phi(p_3), \phi'(p_3) \in \{a, b\}$. Since $|L_\phi(v^*) = 2$, we have $\phi(p_1) \neq \phi(p_3)$, so we get $\phi(b_3) = b$, contradicting the fact that $b \in L_\phi(v^*)$. $\square$

We now prove two short, extremely useful theorems which are consequences of Theorem 1.5.3.

Theorem 1.5.5. *Let H be a planar graph with facial cycle C, and let $P := p_1p_2p_3$ be a subpath of C of length two. Let L be a list-assignment for H where $|L(v)| \geq 3$ for all $v \in V(C \setminus \{p_1, p_2\})$ and $|L(v)| \geq 5$ for all $v \in V(H \setminus C)$. Then, for any color $c \in L(p_1)$, there exists a color $d \in L(p_3)$ such that the following hold.*

1) *If $p_1p_3 \in E(H)$, then $c \neq d$; AND*
2) *For any L-coloring ϕ of $V(P)$ using c, d on p_1, p_3 respectively, ϕ extends to an L-coloring of G.*

Proof. Suppose this does not hold, and let H be a vertex-minimal counterexample to the theorem. Let C, P, L be as in the statement of the theorem, where $P := p_1p_2p_3$, and let $c \in L(p_1)$. If $|V(C)| \leq 4$, then, by Corollary 0.2.4, any L-coloring of $V(C)$ extends to an L-coloring of H, contradicting our assumption that H is a counterexample. Thus, $|V(C)| > 4$. For convenience, we suppose, by applying an appropriate stereographic projection, that C is the outer face of H.

Claim 1.5.6. $p_1p_3 \notin E(H)$.

Proof: Suppose that $p_1p_3 \in E(H)$. Since $|V(C)| > 4$, p_1p_3 is a chord of H. Let $H = H_0 \cup H_1$ be the natural p_1p_3-partition of H, where $p_2 \in V(H_0)$. Let $d \in L(p_3) \setminus \{c\}$. By Theorem 0.2.3, the L-coloring (c, d) of p_1p_3 extends to an L-coloring of H_1. By Corollary 0.2.4, any L-coloring of $p_1p_2p_3$ extends to an L-coloring of H_0. Thus, any L-coloring of $V(P)$ using c, d on the respective vertices p_1, p_3 extends to an L-coloring of H, contradicting our assumption. ■

Since $p_1p_3 \notin E(H)$ and H does not satisfy the claim, it follows that, for any $d \in L(p_3)$, there exists a proper L-coloring σ_d of $V(P)$, with $\sigma_d(p_1) = c$ and $\sigma_d(p_3) = d$, such that σ_d does not extend to an L-coloring of H.

Claim 1.5.7. *H is short-separation-free.*

Proof: Suppose toward a contradiction that there is a separating cycle D of length at most four in H. If necessary, we suppose that D is an induced subgraph of H. This is permissible since if this does not hold, then there is a separating triangle T in H whose vertices lie in $V(D)$, so we replace D with T, and T is an induced subgraph of H. Since D is a separating cycle, we have $|V(\text{Ext}_H(D))| < |V(H)|$, and, by the minimality of H, there is a $d \in L(p_3)$ such that σ_d extends to an L-coloring ψ of $\text{Ext}_H(D)$. Since D is an induced cycle of H, ψ is a proper L-coloring of the subgraph of H induced by $V(\text{Ext}_H(D))$. By Corollary 0.2.4, ψ extends to an L-coloring of $\text{Int}_H(D)$, so σ_d extends to an L-coloring of H, contradicting our assumption. ■

Now we have the following.

Claim 1.5.8. *Every chord of C has p_2 as an endpoint.*

Proof: Suppose toward a contradiction that there is a chord U of C without p_2 as an endpoint. Let $H = H' \cup H''$ be the natural U-partition of H, where $P \subseteq H'$. By the minimality of $|V(H)|$, there is a $d \in L(p_3)$ such that σ_d extends to an L-coloring ϕ of G', and, by Theorem 0.2.3, H'' is L^U_ϕ-colorable, so ϕ extends to an L-coloring of G. But then σ_d extends to an L-coloring of H, which is false. ■

We claim now that H is a broken wheel with principal path P. If this does not hold, then, since every chord of C is incident to p_2 and there at least three colorings of P in $\{\sigma_d : d \in L(p_3)\}$ and , it follows from Theorem 1.5.3 that at least one of these colorings extends to an L-coloring of H, contradicting our assumption.

Claim 1.5.9. *For any distinct colors $d, d' \in L(p_3)$, we have $\sigma_d(p_2) \neq \sigma_{d'}(p_2)$.*

Proof: Suppose toward a contradiction that there is a color f such that $\sigma_d(p_2) = \sigma_{d'}(p_2) = f$. Let L^* be a list-assignment for $V(H)$ where $L^*(p_1) = \{c\}$, $L^*(p_2) = \{f\}$, and $L^*(p_3) = \{d, d', f\}$, where $L^* = L$ otherwise. Since $\sigma_d(p_2) = \sigma_{d'}(p_2) = f$, we have $|\{d, d', f\}| = 3$. Thus, by Theorem 0.2.3, there is an L'-coloring ψ of $V(H)$. Since $\psi(p_3) \neq f$, either σ_d or $\sigma_{d'}$ extends to an L-coloring of H, which is false. ■

Since H is not a triangle, let $H \setminus \{p_2\} = p_1u_1 \cdots u_tp_3$ for some $t \geq 1$. Let $H' := H \setminus \{p_3\}$. Then H' is a broken wheel with principal path $P' := p_1p_2u_t$. Since $|L(u_t)| \geq 3$, there exists a color $d^* \in L(u_t)$ such that $V(P)$ admits a proper L-coloring using c, d^* on p_1, u_t respectively, and, for any proper L-coloring ϕ of $V(P')$ using c, d^* on p_1, u_t respectively, ϕ extends to an L-coloring of $V(H')$. Let $d_1, d_2 \in L(p_3) \setminus \{d^*\}$. By Claim 1.5.9, there exists a $j \in \{1, 2\}$ such that $\sigma_{d_j}(p_2) \neq d^*$. Since the coloring $(c, \sigma_j(p_2), d^*)$ of $V(P')$ extends to an L-coloring of H', the coloring $(c, \sigma_j(p_2), d_j)$ of $V(P)$ extends to an L-coloring of H, which is false. □

We have the following analogue of Theorem 1.5.5 in the case where the endpoints of the specified path have 2-lists.

Theorem 1.5.10. *Let H be a planar graph with facial cycle C, and let $P := p_1p_2p_3$ be a subpath of C of length two. Let L be a list-assignment for H where $|L(v)| \geq 3$ for all $v \in V(C \setminus P)$ and $|L(v)| \geq 5$ for all $v \in V(H \setminus C)$. Suppose further that $|L(p_1)| \geq 2$ and $|L(p_3)| \geq 2$. Then there exists a $c \in L(p_1)$ and a $d \in L(p_3)$ such that the following holds.*

1) *If $p_1p_3 \in E(H)$, then $c \neq d$; AND*
2) *For any L-coloring ϕ of $V(P)$ using c, d on p_1, p_3 respectively, ϕ extends to an L-coloring of G.*

Proof. Suppose that this does not hold, and let H be a vertex-minimal counterexample to the theorem. Let C, P, L be as in the statement of the theorem, where $P := p_1p_2p_3$. As in the proof above, applying Theorem 0.2.3 and Corollary 0.2.4, we immediately have the following from the minimality of H.

Claim 1.5.11. *H is short-separation-free and every chord of C has p_2 as an endpoint.*

The theorem holds trivially if H is a triangle, so H is not a triangle. For convenience, we suppose, by applying an appropriate stereographic projection, that C is the outer face of H. Let $c_1, c_2 \in L(p_1)$ and let $d_1, d_2 \in L(p_3)$. Since H does not satisfy the claim and $p_1p_3 \notin E(H)$, it follows that, for each $i \in \{1,2\}$ and $j \in \{1,2\}$, there is an L-coloring σ_{ij} of $V(P)$ using c_i, d_j on p_1, p_3 respectively, where σ_{ij} does not extend to an L-coloring of H.

We claim now that H is a broken wheel with principal path P. If this does not hold, then, since there are four colorings of P in $\{\sigma_{ij} : 1 \leq i,j \leq 2\}$ and every chord of C is incident to p_2, it follows from Theorem 1.5.3 that at least one of them extends to an L-coloring of H, contradicting our assumption. Thus, H is a broken wheel with principal path P.

Since H is not a triangle, let $H - p_2 = p_1u_1 \cdots u_tp_3$ for some $t \geq 1$. Let $H^* := H - p_3$. Then H^* is a broken wheel with principal path $P^* := p_1p_2u_t$. By Proposition 1.4.4, since none of the four colorings $\{\sigma_{ij} : 1 \leq i,j \leq 2\}$ extends to L-color H, we have $d_1, d_2 \in L(u_t)$ and $|L_(u_t)| = 3$, so let $L(u_t) = \{d_1, d_2, r\}$ for some color r.

Claim 1.5.12. *For each $j = 1,2$, $\sigma_{j1}(p_2) \neq \sigma_{j2}(p_2)$.*

Proof: Suppose there is a $j = 1,2$ such that $\sigma_{j1}(p_2) = \sigma_{j2}(p_2) = f$ for some color f. By Observation 1.4.2, since $|\{d_1, d_2, f\}| = 3$, there is an L-coloring of H in which the edge p_1p_2 is colored with (c_j, f) and p_3 is colored with one of d_1, d_2. But then one of σ_{j1}, σ_{j2} extends to an L-coloring of H, which is false. ■

By the minimality of H, there exist colors $c, c' \in \{c_1, c_2\}$ (possibly $c = c'$) and distinct colors $d, d' \in L(u_t)$ such that $V(P^*)$ admits an L-coloring using c, d on p_1, u_t respectively, and any such L-coloring extends to an L-coloring of H^*, and likewise, $V(P^*)$ admits an L-coloring using c', d' on p_1, u_t respectively, and any such L-coloring extends to an L-coloring of H^*.

Without loss of generality, let $c = c_1$. Since at least one of d, d' lies in $\{d_1, d_2\}$, suppose without loss of generality that $d = d_1$. Since the coloring σ_{12} of P does not extend to an L-coloring of H, we have $\sigma_{12}(p_2) = d_2$. By Claim 1.5.12, there is a $j \in \{1,2\}$ such that $\sigma_{1j}(p_2) \neq r$. Thus, if $d' = r$, then the coloring $(c_1, \sigma_{1j}(p_2), r)$ of P' extends to an L-coloring of H' and leaves the color d_j for p_3. But then σ_{1j} extends to an L-coloring of H, which is false. Thus, we have $d' \neq r$, so $d' = d_2$. Since σ_{12} does not extend to an L-coloring of H, we have $\sigma_{12}(p_2) = d_1$, or else the coloring $(c_1, \sigma_{12}(p_2), d_1)$ of P' extends to L-color H' and leaves d_2 for p_3. Since σ_{ij} does not extend to an L-coloring of H for an $1 \leq i,j \leq 2$, we have $|L(u_1)| = 3$ and $c_1, c_2 \in L(u_1)$.

Claim 1.5.13. *$\{d_1, d_2\} \subseteq L(u_i)$ for each $i = 1, \cdots, t$.*

Proof: Suppose first that $c = c'$. In this case, since $d' = d_2$, we have $\sigma_{11} = d_2$, or else the coloring $(c_1, \sigma_{11}(p_2), d_2)$ of $V(P')$ extends to an L-coloring of H' and leaves the color d_1 for p_3. Since neither σ_{11} nor σ_{12} extends to an L-coloring of H, we have $\{d_1, d_2\} \subseteq L(u_i)$ for each $i = 1, \cdots, t$ by Proposition 1.4.4.

Now suppose that $c \neq c'$. In this case, we have $c' = c_2$. Since $d' = d_2$, we have $\sigma_{21}(p_2) = d_2$, otherwise $(c_2, \sigma_{21}(p_2), d_2)$ is a proper L-coloring of $V(P')$ which extends to an L-coloring of H' and leaves the color d_2 for p_3, so σ_{22} extends to an L-coloring of H, which is false. Thus, we indeed have $\sigma_{21}(p_2) = d_2$. Recall that $\sigma_{12}(p_2) = d_1$.

Thus, as above, we have $\{d_1, d_2\} \subseteq L(u_i)$ for each $i = 1, \cdots, t$ by Proposition 1.4.4, since neither σ_{21} nor σ_{12} extends to an L-coloring of H. $\blacksquare$

Applying Claim 1.5.13, since $|L(u_1)| = 3$ and each of $\{c_1, c_2\}$ and $\{d_1, d_2\}$ is contained in $L(u_1)$, there exist $1 \leq \ell \leq 2$ and $1 \leq k \leq 2$ such that $c_\ell = d_k$. Since $\{d_1, d_2\} \subseteq L(u_i)$ for each $i = 1, \cdots, t$, it follows that t is even, or else, if t is odd, then we color each of $u_2, u_4, \cdots, u_{t-1}, p_3$ with d_k and thus extend $\sigma_{\ell k}$ to an L-coloring of H. This is permissible since there is a color left over for each of $u_1, u_3, \cdots, u_t$, including u_1, u_t, as each endpoint of P is colored with d_k.

Since $\sigma_{12}(p_2) = d_1$, we have $d_1 \neq c_1$. Since t is even, we extend σ_{11} to an L-coloring of H by coloring each of $u_1, u_3, \cdots, u_{t-1}$ with d_1. This is permissible as $c_1 \neq d_1$. For each of $u_2, u_4, \cdots, u_t$, there is a color left over, including for u_t, since each of u_{t-1}, p_3 are colored with d_1. This contradicts the fact that σ_{11} does not extend to an L-coloring of G. $\square$

The final result of Section 1.5 is the following simple fact.

Proposition 1.5.14. *Let G be a planar graph, let C be a facial cycle of G, and let $P := p_1p_2p_3$ be a subpath of C of length two. Let L be a list-assignment for $V(G)$ where $|L(v)| \geq 3$ for all $v \in V(C) \setminus \{p_1, p_2\}$ and $|L(v)| \geq 5$ for all $v \in V(G \setminus C)$. Let $c \in L(p_1)$ and suppose that $|L(p_2) \setminus \{c\}| \geq 3$. Then there exists a $d \in L(p_2) \setminus \{c\}$ such that $|\mathcal{Z}^P_{G,L}(c, d, \bullet)| \geq 2$.*

Proof. Suppose that this does not hold and let G be a vertex minimal counterexample to the claim. For convenience, we suppose that C is the outer face of G. Applying Theorem 0.2.3 and Corollary 0.2.4, it immediately follows from the minimality of G that G is short-separation-free and any chord of C has p_2 as an endpoint.

Claim 1.5.15. $|V(C)| > 3$.

Proof: Suppose not. Thus, $C = p_1p_2p_3$. Since $|L(p_2) \setminus \{c\}| \geq 3$, there is a $d \in L(p_2) \setminus \{c\}$ such that $|L(p_3) \setminus \{c, d\}| \geq 2$. Thus, by Corollary 0.2.4, we have $|\mathcal{Z}^P_{G,L}(c, d, \bullet)| \geq 2$, contradicting our assumption. $\blacksquare$

Since $|V(C)| > 3$ and any chord of C has p_2 as an endpoint, we have $p_1p_3 \notin E(G)$. If G is a broken wheel, then, applying 1) of Proposition 1.4.5, we contradict the fact that G is a counterexample. Thus, G is not a broken wheel. Since $p_1p_3 \notin E(G)$ and $|L(p_2) \setminus \{c\}| \geq 3$, it follows from Theorem 1.5.3 that there exist two colors $c_1, c_2 \in L(p_2) \setminus \{c\}$ such that, for each $i = 1, 2$, we have $\mathcal{Z}^P_{G,L}(c, c_i, \bullet) = L(p_3) \setminus \{c_i\}$, contradicting our assumption that G is a counterexample. $\square$

1.6 Extending Colorings of 3-Paths

This section consists of a result for 3-paths which is an analogue of Theorem 1.5.5. When we delete a path between two faces in a planar graph, we use the results of Sections 1.4 and 1.5 to delete one side of a 2-chord of one of the faces without having to choose a color for the middle vertex of the 2-chord. Under restricted circumstances, we are able to do something similar for 3-chords.

Theorem 1.6.1. *Let H be a planar graph with facial cycle C, and let $P := p_1p_2p_3p_4$ be a subpath of C of length three. Let L be a list-assignment for H such that, for each $v \in V(H \setminus C)$, $|L(v)| \geq 5$, and, for each $v \in V(C) \setminus \{p_1, p_2, p_3\}$,*

$|L(v)| \geq 3$. *Suppose further that* $N(p_3) \cap V(C) = \{p_2, p_4\}$. *Then, for each color* $c \in L(p_1)$*, there exists a color* $d \in L(p_4)$ *such that the following hold.*

1) *If* $p_1p_4 \in E(H)$ *then* $c \neq d$*; AND*
2) *For any* L*-coloring* ϕ *of* $V(P)$ *using* c, d *on* p_1, p_4 *respectively,* ϕ *extends to an* L*-coloring of* H.

Proof. Note that the statement of Theorem 1.6.1 does not specify anything about the number of colors available for p_2, p_3, since it does not matter how many colors are available for these vertices, and the result is vacuously true if p_2p_3 is not L-colorable. Let H be a vertex-minimal counterexample to the theorem. For convenience, we suppose, by applying an appropriate stereographic projection, that C is the outer face of H. By adding edges to H if necessary, we suppose further that every facial subgraph of H, except possibly C, is a triangle. This is permissible as it is always possible to triangulate the interior of C without adding any chord of C with p_3 as endpoint.

Claim 1.6.2. *Any chord of* C *has* p_2 *as an endpoint. Furthermore* $p_1p_4 \notin E(H)$.

Proof: Suppose toward a contradiction that there is a chord U of C which does not have p_2 as an endpoint. By assumption, p_3 is also not an endpoint of U. Let $H := H_0 \cup H_1$ be the natural U-partition of H, where $P \subseteq H_0$. Note that, for each $i = 0, 1$, $|V(H_i)| < |V(H)|$. If $p_1p_4 \in E(H)$, then $p_1p_4 \in E(H_0)$, since, if U has an endpoint in $C \setminus P$, then at least one of p_1, p_4 lies outside of H_1. Furthermore, p_3 has no neighbors in the outer face of H_0, except for p_2, p_4. Thus, by the minimality of H, there is a $d \in L(p_4)$, where, $c \neq d$ if $p_1p_4 \in E(H_0)$, such that, for any L-coloring ϕ of $V(P)$, if ϕ uses c, d on p_1, p_4 respectively, then ϕ extends to an L-coloring of H.

Since H is a counterexample, there is an L-coloring ψ of $V(P)$ using c, d on p_1, p_4 respectively, such that ψ does not extend to an L-coloring of H. Yet ψ extends to an L-coloring ψ^* of H_0, and, by Theorem 0.2.3, H_1 is $L^U_{\psi^*}$-colorable. Thus, ψ extends to an L-coloring of H, a contradiction. We conclude that no such U exists. Now suppose toward a contradiction that $p_1p_4 \in E(H)$. Since p_1p_4 is not a chord of C, we have $C := p_1p_2p_3p_4$. By Corollary 0.2.4, any L-coloring of $V(P)$ extends to an L-coloring of H, contradicting our assumption that H is a counterexample. ■

Since H is a minimal counterexample and $p_1p_4 \notin E(H)$, it follows that, for each $d \in L(p_4)$, there is an L-coloring ϕ_d of $V(P)$ such that $\phi_d(p_1) = c$, $\phi_d(p_4) = d$, and ϕ_d does not extend to an L-coloring of H. Applying Corollary 0.2.4, it immediately follows from the minimality of H that H is short-separation-free.

Claim 1.6.3. $p_2p_4 \notin E(H)$.

Proof: Suppose toward a contradiction that $p_2p_4 \in E(H)$. Let $H = H_0 \cup H_1$ be the natural p_2p_4-partition of H, where $p_1 \in V(H_0)$. Since H is short-separation-free, H_1 is a triangle, and the outer face of H_1 contains the 2-path $P^* := p_1p_2p_4$. By Theorem 1.5.5, since $|L(p_4)| \geq 3$, there is a color $d \in L(p_4)$ such that any L-coloring of P^* using c, d on p_1, p_4 respectively extends to an L-coloring of H_0. Since $V(H) = V(H_0) \cup \{p_3\}$, ϕ_d extends to an L-coloring of H, which is false. ■

Combining Claim 1.6.3 and Claim 1.6.2, it follows that P is an induced subpath of H. Since $p_1p_4 \notin E(H)$, let $C := p_4p_3p_2p_1u_1 \cdots u_t$ for some $t \geq 1$.

Claim 1.6.4. *For any vertex* $w \in V(H \setminus C)$*, if* $|N(w) \cap V(P)| \geq 3$*, then* w *is adjacent to at most one of* p_1, p_4.

Proof: Suppose toward a contradiction that w is adjacent to each of p_1, p_4 and let $H = H_0 \cup H_1$ be the natural p_1wp_4-partition of H, where $P \subseteq H_0$. Since w is adjacent to at least one of p_2, p_3, and H is short-separation-free, we have $V(H) = V(H_1) \cup \{p_2, p_3\}$. By Theorem 1.5.5, since $|L(p_4)| \geq 3$, there is a color $d \in L(p_4)$ such that, for any L-coloring ϕ of p_1wp_4, if ϕ uses c, d on p_1, p_4 respectively, then ϕ extends to an L-coloring of H_1. Since $|L_{\phi_d}(w)| \geq 1$, it follows from our choice of d that ϕ_d extends to an L-coloring of H, which is false. ■

Claim 1.6.5. *There is no common neighbor of* p_2, p_4 *in* $H \setminus C$*.*

Proof: Suppose toward a contradiction that there is a $w \in V(G \setminus C)$ adjacent to each of p_2, p_4. Let $P^* := p_1p_2wp_4$ and let $H = H_0 \cup H_1$ be the natural p_2wp_4-partition of H, where $p_3 \in V(H_0)$. Since H is short-separation-free, we have $V(H) = V(H_1) \cup \{p_3\}$, and H_1 is bounded by outer face $C^* := u_1 \cdots u_tp_4wp_2p_1$. We claim now that there is a chord of C^* with w as an endpoint. Suppose that this does not hold. By the minimality of H, there is a $d \in L(p_4)$ such that any L-coloring of $V(P^*)$ using c, d on p_1, p_4 respectively extends to an L-coloring of H_1. Since $|L_{\phi_d}(w)| \geq 1$, it follows that ϕ_d extends to an L-coloring of H, which is false.

Thus, there is a chord of C^* in H_1 with w as an endpoint. By Claim 1.6.4, $p_1 \notin N(w)$, so w has a neighbor in $\{u_1, \cdots, \cdots, u_t\}$. Let $j \in \{1, \cdots, t\}$ be the minimal index among $\{1 \leq j \leq t : u_j \in N(w)\}$.

Let $H = H' \cup H''$ be the natural p_2wu_j-partition of H, where $p_1 \in V(H')$ and $p_3, p_4 \in V(H'')$. Then H_0 is bounded by outer face $p_1p_2wu_j \cdots u_1$, and, by our choice of j, there is no chord of the outer face of H' with w as an endpoint. Thus, since $|V(H')| < |V(H)|$, and $|L(u_j)| \geq 3$, there is a color $d \in L(u_j)$, where $c \neq d$ if $u_jp_1 \in E(H')$, such that any L-coloring of $p_1p_2wu_j$ using c, d on p_1, u_j respectively extends to an L-coloring of H'.

Since H is short-separation-free, $H'' - p_3$ is bounded by outer face $p_4wu_j \cdots u_t$. By Theorem 1.5.5, since $|L(p_4)| \geq 3$, there is a color $f \in L(p_4)$, where $f \neq d$ if $u_jp_4 \in E(H'' - p_3)$, such that any L-coloring of u_jwp_4 using d, f on u_j, p_4 respectively extends to an L-coloring of $H'' - p_3$. Now, since $p_1 \notin N(w)$, we have $|L_{\phi_f}(w)| \geq 2$, so there is a color $d' \in L_{\phi_f}(w)$ with $d' \neq d$. Coloring w with d' and u_j with d, we then extend ϕ_f to an L-coloring of H by our choice of d, f. This contradicts our assumption. ■

We now rule some more possible 2-chords of C.

Claim 1.6.6. *There are no common neighbors of* p_1, p_3 *in* $H \setminus C$*.*

Proof: Suppose toward a contradiction that there is a $w \in V(H \setminus C)$ adjacent to each of p_1, p_3. Let $H = H_0 \cup H_1$ be the natural p_1wp_3-partition of H, where $p_2 \in V(H_0)$. Since H is short-separation-free, we have $V(H_0) = \{p_1, p_2, p_3, w\}$, and H_1 is bounded by outer cycle $p_4p_3wp_1u_1 \cdots u_t$. Let $P^* := p_1wp_3p_4$. Since there is no chord of the outer face of H_1 with p_3 as an endpoint, it follows from the minimality of H that there is a $d \in L(p_4)$ such that any L-coloring of $V(P^*)$ using c, d on p_1, p_4 respectively extends to an L-coloring of H_1. Since $|L_{\phi_d}(w)| \geq 2$ and $V(H) = V(H_1) \cup \{p_2\}$, ϕ_d extends to an L-coloring of H, which is false. ■

We claim now that there is a chord of C with an endpoint in P. Suppose not, and let $d \in L(p_4)$. By 1) of Proposition 1.5.1, since there is no chord of C with an endpoint in P, and ϕ_d does not extend to an L-coloring of H, there is a vertex $w \in V(H \setminus C)$ with at least three neighbors in P. By Claim 1.6.4, w is adjacent to at most one of p_1, p_4, so $N(w) \cap V(P)$ is either $\{p_1, p_2, p_3\}$ or $\{p_2, p_3, p_4\}$. In the former case, we contradict Claim 1.6.6, and, in the latter case, we contradict Claim 1.6.5.

Thus, there is a chord U of C with an endpoint in P. By Claim 1.6.2, p_2 is an endpoint of U, and, by Claim 1.6.3, there is a $u_m \in \{u_1, \cdots, u_t\}$ such that $U = p_2u_m$. We choose m to be the maximal index among $\{1 \leq j \leq t :$

$u_j \in N(p_2)\}$. Let $H = H_0 \cup H_1$ be the natural U-partition of H, where $p_1 \in V(H_0)$ and $p_3, p_4 \in V(H_1)$. Let $P_1 := u_m p_2 p_3 p_4$. Then H_1 is bounded by outer cycle $C_1 := p_4 p_3 p_2 u_m \cdots u_t$. By the maximality of m, there is no chord of C_1 in H_1 with p_2 as an endpoint. Since $V(C_1) \subseteq V(C)$, it follows from Claim 1.6.2, that C_1 is an induced subgraph of H_1.

Claim 1.6.7. $u_m \neq u_t$.

Proof: Suppose toward a contradiction that $u_m = u_t$. Then H_1 is bounded by outer cycle $u_t p_2 p_3 p_4$, and, since no 4-cycle of H separates p_1 from a vertex of $H_1 \setminus C_1$, we have $V(H_1) = V(C_1) = \{u_t, p_2, p_3, p_4\}$. For each $d \in L(p_4)$, since ϕ_d does not extend to an L-coloring of H, we have $\mathcal{Z}_{H_0}(c, \phi_d(p_2), \bullet) = \{d\}$. In particular, we have $L(p_4) \subseteq L(p_2)$, and, for any distinct $d, d' \in L(p_4)$, the colors $\phi_d(p_2), \phi_{d'}(p_2)$ are distinct. Since $|L(p_4)| \geq 3$, $\{\phi_d(p_2) : d \in L(p_4)\}$ is a set of size at least three in $L(p_2) \setminus \{c\}$. Thus, by 1) of Prop 1.5.5. that there is a $d \in L(p_r)$ such that $\mathcal{Z}_{H_0}(c, \phi_d(p_2), \bullet)| \geq 2$, contradicting the fact that $\mathcal{Z}_{H_0}(c, \phi_d(p_2), \bullet) = \{d\}$. ■

Since $u_m \neq u_t$ and C^1 is an induced cycle of H_1, it follows that, for each $d \in L(p_4)$ and $f \in \mathcal{Z}_{H_0}(c, \phi_d(p_2), \bullet)$, the coloring $(f, \phi_d(p_2), \phi_d(p_3), d)$ is a proper L-coloring of $u_m p_2 p_3 p_4$, and, since ϕ_d does not extend to an L-coloring of H and $f \in \mathcal{Z}_{H_0}(c, \phi_d(p_2), \bullet)$, this coloring of $u_m p_2 p_3 p_4$ does not extend to an L-coloring of H.

Claim 1.6.8. *For each $d \in L(p_4)$, we have $|\mathcal{Z}_{H_0}(c, \phi_d(p_2), \bullet)| = 1$.*

Proof: Suppose there is a $d \in L(p_4)$ with $|\mathcal{Z}_{H_0}(c, \phi_d(p_2), \bullet)| \geq 2$. Since C_1 is an induced subgraph of H_1, each vertex of $u_{m+1}, \cdots, u_{t-1}$ has an L_{ϕ_d}-list of size at least three. By Claim 1.6.5, there is no vertex of $H_1 \setminus C$ adjacent to each of p_2, p_4, so each vertex of the outer face of $H_1 \setminus \{p_2, p_3, p_4\}$, except u_m, u_t, has an L_{ϕ_d}-list of size at least three, and $|L_{\phi_d}(u_t)| \geq 2$. By Claim 1.6.7, $u_m \neq u_t$, and, by Theorem 1.3.4, there is an L_{ϕ_d}-coloring ψ of $H_1 \setminus \{p_2, p_3, p_4\}$ using one of d_1, d_2 on u_m. Thus, ϕ_d extends to an L-coloring of $V(P) \cup V(H_1)$ using one of d_1, d_2 on u_m. Since $d_1, d_2 \in \mathcal{Z}_{H_0}(c, \phi_d(p_2), \bullet)$, ϕ_d extends to an L-coloring of H, contradicting our assumption. ■

We now note that there is a $w^\star \in V(H_1 \setminus C_1)$ with $N(w^\star) \cap V(P_1) = \{u_m, p_2, p_3\}$. To see this, let $d^* \in L(p_4)$ and $f \in \mathcal{Z}_{H_0}(c, \phi_{d^*}(p_2), \bullet)$. Since the coloring $(f, \phi_{d^*}(p_2), \phi_d(p_3), d^*)$ of P_1 does not extend to an L-coloring of H_1, and C^1 is an induced cycle of H_1, it follows from 1) of Proposition 1.5.1 that there is a vertex $w^\star \in V(H_1 \setminus C_1)$ with at least three neighbors in P_1. By Claim 1.6.5, $w^\star$ is adjacent to at most one of p_2, p_4, and since H is short-separation-free and C^1 is an induced subpath of H, it follows from our triangulation conditions that $H[N(w^\star) \cap V(P_1)]$ is a subpath of P_1 of length precisely two. Again applying Claim 1.6.5, this path is $u_m p_2 p_3$, and $w^\star$ is the unique vertex of $H_1 \setminus C_1$ with at least three neighbors in P_1.

We claim now that there are distinct $d, d' \in L(p_4)$ such that $\phi_d(p_2) = \phi_{d'}(p_2)$. Suppose that this does not hold. Then $\{\phi_d(p_2) : d \in L(p_2)\}$ is a set of at least three distinct colors of $L(p_2) \setminus \{c\}$. By 1) of Prop 1.5.5, there is a $d \in \{d_1, d_2, d_3\}$ such that $\mathcal{Z}_{H_0}(c, \phi_d(p_2), \bullet)| \geq 2$, contradicting Claim 1.6.8. Thus, let $d, d' \in L(p_4)$ with $\phi_d(p_2) = \phi_{d'}(p_2) = c^*$ for some color c^*. Applying Claim 1.6.8, let $\mathcal{Z}_{H_0}(c, c^*, \bullet) = \{f^*\}$ for some color f^*. Since $L(p_4)| \geq 3$, let $d'' \in L(p_4) \setminus \{d, d'\}$. Applying Claim 1.6.8, let $\mathcal{Z}_{H_0}(c, \phi_{d''}(p_3), \bullet) = \{f^{**}\}$ for some $f^{**} \in L(u_m)$.

Claim 1.6.9. *$f^{**} \neq f^*$ and $\phi_{d''}(p_2) \neq c^*$.*

Proof: By the minimality of H, there is a color among $d^* \in \{d, d'', d''\}$ such that any L-coloring of P_1 using f^*, d^* on u_m, p_4 respectively extends to an L-coloring of H_1. If $d^* \in \{d, d''\}$, then one of the colorings $(f^*, c^*, \phi_d(p_3), d)$, $(f^*, c^*, \phi_{d'}(p_3), d')$ of $u_m p_2 p_3 p_4$ extends to an L-coloring of H_1, and thus, by our choice of f^*, one of $\phi_d, \phi_{d'}$

extends to an L-coloring of H, which is false. Thus, we have $d^* = d''$. If $f^* = f^{**}$, then the L-coloring $(f^*, \phi_{d''}(p_2), \phi_{d''}(p_3), d'')$ of $u_m p_2 p_3 p_4$ extends to an L-coloring of H_1, and thus, by ouur choice of f^{**}, $\phi_{d''}$ extends to an L-coloring of H, which is false. Thus, we have $f^* \neq f^{**}$. Since $\mathcal{Z}_{H_0}(c, \phi_{d''}(p_3), \bullet) = \{f^{**}\}$ and $\mathcal{Z}_{H_0}(c, c^*, \bullet) = \{f^*\}$, we have $\phi_{d''}(p_3) \neq c^*$. ■

We now have the following.

Claim 1.6.10. $\phi_d(p_3) \neq \phi_{d'}(p_3)$.

Proof: Suppose toward a contradiction that there is a color h such that $\phi_d(p_3) = \phi_{d'}(p_3) = h$. Let L' be a list-assignment for $H_1 - p_2$ where $L'(p_4) = \{d, d', h\}$ and otherwise $L' = L$. Since h is distinct from either of d, d', we have $|L'(p_4)| = 3$. Since $\phi_d(p_2) = \phi_{d'}(p_2) = c^*$, we have $c^* \neq h$ as well. Since H is short-separation-free, $H_1 - p_2$ is bounded by outer cycle $p_4 p_3 w^\star u_m \cdots u_t$. Since $u_t \in V(H_1 - p_2)$ and $p_4 \notin N(w^\star)$, $H_1 - p_2$ is not a broken wheel with principal path $u_m w^\star p_3$. Thus, there is at most one L'-coloring of $u_m w^\star p_3$ which does not extend to an L'-coloring of $H_1 - p_2$. Since $L'(w^\star) \setminus \{f^*, c^*, h\}| \geq 2$, there is an L'-coloring ψ of $H_1 - p_2$ which uses f^* on u_m and h on p_3, where $\psi(w^\star) \neq c^*$. Since $\psi(p_3) = h$, we have $\psi(p_4) \in \{d, d'\}$, so ψ is an L-coloring of $H_1 - p_2$, and ψ extends to an L-coloring of H_1 using c^* on p_2. Thus, by our choice of f^*, one of the two colorings $\phi_d, \phi_{d'}$ of P extend to an L-coloring of H, which is false. ■

We now have the following key claim.

Claim 1.6.11. $N(w^\star) \cap \{u_m, \cdots, u_t\} = \{u_m\}$.

Proof: Suppose that this does not hold, and let $n \in \{m+1, \cdots, t\}$ with $u_n \in N(w^\star)$. Let n be the minimal index among $\{m+1 \leq j \leq t : u_j \in N(w^\star)\}$. Let $K^\dagger$ be the subgraph of H bounded by outer cycle $u_m w^\star u_n \cdots u_{m+1}$. Then the outer face of $K^\dagger$ contains the 2-path $u_m w^\star u_n$.

Now let H' be the subgraph of H bounded by outer cycle $u_n \cdots u_t p_4 p_3 p_2 w^\star$. Then the outer cycle of H' contains the 2-path $P' := w^\star p_3 p_4$. Furthermore, by Claim 1.6.2, every chord of the outer face of H' has $w^\star$ as one of its endpoints. Since $N(p_3) \cap V(C) = \{p_2, p_4\}$ and u_t lies on the outer face of H', H' is not a broken wheel with principal path $w^\star p_3 p_4$. Since there are no chords of the outer face of H' which do not have $w^\star$ as an endpoint, it follows from Theorem 1.5.3 that there is at most one proper L-coloring of $w^\star p_3 p_4$ which does not extend to an L-coloring of H'.

Subclaim 1.6.12. *$K^\dagger$ is a triangle.*

Proof: Suppose toward a contradiction that $K^\dagger$ is not a triangle. By the minimality of n, $K^\dagger$ is not a broken wheel with principal path $u_m w^\star u_n$, and furthermore, there is no chord of the outer face of $K^\dagger$ which does not have $w^\star$ as an endpoint, or else there is a chord of C which does not have p_2 as an endpoint, contradicting Claim 1.6.2. By the minimality of n, $K^\dagger$ is not a broken wheel with principal path $u_m w^\star u_n$. Thus, by Theorem 1.5.3, there is at most one L-coloring of $u_m w^\star u_n$ which does not extend to an L-coloring of $K^\dagger$.

Since $|L(w^\star) \setminus \{c, f\}| \geq 2$, it follows from Theorem 0.2.3 that there is an L-coloring ψ of H' in which $\psi(w^\star) \notin \{c^*, f^*\}$, $\psi(p_3) = \phi_d(p_3)$, and $\psi(p_4) = d$. If ψ extends to an L-coloring of H_1 using c^*, f^* on the respective vertices p_2, u_m, then ϕ_d extends to an L-coloring of H, which is false. Since H is not a triangle, $(f^*, \psi(w^\star, \psi(u_n))$ is a proper L-coloring of the path $u_m w^\star u_n$, so this is the lone coloring of $u_m w^\star u_n$ which does not extend to an L-coloring of $K^\dagger$.

Now, since there is at most one proper L-coloring of $w^\star p_3 p_4$ which does not extend to an L-coloring of H, and $|L(w^\star) \setminus \{c^*, d^*, \psi(w^\star)| \geq 2$, there is an $h \in L(w^\star) \setminus \{c^*, d^*, \psi(w^\star)$ such that any L-coloring of $w^\star p_3 p_4$ using h on $w^\star$ extends to an L-coloring of H'. By Claim 1.6.10, we suppose without loss of generality that $h \neq \phi_d(p_3)$. Since $p_4 \notin N(w^\star)$, $(h, \phi_d(p_3), d)$ is a proper L-coloring of $w^\star p_3 p_4$ which extends to an L-coloring ψ' of H'. Since $\psi'(w^\star) \neq \psi(w^\star)$, ψ' extends to an L-coloring of $H_1 - p_2$ in which u_m is colored with f^*, and the color c^* is left for p_2, so ψ' extends to an L-coloring of H in which P is colored by ϕ_d, contradicting our assumption. ■

Since $K^\dagger$ is a triangle, we have $u_n = u_{m+1}$. We now fix a color $h \in L(u_{n+1}) \setminus \{f^*, f^{**}\}$. Recall that f^{**} is the lone color of $\mathcal{Z}_{H_0}(c, \phi_{d''}(p_2), \bullet)$. Since there is no chord of the outer face of H_1 with p_3 as an endpoint, it follows from the minimality of H that there is a color $d^* \in \{d, d', d''\}$, where $d^* \neq h$ if $u_{m+1} = u_t$, such that any L-coloring of $u_{m+1} w^\star p_3 p_4$ extends to an L-coloring of H'.

Suppose that $d^* \in \{d', d''\}$. Let ψ be an L-coloring of $u_{m+1} w^\star p_3 p_4$ using h, d^* on u_{m+1}, p_4 respectively, where $\psi(p_3) = \phi_{d^*}(p_3)$ and $\psi(w^\star) \in L(w^\star) \setminus \{f^*, c^*, h, \phi_{d^*}(p_2)\}$. Such a ψ exsts, because $|L(w^\star)| \geq 5$. Then ψ extends to an L-coloring ψ' of H'. By our choice of ψ, the colors f^*, c^* are left over for the respective vertices u_m, p_2, so ψ extends to an L-coloring of H whose restriction to P is ϕ_{d^*}. This contradicts our assumption.

Thus, we have $d^* = d''$. Let ψ be an L-coloring of $u_{m+1} w^\star p_3 p_4$ using h, d'' on u_{m+1}, p_4 respectively, where $\psi(p_3) = \phi_{d''}(p_3)$ and $\psi(w^\star) \in L(w^\star) \setminus \{f^{**}, \phi_{d''}(p_2), h, \phi_{d''}(p_3)\}$. Such a ψ exists, because $|L(w^\star)| \geq 5$. Then ψ extends to an L-coloring ψ' of H'. By our choice of ψ, the colors $f^{**}, \phi_{d''}(p_2)$ are left over for the respective vertices u_m, p_2, so ψ extends to an L-coloring of H whose restriction to P is $\phi_{d''}$. This contradicts our assumption ■

Recall that $H_1 - p_2$ is bounded by outer cycle $C_1^* := w^\star p_3 p_4 u_t \cdots u_m$, and $P_1^* := u_m w^\star p_3 p_4$ is a subpath of C_1^* of length three. By assumption, $H_1 - p_2$ has no chord of C_1^* with p_3 as an endpoint, as any such chord of C_1^* is also a chord of C, contradicting our assumption. Likewise, by Claim 1.6.11, there is no chord of C_1^* with $w^\star$ as an endpoint. If there is any other chord of C_1^*, then this chord is also a chord of C with endpoints in $\{u_m, \cdots, u_t, p_4\}$, contradicting Claim 1.6.2. Thus, C_1^* is an induced cycle of $H_1 - p_2$.

We claim now that there is a vertex $v^\star \in V(H_1 - p_2) \setminus V(C_1^*)$ such that $N(v^\star) \cap V(P_1^*) = \{w^\star, p_3, p_4\}$. Let $h \in L(u_\star) \setminus \{f^*, c^*, \phi_d(p_3)\}$. If the coloring $(f^*, h, \phi_d(p_3), d)$ of $u_m w^\star p_3 p_4$ extends to an L-coloring of $H_1 - p_2$, then the color c^* is left for p_2, and thus ϕ_d extends to an L-coloring of H, contradicting our assumption. Since C_1^* is an induced subgraph of $H_1 - p_2$, it follows from 1) of Proposition 1.5.1 that there is a $v^* \in V(H_1 - p_2) \setminus V(C_1^*)$ with at least three neighbors in P_1^*.

Note that $v^\star$ is adjacent to at most one of u_m, p_3, or else H contains a copy of $K_{2,3}$ with bipartition $\{p_2, v^\star\}$, $\{u_m, w^\star, p_3\}$, contradicting the fact that H is short-separation-free. Thus, $v^\star$ is adjacent to both of $w^\star, p_4$. Since $p_4 \notin N(w^\star)$ and H is short-separation-free, it follows from our triangulation conditions that $v^\star$ is adjacent to p_3 as well, so $N(v^\star) \cap V(P_1^*) = \{w^\star, p_3, p_4\}$. Thus $v^\star$ is the unique vertex of $H_1 - p_2$ which lies outside the outer face of $H_1 - p_2$ and has at least three neighbors on P_1^*.

Claim 1.6.13. *There is a color b such that $L(w^\star) = \{c^*, f^*, \phi_d(p_3), \phi_{d'}(p_3), b\}$ and $L(v^\star) = \{d, d', \phi_d(p_3), \phi_{d'}(p_3), b\}$. Furthermore, $d, d' \notin L(w^\star) \setminus \{c^*, f^*\}$.*

Proof: We first show that $\{c^*, f^*, \phi_d(p_3), \phi_{d'}(p_3)\} \subseteq L(w^\star)$ and $|L(w^\star)| = 5$. Suppose at least one of these conditions does not hold. Thus, either $|L(w^\star) \setminus \{c^*, f^*, \phi_d(p_3)\}| \geq 3$ or $|L(w^\star) \setminus \{c^*, f^*, \phi_{d'}(p_3)\}| \geq 3$, so suppose without loss of generality that $|L(w^\star) \setminus \{c^*, f^*, \phi_d(p_3)\}| \geq 3$. Since $w^\star$ is the unique vertex of $H_1 \setminus C_1$ with at least three neighbors on $u_m p_2 p_3 p_4$, and $p_4 \notin N(w^\star)$, it follows from 1) of Proposition 1.5.1 that the L-coloring

$(f^*, c^*, \phi_d(p_3), d)$ of $u_m p_2 p_3 p_4$ extends to an L-coloring of H_1, so ϕ_d extends to an L-coloring of H, contradicting our assumption.

Thus, there is a color b such that $L(w^\star) = \{c^*, f^*, \phi_d(p_3), \phi_{d'}(p_3), b\}$. Now suppose toward a contradiction that one of d, d' lies in $L(w^\star) \setminus \{c^*, f^*\}$, say d without loss of generality. Then $\psi = (f^*, d, \phi_d(p_3), d)$ is a proper L-coloring of $u_m w^\star p_3 p_4$ which leaves c^* for p_2, and since $|L_\psi(v^\star)| \geq 3$, it follows from 1) of Proposition 1.5.1 that ψ extends to an L-coloring of $H_1 - p_2$ and thus an L-coloring of H_1 using c^* on p_2. Thus, ϕ_d extends to an L-coloring of H, contradicting our assumption, so we indeed have $d, d' \notin L(w^\star) \setminus \{c^*, f^*\}$. In particular, we have $\{d, d'\} \cap \{\phi_d(p_3), \phi_{d'}(p_3)\} = \varnothing$.

Now suppose toward a contradiction that $L(v^\star) \neq \{d, d', \phi_d(p_3), \phi_{d'}(p_3), b\}$. In that case, there is a $d^* \in \{d, d'\}$ and a $b' \in L(w^\star) \setminus \{c^*, f^*, d^*, \phi_{d^*}(p_3)\}$ such that $|L(v^\star) \setminus \{b', d^*, \phi_{d^*}(p_3)\}| \geq 3$. Since, it follows that the L-coloring $(f^*, b', \phi_{d^*}(p_3), d^*)$ of $u_m w^\star p_3 p_4$ extends to an L-coloring of $H_1 - p_2$, and thus to an L-coloring of H_1 using c^* on p_2. Thus, ϕ_{d^*} extends to an L-coloring of H, contradicting our assumption. ■

An analogous observation holds for the remaining color d'' of $L(p_4)$.

Claim 1.6.14. *$\{f^{**}, \phi_{d''}(p_2), \phi_{d''}(p_3)\}$ is a subset of $L(w^\star)$ of size three. Furthermore, $L(w^\star) \setminus \{f^{**}, \phi_{d''}(p_2), \phi_{d''}(p_3)\} \subseteq L(v^\star)$ and $\{\phi_{d''}(p_3), d''\} \subseteq L(v^\star)$.*

<u>Proof:</u> If either if these facts do not hold, then the coloring $(\phi_{d''}(p_3), d'')$ of $p_3 p_4$ extends to an L-coloring ψ of $u_m w^\star p_3 p_4$ in which $\psi(u_m) = f^{**}$, $\psi(w^\star) \neq \phi_{d''}(p_2)$, and $|L_\psi(v^\star)| \geq 3$. But then, by 1) of Proposition 1.5.1, ψ extends to an L-coloring of $H_1 - p_2$ and leaves the color $\phi_{d''}(p_2)$ for p_2. Thus, by our choice of f^{**}, ψ extends to an L-coloring of H in which P is colored with $\phi_{d''}$, contradicting our assumption. ■

Recall that $f^* \neq f^{**}$ by Claim 1.6.9. Since $\mathcal{Z}_{H_0}(c, c^*, \bullet) = \{f^*\}$ and $\mathcal{Z}_{H_0}(c, \phi_{d''}(p_2), \bullet) = \{f^{**}\}$, and there is no chord of the outer face of H_0 without p_2 as an endpoint, it follows from Theorem 1.5.3 that H_0 is a broken wheel with principal path $p_1 p_2 u_m$, or else, since $p_1 u_m$ is not a chord of C, one of these two sets has size at least two. It then follows from 1) of Proposition 1.4.7 that $\{c^*, f^*\} = \{\phi_{d''}(p_2), f^{**}\}$. We claim that $d'' \in L(w^\star) \setminus \{c^*, f^*\}$. Suppose not. By Claim 1.6.14, we have $d'' \in L(v^\star)$. Thus, since $d'' \notin L(w^\star) \setminus \{c^*, f^*\}$, we have $d'' \in L(v^\star) \setminus \{\phi_d(p_3), \phi_{d'}(p_3), b\}$ by the first fact of Claim 1.6.13, and thus, by the second fact of Claim 1.6.13, we have $d'' \in \{d, d'\}$, which is false.

Thus, we indeed have $d'' \in L(w^\star) \setminus \{c^*, f^*\}$. Since $\{c^*, f^*\} = \{\phi_{d''}(p_2), f^{**}\}$, it follows that $\psi = (f^{**}, d'', \phi_{d''}(p_3), d'')$ is a proper L-coloring of the path $u_m w^\star p_3 p_4$, and $\phi_{d''}(p_2) \in L_\psi(p_2)$. Since $|L_\psi(v^\star)| \geq 3$, and $v^\star$ is the lone vertex of $V(H_1 - p_2) \setminus V(C_1^*)$ with at least three neighbors on P_1^*, it follows from 1) of Proposition 1.5.1 that ψ extends to an L-coloring of $H_1 - p_2$, and thus to an L-coloring of H_1 in which p_2 is colored with $\phi_{d''}(p_2)$ and u_m is colored with f^{**}. But then, by our choice of f^{**}, $\phi_{d''}$ extends to an L-coloring of H, which is false. □

1.7 Path Reduction

As indicated in Section 1.3, when we delete a path between two cyclic facial subgraphs C_1, C_2 in a short-separation-free graph G with a specified list-assignment L, we do so in a special setting, where the designated cycles C_1, C_2 satisfy the property that, for each $j = 1, 2$ and for sufficiently small values of k, there is no k-chord of C_i which separates vertices of $G \setminus C_i$ with lists of size less than five. Thus, it is natural to introduce the following notation.

Definition 1.7.1. Let G be a planar graph with facial cycle C , let L be a list-assignment for $V(G)$. Let $k \geq 1$ and let H be a connected subgraph of C, where either H is a subpath of C, or, if $H = C$, then at least one vertex of $G \setminus B_{k/2}(C)$

has a list of size less than five. We associate to H a vertex set $\text{Sh}_{k,L}(H,C,G)$, where $v \in \text{Sh}_{k,L}(H,C,G)$ if there is a generalized chord Q of C of length at most k, and with both endpoints in H, such that, letting $G = G_0 \cup G_1$ be the natural (C,Q)-partition of G, there exists an $i \in \{0,1\}$ such that the following hold.

1) $G_i \cap H$ is a connected subgraph of H*; AND*
2) $v \in V(G_i \setminus Q)$ and every vertex of $G_i \setminus C$ has an L-list of size at least five.

If $H = C$, then we just write $\text{Sh}_{k,L}(C,G)$ instead of $\text{Sh}_{k,L}(C,C,G)$. If the list-assignment L is clear from the context then we drop the subscript L from the notation above. If G, C are also clear from the context then we just write $\text{Sh}_k(H)$. Note that, if H is a subpath of C, then, given a proper generalized chord Q of C with both endpoints in H, there is precisely one side of the generalized chord intersecting with H on precisely a subpath of H, i.e there is a precisely one $i \in \{0,1\}$ such that $G_i \cap H$ is connected, although this is not true if $H = C$. In practice, we are interested in the case where there are vertices of $G \setminus C$ with lists of size less than five, and in that case, the definition above uniquely specifies a side of any generalized chord of C.

Definition 1.7.2. Let G be a planar graph with facial cycle C , let L be a list-assignment for $V(G)$, and let $k \geq 1$ be an integer. Let H be a connected subgraph of C, where either H is a subpath of C, or, if $H = C$, then at least one vertex of $G \setminus B_{k/2}(C)$ has a list of size less than five. We then have the following terminology.

1) We say that H is (k,L)-*short* in (C,G) if, for any generalized chord Q of C with both endpoints in H and length at most k, letting $G = G_0 \cup G_1$ be the natural (C,Q)-partition of G, there exists an $i \in \{0,1\}$ such that the following hold.
   a) Every vertex of $G_i \setminus H$ has an L-list of size at least five; *AND*
   b) If Q is not a cycle (i.e Q is a proper generalized chord of C), and at least one endpoint of Q lies in $\mathring{H}$, then $G_i \cap H$ has one connected component.
2) We define $\text{Link}_L(H,C,G)$ to be the set of proper L-colorings ϕ of $V(P) \setminus \text{Sh}_2(H)$ in $G \setminus (E(C) \setminus E(H))$ such that $\text{Sh}_2(H)$ is L_ϕ-inert in $G \setminus \text{dom}(\phi)$.

Recall that $\mathring{H} = H$ if $H = C$. In most uses above, both the graph G and the facial cycle C are clear from the context and we simply say that H is (k,L)-short to mean that H is (k,L)-short in (C,G). In general, if the facial cycle C and graph G are clear from the context, then we just write $\text{Link}_L(H)$. Some care must be taken with the definition above. Given a subpath P of C and a $\phi \in \text{Link}_L(P)$, Definition 1.7.2 does not preclude the possibility that P is a subpath of C consisting of all but a lone edge of C, and ϕ uses the same color on both endpoints of P. In practice, whenever we deal with the situation in which P is a path consisting of all but an edge of the specified facial cycle, we ensure that we obtain an element of $\text{Link}_L(P)$ which does not use the same color on the endpoints of P, so that we obtain a proper L-coloring of $V(P) \setminus \text{Sh}_2(P)$ in G.

This section consists of two results, the first of which consists of some basic properties of the colorings of the form specified in definition 1.7.2, and the second of which provides some conditions under which these colorings exist. We begin with the following definition.

Definition 1.7.3. Given a planar graph G, a cyclic facial subgraph C and a subpath P of C, a vertex $v \in V(P)$ is called a P*-hinge of* C if one of the following holds: Either v is an endpoint of P, or, if v is an internal vertex of P, then, for each $k = 1, 2$, there is no k-chord of C with an endpoint in each connected component of $P - v$. If the facial subgraph C is clear from the context then we just call v a P*-hinge*.

We now state the first of the two results which make up this section. This result consists of the following simple facts.

Theorem 1.7.4. *Let G be a planar graph with facial cycle C and let P be a subpath of C. Let L be a list-assignment for G and suppose that P is $(2, L)$-short*

1) *Let $v \in V(P)$ be a P-hinge and let P_1, P_2 be the two subpaths of P such that $P_1 \cap P_2 = v$ and $P_1 \cup P_2 = P$. For any $\psi_1 \in \mathrm{Link}(P_1)$ and $\psi_2 \in \mathrm{Link}(P_2)$ with $\psi_1(v) = \psi_2(v)$, we have $\psi_1 \cup \psi_2 \in \mathrm{Link}(P)$; AND*
2) *If each vertex of $D_1(C)$ has an L-list of size at least five, then, for any $\phi \in \mathrm{Link}(P)$, each vertex of $D_1(C) \setminus \mathrm{Sh}_2(P))$ has an L_ϕ-list of size at least three.*
3) *Suppose that every vertex of $G \setminus C$ has an L-list of size at least five. Then, for any $\phi \in \mathrm{Link}(P)$, the following holds:*
   a) *For any nonempty subpath Q of $C \setminus P$ of length at most one, if all the vertices of $C \setminus (Q \cup P)$ have L_ϕ-lists of size at least three, then, for any L_ϕ-coloring ψ of Q, $\phi \cup \psi$ extends to an L-coloring of G; AND*
   b) *In particular, if $1 \leq |V(C \setminus P)| \leq 2$ then, for any any extension of ϕ to an L-coloring of $\mathrm{dom}(\phi) \cup V(P)$, ψ extends to an L-coloring of G.*

Proof. 1) is immediate from the definition of $\mathrm{Link}(P)$, so we now prove 2). Suppose there is a $v \in D_1(C) \setminus \mathrm{Sh}_2(P))$ with $|L_\phi(v)| < 3$. Thus, v has at least three neighbors in $\mathrm{dom}(\phi)$. Let S be the set of vertices of $G \setminus C$ with L-lists of size less than five. Since P is $(2, L)$-short, there is a unique pair of vertices $w, w' \in N(v) \cap V(H)$ such that the unique subpath of P with endpoints w, w' contains all the vertices of $N(v) \cap V(P)$. But then, since $|N(v) \cap \mathrm{dom}(\phi)| \geq 3$, there is an element of $\mathrm{dom}(\phi)$ which is separated from an edge of $E(C) \setminus E(P)$ by the 2-chord wvw', contradicting the fact that $\mathrm{dom}(\phi) \cap \mathrm{Sh}_2(P) = \varnothing$.

Now we prove 3). Note that b) follows immediately from a) by setting $Q = C \setminus P$, since $C \setminus (Q \cup P) = \varnothing$ and the conditions of a) are automatically satisfied. Thus, we just need to prove a). Firstly, since $|V(Q)| \geq 1$, P does not consist of all but an edge of C, so any element of $\mathrm{Link}(P)$ is a proper L-coloring of its domain in G. Let $S := V(C \setminus P)$ and let $G' := G \setminus \mathrm{Sh}_2(P)$. Then G' has a unique facial subgraph F' such that $C \setminus P \subseteq F'$ and, by 2), every vertex of $F' \setminus S$ has an L_ϕ-list of size at least three and thus an $L^Q_{\phi \cup \psi}$-list of size at least three. By assumption, every vertex of $S \setminus Q$ has an L_ϕ-list of size at least three and thus an $L^Q_{\phi \cup \psi}$-list of size at least three. Every vertex of $G' \setminus F'$ has an $L^Q_{\phi \cup \psi}$-list of size at least five, and thus, by Theorem 0.2.3, G' admits an $L^Q_{\phi \cup \psi}$-coloring σ. Since $\mathrm{Sh}_2(P)$ is L_ϕ-inert in $G \setminus \mathrm{dom}(\phi)$, it follows that σ extends to an L-coloring of G, so $\phi \cup \psi$ extends to an L-coloring of G, as desired. ☐

The following result, which is the second of the two results which make up this section, provides some conditions under which the colorings defined in Definition 1.7.2 exist.

Theorem 1.7.5. *Let G be a planar graph with facial cycle C and let P be a subpath of C. Let L be a list-assignment for G such that each internal vertex of P has an L-list of size at least three, and suppose further that P is $(2, L)$-short. Let p, p' be the endpoints of P. Then the following hold.*

i) *For any $c \in L(p)$, there is a $\phi \in \mathrm{Link}(P)$ with $\phi(p) = c$. Furthermore, if $p \neq p'$, then, for any $c \in L(p)$ and $A \subseteq L(p')$ with $|A| = 3$, there is a $\phi \in \mathrm{Link}(P)$ such that $\phi(p) = c$ and $\phi(p') \in A$; AND*
ii) *If $|V(P)| \geq 2$ then, for any sets $B \subseteq L(p)$ and $B' \subseteq L(p')$ with $|B| = |B'| = 2$, there is a $\psi \in \mathrm{Link}(P)$ such that $\psi(p) \in B$ and $\psi(p') \in B'$.*

Proof. By some appropriate stereographic projection of G from the sphere onto the plane, we suppose without loss of generality that C is the outer face of G (this is just for notational convenience in the proof of the theorem). We show i) and ii) together by induction on the length of P. This is trivial of P is a singleton so now suppose that $|V(P)| \geq 2$. If P is just an edge, then both i) and ii) trivially hold, since it follows from Corollary 0.2.4 that any L-coloring of P lies in Link(P). Now suppose that $|V(P)| > 2$ and that both i) and ii) hold for any subpath of C of length smaller than $|V(P)|$ which satisfies the specified conditions. Let $P := p_r \cdots p_1$ for some $r \geq 3$, where $p' = p_r$ and $p = p_1$. Let S be the set of vertices of $G \setminus C$ with L-lists of size less than five.

For any $1 \leq k \leq 2$ and any k-chord Q of C with both endpoints in P, we let C_Q^{long} and C_Q^{short} be the two cycles intersecting precisely on p_iwp_j such that $C_Q^{\text{long}} \cup C_Q^{\text{short}} = C \cup Q$ and $C_Q^{\text{short}} \setminus \mathring{Q}$ is a subpath of P. Let $\mathcal{P}_{\text{end}}$ be the set of proper generalized chords Q of C of length at most two such that p_1, p_r are the endpoints of Q. Since P is a path, the above definition uniquely specifies C_Q^{short} and C_Q^{long}. Let $\mathcal{P}$ be the set of generalized chords of C of length at most two with one endpoint in $P - p_1$ and p_1 as the other endpoint. Since each element of $\mathcal{P}$ shares a common endpoint, and this endpoint is also an endpoint of P, we trivially have the following:

Claim 1.7.6. *For any $Q, Q' \in \mathcal{P}$, we have either* $\text{Int}(C_Q^{\text{short}}) \subseteq \text{Int}(C_{Q'}^{\text{short}})$ *or* $\text{Int}(C_{Q'}^{\text{short}}) \subseteq \text{Int}(C_Q^{\text{short}})$.

We now define a subset $\mathcal{P}_{\text{end}}$ of $\mathcal{P}$, where $Q \in \mathcal{P}_{\text{end}}$ if and only if p_1, p_r are the endpoints of Q and there is a vertex with an L-list of size less than five in the open disc bounded by C_Q^{short}. Note that even though P is $(2, L)$-short, Definition 1.7.2 does not preclude the possibility that $\mathcal{P}_{\text{end}}$ is nonempty.

We now fix sets A, B, B' and a color c, where $A, B' \subseteq L(p_r)$, $c \in L(p_1)$ and $B \subseteq L(p_1)$, $|A| = 3$, and $|B| = |B'| = 2$. Note that $P - p_1$ is also $(2, L)$-short, and each internal vertex of $P - p_1$ has an L-list of size at least three. Suppose first that $\mathcal{P} \setminus \mathcal{P}_{\text{end}} = \varnothing$. In that case, we have $\text{Sh}_2(P) = \text{Sh}_2(P - p_1)$. Since $|L(p_2)| \geq 3$, let $d \in L(p_2) \setminus \{c\}$. Since $|V(P)| \geq 3$, it follows from our induction hypothesis that there is a $\phi \in \text{Link}(P - p_1)$ with $\phi(p_r) \in A$ and $\phi(p_2) = d$. Let ϕ' be the extension of ϕ to $\text{dom}(\phi) \cup \{p_1\}$ obtained by coloring p_1 with c. Since there is no chord of C with p_1 as an endpoint and the other endpoint in $P - p_1$, ϕ' is a proper L-coloring of its domain in $G \setminus (E(C) \setminus E(P))$. Since $\text{Sh}_2(P) = \text{Sh}_2(P - p_1)$, we have $\phi' \in \text{Link}(P)$, so P satisfies i). By our induction hypothesis, there is a $\psi \in \text{Link}(P - p_1)$ with $\psi(p') \in B'$. Since $|B| = 2$, let $d \in B \setminus \{\psi(p_2)\}$ and let ψ' be the L-coloring of $\text{dom}(\psi) \cup \{p_1\}$ obtained by coloring p_1 with d. As above, ψ' is a proper L-coloring of its domain in $G \setminus (E(C) \setminus E(P))$. Since $\text{Sh}_2(P) = \text{Sh}_2(P - p_1)$, we have $\psi' \in \text{Link}(P)$, so P satisfies ii). Thus, if $\mathcal{P} \setminus \mathcal{P}_{\text{end}} = \varnothing$, then we are done. Now suppose that $\mathcal{P} \setminus \mathcal{P}_{\text{end}} \neq \varnothing$ and let $Q \in \mathcal{P} \setminus \mathcal{P}_{\text{end}}$ maximize the quantity $|V(\text{Int}(C_Q^{\text{short}})|$. Let $t \in \{2, \cdots, r\}$, where p_1, p_t are the endpoints of Q.

Claim 1.7.7. *There is no chord of C of the form p_1p_j for some $j \in \{t+1, \cdots, r\}$. Furthermore,* $\text{Sh}_2(P) = \text{Sh}_2(p_rPpt) \cup (\text{Int}(C_Q^{\text{short}}) \setminus V(Q))$ *as a disjoint union.*

<u>Proof:</u> If there is a chord of C of the form p_1p_j for some $j \in \{t+1, \cdots, r\}$, then, by Claim 1.7.6, p_1p_j separates p_t from $E(C) \setminus E(P)$, contradicting the maximality of Q. Likewise, there is no 2-chord of C of the form p_1wp_j for some $j \in \{t_1, \cdots, r\}$, or else, by Claim 1.7.6, p_1wp_j separates p_t from $E(C) \setminus E(P)$, contradicting the maximality of Q. Thus, we indeed, have $\text{Sh}_2(P) = \text{Sh}_2(p_rPpt) \cup (\text{Int}_G(C_Q^{\text{short}}) \setminus V(Q))$ as a disoint union. ■

Now we show that P satiisfies i) and ii). Consider the following cases.

Case 1: Q is a chord of C

We break this into two subcases.

Subcase 1.1 $p_r = p_t$

In this case, by Claim 1.7.7, we have $\text{Sh}_2(P) = (\text{Int}(C_Q^{\text{short}}) \setminus V(Q))$. Since $|A| \geq 3$, we choose a color $d \in A \setminus \{c\}$. Let ϕ be the L-coloring of $\{p_1, p_r\}$ using c, d on p_1, p_r respectively. Then, by 0.2.3, ϕ extends to an L-coloring of $\text{Int}(C_Q^{\text{short}})$, so $\text{Sh}_2(P)$ is L_ϕ-inert in $G \setminus \text{dom}(\phi)$, and thus $\phi \in \text{Link}(P)$. Thus, P satisfies i). Likewise, for any $f \in B$ and $f \in B'$ with $f \neq f'$, if ψ is the L-coloring of $\{p_1, p_r\}$ using f, f' on p_1, p_r respectively, $\text{Sh}_2(P)$ is L_ψ-inert in $G \setminus \text{dom}(\psi)$, and $\psi \in \text{Link}(P)$. Thus, P satisfies ii) as well.

Subcase 1.2 $p_r \neq p_t$

Since $|L(p_t)| \geq 3$, we choose a color $d \in L(p_t) \setminus \{c\}$. By our induction hypothesis, since $|V(p_r P p_t)| \geq 2$, there is a $\phi \in \text{Link}(p_r P p_t)$ such that $\phi(p_r) \in A$ and $\phi(p_t) = d$. Let ϕ' be the extension of ϕ to $\text{dom}(\phi) \cup \{p_1\}$ obtained by coloring p_1 with c. By Claim 1.7.7, ϕ' is a proper L-coloring of its domain in $G \setminus (E(C) \setminus E(P))$. Applying Theorem 0.2.3, ϕ' extends to an L-coloring of $\text{Int}(C_Q^{\text{short}})$, and thus, by Claim 1.7.7, $\text{Sh}_2(P)$ is $L_{\phi'}$-inert in $G \setminus \text{dom}(\phi')$, so $\phi' \in \text{Link}(P)$. Thus, P satisfies i).

Applying our induction hypothesis again, since $|L(p_t)| \geq 3$, there is a $\psi \in \text{Link}(p_r P p_t)$ with $\psi(p_r) \in B'$. Let $f \in B \setminus \{\psi(p_t)\}$ and let ψ' be an extension of ψ to $\text{dom}(\psi) \cup \{p_1\}$ obtained by coloring p_1 with c. By Claim 1.7.7, ψ' is a proper L-coloring of its domain in $G \setminus (E(C) \setminus E(P))$. Applying Theorem 0.2.3, ψ' extends to an L-coloring of $\text{Int}(C_Q^{\text{short}})$. Thus, by Claim 1.7.7, $\text{Sh}_2(P)$ is $L_{\psi'}$-inert in $G \setminus \text{dom}(\psi')$, so $\psi' \in \text{Link}(P)$. Thus, P satisfies ii) as well.

Case 2: Q is a 2-chord of C

As above, we break this into two subcases.

Subcase 2.1 $p_r = p_t$

By Theorem 1.5.5, there is a $d \in A$, such that, letting ϕ be the L-coloring of $\{p_1, p_r\}$ using c, d on p_1, p_t respectively, ϕ is a proper L-coloring of its domain in $\text{Int}(C_Q^{\text{small}})$, and any L-coloring of Q using c, d on p_1, p_r respectively extends to an L-coloring of $\text{Int}(C_Q^{\text{small}})$. Thus, $V(\text{Int}(C_Q^{\text{small}})) \setminus V(Q)$ is L_ϕ-inert in $G \setminus \text{dom}(\phi)$. If $p_t = p_r$, then, by Claim 1.7.7, $\text{Sh}_2(P)$ is L_ϕ-inert in $G \setminus \text{dom}(\phi)$, so $\phi \in \text{Link}(P)$. Thus, P satisfies i).

Likewise, by Theorem 1.5.10, there is an $f \in B$ and an $f' \in B'$ such that, letting ψ be the L-coloring of $\{p_1, p_r\}$ using f, f' on p_1, p_r respectively, ψ is a proper L-coloring of its domain in $\text{Int}(C_Q^{\text{small}})$, and any L-coloring of Q using f, f' on p_1, p_r respectively extends to an L-coloring of $\text{Int}(C_Q^{\text{small}})$. Thus, $V(\text{Int}(C_Q^{\text{small}})) \setminus V(Q)$ is L_ψ-inert in $G \setminus \text{dom}(\psi)$. By Claim 1.7.7, $\text{Sh}_2(P)$ is L_ψ-inert in $G \setminus \text{dom}(\psi)$, so $\psi \in \text{Link}(P)$. Thus, P satisfies ii).

Subcase 2.2 $p_r \neq p_t$.

By Theorem 1.5.5 ,since $|L(p_t)| \geq 3$, there is a $d \in L(p_t)$, such that, letting ϕ be the L-coloring of $\{p_1, p_t\}$ using c, d on p_1, p_t respectively, ϕ is a proper L-coloring of its domain in $\text{Int}(C_Q^{\text{small}})$, and any L-coloring of Q using c, d on p_1, p_t respectively extends to an L-coloring of $\text{Int}(C_Q^{\text{small}})$. Thus, $V(\text{Int}(C_Q^{\text{small}})) \setminus V(Q)$ is L_ϕ-inert in $G \setminus \text{dom}(\phi)$. Furthermore, $|V(p_t P p_r)| \geq 2$, and, by our induction hypothesis, there is a $\phi' \in \text{Link}(p_t P p_r)$ with $\phi'(p_t) = d$ and $\phi'(p_r) \in A$. By Claim 1.7.7, the union $\phi \cup \phi'$ is a proper L-coloring of its domain in $G \setminus (E(C) \setminus E(P))$, and $\text{Sh}_2(P)$ is $L_{\phi \cup \phi'}$-inert in $G \setminus \text{dom}(\phi \cup \phi')$. Thus, $\phi \cup \phi' \in \text{Link}(P)$. Thus, P satisfies i).

By Theorem 1.5.10, since $|L(p_t)| \geq 3$, there is a pair of L-colorings ψ_1, ψ_2 of $\{p_1, p_t\}$ using distinct colors on p_t, such that, for each $j = 1, 2$, ψ_j is a proper L-coloring of its domain in $\text{Int}(C_Q^{\text{small}})$, $\psi_j(p_1) \in B$, and any extension of ψ_j to an L-coloring of Q extends to an L-coloring of $\text{Int}(C_Q^{\text{small}})$. By our induction hypothesis, there is a $\psi' \in \text{Link}(p_t P p_r)$ with $\psi'(p_t) \in \{\psi_1(p_t), \psi_2(p_t)\}$ and $\psi'(p_r) \in B'$, say $\psi'(p_t) = \psi_1(p_t)$ without loss of generality. By

Claim 1.7.7, the union $\psi_1 \cup \psi'$ is a proper L-coloring of its domain in $G \setminus (E(C) \setminus E(P))$, and $\mathrm{Sh}_2(P)$ is $L_{\psi_1 \cup \psi'}$-inert in $G \setminus \mathrm{dom}(\psi_1 \cup \psi')$. Thus, $\psi_1 \cup \psi' \in \mathrm{Link}(P)$ and P satisfies ii) as well. This completes the proof of Theorem 1.7.5. $\square$

Chapter 2

Mosaics and Their Properties

We begin by fixing constants N_{mo}, β, where $N_{\text{mo}} \geq 96$ and set $\beta := \frac{17}{15}N_{\text{mo}}^2$. The subscript of N refers to mosaics, which is the term we use for our strengthening of tessellations defined below. We show that every $(\beta + 4N_{\text{mo}}, 1)$-tessellation is colorable, by showing that any tessellation which satisfies some stronger properties stated below is colorable. In particular, we allow our tessellations to contain some precolored faces (i.e closed rings) of length at most N_{mo}, where each precolored face satisfies some additional properties. In Chapters 2-11, we then show that any tessellation satisfying these properties is colorable by showing that no minimal counterexample to colorability exists, where the term minimal counterexample to the claim is made precise in the section below.

2.1 Introduction

In order to state our stronger induction hypothesis, we begin with the following definitions:

Definition 2.1.1. Given a planar embedding G and a cyclic facial subgraph $C \subseteq G$, we say that C is a *highly predictable* facial subgraph of G if, for every induced cycle $K \subseteq G[V(C)]$, the following hold.

1) For every $v \in D_1(K) \setminus V(C)$, the graph $G[N(v) \cap V(K)]$ is either a path of length at most two or K is a triangle with $G[N(v) \cap V(K)] = K$*; AND*
2) There is at most one $v \in D_1(K) \setminus V(C)$ such that $|N(v) \cap V(K)| = 3$.

We have the following simple observation, which is immediate and is stated without proof.

Observation 2.1.2. *Let G be a short-separation-free graph with facial cycle F, where $3 \leq |F| \leq 4$ and $|V(G) \setminus V(F)| > 1$. Suppose further that, for each $v \in V(F)$, every facial subgraph of G containing v, except possibly F, is a triangle. Then F is a highly predictable facial subgraph of G.*

We now introduce the following weakened version of the definition above.

Definition 2.1.3. Given a planar embedding G, a cyclic facial subgraph $C \subseteq G$ and a list-assignment L for $V(G)$, we say that C is an *L-predictable* facial subgraph of G if $V(C) \neq V(G)$, and, for every induced cycle $K \subseteq G[V(C)]$ the following hold.

1) For every $v \in D_1(K, G) \setminus V(C)$, the graph $G[N(v) \cap V(K)]$ is either a proper subpath of K or all of K*; AND*
2) There is a vertex $v \in D_1(K, G) \setminus V(C)$ such that, for any proper L-coloring ϕ of $V(K)$, the following hold.
    i) $|L_\phi(v)| \geq 2$*; AND*
    ii) For each $v' \in D_1(K, G) \setminus (V(C) \cup \{v\})$, $|L_\phi(v')| \geq 3$.

Note that, for any planar embedding G and cyclic facial subgraph $C \subseteq G$, if C is highly predictable facial subgraph of G, then C is also an L-predictable facial subgraph of G for any list-assignment L for $V(G)$.

There are several additional useful properties we want our minimal counterexample to satisfy. To state this, it is necessary to attach a specified orientation to a chart.

Definition 2.1.4. A chart $(G, \mathcal{C}, L)$ is called *oriented* if the embedding of G in the plane is such that there exists a $C_* \in \mathcal{C}$ such that C_* is the outer face of G.

For convenience, an oriented chart $(G, \mathcal{C}, L)$ with outer face $C_* \in \mathcal{C}$ is usually denoted as $(G, \mathcal{C}, L, C_*)$ in order to keep track of the outer face. In order to state the distance conditions we impose on our tessellations, we introduce the following notation.

Definition 2.1.5. Let $\mathcal{T} = (G, \mathcal{C}, L)$ be a chart, let $C \in \mathcal{C}$ and let $\mathbf{P} := \mathbf{P}_{\mathcal{T}}(C)$ be the precolored subgraph of C. We define a subset $w_{\mathcal{T}}(C)$ of $V(C)$ as follows.

$$w_{\mathcal{T}}(C) := \begin{cases} V(C) \text{ if } C \text{ is a closed } \mathcal{T}\text{-ring} \\ V(C \setminus \mathring{\mathbf{P}}) \text{ if } C \text{ is an open } \mathcal{T}\text{-ring} \end{cases}$$

We also introduce a rank function $\text{Rk}(\mathcal{T}|\cdot) : \mathcal{C} \to \mathbb{R}$ defined as follows.

$$\text{Rk}(\mathcal{T}|C) := \begin{cases} |V(C)| \text{ if } C \text{ is a closed } \mathcal{T}\text{-ring} \\ 2N_{\text{mo}} \text{ if } C \text{ is an open } \mathcal{T}\text{-ring} \end{cases}$$

If the underlying chart $\mathcal{T}$ is clear from the context then we drop the symbol $\mathcal{T}$ from $w_{\mathcal{T}}(C)$ or $\text{Rk}(\mathcal{T}|C)$ respectively. We now state our induction hypothesis:

Definition 2.1.6. An oriented chart $\mathcal{T} := (G, \mathcal{C}, L, C_*)$ is called a *mosaic* if $\mathcal{T}$ is a tessellation which satisfies the following conditions:

M0) For each closed $\mathcal{T}$-ring C, we have $|V(C)| \leq N_{\text{mo}}$, and for each open $\mathcal{T}$-ring C', we have $|E(\mathbf{P}_{\mathcal{T}}(C'))| \leq \frac{2N_{\text{mo}}}{3}$.

M1) For each open ring $C \in \mathcal{C}$, letting $\mathbf{P} := \mathbf{P}_{\mathcal{T}}(C)$, there is no chord of C with an endpoint in $\mathring{\mathbf{P}}$, and, for each $v \in D_1(C, G)$, the graph $G[N(v) \cap V(\mathbf{P})]$ is a subpath of $\mathbf{P}$ of length at most one*; AND*

M2) For each closed ring $C \in \mathcal{C}$, C is an L-predictable cyclic facial subgraph of G*; AND*

M3) For each $C \in \mathcal{C} \setminus \{C_*\}$, we have $d(w_{\mathcal{T}}(C_*), w_{\mathcal{T}}(C)) \geq \frac{\beta}{3} + \text{Rk}(C) + \text{Rk}(C_*)$*; AND*

M4) For any distinct $C_1, C_2 \in \mathcal{C} \setminus \{C_*\}$, we have $d(w_{\mathcal{T}}(C_1), w_{\mathcal{T}}(C_2)) \geq \beta + \text{Rk}(C_1) + \text{Rk}(C_2)$.

Chapters 2-11 consist entirely of the proof of the following result.

Theorem 2.1.7. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a mosaic. Then G is L-colorable.*

We begin with the following.

Observation 2.1.8. *For any mosaic $\mathcal{T} = (G, \mathcal{C}, L, C_*)$, $\mathcal{T}$ is a $(\frac{\beta}{3}, N_{\text{mo}})$-tessellation. In particular, we have the following.*

1) For any distinct $C_1, C_2 \in \mathcal{C} \setminus \{C_*\}$, we have $d(C_1, C_2) \geq \beta$; AND

2) For any distinct $C \in \mathcal{C} \setminus \{C_*\}$, if at least one of C, C_* is an open $\mathcal{T}$-ring, then $d(C, C_*) \geq \frac{\beta}{3} + 3N_{\text{mo}}$.

Proof. For any open ring $C \in \mathcal{C}$, since $|E(\mathbf{P}_{\mathcal{T}}(C))| \leq \frac{2N_{\text{mo}}}{3}$, any vertex of the precolored path of C is of distance at most $\frac{N_{\text{mo}}}{3}$ from $C \setminus \mathring{\mathbf{P}}_{\mathcal{T}}(C)$. Thus, for any two distinct elements $C_1, C_2 \in \mathcal{C}$, we have $d(C_1, C_2) \geq d(w_{\mathcal{T}}(C_1), w_{\mathcal{T}}(C_2)) - \frac{2N_{\text{mo}}}{3}$, so the claimed bounds follow immediately from M3) and M4). □

Now we introduce the following terminology for our minimal counterexamples.

Definition 2.1.9. Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a mosaic. We say that $\mathcal{T}$ is *critical* if the following hold.

1) G is not L-colorable*; AND*
2) For any mosaic $(G', \mathcal{C}', L', D)$ with $|V(G')| < |V(G)|$, G' is L'-colorable*; AND*
3) For any mosaic $(G', \mathcal{C}', L', D)$ with $|V(G')| = |V(G)|$ and $\sum_{v \in V(G')} |L(v)| < \sum_{v \in V(G)} |L(v)|$, G' is L'-colorable.

The remainder of the work of Chapters 2-11 consists of showing that there are no critical mosaics. We begin with the following:

Observation 2.1.10. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic. Then the following hold:*

1) G is connected; AND
2) For each $C \in \mathcal{C}$ and $v \in V(C) \setminus V(\mathbf{P}_{\mathcal{T}}(C))$, $|L(v)| = 3$; AND
3) For each $v \in V(G) \setminus \left(\bigcup_{C \in \mathcal{C}} V(C)\right)$, $|L(v)| = 5$ and $\deg_G(v) \geq 5$.

Proof. 1) follows immediately from the minimality of $|V(G)|$. Likewise, 2) and the first part of 3) both follow directly from the minimality of $\sum_{v \in V(G)} |L(v)|$, or else we can remove colors from the lists of some vertices of G. Now suppose toward a contradiction that there is a $v \in V(G) \setminus \left(\bigcup_{C \in \mathcal{C}} V(C)\right)$ such that $\deg_G(v) \leq 4$. Note that every face of G containing v is bounded by a triangle, so $N(v)$ induces a cycle of length at most four and $G[N(v)] = K_4$. Since G is short-separation-free, we have $G = K_4$, and G is trivially L-colorable, contradicting the fact that $\mathcal{T}$ is a counterexample. □

Note that the proof of 3) of Observation 2.1.10 is somewhat atypical. In a minimal counterexample argument involving list-assignments, we usually deal with a vertex v such that $\deg(v) < |L(v)|$ by deleting v to produce a smaller counterexample, but in the context above, the graph $G - v$ does not satisfy the triangulation conditions of Definition 1.3.1, i.e it is not the underlying graph of a tessellation. In general, some care must be taken when constructing smaller counterexamples from critical mosaics.

Definition 2.1.11. Given a chart $\mathcal{T} = (G, \mathcal{C}, L)$ and a subgraph H of G, we let $\mathcal{C}^{\subseteq H}$ denote the set $\{C \in \mathcal{C} : C \subseteq H\}$.

Now we have the following.

Proposition 2.1.12. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic. Then each $H \in \mathcal{C}$ is a cyclic facial subgraph of G.*

Proof. Let $H \in \mathcal{C}$ and suppose towards a contradiction that H is not a cycle. By M2), H is an open $\mathcal{T}$-ring (possibly, G is not connected and H has several components). Since H is a facial subgraph of G and H is not a cycle, there is a set $A \subseteq V(H)$ with $|A| \leq 1$, such that $G \setminus A$ has more than one connected component.

Let $G_1, \cdots, G_r$ be the connected components of $G \setminus A$. For each $j = 1, \cdots, r$, we let $H_j := H \cap (G_j + A)$ and $\mathcal{C}_j := \{H' \in \mathcal{C} \setminus \{H\} : H' \subseteq G_j + A\}$. Note that H_j is a facial subgraph of $G_j + A$. For each $j = 1, \cdots, r$, let C_*^j

be the outer face of $G_j + A$. Note that if $H = C_*$ then $H_j = C_*^j$ for each $j = 1, \cdots, r$. For each $j = 1, \cdots, r$, we set $\mathcal{T}_j := (G_j + A, \mathcal{C}_j \cup \{H_j\}, L, C_*^j)$. Note that $\mathcal{T}_j$ is an oriented tessellation for each $j = 1, \cdots, r$.

Claim 2.1.13. *For each $j \in \{1, \cdots, r\}$, the following hold.*

i) $G_j + A$ is L-colorable; AND

ii) If $V(G_j \cap \mathbf{P}_{\mathcal{T}}(H)) = \varnothing$ and $|A| = 1$ then, letting $A = \{v\}$ and choosing any $c \in L(v)$, the following hold: letting L' be the list-assignment for $V(G)$ in which $L'(v) = \{c\}$ and $L'(x) = L(x)$ for all $x \in V(G_j)$, we get that $\mathcal{T}_j' := (G_j + v, \mathcal{C}_j \cup \{H_j\}, L', C_j^*)$ is a mosaic.

Proof: Let $j \in \{1, \cdots, r\}$. Note that if $\mathcal{T}_j$ is a mosaic, then it immediately follows that $G_j + A$ is L-colorable by the minimality of $\mathcal{T}$, since $|V(G_j + A)| < |V(G)|$. Let $\mathbf{P} := \mathbf{P}_{\mathcal{T}}(H)$.

We first prove ii). Suppose that $V(G_j \cap \mathbf{P}_{\mathcal{T}}(H)) = \varnothing$ and $A \neq \varnothing$. In that case, let $\mathcal{T}_j'$ be as in the statement of ii) and let $A = \{v\}$. Then H_j is an open $\mathcal{T}_j'$-ring. Since $V(G_j \cap \mathbf{P}_{\mathcal{T}}(H)) = \varnothing$, H is an open $\mathcal{T}$-ring, and since H_j has a precolored path in $\mathcal{T}_j'$ consisting of one vertex, $\mathcal{T}_j'$ also satisfies M3), so $\mathcal{T}_j'$ is a mosaic.

Now we prove i). If H_j is an open $\mathcal{T}_j$-ring then we immediately get that $\mathcal{T}_j$ is also a mosaic, so i) holds in this case. The only nontrivial possibility is that H_j is a closed $\mathcal{T}_j$-ring (i.e that $V(H_j) \subseteq V(\mathbf{P})$) so suppose that $V(H_j) \subseteq V(\mathbf{P})$. If H_j contains no cycles, then, since H_j is a facial subgraph of $G_j + A$, we have $V(G_j + A) = V(H_j)$, so $G_j + A$ is L-colorable in that case. Now suppose that the walk H_j contains at least one cycle. Let $U \subseteq \mathbb{R}^2 \setminus H_j$ be an open subset of $\mathbb{R}^2$ with $H_j = \partial(U)$ and $G_j + A \subseteq \mathbb{R}^2 \setminus U$. Now we have the following:

Subclaim 2.1.14. *Let $D \subseteq H_j$ be a cycle and let $U' \subseteq \mathbb{R}^2 \setminus D$ be the unique open connected component of $\mathbb{R}^2 \setminus D$ disjoint to U. Let $G^* \subseteq G_j + v$ be the graph consisting of all edges and vertices of $G_j + A$ in $\mathrm{Cl}(U')$. Then G^* is L-colorable.*

Proof: Let $C^\dagger$ be the outer face of G^* (possibly $C^\dagger = D$). Let $\mathcal{C}^* := \{C \in \mathcal{C} : C \subseteq G^*\}$. We just need to check that $\mathcal{T}^* := (G^*, \mathcal{C}^* \cup \{D\}, L, C^\dagger)$ is a mosaic. Note that $\mathcal{T}^*$ is a tessellation. Since H is an open $\mathcal{T}$-ring it follows from M1) that, for each $v \in V(G^* \setminus D)$, if v has a neighbor in D, then $G^*[N(v) \cap V(D)]$ is a subpath of D of length at most one. Thus, D is a highly predictable cyclic facial subgraph of G^*, and thus an L-predictable cyclic facial subgraph of G^* so $\mathcal{T}^*$ is a mosaic. Since $|V(D)| \leq \frac{2N_{\text{mo}}}{3}$, M0), M1), and M2) are satisfied by $\mathcal{T}^*$, so we just need to check that $\mathcal{T}^*$ satisfies M3) and M4). Suppose toward a contradiction that there is a $C \in \mathcal{C}^*$ such that $d(w_{\mathcal{T}^*}(C), w_{\mathcal{T}^*}(D))$ violates one of M3) or M4).

Let $\beta^* \in \{\beta, \beta/3\}$, where $\beta^* = \beta$ if neither C nor D is the outer face of G^*, and otherwise $\beta^* = \beta/3$. We claim now that we have the following bound:

$$d(w_{\mathcal{T}}(C), w_{\mathcal{T}}(H)) \geq \beta^* + \mathrm{Rk}(\mathcal{T}|C) + 2N_{\text{mo}} \tag{†}$$

If neither H nor C is the outer face of G, then, since $\mathcal{T}$ is a mosaic and H is an open $\mathcal{T}$-ring, we have $d(w_{\mathcal{T}}(C), w_{\mathcal{T}}(H)) \geq \beta + \mathrm{Rk}(\mathcal{T}|C) + 2N_{\text{mo}} \geq \beta^* + \mathrm{Rk}(\mathcal{T}|C) + 2N_{\text{mo}}$, so we are done in that case. If C is the outer face of G, then C is also the outer face of G^*, so $\beta^* = \frac{\beta}{3}$, and, since $\mathcal{T}$ is a mosaic, we have $d(w_{\mathcal{T}}(C), w_{\mathcal{T}}(H)) \geq \beta^* + \mathrm{Rk}(\mathcal{T}|C) + 2N_{\text{mo}}$. Finally, suppose that H is the outer face of G. Then D is the outer face of G^*, so we again have $\beta^* = \frac{\beta}{3}$, and $d(w_{\mathcal{T}}(C), w_{\mathcal{T}}(H)) \geq \beta^* + \mathrm{Rk}(\mathcal{T}|C) + 2N_{\text{mo}}$. This proves (†). Now let $\mathbf{P}' := \mathbf{P}_{\mathcal{T}}(C)$ and consider the following cases.

Case 1: C is an open $\mathcal{T}$-ring

In this case, by inequality (†), we have $d(C \setminus \mathring{\mathbf{P}}', H \setminus \mathring{\mathbf{P}}) \geq \beta^* + 4N_{\text{mo}}$. Since $V(D) \subseteq V(\mathbf{P})$ and any vertex of $\mathbf{P}$ is of distance at most $\left\lfloor \frac{V(P)}{2} \right\rfloor$ from $H \setminus \mathring{P}$, we have $d(C \setminus \mathring{\mathbf{P}}', D) \geq \beta^* + 4N_{\text{mo}} - \left\lfloor \frac{V(\mathbf{P})}{2} \right\rfloor$. Since H is an open $\mathcal{T}$-ring, we have $|E(\mathbf{P})| \leq \frac{2N_{\text{mo}}}{3}$ by M0), so $4N_{\text{mo}} - \left\lfloor \frac{V(\mathbf{P})}{2} \right\rfloor \geq \frac{11N_{\text{mo}}}{3}$. On the other hand, since $V(D) \subseteq V(\mathbf{P})$, we have $2N_{\text{mo}} + |V(D)| \leq \frac{8N_{\text{mo}}}{3}$. Thus, we obtain $d(D, C \setminus \mathring{\mathbf{P}}') \geq \beta^* + 2N_{\text{mo}} + |V(D)|$. Since C has the same rank in $\mathcal{T}$ and $\mathcal{T}^*$, and $w_{\mathcal{T}}(C) = w_{\mathcal{T}^*}(C)$, this contradicts our assumption that $d(w_{\mathcal{T}^*}(C), w_{\mathcal{T}^*}(D)) < \beta^* + \text{Rk}(\mathcal{T}^*|C) + |V(D)|$.

Case 2: C is a closed $\mathcal{T}$-ring

In this case, $w_{\mathcal{T}}(C) = w_{\mathcal{T}^*}(C) = V(C)$, and, again by inequality (†), we have $d(C, H \setminus \mathring{\mathbf{P}}) \geq \beta^* + 2N_{\text{mo}} + |V(C)|$. Since $V(D) \subseteq V(\mathbf{P})$ and any vertex of $\mathbf{P}$ is of distance at most $\left\lfloor \frac{V(\mathbf{P})}{2} \right\rfloor$ from $H \setminus \mathring{\mathbf{P}}$, we obtain

$$d(C, w_{\mathcal{T}^*}(D)) \geq \beta^* + 2N_{\text{mo}} + |V(C)| - \left\lfloor \frac{V(\mathbf{P})}{2} \right\rfloor$$

Since $|V(\mathbf{P})| \leq \frac{2N_{\text{mo}}}{3}$, we have $2N_{\text{mo}} - \left\lfloor \frac{V(\mathbf{P})}{2} \right\rfloor \geq \frac{8N_{\text{mo}}}{3} > |V(D)|$. Thus,. we conclude that $d(C, w_{\mathcal{T}^*}(D)) \geq \beta^* + |V(C)| + |V(D)|$, contradicting our assumption that $d(w_{\mathcal{T}^*}(C), w_{\mathcal{T}^*}(D)) < \beta^* + \text{Rk}(\mathcal{T}^*|C) + |V(D)|$.

Thus, $\mathcal{T}^*$ is a mosaic, as desired. Since $|V(G^*)| < |V(G)|$, G^* is indeed L-colorable by the minimality of $\mathcal{T}$, as desired. This completes the proof of Subclaim 2.1.14. ■

Now we return to the proof of Claim 2.1.13. Since H_j is a facial subgraph of $G_j + A$, it follows that, for each $w \in V(G_j + v) \setminus V(H_j)$, there is a unique cycle $D \subseteq H_j$ such that w lies in the unique open connected component of $\mathbb{R}^2 \setminus D$ disjoint to U. Since $|L(w)| = 1$ for each $w \in V(H_j)$, it follows from Subclaim 2.1.14 that any L-coloring of H_j extends to an L-coloring of $G_j + A$. Since H_j is L-colorable, this completes the proof Claim 2.1.13. ■

Now we retuirn to the proof of Proposition 2.1.12. If $A = \varnothing$, then $G_j + A = G_j$ for each $j = 1, \cdots, r$, and, by Claim 2.1.13, G_j is L-coloring for each $j = 1, \cdots, r$, and thus G is L-colorable, contradicting the fact that $\mathcal{T}$ is critical. Now suppose that $|A| = 1$ and let $A = \{v\}$. We first rule out the possibility that $v \in V(\mathbf{P}_{\mathcal{T}}(H)$. Suppose that $v \in V(\mathbf{P}_{\mathcal{T}}(H))$. By Claim 2.1.13, for each $j = 1, \cdots, r$, there is an L-coloring ϕ_j of $G_j + v$. Since $|L(v)| = 1$ in this case, the union of these colorings is an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. Thus, we have $v \notin V(\mathbf{P}_{\mathcal{T}}(H))$. Since $\mathbf{P}_{\mathcal{T}}(H)$ is a connected subgraph of H, there exists a $j \in \{1, \cdots, r\}$ such that $\mathbf{P}_{\mathcal{T}}(H) \subseteq G_j + v$ and, for each $j' \in \{1, \cdots, r\} \setminus \{j\}$, we have $V(\mathbf{P}_{\mathcal{T}}(H)) \cap V(G_{j'} + v) = \varnothing$.

By Claim 2.1.13, $G_j + v$ admits an L-coloring ϕ, and, for each $j' \in \{1, \cdots, r\} \setminus \{j\}$, the precoloring $\phi(v)$ extends to an L-coloring of $G_{j'} + v$. The union of these colorings is then an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. Thus, our assumption that H contains a cut-vertex of G is false. □

Proposition 2.1.12 has the following corollary:

Corollary 2.1.15. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic. Then G is 2-connected.*

Proof. Applying Proposition 2.1.12, for each $C \in \mathcal{C}$, C is a cyclic facial subgraph of G. Since G is a tessellation, every facial subgraph of G, other than those lying in $\mathcal{C}$, is a triangle, so every facial subgraph of G is cyclic. Thus, G is 2-connected. □

We also have the following very simple bound:

Observation 2.1.16. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, and let $K \subseteq G$ be a separating cycle in G. Let $G = G_0 \cup G_1$ be the natural K-partition of G. Let $i \in \{0, 1\}$, let $m \geq 1$ be an integer and suppose there exists a $C \in \mathcal{C}$ with $C \subseteq G_i$ and $d(K, C) \geq m$. Then, for each $j \in \{1, \cdots, m\}$, we have $|D_j(K, G_i)| \geq 5$. In particular, we have $|V(G_i \setminus K)| > 5(m-1)$ and $|V(G_i)| > 5m$.*

Proof. Firstly, by Corollary 2.1.15, G_i is 2-connected, and furthermore, for each $r \in \{1, \cdots, m-1\}$, each of the three vertex-sets $D_{r-1}(K) \cap G_i$, $D_r(K) \cap V(G_i)$, and $D_{r+1}(K) \cap V(G_i)$ is nonempty, since $d(K, C) \geq m$ and $C \subseteq G_i$. Since G_i is 2-connected, the graph $G[D_r(K) \cap V(G_i)]$ separates $D_{r-1}(K) \cap V(G_i)$ from $D_{r+1}(K) \cap V(G_i)$. Thus, since G is short-separation-free, we have $|D_r(K) \cap V(G_i)| \geq 5$. Summing over $|D_r(K) \cap V(G_i)|$ for each $r = 1, \cdots, m-1$, we have $|V(G_i)| - (|V(K)| + |V(C)|) \geq 5(m-1)$. Thus, we have $|V(G_i \setminus K)| > 5(m-1)$, and since K is a separating cycle in G, we have $|V(K)| \geq 5$. Thus, we have $|V(G_i)| > 5m$. $\square$

We now show that, if we have a separating cycle D of bounded length in a critical mosaic $(G, \mathcal{C}, L, C_*)$, then the subgraph of G consisting of everything on one side of D is L-colorable under certain conditions.

Definition 2.1.17. Let K be a 2-connected outerplanar embedding, where K is bounded by outer cycle D. Given a planar embedding K^*, we say that K^* is a *K-web* if the following conditions are satisfied.

1) $K \subseteq K^*$, and D is the outer face of K^**; AND*
2) For every $v \in D_1(K, K^*)$, the induced graph $K^*[N(v) \cap V(K)]$ is a path in K of length at most one*; AND*
3) Every connected component of $K^*[D_1(K)]$ is an induced cycle of K^*, and furthermore, for any distinct $v, w \in D_1(K, K^*)$, if each of v, w is adjacent to an edge of K, then $vw \notin E(K^*)$*; AND*
4) Every facial subgraph of K^*, except D, is a triangle*; AND*
5) For any two vertices $x, y \in V(K)$, $d_{K^*}(x, y) = d_K(x, y)$.

Now we show the following:

Proposition 2.1.18. *Let K be a 2-connected outerplanar embedding, where K is bounded by an outer cycle $D := v_1 \cdots v_r$. Suppose K satisfies the additional condition that, for any cycle $C \subseteq K$, if $|V(C)| = 4$, then C is not an induced subgraph of K. Then there exists a short-separation-free K-web K^* such that $|V(K^*)| \leq |V(K)|^2$.*

Proof. If $|V(K)| = 3$, then K is a short-separation-free K-web, so we are done in that case. Likewise, if $|V(K)| = 4$, then, by our conditions on K, every facial subgraph of K is a triangle, so K is again a short-separation-free K-web. Now we show that the claim holds for any $|V(K)| \geq 5$ where D is a chordless cycle.

Suppose that $|V(K)| \geq 5$, and let $K := v_1 v_2 \cdots v_r$ for some $r \geq 5$, where $K = D$ is a chordless cycle. We extend K to a short-separation-free annulus K'' defined as follows: Let K' be a graph obtained from K by adding to the open disc bounded by K a length $2r$-cycle $K^\dagger := w_1 w_1^* w_2 w_2^* \cdots w_r w_r^*$ and adding the following edges to the open disc bounded by K. For each $i \in \{1, \cdots, r\}$, we add the edges $\{w_i v_i, w_i v_{i+1}, w_i^* v_i\}$, where the indices are read mo r. Now let K'' be the graph obtained from K' by adding to the open disc bounded by $K^\dagger$ a length-r cycle $K^{\dagger\dagger} := v_1^1 v_2^1 \cdots v_r^1$, and, for each $i \in \{1, \cdots, r\}$, adding the edges $\{v_i^1 w_i^*, v_i^1 w_i, v_i^1 w_{i+1}^*\}$, whre the subscripts are read mod r.

Let $r^* := \lceil r/4 \rceil - 1$. We define a sequence of graphs $K_1, K_2, \cdots, K_{r^*}$, and a sequence of cycles $C_1, \cdots, C_{r^*}$, each of length r, where $C_j := v_1^j v_2^j \cdots v_r^j$ for each $j = 1, \cdots, r^*$, as follows:

1. $K_1 := K''$ and $C_1 := K^{\dagger\dagger}$; *AND*

2. For $1 \leq j < r^*$, K_{j+1} is obtained from K_j by adding to the open disc bounded by C_j a set of vertices $v_1^{j+1}, \cdots, v_r^{j+1}$, and, for each $i \in \{1, \cdots, r\}$, the edges $v_i^{j+1}v_{i+1}^{j+1}$, $v_i^j v_i^{j+1}$ and $v_{i+1}^j v_i^{j+1}$, where the subscripts are read mod r.

Let K^* be a graph obtained from K_{r^*} by adding to the open disc bounded by C_{r^*} a lone vertex z and the edges $zv_i^{r_*}$ for each $i = 1, \cdots, r$. Then $|V(K^*)| \leq 3r + r^*r + 1 \leq r^2$. Furthermore K^* is short-separation-free, and every facial subgraph of K^*, except for K, is a triangle. Now let $v, v' \in V(K)$ and suppose toward a contradiction that $d_{K^*}(v, v') < d_K(v, v')$, and let P be a path in K^* with endpoints v, v' such that $|E(P)| < d_K(v, v')$. By construction of K^*, the path P then contains the vertex z. Thus, we have $|E(P)| = |E(vPz)| + |E(zPv')| \geq 2(r^* + 1) \geq \lfloor r/2 \rfloor \geq d_K(v, v')$, contradicting our assumption. Thus, we have $d_{K^*}(v, v') \geq d_K(v, v')$, and so $d_{K^*}(v, v') = d_K(v, v')$, since $K \subseteq K^*$. We conclude that K^* is a short-separation-free K-web. Since $|V(K^*)| \leq |V(K)|^2$, we are done in this case.

No we show that Proposition 2.1.18 holds in general. We show this by induction on $|V(K)|$. The case where $|V(K)| \leq 4$ is done above. Now let K be a 2-connected outerplanar embedding satisfying the conditions of Proposition 2.1.18, where $|V(K)| \geq 5$. Suppose that, for any 2-connected outerplanar embedding K' satisfying the conditions of Proposition 2.1.18, if $|V(K')| < |V(K)|$, then there exists a short-separation-free K'-web K'' with $|V(K'')| \leq |V(K')|^2$.

Let D be the outer face of K. If $D = K$, then, as shown above, we are done, so now suppose that K contains a chord of of D, and suppose without loss of generality that this chord is of the form v_1v_j for some $j \in \{1, \cdots, r-1\}$. Let $K = K_1 \cup K_2$, where $K_1 \cap K_2 = v_1v_j$, K_1 is bounded by outer cycle $D_1 := v_1 \cdots v_j$, and K_2 is bounded by outer cycle $v_j \cdots v_r$. Note that, for each $i \in \{1, 2\}$ and any cycle $C \subseteq K_i$, if $|V(C)| = 4$, then C is not an induced subgraph of K_i, or else C is an induced subgraph of K. Thus, for each $i = 1, 2$, there exists a short-separation-free K_i-web K_i^* with $|V(K_i^*)| \leq |V(K_i)|^2$.

Now let $K^* = K_1^* \cup K_2^*$. Then K^* is a short-separation-free planar embedding, with outer face D, where $K \subseteq K^*$ and every facial subgraph of K^*, except for D, is a triangle. Furthermore, we have $|V(K^*)| \leq |V(K_1)|^2 + |V(K_2)|^2 = |V(K_1)|^2 + (|V(K \setminus K_1)| + 2)^2$. Thus, we obtain $|V(K^*)| \leq |V(K_1)|^2 + |V(K \setminus K_1)|^2 + 2|V(K \setminus K_1)| + 4$. We have $|V(K)|^2 = (|V(K_1)| + |V(K \setminus K_1)|)^2 = |V(K_1)|^2 + |V(K \setminus K_1)|^2 + 2|V(K_1)||V(K \setminus K_1)|$. Since $|V(K_1)| \geq 3$ and $|V(K \setminus K_1)| \geq 1$, we have $|V(K_1)||V(K \setminus K_1)| \geq |V(K \setminus K_1)| + 2$, so $|V(K^*)| \leq |V(K)|^2$. This completes the proof of Proposition 2.1.18. $\square$

We now note the following.

Observation 2.1.19. *Let G be a short-separation-free planar embedding and let D be a cycle in G with $|V(D)| \geq 5$. Let $D' := G[V(D)] \cap \text{Int}(D)$, and suppose further that, for any 4-cycle T in $\text{Int}(D)$ whose vertices lie in $V(D)$, T is an induced subgraph of D'. Let $G^\dagger$ be the graph obtained from G by deleting all vertices in $\text{Int}(D) \setminus V(D)$ and replacing them with a short-separation-free D'-web D^* in the closed disc bounded by D. Then $G^\dagger$ is short-separation-free.*

Proof. Suppose toward a contradiction that there is a separating cycle $F \subseteq G^\dagger$ with $|V(F)| \leq 4$. Since $\text{Ext}^+(D)$ is short-separation-free, and D^* is short-separation-free, $E(F)$ has nonempty intersection with each of $E(\text{Ext}^+(D)) \setminus E(D)$ and $E(D^*) \setminus E(D')$. In particular, since $|E(F)| \leq 4$, and D is a separating cycle in $G^\dagger$, D^* contains an ℓ-chord of D', where $1 \leq \ell \leq 3$, whose endpoints are non-adjacent in D', contradicting the fact that D^* is a D'-web. $\square$

With the above in hand, we show the following:

Proposition 2.1.20. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, and let D be a cycle in G with $|V(D)| \leq N_{\text{mo}}$, such that D separates an element of $\mathcal{C} \setminus \{C_*\}$ from C_*. Suppose that there exist a $C \in \mathcal{C} \setminus \{C_*\}$ with $C \subseteq \text{Int}(D)$, such that $d(C, D) \geq \frac{|V(D)|^2}{5}$. Then $V(\text{Ext}(D))$ is L-colorable.*

Proof. Set $G^* := \text{Ext}^+(D)$. Let D' be the subgraph of G^* consisting of D and all chords of D in $\text{Int}(D)$. Then D' is an outerplanar embedding, and, since G is short-separation-free and D is a separating cycle, we have $|V(D)| \geq 5$. Furthermore, for any cycle $D'' \subseteq D'$ of length four, D'' is not an induced subgraph of $G[V(D)]$ by our triangulation conditions, since G is short-separation-free. Thus, applying Proposition 2.1.18, Let $G^\dagger$ be the graph obtained from G^* by adding to G^* a D'-web D^* in the closed disc of $\mathbb{R}^2$ bounded by D, where $|V(D^*)| \leq |V(D)|^2$ and the embedding D^* is short-separation-free. Let $L^\dagger$ be a list-assignment for $G^\dagger$ where $L^\dagger(v) = L(v)$ for all $v \in V(G^\dagger) \setminus V(D^* \setminus D)$ and $L^\dagger(v)$ is an arbitrary 5-list for all $v \in V(D^* \setminus D)$.

We claim now that the tuple $\mathcal{T}^\dagger = (G^\dagger, \mathcal{C}^{G^*}, L^\dagger, C_*)$ is a mosaic. By Observation 2.1.19, $G^\dagger$ is short-separation-free, and, by construction of D^*, every face of $G^\dagger$, except those of $\mathcal{C}^{\subseteq G^*}$, is a triangle, so $\mathcal{T}^\dagger$ is an oriented tessellation. M0), M1), and M2) of Definition 2.1.6 are immediate. In particular, by construction of D^*, $\mathcal{T}^\dagger$ still has the property that, for each open $\mathcal{T}^\dagger$-ring C, there is no chord of C with one endpoint in $\mathbf{P}_{\mathcal{T}^\dagger}(C)$, and, for each $v \in D_1(C, G^\dagger)$, $G^\dagger[N(v) \cap V(\mathbf{P}_{\mathcal{T}^\dagger}(C))]$ is a subpath of $\mathbf{P}_{\mathcal{T}^\dagger}(C)$ of length at most one. We just need to check that $\mathcal{T}^\dagger$ satisfies distance conditions M3) and M4) of Definition 2.1.6.

If one of the distance conditions M3), M4) is not satisfied then there exists a pair of distinct rings $C, C' \in \mathcal{C}^{\subseteq G^*}$ and a pair of vertices u, v with $u \in V(C)$ and $v \in V(C')$ such that $d_{G^\dagger}(u, v) < d_G(u, v)$. In that case, there exists a path in D^*, with endpoints x, y in D, such that $d_{D^*}(x, y) < d_D(x, y)$, contradicting the fact that D^* is a D-web. Thus, $(G^\dagger, \mathcal{C}^{\subseteq G^*}, L, C_*)$ is indeed a mosaic.

Suppose toward a contradiction that $|V(G^\dagger)| \geq |V(G)|$. In that case, we have $|V(D^*)| \geq |V(\text{Int}(D))|$. By assumption there is a ring $C^\dagger \in \mathcal{C}$ with $C^\dagger \subseteq \text{Int}(D)$ such that $d(C^\dagger, D) \geq \frac{|V(D)|^2}{5}$. Thus, by Observation 2.1.16, we have $|V(\text{Int}(D))| > |V(D)|^2$, so we have $|V(D^*)| < |V(\text{Int}(D))|$, a contradiction. Since $|V(G^\dagger)| < |V(G)|$, $G^\dagger$ is $L^\dagger$-colorable by the minimality of $\mathcal{T}$, and thus G^* is L-colorable. This completes the proof of Proposition 2.1.20. ☐

We have an analogous fact for the other side:

Proposition 2.1.21. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, and let D be a separating cycle in G with $|V(D)| \leq N_{\text{mo}}$, such that there is a $C \in \mathcal{C}$ with $C \subseteq \text{Int}(D)$ and $d(C, D) \leq \frac{\beta}{3} - \frac{N_{\text{mo}}}{2} - \frac{|V(D)|^2}{5}$. Then $V(\text{Int}(D))$ is L-colorable.*

Proof. Let $\mathcal{C}' := \{C \in \mathcal{C} : C \subseteq \text{Int}(D)\}$ and let G' be a planar embedding of $\text{Int}^+(D)$ obtained by setting C to the outer face of G' by an appropriate stereographic projection. Let D' be the subgraph of G' consisting of the edges and vertices corresponding to D in G. Then we have $V(\text{Int}_{G'}(D')) = V(D')$, and the chords of D in $\text{Ext}(D)$ correspond to the chords of D' in $\text{Int}(D')$. Thus, $G'[V(D')]$ is an outerplanar embedding, and since D is a separating cycle in G, we have $|V(D')| \geq 5$. . Furthermore, for any cycle $D'' \subseteq G'[V(D')]$ of length four, D'' is not an induced subgraph of $G'[V(D')]$ by our triangulation conditions, since G is short-separation-free. Thus, applying Proposition 2.1.18, let G'' be an embedding obtained from G' by adding to G' a short-separation-free $G'[V(D')]$-web $D^\dagger$ in the closed disc bounded by D', with $|V(D^\dagger)| \leq |V(D)|$. By our construction of $D^\dagger$, each face of G'', except those in $\mathcal{C}'$, is a triangle, and thus, by Observation 2.1.19 the tuple $\mathcal{T}^\dagger := (G'', \mathcal{C}', L, C)$ is an oriented tessellation. We claim now that $\mathcal{T}^\dagger := (G'', \mathcal{C}', L, C)$ is a mosaic. As in Proposition 2.1.20, The only nontrivial conditions to check are M3) and M4).

If M4) is not satisfied then, since $C_* \notin \mathcal{C}' \setminus \{C\}$, there exists a pair of distinct rings $C_1, C_2 \in \mathcal{C}' \setminus \{C\}$ and a pair of vertices u, v with $u \in V(C_1)$ and $v \in V(C_2)$ such that $d_{G^\dagger})(u, v) < d_G(u, v)$. In that case, there exists a path in D^*, with endpoints x, y in D, such that $d_{D^\dagger}(x,y) < d_D(x,y)$, contradicting the fact that $D^\dagger$ is a D-web. Likewise, since the distance condition on the new outer face C has only weakened, the same argument shows that M3) is satisfied as well. Thus, $\mathcal{T}^\dagger$ is indeed a mosaic.

We claim now that $|V(G^\dagger)| < |V(G)|$. Suppose that $|V(G^\dagger)| \geq |V(G)|$. In that case, we have $|V(D^\dagger)| \geq |V(\text{Ext}(D))|$. Note that $d(C_*, D) \geq \frac{|V(D)|^2}{5}$, or else, since any two vertices of D are distance at most $\frac{|V(D)|}{2}$ apart, we have $d(C_*, C) \leq \frac{\beta}{3}$, contradicting Observation 2.1.8.

Since $d(D, C_*) \geq \frac{|V(D)|^2}{5}$ and $|V(D^\dagger)| < |V(D)|^2$, we have $|V(D^*)| < |V(\text{Ext}(D))|$ by Observation 2.1.16, a contradiction. Since $|V(G^\dagger)| < |V(G)|$, $G^\dagger$ admits an L-coloring by the minimality of $\mathcal{T}$, so the subgraph of G induced by $\text{Int}(D)$ is L-colorable. $\square$

We also repeatedly use the following basic property of critical mosaics:

Proposition 2.1.22. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic. Then, for each open ring $C \in \mathcal{C}$, the cycle C has no chords in G.*

Proof. Let $C \in \mathcal{C}$ be an open $\mathcal{T}$-ring and let $\mathbf{P} := \mathbf{P}_{\mathcal{T}}(C)$. By Proposition 2.1.12, C is a cycle. Suppose toward a contradiction that there is a chord xy of C. By M1), no endpoint of xy is an internal vertex of P, so let $G = G_0 \cup G_1$ be the natural xy-partition of G, where $\mathbf{P} \subseteq G_0$. For each $i = 0, 1$, let C_i be the cycle $(C \cap G_i) + xy$. Let C_*^0 be the outer face of G_0 and let $\mathcal{T}_0 := (G_0, \mathcal{C}^{\subseteq G_0} \cup \{C_0\}, L, C_*^0)$.

Claim 2.1.23. *$\mathcal{T}_0$ is a mosaic.*

<u>Proof:</u> We begin with the following observation:

Subclaim 2.1.24. *For each $C' \in \mathcal{C}^{\subseteq G_0}$, we have $d(w_{\mathcal{T}_0}(C'), w_{\mathcal{T}_0}(C_0)) \geq d(w_{\mathcal{T}}(C'), w_{\mathcal{T}}C))$.*

<u>Proof:</u> This is an immediate consequence of the fact that, for any $C \in \mathcal{C}^{G^0}$ and any subgraph $H \subseteq C$, any shortest (H, C)-path in G has its C-endpoint in C_0. $\blacksquare$

Now consider the following cases.

Case 1: C_0 is the outer face of G_0

In this case, we have $C_0 = C_*^0$. To check that $\mathcal{T}_0$ is a mosaic, it just suffices to check that M3) holds. Since $\mathcal{T}$ satisfies M3) and M4), we get that $d(w_{\mathcal{T}}(C'), w_{\mathcal{T}}(C)) \geq \frac{\beta}{3} + \text{Rk}(\mathcal{T}|C') + 2N_{\text{mo}}$ for each $C' \in \mathcal{C}^{\subseteq G_0}$. Thus, if C_0 is an open $\mathcal{T}_0$-ring, then applying Subclaim 2.1.24, we immediately get that $d(w_{\mathcal{T}_0}(C_0), w_{\mathcal{T}_0}(C')) \geq \frac{\beta}{3} + \text{Rk}(\mathcal{T}_0|C') + 2N_{\text{mo}}$ for each $C' \in \mathcal{C}^{\subseteq G_0}$. Possibly, C_0 is a closed $\mathcal{T}_0$-ring (i.e x, y are the endpoints of $\mathbf{P}_{\mathcal{T}}(C)$). In that case, since $\text{Rk}(\mathcal{T}_0|C_0) < \text{Rk}(\mathcal{T}|C)$, we again get that $\mathcal{T}_0$ satisfies M3) by Subclaim 2.1.24. Thus, $\mathcal{T}_0$ is a mosaic, as desired.

Case 2: C_0 is not the outer face of G_0

In this case, C_1 separates $V(G_1 \setminus C_1)$ from C_*, and $C_* \subseteq G_0$, so we have $C_*^0 = C_*$. To show that $\mathcal{T}_0$ is a mosaic, just suffices to check that M3) and M4) hold. If M3) does not hold, then we have $d(w_{\mathcal{T}_0}(C_0), w_{\mathcal{T}_0}(C_*)) < \frac{\beta}{3} + \text{Rk}(\mathcal{T}_0|C_0) + \text{Rk}(\mathcal{T}|C_*)$. As above, since C is an open $\mathcal{T}$-ring, we have $\text{Rk}(\mathcal{T}_0|C_0) \leq \text{Rk}(\mathcal{T}|C)$, and since $\text{Rk}(\mathcal{T}_0|C_*) = \text{Rk}(\mathcal{T}|C_*)$, it follows from Subclaim 2.1.24 that $d(w_{\mathcal{T}}(C), w_{\mathcal{T}}(C_*)) < \frac{\beta}{3} + \text{Rk}(\mathcal{T}|C_*) + 2N_{\text{mo}}$, contradicting the fact that $\mathcal{T}$ is a tessellation. The same argument shows that, for each $C' \in \mathcal{C}^{\subseteq G_0} \setminus \{C_*\}$, we have $d(w_{\mathcal{T}_0}(C_0), w_{\mathcal{T}_0}(C')) \geq \beta + \text{Rk}(\mathcal{T}_0|C_0) + \text{Rk}(\mathcal{T}_0|C')$. Thus, $\mathcal{T}_0$ is a mosaic, as desired. $\blacksquare$

Since $\mathcal{T}_0$, since $|V(G_0)| < |V(G)|$, G_0 admits an L-coloring ϕ by the minimality of $\mathcal{T}$. Let C_*^1 be the outer face of G_1. Consider the oriented tessellation $\mathcal{T}_1 := (G_1, \mathcal{C}^{\subseteq G_1} \cup \{C_1\}, L_\phi^{xy}, C_*^1)$. Since C_1 is a cycle with a precolored path of length one in $\mathcal{T}_1$, C_1 is an open $\mathcal{T}_1$-ring. Analogous to Subclaim 2.1.24, we have the following:

Subclaim 2.1.25. *For each $C' \in \mathcal{C}^{\subseteq G_1}$, we have $d(w_{\mathcal{T}_1}(C'), (w_{\mathcal{T}_1}(C_1)) \geq d(w_{\mathcal{T}}(C'), w_{\mathcal{T}}(C))$.*

Proof: This is an immediate consequence of the fact that, for any $C \in \mathcal{C}^{G^1}$ and any subgraph $H \subseteq C$, any shortest (H, C)-path in G has its C-endpoint in C_1. $\blacksquare$

Now we claim that $\mathcal{T}_1$ is a mosaic. Since C_1 is an open $\mathcal{T}_1$-ring and C_1 has a precolored path of length one in $\mathcal{T}_1$, M0), M1), and M2), are trivially satisfied, so as above, we just need to chek M3) and M4). Consider the following cases:

Case 1: C_1 is the outer face of G_1

In this case, we have $C_1 = C_*^1$. To check that $\mathcal{T}_1$ is a mosaic, it just suffices to check that M3) holds. Since $\mathcal{T}$ satisfies M3) and M4), we get that $d(w_{\mathcal{T}}(C'), w_{\mathcal{T}}(C)) \geq \frac{\beta}{3} + \text{Rk}(\mathcal{T}|C') + 2N$. Let $C' \in \mathcal{C}^{\subseteq G_1}$. Since C_1 is an open $\mathcal{T}_1$-ring,we immediately get that $d(w_{\mathcal{T}_1}(C_1), w_{\mathcal{T}_1}(C')) \geq \frac{\beta}{3} + \text{Rk}(\mathcal{T}_1|C') + 2N_{\text{mo}}$ by applying Subclaim 2.1.24, so $\mathcal{T}_1$ is indeed a mosaic in this case.

Case 2: C_1 is not the outer face of G_1

In this case, C_0 separates $V(G_0 \setminus C_0)$ from C_*, and $C_* \subseteq G_1$, so we have $C_*^1 = C_*$. To show that $\mathcal{T}_1$ is a mosaic, just suffices to check M3) and M4). If M3) does not hold, then we have $d(w_{\mathcal{T}_1}(C_1), w_{\mathcal{T}_1}(C_*)) < \frac{\beta}{3} + \text{Rk}(\mathcal{T}_1|C_1) + (\mathcal{T}|C_*)$. Since $\text{Rk}(\mathcal{T}_1|C_1) = 2N_{\text{mo}}$, and $\text{Rk}(\mathcal{T}_1|C_*) = \text{Rk}(\mathcal{T}|C_*)$, it follows from Subclaim 2.1.25 that $d(w_{\mathcal{T}}(C), w_{\mathcal{T}}(C_*)) < \frac{\beta}{3} + \text{Rk}(\mathcal{T}|C_*) + 2N_{\text{mo}}$, contradicting the fact that $\mathcal{T}$ is a tessellation. The same argument shows that, for each $C' \in \mathcal{C}^{\subseteq G_1} \setminus \{C_*\}$, we have $d(w_{\mathcal{T}_1}(C_1, w_{\mathcal{T}_1}(C')) \geq \beta + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}_1|C')$. Thus, $\mathcal{T}_1$ is a mosaic, as desired

Since $\mathcal{T}_1$ is a mosaic and $|V(G_1)| < |V(G)|$, G_1 admits an L_ϕ^{xy}-coloring by the minimality of $\mathcal{T}$, so ϕ extends from G_0 to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is a critical mosaic. $\square$

Now we prove our main results for Section 2.1. We establish some very useful bounds on the distance between separating cycles and rings in a critical mosaic. The first of our two main results for Section 2.1 is the following:

Theorem 2.1.26. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic. Then for any cycle $D \subseteq G$, if $|V(D)| \leq N_{\text{mo}}$ and D separates an element of $\mathcal{C}$ from an edge of $E(C_*)$, then there exists a $C \in \mathcal{C}$ with $C \subseteq \text{Int}(D)$ such that $\max\{d(v, w_{\mathcal{T}}(C)) : v \in V(D)\} < \frac{\beta}{3} + \frac{3}{2}|V(D)| + \text{Rk}(\mathcal{T}|C)$.*

Proof. Given a cycle $D \subseteq G$, we say that D is *bad* if $|V(D)| \leq N_{\text{mo}}$, D separates an element of $\mathcal{C} \setminus \{C_*\}$ from C_*, and, for all $C \in \mathcal{C} \setminus \{C_*\}$ with $C \subseteq \text{Int}(D)$, we have $\max\{d(v, w_{\mathcal{T}}(C)) : v \in V(D)\} \geq \frac{\beta}{3} + \frac{3}{2}|V(D)| + \text{Rk}(\mathcal{T}|C)$.

Suppose towards a contradiction that G contains a bad cycle D, and, among all bad cycles in G, we choose D so as to minimize $|V(\text{Int}(D))$. Since D separates an element of $\mathcal{C} \setminus \{C_*\}$ from C_*, and D is a bad cycle, there exists a $C^\dagger \in \mathcal{C} \setminus \{C_*\}$ such that $d(D, C^\dagger) \geq \frac{\beta}{3}$. This is immediate if $C^\dagger$ is a closed $\mathcal{T}$-ring, and, if $C^\dagger$ is an open $\mathcal{T}$-ring, then, since each vertex of $\mathbf{P}_{\mathcal{T}}(C^\dagger)$ is of distance at most $\lfloor \frac{|V(\mathbf{P}_{\mathcal{T}}(C^\dagger)|}{2}$ from $C^\dagger \setminus \mathring{\mathbf{P}}_{\mathcal{T}}(C^\dagger)$, we have $d(D, C^\dagger) \geq \frac{\beta}{3}$. Since $\frac{N_{\text{mo}}^2}{5} \leq \frac{\beta}{3}$, the graph $\text{Ext}^+(D)$ is L-colorable by Proposition 2.1.20. Let ϕ be an L-coloring of $\text{Ext}^+(D)$. Let $G^\dagger := \text{Int}(D)$. We show now that $\mathcal{T}^\dagger := (G^\dagger, \{D\} \cup \mathcal{C}^{\subseteq G^\dagger}, L_\phi^D, D)$ is a mosaic, where D is the precolored subgraph of D.

Claim 2.1.27. *D is a highly predictable induced $\mathcal{T}^\dagger$-ring.*

Proof: We first show that, for any vertex z, if $z \in V(G^\dagger \setminus D)$ and z has a neighbor in D, then $G^\dagger[N(z) \cap V(D)]$ is a subpath of D of length at most two. Let $D := v_1 \cdots v_r$ for some $5 \leq r \leq N_{\text{mo}}$. We have $r \geq 5$ since G is short-separation-free. Suppose toward a contradiction that there is a $w^* \in V(G^\dagger \setminus D)$ such that w^* has neighbors $v_s, v_t \in V(D)$, where $|s-t| > 2$. Let $G^\dagger = H_1 \cup H_2$, where $H_1 \cap H_2 = v_s w^* v_t$, where H_1 is bounded by outer face $D_1 := v_s v_{s+1} \cdots v_t w^*$ and H_2 is bounded by outer face $D_2 := v_s v_{s-1} \cdots v_t w^*$. Since $|s-t| > 2$, we have $|V(D_i)| < |V(D)|$ for each $i = 1, 2$.

Without loss of generality, let $C^\dagger \subseteq \text{Int}(D_1)$. Thus, D_1 separates an element of $\mathcal{C} \setminus \{C_*\}$ from C_*, and $|V(D_1)| \leq N_{\text{mo}}$. We claim now that D_1 is also a bad cycle. Let $C \in \mathcal{C} \setminus \{C_*\}$ with $C \subseteq \text{Int}(D_1)$. For any subgraph $H \subseteq C$, we have $\max\{d(H,v) : v \in V(D_1)\} \geq \max\{d(H,v) : v \in V(D)\} - 1$. Thus, since $|V(D_1)| < |V(D)|$ and D is a bad cycle, D_1 is also a bad cycle. Since $|V(\text{Int}(D_1)| < |V(\text{Int}(D))|$, this contradicts our assumption.

Thus, we have $|s-t| \leq 2$. If $|s-t| = 2$, then, letting $t = s+2$, w^* is adjacent to each of v_s, v_{s+1}, v_{s+2} by our triangulation conditions, since G is short-separation-free. Thus, in any case, for each $w \in D_1(D, G^\dagger)$, we get that $G^\dagger[N(w) \cap V(D)]$ is a subpath of D of length at most two. To show that D is a highly predictable facial subgraph of $G^\dagger$, it suffices to show that there exists at most one vertex $w \in D_1(D, G^\dagger)$ such that $G^\dagger[N(w) \cap V(D)]$ is a subpath of D of length at precisely two. Suppose toward a contradiction that there are two such vertices w_1, w_2. Without loss of generality, let $G^\dagger[N(w_1) \cap V(D)] = v_1 v_2 v_3$. Thus, there exists an $s \in \{1, \cdots, r\}$ with $s \neq 1$ such that $G^\dagger[N(w_2) \cap V(D)] = v_s v_{s+1} v_{s+2}$.

Let $D' := v_1 w_1 v_3 \cdots v_r$ and let $D'' := v_1 \cdots v_s w_2 v_{s+2} \cdots v_r$. Since G is short-separation-free, we have $\text{Int}(D') = \text{Int}(D) \setminus \{v_2\}$. By the minimality of D, D' is not a bad cycle, so there exists a $C' \in \mathcal{C}$ with $C' \subseteq \text{Int}(D')$ such that $\max\{d(v, w_{\mathcal{T}}(C')) : v \in V(D')\} < \frac{\beta}{3} + \frac{3}{2}|V(D')| + \text{Rk}(\mathcal{T}|C')$. Thus, since $|V(D)| = |V(D')|$, v_2 is the unique vertex of maximal distance from $w_{\mathcal{T}}(C')$ among all vertices of D, and we have $\max\{d(v, w_{\mathcal{T}}(C')) : v \in V(D)\} = d(v_2, w_{\mathcal{T}}(C')) = \frac{\beta}{3} + \frac{3}{2}|V(D)| + \text{Rk}(\mathcal{T}|C')$, and $d(v_{s+1}, w_{\mathcal{T}}(C')) < \max\{d(v, w_{\mathcal{T}}(C')) : v \in V(D)\}$.

Since G is short-separation-free, we have $\text{Int}(D'') = \text{Int}(D) \setminus \{v_{s+1}\}$. Thus, since $v_2 \in V(D'')$, we have $\max\{d(v, w_{\mathcal{T}}(C'))) : v \in V(D")\} = \frac{\beta}{3} + \frac{3}{2}|V(D'')| + \text{Rk}(\mathcal{T}|C')$. By the minimality of D, D'' is not a bad cycle, so there exists a $C'' \in \mathcal{C}$ with $C'' \neq C'$ and $C'' \subseteq \text{Int}(D'')$ such that $\max\{d(v, w_{\mathcal{T}}(C'')) : v \in V(D'')\} < \frac{\beta}{3} + \frac{3}{2}|V(D'')| + \text{Rk}(\mathcal{T}|C'')$. But then we have $d(C', C'') < \frac{2\beta}{3} + 7N_{\text{mo}}$. Since $7N_{\text{mo}} < \frac{\beta}{3}$, this contradicts Observation 2.1.8. ■

Now we return to the proof of Theorem 2.1.26. By Claim 2.1.27, $\mathcal{T}^\dagger$ satisfies M2), since D is highly predictable and thus D is L^D_ϕ-predictable. M0) and M1 are immediate, so, to show that $\mathcal{T}^\dagger$ is a mosaic, it suffices to show that $\mathcal{T}^\dagger$ satisfies the distance conditions of Definition 2.1.6.

Suppose toward a contradiction that $\mathcal{T}^\dagger$ does not satisfy the distance conditions of Definition 2.1.6. Since D is the outer face of $G^\dagger$, there exists a $C \in \mathcal{C}^{\subseteq G^\dagger}$ such that $d(w_{\mathcal{T}^\dagger}(C), D) < \frac{\beta}{3} + \text{Rk}(\mathcal{T}^\dagger|C' + |V(D)|$. Since $w_{\mathcal{T}^\dagger}(C) = w_{\mathcal{T}}(C)$ and C has the same rank in $\mathcal{T}$ and $\mathcal{T}^\dagger$, we have $d(w_{\mathcal{T}}(C), D) < \frac{\beta}{3} + \text{Rk}(\mathcal{T}|C) + |V(D)|$. Since any two vertices of D are of distance at most $\frac{|V(D)|}{2}$ apart, we have $\max\{d(v, w_{\mathcal{T}}(C)) : v \in V(D)\} < \frac{\beta}{3} + \text{Rk}(\mathcal{T}|C) + \frac{3}{2}|V(D)|$, contradicting the fact that D is bad. We conclude that $\mathcal{T}^\dagger$ does indeed satisfy the distance conditions of Definition 2.1.6, so $\mathcal{T}^\dagger$ is a mosaic. Since $|V(G^\dagger)| < |V(G)|$, $G^\dagger$ admits an L^D_ϕ-coloring by the minimality of $\mathcal{T}$, and thus G is L-colorable, contradicting our assumption. Thus there does not exist a bad cycle in G. This completes the proof of Theorem 2.1.26. □

Analogous to the above, we have the following lower bounds. This is the second of two main results for Section 2.1.

Theorem 2.1.28. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic. For any cycle $D \subseteq G$, if $|V(D)| \leq N_{\text{mo}}$ and D separates an element of $\mathcal{C}$ from an edge of $E(C_*)$, then, for each $C \in \mathcal{C}$ with $C \subseteq \text{Int}(D)$, we have $d(w_{\mathcal{T}}(C), D) > \text{Rk}(\mathcal{T}|C) - \frac{3}{2}|V(D)|$.*

Proof. We follow a similar argument to that of Theorem 2.1.28. Given a cycle $D \subseteq G$, we say that D is *defective* if $|V(D)| \leq N_{\text{mo}}$, D separates an element of $\mathcal{C} \setminus \{C_*\}$ from C_*, and there exists a $C \in \mathcal{C}$ with $C \subseteq \text{Int}(D)$ such that $d(w_{\mathcal{T}}(C), D) \leq \text{Rk}(\mathcal{T}|C) - \frac{3}{2}|V(D)|$.

Suppose toward a contradiction that there exists a defective cycle D, and, among all defective cycles in G, we choose D so that $|V(\text{Ext}(D))$ is minimized. Let $G^\dagger := \text{Ext}(D)$ and let $C^\dagger$ be a $\mathcal{T}$-ring with $C^\dagger \subseteq \text{Int}(D)$ with $d(w_{\mathcal{T}}(C^\dagger), D) \leq \text{Rk}(\mathcal{T}|C^\dagger) - \frac{3}{2}|V(D)|$.

Claim 2.1.29. *D is a highly predictable facial subgraph of* $\text{Ext}(D)$*, and an induced cycle of* $\text{Ext}(D)$

Proof: It is immediate from the minimality of D that there is no chord of D in $\text{Ext}(D)$. We now show that, for any vertex $x \in V(G^\dagger \setminus D)$, if x has a neighbor in D, then $G^\dagger[N(w) \cap V(D)]$ is a subpath of D of length at most two. Analogous to the upper bound argument above, if this does not hold, then there exists a cycle $D' \subseteq \text{Ext}(D)$ such that $|V(D')| < |V(D)|$, D' separates C_* from $C^\dagger$, and $d(w_{\mathcal{T}}(C^\dagger), D') \leq d(w_{\mathcal{T}}(C^\dagger), D) + 1$. Thus, D' is also a defective cycle, and since $|V(\text{Ext}(D'))| < |V(\text{Ext}(D))|$, this contradicts our assumption.

Let $D := v_1 \cdots v_r$. To show that D is a highly predictable facial subgraph of $G^\dagger$, it suffices to show that there exists at most one vertex $w \in D_1(D, G^\dagger)$ such that $G^\dagger[N(w) \cap V(D)]$ is a subpath of D of length at precisely two Suppose toward a contradiction that there are two such vertices w_1, w_2. Without loss of generality, let $G^\dagger[N(w_1) \cap V(D)] = v_1v_2v_3$. Thus, there exists an $s \in \{1, \cdots, r\}$ with $s \neq 1$ such that $G^\dagger[N(w_2) \cap V(D)] = v_sv_{s+1}v_{s+2}$.

Let $D' := v_1w_1v_3 \cdots v_r$ and let $D'' := v_1 \cdots v_sw_2v_{s+2} \cdots v_r$. Since G is short-separation-free, we have $\text{Ext}(D') = \text{Ext}(D) \setminus \{v_2\}$. By the minimality of D, D' is not a defective cycle, so we have $d(w_{\mathcal{T}}(C^\dagger), D') > \text{Rk}(C^\dagger) - \frac{3}{2}|V(D')|$. Since $|V(D')| = |V(D)|$, and D is a defective cycle, v_2 is the unique vertex of D of minimal distance to $w_{\mathcal{T}}(C^\dagger)$ among the vertices of D, and thus $d(w_{\mathcal{T}}(C^\dagger), D'') \leq \text{Rk}(C^\dagger) - \frac{3}{2}|V(D)|$. Since $|V(D'')| = |V(D)|$, D'' is also a defective cycle, and since $v_{s+1} \notin V(\text{Ext}(D''))$, this contradicts the minimality of D. ■

By Proposition 2.1.21, there is an L-coloring ϕ of $V(\text{Int}(D))$. Now let $\mathcal{C}^\dagger := \{D\} \cup \mathcal{C}^{\subseteq \text{Ext}(D)}$ and set $\mathcal{T}^\dagger := (\text{Ext}^+(D), \mathcal{C}^\dagger, L_\phi^D, C_*)$. Now consider the tuple $\mathcal{T}^* := (\text{Ext}(D), \{D\} \cup \mathcal{C}^{\subseteq G^\dagger}, L_\phi^D, C_*)$. We claim that $\mathcal{T}^*$ is a mosaic. Since D is a highly predictable cyclic facial subgraph of $\text{Ext}(D)$, D is also an L-predictable closed $\mathcal{T}^*$-ring, so $\mathcal{T}^*$ satisfies M2), and M0), M1) are immediate.

Suppose now that $\mathcal{T}^*$ is not a mosaic. In that case, there exists a $C \in \mathcal{C}^{\subseteq G^\dagger}$ such that $d(w_{\mathcal{T}^*}(C), D)$ either violates M3) or M4) of Definition 2.1.6. Let $\beta^* \in \{\frac{\beta}{3}, \beta\}$, where $\beta^* = \frac{\beta}{3}$ if $C = C_*$ and otherwise $\beta^* = \beta$. Then we have $d(w_{\mathcal{T}^*}(C), D) < \beta^* + \text{Rk}(\mathcal{T}^*|C) + |V(D)|$. Since $w_{\mathcal{T}^*}(C) = w_{\mathcal{T}}(C)$ and C has the same rank in $\mathcal{T}$ and $\mathcal{T}^*$, we have $d(w_{\mathcal{T}}(C), D) < \beta^* + \text{Rk}(\mathcal{T}|C) + |V(D)|$. Since D is defective and any two vertices of D are of distance at most $\frac{|V(D)|}{2}$ apart, we then have $d(w_{\mathcal{T}}(C), w_{\mathcal{T}}(C^\dagger) < \beta^* + \text{Rk}(\mathcal{T}|C) + \text{Rk}(\mathcal{T}|C^\dagger)$.

Since $\mathcal{T}$ and $\mathcal{T}^*$ have the same outer face, this inequality contradicts the fact that $\mathcal{T}$ is a mosaic. Thus, our assumption that $\mathcal{T}^*$ is not a mosaic is false. Since $\mathcal{T}^*$ is a mosaic and $|V(G^*)| < |V(G)|$, G^* admits an L_ϕ^D-coloring by the minimality of $\mathcal{T}$. But then ϕ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. This completes the proof of Theorem 2.1.28. □

To conclude Section 2.1, we have the following useful corollary to the upper bounds in Theorem 2.1.26 and the lower bounds in Theorem 2.1.28.

Corollary 2.1.30. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let $C \in \mathcal{C}$, and let $D \subseteq G$ be a separating cycle. Suppose further that $|V(D)| \leq N_{\text{mo}}$ and $d(D, w_{\mathcal{T}}(C)) \leq \text{Rk}(C) - \frac{3}{2}|V(D)|$. Then there does not exist a $C' \in \mathcal{C}$ with $C' \subseteq \text{Int}(D)$.*

Proof. Suppose toward a contradiction that there exists a $C' \in \mathcal{C}$ with $C' \subseteq \text{Int}(D)$. By Theorem 2.1.28, we have $C \neq C'$. Thus, by Theorem 2.1.26, there exists a $C'' \subseteq \text{Int}(D)$ such that $\max\{d(v, w_{\mathcal{T}}(C'')) : v \in V(D)\} < \frac{\beta}{3} + \frac{3}{2}|V(D) + \text{Rk}(\mathcal{T}|C'')$, so $d(w_{\mathcal{T}}(C), w_{\mathcal{T}}(C'')) < \frac{\beta}{3} + \text{Rk}(C) + \text{Rk}(C'')$, contradicting the distance conditions of Definition 2.1.6. $\square$

2.2 Short Generalized Chords of the Rings of Critical Mosaics

The purpose of the remaining sections of Chapter 2 is to show that, for sufficiently small values of k, there are no k-chords of any of the rings of a critical mosaic which separate the remaining rings of the mosaic. We first introduce the following notation, which we use throughout the remainder of the proof of Theorem 1.1.3.

Definition 2.2.1. Let $\mathcal{T} = (G, \mathcal{C}, L)$ be a chart, and let $\ell \geq 1$ be an integer. For each $C \in \mathcal{C}$, we let $\mathcal{K}^{\ell}C, \mathcal{T})$ be the set of proper ℓ-chords Q of C such that neither endpoint of Q is an internal vertex of the precolored path $\mathbf{P}_{\mathcal{T}}(C)$. Furthermore, given a $Q \in \mathcal{K}^{\ell}(C, \mathcal{T})$, we let G_Q^0, G_Q^1 denote the subgraphs of G such that $G_Q^0 \cup G_Q^1 = G$ is the natural (C, Q)-partition of G, where $\mathbf{P}_{\mathcal{T}}(C) \subseteq G_Q^0$.

Note that if $|E(\mathbf{P}_{\mathcal{T}}(C))| \leq 1$, then, for every $\ell \geq 1$, every k-chord of C lies in $\mathcal{K}^{\ell}(C, \mathcal{T})$. On the other hand, if C is a closed $\mathcal{T}$-ring, then, for every $\ell \geq 1$, $\mathcal{K}^{\ell}(C, \mathcal{T}) = \varnothing$. We are particularly interested in those paths of $\mathcal{K}^{\ell}(C, \mathcal{T})$ for which all the rings in $\mathcal{C} \setminus \{C\}$ lie on the same side as the precolored path of C.

Definition 2.2.2. Let $\mathcal{T} = (G, \mathcal{C}, L)$ be a chart, and let $C \in \mathcal{C}$ and $Q \in \mathcal{K}(C, \mathcal{T})$. We say that Q is $\mathcal{T}$-*non-separating* if $C' \subseteq G_Q^0$ for each $C' \in \mathcal{C} \setminus \{C\}$. Otherwise, we say that Q is $\mathcal{T}$-*separating*. If the underlying chart $\mathcal{T}$ is clear from the context then we drop the symbol $\mathcal{T}$ and say that Q is non-separating or separating respectively.

In some cases, we analyze ℓ-chords of a $C \in \mathcal{C}$ in which the precise value of ℓ is not relevant. Thus, we also introduce the notation $\mathcal{K}(C, \mathcal{T}) := \bigcup_{\ell \geq 1} \mathcal{K}^{\ell}(C, \mathcal{T})$. Lastly, we introduce the following natural definition:

Definition 2.2.3. Let $\mathcal{T} = (G, \mathcal{C}, L)$ be a chart, let $C \in \mathcal{C}$ and let Q be a generalized chord of C in G with $Q \in \mathcal{K}(C, \mathcal{T})$. Then we define two cycles C_Q^0 and C_Q^1 of G, where $C_Q^0 := (C \cap G_Q^0) + Q$ and $C_Q^1 := (C \cap G_Q^1) + Q$.

We now state and prove our lone main result for Section 2.2. The remainder of Section 2.2 consists of the proof of this result and its corollary.

Theorem 2.2.4. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let $C \in \mathcal{C}$. Let Q be a generalized chord of C and let $G = G_0 \cup G_1$ be the natural Q-partition of G. Then the following hold.*

1) *If C is a closed $\mathcal{T}$-ring and Q is a proper generalized chord of C with $|E(Q)| \leq \frac{N_{\text{mo}}}{3}$, then there exists a $j \in \{0, 1\}$ such that $C' \subseteq G_j$ for each $C' \in \mathcal{C} \setminus \{C\}$; AND*
2) *If C is a closed $\mathcal{T}$-ring and Q is not a proper generalized chord of C (i.e Q is a cycle), with $|E(Q)| < \frac{N_{\text{mo}}}{3}$, then there exists a $j \in \{0, 1\}$ such that $C' \subseteq G_j$ for each $C' \in \mathcal{C} \setminus \{C\}$; AND*
3) *If C is an open $\mathcal{T}$-ring, $Q \in \mathcal{K}(C, \mathcal{T})$, and $|E(Q)| \leq \frac{2N_{\text{mo}}}{3}$, then, for each $C' \in \mathcal{C} \setminus \{C\}$, $C' \subseteq G_Q^0$; AND*

4) *If C is an open $\mathcal{T}$-ring with $\mathbf{P} := \mathbf{P}_{\mathcal{T}}(C)$), Q is a proper generalized chord of C, and precisely one endpoint of Q lies in $V(\mathbf{P})$, then, for each $j \in \{0,1\}$, at least one of the following holds: Either $|E(\mathbf{P}_{\mathcal{T}}(C) \cap G_j)| + |E(Q)| > \frac{2N_{\text{mo}}}{3}$ or, for each $C' \in \mathcal{C} \setminus \{C\}$, $C \subseteq G_{1-j}$.*

In the remainder of Section 2.2, we prove several lemmas which we then combine to prove Theorem 2.2.4. We begin with the following:

Lemma 2.2.5. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic. If $C \in \mathcal{C}$ is a closed $\mathcal{T}$-ring and Q is a proper generalized chord of C with $|E(Q)| \leq \frac{N_{\text{mo}}}{3}$. Let $G = G_0 \cup G_1$ be the natural Q-partition of G. Then there exists a $j \in \{0,1\}$ such that, for all $C' \in \mathcal{C} \setminus \{C\}$, $C' \subseteq G_j$.*

Proof. Let $C \in \mathcal{C}$ be a closed ring. Given a path $Q \subseteq G$, we say that Q is *unacceptable* if Q is a proper generalized chord of C such that the following hold.

1) $1 \leq |E(Q)| \leq \frac{N_{\text{mo}}}{3}$; *AND*
2) There exists a pair of rings $D_0, D_1 \in \mathcal{C} \setminus \{C\}$ such that, letting $G = G_0 \cup G_1$ be the natural Q-partition of G, we have $D_i \subseteq G_i$ for each $i = 0, 1$.

Thus, it suffices to show that there does not exist an unacceptable path in G. Consider the following cases.

Case 1: $C = C_*$

In this case, let $G = G_{-} \cup G_1$ be the natural Q-partition of G, where $G_0 \cap G_{‘} = Q$. Let C_*^i be the outer face of G_i for each $i = 0, ‘$. Since Q is a proper generalized chord of C_*, it follows from Proposition 2.1.12 that C_*^i is a cyclic facial subgraph of G_i for each $i = 0, ‘$, and we have $|E(C_*^0)| + |E(C_*^1)| = |E(C_*)| + 2|E(Q)| \leq \frac{5N}{3}$, so there exists an $i \in \{0,1\}$ with $|E(C_*^i)| \leq \frac{2N_{\text{mo}}}{3}$, say $i = 0$ without loss of generality.

Since Q is an unacceptable path, C_*^0 is a cycle of G which separates a ring of $\mathcal{C} \setminus \{C_*\}$ from an edge of $E(C_*)$. By Theorem 2.1.26, there exists a $C^\dagger \in \mathcal{C}$ with $C^\dagger \subseteq \text{Int}(C_*^0)$ such that $\max\{d(v, w_{\mathcal{T}}(C^\dagger) : v \in V(C_*^0)\} < \frac{\beta}{3} + \text{Rk}(\mathcal{T}|C^\dagger)| + \frac{3}{2}|V(C_*^0)|$. For each $i = 0, 1$, let $r_i := E(C_*^i \setminus Q)|$, and let $\ell = |E(Q)|$.

Claim 2.2.6. $r_1 + \ell > N_{\text{mo}}$.

<u>Proof:</u> Suppose toward a contradiction that $r_1 + \ell \leq N_{\text{mo}}$. In that case, since C_*^1 separates an element of $\mathcal{C} \setminus \{C_*\}$ from an edge of $E(C_*^0)|$, and $|E(C_*^1)| \leq N_{\text{mo}}$, it follows from Theorem 2.1.26 that there exists a $C^{\dagger\dagger} \in \mathcal{C} \setminus \{C\}$ with $C^{\dagger\dagger} \subseteq \text{Int}(C_*^1)$ and $\max\{d(v, w_{\mathcal{T}}(C^{\dagger\dagger}) : v \in V(C_*^1)\} < \frac{\beta}{3} + \text{Rk}(\mathcal{T}|C^{\dagger\dagger})| + \frac{3}{2}|V(C_*^1)|$. Since $C^{\dagger\dagger}$ is a cycle contained in $\text{Int}(C_*^1)$, we have $C^\dagger \neq C^{\dagger\dagger}$, and since $C_*^0 \cap C_*^1 = Q$, we have

$$d(C^\dagger, C^{\dagger\dagger}) < \frac{2\beta}{3} + \frac{3}{2}|V(C_*^0)| + \frac{3}{2}|V(C_*^1)| + 4N_{\text{mo}} < 2\left(\frac{\beta}{3} + \frac{7}{2}N_{\text{mo}}\right)$$

Since $7N_{\text{mo}} < \frac{\beta}{3}$, this contradicts 1) of Observation 2.1.8. Thus, we have $r_1 + \ell > N_{\text{mo}}$. ■

Since C_* is a closed $\mathcal{T}$-ring in this case, we have $d(C_*, w_{\mathcal{T}}(C^\dagger)) \geq \frac{\beta}{3} + |V(C_*)| + \text{Rk}(\mathcal{T}|C^\dagger)$. On the other hand, we have $\max\{d(v, w_{\mathcal{T}}(C^\dagger) : v \in V(C_*^0)\} < \frac{\beta}{3} + \frac{3}{2}|V(C_*^0)| + \text{Rk}(\mathcal{T}|C^\dagger)$ so we have $\frac{3}{2}|V(C_*^0)| > |V(C_*)|$. Thus, we obtain $\frac{3}{2}(r_0 + \ell) > r_0 + r_1$.

Since $\frac{3}{2}(r_0+\ell) > r_0+r_1$, we have $\frac{r_0}{2} + \frac{3}{2}\ell > r_1$. Furthermore, we have $\ell > r_0$, or else if $r_0 \geq \ell$, then, by Claim 2.2.6, we have $|E(C_*)| = r_0 + r_1 > N_{\text{mo}}$, which is false. Thus, we have $2\ell > r_1$. Since $\ell \leq \frac{N_{\text{mo}}}{3}$, we obtain $r_1 < \frac{2N_{\text{mo}}}{3}$. But then $r_1 + \ell \leq N_{\text{mo}}$, contradicting Claim 2.2.6. This completes the case where $C = C_*$.

Case 2: $C \neq C_*$

In this case, for any proper generalized chord Q of C with $1 \leq |E(Q)| \leq N_{\text{mo}}$, there exists a partition $G = G_{\text{in}} \cup G_{\text{out}}$ of G', and a pair of cycles C_{in}, C_{out}, such that $G_{\text{in}} \cap G_{\text{out}} = Q$, $G_{\text{in}} = \text{Int}(C_{\text{in}})$, and $G_{\text{out}} = \text{Ext}(C_{\text{out}})$. Among all unacceptable paths of C, choose Q so that $|V(G_{\text{out}})|$ is minimized. Since Q is an unacceptable path in G, let $D_{\text{in}} \in \mathcal{C} \setminus \{C\}$, where $D_{\text{in}} \subseteq \text{Int}(C_{\text{in}})$. Note that $C \subseteq \text{Ext}(C_{\text{in}})$.

Claim 2.2.7. $|V(C_{\text{in}})| > N_{\text{mo}}$ *and* $|V(C_{\text{out}})| < \frac{2N_{\text{mo}}}{3}$.

<u>Proof:</u> Suppose toward a contradiction that $|V(C_{\text{in}})| \leq N_{\text{mo}}$. Since $D_{\text{in}} \subseteq \text{Int}(C_{\text{in}})$ and $C \subseteq \text{Ext}(C_{\text{in}})$, it follows from Theorem 2.1.26 that there exists a $D^\dagger \in \mathcal{C} \setminus \{C\}$ with $D^\dagger \subseteq \text{Int}(C_{\text{in}})$ such that $\max\{d(v, w_{\mathcal{T}}(D^\dagger) : v \in V(C_{\text{in}}\} < \frac{\beta}{3} + \frac{3}{2}|V(C_{\text{in}})| + \text{Rk}(\mathcal{T}|D^\dagger)$. Thus, we have $d(C, w_{\mathcal{T}}(D^\dagger)) < \frac{\beta}{3} + \frac{7}{2}N_{\text{mo}} < \beta$, contradicting Observation 2.1.8.

Thus, we have $|V(C_{\text{in}})| > N_{\text{mo}}$, as desired. Now suppose toward a contradiction that $|V(C_{\text{out}})| \geq \frac{2N_{\text{mo}}}{3}$. In that case, we have $|E(C_{\text{out}}) \setminus E(Q)| > \frac{N_{\text{mo}}}{3}$. Likewise, since $|E(C_{\text{in}})| > N_{\text{mo}}$, we have $|E(C_{\text{in}}) \setminus E(Q)| > \frac{2N_{\text{mo}}}{3}$, so $|E(C)| > N_{\text{mo}}$, which is false. ■

Now, we have $C \subseteq \text{Int}(C_{\text{out}})$, and C_{out} separates D_{in} from C_*, since $\text{Int}(C_{\text{in}}) \subseteq \text{Int}(C_{\text{out}})$. Since C and C_{out} share a vertex, it immediately follows from Proposition 2.1.21 that $V(\text{Int}(C_{\text{out}}))$ is L-colorable, so let ϕ be an L-coloring of $V(\text{Int}(C_{\text{out}}))$. Let $G^* := \text{Ext}(C_{\text{out}})$ consider the chart $\mathcal{T}^* := (G^*, \{C_{\text{out}}\} \cup \mathcal{C}^{\subseteq G^*}, L_\phi^{C_{\text{out}}}, C_*)$. We claim now that $\mathcal{T}^*$ is a mosaic. Firstly, $\mathcal{T}^*$ is a tessellation in which C_{out} is a closed ring. By Claim 2.2.7 we have $|E(C_{\text{out}})| \leq N_{\text{mo}}$, so M0) is satisfied, and M1) is immediate. We now have the following:

Claim 2.2.8. C_{out} *is an L-predictable $\mathcal{T}^*$-ring.*

<u>Proof:</u> Let $C := v_1 \cdots v_n$ and suppose without loss of generality that v_1 is an endpoint of Q. Since Q is a proper generalized chord, let $1 < i \leq n$, where $Q := v_1 u_1 \cdots u_{\ell-1} v_i$. Without loss of generality, let $C_{\text{in}} := v_1 Q v_i v_i v_{i-1} \cdots v_1$ and let $C_{\text{out}} := v_1 Q v_i v_{i+1} \cdots v_1$. We first show that, for any $w \in V(G^* \setminus C_{\text{out}})$, if w has a neighbor $u \in \{u_1, \cdots, u_{\ell-1}\}$, then for any vertex $y \in N(w) \cap V(C_{\text{out}})$, if $y \neq u$, then uy is an edge of C_{out}.

Suppose toward a contradiction that there exists a vertex $w' \in V(G^* \setminus C_{\text{out}})$, a neighbor u of w', with $u' \in \{u_1, \cdots u_{\ell-1}\}$, and a neighbor y of w' such that $y \in V(C_{\text{out}}) \setminus \{u\}$ but u, y are not adjacent vertices of C_{out}.

Let P_1, P_2 be the unique subpaths of C_{out} such that $E(P_1) \cup E(P_2) = E(C_{\text{out}})$ and P_1, P_2 intersect precisely on the vertices u, y. If $y \notin V(Q)$, then each of the two paths $ywuQv_1$, $wyuQv_i$ is a proper generalized chord of C of length at most $\frac{N_{\text{mo}}}{3}$, at least one of which is an unacceptable path, contradicting the minimality of Q. Thus, we have $y \in V(Q)$, so precisely one of P_1, P_2 is a subpath of Q, so suppose without loss of generality that $P_1 \subseteq Q$, and that $y \in V(uQv_i)$. Let $Q' := v_1 Q u w' y Q v_i$. Since u, y are not adjacent in C_{out}, we have $|E(Q')| \leq |E(Q)| \leq \frac{N_{\text{mo}}}{3}$. Note that Q' is a proper generalized chord of C with the same endpoints as Q. We claim now that Q' is an unacceptable path G. We just need to show that the generalized chord Q' of C separates D_{in} from C_*.

Suppose not. In that case, D_{in} lies in the closed disc bounded by the cycle $uQyw'$, and thus C also lies in the closed disc bounded by $uQyw'$. Furthermore, note that the two vertices u, y are of distance at least three apart on Q, or else $uQyw'$ is a cycle of length at most four which separates D_{in} from C_*, contradicting short-separation-freeness.

Now we apply Theorem 2.1.28. Let $A := uQyw'$. We have $|V(A)| \leq \frac{N_{\text{mo}}}{3} + 2$, and, by Claim 2.2.7, we have $|E(C_{\text{in}})| > N_{\text{mo}}$, and thus $|V(C)| \geq \frac{2N_{\text{mo}}}{3} + 1$. By Theorem 2.1.28, we have $d(A, C) > |V(C)| - \frac{3}{2}|V(A)|$, so we have $d(A, C) > (\frac{2N_{\text{mo}}}{3} + 1) - (\frac{N_{\text{mo}}}{2} + 3) = \frac{N_{\text{mo}}}{6} - 2$. However, since $|E(Q)| \leq \frac{N_{\text{mo}}}{3}$, and u, y are of distance at least three apart on Q, at least one of u, y is of distance at most $\frac{N_{\text{mo}}}{6} - 3$ from $\{v_1, v_i\}$, so we have a contradiction.

Thus, Q' is indeed an unacceptable path in G', contradicting the minimality of Q, so our original assumption on the vertex w' is false. We conclude that, for each vertex $w \in V(G^* \setminus C_{\text{out}})$, if w has a neighbor $u \in \{u_1, \cdots, u_{\ell-1}\}$, then for any vertex $y \in N(w) \cap V(C_{\text{out}})$, if $y \neq u$, then uy is an edge of C_{out}. An identical argument shows that there does not exist a chord xy of C_{out} in $E(G^*)$ such that xy has at least one endpoint in $\{u_1, \cdots, u_{\ell-1}\}$.

Now suppose toward a contradiction that C_{out} is not an L-predictable $\mathcal{T}^*$-ring. Combining the above facts, there exists a $w \in V(G^* \setminus C_{\text{out}})$ such that w is adjacent to each of v_1, u_1, v_i. By the minimality of Q, $v_1 w v_i$ is not an unacceptable path, and thus the 2-chord $v_1 w v_i$ of C does not separate D_{in} from C_*, and thus the cycle $v_1 w v_i u$ separates D_{in} from C_*, contradicting the fact that G is short-separation-free. This completes the proof of Claim 2.2.8. ■

Since C_{out} is an L-predictable cyclic facial subgraph of G^*, $\mathcal{T}^*$ also satisfied M2). To finish showing that $\mathcal{T}^*$ is a mosaic, it suffices to check that $\mathcal{T}^*$ satisfies the distance conditions of Definition 2.1.6.

Claim 2.2.9. $|V(C)| \geq |V(C_{\text{out}})| + \frac{|E(Q)}{2}$.

Proof: Let $r_{\text{in}} := |E(C_{\text{out}}) \setminus E(Q)|$ and let $r_{\text{out}} := |E(C_{\text{out}}) \setminus E(Q)|$. Suppose toward a contradiction that $|V(C)| < |V(C_{\text{out}})| + \frac{|E(Q)}{2}$. Then we have $r_{\text{in}} + r_{\text{out}} < r_{\text{out}} + \frac{|E(Q)}{2}$, so $r_{\text{in}} < \frac{|E(Q)|}{2}$. But then $|V(C_{\text{in}})| < \frac{3|E(Q)}{2}$ and thus $|V(C_{\text{in}})| < \frac{N_{\text{mo}}}{2}$, contradicting Claim 2.2.7. ■

Now, for any $C' \in \mathcal{C}^{\subseteq G^*}$, we have $d(w_{\mathcal{T}}(C'), C) \leq d(w_{\mathcal{T}}(C'), C_{\text{out}}) + \frac{|E(Q)}{2}$. For any such C', $w_{\mathcal{T}}(C') = w_{\mathcal{T}^*}(C')$, and C' has the same rank in $\mathcal{T}$ and $\mathcal{T}^*$. By Claim 2.2.9, we have $\text{Rk}(\mathcal{T}^*|C_{\text{out}}) \leq \text{Rk}(\mathcal{T}|C) - \frac{|E(Q)}{2}$. Thus, since $\mathcal{T}$ satisfies the distance conditions of Definition 2.1.6, and $\mathcal{T}$ and $\mathcal{T}^*$ have the same outer face, $\mathcal{T}^*$ also satisfies the distance conditions of Definition 2.1.6, so $\mathcal{T}^*$ is indeed a mosaic, as desired. Since $|V(\text{Ext}(C_{\text{out}}))| < |V(G)|$, $\text{Ext}(C_{\text{out}})$ is $L_\phi^{C_{\text{out}}}$-colorable by the minimality of $\mathcal{T}$. Thus, ϕ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. This completes the proof of Lemma 2.2.5. □

Lemma 2.2.5 has the following corollary.

Corollary 2.2.10. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let $C \in \mathcal{C}$. Then C is an induced cycle.*

Proof. By Proposition 2.1.22, it suffices to show that C is an induced cycle in G in the case where C is closed $\mathcal{T}$-ring. Suppose toward a contradiction that C is not an induced cycle, and let $xy \in E(G) \setminus E(C)$ be a chord of C. Let $G = G_0 \cup G_1$ be the natural xy-partition of G. Applying Lemma 2.2.5, let $C' \subseteq G_0$ for each $C' \in \mathcal{C} \setminus \{C\}$. Let $C_0 := (C \cap G_0) + xy$, let C_0^{out} be the outer face of G_0, and let $\mathcal{T}_0 := (G_0, \mathcal{C} \setminus \{C\} \cup \{C_0\}, L, C_0^{\text{out}})$. Note that, since $|V(C_0)| < |V(C)|$, and C_0 is a closed $\mathcal{T}_0$-ring, $\mathcal{T}_0$ satisfies the distance conditions of Definition 2.1.6. Thus, $\mathcal{T}_0$ is a mosaic, and since $|V(G_0)| < |V(G)|$, G_0 admits an L-coloring ϕ by the minimality of $\mathcal{T}$.

Let $C_1 := (G_1 \cap Q) + xy$. We claim now that ϕ extends to L-color G_1. By definition, we have $|L(p)| = 1$ for each $p \in V(C)$, and since $V(C)$ is L-colorable, ϕ extends to an L-coloring ϕ' of $V(G_0) \cup V(C_1)$. Since $|L(v)| \geq 5$ for all $v \in V(G_1) \setminus V(C_1)$, the graph $G_1 \setminus C_1$ contains a lone facial subgraph F such that F contains every vertex of $G_1 \setminus C_1$ with an $L_{\phi'}$-list of size less than five. Since C is L-predictable, there exists a vertex $w \in V(F)$ such that $|L_{\phi'}(v)| \geq 3$ for all $v \in V(F - w)$, and $|L_\phi(w)| \geq 2$. Thus, $G_1 \setminus C_1$ is $L_{\phi'}$-colorable, so ϕ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. □

The above work proves 1) of Theorem 2.2.4. With the lemma below, we prove 2) of Theorem 2.2.4.

Lemma 2.2.11. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let $C \in \mathcal{C}$ be a closed ring. For any generalized chord Q of C with $|E(Q)| < \frac{N_{\text{mo}}}{3}$ and $V(Q \cap C)| = 1$, the cycle Q does not separate two elements of $\mathcal{C} \setminus \{C\}$.*

Proof. Given a chord Q of C with $|E(Q)| < \frac{N_{\text{mo}}}{3}$ and $V(Q \cap C)| = 1$, we say that Q is *undesirable* if Q separates two elements of $\mathcal{C} \setminus \{C\}$. Suppose toward a contradiction that there exists an undesirable cycle Q. We now break the proof of Lemma 2.2.11 into two cases:

Case 1: $C = C_*$

In this case, choose Q so that, among all undesirable cycles, $|V(\text{Ext}(Q))|$ is minimized. Let v be the lone vertex of $V(Q \cap C)$. By assumption, there exist $D_1, D_2 \in \mathcal{C} \setminus \{C_*\}$ with $D_1 \subseteq \text{Int}(Q)$ and $D_2 \subseteq \text{Ext}(Q)$.

Claim 2.2.12.

1) Q does not have a chord in $\text{Ext}(Q)$; AND

2) There is no edge of $\text{Ext}(Q)$ with one endpoint in $Q \setminus \{v\}$ and one endpoint in $C_* \setminus \{v\}$, and likewise, there does not exist a vertex of $V(\text{Ext}(Q)) \setminus V(C_* \cup Q)$ with one neighbor in $V(Q \setminus \{v\})$ and one neighbor in $V(C_* \setminus \{v\})$; AND

3) For any vertex $x \in V(\text{Ext}(Q)) \setminus V(C_* \cup Q)$, the graph $G[V(Q) \cap N(x)]$ is a subpath of Q of length at most one.

Proof: 1) and 3) follow immediately from the minimality of $|V(\text{Ext}(Q))|$. Likewise, if 2) does not hold, then there exists a proper generalized chord Q' of C, which separates D_1 from D_2, such that $|E(Q')| \leq |E(Q)| + 1$, and thus $|E(Q')| \leq \frac{N_{\text{mo}}}{3}$. This contradicts Lemma 2.2.5. ■

We also have the following.

Claim 2.2.13. *For each $D \in \mathcal{C} \setminus \{C_*\}$ with $D \subseteq \text{Ext}(Q)$ we have $d(D, Q) > \frac{2\beta}{3} - 3N_{\text{mo}}$.*

Proof: By Theorem 2.1.26 applied to the cycle Q, there exists a $D' \in \mathcal{C} \setminus \{C_*\}$ with $D' \subseteq \text{Int}(Q)$, such that $d(D', Q) < \frac{\beta}{3} + |V(Q)| + 2N_{\text{mo}}$. By Observation 2.1.8, we have $d(D, D') \geq \beta$. Since any two vertices of Q are of distance at most $\frac{N_{\text{mo}}}{6}$ apart, we then have $d(D, Q) \geq \beta - \frac{N_{\text{mo}}}{6} - \left(\frac{\beta}{3} + |V(Q)| + 2N_{\text{mo}}\right)$. Thus, we have $d(D, Q) > \frac{2\beta}{3} - 3N_{\text{mo}}$, as desired. ■

We now write $Q := vu_1 \cdots u_m$ and $C_* := vb_1 b_2 \cdots b_n$ for some integers m, n. We then have the following:

Claim 2.2.14. *$V(\text{Int}(Q))$ is L-colorable.*

Proof: Let $G' := G \setminus (\text{Ext}(Q) \setminus V(C \cup Q))$, and let G_* be a graph obtained from G' by adding to G' an edge $u_i b$ with one endpoint in $Q \setminus \{v\}$ and one endpoint in $C_* - v$ in the closed region bounded by the graph $C_* \cup Q$. Then G^* is 2-connected. Furthermore, there exist cycles C_*^1, C_*^2 of G_* which intersect precisely on the vertices v, u_i, b, such that $vu_1 \cdots u_i b \subseteq C_*^1$, $vu_m \cdots u_i b \subseteq C_*^2$, and $G_* \setminus (\text{Int}(Q) \setminus Q) = C_*^1 \cup C_*^2$. For each $i = 1, 2$, let $G_*^i := \text{Int}_{G_*}(C_*^i)$ and let $\ell_i := |V(C_*^i)|$.

Now we apply Proposition 2.1.18 to the closed disc bounded by C_*^i for each $i = 1, 2$. For each $i = 1, 2$, we embed a short-separation-free C_*^i-web K^i in the closed disc bounded by C_*^i, where $|V(K^i)| \leq \ell_i^2$ for each $i = 1, 2$. Let $G^\dagger$ be the graph obtained from G_* in this way, and let $\mathcal{C}^\dagger := \{C_*\} \cup \{D \in \mathcal{C} \setminus \{C_*\} : D \subseteq \text{Int}(Q)\}$. Then $G^\dagger$ is short-separation-free. Let $\mathcal{T}^\dagger := (G^\dagger, \mathcal{C}^\dagger, L, C_*)$. Then $\mathcal{T}^\dagger$ is a tessellation, where C_* is a closed $\mathcal{T}^\dagger$-ring. We claim that $\mathcal{T}^\dagger$ is a mosaic as well. Note that, by definition of a $C_*^i i$-web for each $i = 1, 2$, C_* is a highly predictable cyclic facial subgraph of $G^\dagger$, and thus an L-predictable $\mathcal{T}^\dagger$-ring, so M2) is satisfied. M0) and M1) are immediate, and, since $\mathcal{T}$ satisfies distance conditions M3)-M4), it follows from the construction of K^1, K^2 that $\mathcal{T}^\dagger$ does as well. Thus, $\mathcal{T}^\dagger$ is indeed a mosaic. To finish the proof of Claim 2.2.14, it just suffices to check that $|V(G^\dagger)| < |V(G)|$.

Suppose toward a contradiction that $|V(G^\dagger)| \geq |V(G)|$. Now, since Q is an undesirable cycle, there exists a $D \in \mathcal{C} \setminus \{C_*\}$ with $D \subseteq \text{Ext}(Q)$. By Claim 2.2.13, we have $d(D, Q) > \frac{2\beta}{3} - 3N_{\text{mo}}$. By Observation 2.1.16 applied to $\text{Ext}(Q)$, we have $|V(\text{Ext}(Q) \setminus Q)| \geq 5\left(\frac{2\beta}{3} - 3N_{\text{mo}}\right)$. On the other hand, we have $|V(G^\dagger) \setminus V(\text{Int}(Q) \setminus Q))| \leq \ell_1^2 + \ell_2^2$. Since $|V(G^\dagger)| \geq |V(G)|$, we thus have $\frac{1}{5}(\ell_1^2 + \ell_2^2) \geq \frac{2\beta}{3} - 3N_{\text{mo}}$. Now, we have $\ell_1^2 + \ell_2^2 \leq (\ell_1 + \ell_2)^2 \leq (|E(C)| + |E(Q)| + 1)^2 \leq \left(\frac{4N_{\text{mo}}}{3}\right)^2$. Thus, we have $3N + \frac{16N_{\text{mo}}^2}{45} \geq \frac{2}{3}\left(\frac{17N_{\text{mo}}^2}{15}\right)$, which is false.

Thus, $|V(G^\dagger)| < |V(G)|$, so $G^\dagger$ admits an L-coloring by the minimality of $\mathcal{T}$. Since Q has no chord in $\text{Ext}(C)$, there is a proper L-coloring of the subgraph of G induced by $V(\text{Int}(Q))$. ■

Applying Claim 2.2.14, let ϕ be an L-coloring of $V(\text{Int}(Q))$. By Observation 2.2.12, there is no edge of $\text{Ext}(C)$ with one endpoint in $Q - v$ and one endpoint in $C_* - v$. Thus, since $|L(p)| = 1$ for each $p \in V(C)$, ϕ extends to a proper L-coloring ϕ' of $V(\text{Int}(Q')) \cup V(C)$. Since $C \cup Q$ is connected, the graph $\text{Ext}(Q) \setminus V(C \cup Q)$ contains a face F containing all the vertices of distance one from $V(C \cup Q)$. Combining 2) of Observation 2.2.12 with the fact that C_* is an L-predictable $\mathcal{T}$-ring, there is a lone vertex $w \in V(F)$ such that $|L_{\phi'}(w)| \geq 2$, and $|L_{\phi'}(x)| \geq 3$ for all $x \in V(F)$.

Now, let H be a connected component of $\text{Ext}(Q) \setminus V(C \cup Q)$, let $F' \subseteq F$ be the outer face of H, and let $\mathcal{C}' := \{F'\} \cup \{D \in \mathcal{C} \setminus \{C_*\} : D \subseteq H\}$. We claim now that $\mathcal{T}' := (H, \mathcal{C}' L_{\phi'}, F')$ is a mosaic. Note that if $w \in V(F')$ then we set $\mathbf{P}_{\mathcal{T}'}(F') = w$, otherwise all the vertices of F' have $L_{\phi'}$-lists of size at least three, and then we just choose any edge of F' to be $\mathbf{P}_{\mathcal{T}'}(F')$. In either case, $\mathcal{T}'$ satisfies M0) and M1), and M2) is immediate. It also immediately follows from Claim 2.2.13 that $\mathcal{T}'$ satisfies the distance conditions of Definition 2.1.6, so $\mathcal{T}'$ is a mosaic. Thus, by the minimality of $\mathcal{T}$, H is $L_{\phi'}$-colorable. Since this holds for each connected component of $\text{Ext}(Q) \setminus V(C \cup Q)$, ϕ' extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. This completes the case of Lemma 2.2.11 in which C is the outer face.

Case 2: $C \neq C_*$

In this case, we choose Q so that, among undesirable cycles, the quantity $|V(\text{Int}(Q))|$ is minimized. We now have the following:

Claim 2.2.15. $C \subseteq \text{Int}(Q)$

<u>Proof:</u> Suppose not. Then, since C is a facial subgraph of G, we have $C \subseteq \text{Ext}(D)$. Since Q is an undesirable cycle, it follows from Theorem 2.1.26 that there exists a $D \in \mathcal{C}$ with $D \subseteq \text{Int}(Q)$ such that $d(D, Q) < \frac{\beta}{3} + |V(Q)| + 2N_{\text{mo}}$. Since $C \subseteq \text{Ext}(Q)$, we have $C \neq D$. Furthermore, since any two vertices of Q are of distance at most $\frac{N_{\text{mo}}}{6}$ apart, we have $d(C, D) < \frac{\beta}{3} + \frac{N_{\text{mo}}}{2} + 2N_{\text{mo}} < \beta$, contradicting Observation 2.1.8. ■

Now the same argument as in Case 1, with the roles of $\text{Ext}(Q)$ and $\text{Int}(Q)$ interchanged, shows that there is an L-coloring ϕ of $V(\text{Ext}(Q))$ which extends to an L-coloring ϕ' of $V(\text{Ext}(Q) \cup C)$, such that ϕ' extends to the interior of Q. This contradicts the fact that $\mathcal{T}$ is not L-colorable. This completes the proof of Lemma 2.2.11. □

We now prove 3) and 4) of Theorem 2.2.4. We begin by showing that, for small values of k, if we have a k-chord of an open ring in a critical mosaic, then one side of the k-chord is colorable under certain conditions:

Proposition 2.2.16. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic. Let $C \in \mathcal{C}$ be an open $\mathcal{T}$-ring and let $\mathbf{P} = p_1 \cdots p_m$. Then the following hold.*

1) *Let Q be a proper generalized chord of C with $|E(Q)| \leq \frac{2N_{\text{mo}}}{3}$. If $Q \in \mathcal{K}(C, \mathcal{T})$, Q is $\mathcal{T}$-separating, and C^1_Q does not have a chord in G^1_Q, then the subgraph of G induced by $V(G^0_Q)$ is L-colorable; AND*

2) *Let Q be a proper generalized chord of C with precisely one endpoint in $\mathring{\mathbf{P}}$, and let $G = G_0 \cup G_1$ be the natural Q-partition of G, where $p_1 \in V(G_0)$ and $p_m \in V(G_1)$. For each $i = 0, 1$, let C_i be the cycle $(C \cap G_i) + Q$. If the following conditions hold, then the subgraph of G induced by $V(G_1)$ is L-colorable.*

   i) *$|E(P \cap G_1)| + |E(Q)| \leq \frac{2N_{\text{mo}}}{3}$, and there exists a $C' \in \mathcal{C} \setminus \{C\}$ with $C' \subseteq G_1$; AND*

   ii) *There is no chord of C_1 in G_1.*

Proof. We first show the following result analogous to Proposition 2.1.18 from Section 2.1:

Claim 2.2.17. *Let K be a planar embedding of a chordless cycle, with $|V(K)| \geq 5$. Let P be a subpath of K with $|E(P)| > 1$ (possibly $P = K$). Then there exists a short-separation-free planar embedding K^* such that the following hold.*

1) *$K \subseteq K^*$, and K is the outer face of K^*; AND*
2) *For every $v \in D_1(P, K^*)$, the induced graph $K^*[N(v) \cap V(P)]$ is a subpath of P of length one; AND*
3) *For any distinct $v, w \in D_1(P, K^*)$, if each of v, w is adjacent to an edge of P, then $vw \notin E(K^*)$; AND*
4) *Every facial subgraph of K^*, except K, is a triangle; AND*
5) *For any vertex $x \in V(\mathring{P})$, we have $d_K(x, K \setminus P) = d_{K^*}(x, K \setminus \mathring{P})$; AND*
6) *$|V(K^* \setminus K)| \leq 4|V(P)|^2$.*

<u>Proof:</u> We break this into two cases:

Case 1: $|V(P)| \geq \frac{|V(K)|}{2}$

In this case, we let K^* be a K-web. Then $|V(K^*)| \leq |V(C)|^2$ and $|V(C)| \leq 2|V(P)|$, so $|V(K^*)| \leq 4|V(P)|^2$, and thus $|V(K^* \setminus K)| \leq 4|V(P)|^2$, as desired.

Case 2: $|V(P) < \frac{|V(K)|}{2}$

In this case, we write $K := v_1 \cdots v_r$ for some $r \geq 5$, and let $1 \leq \ell < \frac{r}{2}$, where $P := v_1 \cdots v_\ell$, and let K' be a graph obtained from K by adding to K a lone vertex x to the interior of the open disc bounded by K, and adding edges incident to z so that z is adjacent to each of $v_{\ell+1}, v_{\ell+2}, \cdots, v_r, v_1\}$. Let $C := v_1 v_2 \cdots v_{\ell+1} x$. Then $|V(C)| \geq 5$, since $\ell \geq 3$, and thus, by Proposition 2.1.18, there exists short-separation a C-web H with $|V(H)| \leq |V(C)|^2$. Let K^* be a graph obtained from K' by embedding the C-web H into the open disc bounded by C. Note that K^* is short-separation-free as well. We claim now that K^* satisfies the desired properties. Properties 1), 2), 3), and 4) are immediate from the definition of a C-web.

Now let $x \in V(\mathring{P})$, and suppose toward a contradiction that $d_{K^*}(x, K \setminus \mathring{P}) < d_K(x, K \setminus \mathring{P})$. In that case, there is a shortest $(x, K \setminus \mathring{P})$-path in K^* containing the vertex z, and $d_{K*}(x, z) + 1 < d_K(x, K \setminus \mathring{P})$. Now, note that $d_{K^*}(x, z) = d_H(x, z)$, and $d_H(x, z) = d_C(x, z)$, since H is a C-web. Thus, we have $d_C(x, z) < d_K(x, K \setminus \mathring{P})$, which is false. Thus, $d_{K^*}(x, K \setminus \mathring{P}) \geq d_K(x, K \setminus \mathring{P})$, and so $d_{K^*}(x, K \setminus \mathring{P}) = d_K(x, K \setminus \mathring{P})$, since $K \subseteq K^*$. Thus,

K^* satisfies 5) . Furthermore, $|V(K^* \setminus K)| = |V(H \setminus P)| \leq (|V(P)|+2)^2 - |V(P)| \leq 4|V(P)|^2$, so 6) is satisfied as well. ■

Given a planar embedding K of a chordless cycle with $|V(K)| \geq 5$, and a subpath $P \subseteq K$, if K^* is a planar embedding satisfying Claim 2.2.17, then we call K^* a *P-partial K-web*. Analogous to Observation 2.1.19, the following is immediate:

Claim 2.2.18. *Let G be a short-separation-free graph, let C be a cyclic facial subgraph of G, and let Q be a proper generalized chord of C. Let $G = G_0 \cup G_1$ be the natural Q-partition of G. For each $i = 0, 1$, let C_i by the cycle $(G_i \cap C) + Q$. Suppose that $G_1 = \text{Int}(C_1)$, and let $G^* := G \setminus (G_1 \setminus C_1)$. Let Q' be a subpath of C_1, with $Q \subseteq Q'$, and let $G^\dagger$ be a graph obtained from G^* by adding to G^* a Q'-partial C_1-web in the closed disc bounded by C_1. Then $G^\dagger$ is short-separation-free.*

Proof: Suppose toward a contradiction that there is a separating cycle $D \subseteq G^\dagger$ with $|V(D)| \leq 4$. Since G_0 is short-separation-free, and K^* is short-separation-free, $E(D)$ has nonempty intersection with each of $E(G_0) \setminus E(Q)$ and $E(K^*) \setminus E(Q)$. Since $|E(D)| \leq 4$, and D is a separating cycle in $G^\dagger$, K^* contains an ℓ-chord of Q, where $1 \leq \ell \leq 4$, whose endpoints are non-adjacent in Q. Since $Q \subseteq Q'$, this contradicts the fact that K^* is a Q'-partial C_1-web. ■

Now we return to the proof of Proposition 2.2.16. Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let $C \in \mathcal{C}$ be an open ring. Let $\mathbf{P} := \mathbf{P}_{\mathcal{T}}(C)$. We first prove 1). Let Q be a $\mathcal{T}$-separating proper generalized chord of C with $1 \leq |E(Q)| \leq \frac{2N_{\text{mo}}}{3}$. C is a cycle by Proposition 2.1.12, so let $C := v_1 v_2 \cdots v_\ell$ for some $\ell \geq 3$. Let $Q := v_i w_1 \cdots w_t v_j$ for some $j \geq i$, where $t = |V(Q)| - 1$ and $C_Q^1 = v_i \cdots v_j w_1 \cdots w_t$. Consider the following cases:

Case 1: $C_* = C$

In this case, let $G_* := G \setminus (G_Q^1 \setminus C_Q^1)$. Then G_* admits a partition $G_* = G_*^0 \cup G_*^1$, where $G_*^0 = G_Q^0$ and $G_*^1 = C_Q^1$. Furthermore, by Proposition 2.1.22, we have $|V(Q)| \geq 3$. We construct a new mosaic from G^* as follows. Let $G^\dagger$ be a graph obtained from G^* by embedding a Q-partial C_Q^1-web K^* in the closed disc of $\mathbb{R}^2$ bounded by C_Q^1. Let $L^\dagger$ be a list-assignment for $G^\dagger$ obtained by setting $L^\dagger(v) = L(v)$ for all $v \in V(G_Q^0) \cup V(C)$, and letting $L^\dagger(v)$ be an arbitrary 5-list for any $v \in V(G^*) \setminus V(G_Q^0 \cup C)$. Note that C is a facial subgraph of $G^\dagger$.

We claim that $\mathcal{T}^\dagger := (G^\dagger, \{C\} \cup \mathcal{C}^{\subseteq G_Q^0}, L^\dagger, C)$ is a mosaic. By Claim 2.2.18, $G^\dagger$ is short-separation-free, and, since every facial subgraph of G, except those of $\mathcal{C}$, is a triangle, it follows from the construction of K^* that every facial subgraph of G, except those of $\{C\} \cup \mathcal{C}^{\subseteq G_Q^0}$, is a triangle. Thus, $\mathcal{T}^\dagger$ is a tessellation, and clearly satisfies M0), M1), and M2) of Definition 2.1.6. It just suffices to check that distance conditions M3 and M4 hold for $\mathcal{T}^\dagger$. If these do not hold, then, there exists a $C' \in \mathcal{C}^{\subseteq G_Q^0}$ and a subgraph H of C' such that $d_{G^\dagger}(H, C \setminus \mathring{\mathbf{P}}) < d_G(H, C \setminus \mathring{\mathbf{P}})$. In that case, there exists an $x \in V(\mathring{Q})$ such that $d_{G^\dagger}(x, C_Q^1 \setminus \mathring{Q}) < d_G(x, C_Q^1 \setminus \mathring{Q})$. On the other hand, since K^* is a Q-partial C_Q^1-web, we have $d_G(x, C_Q^1 \setminus \mathring{Q}) \leq d_{G^\dagger}(x, C_Q^1 \setminus \mathring{Q})$, so we have a contradiction.

Thus, $\mathcal{T}^\dagger$ is indeed a mosaic. To finish, we just need to check that $|V(G^\dagger)| < |V(G)|$. Suppose toward a contradiction that $|V(G^\dagger)| \geq |V(G)|$. In that case, we have $|V(G^\dagger \setminus G_Q^0)| \geq |V(G_Q^1 \setminus Q)|$. Now, $V(G^\dagger \setminus G_Q^0) = V(K^* \setminus Q)$, so we have $|V(K^* \setminus Q)| \geq |V(G_Q^1 \setminus Q)|$. Since $Q^* \subseteq K^*$ and $Q \subseteq G_Q^1$, we then have $|V(K^* \setminus G_*^1)| \geq |V(G_Q^1 \setminus C_Q^1)|$.

Since K^* is a Q-partial G_1^*-web, we have $|V(K^* \setminus Q)| \leq 4|V(Q)|^2 \leq \frac{16}{9}N_{\text{mo}}^2$. On the other hand, since Q is a $\mathcal{T}$-separating generalized chord of C, there exists a $C^\dagger \in \mathcal{C}^{\subseteq G_Q^1}$, and thus $d(C^\dagger, C_Q^1) \geq d(C^\dagger, C) - \frac{|V(Q)|}{2}$. Thus, since C is an open $\mathcal{T}$-ring, G_Q^1 contains a $(C^\dagger, C_Q^1)$-path of length greater than $\frac{\beta}{3}$ by Observation 2.1.8. Thus, by Observation 2.1.16, we have $|V(G_Q^1 \setminus C_Q^1)| > \frac{5\beta}{3}$. Combining these, we have $\frac{16}{9}N_{\text{mo}}^2 > \frac{5\beta}{3}$, and thus $\frac{16}{15}N_{\text{mo}}^2 > \beta$, whcih is false. We conclude that $|V(G^\dagger)| < |V(G)|$, and since $\mathcal{T}^\dagger$ is a mosaic we get that $G^\dagger$ is $L^\dagger$-colorable by the minimality of $\mathcal{T}$, so the subgraph of G induced by $V(G^0_Q)$ is L-colorable, as desired. This completes Case 1 of Fact 1) of Proposition 2.2.16.

Case 2.1 $C \neq C_*$ and Q does not separate $\mathbf{P}$ from C_*

In this case, we have $G^1_Q = \text{Int}(C^1_Q)$. As in Case 1, we create a mosaic $\mathcal{T}^\dagger := (G^\dagger, \{C\} \cup \mathcal{C}^{\subseteq G^0_Q}, L^\dagger, C_*)$, where $G^\dagger$ is obtained from $\text{Ext}(C^1_Q)$ by embedding a Q-partial C^1_Q-web K^* in the closed disc of $\mathbb{R}^2$ bounded by C^1_Q, and $L^\dagger$ is a list-assignment for $G^\dagger$ obtained by setting $L^\dagger(v) = L(v)$ for all $v \in V(G^0_Q) \cup V(C)$, and letting $L^\dagger(v)$ be an arbitrary 5-list for any $v \in V(G^*) \setminus V(G^0_Q \cup C)$. An identical argument to the case above then shows that G^0_Q is L-colorable. Since C^1_Q does not have a chord in G^1_Q, the subgraph of G induced by $\text{Int}(C^0_Q)$ is L-colorable, so we are done.

Case 2.2 $C \neq C_*$ and Q separates $\mathbf{P}$ from C_*

This case is easier. In this case we have $G^0_Q = \text{Int}(C^0_Q)$, and we note the following:

Claim 2.2.19. *$\mathcal{T}^0 := (G^0_Q, \{C^0_Q\} \cup \mathcal{C}^{\subseteq G^0_Q}, L, C^0_Q)$ is a mosaic.*

<u>Proof:</u> It suffices to show that $\mathcal{T}^0$ satisfies M3). The other conditions are immediate. If not, there is a $C' \in \mathcal{C}$ with $C' \subseteq \text{Int}(C^0_Q)$ such that $d(w_{\mathcal{T}^0}(C'), C \setminus \mathring{\mathbf{P}}) < \frac{\beta}{3} + \text{Rk}(\mathcal{T}^0|C') + 2N_{\text{mo}}$. Note that $w_{\mathcal{T}^0}(C') = w_{\mathcal{T}}(C')$, and $C \neq C_*$. Since $\mathcal{T}$ is a mosaic, we get $d(w_{\mathcal{T}}(C'), C \setminus \mathring{P}) \geq \beta + \text{Rk}(\mathcal{T}|C') + 2N_{\text{mo}}$. Since $\text{Rk}(\mathcal{T}|C') = \text{Rk}(\mathcal{T}^0|C')$, we have a contradiction. $\blacksquare$

Since $|V(G^0_Q)| < |V(G)|$ and $\mathcal{T}^0$ is a mosaic, $\text{Int}(C^0_Q)$ is L-colorable. Since C^1_Q does not have a chord in G^1_Q, the subgraph of G induced by $\text{Int}(C^0_Q)$ is L-colorable, so we are done. This proves 1) of Proposition 2.2.16.

Now we prove 2). We now apply a similar argument to that of 1). Let $\mathbf{P} := p_1 \cdots p_m$ and let Q be a proper generalized chord of C with precisely one endpoint in $\mathring{P}$, and let $r \in \{2, \cdots, m-1\}$, where p_r is the $\mathring{\mathbf{P}}$-endpoint of Q. Let $G = G_0 \cup G_1$, where G_0, G_1 are as in the statement of 2) of Proposition 2.2.16, and, for each $i = 0, 1$, let $C_i := (G_i \cap C) + Q$. Finally, let w be the lone endpoint of Q in $V(C \setminus \mathring{\mathbf{P}})$ and set $Q^\dagger := wQp_rp_{r+1} \cdots p_m$. Note that each of C_0, C_1 is a cycle by Proposition 2.1.12. Consider the following cases:

Case 1: $C_* = C$

In this case, let $G^\star := G \setminus (G_1 \setminus C_1)$. Then $G^\star$ admits a partition $G^\star = G^\star_0 \cup G^\star_1$, where $G^\star_0 = G_0$ and $G^\star_1 = C_1$. Furthermore, by Proposition 2.1.22, we have $|V(Q)| \geq 3$. We construct a new mosaic from $G^\star$ as follows. Let $G^\dagger$ be a graph obtained from $G^\star$ by embedding a $Q^\dagger$-partial C^1-web K^* in the closed disc of $\mathbb{R}^2$ bounded by C^1. Let $L^\dagger$ be a list-assignment for $G^\dagger$, where $L^\dagger(x) = L(x)$ for each $x \in V(G^*))$, and $L^\dagger(x)$ is an arbitrary 5-list for each $x \in V(G^\dagger) \setminus V(G^\star)$. By Claim 2.2.18, $G^\dagger$ is short-separation-free, since $Q \subseteq Q^\dagger$, and, by construction of K^*, every facial subgraph of $G^\dagger$, except those of $\mathcal{C}^{\subseteq G_0} \cup \{C\}$, is a triangle. Thus, let $\mathcal{T}^\dagger)$ be the oriented tessellation $(G^\dagger, \{C\} \cup \mathcal{C}^{\subseteq G_0}, L^\dagger, C_*)$. We claim that $\mathcal{T}^\dagger$ is a mosaic. M0) and M2) are trivial, so we now check M1). By construction of K^*, the open $\mathcal{T}^\dagger$-ring C still satisfies the property that there is no chord of C in $G^\dagger$ with an endpoint in $\mathring{\mathbf{P}}_{\mathcal{T}^\dagger}$ (indeed, by construction of K^*, C is still an induced subgraph of $G^\dagger$), and, again by construction of K^*, for each $v \in D_1(C, P_{\mathcal{T}^\dagger}(C))$, the subgraph of $G^\dagger$ induced by $N(v) \cap V(P_{\mathcal{T}^\dagger}(C)$ is a subpath of P of length at most one. Now we just need to check that distance conditions M3) and M4) hold.

If these do not hold, then there exists a $C' \in \mathcal{C}^{\subseteq G_0}$ and a subgraph H of C' such that $d_{G^\dagger}(H, C \setminus \mathring{\mathbf{P}}) < d_G(H, C \setminus \mathring{\mathbf{P}})$. In that case, there exists an $x \in V(\mathring{Q})$ such that $d_{G^\dagger}(x, C^1 \setminus \mathring{Q}^\dagger) < d_G(x, C_1 \setminus \mathring{Q}^\dagger)$. On the other hand, since K^* is a $Q^\dagger$-partial C_1-web, we have $d_G(x, C_1 \setminus \mathring{Q}^\dagger) \leq d_{G^\dagger}(x, C_1 \setminus \mathring{Q}^\dagger)$, so we have a contradiction.

Thus, $\mathcal{T}^\dagger$ is indeed a mosaic. To finish, we just need to check that $|V(G^\dagger)| < |V(G)|$. Suppose toward a contradiction that $|V(G^\dagger)| \geq |V(G)|$. In that case, we have $|V(G^\dagger \setminus G_0)| \geq |V(G_1 \setminus Q)|$. Now, $V(G^\dagger \setminus G_0) = V(K^* \setminus Q)$, so we have $|V(K^* \setminus Q)| \geq |V(G_1 \setminus Q)|$. Since $Q \subseteq K^*$ and $Q \subseteq C_1$, we then have $|V(K^* \setminus C_1)| \geq |V(G_1 \setminus C_1)|$. Since K^* is a $Q^\dagger$-partial G_1^*-web, we have $|V(K^* \setminus Q^\dagger)| \leq 4|V(Q^\dagger)|^2$. Thus, we have $|V(K^* \setminus Q)| \leq |V(Q^\dagger \setminus Q)| + 4|V(Q^\dagger)|^2$. Since $|V(Q^\dagger| \leq \frac{2N_{\text{mo}}}{3}$, we have $|V(K^* \setminus Q))| \leq \frac{2N_{\text{mo}}}{3} + \frac{16N_{\text{mo}}^2}{9}$

By assumption, there exists a $C^\dagger \in \mathcal{C}$ with $C^\dagger \subseteq G_1$, and thus $d(C^\dagger, C_1) \geq d(C^\dagger, C) - \frac{|V(Q)|}{2}$. Since C is an open $\mathcal{T}$-ring, G_1 contains a $(C^\dagger, C_1)$-path of length greater than $\frac{\beta}{3}$. By Observation 2.1.16, we have $|V(G_1 \setminus C_1)| > \frac{5\beta}{3}$. Combining these, we obtain $\frac{2N_{\text{mo}}}{3} + \frac{16N_{\text{mo}}^2}{9} > \frac{5\beta}{3}$, so $\frac{16N_{\text{mo}}^2}{15} + \frac{2N_{\text{mo}}}{5} > \beta$. Since $\frac{2N_{\text{mo}}}{5} \leq \frac{N}{15}$, we have $\frac{17}{15}N_{\text{mo}}^2 > \beta$, which is false. We conclude that $|V(G^\dagger)| < |V(G)|$, and since $\mathcal{T}^\dagger$ is a mosaic we get that $G^\dagger$ is $L^\dagger$-colorable by the minimality of $\mathcal{T}$, so the subgraph of G induced by $V(G_0)$ is L-colorable. This completes Case 1 of Fact 2) of Proposition 2.2.16.

Case 2: $C_* \neq C$

We break this into two subcases.

Case 2.1 $G_0 = \text{Ext}(C_0)$

In this case, we have $G_1 = \text{Int}(C_1)$, and, as in Case 1, we create a mosaic $\mathcal{T}^\dagger := (G^\dagger, \{C\} \cup \mathcal{C}^{\subseteq G_0}, L^\dagger, C_*)$, where $G^\dagger$ is obtained from $\text{Ext}(C_1)$ by embedding a Q-partial C_1-web K^* in the closed disc of $\mathbb{R}^2$ bounded by C_1, and $L^\dagger$ is a list-assignment for $G^\dagger$ obtained by setting $L^\dagger(v) = L(v)$ for all $v \in V(G_0) \cup V(C)$, and letting $L^\dagger(v)$ be an arbitrary 5-list for any $v \in V(G^\dagger) \setminus V(G_0 \cup C)$. An identical argument to the case above then shows that G_0 is L-colorable. Since C_1 does not have a chord in G_1, the subgraph of G induced by $\text{Ext}(C_0)$ is L-colorable, so we are done.

Case 2.2 $G_0 = \text{Int}(C_0)$

Let $\mathcal{T}_0 := (G_0, \{C_0\} \cup \mathcal{C}^{\subseteq G_0}, L, C_0)$. Analogous to Claim 2.2.19, it is immediate that $\mathcal{T}_0$ is a mosaic. Thus, since $|V(G_0)| < |V(G)|$, $\text{Int}(C_0)$ is L-colorable by the minimality of $\mathcal{T}$. Since C_1 does not have a chord in G_1, the subgraph of G induced by $\text{Int}(C_0)$ is L-colorable, so we are done. This completes the proof of Proposition 2.2.16. $\square$

Now we prove 3) of Theorem 2.2.4:

Lemma 2.2.20. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let C be an open $\mathcal{T}$-ring. Let Q be a proper generalized chord of C with $|E(Q)| \leq \frac{2N_{\text{mo}}}{3}$ and suppose that Q does not have an endpoint in $\mathring{\mathbf{P}}$. Then, for each $C' \in \mathcal{C} \setminus \{C\}$, we have $C' \subseteq G_Q^0$.*

Proof. We follow a similar argument to that of Lemma 2.2.5. Given a $Q \in \mathcal{K}(C, \mathcal{T})$, we say that Q is *unacceptable* if $|E(Q)| \leq \frac{2N_{\text{mo}}}{3}$ and there exists a $C' \in \mathcal{C}$ with $C' \subseteq G_Q^1$. We claim that there does not exist an unacceptable generalized chord in $\mathcal{K}(C, \mathcal{T})$. Suppose toward a contradiction that there is an unacceptable $Q \in \mathcal{K}(C, \mathcal{T})$, and, among all unacceptable elements of $\mathcal{K}(C, \mathcal{T})$, we choose Q so that $|V(G_Q^1)|$ is minimized. By Corollary 2.1.30, since $V(Q)$ has nonempty intersection with $V(C \setminus \mathring{\mathbf{P}})$ and C has a rank of $2N_{\text{mo}}$, $E(C_Q^1) \setminus E(Q)$ does not consist of only an edge. Let $Q := w_1 \cdots w_r$, where w_1, w_r are distinct elements of $V(C \setminus \mathring{\mathbf{P}})$ and $\{w_2, \cdots, w_{r-1}\} \cap V(C) = \varnothing$.

Claim 2.2.21. *There does not exist a chord of C_Q^1 with both endpoints in Q, and furthermore, for any $v \in V(G_Q^1) \setminus V(C_Q^1)$, if v has a neighbor in $V(Q)$, then $G_Q^1[V(Q) \cap N(v)]$ is a subpath of Q of length at most one.*

<u>Proof:</u> If this does not hold, then there is a pair of indices $1 \leq i < j \leq r$, where $|j - i| > 1$, and a path $Q^* \subseteq G_Q^1$ of length at most two, such that Q^* has endpoints w_i, w_j, and Q^* is otherwise disjoint to C_Q^2. Then Q^* is a proper

k-chord of C^1_Q, where $1 \leq k \leq 2$. Let $Q^{**} := w_1 \cdots w_i Q^* w_j \cdots w_r$. Then Q^{**} is a proper generalized chord of C with $|E(Q^{**})| \leq |E(Q)|$ and $Q^{**} \in \mathcal{K}(C, \mathcal{T})$, since neither endpoint of Q^{**} lies in $\mathring{P}$. Furthermore, G^1_Q contains the cycle $D := w_i w_{i+1} \cdots w_j Q^* w_i$.

Subclaim 2.2.22. $C \subseteq \text{Ext}(D)$.

Proof: Suppose not. Then $C \neq C_*$ and, since C, C_* are facial subgraphs of G, we have $C \subseteq \text{Int}(D)$. By Theorem 2.1.28, we have $d(C \setminus \mathring{\mathbf{P}}, D) > 2N_{\text{mo}} - \frac{3}{2}|V(D)|$. Since $|V(D)| \leq \frac{2N_{\text{mo}}}{3} + 1$, we have $d(C \setminus \mathring{\mathbf{P}}, D) > 2N_{\text{mo}} - (N_{\text{mo}} + 2) = N_{\text{mo}} - 2$, contradicting the fact that each vertex of $V(D \cap Q)$ is of distance at most $\frac{N_{\text{mo}}}{3}$ from $C \setminus \mathring{\mathbf{P}}$. ■

Thus, we have $C \subseteq \text{Ext}(D)$. Since $|V(G^1_{Q^{**}})| < |V(G^1_Q)|$ and $|E(Q^{**})| < |E(Q)|$, it follows from the minimality of Q that there exists a $C' \in \mathcal{C}^{\subseteq G^1_Q}$ with $C' \subseteq \text{Int}(D)$, or else Q^{**} is also an unacceptable element of $\mathcal{K}(C, \mathcal{T})$. Thus, by Theorem 2.1.26, there exists a $C'' \in \mathcal{C}$ with $C'' \subseteq \text{Int}(D)$ such that $\max\{d(v, w_{\mathcal{T}}(C'')) < \frac{\beta}{3} + \frac{3}{2}|V(D)| + \text{Rk}(\mathcal{T}|C'')$. Since $V(D)$ has nonempty intersection with $V(C \setminus \mathring{P})$, we then have $d(w_{\mathcal{T}}(C), w_{\mathcal{T}}(C'')) < \frac{\beta}{3} + \frac{3}{2}|V(D)| + \text{Rk}(\mathcal{T}|C'')$. But since C has a rank of $2N_{\text{mo}}$ in $\mathcal{T}$ and $|V(D)| \leq N_{\text{mo}}$, this contradicts the distance conditions of Definition 2.1.6 applied to $\mathcal{T}$. This completes the proof of Claim 2.2.21. ■

Now we have the following:

Claim 2.2.23. *C^1_Q is an induced cycle in G^1_Q, and furthermore, $V(G^0_Q)$ is L-colorable.*

Proof: Suppose toward a contradiction that there is a chord xy of C^1_Q in G^1_Q. By Proposition 2.1.22, at least one endpoint of xy lies in $V(Q) \setminus V(C^1_Q)$. By Claim 2.2.21, xy has precisely one endpoint in $V(Q)$, so suppose without loss of generality that $x \in V(\mathring{Q})$ and $y \in V(C^1_Q \setminus Q)$. Thus, there exists an $i \in \{2, \cdots, r-1\}$ with $x = w_i$. Let $Q_1 := w_1 Q w_i y$ and let $Q_2 := w_r Q w_i y$. Each of Q_1, Q_2 is a proper generalized chord of C with both endpoints in $C \setminus \mathring{P}$, and $|E(Q_i)| \leq |E(Q)|$ for each $i = 1, 2$. Furthermore, for each $i \in \{1, 2\}$, we have $|V(G^1_{Q_i})| < |V(G^1_Q)|$. Yet there is at least one $i \in \{1, 2\}$ such that Q_i separates an element of $\mathcal{C} \setminus \{C\}$ from $\mathbf{P}$, contradicting the minimality of Q. Thus, we conclude that C^1_Q is an induced cycle of G^1_Q, as desired. Since C^1_Q is an induced cycle of G^1_Q, it immediately follows from Proposition 2.2.16 that $V(G^0_Q)$ is L-colorable. ■

Applying Claim 2.2.23, let ϕ be an L-coloring of $V(G^0_Q)$ and let C^1_* be the outer face of G^1_Q (possibly $C^1_* = C^1_Q$). Let $\mathcal{T}^* := (G^1_Q, \mathcal{C}^{\subseteq G^1_Q} \cup \{C^1_Q\}, L^Q_\phi, C^1_*)$. We claim that $\mathcal{T}^*$ is a mosaic. Firstly, since $V(Q) \neq V(C^1_Q)$, C^1_Q is an open $\mathcal{T}^*$-ring with precolored path Q, and, by Claim 2.2.23, there is no chord of C^1_Q in G^1_Q. Combining this with Claim 2.2.21, we get that $\mathcal{T}^*$ satisfies M1) of Definition 2.1.6, and M0) and M2) are immediate. Finally, since $C^1_Q \setminus \mathring{Q} \subseteq C \setminus \mathring{\mathbf{P}}$, $\mathcal{T}^*$ also satisfies the distance conditions of Definition 2.1.6.

Thus, $\mathcal{T}^*$ is indeed a mosaic, as desired. Since $|V(G^1_Q)| < |V(G)|$, G^1_Q admits an L^Q_ϕ-coloring by the minimality of $\mathcal{T}$, so ϕ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. This completes the proof of Lemma 2.2.20. □

Lemma 2.2.20 proves 3) of Theorem 2.2.4. To prove 4), we first make the following definition:

Definition 2.2.24. Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let $C \in \mathcal{C}$ be an open $\mathcal{T}$-ring. Let $\mathbf{P}_{\mathcal{T}}(C) := \mathbf{P} = p_1 \cdots p_m$. Given a proper generalized chord Q of C, we say that Q is *$\mathcal{C}$-splitting* if the following hold.

1) $|E(Q)| \leq \frac{N_{\text{mo}}}{3}$*; AND*

2) Q has precisely one endpoint in $\mathring{\mathbf{P}}$*; AND*

3) Letting $G = G_0 \cup G_1$ be the natural Q-partition of G, there exists a $j \in \{0,1\}$ such that $|E(\mathbf{P} \cap G_j)| + |E(Q)| \leq \frac{2N_{\text{mo}}}{3}$ and there exists a $C' \in \mathcal{C} \setminus \{C\}$ with $C' \subseteq G_j$.

Thus, to prove 4) of Theorem 2.2.4, it suffices to prove the following:

Lemma 2.2.25. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let $C \in \mathcal{C}$ be an open $\mathcal{T}$-ring. Then there does not exist a $\mathcal{C}$-splitting proper generalized chord of C.*

Proof. Let $\mathbf{P} = \mathbf{P}_{\mathcal{T}}(C)$, where $\mathbf{P} = p_1 \cdots p_m$ for some $m \geq 2$. Given a proper generalized chord Q of C, we say that Q is *left-splitting* if Q has precisely one endpoint in $\mathbf{P}$, and, letting $G = G_0 \cup G_1$ be the natural Q-partition of G, where $p_1 \in V(G_0)$, we have $|E(\mathbf{P} \cap G_0)| + |E(Q)| \leq \frac{2N_{\text{mo}}}{3}$, and there exists a $C' \in \mathcal{C} \setminus \{C\}$ with $C' \subseteq G_0$.

Likewise, given a proper generalized chord Q of C, we say that Q is *right-splitting* if Q has precisely one endpoint in $\mathring{\mathbf{P}}$, and, letting $G = G_0 \cup G_1$ be the natural Q-partition of G, where $p_m \in V(G_1)$, we have $|E(\mathbf{P} \cap G_1)| + |E(Q)| \leq \frac{2N_{\text{mo}}}{3}$, and there exists a $C' \in \mathcal{C} \setminus \{C\}$ with $C' \subseteq G_1$.

We claim now that there does not exist a right-splitting proper generalized chord of C. An identical argument then shows that there is no left-splitting proper generalized chord of C. Suppose toward a contradiction that there is a right-splitting proper generalized chord Q of C, and among all such proper generalized chords Q, we choose Q so that $|V(G_1)|$ is minimized, where $G = G_0 \cup G_1$ is as in the definition of right-splitting generalized chords of C above. Let $r \in \{2, \cdots, m-1\}$, where p_r is the $\mathring{\mathbf{P}}$-endpoint of Q, and let $w \in V(C)$ be the other endpoint of Q. Let $Q' := p_m p_{m-1} \cdots p_r Q w$. By assumption, we have $|E(Q')| \leq \frac{2N_{\text{mo}}}{3}$. Let $C_1 := (C \cap G_1) + Q$. If $V(C_1) = V(Q')$, then C_1 is a separating cycle of length at most $\frac{2N_{\text{mo}}}{3}$, and since Q is a right-splitting chord of C, and $d(Q', C \setminus \mathring{\mathbf{P}}) = 0$, this contradicts Corollary 2.1.30. Thus, $E(C_1) \setminus E(Q')$ does not consist of only one edge. Now we have the following:

Claim 2.2.26. *There does not exist a chord of C_1 with both endpoints in Q, and furthermore, for any $v \in V(G_1 \setminus Q)$, if v has a neighbor in $V(Q)$, then $G_1[V(Q) \cap N(v)]$ is a subpath of Q of length at most one.*

<u>Proof:</u> Let $Q := w_1 \cdots w_s$, where $w_1 = p_r$ and $w_s = w$. If this does not hold, then there is a pair of indices $1 \leq i < j \leq s$, with $|j - i| > 1$, and a path $Q^* \subseteq G_1$ of length at most two such that Q^* has endpoints w_i, w_j, and Q^* is otherwise disjoint to Q^*. Let $Q^\dagger := w_1 Q w_i Q^* w_j Q w_s$. Then $Q^\dagger$ is also a proper generalized chord of C, with precisely one endpoint in $\mathring{\mathbf{P}}$, and $|E(Q^\dagger)| < |E(Q)|$. Let $G_1 = H \cup H'$ be the natural $Q^\dagger$-partition of G_1, where $w_{i+1} \in V(H \setminus Q^\dagger)$. If there exists a $C' \in \mathcal{C} \setminus \{C\}$ with $C' \subseteq H'$, then, since $p_m \in V(H')$, $Q^\dagger$ is also a right-splitting proper generalized chord of C. Since $|V(H')| < |V(G_1)|$, this contradicts the minimality of Q. Thus, since Q is right-splitting, there is a $C' \in \mathcal{C} \setminus \{C\}$ with $C' \subseteq H$, so the cycle $D := w_i Q w_j Q^* w_i$ separates C' from $C \setminus \{p_r\}$. Since $|E(Q)| \leq \frac{2N_{\text{mo}}}{3}$ and Q has an endpoint in $C \setminus \mathring{\mathbf{P}}$, we have $d(D, C \setminus \mathring{\mathbf{P}}) \leq \frac{N_{\text{mo}}}{3}$ and $|V(D)| \leq N_{\text{mo}}$, so, since Q is right-splitting, we contradict Corollary 2.1.30. ■

Now we deal with the case of chords with an endpoint in the precolored path and the other endpoint in Q:

Claim 2.2.27. *There is no chord of C_1 in G_1 with one endpoint in $\{p_{r+1}, \cdots, p_m\}$ and one endpoint in $V(Q)$. Likewise, there is no vertex $v \in V(G_1 \setminus C_1)$ with one neighbor in $\{p_{r+1}, \cdots, p_m\}$ and one neighbor in $V(Q \setminus \{p_r\})$.*

<u>Proof:</u> Suppose not. Then G_1 contains a proper k-chord $Q^* := x_1 \cdots x_k$ of C_1, where $k \leq 3$, $x_1 \in V(Q \setminus \{p_r\})$ and $x_k \in \{p_{r+1}, \cdots, p_m\}$. Let $Q^\dagger := x_k Q^* x_1 Q w$. Then $Q^\dagger$ is a proper generalized chord of C. Furthermore, we have $|E(Q^\dagger)| \leq (|E(Q)| - 1) + (k - 1) \leq |E(Q)| + 1$. Since Q is right-splitting, we have $|E(Q)| + 1 \leq \frac{2N_{\text{mo}}}{3}$, as one endpoint of Q lies in $\mathring{\mathbf{P}}$.

Case 1: $x_k = p_m$

In this case, we have $Q^\dagger \in \mathcal{K}(C, \mathcal{T})$. Since $|E(Q^\dagger)| \leq \frac{2N_{\text{mo}}}{3}$, it follows from Lemma 2.2.20 that $C' \subseteq G^0_{Q^\dagger}$ for each $C' \in \mathcal{C} \setminus \{C\}$. Thus, since Q is right-splitting, there is a $C' \in \mathcal{C} \setminus \{C\}$ such that the cycle $p_r\mathbf{P}p_mQ^*x_1Q$ separates C' from p_1. Since Q is right-splitting, this contradicts Corollary 2.1.30.

Case 2: $x_k \neq p_m$

In this case, $Q^\dagger$ has precisely one endpoint in $\mathring{\mathbf{P}}$. Let $G_1 = H \cup H'$ be the natural $Q^\dagger$-partition of G_1, where $p_r \in V(H)$. If there exists a $C' \in \mathcal{C} \setminus \{C\}$ with $C' \subseteq H'$, then, since Q^* has precisely one endpoint in $\mathring{\mathbf{P}}$ and $|E(Q^\dagger)| \leq \frac{2N_{\text{mo}}}{3}$, $Q^\dagger$ is also right-splitting. Since $|V(H')| < |V(G_1)|$, this contradicts the minimality of Q. Thus, since Q is right-splitting, there is a $C' \in \mathcal{C} \setminus \{C\}$ with $C' \subseteq H$. But then the cycle $p_r\mathbf{P}p_mQ^*x_1Q$ separates C' from p_1. Since Q is right-splitting, this contradicts Corollary 2.1.30. This completes the proof of Claim 2.2.27. ■

Lastly, we have the following claim:

Claim 2.2.28. *C_1 is an induced cycle in G_1, and furthermore, $V(G_0)$ is L-colorable.*

<u>Proof:</u> Suppose toward a contradiction that there is a chord xy of C_1 in G_1. By Proposition 2.1.22, at least one endpoint of xy lies in $V(Q') \setminus V(C)$, and thus lies in $V(\mathring{Q})$. Combining Claim 2.2.26 and Claim 2.2.27, xy has precisely one endpoint in $V(Q')$ and the other endpoint lies in $C \setminus \mathring{\mathbf{P}}$, so suppose without loss of generality that $x \in V(\mathring{Q})$ and $y \in V(C \setminus \mathring{\mathbf{P}})$.

Let $G_1 = H \cup H'$ be the natural xy-partition of G_1, where $p_r \cdots p_m \subseteq H$. If there exists a $C' \in \mathcal{C}^{\subseteq G_1}$ with $C' \subseteq H'$, then $wQxy$ is a proper generalized chord of C separating C' from $\mathbf{P}$. Since x is an internal vertex of Q, we have $|E(wQxy)| \leq \frac{2N_{\text{mo}}}{3}$, so $wQxy$ contradicts Lemma 2.2.20. Since Q is right-splitting, there is a $C' \in \mathcal{C}^{\subseteq G_1}$ with $C' \subseteq H$. Since x is an internal vertex of Q, we have $|E(p_r\mathbf{P}p_m)| + |E(p_rQx)| + 1 \leq |E(p_r\mathbf{P}p_m)| + |E(Q)| \leq \frac{2N_{\text{mo}}}{3}$. But then the proper generalized chord p_rQxy of C is also right-splitting. Since $|V(H)| < |V(G_1)|$, this contradicts the minimality of Q. Thus, we conclude that C_1 is an induced cycle of G_1, as desired. Since C^1 is an induced cycle of G_1, it follows from 2) of Proposition 2.2.16 that $V(G_0)$ is L-colorable. ■

Applying Claim 2.2.28, let ϕ be an L-coloring of $V(G_0)$ and let C^1_* be the outer face of G_1 (possibly $C^1_* = C^1$). Let $\mathcal{T}^* := (G_1, \mathcal{C}^{\subseteq G_1} \cup \{C_1\}, L^Q_\phi, C^1_*)$. We claim that $\mathcal{T}^*$ is a mosaic. Firstly, since $V(Q') \neq V(C_1)$, C_1 is an open $\mathcal{T}^*$-ring with precolored path Q', and, by Claim 2.2.28, there is no chord of C_1 in G_1. Combining this with Claim 2.2.26 and Claim 2.2.27, we get that $\mathcal{T}^*$ satisfies M1) of Definition 2.1.6, and M0) and M2) are immediate. Finally, since $C_1 \setminus \mathring{Q}' \subseteq C \setminus \mathring{\mathbf{P}}$, $\mathcal{T}^*$ also satisfies the distance conditions of Definition 2.1.6. Thus, $\mathcal{T}^*$ is indeed a mosaic. Since $|V(G_1)| < |V(G)|$, G_1 admits an L^Q_ϕ-coloring by the minimality of $\mathcal{T}$, so ϕ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. This completes the proof of Lemma 2.2.25. □

The above completes the proof of Theorem 2.2.4. We also have the following useful corollary to this result.

Corollary 2.2.29. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let $C \in \mathcal{C}$. Then the following hold.*

1) *C is an induced subgraph of G; AND*
2) *For any subgraph H of G, if $H \cap C' = \varnothing$ for all $C' \in \mathcal{C} \setminus \{C\}$, then H is L-colorable. In particular, $|\mathcal{C}| > 1$.*

Proof. It suffices to prove that the claim on H holds in the case where $C \subseteq H$. By Corollary 2.2.10 C is an induced cycle of G If C is a closed $\mathcal{T}$-ring, then, since C is an L-predictable $\mathcal{T}$-ring, so there exists a $w \in V(C)$ and an

L-coloring ϕ of $V(C-w)$ such that $|L_\phi(w)| \geq 1$ and $|L_\phi(v)| \geq 3$ for all $v \in D_1(C) \setminus \{w\}$. Thus, $H \setminus (C-w)$ contains a lone facial subgraph F such that every vertex of $H \setminus (C-w)$ with an L_ϕ-list of size less than five lies in F, and F contains a a lone vertex w such that $|L_\phi(w)| \geq 1$ and $|L_\phi(v)| \geq 3$ for all $v \in V(F-w)$. Thus, $H \setminus (C-w)$ is L_ϕ-colorable, so H is L-colorable.

Now suppose that C is an open $\mathcal{T}$-ring, let $\mathbf{P} := \mathbf{P}_{\mathcal{T}}(C)$ and let ϕ be an L-coloring of $\mathbf{P}$. By M1), we have $|L_\phi(v)| \geq 3$ for all $v \in D_1(P,G) \setminus V(C)$. Since C is an induced cycle in G, each vertex of $C \setminus \mathbf{P}$, except those adjacent to $\mathbf{P}$ in C, has an L_ϕ-list of size at least three. Thus, $H \setminus \mathbf{P}$ contains a lone facial subgraph F containing all vertices with L_ϕ-lists of size less than five. If $|V(C \setminus P)| = 1$, then there is a $w \in V(F)$ such that $|L_\phi(w)| \geq 1$ and $|L_\phi(v)| \geq 3$ for all $v \in V(F-w)$, so H is L-colorable by Theorem 0.2.3 in that case. If $|V(C \setminus \mathbf{P})| \geq 2$, then there exist two vertices $w_1, w_2 \in V(F)$ such that $|L_\phi(w)| \geq 3$ for all $w \in V(F) \setminus \{w_1, w_2\}$ and $|L_\phi(w_i)| \geq 2$ for each $i = 1, 2$. Thus, by Theorem 1.3.4, $H \setminus \mathbf{P}$ is L_ϕ-colorable, and thus H is L-colorable. Since $\mathcal{T}$ is critical, we thus have $|\mathcal{C}| > 1$. □

2.3 Bands of Open Rings in Critical Mosaics

This section consists of a lone main result and two useful corollaries. Ituitively, this main result states that in a critical mosaic $\mathcal{T} = (G, \mathcal{C}, L, C_*)$, if $C \in \mathcal{C}$ is an open $\mathcal{T}$-ring, then G does not contain any "shortcuts" of the precolored path, which is made precisely below. We begin with the following definition:

Definition 2.3.1. Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let $C \in \mathcal{C}$ be an open $\mathcal{T}$-ring. Let $\mathbf{P}_{\mathcal{T}}(C) := \mathbf{P} = p_1 \cdots p_m$ and let Q be a generalized chord of $\mathbf{P}$. Given such a Q, we associate to Q a cycle Q_{ext} as follows: Let $p_i, p_j \in V(\mathbf{P})$, with $1 \leq i \leq j \leq m$, where p_i, p_j are the endpoints of Q in $\mathbf{P}$, and set $Q_{\text{ext}} := p_i Q p_j \mathbf{P} p_i$. Note that Q_{ext} is not necessarily a generalized C-chord, as it possibly intersects with C on many vertices.

We call call Q a *C-band* if, letting $G = G_0 \cup G_1$ be the natural Q_{ext}-partition of G, the following hold.

1) $E(p_i \mathbf{P} p_j) \subseteq E(G_0)$ and $E(P) \setminus E(p_i \mathbf{P} p_j) \subseteq E(G_1)$*; AND*
2) For any $x \in D_1(C, G)$, if $G[N(x) \cap V(P)]$ is an edge of $p_i \mathbf{P} p_j$, then $x \in V(G_0)$*; AND*
3) There exists a $C' \in \mathcal{C} \setminus \{C\}$ with $C' \subseteq G_0$.

We say that a C-band Q is *short* if $|E(Q_{\text{ext}})| \leq \frac{11 N_{\text{mo}}}{12}$.

Note that, since $|E(\mathbf{P})| \geq 1$,and $\mathcal{T}$ is a tessellation, the partition $G = G_0 \cup G_1$ satisfying the conditions above uniquely specifies G_0 and G_1, even if $E(\mathbf{P}) \setminus E(p_i \mathbf{P} p_j) = \varnothing$, since, if p_i, p_j are the endpoints of $\mathbf{P}$, then there is at least one vertex x of $D_1(C)$ such that $G[N(x) \cap V(\mathbf{P})]$ is an edge of $p_i \mathbf{P} p_j$. Possibly, we have $Q_{\text{ext}} = C$, since the unique path Q of length greater than zero in $C \setminus E(\mathbf{P})$ is a C-band. In that case, we have $G_0 = G$ and $G_1 = Q_{\text{ext}}$.

Our main result for Section 2.3 is the following.

Theorem 2.3.2. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let $C \in \mathcal{C}$ be an open $\mathcal{T}$-ring. Let $\mathbf{P} := \mathbf{P}_{\mathcal{T}}(C)$, where $\mathbf{P} = p_1 \cdots p_m$. Then the following hold.*

1) *G does not contain a short C-band. In particular, for any C-band Q, we have $|E(Q)| > \frac{N_{\text{mo}}}{4}$; AND*
2) *Let Q be a proper generalized chord of C with its C-endpoints in $V(\mathbf{P})$, and let $1 \leq i < j \leq m$, where p_i, p_j are the endpoints of Q. Suppose further that G, letting $G = G_0 \cup G_1$ be the natural Q-partition of G, we have $p_i \mathbf{P} p_j \subseteq G_0$, and, for each $C' \in \mathcal{C} \setminus \{C\}$, $C' \subseteq G_1$. Then $|E(p_1 \mathbf{P} p_i)| + |E(Q)| + |E(p_j \mathbf{P} p_m)| + |E(Q)| > \frac{2N_{\text{mo}}}{3}$.*

We begin by proving 1):

Lemma 2.3.3. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let $C \in \mathcal{C}$ be an open $\mathcal{T}$-ring. Then G does not contain a short C-band. In particular, for any C-band Q, we have $|E(Q)| > \frac{N_{\text{mo}}}{4}$.*

Proof. Suppose toward a contradiction that there is a short C-band Q. Let $G = G_0 \cup G_1$, where $G_0 \cap G_1 = Q_{\text{ext}}$ and G_0, G_1 are as in Definition 2.3.1. By definition, there is a $C^\dagger \in \mathcal{C} \setminus \{C\}$ with $C^\dagger \subseteq G_0$. Let $\mathbf{P}_{\mathcal{T}}(C) := \mathbf{P} = p_1 \cdots p_m$.

Claim 2.3.4. $C \subseteq G_1$

Proof: Suppose not. Since C is a facial subgraph of G and Q_{ext} is a cycle, we have $C \subseteq G_0$. Since Q is a C-band, the $\mathbf{P}$-endpoints of Q are the endpoints of $\mathbf{P}$, or else there is a vertex p of $\mathbf{P}$ in $V(G_1 \setminus Q_{\text{ext}})$ and thus $p \notin V(C_0)$. Thus, we have $\mathbf{P} \subseteq Q_{\text{ext}}$, and since $C \subseteq G_0$, the short C-band Q is the unique subpath of $C \setminus E(P)$ with endpoints p_1, p_m and containing all vertices of $C \setminus \mathring{\mathbf{P}}$. Thus, we have $G_1 = C$, contradicting our assumption that $C \not\subseteq G_1$. ■

We now have the following:

Claim 2.3.5. *Q_{ext} has no chord in G_0, and furthermore, for each vertex $v \in D_1(Q_{\text{ext}}, G_0)$, $G[N(v) \cap V(Q_{\text{ext}})]$ is a subpath of Q_{ext} of length at most two.*

Proof: Suppose not. Thus, there is a k-chord R of Q_{ext} in G_0, where $1 \leq k \leq 2$. Let $G_0 = H \cup H'$ be the natural R-partition of H, and note that $|V(H)| < |V(G_0)|$, and $|V(H')| < |VG_0)|$. Let $K := (H \cap Q_{\text{ext}}) + R$ and $K' := (H' \cap Q_{\text{ext}}) + R$. Note that each of K, K' is a cycle of length at most $|E(Q_{\text{ext}})|$, since R is either a chord of Q_{ext} or a 2-chord of Q_{ext} whose endpoints are not adjacent in Q_{ext}. Consider the following cases:

Case 1: $G_0 = \text{Int}(Q_{\text{ext}})$

In this case, one of the two cycles K, K' separates $C^\dagger$ from C_*. Since each of K, K' is of distance at most $\frac{N_{\text{mo}}}{4}$ from $V(\mathring{P})$, and each vertex of $\mathring{\mathbf{P}}$ is of distance at most $\frac{N_{\text{mo}}}{3}$ from $C \setminus \mathring{\mathbf{P}}$, each of K, K' is of distance at most $\frac{N_{\text{mo}}}{3} + \frac{N_{\text{mo}}}{4}$ from $C \setminus \mathring{\mathbf{P}}$, so we contradict Corollary 2.1.30.

Case 2: $G_0 \neq \text{Int}(Q_{\text{ext}})$

In this case, we have $G_0 = \text{Ext}(Q_{\text{ext}})$ and $C \neq C_*$. By Claim 2.3.4, we have $C \subseteq \text{Int}(Q_{\text{ext}})$, and one of the two cycles K, K' topoologically separates C from C_*. Since each of K, K' has length at most $|E(Q_{\text{ext}})$, we get that each of K, K' has length at most $\frac{11N_{\text{mo}}}{12}$. Now, $2N_{\text{mo}} - \left(\frac{3}{2}\right)\left(\frac{11N_{\text{mo}}}{12}\right) = \frac{15N}{24}$, and each of K, K' is of distance at most $\frac{N_{\text{mo}}}{3} + \frac{N_{\text{mo}}}{4}$ from $C \setminus \mathring{\mathbf{P}}$. Since $\frac{N_{\text{mo}}}{3} + \frac{N_{\text{mo}}}{4} = \frac{14N_{\text{mo}}}{24} < \frac{15N_{\text{mo}}}{24}$, we contradict Corollary 2.1.30. ■

Now we return to the proof of Lemma 2.3.3. We have the following:

Claim 2.3.6. *$V(G_1)$ is L-colorable*

Proof: Suppose that, for each $C' \in \mathcal{C} \setminus \{C\}$, we have $V(C' \cap G_1) = \varnothing$. In that case, by Corollary 2.2.29, we immediately get that $V(G_1)$ is L-colorable, so now we suppose there is a $C' \in \mathcal{C} \setminus \{C\}$ with $V(C' \cap G_1) \neq \varnothing$, so we have $C' \subseteq G_1$, since $C' \cap Q_{\text{ext}} = \varnothing$. Note that, if Q_{ext} is not a separating cycle in G, then we have $V(G_1) = V(Q_{\text{ext}})$, contradicting our assumption that $C' \subseteq G_1$, so Q_{ext} is a separating cycle. Furthermore, by Observation 2.1.8 we have $d(C^\dagger, Q_{\text{ext}}) \geq d(C^\dagger, C) - \frac{N_{\text{mo}}}{8} \geq \frac{\beta}{3} + 3N_{\text{mo}} - \frac{N_{\text{mo}}}{8} > \frac{\beta}{3}$.

If $G_1 = \text{Ext}(Q_{\text{ext}})$, then we have $C^\dagger \subseteq \text{Int}(Q_{\text{ext}})$. Since $\beta \geq \frac{3N_{\text{mo}}^2}{5}$, we get that $V(G_1)$ is L-colorable by Proposition 2.1.20. Now suppose that $G_1 = \text{Int}(Q_{\text{ext}})$. By Claim 2.3.4, we have $C \subseteq \text{Int}(Q_{\text{ext}})$, and since $d(C, Q_{\text{ext}}) = 0$, it follows from Proposition 2.1.21 that $V(G_1)$ is L-colorable. ■

Applying Claim 2.3.6, let ϕ be an L-coloring of $V(G_1)$, let C_*^0 be the outer face of G_0 (possibly $C_*^0 = C_*$), and consider the oriented tessellation $\mathcal{T}' := (G_0, \{Q_{\text{ext}}\} \cup \mathcal{C}^{\subseteq G_0}, L_\phi^Q, C_*^0)$.

Claim 2.3.7. *$\mathcal{T}'$ is a mosaic.*

Proof: Since $|V(Q_{\text{ext}})| \leq N_{\text{mo}}$, we immediately have M0) and M1). By Claim 2.3.5, Q_{ext} is a highly predictable facial subgraph of G_0, so, since Q_{ext} is a closed $\mathcal{T}'$-ring, we have M2) as well. To finish, we just need to check that the distance conditions M3) and M4) still hold. If these distance conditions do not hold, then, since $\mathcal{T}$ is a mosaic and C is an open $\mathcal{T}$-ring, there exists a $\beta^* \in \{\frac{\beta}{3}, \beta\}$, and a $C' \in \mathcal{C} \setminus \{C\}$ such that $d(C, w_{\mathcal{T}'}(C')) < \beta^* + \text{Rk}(\mathcal{T}'|C') + |V(C)|$ and $d(C, w_{\mathcal{T}}(C')) \geq \beta^* + \text{Rk}(\mathcal{T}|C') + 2N_{\text{mo}}$. Note that $\text{Rk}(\mathcal{T}'|C') = \text{Rk}(\mathcal{T}|C')$ and $w_{\mathcal{T}}(C') = w_{\mathcal{T}'}(C')$, so $2N_{\text{mo}} < |V(C)|$, which is false. We conclude that $\mathcal{T}'$ is also a mosaic, as desired. ■

Now we finish the proof of Lemma 2.3.3. Consider the following cases:

Case 1: Q_{ext} is a separating cycle in G

In this case, we have $|V(G_0)| < |V(G)|$. Since $\mathcal{T}'$ is a mosaic by Claim 2.3.7, we get that $V(G_0)$ is L_ϕ^Q-colorable by the minimality of $\mathcal{T}$. Thus, ϕ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical.

Case 2: Q_{ext} is not a separating cycle in G

In this case, we have $V(G_1) = V(Q_{\text{ext}})$ and $V(G_0) = V(G)$. By the minimality of $\sum_{v \in V(G)} |L(v)|$, the mosaic $\mathcal{T}'$ is colorable, so G is L_ϕ^Q-colorable, and thus G is L-colorable, contradicting the fact that $\mathcal{T}$ is critical. Thus, there does not exist a short C-band. Since $\frac{2N_{\text{mo}}}{3} + \frac{N_{\text{mo}}}{4} = \frac{11N_{\text{mo}}}{12}$, it immediately follows that, for any C-band Q, we have $|E(Q)| > \frac{N_{\text{mo}}}{4}$. This completes the proof of Lemma 2.3.3. □

The above proves 1) of Theorem 2.3.2. This result, together with the results of Section 2.2, yields the following corollary:

Corollary 2.3.8. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$, let $C \in \mathcal{C}$, and let Q be a generalized chord of Q with $|E(Q)| \leq \frac{N_{\text{mo}}}{4}$. Let $G = G_0 \cup G_1$ be the natural Q-partition of G. Then there exists a $j \in \{0, 1\}$ such that, for each $C' \in \mathcal{C} \setminus \{C\}$, $C' \subseteq G_j$.*

Proof. Suppose toward a contradiction that there exist $C_0, C_1 \in \mathcal{C} \setminus \{C\}$ such that $C_j \subseteq G_j$ for each $j = 0, 1$. Suppose first that C is a closed ring. By 1) of Theorem 2.2.4, Q is not a proper generalized chord of C, so Q is a cycle, contradicting 2) of Theorem 2.2.4. Thus, C is an open ring. Let $\mathbf{P} := \mathbf{P}_{\mathcal{T}}(C)$. If Q is a not a proper generalized chord of C, then Q is a cycle and we contradict Corollary 2.1.30. Thus, Q is a proper generalized chord of C.

By 3) of Theorem 2.2.4, we have $Q \notin \mathcal{K}(C, \mathcal{T})$. By Lemma 2.3.3, Q does not have both endpoints in $\mathbf{P}$. Thus, precisely one endpoint of Q lies in $V(\mathring{\mathbf{P}})$ and one endpoint of Q lies in $V(C \setminus \mathbf{P})$. Let $\mathbf{P} := p_1 \cdots p_m$ and let $r \in \{2, \cdots, m-1\}$, where p_r is the $\mathring{\mathbf{P}}$-endpoint of Q. Since $|E(\mathbf{P})| \leq \frac{2N_{\text{mo}}}{3}$, we have $|E(p_1\mathbf{P}p_r)| + |E(p_r\mathbf{P}p_m)| + 2|E(Q)| < \frac{4N_{\text{mo}}}{3}$, so either $|E(p_1\mathbf{P}p_r)| + |E(Q)| < \frac{2N_{\text{mo}}}{3}$ or $|E(p_r\mathbf{P}p_m)| + |E(Q)| < \frac{2N_{\text{mo}}}{3}$. In either case, we contradict 4) of Theorem 2.2.4. □

Given the results of Corollary 2.3.8, it is natural to introduce the following notation, which we use throughout the remaining chapters.

Definition 2.3.9. Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let $C \in \mathcal{C}$, and let Q be a generalized chord of C with $|E(Q)| \leq \frac{N_{\text{mo}}}{4}$. We define two graphs G_Q^{small} and G_Q^{large}, each a subgraph of G, in the following way. We let

$G = G_Q^{\text{small}} \cup G_Q^{\text{large}}$ be the natural Q-partition of G, where $C' \subseteq G_Q^{\text{large}}$ for all $C' \in \mathcal{C} \setminus \{C\}$. In particular, note that if C is an open $\mathcal{T}$-ring and $Q \in \mathcal{K}(C, \mathcal{T})$ with $|E(Q)| \leq \frac{2N_{\text{mo}}}{3}$, then $G_Q^{\text{small}} = G_Q^1$ and $G_Q^{\text{large}} = G_Q^0$. By Corollary 2.2.29, the graphs G_Q^{small} and G_Q^{large} are uniquely defined, since there exists a $C' \in \mathcal{C} \setminus \{C\}$ with $C' \subseteq G_Q^{\text{large}}$.

We now prove 2) of Theorem 2.3.2, which we restate below.

Lemma 2.3.10. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let $\mathbf{P} := p_1 \cdots p_m$. Let Q be a proper generalized chord of C with its C-endpoints in $V(\mathbf{P})$. Let $1 \leq i < j \leq m$, where p_i, p_j are the endpoints of Q. Let $G = G_0 \cup G_1$ be the natural Q-partition of G, where $p_i \mathbf{P} p_j \subseteq G_0$, and, for each $C' \in \mathcal{C} \setminus \{C\}$, $C' \subseteq G_1$. Then $|E(p_1 \mathbf{P} p_i)| + |E(Q)| + |E(p_j \mathbf{P} p_m)| + |E(Q)| > \frac{2N_{\text{mo}}}{3}$.*

Proof. Given a proper generalized chord Q of C, we say that Q is *defective* if Q has both endpoints in P and furthermore, letting $G = G_0 \cup G_1$ be the natural Q-partition of G and letting $1 \leq i < j \leq m$, where p_i, p_j are the endpoints of Q, the following hold.

1) $p_i \mathbf{P} p_j \subseteq G_0$, and, for each $C' \in \mathcal{C} \setminus \{C\}$, $C' \subseteq G$); *AND*
2) $|E(p_1 \mathbf{P} p_i)| + |E(Q)| + |E(p_j \mathbf{P} p_m)| \leq \frac{2N_{\text{mo}}}{3}$.

Suppose toward a contradiction that there is a defective generalized chord Q of C. Among all defective generalized chords of C, we choose Q so that $|V(G_Q^{\text{large}})|$ is minimized. By 1) of the definition above, G_Q^{large} and G_Q^{small} are well-defined. Let $1 \leq i < j \leq m$, where p_i, p_j are the endpoints of Q. Since $\mathcal{C} \setminus \{C\} \neq \varnothing$, we fix a $C' \in \mathcal{C} \setminus \{C\}$.

Claim 2.3.11. *G_Q^{small} is L-colorable.*

<u>Proof:</u> Each vertex of $G_Q^{\text{small}} \setminus \{p_i, \cdots, p_j\}$ has an L-list of size five. Since $V(\mathbf{P})$ is L-colorable, let ϕ be an L-coloring of $p_i \mathbf{P} p_j$. By M1), each vertex of $G_Q^{\text{small}} \setminus \{p_i, \cdots, p_j\}$ with a neighbor in $p_i \mathbf{P} p_j$ has an L_ϕ-list of size at least three. Thus, $G_Q^{\text{small}} \setminus \{p_i, \cdots, p_j\}$ has a lone facial subgraph F containing all vertices of $G_Q^{\text{small}} \setminus \{p_i, \cdots, p_j\}$ with L_ϕ-lists of size less than five, and each vertex of F has an L_ϕ-list of size at least three. By Theorem 0.2.3, ϕ extends to an L-coloring of G_Q^{small}. ■

Let $\mathbf{P}^\dagger := p_1 \mathbf{P} p_i Q p_j \mathbf{P} p_m$ and let $C^\dagger := C \cap G_Q^{\text{large}} + Q$. Now we note the following:

Claim 2.3.12. *$V(\mathbf{P}^\dagger) \neq V(C^\dagger)$ and there does not exist a chord xy of $C^\dagger$ with $x \in V(\mathbf{P}^\dagger)$ and $y \in V(C^\dagger \setminus P^\dagger)$.*

<u>Proof:</u> Firstly, since $p_1 p_m$ is not an edge of C, we have $V(\mathbf{P}^\dagger) \neq V(C^\dagger)$. Now suppose toward a contradiction that there is a chord xy of $C^\dagger$, where $x \in V(\mathbf{P}^\dagger)$ and $y \in V(C^\dagger \setminus \mathbf{P}^\dagger)$. Note that $y \in V(C \setminus \mathbf{P})$ and G contains the paths Q_1, Q_2, where $Q_1 := p_i Q x y$ and $Q_2 := p_j Q x y$. Each of Q_1, Q_2 is a proper generalized chord of C with precisely one endpoint in $\mathbf{P}$ and one endpoint in $C \setminus \mathbf{P}$. Note that $|E(p_1 \mathbf{P} p_i)| + |E(Q_1)| \leq \frac{2N_{\text{mo}}}{3}$ and $|E(p_j \mathbf{P} p_m)| + |E(Q_2)| \leq \frac{2N_{\text{mo}}}{3}$. If each of p_i, p_j is an endpoint of $\mathbf{P}$, then we have $i = 1$ and $j = m$, and $Q_1, Q_2 \in \mathcal{K}(C, \mathcal{T})$. By 3) of Theorem 2.2.4 applied to Q_2, we have $C' \subseteq G_{Q_2}^0$. Thus, Q_1 separates C' from P, contradicting 3) of Theorem 2.2.4 applied to Q_1.

Thus, suppose without loss of generality that p_i is an internal vertex of P. If $p_j = p_m$, then, by Theorem 2.2.4, $C' \subseteq G_{Q_2}^0$. But then, by 4) of Theorem 2.2.4, we have $|E(p_1 \mathbf{P} p_i)| + |E(Q_1)| > \frac{2N_{\text{mo}}}{3}$, which is false. Thus, p_j is also an internal vertex of P. Since $|E(p_1 \mathbf{P} p_i)| + |E(Q_1)| \leq \frac{2N_{\text{mo}}}{3}$, it follows from 4) of Theorem 2.2.4 that Q_1 separates C' from p_1. But since $|E(p_j \mathbf{P} p_m)| + |E(Q_2)| \leq \frac{2N_{\text{mo}}}{3}$ as well, it again follows from 4) of Theorem 2.2.4 that Q_2 separates C' from p_m, so $C' \subseteq G_Q^{\text{small}}$, which is false. This completes the proof of Claim 2.3.12. ■

Finally, we show the following:

Claim 2.3.13. $\mathbf{P}^\dagger$ *is a chordless path, and furthermore, for each vertex* $v \in V(G^1 \setminus C^\dagger)$*,* $G[N(v) \cap V(\mathbf{P}^\dagger)]$ *is a subpath of* $\mathbf{P}^\dagger$ *of length at most one.*

Proof: If this does not hold, then there is a k-chord R of $\mathbf{P}^\dagger$, where $1 \leq k \leq 2$, and either $k = 1$ or, if $k = 2$, then the endpoints of R are not adjacent in $\mathbf{P}^\dagger$. Let u, v be the endpoints of R. Since $\mathcal{T}$ satisfies M1) and C is an induced subgraph of G, at least one of u, v lies in $V(Q) \setminus \{p_i, p_j\}$, so let $v \in V(Q) \setminus \{p_i, p_j\}$. Consider the following cases:

Case 1: $u \in \{p_1, \cdots, p_i\} \cup \{p_j, \cdots, p_m\}$

In this case, suppose without loss of generality that $u \in \{p_1, \cdots, p_i\}$. Let $Q' := uRvQp_i$ and let $Q'' := uRvQp_j$. Then Q'' is a proper generalized chord of C with endpoints u, p_j in $\mathbf{P}$. Let $G = G_0'' \cup G_1''$ be the natural Q''-partition of G, where $u\mathbf{P}p_j \subseteq G_0''$. Then G_1'' is a proper subgraph of G_Q^{large}. Furthermore, since R is either a chord of $P^\dagger$ or a 2-chord of $\mathbf{P}^\dagger$ with endpoints which are not adjacent in $\mathbf{P}^\dagger$, we have

$$|E(p_1\mathbf{P}u)| + |E(Q'')| + |E(p_j\mathbf{P}p_m)| \leq |E(p_1\mathbf{P}p_i)| + |E(Q)| + |E(p_j\mathbf{P}p_m)| \leq \frac{2N_{\text{mo}}}{3}$$

Thus, there exists a $C'' \in \mathcal{C} \setminus \{C\}$ with $C'' \subseteq G_0''$. Since $C'' \subseteq G_0''$ and Q does not separate C'' from $C \setminus P$, the generalized chord Q' of C separates C'' from $C \setminus \mathbf{P}$ as well. By Lemma 2.3.3, we have the following two inequalities:

$$|E(R)| + |E(p_iQv)| + |E(u\mathbf{P}p_i)| > \frac{11N_{\text{mo}}}{12}$$

$$|E(R)| + |E(vQp_j)| + |E(u\mathbf{P}p_j)| > \frac{11N_{\text{mo}}}{12}$$

Combining these, since $|E(R)| \leq 2$, we have $|E(Q)| + 2|E(u\mathbf{P}p_i)| + |E(p_i\mathbf{P}p_j)| + 2 \geq \frac{22N_{\text{mo}}}{12}$. On the other hand, since Q is defective, we have $|E(Q) \leq \frac{2N_{\text{mo}}}{3} - |E(p_1\mathbf{P}p_i)| - |E(p_j\mathbf{P}p_m)|$. Thus, we have the following:

$$\left(\frac{2N_{\text{mo}}}{3} - |E(p_1\mathbf{P}p_i)| - |E(p_j\mathbf{P}p_m)|\right) + 2|E(u\mathbf{P}p_i)| + |E(p_i\mathbf{P}p_j)| + 2 \geq \frac{22N_{\text{mo}}}{12}$$

Since $|E(\mathbf{P})| \leq \frac{2N_{\text{mo}}}{3}$, we have $|E(p_i\mathbf{P}p_j)| \leq \frac{2N_{\text{mo}}}{3} - |E(p_1\mathbf{P}p_i)| - |E(p_j\mathbf{P}p_m)|$. Thus, we have

$$2\left(\frac{2N_{\text{mo}}}{3} - |E(p_1\mathbf{P}p_i)| - |E(p_j\mathbf{P}p_m)|\right) + 2|E(u\mathbf{P}p_i)| + 2 \geq \frac{22N_{\text{mo}}}{12}$$

Since $|E(p_1\mathbf{P}p_i)| = |E(p_1\mathbf{P}u)| + |E(u\mathbf{P}p_i)|$, we have $\frac{4N_{\text{mo}}}{3} + 2 \geq \frac{22N_{\text{mo}}}{12}$, which is false. This completes Case 1.

Case 2: $u \in V(Q) \setminus \{p_i, p_j\}$

This case is easier than Case 1. In this case, let $Q' := p_iQuRvQp_j$. Let $G = G_0' \cup G_1'$, where $G_0' \cap G_1' = Q'$ and $p_i\mathbf{P}p_j \subseteq G_0'$. NOte that, since R us either a chord of Q or a 2-chord of Q with endpoints which are not adjacent in Q, we have $|E(p_1\mathbf{P}p_i)| + |E(Q')| + |E(p_j\mathbf{P}p_m)| \leq \frac{2N_{\text{mo}}}{3}$. Furthermore, since there is an internal vertex of xQy lying in $G_0' \setminus V(Q')$, we have $|V(G_1') < |V(G_Q^{\text{large}})|$. Thus, there exists a $C'' \in \mathcal{C} \setminus \{C\}$ such that Q' separates C'' from $C \setminus \mathbf{P}$, or else Q' is a defective path, contradicting the minimality of Q. In that case, since $C'' \not\subseteq G_Q^{\text{small}}$, the cycle $D := xQyRx$ separates C'' from C. Note that, since u, v are not adjacent in Q and $|E(Q)| \leq \frac{2N_{\text{mo}}}{3}$, each vertex of D is of distance at most $\frac{N_{\text{mo}}}{3}$ from $\{p_i, p_j\}$, and thus $d(D, C \setminus \mathbf{P}) \leq \frac{2N_{\text{mo}}}{3}$. Furthermore, $|E(D)| \leq \frac{2N_{\text{mo}}}{3} + 1$. But then, since D separates C from C'', we contradict Corollary 2.1.30. This completes the proof of Claim 2.3.13. ■

Combining Claim 2.3.12 and Claim 2.3.13, since C is an induced subgraph G, it follows that $C^\dagger$ is an induced subgraph of $G^\dagger$. By Claim 2.3.11, there is an L-coloring ϕ of G_Q^{small}. Since $C^\dagger$ has no chord in G_Q^{large}, ϕ is an L-coloring of the subgraph of G induced by G_Q^{small}.

Let L' be a list-assignment for $V(G_Q^{\text{large}})$, where $L'(x) = \{\phi(x)\}$ for each $x \in V(\mathbf{P}^\dagger)$ and $L'(x) = L(x)$ for all $x \in V(G_1 \setminus \mathbf{P}^\dagger)$. Note that $V(\mathbf{P}^\dagger)$ is L'-colorable, since $|L(p)| = 1$ for each $p \in V(\mathbf{P})$, $C^\dagger$ is a chordless cycle in G_Q^{large} and $V(\mathbf{P}^\dagger) \neq V(C^\dagger)$. Let C_{**} be the outer face of G_Q^{large} (i.e C_{**} is either C_* or $C^\dagger$). Let $\mathcal{T}' := (G_Q^{\text{large}}, \{C^\dagger\} \cup (\mathcal{C} \setminus \{C\}, L', C_{**})$. Since $V(\mathbf{P}^\dagger)$ is L'-colorable, $\mathcal{T}'$ is a tessellation. Since $V(C^\dagger) \neq V(\mathbf{P}^\dagger)$, $C^\dagger$ is an open $\mathcal{T}'$-ring. By assumption on Q, we have $|E(\mathbf{P}^\dagger)| \leq \frac{2N_{\text{mo}}}{3}$, so $\mathcal{T}'$ satisfies M0). M2) is immediate, and, by Claim 2.3.13, $\mathcal{T}'$ satisfies M1) as well. Since each vertex of $C^\dagger \setminus \mathring{\mathbf{P}}^\dagger$ lies in $V(C \setminus \mathring{\mathbf{P}})$, we immediately get that $\mathcal{T}'$ also satisfies distance conditions M3) and M4) of Definition 2.1.6 as well. Thus, $\mathcal{T}'$ is indeed a mosaic. Since $|V(G_Q^{\text{large}})| < |V(G)|$, G_Q^{large} admits an L-coloring, so ϕ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. This completes the proof of Lemma 2.3.10. $\square$

The above proves 2) of Theorem 2.3.2, and thus completes the proof of Theorem 2.3.2. This result has the following useful corollary:

Corollary 2.3.14. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let C be an open $\mathcal{T}$-ring, and let $\mathbf{P} := \mathbf{P}_{\mathcal{T}}(C)$. Let $\mathbf{P} := p_1 \cdots p_m$. Then the following hold.*

1) *Let Q be a proper generalized chord of C with its C-endpoints in $V(\mathbf{P})$, where $|E(Q)| \leq \frac{N_{\text{mo}}}{4}$, and let $1 \leq i < j \leq m$, where p_i, p_j are the endpoints of Q. Then $|E(p_1\mathbf{P}p_i)| + |E(Q)| + |E(p_j\mathbf{P}p_m)| > \frac{2N_{\text{mo}}}{3}$; AND*
2) *$|E(\mathbf{P})| = \lfloor \frac{2N_{\text{mo}}}{3} \rfloor$*

Proof. 1) is an immediate consequence of Theorem 2.3.2, since Q separates the path $p_i\mathbf{P}p_j$ from each element of $\mathcal{C} \setminus \{C\}$. Now we prove 2). Suppose $|E(\mathbf{P})| \neq \lfloor \frac{2N_{\text{mo}}}{3} \rfloor$. Thus, since $E(\mathbf{P}) \leq \frac{2N_{\text{mo}}}{3}$ we have $|E(\mathbf{P})| < \lfloor \frac{2N_{\text{mo}}}{3} \rfloor$. By our triangulation conditions, p_1, p_2 have a unique common neighbor $x \in D_1(C)$. Applying 1) to $Q := p_1xp_2$, we have $2 + (|E(\mathbf{P})| - 1) > \frac{2N_{\text{mo}}}{3}$, contradicting the fact that $|E(\mathbf{P})| < \lfloor \frac{2N_{\text{mo}}}{3} \rfloor$. $\square$

Chapter 3

Vertices of Distance One From Open Rings

Given a critical mosaic $\mathcal{T} = (G, \mathcal{C}, L, C_*)$, the goal of this chapter is to characterize the ball of distance one from each open ring $C \in \mathcal{C}$. In order to state our main theorem for Chapter 3, we first introduce the following definition.

Definition 3.0.1. Let $\mathcal{T} = (G, \mathcal{C}, L)$ be a chart, let $C \in \mathcal{C}$ be an open $\mathcal{T}$-ring, and let $\mathbf{P} := \mathbf{P}_{\mathcal{T}}(C)$. Given an $x \in D_1(C \setminus \mathring{\mathbf{P}})$, we say that x is a *$\mathcal{C}$-shortcut* if one of the following conditions holds.

1) There exists a $C' \in \mathcal{C} \setminus \{C\}$ such that $d(x, w_{\mathcal{T}}(C')) < d(C \setminus \mathring{\mathbf{P}}, w_{\mathcal{T}}(C'))$*; OR*
2) x has a neighbor in $\mathring{\mathbf{P}}$.

We now state our main result for Chapter 3, which we prove over the next three sections.

Theorem 3.0.2. *Let $\mathcal{T}$ be a critical mosaic, let C be an open $\mathcal{T}$-ring, and let $\mathbf{P} := \mathbf{P}_{\mathcal{T}}(C)$. Then G contains a unique cycle $C^1 := w_1 \cdots w_r$ such that $V(C^1) = D_1(C, G)$, and such that, letting $G = G_0 \cup G_1$ be the natural C^1-partition of G, where $C \subseteq G_0$, the following hold.*

1) For each $v \in V(C^1)$, the subgraph of G_0 induced by $\{v\} \cup N(v)$ is either an edge or a broken wheel with principal vertex v; AND
2) If $w_i w_j$ is a chord of C^1, and each of w_i, w_j has a neighbor in $C \setminus \mathring{\mathbf{P}}$, then the following hold.
   i) $|i - j| = 2$ and $w_i w_j \in E(G_1)$. In particular, $w_i w_j$ does not separate two vertices of $G_1 \setminus C^1$; AND
   ii) Neither of w_i, w_j has a neighbor in $\mathbf{P}$; AND
   iii) Each of w_i, w_j is a $\mathcal{C}$-shortcut.

3.1 2-Chords on One Side of the Precolored Path

The main result of this section is the following:

Lemma 3.1.1. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let $C \in \mathcal{C}$ be an open ring. Let $Q := v_1 v_2 v_3$ be a 2-chord of C with $v_1 v_2 v_3 \in \mathcal{K}^2(C, \mathcal{T})$. Then $V(G_Q^1) = \{v_2\} \cup V(C \cap G_Q^1)$, and G_Q^1 is a broken wheel with principal path $v_1 v_2 v_3$.*

Proof. Given a path $xwy \in \mathcal{K}^2(\mathcal{T}, C)$, we call xwy a *bad path* if $V(G^1_{xwy}) \neq \{v\} \cup V(C \cap G^1_{xwy})$. It suffices to prove that there does not exist a bad path. Fix $\mathbf{P} := \mathbf{P}_{\mathcal{T}}(C)$ and suppose toward a contradiction that there exists a bad path $xwy \in \mathcal{K}^2(C, \mathcal{T})$, and let xwy be chosen so as to minimize $|V(G^1_{xwy})|$ over all bad paths. By 3) of Theorem 2.2.4, each vertex of $G^1_{xwy} \setminus C$ has an L-list of size five. Let $S := V(G^1_{xwy}) \setminus (\{w\} \cup V(C))$.

Claim 3.1.2.

1) $|V(G^1_{xwy} \cap C)| \geq 4$; AND

2) For every vertex $v \in V(G^1_{xwy} \cap C) \setminus \{x, y\}$, v is not adjacent to w; AND

3) $x, y \notin V(\mathbf{P})$; AND

4) G^0_{xwy} is L-colorable.

Proof: Let $G^1_{xwy} \cap C = xv_1 \cdots v_t y$ for some integer t. If $t \leq 1$, then the cycle $xwyv_1 \cdots v_t$ has length at most 4. However, since xwy is a bad path, we have $V(G^1_{xwy}) \neq \{w, x, y\} \cup \{v_1, \cdots, v_t\}$, so $xwyv_1 \cdots v_t$ separates a vertex of $V(G^1_{xwy}) \setminus \{w, x, y, v_1, \cdots, v_t\}$ from G^0_{xwy}, contradicting the fact that $\mathcal{T}$ is a tessellation. Thus, we have $t \geq 2$, so $|V(G^1_{xwy} \cap C)| \geq 4$. This proves Fact 1.

Now let $i \in \{1, \cdots, t\}$ and suppose toward a contradiction that $v_i w \in E(G)$. Now consider the two paths xwv_i and $v_i wy$. Note that $|V(G^1_{xwv_i})| < |V(G^1_{xwy})|$, since $G^1_{xwv_i} \subseteq G^1_{xwy}$ and $y \notin V(G^1_{xwv_i})$. Likewise, $|V(G^1_{v_i wy})| < |V(G^1_{xwy})|$. Thus, by the minimality of xwy, we have $V(G^1_{xwv_i}) = \{w\} \cup \{x, v_1, \cdots, v_i\}$, and $V(G^1_{v_i wy}) = \{w\} \cup \{v_i, \cdots, v_t, y\}$. But $G^1_{xwy} = G^1_{xwv_i} \cup G^1_{v_i wy}$, so we then we have $V(G^1_{xwy}) = \{w\} \cup \{x, v_1, \cdots, v_t, y\}$, contradicting the fact that xwy is bad. Thus, for each $i \in \{1, \cdots, t\}$, v_i is not adjacent to w_i. This proves Fact 2.

Let $\mathbf{P} = p_1 p_2 \cdots p_m$. Suppose toward a contradiction that $\{x, y\} \cap \{p_1, p_m\} \neq \varnothing$, and suppose without loss of generality that $x = p_1$. Since $p_1 \cdots p_m \subseteq G^0_{xwy}$, we have $v_1 \notin V(\mathbf{P})$. Let $L(p_1) = \{c\}$ and let $a, b \in L(v_1) \setminus \{c\}$. Let L^* be a list-assignment for $G \setminus \{v_1\}$ where $L^*(u) = L(u) \setminus \{a, b\}$ for all $u \in N(v_1) \setminus V(C)$, and $L^*(u) = L(u)$ for all $u \in V(G \setminus \{v_i\}) \setminus (N(v_1) \cap V(C))$. Furthermore, there is a facial subgraph $C^\dagger$ of $G \setminus \{v_1\}$ such that $V(C^\dagger) = (V(C) \setminus \{v_1\}) \cup N(v_1)$. We claim now that $G \setminus \{v_1\}$ is L^*-colorable. We just need to check that the tessellation $\mathcal{T}' := (G \setminus \{v_1\}, (\mathcal{C} \setminus \{C\}) \cup \{C^\dagger\}, L^*)$ is a mosaic. Note that $C^\dagger$ is an open $\mathcal{T}'$-ring, and furthermore, since xwy separates $(\{v_1\} \cup N(v_1)) \setminus \{x, w, y\}$ from $G^0_{xwy} \setminus \{x, w, y\}$, $\mathcal{T}'$ still satisfies M1).

Thus, if $\mathcal{T}'$ is not a mosaic, then there exists a $C' \in \mathcal{C} \setminus \{C\}$ such that $d(w_{\mathcal{T}'}(C'), w_{\mathcal{T}'}(C^\dagger))$ violates either M3) or M4) of Definition 2.1.6. For any subgraph H of C' and any shortest $(H, C \setminus \mathring{\mathbf{P}})$-path P^* in G, P^* does not have v_1 as an endpoint, since any such P^* has nonempty intersection with $\{x, w, y\}$, and w is not adjacent to v_1. Since P^* does not have v_1 as an endpoint, we have $d(w_{\mathcal{T}'}(C'), w_{\mathcal{T}'}(C^\dagger)) \geq d(w_{\mathcal{T}}(C'), w_{\mathcal{T}}(C))$, so $\mathcal{T}'$ is indeed a mosaic.

Thus, by the minimality of $\mathcal{T}$, $G \setminus \{v_1\}$ admits an L^*-coloring ϕ. Then there is a color of $\{a, b\}$ not used among the vertices of $N(v_1) \setminus \{v_2\}$, so there is a color left over for v_1. Thus, ϕ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. This proves Fact 3 of Claim 3.1.2.

Now we show that G^0_{xwy} is L-colorable. Let G^* be a graph with obtained from $G \setminus S$ by adding to $G \setminus S$ the edges $\{wv_i : i = 1, \cdots, t\}$ in $G^1_{xwy} \setminus S$. The chart $\mathcal{T}' := (G^*, \mathcal{C}, L)$ does not violate distance conditions M3) or M4) of Definition 2.1.6, and, since $\{x, w, y\}$ separates P from $G^* \setminus G^0_{xwy}$, M1) is still satisfied as well, so $\mathcal{T}'$ is a mosaic. By assumption, $S \neq \varnothing$ and thus $|V(G^*)| < |V(G)|$. Since $(G^*, \mathcal{C}, L)$ is a mosaic, G^* is L-colorable by the minimality of $\mathcal{T}$. Since $G^0_{xwy} \subseteq G^*$, we get that G^0_{xwy} is L-colorable, as desired. This completes the proof of Claim 3.1.2. ■

By Fact 4 of Claim 3.1.2, there exists an L-coloring of G^0_{xwy}. Since this L-coloring of G^0_{xyw} does not extend to L-color G, and C is an induced subgraph of G, it follows from 1) of Proposition 1.5.1 that there exists a vertex $v^* \in S$ such that each of x, w, y is adjacent to v^*.

Since v^* is adjacent to x, y and $v^* \notin V(C)$, we have $v^* \in D_1(C, G)$. Furthermore, note that $G^1_{xv^*y} \subseteq G^1_{xwy}$, and thus $|G^1_{xv^*y}| < |V(G^1_{xvwy})|$, since $w \notin V(G^1_{xv^*y})$. By the minimality of xwy, the path xv^*y is not bad. Thus, $V(G^1_{xv^*y}) = \{v^*\} \cup V(G^1_{xv^*y} \cap C)$. In particular, since G is short-separation-free-, G^1_{xy} is a wheel with central vertex v^* adjacent to all the vertices of the cycle $xwyv_1 \cdots v_t$. Now we let ψ be an L-coloring of G^0_{xwy}.

Claim 3.1.3. *There is a set of two colors a, b such that the following hold.*

1) $L(v^*) = \{a, b\} \cup \{\psi(x), \psi(w), \psi(y)\}$; AND

2) $\{a, b\} \subseteq L(v_i)$ for each $i = 1, \cdots, t$.

Proof: Since ψ does not extend to an L-coloring of G, we have $|L_\psi(v^*)| = 2$ by 1) of Proposition 1.5.1. This proves 1). Furthermore, each of the two colors in $L_\psi(v^*)$ lies in $L(v_i)$ for each $i = 1, \cdots, t$, or else, applying Proposition 1.4.4, we obtain an L-coloring of $G^0_{xv^*y}$ which extends to an L-coloring of G. ■

Let L^* be a list-assignment for $G^0_{xv^*y}$ such that $L^*(v^*) = L(v) \setminus \{a, b\}$, and $L^*(u) = L(u)$ for all $u \in V(G^1_{xv^*y}) \setminus \{v^*\}$. Let $C'' := (C \cap G^0_{xv^*y}) + xv^*y$. Consider the tessellation $\mathcal{T}^* := (G^0_{xv^*y}, (\mathcal{C} \setminus \{C\}) \cup \{C''\}, L^*)$.

We claim that $\mathcal{T}^*$ is a mosaic. Since $\{x, w, y\}$ separates v^* from $\mathbf{P}$, $\mathcal{T}^*$ still satisfies M1), and M0) and M2) are immediate. Thus, if $\mathcal{T}^*$ is not a mosaic, then there exists a $C' \in \mathcal{C} \setminus \{C\}$ such that $d(w_{\mathcal{T}^*}(C'), w_{\mathcal{T}^*}(C''))$ violates either M3) or M4) of Definition 2.1.6. For any subgraph H of C' and any shortest (H, C)-path P in G, we have $v^* \notin V(P)$, since $v^* \notin V(C)$ and the deletion of $\{x, w, y\}$ separates v^* from C'. Thus, we have $d(w_{\mathcal{T}^*}(C'), w_{\mathcal{T}^*}(C'')) \geq d(w_{\mathcal{T}}(C'), w_{\mathcal{T}}(C))$. Since $\mathcal{T}$ is a mosaic, $\mathcal{T}^*$ is also a mosaic.

Since $|V(G^0_{xv^*y})| < |V(G)|$, there is an L^*-coloring ϕ of $G^0_{xv^*y}$ by the minimality of $\mathcal{T}$. Furthermore, since $\mathcal{T}$ is critical, ϕ does not extend to an L-coloring of G. Thus, since $\phi(v^*) \notin \{a, b\}$, we have $\phi(x), \phi(y) \in \{a, b\}$, or else we extend ϕ to an L-coloring of G by coloring the vertices of $v_1 \cdots v_t$ with the colors of $\{a, b\}$. Now let ϕ' be the restriction of ϕ to $V(G^0_{xwy})$.

Claim 3.1.4. *There is a pair of colors $r, s \in L_{\phi'}(v^*)$ with $r, s \notin \{a, b\}$.*

Proof: If $\phi'(x) = \phi'(y)$, then suppose without loss of generality that $\phi'(x) = \phi'(y) = a$. In this case, we have $|L_{\phi'}(v^*)| \geq 3$, since $|\{\phi'(x), \phi'(w), \phi'(y)\}| = 2$. Since $|L_{\phi'}(v^*) \setminus \{b\}| \geq 2$, we have our desired r, s. On the other hand, if $\phi'(x) \neq \phi'(y)$, then $\{\phi'(x), \phi'(y)\} = \{a, b\}$, and thus $L_{\phi'}(v^*)$ is disjoint to $\{a, b\}$. Since $|L_{\phi'}(v^*)| \geq 2$ in this case, we again have our desired r, s. ■

Let $r, s \in L_{\phi'}(a)$ be as in Claim 3.1.4. Since $\{a, b\} \subseteq L(v_1)$ by Claim 3.1.3 and $|L(v_1)| = 3$, and least one of r, s does not lie in $L(v_1)$. Suppose without loss of generality that $r \notin L(v_1)$. Then the coloring $(\phi'(x), r, \phi'(y))$ of xv^*y extends to an L-coloring of the broken wheel $G^1_{xv^*y}$, and thus ϕ' extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. Thus, there is no bad path in $\mathcal{K}^2(C, \mathcal{T})$, as desired.

Now let $v \in D_1(C, G)$ with $|N(v) \cap V(C)| \geq 2$ and let $xvy \in \mathcal{K}^2(C, \mathcal{T})$. Since xvy is not a bad path, we have $V(G^2_{xvy}) = \{v\} \cup V(G^1_{xvy} \cap C)$. Let $G^1_{xvy} \cap C = xv_1 \cdots v_t y$ for some $t \geq 0$. Since C is an induced cycle of G, it follows from our triangulation conditions that $vv_i \in E(G^1_{xvy})$ for each $i = 1, \cdots, t$, so G^1_{xvy} is indeed a broken wheel with principal path xvy. This completes the proof proves Lemma 3.1.1. □

3.2 3-Chords on One Side of the Precolored Path

In this section, we prove an analogue to Lemma 3.1.1 for 3-chords of C.

Lemma 3.2.1. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let $C \in \mathcal{C}$ be an open ring, and let $Q := x_1x_2x_3x_4$ be a 3-chord of C with $Q \in \mathcal{K}(C, \mathcal{T})$. Then the following hold.*

1) *If $|V(G_Q^1 \setminus Q)| > 1$, then G_Q^0 is L-colorable; AND*
2) *If $|V(G_Q^0)| < |V(G)$, $N(x_2) \cap V(C \cap G_Q^1) = \{x_1\}$, and $N(x_3) \cap V(C \cap G_Q^1) = \{x_4\}$ then there exists a $v \in V(G_Q^1) \setminus V(C)$ with at least three neighbors on Q; AND*
3) *$V(G_Q^1) \subseteq B_1(C)$; AND*
4) *If there exists a $j \in \{2, 3\}$ such that x_j is not a $\mathcal{C}$-shortcut, then $V(G_Q^1) = V(Q) \cup V(C \cap G_Q^1)$.*

Proof. We first prove 1). Consider the following cases:

Case 1: For each $j \in \{2, 3\}$, x_j is not a $\mathcal{C}$-shortcut.

In this case, we simply let D be the cycle $(C \cap G_Q^0) + Q$. Then, since neither x_2 nor x_3 is a $\mathcal{C}$-shortcut, the tessellation $(G_Q^0, (\mathcal{C} \setminus \{C\}) \cup \{D\}, L)$ is also a mosaic. Since $|V(G_Q^0)| < |V(G)|$, G_Q^0 is L-colorable by the minimality of $\mathcal{T}$.

Case 2: There exists a $j \in \{2, 3\}$ such that x_j is a $\mathcal{C}$-shortcut.

In this case, suppose without loss of generality that x_3 is a $\mathcal{C}$-shortcut. We break this into two subcases/

Case 2.1 x_2 is not a $\mathcal{C}$-shortcut.

In this case, let $G^\dagger$ be a graph obtained from G_Q^0 by adding to G_Q^0 a lone vertex v^* adjacent to each vertex of $\{x_2, x_3, x_4\}$. Set $P^\dagger := x_1x_2v^*x_4$, and let $C^\dagger := (G_Q^1 \cap C) + P^\dagger$. Let $L^\dagger$ be a list-assignment for $V(G^\dagger)$ where $L^\dagger(w) = L(w)$ for all $w \in V(G^\dagger) \setminus \{v^*\}$, and $L^\dagger(v^*)$ is an arbitrary 3-list. We claim that $\mathcal{T}' := (G^\dagger, (\mathcal{C} \setminus \{C\}) \cup \{C^\dagger\}, L^\dagger)$ is a mosaic.

M0) and M2) are immediate. Since x_2 is not a $\mathcal{C}$-shortcut, $\mathcal{T}'$ satisfies M1) and distance conditions M3) and M4) of 2.1.6. It thus suffices to check that $G^\dagger$ is short-separation-free. If not, then G_Q^0 contains a k-chord Q^* of Q, where Q^* has endpoints x_2, x_4 and $1 \leq k \leq 2$. Since G is short-separation-free, the deletion of Q^* leaves C' and x_3 in different connected components of $G \setminus Q^*$ for each $C' \in \mathcal{C}$, and also leaves $V(\mathring{\mathbf{P}}_{\mathcal{T}}(C))$ and x_3 in different connected components of $G \setminus Q^*$. But then, since each vertex of Q^* lies in $B_1(C, G)$, x_3 is not a $\mathcal{C}$-shortcut, contradicting our assumption. Thus, $\mathcal{T}'$ is indeed a mosaic. By assumption, we have $|V(G^\dagger)| < |V(G)|$, and thus $G^\dagger$ is $L^\dagger$-colorable by the minimality of $\mathcal{T}$, so G_Q^0 is indeed L-colorable, as desired.

Case 2.2 x_2 is a $\mathcal{C}$-shortcut.

In this case, let $G^\dagger$ be a graph obtained from G_Q^0 by adding to G_Q^0 a lone vertex v^* adjacent to each vertex of $\{x_1, x_2, x_3, x_4\}$. Set $P^\dagger := x_1v^*x_4$, and let $C^\dagger := (G_Q^0 \cap C) + P^\dagger$. Let $L^\dagger$ be a list-assignment for $V(G^\dagger)$ where $L^\dagger(w) = L(w)$ for all $w \in V(G^\dagger - v^*)$, and $L^\dagger(v^*)$ is an arbitrary 3-list. We claim that $\mathcal{T}' := (G^\dagger, (\mathcal{C} \setminus \{C\}) \cup \{C^\dagger\}, L^\dagger)$ is a mosaic. The distance conditions M3) and M4) are clearly satisfied in this case, and, since there is no chord of $C^\dagger$ with one endpoint in $V(\mathring{\mathbf{P}}_{\mathcal{T}}(C))$ where v^* is the other endpoint, we have M1). M0) and M2) are immediate. We just need to check that $\mathcal{T}'$ is a tessellation. It suffices to check that $G^\dagger$ is short-separation-free.

If $G^\dagger$ is not short-separation-free, then G_Q^0 contains a k-chord Q^* of Q, where $1 \leq k \leq 2$. Since G is short-separation-free, the deletion of Q^* leaves a vertex $x' \in \{x_2, x_3\}$ in a different connected component of $G \setminus Q^*$ from any vertex of $G_Q^0 \setminus Q^*$. Since each vertex of Q^* lies in $B_1(C)$, this contradicts our assumption that each of x_2, x_3 is a $\mathcal{C}$-shortcut. Thus, $\mathcal{T}'$ is indeed a mosaic. By assumption, we have $|V(G^\dagger)| < |V(G)|$, and thus $G^\dagger$ is $L^\dagger$-colorable by the minimality of $\mathcal{T}$. Thus, G_Q^0 is indeed L-colorable, as desired. This proves 1).

Now we prove 2). Let $N(x_2) \cap V(C \cap G_Q^1) = \{x_1\}$ and $N(x_3) \cap V(C \cap G_Q^1) = \{x_4\}$. By 1), G_Q^0 is L-colorable, since $|V(G_Q^1)| < |V(G)$ by assumption. Since $|V(G_Q^0)| < |V(G)|$, the path $C \cap G_Q^1$ has length at least two, or else G contains a separating cycle of length at most four. Thus, Q does not have a chord in G_Q^1, so the subgraph of G induced by G_Q^0 is L-colorable. Let ϕ be an L-coloring of $V(G_Q^0)$. Since $\mathcal{T}$ is critical, ϕ does not extend to L-color G. Since $N(x_2) \cap V(C \cap G_Q^1) = \{x_1\}$ and $N(x_3) \cap V(C \cap G_Q^1) = \{x_4\}$, it follows from 1) of Proposition 1.5.1 that there is a vertex of $V(G_Q^1) \setminus V(Q \cup C)$ with at least three neighbors in Q.

Now we prove 3). We proceed analogously to the proof of Lemma 3.1.1. Given a 3-chord Q of C with $Q \in \mathcal{K}(C, \mathcal{T})$, we call Q *defective* if $V(G_Q^1) \setminus V(Q \cup C) \not\subseteq B_1(C)$. Suppose toward a contradiction that a defective 3-chord Q of C exists, and, among all defective 3-chords of C, choose Q so that $|V(G_Q^1)|$ is minimized. By Proposition 2.1.22, the graph $G[V(C \cap G_Q^1)]$ is a chordless path with endpoints x_1, x_4. Furthermore, this path has length at least two, or else, since $V(G_Q^1) \setminus V(Q \cup C) \neq \varnothing$, the cycle $x_1x_2x_3x_4$ separates a vertex of $V(G_Q^1) \setminus V(Q)$ from $G_Q^0 \setminus Q$, contradicting short-separation-freeness. Thus, let $G[V(C \cap G_Q^1)] = q_0q_1 \cdots q_{t+1}$, where $t \geq 1$, $q_0 = x_1$ and $q_{t+1} = x_4$.

Since $V(G_Q^1) \setminus B_1(C, G) \neq \varnothing$, let $v^\dagger \in V(G_Q^1) \setminus B_1(C)$. Note now that $N(x_2) \cap V(C \cap G_Q^1) = \{x_1\}$. To see this, suppose toward a contradiction that there exists an $s \in \{1, \cdots, t+1\}$ such that $x_2q_s \in E(G)$. Suppose that $s = t+1$, and let $Q^* := x_1x_2x_4$. By Lemma 3.1.1, we have $V(G_{Q^*}^1) \setminus V(Q^* \cup C) = \varnothing$, and thus, the 3-cycle $x_2x_3x_4$ separates $v^\dagger$ from $G_Q^0 \setminus Q$, contradicting short-separation-freeness. Thus, we have $s \in \{1, \cdots, t\}$. Let $Q^* := x_1x_2q_s$ and let $Q^{**} = q_sx_2x_3x_4$. By Lemma 3.1.1, we have $V(G_{Q^*}^1) = \{x_1, x_2\} \cup \{q_1, \cdots, q_s\}$, and thus $v^\dagger \in V(G_{Q^*}^1)$. By the minimality of Q, we have $V(G_{Q^{**}}^1) = \{v^\dagger, x_2, x_3\} \cup \{q_s, \cdots, q_{t+1}\}$, and thus $V(G_Q^1) = V(Q) \cup \{v^\dagger\} \cup \{q_1, \cdots, q_t\}$, contradicting the fact that Q is defective. The same argument as above shows that $N(x_3) \cap V(C \cap G_Q^1) = \{x_4\}$.

By 2), there exists $v^* \in V(G_Q^1) \setminus V(Q \cup C)$ such that v^* has three neighbors among the vertices of Q. Since v^* has at least one neighbor in $\{x_1, x_4\}$, we have $v^* \in B_1(C, G)$. In particular, $v^* \neq v^\dagger$. Suppose without loss of generality that $x_1, x_3 \in N(v^*)$ and let $Q^* := x_1v^*x_3x_4$. If $v^\dagger \in V(G_{Q^*}^1)$, then Q^* is a defective path of $\mathcal{K}^3(C, \mathcal{T})$. Since $x_2 \notin V(G_{Q^*}^1)$, we have $|V(G_{Q^*}^1)| < |V(G_Q^1)|$. Thus, we have $v^\dagger \notin V(G_{Q^*}^1)$, and thus the 4-cycle $x_1v^*x_3v_4$ separates v^* from $v^\dagger$, contradicting short-separation-freeness.

Now we prove 4). We proceed analogously to the proof of 2). Given a 3-chord Q of C, we call Q a *bad* 3-chord if the following hold.

1) $Q \in \mathcal{K}(C, \mathcal{T})$*; AND*
2) There exists an $x \in V(\mathring{Q})$ such that x is not a $\mathcal{C}$-shortcut*; AND*
3) $V(G_Q^1) \neq V(Q) \cup V(C \cap G_Q^1)$.

Suppose toward a contradiction that there exists a $Q \in \mathcal{K}^3(C, \mathcal{T})$ which is bad, and, among all such elements of $\mathcal{K}^3(C, \mathcal{T})$, choose Q so that $|V(G_Q^1)|$ is minimized. Let $Q := x_1x_2x_3x_4$ and suppose without loss of generality that x_2 is not a $\mathcal{C}$-shortcut. By Proposition 2.1.22, the graph $G[V(C \cap G_Q^1)]$ is a chordless path with endpoints x_1, x_4. Furthermore, this path has length at least two, or else, since Q is bad, the cycle $x_1x_2x_3x_4$ separates a vertex of $V(G_Q^1) \setminus V(Q)$ from C', contradicting short-separation-freeness. Thus, let $G[V(C \cap G_Q^1)] = q_0q_1 \cdots q_{t+1}$, where $t \geq 1$, $q_0 = x_1$ and $q_{t+1} = x_4$. Now we note the following :

Claim 3.2.2. *Q is an induced subpath of G_Q^1.*

Proof: Suppose towards a contradiction that G_Q^1 contains a chord of Q. Since C is a chordless cycle, G_Q^1 either contains the edge x_2x_4 or x_1x_3. Suppose that $x_1x_3 \in E(G_Q^1)$ and let $Q^* := x_1x_3x_4$. By Lemma 3.1.1, x_3 is adjacent to each vertex of $\{q_0, q_1, \cdots, q_{t+1}\}$. Thus, we have $V(G_Q^1) = V(Q) \cup \{q_1, \cdots, q_t\}$, or else the triangle $x_1x_2x_3$ separates a vertex of $G_Q^1 \setminus V(C \cup Q)$ from G_Q^0, contradicting the fact that G is short-separation-free. Since $V(G_Q^1) = V(Q) \cup \{q_1, \cdots, q_t\}$, we contradict the fact that Q is bad. The same argument shows that $x_2x_3 \notin E(G_Q^1)$. Thus, Q is indeed an induced subpath of G_Q^1, as desired. ■

Since Q is bad, we have $|V(G_Q^0)| < |V(G)|$, and thus G_Q^0 is L-colorable by 1). Since Q is an induced subpth of G_Q^1, and $C \cap G_Q^1$ is a chordless path of length at least two, it follows that the subgraph of G induced by $V(G_Q^0)$ is L-colorable. We claim now that $N(x_2) \cap V(C \cap G_Q^1) = \{x_1\}$, and, likewise, $N(x_3) \cap V(C \cap G_Q^1) = \{x_4\}$. Suppose toward a contradiction that there exists a $j \in \{1, \cdots, t+1\}$ such that $x_2q_j \in E(G)$. Since Q is an induced subpath of G_Q^1 by Claim 3.2.2, we have $j \neq t+1$. Let $Q^* := x_1x_2q_j$ and let $Q^{**} := q_jx_2x_3x_4$. By Lemma 3.1.1, we have $V(G_{Q^*}^1) = \{q_1, \cdots, q_j\} \cup \{x_1, x_2\}$. Since $V(G_Q^1) \neq V(Q) \cup V(C \cap G_Q^1)$, there exists a $w \in V(G_Q^1) \setminus V(C \cup Q)$ with $w \in V(G_{Q^*}^1)$.

Since $x_2$is not a $\mathcal{C}$-shortcut, Q^{**} is also a bad path of $\mathcal{K}^3(C, \mathcal{T})$, yet $|V(G_{Q^{**}}^1)| < |V(G_Q^1)|$, so this contradicts the minimality of Q. We conclude that $N(x_2) \cap V(C \cap G_Q^1) = \{x_1\}$, as desired. Now suppose toward a contradiction that there exists a $j \in \{0, 1, \cdots, t\}$ such that $x_3q_j \in E(G)$. Since Q is an induced subpath of G_Q^1 by Claim 3.2.2, we have $j \neq 0$. Let $Q^* := x_4x_3q_j$ and let $Q^{**} := x_1x_2x_3q_j$. Since $x_2 \in V(Q^{**})$, an identical argument to the above shows that Q^{**} is also a bad element of $\mathcal{K}^3(C, \mathcal{T})$ with $|V(G_{Q^{**}}^1)| < |V(G_Q^1)|$, contradicting the minimality of Q.

By 2), there exists a $v^* \in V(G_Q^1) \setminus V(Q \cup C)$ such that v^* has at least three neighbors among the vertices of Q. We claim now that $V(G_Q^1) = V(Q) \cup \{v^*\} \cup \{q_1, \cdots, q_t\}$. To see this, suppose toward a contradiction that there is a $v^\dagger \in V(G_Q^1) \setminus V(Q \cup C)$ with $v^\dagger \neq v^*$. Note that, since $N(v^*) \cap V(Q)| \geq 3$, G contains a 4-chord Q^* of C with endpoints x_1, x_4, such that $Q^* \setminus \{x_1, x_4\} = v^*x_j$ for some $j \in \{2, 3\}$. Then $|V(G_{Q^*}^1)| < |V(G_Q^1)|$, since at least one of $\{x_2, x_3\}$ lies outside of $V(G_{Q^*}^1)$.

Now, v^* is not a $\mathcal{C}$-shortcut, since $V(Q) \subseteq B_1(C, G)$ and the deletion of Q leaves v^* in a different connected component of $G \setminus Q$ from $V(\mathring{\mathbf{P}}_{\mathcal{T}}(C)$ and from every $C' \in \mathcal{C} \setminus \{C\}$. Furthermore, we have $v^\dagger \in V(G_{Q^*}^1)$, or else $G_{Q^*}^0 \cap G_Q^1$ contains a cycle of length at most four which separates $v^\dagger$ from $G_Q^0 \setminus Q$, contradicting short-separation-freeness. But since v^* is not a $\mathcal{C}$-shortcut and $|V(G_{Q^*}^1)| < |V(G_Q^1)|$, this contradicts the minimality of Q.

Thus, we conclude that $V(G_Q^1) = V(Q) \cup \{v^*\} \cup \{q_1, \cdots, q_t\}$. Since Q is an induced subpath of G_Q^1, v^* is adjacent to all four vertices of Q by our triangulation condition. Applying Lemma 3.1.1, v^* is adjacent to each vertex of $V(C \cap G_Q^1)$, and furthermore, $V(G_Q^1) = V(Q) \cup V(C \cap G_Q^1) \cup \{v^*\}$. Since v^* is adjacent to each of x_1, x_4, let $Q^* := x_1v^*x_4$.

Claim 3.2.3. *There exist three L-colorings ϕ_1, ϕ_2, ϕ_3 of $G_{Q^*}^0$ such that the following hold.*

1) $|\{\phi_1(v^*), \phi_2(v^*), \phi_3(v^*)\}| = 3$*; AND*
2) $\{\phi_1(v^*), \phi_2(v^*), \phi_3(v^*)\} \subseteq L(q_j)$ *for each* $j = 1, \cdots, t$.

Proof: Let $C^\dagger := (C \cap G_Q^0) + Q^*$ and let $G^\dagger = G \setminus \{q_1, \cdots, q_t\}$. We claim that $\mathcal{T}^\dagger := (G^\dagger, (\mathcal{C} \setminus \{C\}) \cup \{C^\dagger\}, L)$ is a mosaic. To see this, just note that, since v^* is not a $\mathcal{C}$-shortcut, $\mathcal{T}^\dagger$ satisfies the distance conditions of Definition 2.1.6 and satisfies M1) as well. Since $|V(G^\dagger)| < |V(G)|$, $G^\dagger$ admits an L-coloring ϕ_1 by the minimality of $\mathcal{T}$.

Let L^* be a list-assignment for $G^\dagger$ where $L^*(w) = L(w)$ for all $w \in V(G^\dagger) \setminus \{v^*\}$ and $L^*(v^*) = L(v^*) \setminus \{\phi_1(v^*)\}$.

Then $(G^\dagger, (\mathcal{C} \setminus \{C\}) \cup \{C^\dagger\}, L^*)$ is again a mosaic by the minimality of $\mathcal{T}$ and thus admits an L^*-coloring ϕ_2. Removing $\phi_2(v^*)$ from $L^*(v^*)$ and applying the argument a second time, we obtain our desired three colorings ϕ_1, ϕ_2, ϕ_3 of $G^1_{Q^*}$ satisfying 1). We claim that ϕ_1, ϕ_2, ϕ_3 also satisfy 2), 3), and 4).

Suppose toward a contradiction that there exists a $j \in \{1, \cdots, t\}$ and an $i \in \{1, 2, 3\}$ such that $\phi_i(v^*) \notin L(q_j)$. Then, since $t \geq 1$, ϕ_i extends to an L-coloring of G, contradicting our assumption. Furthermore, if there exists an $i \in \{1, 2, 3\}$ such that $\phi_i(x_1) \notin \{\phi_1(v^*), \phi_2(v^*), \phi_3(v^*)\}$, then, by 2), we have $|L(q_1) \setminus \{\phi_i(x_1)| \geq 3$, and thus ϕ_i extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. The same argument shows that $\phi_i(x_4) \in \{\phi_1(v^*), \phi_2(v^*), \phi_3(v^*)\}$ for each $i = 1, 2, 3$. ■

Let $G^\dagger$ be the graph obtained from G by deleting the vertices $\{q_1, \cdots, q_t\}$ and the edge x_1v^*. Let $P^\dagger := x_2v^*x_4$ and let $C^\dagger := (C \cap G^0_Q) + P^\dagger$. Let $C^\dagger_*$ be the outer face of $G^\dagger$ and let ϕ_1, ϕ_2, ϕ_3 be as in Claim 3.2.3.Since $|L(v^*)| \geq 5$, let $r, s \in L(v^*) \setminus \{\phi_1(v^*), \phi_2(v^*), \phi_3(v^*)\}$. Let $L^\dagger$ be a list-assignment for $G^\dagger$ where $L^\dagger(x_2) = L(x_2) \setminus \{r, s\}$, and $L^\dagger(w) = L(w)$ for all $w \in V(G^\dagger) \setminus \{x_2\}$. Consider the tuple $\mathcal{T}^\dagger := (G^\dagger, (\mathcal{C} \setminus \{C\}) \cup \{C^\dagger, L^\dagger, C^\dagger_*)$.

By assumption, x_2 is not a $\mathcal{C}$-shortcut, and since v^* is not a $\mathcal{C}$-shortcut, $\mathcal{T}^\dagger$ is a mosaic. Since $t \geq 1$, we have $|V(G^\dagger)| < |V(G)|$, and thus $G^\dagger$ admits an L-coloring ϕ^* by the minimality of $\mathcal{T}$. Note that $\phi^*(x_1) \in \{\phi_1(v^*), \phi_2(v^*), \phi^*(v_3)\}$, or else, by Claim 3.2.3, we have $|L(q_1) \setminus \{\phi^*(x_1)\}| \geq 3$, and thus ϕ^* extends to an L-coloring of G, contradicting the minimality of $\mathcal{T}$. The same argument shows that $\phi^*(x_4) \in \{\phi_1(v^*), \phi_2(v^*), \phi^*(v_3)\}$. Thus, the coloring ϕ^* uses at most one of $\{r, s\}$ among the vertices of $V(Q)$. Suppose without loss of gnerality that the color r is not used by ϕ^* on the vertices of $V(Q)$. Let ϕ^{**} be the restriction of ϕ^* to $V(G^0_Q)$. We extend ϕ^{**} to an L-coloring of $G^0_{Q^*}$ by coloring v^* with r. By Claim 3.2.3, we have $|L(q_j) \setminus \{r\}| \geq 3$ for each $j = 1, \cdots, t$, and thus ϕ^{**} extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. This completes the proof of Lemma 3.2.1. □

Applying the above, we have the following.

Proposition 3.2.4. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let $C \in \mathcal{C}$, and let $Q := x_1x_2x_3x_4$ be a 3-chord of C with $Q \in \mathcal{K}(C, \mathcal{T})$. Then the following hold.*

1) *There exists a path $P \subseteq G^1_Q$ with endpoints x_2, x_3 such that $V(P) = V(G^1_Q) \setminus V(C)$ and $V(P) \subseteq B_1(C)$, and $|V(P)| \leq 3$; AND*
2) *If $V(G^1_Q \setminus Q) \not\subseteq V(C)$, then $x_1, x_4 \notin V(\mathbf{P}_{\mathcal{T}}(C))$.*

Proof. Suppose towards a contradiction that there exists a 3-chord Q of C satisfying the conditions of Proposition 3.2.4 such that G^1_Q does not admit a path with endpoints x_2, x_3 visiting all the vertices of $G^1_Q \setminus C$. Choose Q such that, with respect to this property, $V(G^1_Q)$ is minimized. Note that $|V(G^1_Q \cap C)| \geq 3$, or else, since G is short-separation-free, we have $V(G^1_Q) = V(Q)$, and the path x_2x_3 contains all the vertices $G^1_Q \setminus C$, contradicting our assumption. Thus, let $G^1_Q \cap C = x_1v_1 \cdots v_\ell x_4$ for some $\ell \geq 2$.

Suppose towards a contradiction that $N(x_2) \cap V(G^1_Q \cap C) \neq \{x_1\}$. Since C is an induced cycle of G, and $x_1x_4 \notin E(G)$, there exists a $j \in \{1, \cdots, \ell\}$ such that $x_2v_j \in E(G^1_Q)$. Let $Q^* := v_jx_2x_3x_4$. By the minimality of Q, there exists a path $P^* \subseteq G^1_{Q^*}$ with endpoints x_2, x_3, such that $V(P^*) = V(G^1_{Q^*} \setminus C)$. By Lemma 3.1.1, we have $(VG^1_{x_1x_2v_j}) \setminus V(C) = \{v_2\}$, and thus $V(P^*) = V(G^1_Q \setminus C)$, contradicting our assumption.

Thus, we have $N(x_2) \cap V(G^1_Q \cap C) = \{x_1\}$, and an identical argument shows that $N(x_3) \cap V(G^1_Q \cap C) = \{x_4\}$. Since $V(G^1_Q) \neq V(Q)$, we get by 2) of Lemma 3.2.1 that there exists a $v^* \in V(G^1_Q) \setminus V(Q \cup C)$ such that v^* has at least three neighors on Q. If $V(Q) \subseteq N(v^*)$, then, since G is short-separation-free, we have $V(G^1_Q) = \{x_2, x_3\} \cup V(G^1_{x_1v^*x_4})$.

By Lemma 3.1.1, we have $V(G^1_{x_1v^*x_4}) \setminus V(C) = \{v^*\}$, and thus $V(G^1_Q) \setminus V(C) = \{v^*, x_2, x_3\}$. Since G^1_Q contains the path $x_2x_3v^*$, this contradicts our assumption.

Thus, we have $N(v^*) \cap V(Q)| = 3$. If $G[N(v^*) \cap V(Q)]$ is not a subpath of Q of length two, then, since G is short-separation-free, it follows from our triangulation conditions that G^1_Q contains one of the edges x_1x_3, x_2x_4, which is false. Thus, suppose without loss of generality that $V(Q) \cap N(v^*) = \{x_1, x_2, x_3\}$ and let $Q^* := x_1v^*x_3x_4$.

By the minimality of Q, there is a path $P^* \subseteq G^1_{Q^*}$ with endpoints v^*, x_3 such that $V(P^*) = V(G^1_{Q^*}) \setminus V(C)$. Since $x_2 \in V(G^1_{Q^*} \setminus Q^*)$, we have $x_2 \notin V(P^*)$. Since G is short-separation-free, we have $V(G^1_Q) \setminus V(G^1_{Q^*}) = \{x_2\}$, and thus the path $x_2v^*P^*x_3$ contains alal the vertices of $G^1_Q \setminus C$, contradicting our assumption. Thus, for any 3-chord Q of C satisfying the conditions of Proposition 3.2.4, Q admits a path P with endpoints x_2, x_3 visiting all the vertices of $G^1_Q \setminus C$. By Lemma 3.2.1, we have $V(P) \subseteq B_1(C, G)$. We claim now that $|V(P)| \leq 3$. If $P = x_2x_3$, we are done, so now suppose that $P := x_2v_1 \cdots v_sx_3$ for some $s \geq 1$ and suppose toward a contradiction that $s > 1$.

Note that no vertex of $\{v_1, \cdots, v_s\}$ is a $\mathcal{C}$-shortcut, since the deletion of Q separates each element of $\{v_1, \cdots, v_s\}$ from each $C' \in \mathcal{C} \setminus \{C\}$ and from $V(\mathring{\mathbf{P}}_{\mathcal{T}}(C))$. We now note that there does to exist a chord of P of the form v_iv_j for some $1 \leq i < j-1 \leq s-1$, or else there exists a 3-chord Q^* of C with $Q^* \subseteq G^1_Q$, $v_iv_j \subseteq Q^*$, and $v_{i+1} \in V(G^1_{Q^*}) \setminus V(Q^*)$. Since no vertex of $\{v_1, \cdots, v_s\}$ is a $\mathcal{C}$-shortcut, this contradicts Lemma 3.2.1

Thus, $P - x_2x_3$ is a chordless path in G. By Lemma 3.2.1, there is a vertex v_i of $\{v_1, \cdots, v_s\}$ with at least three neighbors on Q. If v_i is adjacent to each vertex of Q, then, by Lemma 3.1.1, we have $V(G^1_{x_1v_ix_4}) \setminus V(C) = \{v_i\}$ and thus, by short-separation-freeness, we have $V(G^1_Q) \setminus V(C) = \{v_i, x_2, x_3\}$, contradicting our assumption that $s > 1$. Thus, again applying our triangulation conditions, there is an $i \in \{1, \cdots, s\}$ such that $G[N(v_i) \cap V(Q)]$ is a subpath of Q of length precisely two, so suppose without loss of generality that $G[N(v_i) \cap V(Q)] = x_1x_2x_3$ and let $Q^* := x_1v_ix_3x_4$. Then $G^1_{Q^*}$ contains each vertex of $\{v_1, \cdots, v_s\} \setminus \{v_i\}$. Since v_i is not a $\mathcal{C}$-shortcut, this contradicts Lemma 3.2.1. Thus, we have $s = 1$, as desired. This proves 1).

Now we prove 2). Suppose there exists a 3-chord $Q := x_1x_2x_3x_4$ of C with $Q \in \mathcal{K}(C, \mathcal{T})$, where x_1 is an endpoint of $\mathbf{P}_{\mathcal{T}}(C)$ and $V(G^1_Q \setminus Q) \not\subseteq V(C)$. Among all such 3-chords of C, we choose Q so that $|V(G^1_Q)|$ is minimized. Since x_4 is not an internal vertex of $\mathbf{P}_{\mathcal{T}}(C)$, we have $x_4 \in V(C \setminus P)$, or else we we contradict 1) of Theorem 2.3.2. By 1), there is a lone vertex v^* such that $G^1_Q \setminus C$ is the triangle $x_2x_3v^*$. If $x_3x_1 \in E(G^1_Q)$, then the cycle $x_1x_3x_2$ separates v^* from G^0_Q, which is false. The same argument shows that $x_2x_4 \notin E(G^1_Q)$.

If x_3 has a neighbor $u \in V(C \cap G^1_Q) \setminus \{x_1\}$, then we have $v^* \in V(G^1_{x_1x_2x_3u})$, contradicting the minimality of Q. Thus, we have $N(x_3) \cap V(C \cap G^1_Q) = \{x_4\}$. Since $x_2x_4 \notin E(G)$ and C is induced, we have $v^*x_4 \in E(G)$ by our triangulation conditions. Since x_1 is precolored, let $L(x_1) = \{c\}$. Furthermore, let $C \cap G^1_Q = x_1u_1 \cdots u_t$ for some $t \geq 1$, where $u_t = x_4$. We have $t > 1$, or else the cycle $x_1x_2x_3x_4$ separates v^* from G^0_Q, which is false. In particular, since $x_1x_3, x_2x_4 \notin E(G^1_Q)$ and C is an induced subgraph of G, Q is an induced subgraph of G^1_Q. Let a, b be two colors in $L(u_1) \setminus \{c\}$. Now consider the following cases:

Case 1: $v^*x_1 \in E(G)$

In this case, let $Q^* := x_1v^*x_4$ and let L' be a list-assignment for $G^1_{Q^*}$ where $L'(v) = L(v)$ for all $v \in V(G^0_{Q^*}) \setminus \{v^*\}$ and $L'(v^*) = L(v^*) \setminus \{a, b\}$. Let $C' := (C \cap G^0_{Q^*}) + Q^*$ and let C^{**} be the outer face of $G^0_{Q^*}$. Since v^* is not a C-shortcut, the tuple $\mathcal{T}' := (G^0_{Q^*}, (\mathcal{C} \setminus \{C\}) \cup \{C'\}, L', C^{**})$ is a mosaic, and since $|V(G^0_{Q^*})| < |V(G)|$, $G^0_{Q^*}$ admits an L'-coloring ϕ'. Furthermore, $G^1_{Q^*}$ consists of a broken wheel with principal path $x_4v^*x_1$, and since $c \neq \phi'(v^*)$, either $c \notin L(u_1)$ or $\phi'(v^*) \notin L(u_1)$. In either case, ϕ' extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical.

Case 2: $v^*x_1 \notin E(G)$

In this case, by our triangulation conditions, together with Lemma 3.1.1, there is an $i \in \{1, \cdots, t-1\}$ such that $N(x_2) \cap (C \cap G_Q^1) = \{x_1, u_1, \cdots, u_i\}$ and $N(v^*) \cap V(C \cap G_Q^1) = \{u_i, \cdots, u_t\}$. Since $\deg(v^*) > 4$ we have $i < t-1$. Let $Q^\dagger := x_4v^*u_i$.

Claim 3.2.5. *Let $S \subseteq L(v^*)$ with $|S| = 3$ and let $L^\dagger$ be a list-assignment for $G_{Q^\dagger}^0$, where $L^\dagger(v^*) = S$ and otherwise $L^\dagger = L$. Then $G_{Q^\dagger}^0$ is $L^\dagger$-colorable.*

Proof: Let $C^\dagger := (C \cap G_{Q^\dagger}^0) + Q^\dagger$ and let $C_*^\dagger$ be the outer face of $G_{Q^\dagger}^0$. Since v^* is not a C-shortcut, $\mathcal{T}'' := (G_{Q^\dagger}^0, (\mathcal{C} \setminus \{C\}) \cup \{C^\dagger\}, L^\dagger, C_*^\dagger)$ is a mosaic. Since $u_{i+1} \in V(G) \setminus V(G_{Q^\dagger}^0)$, we have $|V(G_{Q^\dagger}^0)| < |V(G)|$, so $G_{Q^\dagger}^0$ admits an $L^\dagger$-coloring by the minimality of $\mathcal{T}$. ■

Applying the above, since $|L(v^*)| = 5$, let ψ_1, ψ_2, ψ_3 be L-colorings of $G_{Q^\dagger}^0$ using different colors on v^*. Let H_1 be the broken wheel with principal path $x_1x_2u_i$, where $H_1 \setminus \{x_2\} = x_1u_1 \cdots u_i$, and let H_2 be the broken wheel with principal path $u_iv^*u_t$, where $H_2 \setminus \{v^*\} = u_i \cdots u_t$.

For each $j = 1, 2, 3$, let $d_j := \psi_j(v^*)$. Note that $L(u_m) = \{d_1, d_2, d_3\}$ for each $m = i+1, \cdots, t-1$, or else there exists a $j \in \{1, 2, 3\}$ such that ψ_j extends to an L-coloring of G, which is false. Likewise, we have $\{\psi_1(u_i), \psi_2(u_i), \psi_3(u_i)\} \subseteq L(u_{i+1})$ and $\{\psi_1(u_t), \psi_2(u_t), \psi_3(u_t)\} \subseteq L(u_{t-1})$, or else there exists a $j \in \{1, 2, 3\}$ such that ψ_j extends to H_2 and thus to G, which is false. Let r, s be two colors in $L(v^*) \setminus \{d_1, d_2, d_3\}$. Note now that the colorings $\{\psi_1, \psi_2, \psi_3\}$ use at least two colors of $\{d_1, d_2, d_3\}$ on u_i, since $\{\psi_i(v^*), \psi_2(v^*), \psi_3(v^*)\} = \{d_1, d_2, d_3\}$. Likewise, the colorings $\{\psi_1, \psi_2, \psi_3\}$ use at least two colors of $\{d_1, d_2, d_3\}$ on u_t.

Claim 3.2.6. *For each $j = 1, 2, 3$, $\{\psi_j(x_2), \psi_j(x_3)\} = \{r, s\}$. Furthermore, each of r, s appears at least once among $\{\psi_1(x_2), \psi_2(x_2), \psi_3(x_2)\}$.*

Proof: If there exists a $j \in \{1, 2, 3\}$ such that this does not hold, then, starting with ψ_j and uncoloring v^*, there is a color among $\{r, s\}$ left over for v^*, since $\psi(x_4) \in \{d_1, d_2, d_3\}$ and $\psi(u_1) \in \{d_1, d_2, d_3\}$. But then the resulting L-coloring extends to H_2 and thus to G, which is false.

Now suppose that $\{\psi_1(x_2), \psi_2(x_2), \psi_3(x_2)\} = \{r\}$. Thus, for each $j \in \{1, 2, 3\}$, the coloring $(cr, \psi_j(u_i),)$ of $x_1x_2u_i$ extends to an L-coloring of H_1, since ψ_j is an L-coloring of $G_{Q^\dagger}^0$. Thus, since $\mathcal{Z}_{H_1}(c, r, \bullet) \neq \varnothing$, there exists a $q \in \mathcal{Z}_{H_1}(c, r, \bullet)$ with $q \notin \{d_1, d_2, d_3\}$. Now we simply choose a $j \in \{1, 2, 3\}$ and restrict ψ_j to G_Q^0. Then we 2-color the path $u_t \cdots u_{i+1}$ with colors from $\{d_1, d_2, d_3\}$, starting with $\psi_j(u_t)$. Since $q \notin \{d_1, d_2, d_3\}$, there is a color left over for v^* after coloring u_i with q, and the resulting coloring extends to G, since $q \in \mathcal{Z}_{H_1}(c, r, \bullet)$. This contradicts the fact that $\mathcal{T}$ is critical. ■

Now we are ready to finish the proof of 2) of Proposition 3.2.4. Since $\{\psi_1, \psi_2, \psi_3\}$ use at least two colors of d_1, d_2, d_3 on u_i, at least one of r, s does not lie in u_i, so let $r \notin L(u_i)$. By Claim 3.2.6, there is a $j \in \{1, 2, 3\}$ with $\psi_j(x_2) = r$, so, without loss of generality, let $\psi_1(x_2) = r$. Since $r \notin L(u_i)$, we have $|\mathcal{Z}_{H_1}(c, r, \bullet)| \geq 2$. Furthermore, for each $d \in \{d_1, d_2, d_3\} \setminus \{\psi_1(u_t)\}$, we have $\mathcal{Z}_{H_1}(c, r, \bullet) \cap \mathcal{Z}_{H_2}(\bullet, d, \psi_1(u_t)) = \varnothing$, or else, uncoloring v^* and then coloring it with d, we produce an L-coloring of $V(G_Q^0) \cup \{v^*\}$ which extends to G, which is false. Without loss of generality, let $d_2 = \psi_1(u_t)$. Thus, $\mathcal{Z}_{H_2}(\bullet, d_1, d_2) \cup \mathcal{Z}_{H_2}(\bullet, d_3, d_2)$ consists of a lone color, so we have $d_1, d_3 \in L(u_i)$, and, letting $\{r'\} = \mathcal{Z}_{H_2}(\bullet, d_1, d_2) \cup \mathcal{Z}_{H_2}(\bullet, d_3, d_2)$, we have $L(u_i) = \{r', d_1, d_3\}$ and $\mathcal{Z}_{H_1}(c, r, \bullet) = \{d_1, d_3\}$.

Since $\{\psi_1, \psi_2, \psi_3\}$ use at least two colors of d_1, d_2, d_3 on u_t, there is a coloring among ψ_1, ψ_2, ψ_3 using one of d_1, d_3 on u_t, so, without loss of generality, suppose that $\psi_2(u_t) = d_3$. If $\psi_2(x_2) = r$, then, since d_1, d_3 lie in $L(u_i)$, we

2-color the path $u_i \cdots u_t$ with d_1, d_3, starting with the precoloring $\psi_2(u_t)$ of u_t. Since $\mathcal{Z}_{H_1}(c, r, \bullet) = \{d_1, d_3\}$, this coloring extends to $G \setminus \{v^*\}$, and the color d_2 is left over for v^*, so we have an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. Thus, we have $\psi_2(x_2) = s$ and $\{d_1, d_3\} \not\subseteq \mathcal{Z}_{H_1}(c, s, \bullet)$. If H_1 is a triangle, then we have $c \in \{d_1, d_3\}$, contradicting the fact that $\mathcal{Z}_{H_1}(c, r, \bullet) = \{d_1, d_3\}$. Thus, H_1 is not a triangle, and $s \in L(u_m)$ for each $m = 1, \cdots, i-1$. Furthermore, $r \in L(u_m)$ for each $m = 1, \cdots, i-1$, or else $|\mathcal{Z}_{H_1}(c, r, \bullet)| = 3$, contradicting the fact that $\mathcal{Z}_{H_1}(c, r, \bullet) = \{d_1, d_3\}$.

As above, $\mathcal{Z}_{H_2}(\bullet, d_1, d_3) \cup \mathcal{Z}_{H_2}(\bullet, d_2, d_3)$ is disjoint to $\mathcal{Z}_{H_1}(c, s, \bullet)$, or else $\psi_2|_{G_Q^0}$ extends to an L-coloring of G. But since $d_2 \notin L(u_i)$, we have $\{r', d_3\} = \mathcal{Z}_{H_2}(\bullet, d_2, d_3)$, and $\mathcal{Z}_{H_1}(c, s, \bullet) = \{d_1\}$. Thus, $s \in L(u_i)$ so $s' = r$. d

Now, if $u_1 \cdots u_i$ is a path of even length then, for each odd $m \in \{1, \cdots, i-1\}$, we color u_i with r and each of d_1, d_3 is left over for u_i, so we have $d_3 \in \mathcal{Z}_{H_1}(u_i, s, c)$, contradicting the fact that $\mathcal{Z}_{H_1}(c, s, \bullet) = \{d_1\}$. Thus, $u_1 \cdots u_i$ is a path of odd length. But now, since $s \neq c$, we color u_m with s for each odd $m \in \{1, \cdots, i\}$ and color x_2 with r, leaving a color left over for each of $\{u_1, u_3, \cdots, u_{i-1}\}$, so we get $s \in \mathcal{Z}_{H_1}(c, r, \bullet)$, contradicting the fact that $\mathcal{Z}_{H_1}(c, r, \bullet) = \{d_1, d_3\}$. This completes the proof of Proposition 3.2.4. $\square$

3.3 2-Chords Incident to an Internal Vertex of the Precolored Path

In this section, we analyze 2-chords Q of the open rings of a critical mosaic, where Q has precisely one endpoint which is an internal vertex of the precolored path of the ring. Recalling the notation of Definition 2.3.9, we have the following result, which is the main result of Section 3.3. We prove this result and them combine the work of Section 3.3 with the work of the previous two sections to prove Theorem 3.0.2.

Lemma 3.3.1. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic. Let C be an open ring, and let $\mathbf{P} = p_1 \cdots p_m$. Let $Q := v_1 x v_2$ be a 2-chord of C, where $v_1 \in V(\mathring{\mathbf{P}})$ and $v_2 \in V(C \setminus \mathring{\mathbf{P}})$. Then the following hold.*

1) $v_1 \in \{p_2, p_{m-1}\}$ and $\mathbf{P} \cap G^{\text{small}}_{v_1 x v_2}$ is a path of length one; AND

2) $V(G^{\text{small}}_Q) = \{p_2, x, p_{m-1}\} \cup V(C \cap G^{\text{small}}_Q)$; AND

3) G^{small}_Q is a broken wheel with principal path $v_1 x v_2$.

Proof. We first note the following:

Claim 3.3.2. *For any 2-chord $v_1 x v_2$ of C with $v_1 \in V(\mathring{\mathbf{P}})$ and $v_2 \in V(C \setminus \mathring{\mathbf{P}})$, we have $v_1 \in \{p_2, p_{m-1}\}$, and $\mathbf{P} \cap G^{\text{small}}_{v_1 x v_2}$ is a path of length one.*

Proof: If $v_1 \notin \{p_2, p_{m-1}\}$, then, since $|E(\mathbf{P})| \leq \frac{2N_{\text{mo}}}{3}$, each of the two paths $\mathbf{P} \cap G^{\text{small}}$ and $\mathbf{P} \cap G^{\text{large}}$ has length at most $\frac{2N_{\text{mo}}}{3}$, contradicting 4) of Theorem 2.2.4. Thus, we have $v_1 \in \{p_2, p_{m-1}\}$. Suppose without loss of generality that $v_1 = p_2$. If $m > 3$ and $G^{\text{small}}_Q \cap \mathbf{P} = p_2 \cdots p_m$, then we again contradict 4) of Theorem 2.2.4, so we are done. $\blacksquare$

This proves 1) of Lemma 3.3.1. Now we prove 2). Given a 2-chord $v_1 x v_2$ of C, where $v_1 \in V(\mathring{\mathbf{P}})$ and $v_2 \in V(C \setminus \mathring{\mathbf{P}})$, we say that $v_1 x v_2$ is *bad* if $V(G^{\text{small}}_{v_1 x v_2} \setminus C) \neq \{x\}$. We prove that there is no bad 2-chord of C with p_2 as an endpoint. An identical argument then shows that there is no bad 2-chord of C with p_{m-1} as an endpoint. By Claim 3.3.2, this implies that there are no bad 2-chords of C.

Suppose toward a contradiction that there is a bad 2-chord Q of C with p_2 as an endpoint, and, among all such bad 2-chords of C, we choose Q so that $|V(G_Q^{\text{small}})$ is minimized. Let $Q := p_2xv$ for some $v \in V(C \setminus \mathring{\mathbf{P}})$. Applying Claim 3.3.2, suppose without loss of generality that $\mathbf{P} \cap G_Q^{\text{small}} = p_1p_2$. Let $G_Q^{\text{small}} \cap C = p_2p_1v_1 \cdots v_t$, where $v_t = v$.

Claim 3.3.3. $|V(G_Q^{\text{small}} \cap C)| > 3$ *and* $v \notin V(\mathbf{P})$

Proof: If $|V(G_Q^{\text{small}} \cap C)| \leq 3$, then the cycle $(G_Q^{\text{small}} \cap C) + p_2xv$ has length at most four and separates an element of S from $G_Q^{\text{large}} \setminus Q$, contradicting short-separation-freeness. That is, we have $t > 1$. Furthermore, if $v \in V(\mathbf{P})$, then, since $v \notin V(\mathring{\mathbf{P}})$, we have $v = p_m$, and p_2xp_m is a short C-band, contradicting 1) of Theorem 2.3.2. ■

Now we have the following:

Claim 3.3.4. *For each* $z \in V(C \cap G_Q^{\text{small}}) \setminus \{p_2, v\}$*,* $xz \notin E(G)$*.*

Proof: Suppose toward a contradiction that $xp_1 \in E(G)$. Then G contains the 2-chord p_1xv_2 of G, and, by Lemma 3.1.1, we have $V(G^1_{p_1xv_2} \setminus C) = \{x\}$. Since G is short-separation-free, we have $V(\setminus G^1) = \{p_2\}$, so $V(G_Q^{\text{small}}) \setminus C) = \{x\}$, contradicting our assumption that p_2xv_2 is a bad 2-chord of C. Thus, $xp_1 \notin E(G)$.

Now suppose toward a contradiction that there is an $i \in \{1, \cdots, t-1\}$ such that $xv_i \in E(G)$. Then $|V(G^{\text{small}}_{p_2xv_i})| < |V(G_Q^{\text{small}})|$. Thus, by assumption, we have $V(G^{\text{small}}_{p_2xv_i}) \setminus V(C) = \{x\}$. Furthermore, we have $v_ixv_t \in \mathcal{K}(C, \mathcal{T})$, and $G^{\text{small}}_{v_ixv_t} = G^1_{v_ixv_t}$. By Lemma 3.1.1, we have $V(G^{\text{small}}_{v_ixv_t}) \setminus V(C) = \{x\}$, and thus $V(G^{\text{small}}_{p_2xv}) \setminus V(C) = \{x\}$, contradicting our assumption. ■

Now we have the following:

Claim 3.3.5. $V(G_Q^{\text{large}})$ *is L-colorable.*

Proof: Let C_1 be the cycle $(C \cap G_Q^{\text{small}}) + Q$, and consider the following cases:

Case 1: p_2, v have a common neighbor in $V(G_Q^{\text{large}}) \setminus (V(C) \cup \{x\})$

In this case, let C_*^{large} be the outer face of G_Q^{large}. We claim that the tessellation $\mathcal{T}' := (G_Q^{\text{small}}, \{C_1\} \cup (\mathcal{C} \setminus \{C\}), L, C_*^{\text{large}})$ is a mosaic. Let $\mathbf{P}' := \mathbf{P}_{\mathcal{T}}(C) \cap G_Q^{\text{large}}$. There is no chord of C_1 with an endpoint in $V(\mathring{\mathbf{P}}')$, or else, since C is an induced cycle of G, there is a neighbor of x in $\{p_3, \cdots, p_{m-1}\}$, and thus, letting $p \in N(x) \cap \{p_3, \cdots, p_{m-1}\}$, the 2-chord pxv of C contradicts Claim 3.3.2. Thus, $\mathcal{T}'$ satisfies M1) of Definition 2.1.6, and M0) and M2) are immediate. Let y be a common neighbor of p_2, v in $V(G_Q^{\text{large}}) \setminus (V(C) \cup \{x\})$. Note that $G^{\text{large}}_{p_2yv} \subseteq G_Q^{\text{large}}$, or else the 4-cycle p_2xvy separates an element of $\mathcal{C} \setminus \{C\}$ from p_m, which is false. Suppose toward a contradiction that there is a $C' \in \mathcal{C} \setminus \{C\}$ and a subgraph $H \subseteq C'$ such that $d(H, C_1 \setminus \mathring{\mathbf{P}}') < d(H, C \setminus \mathring{\mathbf{P}})$ and let R be a shortest $(H, C_1 \setminus \mathring{\mathbf{P}}')$-path with $|E(R)| < d(H, C \setminus \mathring{\mathbf{P}})$. Since $\mathring{\mathbf{P}}' \subseteq \mathring{\mathbf{P}}$, R has x as an endpoint. But since $\{p_2, y, v\}$ separates x from H, and $p_2, v \in V(C_1 \setminus \mathring{\mathbf{P}}')$, $R \setminus \{x\}$ has y as an endpoint, and since $yv \in E(G)$, G contains an $(H, C \setminus \mathring{\mathbf{P}})$-path of length R, contradicting our assumption. Thus, no such H exists, so $\mathcal{T}'$ satisfies the distance conditions of Definition 2.1.6, and thus $\mathcal{T}'$ is indeed a mosaic. Since $|V(G_Q^{\text{large}})| < |V(G)|$, G_Q^{large} is L-colorable by the minimality of $\mathcal{T}$.

Case 2: p_2, v_2 do not have a common neighbor in $V(G_Q^{\text{large}}) \setminus (V(C) \cup \{x\})$.

In this case, let $G^\dagger$ be a graph obtained from G by first deleting the vertices of $G_Q^{\text{small}} \setminus \{p_1, p_2, x, v\}$ and replacing them with a lone vertex v^* adjacent to each of $\{p_1, p_2, x, v\}$. Let $C^\dagger := (C \cap G_Q^{\text{large}}) + p_1v^*v$. Then $C^\dagger$ is a facial subgraph of $G^\dagger + p_1x$.

We claim now that $\mathcal{T}^\dagger := (G^\dagger + p_1x, \{C^\dagger\} \cup (\mathcal{C} \setminus \{C\}), L, C_*^\dagger)$ is a mosaic. Firstly, we note that $G^\dagger + p_1x$ is short-separation-free, or else G_Q^{large} contains a 2-chord of C_1 with endpoints p_2, v_2. Since C is an induced cycle, p_2v_2 is not a chord of C_1, so p_2, v_2 have a common neighbor in $V(G_Q^{\text{large}}) \setminus (V(C) \cup \{x\})$, contradicting our assumption. Thus, $\mathcal{T}^\dagger$ is a tessellation. We claim now that $\mathcal{T}^\dagger$ is a mosaic. Since x is not adjacent to any vertex of $\{p_3, \cdots, p_m\}$, $\mathcal{T}^\dagger$ also satisfies M1). M0) and M2) are immediate.

Now suppose toward a contradiction that there is a $C' \in \mathcal{C} \setminus \{C\}$ and a subgraph H of C' such that $d_{G^\dagger}(H, C^\dagger \setminus \mathring{\mathbf{P}}) < d_G(H, C \setminus \mathring{\mathbf{P}})$. Then there is a shortest $(H, C^\dagger \setminus \mathring{\mathbf{P}})$-path R with $|E(R)| < d_G(H, C \setminus \mathring{\mathbf{P}})$, so R has v^* as an endpoint. But then $R \setminus \{v^*\}$ has one of p_2, x as its endpoint, and, in G, each of these vertices had a neighbor in $C \setminus \mathring{\mathbf{P}}$, so we have a contradiction. Thus, $\mathcal{T}^\dagger$ also satisfies the distance conditions of Definition 2.1.6, so $\mathcal{T}^\dagger$ is a mosaic. Note that $|V(G^\dagger + p_1x)| < |V(G)|$, or else, since $|V(G^{\text{small}} \cap C)| > 3$, we contradict the fact that $S \neq \varnothing$. Thus, by the minimality of $\mathcal{T}$, $G^\dagger + p_1x$ is L-colorable, so G_Q^{large} is L-colorable. ■

Now we return to the proof of Lemma 3.3.1. Let $S := V(G_Q^{\text{small}}) \setminus (V(C) \cup \{x\})$. By assumption, $S = \varnothing$. Applying Claim 3.3.5, let ϕ be an L-coloring of G_Q^{large}. Since C is an induced subgraph of G and $|V(C \cap G_Q^{\text{small}})| > 3$, ϕ is also a proper L-coloring of the subgraph of G induced by $V(G_Q^{\text{large}})$. By Claim 3.3.4, we have $p_1x \notin E(G)$. Thus, since $\mathbf{P}$ is L-colorable and $|L(p_2)| = 1$, ϕ extends to an L-coloring ϕ' of $V(G_Q^{\text{large}}) \cup \{p_1\}$.

Claim 3.3.6. *There is a lone vertex $v^* \in V(G_Q^{\text{small}} \setminus C) \setminus \{x\}$ such that $S = \{v^*\}$, and v^* is adjacent to each vertex of $G_Q^{\text{small}} \setminus \{v^*\}$.*

Proof: We first note that there is a vertex $v^* \in S$ with at least three neighbors among $\{p_1, p_2, x, v\}$. To see this, suppose toward a contradiction that no such vertex exists. By Claim 3.3.4, x has no neighbors in $V(C \cap G_Q^{\text{small}}) \setminus \{p_2, v\}$. Thus, since C is an induced subgraph of G, it follows from 1) of Proposition 1.5.1 that ϕ' extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. Thus, S contains a vertex v^* with at least three neighbors among $\{p_1, p_2, x, v\}$. We claim now that v^* is adjacent to each of $\{p_1, p_2, x, v\}$. Suppose not. Then v^* has precisely three neighbors among $\{p_1, p_2, x, v\}$. Consider the following cases:

Case 1: $\{p_1, v\} \subseteq N(v^*)$

In this case, if v^* is also adjacent to x, then, since $xp_1 \notin E(G)$ and G is short-separation-free, we have $p_2 \in N(v^*)$ by our triangulation conditions, contradicting our assumption. On the other hand, if $p_2 \in N(v^*)$, then, again since G is short-separation-free, $E(G)$ either contains p_2v or v^*x by our triangulation conditions. Since C is an induced subgraph of G, we have $x \in N(v^*)$, contradicting our assumption

Case 2: $\{p_1, v\} \not\subseteq N(v^*)$

In this case, we have either $N(v^*) \cap \{p_1, p_2, x, v\} = \{p_1, p_2, x\}$ or $N(v^*) \cap \{p_1, p_2, x, v\} = \{p_2, x, v\}$. Suppose that $N(v^*) \cap \{p_1, p_2, x, v\} = \{p_2, x, v\}$. In that case, by the minimality of Q, we have $V(G_{p_2v^*v}^{\text{small}} \setminus V(C) = \{v^*\}$, and thus, since C is an induced subgraph of G, it follows from our triangulation conditions that $v^*p_1 \in E(G)$, contradicting our assumption.

The only possibility left to rule out is that $N(v^*) \cap \{p_1, p_2, x, v\} = \{p_1, p_2, x\}$. In this case, G contains the 3-chord $Q^* := p_1v^*xv$ of C. Note that $Q^* \in \mathcal{K}(C, \mathcal{T})$. By Proposition 3.2.4, since $v^*v \notin E(G)$, there is a lone vertex $z \in V(G_{Q^*}^1)$ such that $G_{Q^*}^1 \setminus C$ consists of the triangle v^*zx. Since $\deg_G(v^*) \geq 5$ and C is an induced subgraph of G, it follows from our triangulation conditions that there exists an $r \in \{1, \cdots, t-1\}$ such that $G[V(C) \cap N(v^*)] = p_2p_1v_1 \cdots v_r$ and $G[C \cap N(z)] = v_rv_{r+1} \cdots v_t$.

Let H_1 be the broken wheel with principal path $p_1v^*v_r$, where $H_1 \setminus \{v^*\} = p_1v_1\cdots p_r$. Let H_2 be the broken wheel with principal path v_rzv, where $H_2 \setminus \{z\} = v_rv_{r+1}\cdots v_t$. Let a, b be two colors in $L(v_1) \setminus \{\psi'(p_1)\}$ and let q be a color in $L(v^*) \setminus \{a, b, \psi'(p_1), \psi'(p_2)\}$. Let $Q^\dagger$ be the 3-chord p_2v^*zv of C and let $L^\dagger$ be a list-assignment for $G^{\text{small}}_{Q^\dagger}$ where $L^\dagger(v^*) = q$ and $L^\dagger(z) = L(z)$ otherwise. Let $C^\dagger$ be the cycle $(C \cap G^{\text{large}}_{Q^\dagger} + p_2v^*zv$ and let D be the outer face of $G^{\text{large}}_{Q^\dagger}$. Let $\mathcal{T}^\dagger := (G^{\text{large}}_{Q^\dagger}, \{C^\dagger\} \cup (\mathcal{C} \setminus \{C\}), L^\dagger, D)$ and let $\mathbf{P}^\dagger := p_m \cdots p_2v^*$. Then $\mathbf{P}^\dagger$ is a proper subpath of $C^\dagger$, since $z \notin V(\mathbf{P}^\dagger)$, and $\mathbf{P}^\dagger$ is a chordless subpath of $C^\dagger$. Thus $\mathbf{P}^\dagger$ is $L^\dagger$-colorable, and $\mathcal{T}^\dagger$ is a tessellation in which $C^\dagger$ is an open ring.

We claim now that $\mathcal{T}^\dagger$ is a mosaic. Since $N(x) \cap \{p_3, \cdots, p_m\} = \varnothing$, each of v^*, z is adjacent to a subpath of $\mathbf{P}^\dagger$ of length at most one, and there is no chord of $C^\dagger$ with an endpoint in $\mathring{\mathbf{P}}^\dagger$ (indeed, $C^\dagger$ is an induced subgraph of $G^{\text{large}}_{Q^\dagger}$, so M1) is satisfied. Since $|E(\mathbf{P}^\dagger)| = |E(\mathbf{P})|$, we immediately have M0), and M2) is trivial. Now suppose toward a contradiction that there is a $C' \in \mathcal{C} \setminus \{C\}$ and a subgraph H of C' such that $d(H, C^\dagger \setminus \mathring{\mathbf{P}}^\dagger) < d(H, C \setminus \mathring{\mathbf{P}})$. Since $\mathring{\mathbf{P}}^\dagger = \mathring{\mathbf{P}}$, there is a shortest $(H, C^\dagger \setminus \mathring{\mathbf{P}})$-path R with $|E(R)| < d(H, C \setminus \mathring{\mathbf{P}})$. Thus, R has one of $\{v, z\}$ as an endpoint. But since each of $\{p_2, x, v\}$ is of distance at most one from $C \setminus \mathring{\mathbf{P}}$, we have a contradiction. Thus, $\mathcal{T}^\dagger$ also satisfies the distance conditions of Definition 2.1.6, so $\mathcal{T}^\dagger$ is a mosaic.

Since $|V(G^{\text{large}}_{Q^\dagger})| < |V(G)|$, $G^{\text{large}}_{Q^\dagger}$ is $L^\dagger$-colorable by the minimality of $\mathcal{T}$. Thus, let ψ be an $L^\dagger$-coloring of $G^{\text{large}}_{Q^\dagger}$. By definition of $L^\dagger$, we have $\{\psi(v^*)\} \neq L(p_1)$, so ψ extends to an L-coloring ψ' of $G \setminus \{v_1, \cdots, v_{t-1}\}$.

Subclaim 3.3.7. *H_1 is a triangle.*

Proof: Suppose that H_1 is not a triangle, so $H_1 - v^* = p_1v_1\cdots v_r$ for some $r > 1$. Since the coloring $(\psi'(z), \psi'(v))$ of the edge zv extends to the broken wheel H_2, there is an extension of ψ' to an L-coloring ψ'' of $G \setminus \{v_1, \cdots, v_{r-1}\}$. Since $|L(v_1)| = 3$ and $|L(p_1)| = 1$, we have by definition of $L^\dagger$ that either $\psi''(p_1) \notin L(v_1)$ or $\psi''(v^*) \notin L(v_1)$. In either case, ψ'' extends to the path $v_1\cdots v_{r-1}$, contradicting the fact that $\mathcal{T}$ is critical. ■

Since H_1 is a triangle, we have $H_2 \setminus \{z\} = v_1\cdots v_t$. By definition of $L^\dagger$, we have $|L_{\psi'}(v_1)| \geq 2$. Since each internal vertex of $v_1\cdots v_t$ has an $L_{\psi'}$-list of size at least two, ψ' extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. This completes Case 2. Thus, our assumption that $|N(v^*) \cap \{p_1, p_2, v, x\}| = 3$ is false, so v^* is adjacent to each vertex of $\{p_1, p_2, x, v\}$, as desired. By Lemma 3.1.1, $G^1_{p_1v^*v}$ is a broken wheel with principal path p_1v^*v, and thus v^* is adjacent to each vertex of the cycle $p_2p_1v_1\cdots v_tx$. Thus, since G is short-separation-free, we have $V(G^{\text{small}}_Q) = \{v^*\} \cup \{p_2, p_1, v_1, \cdots, v_t, x\}$, so we are done. This completes the proof of Claim 3.3.6. ■

Applying Claim 3.3.6, let $S = \{v^*\}$. Since $|L(p_1)| = 1$, let $L(p_1) = \{c\}$ for some color c. Since v^* has four neighbors in dom(ϕ'), $|L_{\phi'}(v^*)| \geq 1$, so ϕ' extends to an L-coloring ϕ'' of $G \setminus \{v_1, \cdots, v_{t-1}\}$. Note that $c \in L(v_1)$, or else, since $G^1_{p_1v^*v}$ is a broken wheel with principal path p_1v^*v, ϕ'' extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. Let $L(v_1) = \{a, b, c\}$ and let $L(p_2) = \{d\}$ (possibly $d \in \{a, b\}$). Since $|L(v^*)| \geq 5$, let q be a color in $L(v^*) \setminus \{a, b, c, d\}$, and let L'' be a list-assignment for $G^{\text{small}}_{p_2v^*v}$, where $L''(v^*) = \{q\}$ and $L''(z) = L(z)$. Let C'' be the cycle $(C \cap G^{\text{small}}_{p_2v^*v}) + p_2v^*v$, and let C''_* be the outer face of $G^{\text{small}}_{p_2v^*v}$ (that is, either $C''_* = C_*$, or, if $C_* = C$, then $C''_* = C''$).

Let $\mathcal{T}'' := (G^{\text{small}}_{p_2v^*v}, \{C''\} \cup (\mathcal{C} \setminus \{C\}), L'', C''_*)$ and let $\mathbf{P}'' := p_mp_{m-1}\cdots p_2v^*$. By Claim 3.3.4, $v \notin V(P)$, so $\mathbf{P}''$ is a proper subpath of C''. Thus the subgraph of $G^{\text{small}}_{p_2v^*v}$ induced by P'' is L''-colorable, and $\mathcal{T}''$ is a tessellation in which C'' is an open ring. We claim now that $\mathcal{T}''$ is a mosaic.

Firstly, since $\mathbf{P}$ is a chordless subpath of C, $\mathbf{P}''$ is a chordless subpath of C''. Likewise, since $N(v^*) \cap V(C'') = \{v, p_2\}$, there is no chord of C'' with an endpoint in $\mathring{\mathbf{P}}''$ (indeed, C'' is an induced subgraph of $G^{\text{small}}_{p_2v^*v}$. Finally, by Claim 3.3.2, x does not have a neighbor among $\{p_3, \cdots, p_m\}$ so $G[N(x) \cap V(\mathbf{P}'')]$ consists of the edge v^*p_2. Thus,

$\mathcal{T}''$ satisfies M1), and since $|E(\mathbf{P}'')| = |E(\mathbf{P})|$, M0) and M2) are immediate as well.

Now suppose toward a contradiction that there is a $C' \in \mathcal{C} \setminus \{C\}$ and a subgraph $H \subseteq C'$ such that $d(H, C'' \setminus \mathring{\mathbf{P}}'') < d(H, C \setminus \mathring{\mathbf{P}})$. Since $\mathring{\mathbf{P}}'' = \mathring{\mathbf{P}}$, there is an $(H, C'' \setminus \mathring{\mathbf{P}})$-path R with $|E(R)| < d(H, C \setminus \mathring{\mathbf{P}})$. Thus, R has endpoint v^*, and $R \setminus \{v^*\}$ has one of p_2, x as its endpoint. Since each of p_2, x has a neighbor in $C \setminus \mathring{\mathbf{P}}$, this contradicts the fact that $|E(R)| < d(H, C \setminus \mathring{\mathbf{P}})$, so no such H exists . Thus,$\mathcal{T}''$ also satisfies the distance conditions of Definition 2.1.6

Thus, $\mathcal{T}''$ is a mosaic. Since $G \setminus G^{\text{small}}_{p_2v^*v} = p_1v_1 \cdots v_{t-1}$, $|V(G^{\text{small}}_{p_2v^*v})| < |V(G)|$, so $G^{\text{small}}_{p_2v^*v}$ is L''-colorable by the minimality of $\mathcal{T}$. Let ψ be an L''-coloring of $G^{\text{small}}_{p_2v^*v}$. Note that ψ extends to an L-coloring ψ' of $G \setminus \{v_1, \cdots, v_{t-1}\}$, since $\{\psi''(v^*)\} \neq L(p_1)$ and $p_1v \notin E(G)$. Since $\psi''(v^*) \notin L(v_1)$, ψ' extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. This completes the proof of Lemma 3.3.1. $\square$

We now combine the result above with the results of sections 4.1 and 4.2 to prove Theorem 3.0.2. Firstly, combining condition M1) with our triangulation conditions, there is a path $P' \subseteq G$ such that $V(P') = D_1(\mathbf{P}, G) \setminus V(C)$, where, for each $w \in V(P')$, $G[N(w) \cap V(\mathbf{P})]$ is a path of length at most one, as G is short-separation-free. Combining this with Lemma 3.1.1 and Lemma 3.3.1, there is a unique cycle C^1 such that $V(C^1) = D_1(C)$, and such that, letting $G = G_0 \cup G_1$ be the nautral C^1-partition of G, where $C \subseteq G_0$, the graph $G_0[(\{v\} \cup N(v)]$ is either an edge or broken wheel with principal vertex v. Thus, C^1 satisfies 1) of Theorem 3.0.2.

To finish the proof of Theorem 3.0.2, we check that C^1 also satisfies 2). Suppose toward a contradiction that C^1 does not satisfy 2) of Theorem 3.0.2. In that case, there exists a chord xx' of C^1, where each of x, x' has a neighbor in $C \setminus \mathring{\mathbf{P}}$, and the chord xx' violates at least one of i)-iii) of 2) of Theorem 3.0.2. Since each of x, x' has a neighbor in $C \setminus \mathring{\mathbf{P}}$, there exists a 3-chord Q of C whose middle edge is xx', where $Q \in \mathcal{K}(C, \mathcal{T})$.

By 3) of Lemma 3.2.1, we have $V(G^1_Q) = V(G^1_Q \cap C) \cup V(G^1_Q \cap C^1)$. Since xx' is a chord of C^1, we have $V(G^1_Q) \neq V(C \cap G^1_Q) \cup \{x, x'\}$. In particular, $G^1_Q \cap C^1$ is a path of length at least two. By 1) of Proposition 3.2.4, this path has length precisely two, so xx' satisfies i) of 2) of Theorem 3.0.2 and thus violates either ii) or iii) of 2) of Theorem 3.0.2.

Suppose that iii) is violated. In that case, at least one of x, x' is not a $\mathcal{C}$-shortcut, contradicting 4) of Lemma 3.2.1. Thus, ii) of 2) is violated, and thus one of x, x' has a neighbor in $\mathbf{P}$. Since xx' is a chord of C^1, we have $V(G^1_Q \setminus Q) \not\subseteq V(C)$, contradicting 2) of Proposition 3.2.4. This completes the proof of Theorem 3.0.2, and motivates the following natural terminology:

Definition 3.3.8. Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let $C \in \mathcal{C}$ be an open ring, and let C^1 be the unique cycle in G with $V(C^1) = B_1(C)$. We call C^1 the *1-necklace* of C.

It is also very natural to introduce some notation for two special subpath of the 1-necklace of an open ring, the first of which consists of all the vertices of the 1-necklace with a neighbor in the precolored path and the second of which consists of all the vertices of the 1-necklace without any neighbors outside of the precolored path. By 2) of Corollary 2.3.14, the precolored path of an open ring in a critical mosaic has length $\lfloor \frac{2N_{\text{mo}}}{3} \rfloor$, so, by M2), these two paths are distinct.

Definition 3.3.9. Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let $C \in \mathcal{C}$ be an open ring, and let C^1 be the 1-necklace of C. We then define two paths $\mathbf{P}^1_{\mathcal{T}}(C)$ and $\mathbf{P}^{1+}_{\mathcal{T}}(C)$ to be the unique subpaths of C^1 such that the following hold.

1) $V(\mathbf{P}^1_{\mathcal{T}}(C)) = \{v \in V(C^1) : N(v) \cap V(C) \subseteq V(\mathbf{P})\}$*; AND*

2) $V(\mathbf{P}^{1+}_{\mathcal{T}}(C)) = \{v \in V(C^1) : N(v) \cap V(\mathbf{P}) \neq \varnothing\}$*; AND*

3) $\mathbf{P}^1_{\mathcal{T}}(C) \subsetneq \mathbf{P}^{1+}_{\mathcal{T}}(C)$.

As with the precolored subgraph of C, we usually just drop the $\mathcal{T}$ and the C from the notation above and just write $\mathbf{P}^1$, if the underlying tessellation or ring, or both, are clear from the context. In Chapter 4, we perform a similar analysis to that of Chapter 3 on the vertices of distance two from an open ring in a critical mosaic, and we also analyze 3-chords of open rings with one endpoint outside the precolored path and one endpoint which is an internal vertex of the precolored path.

Chapter 4

Vertices of Distance Two From Open Rings

We begin by stating the main result which we prove in this chapter. The proof of this result consists of the entirety of Chapter 4. This result, together with Theorem 3.0.2, contains all the analysis of the structure of a critical mosaic near each open ring that we need in order to begin coloring and deleting a path between two rings in a critical mosaic.

Theorem 4.0.1. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let C be an open $\mathcal{T}$-ring. Then G contains a cycle C^2 such that, letting $G = G' \cup G''$ be the natural C^2-partition of G, where $C \subseteq G'$, the following hold.*

1) *$C^2 \cap C^1 = \mathbf{P}^1$ and $V(G') = V(C \cup C^1 \cup C^2)$; AND*
2) *$V(C^2 \setminus \mathbf{P}^1) = D_2(C \setminus \mathbf{P}) \setminus V(C^1)$.*

Furthermore, for any chord uv of C^1, where $u \in V(\mathbf{P}^1)$ and $v \in V(C^1 \setminus \mathbf{P}^1)$, and any $w \in N(u) \cap V(\mathring{\mathbf{P}})$ and $w' \in N(v) \cap V(C)$, letting $R := uww'v$ and $\mathbf{P} := p_1 \cdots p_m$, the following hold.

1) *$|V(G_R^{\text{small}}) \setminus V(C \cup R)| = 1$, and w is of distance precisely one from an endpoint of $\mathbf{P}$; AND*
2) *$G_R^{\text{small}} \setminus \{p_3, p_{m-2}\}$ is a wheel whose central vertex is the lone vertex of $V(G_R^{\text{small}}) \setminus V(C \cup R)$.*

To prove this, we first need to analyze graphs consisting of sequences of broken wheels.

4.1 Sequences of Broken Wheels

In this section, we prove a fact about about graphs consisting of broken wheels in sequence. We begin with the following definition.

Definition 4.1.1. A graph G is called a *wheel sequence* if G is a connected graph, and, for some integer $k \geq 1$, G contains subgraphs $H_1, \cdots, H_k$, and 2-paths $P_1, \cdots, P_k$, where $P_i := x^i y^i z^i$ for each $i = 1, \cdots, k$, such that the following hold.

1) For each $i = 1, \cdots, k$, H_i is a broken wheel with principal path P_i; *AND*
2) For each $i = 2, \cdots, k$, $H_{i-1} \cap H_i = \{x^i\}$; *AND*
3) $E(G) = E(H_1) \cup \cdots E(H_k) \cup \{y^1 y_2, \cdots, y^{k-1} y^k\}$ as a disjoint union.

Given a wheel sequence G as above, we associate to G the following terminology and notation:

1) The path $y^1 \cdots y^k$ is called the *apex path* of G and k is called the *length* of G.

2) The vertices x^1, z^k are called the *wheel terminals* of G.

3) $\mathcal{H}(G)$ denotes the k-tuple $(H_1, \cdots, H_k)$, and $\mathcal{P}(G)$ denotes the k-tuple $(P_1, \cdots, P_k)$.

In Section 4.2, when we analyze the structure of a critical mosaic in the ball of distance two from an open ring, we are particularly interested in graphs which consit of a wheel sequence together with a lone vertex adjacent to each vertex on the apex path of the wheel sequence. We introduce one more definition and then state our main result for Section 4.1.

Definition 4.1.2. A *crown* is a 4-tuple (G, w, P, L) such that the following hold.

1) G is a graph and w is a vertex of G such $N(w) = V(P)$ and $G - w$ is a wheel sequence with apex path P*; AND*

2) L is a list-assignment for $V(G)$ such that the following hold.

   i) $|L(w)| \geq 2$, each endpoint of P has an L-list of size at least four, and each internal vertex of P ha an L-list of size at least five*; AND*

   ii) Each wheel terminal of $G - w$ is precolored*; AND*

   iii) All remaining vertices of G have L-lists of size three.

Our lone main result for Section 4.1 is the following.

Theorem 4.1.3. *Let (G, w, P, L) be a crown and let $\mathcal{H}(G\backslash\{w\}) = (H_1, \cdots, H_k)$ and $\mathcal{P}(G\backslash\{w\}) = (P_1, P_2, \cdots, P_k)$, where $P_i = x^i y^i z^i$ for each $i = 1, \cdots, k$.Furthermore, let $L(x^1) = \{c\}$ and $L(z^k) = \{c'\}$. Then the following hold.*

1) If $|V(P)| \geq 3$, then G is L-colorable; AND

2) If $|V(P)| = 2$, then, letting $X := \bigcap_{u \in V(H_1 \backslash P_1)} L(u)$ and $X' := \bigcap_{u \in V(H_2 \backslash P_2)} L(u)$, one of the following three statements holds.

   a) G is L-colorable; OR

   b) There is a set of two colors such common to the lists of each vertex of $V(G) \setminus \{w, x^1, z^2, y^1, y^2\}$; OR

   c) There is a set $S = \{a, b, r\}$ of three colors, where $L(w) = \{a, b\}$, such that $L(y^1) \setminus \{c\} = L(y^2) \setminus \{c'\} = S$, and furthermore, $\{b, r\} \subseteq X$, $\{a, r\} \subseteq X'$, and $|L(z^1) \cap S| \geq 2$.

We begin by proving the first half of Theorem 4.1.3.

Proposition 4.1.4. *Let (G, w, P, L) be a crown with $V(P)| \geq 3$. Then G is L-colorable.*

Proof. Let $\mathcal{H}(G \backslash \{w\}) = (H_1, \cdots, H_k)$ and $\mathcal{P}(G \backslash \{w\}) = (P_1, \cdots, P_k)$, where $P_i = x^i y^i z^i$ for each $i = 1, \cdots, k$. We prove the proposition by induction on the length of P. The base case is $|V(P)| = 3$. Let $L(x^1) = \{c\}$ and $L(z^3) = \{c'\}$. Let $a, b \in L(w)$. Possibly, one or both of c, c' lies in $\{a, b\}$. Suppose towards a contradiction that G is not L-colorable. Now we have the following:

Claim 4.1.5. *If there is a pair $(r, r') \in L(y^1) \times L(y^2)$ such that $|\mathcal{Z}_{H_1}(c, r, \bullet)| \geq 2$ and $|\mathcal{Z}_{H_3}(\bullet, r', c')| \geq 2$, then $\{a, b\} \cap \{r, r'\} \neq \varnothing$.*

Proof: Suppose there a pair (r, r') with $|\mathcal{Z}_{H_1}(c, r, \bullet)| \geq 2$ and $|\mathcal{Z}_{H_3}(\bullet, r', c')| \geq 2$. Suppose that $\{a, b\} \cap \{r, r'\} = \varnothing$. Now let L^* be a list-assignment for H_2, where we set $L^*(y^2) := L(y^2) \backslash \{r, r'\}$, $L^*(z^1) := \mathcal{Z}_{H_1}(c, r, \bullet)$, and, likewise, $L^*(z^2) := \mathcal{Z}_{H_3}(\bullet, r', c')$. Finally, we set $L^*(u) := L(u)$ for all $u \in V(H_2) \setminus \{x^2, y^2, z^2\}$.

Note that H_2 is L^*-colorable by Theorem 1.3.4, since every vertex in H_2 has an L^*-list of size at least 3, except possibly the vertices $\{x^2, z^2\}$, which have L^*-lists of size at least 2. Thus, let ψ be an L^*-coloring of H_2. Now, ψ extends to an L-coloring ψ' of $G \setminus \{w\}$ in which $\psi'(y^1) = r$ and $\psi'(y^3) = r'$, since $\psi(z^1) \in \mathcal{Z}_{H_1}(c, r, \bullet))$ and $\psi(x^3) \in \mathcal{Z}_{H_3}(\bullet, r', c')$. Now, ψ' leaves an color over for w, since $|L(w) \setminus \{r, r'\}| \geq 2$. Thus, G is indeed L-colorable, contradicting our assumption. ■

By Proposition 1.4.5, the sets $\{r \in L(y^1) : |\mathcal{Z}_{H_1}(c, r, \bullet)| \geq 2\}$ and $\{r' \in L(y^2) : |\mathcal{Z}_{H_3}(\bullet, r', c')| \geq 2\}$ are both nonempty. Thus, for the remainder of the proof, we may suppose that one of these two sets is a subset of $\{a, b\}$. So now suppose without loss of generality that $\{r \in L(y^1) : |\mathcal{Z}_{H_1}(c, r, \bullet)| \geq 2\}$ is a nonempty subset of $\{a, b\}$, and that $a \in L(y^1) \setminus \{c\}$ with $|\mathcal{Z}_{H_1}(c, a, \bullet)| \geq 2$. Consider the following cases.

Case 1: H_2 is a triangle.

We break this case into subcases:

Case 1.1 $a \in L(y^3) \setminus \{c'\}$

In this case, let ψ be an L-coloring of H_3 with $\psi(y^1) = a$. Such an L-coloring of H_3 exists by Thomassen. Since $|L(y^2) \setminus \{a, b, \psi(x^3)\}| \geq 2$, and $|\mathcal{Z}_{H_1}(c, a, \bullet)| \geq 2$, there is a color $d \in L(y^2) \setminus \{a, b, \psi(x^3)\}$ such that $\mathcal{Z}_{H_1}(c, a, \bullet) \neq \{d, \psi(x^3)\}$. Now we extend ψ to an L-coloring ψ' of $V(H_3) \cup \{y^1, y^2\}$ by setting $\psi'(y^1) = a$ and $\psi'(y^2) = d$. Since there is a color left over in $\mathcal{Z}_{H_1}(c, a, \bullet)$, ψ' extends to an L-coloring ψ'' of $G \setminus \{w\}$. Since $b \notin \{\psi''(y^1), \psi''(y^2), \psi''(y^3)\}$, there is a color left over in $L(w)$, so G is L-colorable, contradicting our assumption.

Case 1.2 $a \notin L(y^3) \setminus \{c'\}$.

In this case, there are at least two colors $d_0, d_1 \in L(y^3) \setminus \{a, b, c'\}$. We claim that, for some $i \in \{0, 1\}$, there is an L-coloring ψ of $G \setminus \{w\}$ with $\psi(y^1) = a$ and $\psi(y^3) = d_i$. For each $i \in \{0, 1\}$, let ψ_i be an L-coloring of H_3 with $\psi_i(y^1) = d_i$. Such an L-coloring of H_3 exists for each $i \in \{0, 1\}$ by Theorem 0.2.3. Consider the following subcases.

Case 1.2.1 For each $i \in \{0, 1\}$, $|L(y^2) \setminus \{a, b, d_i, \psi(z^2)\}| = 1$

In this case, since $|L(y^2)| \geq 5$, we have $|L(y^2)| = 5$ and $\{a, b, d_0, d_1\} \subseteq L(y^2)$. Let e be the lone color of $L(y^2) \setminus \{a, b, d_0, d_1\}$.

Claim 4.1.6. *If there is an $i \in \{0, 1\}$ such that $\psi_i(z^2) \neq e$ then $\mathcal{Z}_{H_1}(c, a, \bullet) = \{\psi_i(z^2), e\}$. On the other hand, if there is an $i \in \{0, 1\}$ such that $\psi_i(z^2) = e$, then $\mathcal{Z}_{H_1}(c, a, \bullet) = \{d_{1-i}, e\}$.*

<u>Proof:</u> Let $i \in \{0, 1\}$ and suppose that $\psi_i(z^2) \neq e$. Suppose further that $\mathcal{Z}_{H_1}(c, a, \bullet) \neq \{\psi_i(z^2), e\}$. We claim that there is an extension of ψ_i to an L-coloring of G. We first extend ψ_i to an L-coloring ψ_i' of $V(H_3) \cup \{y^1, y^2\}$ by setting $\psi_i'(y^2) = e$ and $\psi_i'(y^1) = a$. Since $\mathcal{Z}_{H_1}(c, a, \bullet) \neq \{\psi_i'(z^2), \psi_i'(y^2)\}$ and $|\mathcal{Z}_{H_1}(c, a, \bullet)| \geq 2$, there is a color left in $\mathcal{Z}_{H_1}(c, a, \bullet)$ for z^1, so ψ' extends to an L-coloring of $G \setminus \{w\}$. Since $b \notin \{\psi'(y^1), \psi'(y^2), \psi'(y^3)\}$, there is a color left over for w, so G is L-colorable, contradicting our assumption.

Now let $i \in \{0, 1\}$ and suppose that $\psi_i(z^2) = e$. Suppose further that $\mathcal{Z}_{H_1}(c, a, \bullet) \neq \{d_{1-i}, e\}$. We claim that there is an extension of ψ_{1-i} to an L-coloring of G. We first extend ψ_{1-i} to an L-coloring ψ_{1-i}' of $V(H_3) \cup \{y^1, y^2\}$ by setting $\psi_{1-i}'(y^2) = d_{1-i}$ and $\psi_{1-i}'(y^1) = a$. This is permissible as $d_i \neq a$ and $d_{1-i} \in L(y^2)$. Since $\mathcal{Z}_{H_1}(c, a, \bullet) \neq \{\psi_{1-i}'(y^2), \psi_{1-i}'(z^2)\}$ and $|\mathcal{Z}_{H_1}(c, a, \bullet)| \geq 2$, there is a color of $\mathcal{Z}_{H_1}(c, a, \bullet)$ in $L(z^1) \setminus \{a, \psi_{1-i}'(y^2), \psi_{1-i}'(z^2)$. Thus, ψ_{1-i}' extends to an L-coloring of $G \setminus \{w\}$. Since $b \notin \{\psi_{1-i}'(y^1), \psi_{1-i}'(y^2), \psi_{1-i}'(y^3)\}$, ψ_{1-i}' extends to an L-coloring of G, contradicting our assumption. ■

Now, suppose towards a contradiction that $\psi_0(z^2) \neq e$ and $\psi_1(z^2) \neq e$. In that case, by Claim 4.1.6, there is a color $f \in L(z^1)$ such that $f = \psi_0(z^2) = \psi_1(z^1)$ and $\mathcal{Z}_{H_1,L}(c, a, \bullet) = \{e, f\}$. Note that $f \notin \{d_0, d_1\}$, since $\psi_0(z^2) \neq d_0$ and $\psi_1(z^2) \neq d_1$. Now extend ψ_0 to an L-coloring ψ_0' of $V(H_3) \cup \{y^1, y^2\}$ by setting $\psi_0'(y^1) = e$ and $\psi_0'(y^2) = d_1$. Then ψ_0' extends to an L-coloring of $G \setminus \{w\}$, since there is a color left over for z^1 in $\mathcal{Z}_{H_1}(c, a, \bullet)$. The resulting coloring of $G - w$ extends to an L-coloring of G, since $b \notin \{\psi_0'(y^1), \psi_0'(y^2), \psi_0'(y^3)\}$. This contradicts our assumption.

We conclude that $e \in \{\psi_0(z^2), \psi_1(z^2)\}$. If $e = \psi_0(z^2) = \psi_1(z^2)$, then there exists $i \in \{0, 1\}$ such that $\mathcal{Z}_{H_1}(c, a, \bullet) \neq \{e, d_{1-i}\}$, contradicting Claim 4.1.6. Thus, if either $e \notin \{\psi_0(z^2), \psi_1(z^2)\}$ or $e = \psi_0(z^0) = \psi_1(z^2)$, then G is L-colorable, contradicting our assumption. So now suppose without loss of generality that $\psi_0(z^2) = e$ and $\psi_1(z^2) \neq e$. In that case, by Claim 4.1.6, we have $\mathcal{Z}_{H_1}(c, a, \bullet) = \{d_1, e\} = \{\psi_1(z^2), e\}$. Yet $d_1 \neq \psi_1(z^2)$, since ψ_1 is a proper L-coloring of H_3 in which y^3 is colored with d_1, so G is indeed L-colorable, contradicting our assumption. This completes Subcase 1.2.1.

Case 1.2.2 For some $i \in \{1, 2\}$, $L(y^2) \setminus \{a, b, d_i, \psi_i(z^2)\}| \geq 2$.

In this case, let e_1, e_2 be two colors in $L(y^2) \setminus \{a, b, d_i, \psi_i(z^2)\}$. Then there exists a $j \in \{1, 2\}$ such that $\mathcal{Z}_{H_1}(c, a, \bullet) \neq \{\psi_i(z^2), e_j\}$. Now extend ψ_i to an L-coloring ψ_i' of $H_3 \cup \{y_1, y_2\}$ by setting $\psi_i'(y^1) = a$ and $\psi_i'(y^2) = e_j$. Then ψ_i' extends to an L-coloring ψ of $G \setminus \{w\}$, since there is a color left over for z^1 in $\mathcal{Z}_{H_1}(c, a, \bullet)$. Since $b \notin \{\psi(y^1), \psi(y^2), \psi(y^3)\}$, this coloring ψ extends to an L-coloring of G, contradicting our assumption. This completes the case where H_2 is a triangle.

Case 2: H_2 is not a triangle.

In this case, let $H_2 \setminus \{y^2\} := x^2 w_1 \cdots w_t z^2$ for some $t \geq 1$. We break this into the following subcases:

Case 2.1: $L(y^2) \setminus \{a, b\} = L(w_1)$

In this case, since $|L(w_1)| = 3$, we have $L(y^2) = L(w_1) \cup \{a, b\}$ as a disjoint union.

Claim 4.1.7. *Let $r \in L(y^1) \setminus \{a, b, c\}$ and $r' \in L(y^3) \setminus \{a, b, c'\}$. Then $|\mathcal{Z}_{H_1}(r, c, \bullet))| = |\mathcal{Z}_{H_3}(\bullet, r', c')| = 1$ and $\{a, b\} = \mathcal{Z}_{H_1}(c, r, \bullet) \cup \mathcal{Z}_{H_3}(\bullet, r', c')$.*

Proof: Let $x \in \mathcal{Z}_{H_1}(c, r, \bullet)$ and $x' \in \mathcal{Z}_{H_3}(\bullet, r', c')$. Suppose towards a contradiction that $\{x, x'\} \neq \{a, b\}$. Let $s \in \{a, b\} \setminus \{x, x'\}$. Since $\{a, b\} \subseteq L(y^2)$, we have $s \in L(y^2)$. Consider the L-coloring (r, s, r') of $y^1y^2y^3$. We claim that this extends to an L-coloring of $G \setminus \{w\}$. It suffices to show that the coloring (x, s, x') of $x^2y^2z^2$ extends to an L-coloring of H. Since $s \notin L(w^1)$, this coloring of the principal path of H_2 does indeed extend to an L-coloring of H_2. Thus, the coloring (r, s, r') of $y^1y^2y^3$ extends to an L-coloring ψ of $G - w$, and since there is a color of $\{a, b\}$ left over for w, ψ extends to an L-coloring of G, contradicting our assumption. ■

Now, as above, let $r \in L(y^1) \setminus \{c\}$ and $r' \in L(y^3) \setminus \{c'\}$. Applying Claim 4.1.7, we have $|\mathcal{Z}_{H_1}(c, r, \bullet) = |\mathcal{Z}_{H_3}(\bullet, r', c')| = 1$ and $\mathcal{Z}_{H_1}(c, r, \bullet) \cup \mathcal{Z}_{H_3}(\bullet, r', c') = \{a, b\}$. We now choose a color $t \in L(y^2) \setminus \{a, b, r, r'\}$, and we claim that the coloring (r, t, r') of $y^1y^2y^3$ extends to an L-coloring of $G \setminus \{w\}$. If we show this, then we are done, since $a, b \notin \{r, t, r'\}$. Let $\mathcal{Z}_{H_1}(c, r, \bullet) = \{x\}$ and $\mathcal{Z}_{H_3}(\bullet, r', c') = \{x'\}$, where $\{x, x'\} = \{a, b\}$. It just suffices to show that the coloring (x, t, x') of $x^2y^2z^2$ extends to an L-coloring of H_2. This holds since $x \notin L(w_1)$, so we are done. This completes Case 2.1.

Case 2.2 $L(y^2) \setminus \{a, b\} \neq L(w_1)$

In this case, since $|L(y^2)| \geq 5$, there is a color $e \in L(y^2) \setminus \{a, b\}$ with $e \notin L(w^1)$.

Claim 4.1.8. *For each color $r' \in L(y^3) \setminus \{b, c'\}$, we have $\mathcal{Z}_{H_3}(\bullet, r', c') = \{e\}$.*

Proof: Suppose there is an $r' \in L(y^3) \setminus \{b, c'\}$ with $\mathcal{Z}_{H_3}(\bullet, r', c') \neq \{e\}$. Since $\mathcal{Z}_{H_3}(\bullet, r', c') \neq \varnothing$ by Theorem 0.2.3, let $x' \in \mathcal{Z}_{H_3}(\bullet, r', c')$ with $x' \neq e$. Now we claim that the coloring (a, e, r') of $y^1y^2y^3$ extends to an L-coloring of G. Since $|\mathcal{Z}_{H_1}(c, a, \bullet)| \geq 2$, there is a color $x \in \mathcal{Z}_{H_1}(c, r, \bullet) \setminus \{e\}$. The coloring (x, e, x') of $x^2y^2z^2$ extends to an L-coloring of H_2 since $e \notin L(w_1)$. Thus, since $x \in \mathcal{Z}_{H_1}(c, a, \bullet)$ and $x' \in \mathcal{Z}_{H_3}(\bullet, r', c')$, the coloring (a, e, r') of $y^1y^2y^3$ extends to an L-coloring ψ of $G \setminus \{w\}$. Finally, since $e, r' \neq b$, there is a color left over in $L(w)$, so ψ extends to an L-coloring of G, contradicting our assumption. ■

Now, since $|L(y^3)| \geq 4$, we let $d_0, d_1 \in L(y^3) \setminus \{b, c'\}$. Applying Claim 4.1.8, we have $\mathcal{Z}_{H_3}(\bullet, d_0, c) = \mathcal{Z}_{H_3}(\bullet, d_1, c) = \{e\}$. Now let $r \in L(y^1) \setminus \{a, b, c\}$ and let ψ be an L-coloring of H_1 with $\psi(y^1) = r$. Such a ψ exists by Theorem 0.2.3. Now we have the following:

Claim 4.1.9. $L(w_t) = L(y^2) \setminus \{r, \psi(z^1)\}$ *and* $e \in L(w_t)$.

Proof: Suppose that $e \notin L(w_t)$. Since $d_0, d_1 \notin \{b, c'\}$, choose an $i \in \{0, 1\}$ with $d_i \notin \{a, b, c'\}$. Then choose a color $s \in L(y^2) \setminus \{a, b, d_i, e\}$, and consider the coloring (a, s, d_i) of $y^1y^2y^3$. We claim that this extends to an L-coloring of G. Firstly, since $|\mathcal{Z}_{H_1}(c, a, \bullet)| \geq 2$, let $s' \in \mathcal{Z}_{H_1}(c, a, \bullet) \setminus \{s\}$. Now consider the coloring (s', s, e) of $x^2y^2z^2$. This extends to an L-coloring of H_2, since $e \notin L(w_t)$. Thus, since $s' \in \mathcal{Z}_{H_1}(c, a, \bullet)$ and $e \in \mathcal{Z}_{H_3}(\bullet, d_i, c')$, the coloring (a, s, d_i) of $y^1y^2y^3$ extends to an L-coloring ϕ of $G \setminus \{w\}$. Finally, since $s, d_i \notin \{a, b\}$, the color b is left over for w, so ϕ extends to an L-coloring of G, contradicting our assumption.

Thus, we conclude that $e \in L(w_t)$. Now suppose towards a contradiction that $L(y^2) \setminus \{r, \psi(z^1)\} \neq L(w_t)$. In that case, since $|L(w_t)| = 3$ and $|L(y^2)| \geq 5$, there is a color $s \in L(y^2) \setminus \{r, \psi(z^1)\}$ with $s \notin L(w_t)$. Consider the following cases:

Case 1: $s \in \{a, b\}$.

In this case, we simply choose an $i \in \{0, 1\}$ such that $d_i \neq a$. At least one such d_i exists. Since $d_i \notin \{b, c'\}$, we then have $d_i \notin \{a, b, c'\}$. Now we extend ψ to an L-coloring ψ' of $V(H_1) \cup \{y^2, y^3\}$ by setting $\psi'(y^2) = s$ and $\psi'(y^3) = d_i$. This is a proper coloring of $V(H_1) \cup \{y^2, y^3\}$, since $s \in \{a, b\}$ and $r, d_i \notin \{a, b\}$, and $s \neq \psi(z^1)$. To see that ψ' extends to an L-coloring of $G \setminus \{w\}$, we just note that the coloring $(\psi(z^1), s, e)$ of $x^2y^2z^2$ extends to an L-coloring of H_2, since $s \notin L(w_t)$. Thus, since $e \in \mathcal{Z}_{H_3}(d_i)$, ψ extends to an L-coloring of $G \setminus \{w\}$. Finally, since $\psi'(y_1), \psi'(y^3) \notin \{a, b\}$, there is a color left over for w, so ψ extends to an L-coloring of G, contradicting our assumption.

Case 2: $s \notin \{a, b\}$.

As above, we choose $i \in \{0, 1\}$ with $d_i \neq s$. At least one such i exists. Then we take the coloring (a, s, d_i) of $y^1y^2y^3$. This is a proper coloring of $y^1y^2y^3$. Since $\mathcal{Z}_{H_1}(c, a, \bullet)| \geq 2$, let $s' \in \mathcal{Z}_{H_1}(c, a, \bullet) \setminus \{s\}$. The coloring (s', s, e) of $x^2y^2z^2$ extends to an L-coloring of H_2, since $s \notin L(w_t)$, and thus, since $s' \in \mathcal{Z}_{H_1}(c, a, \bullet)$ and $e \in \mathcal{Z}_{H_3}(\bullet, d_i, c')$, the coloring (a, s, d_i) of $y^1y^2y^3$ extends to an L-coloring ϕ of $G \setminus \{w\}$. Since $s \notin \{a, b\}$ and $a, d_i \neq b$, the color b is left over for w, so ϕ extends to an L-coloring of G, contradicting our assumption. Thus, we conclude that $e \in L(w_t)$ and $L(y^2) \setminus \{r, \psi(z^1)\} = L(w_t)$. This completes the proof of Claim 4.1.9. ■

Applying Claim 4.1.9, we have $L(w_t) = L(y^2) \setminus \{r, \psi(z^1)\}$ and $e \in L(w_t)$. Since $|L(y^2)| \geq 5$, we have $r \in L(y^2)$ and $r \neq e$. Now, there is at least one $i \in \{0, 1\}$ such that $d_i \neq r$. Given this d_i, consider the coloring (a, r, d_i) of $y^1y^2y^3$. We claim that this coloring of $y^1y^2y^3$ extends to an L-coloring of G. Let $x \in \mathcal{Z}(H_1, a)$. Then the coloring

(x, r, e) of $x^2y^2z^2$ extends to an L-coloring of H_2, since $r \notin L(w_t)$. Since $e \in \mathcal{Z}_{H_3,L}(\bullet, d_i, c')$ and $x \in \mathcal{Z}_{H_1}(c, r, \bullet)$, the coloring (a, r, d_i) of $y^1y^2y^3$ extends to an L-coloring ψ of $G \setminus \{w\}$. Since $b \neq r, d_i$, there is a color in $L(w)$ left over, so ψ extends to an L-coloring of G, contradicting our assumption. This completes Case 2.2, and thus completes the base case of Proposition 4.1.4.

Now let $k \geq 3$, and suppose that, for any crown (G', w', P', L') with $|V(P)| = k$, G' is L'-colorable. Suppose now that the crown (G, w, P, L) satisfies $|V(P)| = k+1$. Let $a, b \in L(w)$, and let $L(x^1) = \{c\}$ and $L(z^{k+1}) = \{c'\}$ for some colors c, c' (possibly one or both of c, c' lies in $\{a, b\}$).

Since $|L(y^{k+1})| \geq 4$, let $d \in L(y^{k+1}) \setminus \{a, b, c'\}$. By Thomassen, there is an L-coloring ψ of H_{k+1} such that $\psi(y^{k+1}) = d$. Now let L^* be a list-assignment for $G \setminus (H_{k+1} \setminus \{x^{k+1}\})$ defined as follows: Set $L^*(x^{k+1}) := \psi(x^{k+1})$ and $L^*(y^k) = L(y^k) \setminus \{d\}$. Then set $L^*(w) := \{a, b\}$, and, finally, $L^*(u) := L(u)$ for all $u \in V(G) \setminus (V(H_{k+1}) \cup \{y^k, w\})$.

Let $G^* := G \setminus (H_{k+1} \setminus \{x^{k+1}\})$. Then $G^* - w$ is a wheel sequence with $\mathcal{H}(G^* - w) = (H_1, \cdots, H_k)$ and $\mathcal{P}(G^* - w) = (P_1, \cdots, P_k)$. In particular, (G^*, w, p_1Pp_k, L^*) is a crown, and thus, since $|V(p_1Pp_k)| = k$, G^* is L^*-colorable.

Let ϕ be an L^*-coloring of G^*. Note that $\phi \cup \psi$ is well-defined, since ϕ, ψ agree on their common domain of $\{x^{k+1}\}$, so we just need to check that $\phi \cup \psi$ is a proper L-coloring of G. We have $\phi(w) \neq \psi(y^{k+1})$, since $\psi(y^{k+1}) \notin \{a, b\}$, and we have $\phi(y^k) \neq \psi(y^{k+1})$, since $\phi(y^k) \in L(y^k) \setminus \{d\}$. Thus, for any edge e of G with one endpoint in dom(ψ) and the other in dom(ϕ), the endpoints of e are assigned different colors by $\phi \cup \psi$, so $\phi \cup \psi$ is indeed a proper L-coloring of G, as desired. This completes the proof of Proposition 4.1.4 and thus proves 1) of Theorem 4.1.3. $\square$

Now we prove the second half of Theorem 4.1.3, i.e we deal with the special case where the apex path has length one. We restate this with the proposition below.

Proposition 4.1.10. *Let (G, w, P, L) be a crown with $|V(P)| = 2$. Let $\mathcal{H}(G - w) = (H_1, H_2)$ and $\mathcal{P}(G - w) = (P_1, P_2)$, where $P_i = x^iy^iz^i$ for each $i = 1, 2$. Let $L(x^1) = \{c\}$ and $L(z^2) = \{c'\}$. Set $X := \bigcap_{u \in V(H_1 \setminus P_1)} L(u)$ and $X' := \bigcap_{u \in V(H_2 \setminus P_2)} L(u)$ Then one of the following three statements holds.*

a) G is L-colorable; OR

b) There is a set T of two colors such that $T \subseteq L(u)$ for each $u \in V(G) \setminus (\{w, x^1, z^2, y^1, y^2\})$; OR

c) There is a set $S = \{a, b, r\}$ of three colors such that $L(w) = \{a, b\}$, $L(y^1) \setminus \{c\} = L(y^2) \setminus \{c'\} = S$, and furthermore, $\{b, r\} \subseteq X \setminus \{z^1\}$, $\{a, r\} \subseteq X' \setminus \{z^1\}$, and $|L(z^1) \cap S| \geq 2$.

Proof. Let $a, b \in L(w)$ (possibly, one or both of c, c' lies in $\{a, b\}$). Now we partition $L(y^1) \setminus \{c\}$ into two sets T_1 and T_2, where $T_1 := \{r \in L(y^1) \setminus \{c\} : |\mathcal{Z}_{H_1}(c, r, z^1)| = 1\}$ and $T_2 := \{r \in L(y^1) \setminus \{c\} : |\mathcal{Z}_{H_1}(c, r, \bullet)| \geq 2\}$. Note that $L(y^1) \setminus \{c\} = T_1 \cup T_2$ as a disjoint union.

Likewise, we partition $L(y^2) \setminus \{c'\}$ into two sets T_1' and T_2' where $T_1' := \{r \in L(y^2) \setminus \{c'\} : |\mathcal{Z}_{H_2}(\bullet, r', c')| = 1\}$ and $T_2' := \{r \in L(y^2) \setminus \{c'\} : |\mathcal{Z}_{H_2}(z^1, r', c')| \geq 2\}$. As above, $L(y^2) \setminus \{c'\} = T_1' \cup T_2'$ is a disjoint union. Note that $T_1 \subseteq X$ and $T_1' \subseteq X'$ by Proposition 1.4.4. Suppose now for the remainder of the proof of Proposition 4.1.10 that G is not L-colorable. We now have the following fact:

Claim 4.1.11. *If there are colors $r \neq r'$, where $r \in$ and $r' \in L(y^1)$ such that $\{r, r'\} \neq \{a, b\}$ and $\mathcal{Z}_{H_1}(c, r, \bullet) \cap \mathcal{Z}_{H_2}(\bullet, r', c') \neq \varnothing$, then the L-coloring (r, r') of the edge y^1y^2 extends to an L-coloring of G.*

Proof: Suppose there such a pair r, r'. Since $\mathcal{Z}_{H_1}(c, r, \bullet) \cap \mathcal{Z}_{H_2}(\bullet, r', c') \neq \varnothing$, there is an L-coloring ϕ of $G \setminus \{w\}$ such that $\phi(y^1) = r$ and $\phi(y^2) = r'$. Since $\{r, r'\} \neq \{a, b\}$, there is a color left over in $L(w)$, so ϕ extends to an L-coloring of G. ■

Let $U_{G,L}(y^1y^2) \subseteq \Phi_{G,L}(y^1y^2)$ be the set of L-colorings (s_1, s_2) of the edge y^1y^2 such that $\mathcal{Z}_{H_1}(c, s_1, \bullet) \cap \mathcal{Z}_{H_2}(\bullet, s_2, c') \neq \varnothing$ and $\{s_1, s_2\} \neq \{a, b\}$. Thus, by Claim 4.1.11, if $U_{G,L}(y^1y^2) \neq \varnothing$, then G is L-colorable. For any $s_1 \in L(y^1) \setminus \{c\}$ and $s_2 \in L(y^2) \setminus \{c'\}$, we set $I(s_1, s_2) := \mathcal{Z}_{H_1}(c, s_1, \bullet) \cap \mathcal{Z}_{H_2}(\bullet, s_2, c')$.

Claim 4.1.12.

1) *If $\{a, b\} \subseteq L(y^1) \setminus \{c\}$ and $\mathcal{Z}_{H_1}(c, a, \bullet) \cup \mathcal{Z}_{H_1}(c, b, \bullet) = L(z^1)$, then $U_{G,L}(y^1y^2) \neq \varnothing$. Likewise, if $\{a, b\} \subseteq L(y^2)$ and $\mathcal{Z}_{H_2}(\bullet, a, c') \cup \mathcal{Z}_{H_2}(\bullet, b, c') = L(z^1)$, then $U_{G,L}(y^1y^2) \neq \varnothing$; AND*
2) *If there is a color $r \in L(y^1) \setminus \{c\}$ such that $\mathcal{Z}_{H_1}(c, r, \bullet)| = 3$, then G is L-colorable. Likewise, if there is a color $r' \in L(y^2) \setminus \{c\}$ such that $|\mathcal{Z}_{H_2}(\bullet, r', c')| = 3$, then G is L-colorable.*

Proof: Let $\{a, b\} \subseteq L(y^1)$ and $r' \in L(y^2) \setminus \{a, b, c'\}$. If $\mathcal{Z}_{H_1}(c, a, \bullet) \cup \mathcal{Z}_{H_1}(c, b, \bullet) = L(z^1)$ then $I(a, r') \cup I(b, r') \neq \varnothing$ and thus either $(a, r') \in U_{G,L}(y^1y^2)$ or $(b, r') \in U_{G,L}(y^1y^2)$. An identical argument shows the analogous statement for the case where $\{a, b\} \subseteq L(y^2)$. This proves Fact 1.

Now let $r \in L(y^1) \setminus \{c\}$ and suppose that $\mathcal{Z}_{H_1}(c, r, \bullet)| = 3$, so $\mathcal{Z}_{H_1}(c, r, \bullet) = L(z^1)$. Now, if $r \in \{a, b\}$, then we just choose an $r' \in L(y^2) \setminus \{a, b, c'\}$. We have $I(r, r') \neq \varnothing$ since $\mathcal{Z}_{H_1}(c, r, \bullet) = L(z^1)$, and $(r, r') \in U_{G,L}(y^1y^2)$, since $r \neq r'$ and $r \notin \{a, b\}$. On the other hand, if $r \notin \{a, b\}$, then we simply let $r' \in L(y^2) \setminus \{c', r\}$. Then $(r, r') \in U_{G,L}(y^1y^2)$, so G is L-colorable. An identical argument shows that, if there is a $r' \in L(y^2) \setminus \{c\}$ such that $|\mathcal{Z}_{H_2}(\bullet, r', c')| = 3$, then G is L-colorable. This completes the proof of Fact 2. ■

Now we have the following:

Claim 4.1.13. *If either $|L(y^1) \setminus \{a, b, c\}| \geq 2$ or $|L(y^2) \setminus \{a, b, c'\}| \geq 2$ then Proposition 4.1.10 is satisfied.*

Proof: Suppose without loss of generality that $|L(y^1) \setminus \{a, b, c\}| \geq 2$. In that case, there are two colors $d_1, d_2 \in L(y^1) \setminus \{a, b, c\}$. Since G is not L-colorable, we have $U_{G,L}(y^1y^2) = \varnothing$. Now consider the following cases:

Case 1: $|\mathcal{Z}_{H_1}(c, d_1, \bullet) \cup \mathcal{Z}_{H_1}(c, d_2, \bullet)| = 1$

In this case, there is a color $e \in L(z^1) \setminus \{d_1, d_2\}$ such that $\{e\} = \mathcal{Z}_{H_1}(c, d_1, \bullet) = \mathcal{Z}_{H_1}(c, d_2, \bullet)$. Furthermore, since $d_1, d_2 \in T_1$, we have $d_1, d_2 \in X$, and thus $L(z^1) = \{e, d_1, d_2\}$. We also have $\{d_1, d_2\} \subseteq X$ by Proposition 1.4.4. We also note that H_1 is not a triangle, or else $d_1 \in \mathcal{Z}_{H_1}(c, d_2, \bullet)$ and $d_2 \in \mathcal{Z}_{H_1}(c, d_2, \bullet)$, contradicting the fact that $\{e\} = \mathcal{Z}_{H_1}(c, d_1, \bullet) = \mathcal{Z}_{H_1}(c, d_2, \bullet)$. Since H_1 is not a triangle let $H_1 \setminus \{y^1\}$ be the path $x^1 w_1 \cdots w_t z^1$ for some $t \geq 1$.

Now, if there is a color $r' \in L(y^2) \setminus \{c\}$ such that $e \in \mathcal{Z}_{H_2}(\bullet, r', c')$, then G is L-colorable. To see thus, let r' be such a color in $L(y^2) \setminus \{c'\}$ and let $i \in \{1, 2\}$ be such that $d_i \neq r'$. Then $e \in I(d_i, r')$ and thus $(d_i, r') \in U_{G,L}(y^1y^2)$, contradicting our assumption. Thus, we have $\mathcal{Z}_{H_2}(z^1, r', c') \subseteq \{d_1, d_2\}$ for each $r' \in L(y^2) \setminus \{c'\}$. Let r be a color of $L(y^1) \setminus \{d_1, d_2, c\}$ (possibly $r \in \{a, b\}$). Since $|\mathcal{Z}_{H_1}(c, d_1, \bullet)| = |\mathcal{Z}_{H_1}(c, d_2, \bullet)| = 1$, we have $|\mathcal{Z}_{H_1}(c, r, \bullet)| \geq 2$ by Proposition 1.4.5. Now consider the following subcases.

Case 1.1 $|\{d_1, d_2\} \cap L(y^2) \setminus \{c'\}| \geq 1$.

In this case, suppose without loss of generality that $d_1 \in L(y^2)$. Thus, since $\mathcal{Z}_{H_2}(\bullet, d_1, c') \subseteq \{d_1, d_2\}$ we have $\mathcal{Z}_{H_2}(\bullet, d^1, c') = \{d_2\}$. If $\mathcal{Z}_{H_1}(c, r, \bullet) \cap \mathcal{Z}_{H_2}(\bullet, d_1, c') \neq \varnothing$, then $(r, d_1) \in U_{G,L}(y^1y^2)$, contradicting our assumption. Thus, we have $\mathcal{Z}_{H_1}(c, r, z^1) = \{d_1, e\}$. In that case, we have $r \neq e$ and thus $r \notin L(z^1)$, since $L(z^1) = \{d_1, d_2, e\}$. Note that $r \in \bigcap_{j=1}^{t} L(w_j)$, or else, if there is a $j \in \{1, \cdots, t\}$ such that $r \notin L(w_j)$, then $d_2 \in \mathcal{Z}_{H_1}(c, r, \bullet)$, contradicting our assumption. Since $\{d_1, d_2\} \subseteq X$, we have $L(w_j) = \{d_1, d_2, r\}$ for each $j = 1, \cdots, t$. But then $c \notin L(w_1)$, and thus $\mathcal{Z}_{H_1}(c, r, \bullet) = \{d_1, d_2, e\}$, contradicting our assumption. This completes Case 1.1.

Case 1.2 $\{d_1, d_2\} \cap (L(y^2) \setminus \{c'\}) = \varnothing$

In this case, $L(y^2) \setminus \{c'\}$ contains three colors ℓ_1, ℓ_2, ℓ_3 such that $\{d_1, d_2\} \cap \{\ell_1, \ell_2, \ell_3\} = \varnothing$. Suppose without loss of generality that $\ell_1 \neq e$. Now, if H_2 is a triangle, then $e \in \mathcal{Z}_{H_1}(z^1, \ell_1, c')$, contradicting the fact that $\mathcal{Z}_{H_1}(\bullet, \ell_1, c') \subseteq \{d_1, d_2\}$. Thus, H_2 is not a triangle, so let $H_2 \setminus \{y^2\} = z^1 v_1 \cdots v_{t'} z^2$ for some $t' \geq 1$. Now, since $L(z^1) = \{d_1, d_2, e\}$, we suppose without loss of generality that $\ell_1, \ell_2 \notin L(z^1)$. In that case, we have $\{\ell_1, \ell_2\} \subseteq L(v_j)$ for each $j = 1, \cdots, t'$, or else, for some $k \in \{1, 2\}$, we have $\mathcal{Z}_{H_2}(z^1, \ell_k, c')| = 3$, and thus G is L-colorable by Claim 4.1.12, contradicting our assumption.

Now, if $e \notin L(v_1)$, then we have $e \in \mathcal{Z}_{H_2}(\bullet, \ell_k, c')$ for each $k = 1, 2$, since $e \notin \{ell_1, \ell_2\}$, contradicting the fact that $\mathcal{Z}_{H_2}(z_1, r', c') \subseteq \{d_1, d_2\}$ for each $r' \in L(y^2) \setminus \{c'\}$. Thus, we have $L(z^1) = \{\ell_1, \ell_2, e\}$. In particular, $d_1, d_2 \notin L(v_1)$. Now, since $\mathcal{Z}_{H_1}(r) \geq 2$, there is a $k \in \{1, 2\}$ such that $d_k \in \mathcal{Z}_{H_1}(r)$. Suppose that $r \notin \{a, b\}$. In that case, we simply choose a color $\ell_j \in \{\ell_1, \ell_2, \ell_3\} \setminus \{r\}$. Then $d_k \in \mathcal{Z}_{H_2}(\bullet, \ell_j, c')$ since $d_k \notin L(v_1)$. But then $(r, \ell_j) \in U_{G,L}(y^1y^2)$ since $r \notin \{a, b\}$. This contradicts our assumption. Now suppose that $r \in \{a, b\}$, and let $j \in \{1, 2, 3\}$ such that $\ell_j \notin \{a, b\}$. Thus, $r \neq \ell_j$. But since $r \in T_2$, there is a $k \in \{1, 2\}$ such that $d_k \in \mathcal{Z}_{H_1}(c, r, \bullet)$. Furthermore, we have $\ell_j \neq d_k$, and $d_k \notin L(v_1)$, so $d_k \in \mathcal{Z}_{H_2}(\bullet, \ell_j, c')$. But then, since $\ell_j \notin \{a, b\}$, we have $(r, \ell_j) \in U_{G,L}(y^1y^2)$, contradicting our assumption. This completes Case 1.

Case 2: $|\mathcal{Z}_{H_1}(c, d_1, \bullet) \cup \mathcal{Z}_{H_1}(c, d_2, \bullet)| \geq 2$

In this case, we note that $T_2' \subseteq \{d_1, d_2\}$. To see this, suppose there is a color $r' \in T_2'$ with $r' \notin \{d_1, d_2\}$. Since $|\mathcal{Z}_{H_2}(\bullet, r', c')| \geq 2$, we have $I(d_1, r') \cup I(d_2, r') \neq \varnothing$, so either (d_1, r') or (d_2, r') lies in $U_{G,L}(y^1y^2)$, contradicting our assumption. So we have $T_2' \subseteq \{d_1, d_2\}$. Suppose without loss of generality that $d_1 \in T_2'$, and let $r \in L(y^1) \setminus \{d_1, d_2, c\}$. We also note that $|\mathcal{Z}_{H_1}(c, d_1, z^1) \cup \mathcal{Z}_{H_1}(c, d_2, \bullet)| = 2$, or else, if $\mathcal{Z}_{H_1}(c, d_1, z^1) \cup \mathcal{Z}_{H_1}(c, d_2, \bullet) = L(z^1)$, then, for any $r' \in L(y^2) \setminus \{d_1, d_2, c'\}$, either (d_1, r') or (d_2, r') lies in $U_{G,L}(y^1y^2)$, contradicting our assumption.

Case 2.1 $d_1 \in L(z^1)$

If $d_1 \in L(z^1)$, then we have $\mathcal{Z}_{H_2}(\bullet, d_1, c') = L(z^1) \setminus \{d_1\}$. If $I(r, d_1) \cup I(d_2, d_1) \neq \varnothing$, then $U_{G,L}(y^1y^2) \neq \varnothing$ since $d_1 \notin \{a, b\}$. This contradicts our assumption. Thus, we have $\mathcal{Z}_{H_1}(c, r, \bullet) = \mathcal{Z}_{H_1}(c, d_2, \bullet) = \{d_1\}$. We thus have $d_1 \in T_1'$ by Proposition 1.4.5, so $\mathcal{Z}_{H_1} = L(z^1) \setminus \{d_1\}$.

Since $r, d_2 \in T_1$, we have $\{r, d_2\} \subseteq L(z^1)$, so $L(z^1) = \{d_1, d_2, r\}$. Now, we note that $L(y^2) \setminus \{d_1, d_2, c'\} = \{r\}$. To see this, suppose there is an $r' \in L(y^2) \setminus \{d_1, d_2, c'\}$ with $r' \neq r$. Then $r' \in T_2'$, since $r' \notin L(z^1)$, contradicting the fact that $T_2' \subseteq \{d_1, d_2\}$. Thus, we have $L(y^2) \setminus \{d_1, d_2, c'\} = \{r\}$, so $L(y^2) = \{d_1, d_2, c', r\}$.

Now, since $\mathcal{Z}_{H_1}(\bullet, d_1, c') = L(z^1) \setminus \{d_1\}$, we have $\mathcal{Z}_{H_2}(z^1, d_2, c') = \mathcal{Z}_{H_2}(\bullet, r, c') = \{d_1\}$, or else either (d_1, r) or (d_1, d_2) lies in $U_{G,L}(y^1y^2)$, contradicting our assumption. But then $d_1 \in I(d_2, r)$, so $(d_2, r) \in U_{G,L}(y^1y^2)$, contradicting our assumption. This completes Case 2.1.

Case 2.2 $d_1 \notin L(z^1)$

In this case, we have $d_1 \in T_2 \cap T_2'$. Let $L(z^1) = \{\ell_1, \ell_2, \ell_3\}$ and let $\ell_1, \ell_2 \in \mathcal{Z}_{H_2}(\bullet, d_1, c')$. Thus, we have $\mathcal{Z}_{H_1}(d_2) =$

$\mathcal{Z}_{H_1}(r) = \{\ell_3\}$, or else $(d_2, d_1) \cup I(r, d_1) \neq \varnothing$, and thus either (d_2, d_1) or (r, d_1) lies in $U_{G,L}(y^1y^2)$, contradicting our assumption. So we have $r, d_2 \in T_1$ and thus $r, d_2 \in L(z^1)$. Thus, we have $\{r, d_2\} = \{\ell_1, \ell_2\}$, since $r, d_2 \neq \ell_3$, and so $L(z^1) = \{r, d_2, \ell_3\}$. Since $d_1 \in T'_2$, we have $\mathcal{Z}_{H_2}(\bullet, d_1, c') = \{r, d_2\}$, or else $\ell_3 \in I(d_2, d_1) \cup I(r, d_1)$, and thus either (d_2, d_1) or (r, d_1) lies in $U_{G,L}(y^1y^2)$, contradicting our assumption.

Now, if $L(y^2)\setminus\{r, d_2\ell_3, c'\} \neq \varnothing$, then let $r' \in L(y^2)\setminus\{r, d_2, \ell_3, c'\}$. Since $r' \notin L(z^1)$, we have $r' \in T'_2$, contradicting the fact that $T'_2 \subseteq \{d_1, d_2\}$. Thus, we have $L(y^2) = \{r, d_2, \ell_3, c'\}$. But we also have $\ell_3 \neq d_1$, since $d_1 \notin L(z^1)$, so we get $d_1 \notin L(y^2)$, contradicting the fact that $d_1 \in T'_2$. This completes Case 2.2 and thus completes the proof of Claim 4.1.13. ■

Thus, we may assume for the remainder of the proof of Proposition 4.1.10 that $|L(y^1) \setminus \{a, b, c\}| = 1$ and $|L(y^2) \setminus \{a, b, c'\}| = 1$. If $\{a, b\} \subseteq X \cap X'$, then we are done. So now suppose that $\{a, b\} \not\subseteq X \cap X'$, and suppose without loss of generality that $a \notin X$. Thus, $a \notin T_1$, since $T_1 \subseteq X$. Since $a \in L(y^1) \setminus \{c\}$, we have $a \in T_2$. Suppose that G is not L-colorable. Thus, we have $U_{G,L}(y^1y^2) = \varnothing$. We show that either Statement b) or Statement c) of Proposition 4.1.10 is satisfied.

Claim 4.1.14. $|L(w)| = 2$.

Proof: Let f be be the L-coloring of $\{x^1, z^2\}$ obtained by coloring x_1 with c and z^2 with c'. Let u be the lone vertex of $N(x^1) \setminus \{y^1\}$ and let u' be the lone vertex of $N(z^2) \setminus \{y^2\}$. If $|L(w)| \geq 3$, then $\{u, u'\}$ either consists of a lone vertex with an L_f-list of size at least one, or two vertices with L_f-lists of size at least two. In the first case $G \setminus \{x^1, z^2\}$ is L_f-colorable by Theorem 0.2.3, and in the second case, $G \setminus \{x^1, z^2\}$ is L_f-colorable by Theorem 1.3.4. Thus, G is L-colorable, contradicting our assumption. So we have $|L(w)| = 2$. ■

By Claim 4.1.11, if $T'_2 \not\subseteq \{a, b\}$, then G is L-colorable, contradicting our assumption. So we have $T'_2 \subseteq \{a, b\}$. Thus, we have $T_2 \subseteq \{a, b\}$ as well, or else $U_{G,L}(y^1y^2) \neq \varnothing$, contradicting our assumption. Let $r \in L(y^1) \setminus \{a, b, c\}$ and $r' \in L(y^2) \setminus \{a, b, c'\}$.

Case 1: $\{a, b\} \subseteq L(z^1)$

In this case, since $a \notin X$, we have $\mathcal{Z}_{H_1}(c, a, \bullet) = L(z^1) \setminus \{a\}$. Now, let $r \in L(y^1) \setminus \{a, b, c'\}$ and $r' \in L(y^2) \setminus \{a, b, c'\}$. Thus, we have $\mathcal{Z}_{H_2}(\bullet, r', c') = \{a\}$, or else the coloring $(a, r') \in U_{G,L}(y^1y^2)$, contradicting our assumption. Since $r' \in T'$, we have $r' \in X'$ and thus $L(z^1) = \{a, b, r'\}$. Furthermore, since $r \in X$ we have $r \in L(z^1)$ and thus $r = r'$.

Subcase 1.1 $\mathcal{Z}_{H_2}(z^1, b, c') = \{a\}$

In this case, by Proposition 1.4.5, we have $\mathcal{Z}_{H_2}(\bullet, a, c') = L(z^1) \setminus \{a\} = \{b, r'\}$. Thus, we have $\mathcal{Z}_{H_1}(c, r, \bullet) = \{a\}$, or else the coloring $(r, a) \in U_{G,L}(y^1y^2)$, contradicting our assumption. But then, the coloring $(r, b) \in U_{G,L}(y^1y^2)$, which, again, contradicts our assumption. This completes Subcase 1.1.

Case 1.2: $\mathcal{Z}_{H_2}(\bullet, b, c') \neq \{a\}$

We break this into two subcases:

Case 1.2.1 $a \in \mathcal{Z}_{H_2}(\bullet, b, c')$

In this case, we have $\mathcal{Z}_{H_2}(\bullet, b, c') = \{a, r\}$. Thus, we have $\mathcal{Z}_{H_1}(c, r, \bullet) = \{b\}$, or else $(r, b) \in U_{G,L}(y^1y^2)$, contradicting our assumption. Furthermore, we have $\mathcal{Z}_{H_2}(\bullet, a, c') = \{r\}$, or else $(r, a) \in U_{G,L}(y^1y^2)$, contradicting our assumption. Putting these facts together, we have $b \in X$ and $a, r \in X'$. Recall that $r \in X$ as well. Applying

Claim 4.1.14, we have $L(y^1) \setminus \{c\} = L(y^2) \setminus \{c'\} = \{a, b, r\}$ and $L(w) = \{a, b\}$. Furthermore, $\{a, r\} \subseteq X'$, and $\{b, r\} \subseteq X$. Thus, in Case 1.2.1, Statement c) of Proposition 4.1.10 is satisfied, so we are done. This completes Case 1.2.1.

Case 1.2.2 $a \notin \mathcal{Z}_{H_2}(\bullet, b, c')$

In that case, we have $\mathcal{Z}_{H_2}(\bullet, b, c') = \{r'\}$. Since $\mathcal{Z}_{H_2}(\bullet, r', c') = \{a\}$, we then have $\mathcal{Z}_{H_2}(\bullet, a, c') = \{b, r'\}$ by Proposition 1.4.5. Thus, we have $\mathcal{Z}_{H_1}(c, r, \bullet) = \{a\}$, or else $I(r, a) \cap \{b, r'\} \neq \varnothing$ and thus $(r, a) \in U_{G,L}(y^1y^2)$, contradicting our assumption. Since $\mathcal{Z}_{H_1}(c, r, \bullet) = \{a\}$, we have $r = r'$. To see this, note that if $r \neq r'$, then $(r, r') \in U_{G,L}(y^1y^2)$, since $a \in I(r, r')$. This contradicts our assumption. So we have $r \in T_1 \cap T_1'$, and $b \in T_1'$ as well. Now, if $b \in T_1$, then we have $\{b, r\} \subseteq X \cap X'$, and then Statement 2 is satisfied, so we are done. So suppose now that $b \notin T_1$. Then $a \in I(b, r')$ and $(b, r') \in U_{G,L}(y^1y^2)$, contradicting our assumption. This completes Case 1.

Case 2: $\{a, b\} \not\subseteq L(z^1)$.

In this case, suppose without loss of generality that $a \notin L(z^1)$. Thus, we have $a \in T_2 \cap T_2'$. Now, if both H_1 and H_2 are triangles, then Statement b) of Proposition 4.1.10 is trivially satisfied, so suppose without loss of generality that H_1 is not a triangle, and let $H_1 \setminus \{y^1\} = x^1w_1 \cdots w_tz^1$ for some $t \geq 1$.

Now, if either $a \notin L(w_1)$ or $c \notin L(w_1)$, then $\mathcal{Z}_{H_1}(c, a, \bullet 1) = L(z^1)$, and thus $(a, r) \in U_{G,L}(y^1y^2)$, contradicting our assumption. If $r \notin L(w_1)$ then $r \in T_2$, contradicting our assumption. Thus, we have $L(w^1) = \{a, r, c\}$. Note then that $\mathcal{Z}_{H_1}(c, b, \bullet) = L(z^1) \setminus \{b\}$, since $b \notin L(w_1)$. This implies that $b \in L(z^1)$, or else, if $b \notin L(z^1)$, then $\mathcal{Z}_{H_1}(c, b, z^1) = L(z^1)$, since $b \notin L(w^1)$. But then $(b, r') \in U_{G,L}(y^1y^2)$, contradicting our assumption. Thus, we have $b \in L(z^1)$ and $\mathcal{Z}_{H_1}(c, b, \bullet) = L(z^1) \setminus \{b\}$. Furthermore, since $a \in T_2$, we have $\mathcal{Z}_{H_1}(c, a, \bullet) = L(z^1) \setminus \{b\}$ as well, or else $U_{G,L}(y^1y^2) \neq \varnothing$ by Claim 4.1.12, contradicting our assumption.

Case 2.1 $r \neq r'$

In this case, since $r, r' \in T_1 \cap T_1'$, we have $L(z^1) = \{r, r', b\}$. It follows that H_2 is also not a triangle, or else, if H_2 is a triangle, then we have $\mathcal{Z}_{H_2}(\bullet, a, c') = L(z^1)$, since $c' \notin L(z^1)$. But then $(r, a) \in U_{G,L}(y^1y^2)$, contradicting our assumption. Thus, let $H_2 - y^2 = z^1v_1 \cdots v_{t'}z^2$, for some $t' \geq 1$. Since $r' \in T_1'$, we have $r', c' \in L(v_{t'})$. If $a \notin L(v_{t'})$, then $\mathcal{Z}_{H_2}(\bullet, a, c') = L(z^1)$, since $a \notin L(z^1)$. But then $(r, a) \in U_{G,L}(y^1y^2)$, contradicting our assumption. Thus, we have $L(v_{t'}) = \{a, r', c'\}$, so $b \notin L(v_{t'})$. Since $b \notin L(v_{t'})$, we have $\mathcal{Z}_{H_2}(\bullet, b, c') = \{r, r'\}$. Furthermore, we have $\mathcal{Z}_{H_2}(\bullet, r', c') = \{b\}$, or else $r \in I(b, r')$ and thus $(b, r') \in U_{G,L}(y^1y^2)$, contradicting our assumption.

Now, since $\mathcal{Z}_{H_2}(\bullet, r', c') = \{b\}$, we have $\mathcal{Z}_{H_1}(c, r, \bullet) = \{r'\}$, or else $b \in I(r, r')$ and thus $(r, r') \in U_{G,L}(y^1y^2)$, contradicting our assumption. But then $r' \in I(r, b)$ and thus $(r, b) \in U_{G,L}(y^1y^2)$, contradicting our assumption. This Completes Case 2.1.

Case 2.2 $r = r'$

In this case, we have $r \in X \cap X'$, since $r \in T_1 \cap T_1'$. Furthermore, $L(z^1) = \{r, b, s\}$ for some $s \neq a$. Recall that $\mathcal{Z}_{H_1}(c, a, \bullet) = \mathcal{Z}_{H_1}(c, b, \bullet) = L(z^1) \setminus \{b\} = \{r, s\}$. Thus, we have $a \in L(w_j)$ for each $j = 1, \cdots, t$, or else $\mathcal{Z}_{H_1}(a) = L(z^1)$. Thus, since $r \in X$, we have $\{a, r\} \subseteq L(w_j)$ for each $j = 1, \cdots, t$.

Now, if $\{b, r\} \subseteq L(u)$ for each $u \in V(H_2) \setminus \{y^2\}$, then Statement c) of Proposition 4.1.10 is satisfied, with $S = \{a, b, r\}$, so we are done in that case. So now suppose that $\{b, r\} \not\subseteq L(u)$ for each $u \in V(H_2) \setminus \{y^2\}$. In that case, since $\{b, r\} \subseteq L(z^1)$, H_2 is not a triangle, and, letting $H_2 \setminus \{y^2\} = z^1v_1 \cdots v_{t'}z^2$ for some $t' \geq 1$, we have $\{b, r\} \not\subseteq L(v_j)$ for some $j = 1, \cdots, t'$. Since $r' \in X'$, we have $b \notin L(v_j)$ for some $j \in \{1, \cdots, t'\}$. In that case, we have $\mathcal{Z}_{H_2}(\bullet, b, c') = \{r, s\}$.

We have $\{a, r\} \subseteq L(w_t)$, as indicated above. Furthermore, we have $b \in L(w_t)$ as well, or else $b \in \mathcal{Z}_{H_1}(c, a, \bullet)$, contradicting the fact that $\mathcal{Z}_{H_1}(c, a, \bullet) = L(z^1) \setminus \{b\}$. So we get $L(w_t) = \{a, b, r\}$, and thus $s \in \mathcal{Z}_{H_1}(c, r, \bullet)$, since $s \notin L(w_t)$. But then we have $s \in I(r, b)$, so $(r, b) \in U_{G,L}(y^1y^2)$, contradicting our assumption.This completes Case 2 and thus completes the proof of Proposition 4.1.10 and Theorem 4.1.3. $\square$

We now use the results above to analyze the vertices of distance two from an open ring in a critical mosaic.

4.2 4-Chords on One Side of the Precolored Path

The purpose of this section is to prove the following result.

Lemma 4.2.1. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let $C \in \mathcal{C}$ be an open ring, and let $Q := x_1y_1wy_2x_2$ be a 4-chord of C, with $Q \in \mathcal{K}(C, \mathcal{T})$. Then $V(G^1_Q - w) \subseteq B_1(C)$.*

Proof. We proceed analogously to the proof of Lemma 3.1.1. Given a $Q \in \mathcal{K}^4(C, \mathcal{T})$, with $Q = x_1y_1wy_2x_2$, we call Q a *bad path* if $V(G^1_Q - w) \not\subseteq B_1(C)$. Analogous to Section 3.1, our goal is to show that there does not exist a bad path in $\mathcal{K}^4(C, \mathcal{T})$. Let C^1 be the 1-necklace of C. We first gather the following facts.

Claim 4.2.2. *For any bad path Q, the following hold.*

1) *There is no chord of Q in G, except possibly x_1x_2; AND*
2) *The middle vertex of Q has no neighbors in $V(C \cap G^1_Q)$; AND*
3) *For any $v \in V(G^0_Q \setminus Q)$, if v has a neighbor in $V(Q)$, then $G[N(v) \cap V(Q)]$ is a subpath of Q of length at most two.*

Proof: Let $Q := x_1y_1wy_2x_2$ and let $S := V(G^1_Q - w) \setminus B_1(C)$. Since Q is bad, $S \neq \varnothing$. Suppose toward a contradiction that G contains an edge $e \in \{x_1w, x_2w, y_1y_2, x_1y_2, x_2y_1\}$. Thus, for some $2 \leq k \leq 3$, there is a k-chord Q' of C such that Q' has endpoints x_1, x_2 and $E(Q') \setminus E(Q) = \{e\}$. Note that $Q' \in \mathcal{K}(C, \mathcal{T})$. Since Q' is a 2-chord or a 3-chord of C, we have $S \cap V(G^1_{Q'}) = \varnothing$, or else we contradict Theorem 3.0.2. If $e \in E(G^0_Q)$, then $G^1_Q \subseteq G^1_{Q'}$, then $S \subseteq V(G^1_{Q'})$, which is false. Thus, we have $e \in E(G^1_Q) \setminus E(Q)$. Since $S \cap V(G^1_{Q'}) = \varnothing$, there is a cycle of length at most four which separates S from $G^0_Q \setminus Q$, contradicting short-separation-freeness. This proves 1).

Now we prove 2). Suppose toward a contradiction that w has a neighbor u in $V(C \cap G^1_Q)$. By 1), $x_1, x_2 \notin N(w)$, so G contains the two 3-chords $Q' := x_1y_1wu$ and $Q'' := x_2y_2wu$ of C, and u is an internal vertex of $C \cap G^1_Q$. Furthermore, each of Q', Q'' lies in $\mathcal{K}(C, \mathcal{T})$, and we have either $S \subseteq V(G^1_{Q'})$ or $S \subseteq V(G^1_{Q''})$. In either case, we contradict Theorem 3.0.2.

Now we prove 3). Let $v \in V(G^0_Q \setminus Q)$, where v has a neighbor in Q. The claim is trivial if $|N(v) \cap V(Q)| = 1$, so suppose that $|N(v) \cap V(Q)| \geq 2$. If v has two neighbors which are of distance more than two apart on Q, then, for some $2 \leq k \leq 3$, there is a k-chord Q' of C with $v \in V(Q')$, where Q' has endpoints x_1, x_2 and $G^1_Q \subseteq G^1_{Q'}$. In that case, we have $S \subseteq V(G^1_{Q'})$. Since $Q' \in \mathcal{K}(C, \mathcal{T})$, this again contradicts Theorem 3.0.2. Thus, any two neighbors of v on Q are either adjacent on Q or are of distance precisely two apart on Q. By 1), there is no chord of Q in G, except possibly x_1x_2. Thus, if v has two neighbors u, u' which are of distance precisely two apart on Q, then, since G is short-separation-free, it follows from our triangulation conditions that v is also adjacent to the midpoint of the 2-path uQu', so we are done. ■

We now have the following.

Claim 4.2.3. *For any bad path Q, $V(G_Q^0)$ is L-colorable and $C \cap G_Q^1$ is a path of length at least two. In particular, Q is an induced subgraph of G.*

Proof: We have to be somewhat careful because we need to deal with the possibility that we have a bad path Q such that $C \cap G_Q^1$ is just an edge. In that case, a proper L-coloring of G_Q^0 is not necessarily a proper L-coloring of $V(G_Q^0)$. Suppose toward a contradiction that there exists a bad path Q^m such that $V(G_{Q^m}^0)$ is not L-colorable. Among all bad paths Q such that $V(G_Q^0)$ is not L-colorable, we choose Q^m so that $|V(G_{Q^m}^0)|$ is minimized.

> **Subclaim 4.2.4.** *For any $v \in V(G_{Q^m}^0 \setminus Q^m)$ with a neighbor in Q^m, the graph $G[N(v) \cap V(Q^m)]$ is a subpath of Q^m of length at most one.*
>
> Proof: Suppose there is a v for which this does not hold. By 3) of Claim 4.2.3, $G[N(v) \cap V(Q^m)]$ is a subpath of Q^m of length precisely two. Let u be the midpoint of $G[N(v) \cap V(Q^m)]$ and let Q be the 4-chord of C obtained from Q^m by replacing u with v. Since G is short-separation-free, we have $V(G_{Q^m}^0) = V(G_Q^0) \cup \{u\}$ as a disjoint union. By the minimality of Q^m, we get that $V(G_Q^0)$ admits an L-coloring ψ. Since $|L_\psi(u)| \geq 2$, ψ extends to an L-coloring of $V(G_Q^0) \cup \{u\}$, contradicting our assumption that $V(G_{Q^m}^0)$ is not L-colorable. ■

Let $S := V(G_{Q^m}^1 \setminus Q^m) \setminus B_1(C)$. Since Q^m is bad, S is nonempty. If $|S| = 1$, then, since the lone vertex of S has degree at least five, we contradict the fact that $S \subseteq V(G) \setminus B_1(C)$. Thus, we have $|S| > 1$.

Now we construct a smaller mosaic in the following way. Let $G^\dagger$ be a graph obtained from G by deleting all the vertices of $V(G_{Q^m}^1) \setminus V(C \cup Q^m)$ and replacing them with a lone vertex $w^\dagger$ adjacent to all the vertices in the cycle $(C \cap G_{Q^m}^1) + Q^m$. Let $\mathcal{T}^\dagger := (G^\dagger, \mathcal{C}, L, C_*)$. We claim now that $\mathcal{T}$ is a mosaic.

We first show that $\mathcal{T}$ is short-separation-free. Possibly $x_1x_2 \in E(G)$, but in that case, x_1x_2 is the lone edge of $C \cap G_{Q^m}^1$. By 1), G has no other chords of Q^m, so, since G is short-separation-free, it follows from Subclaim 4.2.4 that $G^\dagger$ is also short-separation-free. Thus, $\mathcal{T}^\dagger$ is a tessellation.

Since $Q^m \in \mathcal{K}(C, \mathcal{T})$ and every vertex of Q has distance at most two from $C \setminus \mathring{\mathbf{P}}$, it immediately follows that $\mathcal{T}^\dagger$ satisfies the distance conditions of Definition 2.1.6, and M0)-M2) are trivially satisfied. Thus, $\mathcal{T}^\dagger$ is a mosaic. Since $|S| > 1$, we have $|V(G^\dagger)| < |V(G)|$. By the minimality of $\mathcal{T}$, it follows that $G^\dagger$ admits an L-coloring, which restricts to an L-coloring of the subgraph of G induced by $V(G_{Q^m}^0)$, contradicting our assumption that $V(G_{Q^m}^0)$ is not L-colorable.

Now we prove the second part of Claim 4.2.3. Let $Q := x_1y_1wy_2x_2$ be a bad path and suppose toward a contradiction that $C \cap G_Q^1$ has length less than two. Thus, $C^1 \cap G_Q^1 = x_1x_2$ and G_Q^1 is bounded by the 5-cycle $x_1y_1wy_2x_2$. By 4), there is a proper L-coloring ψ of $V(G_Q^0)$. Since G is not L-colorable, ψ does not extend to an L-coloring of $V(G_Q^1)$, so it follows from Theorem 1.3.5 that $V(G_Q^1) \setminus V(Q)$ consists of a lone vertex adjacent to all five vertices of Q, so $V(G_Q^1 - w) \setminus B_1(C) = \varnothing$, contradicting the fact that Q is bad. Thus, we have $C \cap G_Q^1 \neq x_1x_2$. We also have $C \cap G_Q^0 \neq x_1x_2$, or else we contradict 2) of Corollary 2.3.14. Since C is an induced subgraph of G, it follows that $x_1x_2 \notin E(G)$, so Q is indeed an induced subgraph of G. ■

Now we return to the main proof of Lemma 4.2.1. Suppose toward a contradiction that there exists a bad path $Q \in \mathcal{K}^4(C, \mathcal{T})$, and let Q be chosen so as to minimize $|V(G_Q^1)|$ over all bad paths. Let $Q := x_1y_1wy_2x_2$, and let $S := V(G_Q^1 - w) \setminus B_1(C)$. By 3) of Theorem 2.2.4, each vertex of $G_Q^1 \setminus C$ has an L-list of size five. Now, $C \cap G_Q^1$ is a chordless path with endpoints x_1, x_2, and we denote this path by R^0. By 2) of Claim 4.2.3, $N(w) \cap V(R^0) = \varnothing$, so there is a subpath R^1 of C^1 with endpoints y_1, y_2 such that $V(R^1) = V(G_Q^1) \cap D_1(C)$.

Claim 4.2.5. *R^1 is an induced subgraph of G.*

Proof: We first note the following:

> **Subclaim 4.2.6.** *For each $z \in V(R^1) \setminus \{y_1, y_2\}$, z is not a $\mathcal{C}$-shortcut.*
>
> Proof: Suppose toward a contradiction that there is a $z \in V(R^1) \setminus \{y_1, y_2\}$ which is a $\mathcal{C}$-shortcut. Since Q separates each vertex of $V(R^1) \setminus \{y_1, y_2\}$ from each internal vertex of $\mathbf{P}$, there is a $C' \in \mathcal{C} \setminus \{C\}$ and a subgraph $H \subseteq C'$ such that $d(w, H) < d(C \setminus \mathring{\mathbf{P}}, H)$. Since $d(y_j, C') \leq d(C \setminus \mathring{\mathbf{P}}, C')$ for each $j = 1, 2$, we have $wz \in E(G)$, as the deletion of Q leaves C' and z in different connected components. Since $z \in V(R^1) \setminus \{y_1, y_2\}$, there is a $q \in V(R^0) \cap N(z)$.
>
> If q is an internal vertex of R^0, then at least one of the paths x_1y_1wzq, x_2y_2wzq separates S from $\mathbf{P}$ and is thus a bad path of $\mathcal{K}^4(C, \mathcal{T})$. This contradicts the minimality of Q. Thus, we have $q \in \{y_1, y_2\}$, so suppose without loss of generality that $q = y_1$. Let $Q^* := x_1zwy_2x_2$. Since G is short-separation-free, we have $V(G^1_{Q^*}) \cup \{y_1\} = V(G^1_Q)$ as a disjoint union, and Q^* separates S from $\mathbf{P}$, contradicting the minimality of Q, so our assumption that z is a $\mathcal{C}$-shortcut is false. ■

Now, if R^1 has a chord in G, then this chord has endpoints y_1, y_2, or else, since no vertex of $R^1 \setminus \{y_1, y_2\}$ is a $\mathcal{C}$-shortcut, we contradict Theorem 3.0.2. By Claim 4.2.3, $y_1y_2 \notin E(G)$, so R^1 is indeed an induced subgraph of G. This completes the proof of Claim 4.2.5. ■

Analogous to Section 3.1, we have the following facts.

Claim 4.2.7.

1) y_1 has no neighbors in $R^0 - x_1$. Likewise, y_2 has no neighbors in $R^0 - x_2$; AND

2) No internal vertex of R^1 is adjacent to w.

Proof: The two parts of 1) are symmetric so we just prove that y_1 has no neighbors in $R^0 - x_1$. Suppose toward a contradiction that y_1 has a neighbor $q \in V(R^0 - x_1)$. By Claim 4.2.3, $q \neq y_2$, so q is an internal vertex of R^0. By Theorem 3.0.2, we have $S \cap V(G^1_{x_1y_1q}) = \varnothing$, so the 4-chord $Q' := qy_1wy_2x_2$ of C separates S from $\mathbf{P}$. Since $Q' \in \mathcal{K}(C, \mathcal{T})$ and $|V(G^1_{Q'})| < |V(G^1_Q)|$, we contradict the minimality of Q. This proves 1).

Now we prove 2), Let y be an internal vertex of R^1. Thus, y has a neighbor $q \in V(R^0)$. If q is an internal vertex of R^0, then, letting $Q' := qywy_2x_2$ and $Q'' := qywy_1x_1$, each of Q', Q'' is an element of $\mathcal{K}(C, \mathcal{T})$, ando ne of Q', Q'' separates S from $\mathbf{P}$, contradicting the minimality of Q. Thus, we have $q \in \{x_1, x_2\}$. Suppose without loss of generality that $q = x_1$. Since G is short-separation-free, it follows that 4-chord $x_1ywy_2x_2$ separates S from both y_1 and $\mathbf{P}$, contradicting the minimality of Q. ■

We let $R^0 = q_0q_1 \cdots q_{t+1}$ for some $t \geq 0$, where $q_0 = x_1$ and $q_{t+1} = x_2$. Likewise, we let $R^1 = p_0 \cdots p_{s+1}$ for some integer $s \geq 0$, where $p_0 = y_1$ and $p_{s+1} = y_2$. Finally, we set $R^* := x_1y_1R^1y_2x_2$. Since $N(p_0) \cap V(R^0) = \{q_0\}$ by Fact 1 of Claim 4.2.7, and the two vertices p_0, p_1 have a unique common neighbor on R^0, we have $p_1q_0 \in E(G)$. Likewise, we have $p_sq_{t+1} \in E(G)$. Thus, we make the following definitions, which we retain for the remainder of the proof of Lemma 4.2.1. We set $R^\dagger := q_0p_1R^1p_sq_{t+1}$ and $C^\dagger := (C \cap G^1_Q) \cup R^\dagger$. Note that $R^\dagger \in \mathcal{K}(C, \mathcal{T})$. Now we have the following:

Claim 4.2.8. *Let $\{a, b\}$ be a set of two colors, and let L^* be a list-assignment for $G^0_{R^\dagger}$ where $L^*(p_i) = L(p_i) \setminus \{a, b\}$ for all $1 \leq i \leq s$, and otherwise $L^* = L$. Then $G^0_{R^\dagger}$ is L^*-colorable.*

Proof: Let $C_*^\dagger$ be the outer face of $G^0_{R^\dagger}$. We first show that the tuple $(G^0_{R^\dagger}, (\mathcal{C}\setminus\{C\})\cup\{C^\dagger\}), L^*, C_*^\dagger)$ is a mosaic. Since no internal vertex of $R^\dagger$ is adjacent to an an internal vertex of $\mathbf{P}$, it just suffices to check that conditions M3) and M4) of Definition 2.1.6 hold. If these conditions do not hold, then there exists an index $i\in\{1,\cdots,s\}$, a $C'\in\mathcal{C}\setminus\{C\}$ and a subgraph $H\subseteq C'$ be such that $d(H,p_i)<d(H,C\setminus\mathring{\mathbf{P}})$. Since Q separates $p_1\cdots,p_s$ from C', and each vertex of Q is of distance at least $d(H,C\setminus\mathring{\mathbf{P}})-2$ from H, there is a vertex of Q of distance $d(H,C\setminus\mathring{\mathbf{P}})-2$ from H which is adjacent to a vertex of p_i. But then p_i is adjacent to w, contradicting Fact 2. Thus, $(G^0_{R^\dagger}, (\mathcal{C}\setminus\{C\})\cup\{C^\dagger\}), L^*)$ is indeed a mosaic Furthermore, $|V(G^0_{R^\dagger})|<|V(G)|$, since $|V(R^0)|\geq 3$ and each internal vertex of R^0 lies outside of $G^0_{R^\dagger}$. Thus, by the minimality of $\mathcal{T}$, $G^0_{R^\dagger}$ is L^*-colorable. ■

Also analogous to Section 3.1 is the fact that S consists of a lone vertex.

Claim 4.2.9. *There exists a $v^*\in S$ adjacent to each of p_0,p_{s+1},w.*

Proof: Applying Claim 4.2.3, let ϕ be an L-coloring of $V(G^0_Q)$. By 1) of Proposition 1.5.1, there is a vertex $v^*\in V(G^1_Q\setminus C)$ adjacent to at least three vertices of Q. By Claim 4.2.7, the neighborhood if this vertex on Q is p_0,p_{s+1},w. Since v^* is adjacent to each of p_0,p_{s+1},w, we have $v^*\in B_2(C)\setminus V(C)$. If $v^*\in V(R^1)$, then R^1 has a chord in G, since $|V(R^1)|\geq 4$ and v^* is an internal vertex of R^1 adjacent to both of p_0,p_{s+1}, contradicting the fact that R^1 is a chordless path. Thus, $v^*\in(D_2(C)\cap V(G^1_Q-w))$, so $v^*\in S$, as desired. ■

We can now apply the work of Section 4.1. Let $G^*:=G^1_Q\setminus\{w,p_0,p_{s+1}\}$. We retain this notation for the remainder of the proof of Lemma 4.2.1.

Claim 4.2.10.

1) *$V(G^1_Q)=V(Q)\cup V(R^0)\cup V(R^1)\cup\{v^*\}$, and $N(v^*)=\{w\}\cup\{p_0,\cdots,p_{s+1}\}$; AND*
2) *G^*-v^* is a wheel sequence with apex path $p_1\cdots p_s$.*

Proof: Let $Q^*:=q_0p_0v^*p_{s+1}q_{t+1}$. Since $Q^*\subseteq G^1_Q$, we have $Q^*\in\mathcal{K}^4(C,\mathcal{T})$ and $G^1_{Q^*}\subseteq G^2_Q$. Since $w\notin V(G^1_{Q^*})$, we have $|V(G^1_{Q^*})|<|V(G^2_Q)|$. Since $v^*\in S$, we have $v^*\in D_2(C)$, and thus, by the minimality of Q, we have $V(G^1_{Q^*})\setminus B_1(C)=\{v^*\}$. Since G is short-separation-free, we have $V(G^0_{Q^*})=V(G^0_Q)\cup\{v^*\}$, and $V(G^1_Q)=V(G^1_{Q^*})\cup\{w\}$. Thus, we get $V(G^1_Q)\setminus B_1(C)=\{w,v^*\}$, so $V(G^1_Q)=V(Q)\cup V(R^0)\cup V(R^1)\cup\{v^*\}$.

Furthermore, since R^1 is a chordless path, and every facial subgraph of G containing v^* is a triangle, every vertex of R^1 is adjacent to v^*. Since $v^*\in V(G^1_Q\setminus Q)$, every neighbor of v^* in G lies in G^1_Q, so $N(v^*)=\{p_0,\cdots,p_{s+1}\}\cup\{w\}$. This proves Fact 1.

Now we show Fact 2. Since R^1 is a chordless path and $G^2_{R^1}=V(R^1)\cup V(R^0)$, it suffices to show that every vertex in $R^1\setminus\{p_0,p_{s+1}\}$ has at least two neighbors on $q_0\cdots q_{t+1}$. Suppose there exists an $i\in\{1,\cdots,s\}$ such that $|N(p_i)\cap V(C\cap G^1_Q)|=1$. Thus, for some $j\in\{0,\cdots,t+1\}$, q_j is the lone vertex of $N(p_i)\cap V(C)$. But then, since C is chordless, and every facial subgraph of q_j, except possibly C, is a triangle, q_j is adjacent to both p_{i-1} and p_{i+1}, so, by Fact 1 above, G has a $K_{2,3}$ with bipartition $\{v^*,q_j\},\{p_{i-1},p_i,p_{i+1}\}$, contradicting the fact that $\mathcal{T}$ is a tessellation. Thus, for each $i\in\{1,\cdots,s\}$, we have $|N(p_i)\cap V(C)|\geq 2$. Thus, G^* is a wheel sequence with apex path R^1, as desired. This proves Fact 2. ■

Now let $T:=\{q_0,q_{t+1}\}$. Then the following holds:

Claim 4.2.11. *For any L-coloring ϕ of G_Q^0, $(G^*, v^*, R^1 \setminus \{p_0, p_{s+1}\}, L_\phi^T)$ is a crown. Furthermore, we have $s = 2$, and $(G^*, v^*, R^1 \setminus \{p_0, p_{s+1}\}, L_\phi^T)$ satisfies either Statement 2b) or Statement 2c) of Theorem 4.1.3.*

Proof: By Claim 4.2.3, there is an L-coloring ϕ of $V(G_Q^0)$. We have $|L_\phi^T(v^*)| \geq 2$, since $|V(Q)\setminus T| = 3$. Furthermore, applying Facts 1 and 2 of Claim 4.2.7, together with the fact that R^1 is a chordless path, we have $N(p_i) \cap (V(Q)\setminus T) = \varnothing$ for each $1 < i < s$. Furthermore, we have $N(p_1) \cap V(Q)\setminus T) = \{p_0\}$, and likewise, $N(p_s) \cap (V(Q)\setminus T) = \{p_{s+1}\}$. Combining these, we obtain $|L_\phi^T(p_1)| \geq 4$ and $|L_\phi^T(p_s)| \geq 4$, and $|L_\phi^T(p_i)| \geq 5$ for each $1 < i < s$. Thus, since $G^* - v^*$ is a wheel sequence with apex path $p_1 \cdots p_s$, the tuple $(G^*, v^*, R^1 \setminus \{p_0, p_{s+1}\}, L_\phi^T)$ is indeed a crown, as $s \geq 2$.

Now, note that $s \leq 2$, or else it follows from Theorem 4.1.3 that G^* is L_ϕ^T-colorable. But then ϕ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. Thus, we indeed have $s \leq 2$, so $s = 2$. If the crown $(G^*, v^*, R^1 \setminus \{p_0, p_{s+1}\}, L_\phi^T)$ satisfies neither Statement 2b) nor Statement 2c) of Theorem 4.1.3, then G^* is L_ϕ^T-colorable so, again, we contradict the fact that $\mathcal{T}$ is critical. This proves Claim 4.2.11. ■

Since $G^* - v^*$ is a wheel sequence of length 2, we let $\mathcal{H}(G^*) = (H_1, H_2)$. Thus, there is an index $n \in \{1, \cdots, t\}$ such that H_1 has principal path $q_0 p_1 q_n$ and H_2 has principal path $q_n p_s q_{t+1}$. The following definition is useful for the remainder of the proof of Lemma 4.2.1.

Definition 4.2.12. A triple $[A_1, A_2, \psi]$ is called a *pointer* of G_Q^1 if the following hold.

1) ψ is an L-coloring of G_Q^1, and, for each $i \in \{1, 2\}$, A_i is a nonempty set of colors with $A_i \subseteq L_\psi(p_i)$ for each $i \in \{1, 2\}$*; AND*
2) $A_1 \cap A_2 = \varnothing$*; AND*
3) For some $i \in \{1, 2\}$, $|L_\psi(v^*) \setminus \{r\}| \geq 2$ for each $r \in A_i$*; AND*
4) There exists a pair $(r_1, r_2) \in A_1 \times A_2$ such that $\mathcal{Z}_{H_1}(\psi(q_0), r_1, \bullet) \cap \mathcal{Z}_{H_2}(\bullet, r_2, \psi(q_{t+1})) \neq \varnothing$.

If $[A_1, A_2, \psi]$ is a pointer of G_Q^1 and $A_1 = \{a\}$ for some $a \in L_\psi(p_1)$, then we write this as $[a, A_2, \psi]$. Likewise if A_2 is a singleton. We use the following fact repeatedly.

Claim 4.2.13. *There does not exist a pointer of G_Q^1.*

Proof: Suppose toward a contradiction that there exists a pointer $[A_1, A_2, \psi]$ of G_Q^1, and let $(r_1, r_2) \in A_1 \times A_2$ such that $\mathcal{Z}_{H_1}(\psi(q_0), r_1, q_n) \cap \mathcal{Z}_{H_2}(q_n, r_2, \psi(q_{t+1})) \neq \varnothing$. Suppose without loss of generality that $|L_\psi(v^*) \setminus \{r\}| \geq 2$ for each $r \in A_1$. Since $L_\psi(v^*) \setminus \{r_1\}| \geq 2$, there is an extension ψ' of ψ to an L-coloring of $G_{R^\dagger}^1$, in which $\psi'(p_1) = r_1$ and $\psi'(p_2) = r_2$. Since $\mathcal{Z}_{H_1,L}(\psi(q_0), r_1, q_n) \cap \mathcal{Z}_{H_2,L}(q_n, r_2, \psi(q_{t+1})) \neq \varnothing$, ψ' extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. ■

We now rule out the possibility that $n = 1 = t$.

Claim 4.2.14. *At least one of H_1, H_2 is not a triangle.*

Proof: Suppose toward a contradiction that $n = 1 = t$. In that case, we have $H_1 = q_0 p_1 q_1$ and $H_2 = q_1 p_2 q_2$. Let $L(q_1) = \{a, b, c\}$.

Subclaim 4.2.15. *For any L-coloring ϕ of G_Q^0, the following hold:*

1) $|\{\phi(q_0), \phi(q_2)\} \cap \{a, b, c\}| = 2$*; AND*

2) $\phi(p_3) = \phi(q_0)$ *and* $\phi(p_0) = \phi(q_2)$*; AND*

3) There is a pair of colors x, y *such that* $L(p_1) = L(p_2) = \{a, b, c, x, y\}$ *and* $L(v^*) = \{\phi(p_0), \phi(w), \phi(p_3), x, y\}$*; AND*

4) $\{\phi(q_0), \phi(q_2), \phi(w)\} = \{a, b, c\}$.

Proof: Let ϕ be an L-coloring of G^0_Q. Firstly, we have $|\{\phi(q_0), \phi(q_2)\} \cap \{a, b, c\}| = 2$, or else ϕ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. To see this, suppose toward a contradiction that $|\{\phi(q_0), \phi(q_2)\} \cap \{a, b, c\}| \leq 1$. Now, $G^0_{R^\dagger} \setminus G^0_Q$ is the triangle $p_1p_2v^*$, where $L_\phi(p_i)| \geq 3$ for each $i \in \{1, 2\}$ and $|L_\phi(v^*)| \geq 2$. Thus, ϕ extends to an L-coloring of $G^0_{R^\dagger}$.

If $\{\phi(q_0), \phi(q_2)\} \cap \{a, b, c\} = \varnothing$, then any extension of ϕ to an L-coloring of $G^0_{R^1}$ then extends to G, since there is a color left over for q_1. This contradicts the fact that $\mathcal{T}$ is critical. Thus, suppose without loss of generality that $\phi(q_0) = a$ and $\phi(q_2) \notin \{b, c\}$. Since $|L_\phi(p_1)| \geq 3$ and $\phi(p_0) = a$, there is a color $x \in L_\phi(p_1) \setminus \{a, b, c\}$, and there exists an extension ϕ' of ϕ in which $\phi'(p_1) = x$. But then there is at least one color left over for q_1, so ϕ' extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical.

This proves Fact 1. Thus, suppose without loss of generality that $\phi(q_0) = a$ and $\phi(q_2) = b$, and let ϕ' be an extension of ϕ to $G^0_Q \cup (q_0q_1q_2)$ in which $\phi'(q_1) = c$. Now, the triangle $v^*p_1p_2$ is not $L_{\phi'}$-colorable, or else ϕ' extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical.

Thus, we have $|L_{\phi'}(p_1)| = |L_{\phi'}(p_2)| = |L_{\phi'}(v^*)| = 2$, and $L_{\phi'}(p_1) = L_{\phi'}(p_2) = L_{\phi'}(v^*)$. It follows then that $\phi(p_0) = b$, $\phi(p_3) = a$, and there is a pair of colors x, y such that both of the following hold.

1. $L(p_1) = L(p_2) = \{a, b, c, x, y\}$; *AND*
2. $L(v^*) = \{a, b, \phi(w), x, y\}$.

The above argument shows that, for any L-coloring ϕ of G^0_Q, Facts 1), 2), and 3) of Subclaim 4.2.15 hold. To finish, it suffices to show, for our given coloring ϕ, that $\phi(w) = c$. Suppose towards a contradiction that $\phi(w) \neq c$. In that case, we have $c \notin L(v^*)$. Now let L^* be a list-assignment for $G^0_{R^\dagger}$ where $L^*(p_i) = L(p_i) \setminus \{a, c\}$, and $L^*(u) = L(u)$ for all $u \in V(G^0_{R^\dagger}) \setminus \{p_1, p_2\}$. By Claim 4.2.8, $G^0_{R^\dagger}$ admits an L^*-coloring ψ.

Now, $V(G^1_{R^\dagger}) \setminus V(R^\dagger) = \{q_1\}$, and thus ψ uses the color b on one of p_1, p_2, or else ψ uses a color of $\{a, b, c\}$ on at most two vertices of $N(q_1)$, in which case, ψ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. Let ψ' be the restriction of ψ to G^0_Q. Applying Facts 1 and 2 to ψ', we have $\{\psi'(q_0), \psi'(p_0)\} = \{\psi'(p_3), \psi'(q_2)\} = \{a, c\}$, and $\psi'(q_0) \neq \psi'(q_2)$.

Thus, we may suppose without loss of generality that $\psi'(q_0) = a$ and $\psi'(q_2) = c$. Thus, we have $\psi'(p_3) = a$ and $\psi'(p_0) = c$. Now let ψ'' be an extension of ψ' to $V(G^0_Q) \cup \{p_1, p_2\}$ obtained by coloring the edge p_1p_2 with (x, y). Then there is a color left over for q_1, since at most two neighbors of q_1 are colored with colors among $\{a, b, c\}$. Furthermore, since $\psi''(p_0) = c$, and $c \notin L(v^*)$, there is also a color left over for v^*, as v^* has degree 5. Thus, ψ'' extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. So we have $\phi(w) = c$, as desired. ■

Now we have enough to finish. Combining the facts above, there exists a pair of colors x, y such that $L(p_1) = L(p_2) = L(v^*) = \{a, b, c, x, y\}$, and, furthermore, for any L-coloring ϕ of G^0_Q, we have $\{\phi(q_0), \phi(p_0), \phi(w)\} = \{\phi(w), \phi(p_3), \phi(q_2)\} = \{a, b, c\}$.

Now, let L^* be a list-assignment for $G^0_{R^\dagger}$ where $L^*(p_i) = L(p_i)\setminus\{x, y\}$ for each $i \in \{1, 2\}$ and $L^*(u) = L(u)$ for each $u \in V(G^0_{R^\dagger}) \setminus \{p_1, p_2\}$. By Claim 4.2.8, there is an L^*-coloring ψ of $G^0_{R^\dagger}$. Since $L(p_1) \setminus \{x, y\} = L(p_2) \setminus \{x, y\} = \{a, b, c\}$, suppose without loss of generality that $\psi(p_1) = a$ and $\psi(p_2) = b$. Thus, we have $c \in \{\psi(q_0), \psi(q_2)\}$, or else c is left over for q_1, and then ϕ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. Thus, suppose without loss of generality that $\psi(q_0) = c$. Since the restriction of ψ to G^0_Q is an L-coloring of G^0_Q, we have $\psi(p_3) = \psi(q_0) = c$ and $\psi(q_2) \in \{a, b\}$ by Subclaim 4.2.15. Since $\psi(p_2) = b$, we have $\psi(q_2) = a$. By Fact 2, we have $\psi(p_0) = a$. Yet $\psi(p_1) = a$, so we contradict the fact that ψ is a proper L^*-coloring of $G^0_{R^\dagger}$. This completes the proof of Claim 4.2.14. ■

By Claim 4.2.14, we suppose without loss of generality that H_1 is not a triangle. Thus, $H_1 \setminus \{p_1\} = q_0 \cdots q_n$, where $n > 1$. Now fix an L-coloring ϕ of G^0_Q. By Claim 4.2.11, $(G^*, v^*, p_1p_2, L^T[\phi])$ is a crown which satisfies either Statement 2b) or Statement 2c) of Theorem 4.1.3. Thus we have the following two subcases.

Subcase 2.1: There is a set $\{a, b\}$ of two colors such that $\{a, b\} \subseteq L^T[\phi](q_j)$ for each $j = 1, \cdots, t$

In this case, let L^* be a list-assignment for $G^0_{R^\dagger}$, where $L^*(p_i) = L(p_i) \setminus \{a, b\}$ for each $i = 1, 2$, and $L^*(u) = L(u)$ for all $u \in V(G^0_{R^\dagger}) \setminus \{p_1, p_2\}$. By Claim 4.2.8, there is an L^*-coloring ψ of $G^1_{R^\dagger}$.

Note that $\{\psi(q_0), \psi(q_{t+1})\} \subseteq \{a, b\}$, or else we extend ψ to an L-coloring of G by coloring the path $q_1 \cdots q_t$ with the colors of $\{a, b\}$, contradicting the fact that $\mathcal{T}$ is critical. Now, let $\psi' = \psi|_{G^0_Q}$. We show that ψ' extends to an L-coloring of G. Since $\psi'(q_0) \in \{a, b\}$, we have $|L_{\psi'}(p_1) \setminus \{a, b\}| \geq 2$. Thus, there is a $d \in L_{\psi'}(p_1) \setminus \{a, b\}$ and a $j \in \{1, \cdots, n-1\}$ such that $d \notin L(q_j)$.

Note now that $d \in L(q_n)$. To see this, suppose toward a contradiction that $d \notin L(q_n)$. In that case, since $d \notin L(q_j)$, we have $\mathcal{Z}_{H_1}(\psi'(q_0), d, \bullet) = L(q_n)$. Since $|L_{\psi'}(p_2) \geq 3$ and $|L_{\psi'}(v^*)| \geq 2$, let $d^* \in L_{\psi'}(p_2)$ with $|L_{\psi'}(v^*) \setminus \{d^*\}$. But then $[d, d^*, \psi']$ is a pointer for G^1_Q, contradicting Claim 4.2.13. Thus, we indeed have $d \in L(q_n)$, so $L(q_n) = \{a, b, d\}$.

Now, since $\psi(q_{t+1}) \in \{a, b\}$, we have $|L_\phi(p_2) \setminus \{a, b, d\}| \geq 1$. Thus, for each $x \in L_{\psi'}(p_2) \setminus \{a, b, d\}$, we get $L_\phi(v^*) = \{d, x\}$, or else $[d, x, \phi]$ is a pointer for G^1_Q, contradicting Claim 4.2.13. Suppose without loss of generality that $\psi(q_{t+1}) = a$. In particular, we then have $L_\phi(p_2) = \{b, d, x\}$, since $L_{\psi'}(p_2) \setminus \{a, b, d\} = \{x\}$. We then have $\mathcal{Z}_{H_2}(q_n, b, \psi(q_{t+1})) = \{d\}$, or else $[d, b, \psi'(q_{t+1})]$ is a pointer for G^1_Q, contradicting Claim 4.2.13.

As above, we have $|L_{\psi'}(p_1)\setminus\{a, b, d\}| \geq 1$, since $\psi'(q_0) \in \{a, b\}$. Let $x' \in L_{\psi'}(p_1)\setminus\{a, b, d\}$. Then $|\mathcal{Z}_{H_1}(\psi(q_0), x', \bullet)| \geq 2$ since $x' \notin L(q_n)$. Thus $\mathcal{Z}_{H_1}(\psi(q_0), x', \bullet) = \{a, b\}$, or else $[x', b, \psi']$ is a pointer for G^1_Q, contradicting Claim 4.2.13. Since H_1 is not a triangle, we have $x' \in \bigcap_{i=1}^{n-1} L(q_i)$, or else $\mathcal{Z}_{H_1}(\psi(q_0), x', \bullet) = L(q_n)$. Thus, we get $L(q_{n-1}) = \{a, b, x'\}$. But then $d \in \mathcal{Z}_{H_1}(\psi(q_0), x', \bullet)$, contradicting the fact that $\mathcal{Z}_{H_1}(\psi(q_0), x', \bullet) = \{a, b\}$.

This completes Subcase 2.1, so we have ruled out the possibility that Statement 2b) of Theorem 4.1.3 holds when applied to the crown $(G^*.v^*, p_1p_2, L^T_\phi)$. The only case left to consider is the possibility that Statement 2c) of Theorem 4.1.3 holds:

Subcase 2.2: There is a set $S = \{a, b, r\}$ of three colors such that $L^T_\phi(v^*) = \{a, b\}$, $L(p_1) \setminus \{\phi(q_0)\} = L(p_2) \setminus \{\phi(q_{t+1})\} = S$, and $|L(q_n) \cap S| \geq 2$. Furthermore, we have $\{b, r\} \subseteq \bigcap_{i=1}^{n-1} L(q_i)$ and $\{a, r\} \subseteq \bigcap_{i=n+1}^{t} L(q_i)$.

In this case, let S' be a set of two colors in $L(q_n) \cap S$, and let L^* be a list-assignment for $G^0_{R^1}$ where $L^*(p_1) = L(p_1) \setminus S'$, $L^*(p_2) = L(p_2) \setminus S'$, and $L^*(u) = L(u)$ for all $u \in V(G^1_{R^1}) \setminus \{p_1, p_2\}$. Applying Claim 4.2.8, let ψ be an L^*-coloring of $G^0_{R_1}$ and let $\psi' = \psi|_{G^1_Q}$. Furthermore, since $|L(q_n)| = 3$, let d' be a color such that $L(q_n) = S' \cup \{d'\}$.

By Claim 4.2.11, $(G^*, v^*, p_1p_2, L^T_{\psi'})$ is a crown which satisfies either Statement 2b) or Statement 2c) of Theorem 4.1.3. If $(G^*, v^*, p_1p_2, L^T_{\psi'})$ satisfies Statement 2b) of Theorem 4.1.3, then we are back to Subcase 2.1 with the roles of ψ and ϕ interchanged, so we are done in that case. So suppose toward a contradiction that $(G^*, v^*, p_1p_2, L^T_{\psi'})$ satisfies Statement 2c) of Theorem 4.1.3. In that case, we have $L_{\psi'}(p_1) = L_{\psi'}(p_2)$, and $|L_{\psi'}(p_1)| = |L_{\psi'}(p_1)| = 3$. Since $\psi(p_1) \in L_{\psi'}(p_1)$ and $\psi(p_2) \in L_{\psi'}(p_2)$, there is a color d such that $L_{\psi'}(p_1) = L_{\psi'}(p_2) = \{\psi(p_1), \psi(p_2), d\}$.

Now, there is an $i \in \{1, 2\}$ such that $\psi(p_i) \notin L(q_n)$. If no such i exists, then we have $\{\psi(p_1), \psi(p_2)\} \subseteq L(q_n)$ and $S' \subseteq L(q_n)$. Yet $\{\psi(p_1), \psi(p_2)\}$ and S' are disjoint and $|L(q_n)| = 3$. So suppose without loss of generality that $\psi(p_2) \notin L(q_n)$. Now we gather the following facts:

Claim 4.2.16.

1) $d' = \psi(p_1)$*; AND*

2) $d \in S'$*; AND*

3) $\psi(p_1) \in L(q_j)$ *for each* $j = 1, \cdots, n$*; AND*

4) *For some* $j \in \{1, \cdots, t\} \setminus \{n\}$*, we have* $d \notin L(q_j)$*.*

Proof: Firstly, we have $d' \in \{\psi(p_1), \psi(p_2)\}$, or else $|L(q_n) \cap L_{\psi'}(p_i)| \leq 1$ for each $i \in \{1, 2\}$, contradicting the fact that $(G^*, v^*, p_1p_2, L^T_{\psi'})$ satisfies Statement 2c) of Theorem 4.1.3. Likewise, we have $d \in S'$, or else, again, $|L(q_n) \cap L_{\psi'}(p_i)| \leq 1$. Since $\psi(p_2) \notin L(q_n)$ and $d' \in L(q_n)$, we have $d' = \psi(p_1)$. This proves Facts 1 and 2.

Now, suppose toward a contradiction that there exists a $j \in \{1, \cdots, n\} \setminus \{t\}$ such that $\psi(p_1) \notin L(q_j)$. Now, if $j \in \{1, \cdots, n-1\}$, then $\mathcal{Z}_{H_1}(\psi(q_0), \psi(p_1), \bullet)| \geq 2$, since $\psi(p_1) \notin L(q_j)$, and we have $\mathcal{Z}_{H_2}(\bullet, \psi(p_2), \psi(q_{t+1}))| \geq 2$ as well, since $\psi(q_2) \notin L(q_n)$. But then $[\psi(p_1), \psi(p_2), \psi']$ is a pointer for G^1_Q, contradicting Claim 4.2.13. So now suppose instead that $j \in \{n+1, \cdots, t\}$. But then, since $L_{\psi'}(p_1) = L_{\psi'}(p_2)$, we simply interchange the colors on p_1 and p_2, and we obtain the pointer $[\psi(p_2), \psi(p_1), \psi']$, again contradicting Claim 4.2.13. Thus, our assumption that there exists a $j \in \{1, \cdots, t\} \setminus \{n\}$ such that $\psi(p_1) \notin L(q_j)$ is false. This proves Fact 3.

Now, since $d \neq d'$ and $\{d, \psi(p_1)\} \subseteq L(q_n)$, there exists a $j \in \{1, \cdots, t\} \setminus \{n\}$ such that $\{d, \psi(p_1)\} \not\subseteq L(q_j)$. If no such j exists, then $\{d, \psi(p_1)\} \subseteq L(q_j)$ for each $j = 1, \cdots, t$, and then we are back to Case 2.1 with the roles of ϕ and ψ interchanged, and this case has already been ruled out. Thus, there is a $j \in \{1, \cdots, t\} \setminus \{n\}$ with $\{d, \psi(p_1)\} \not\subseteq L(q_j)$. Since $\psi(p_1) \in L(q_j)$ by Fact 3, we have $d \notin L(q_j)$. This proves Fact 4, and completes the proof of Claim 4.2.16. ■

Applying Facts 1 and 2, there is a color $\ell \neq \psi(p_2)$ such that $L(q_n) = \{\psi(p_1), d, \ell\}$. By Fact 4, there is a $j \in \{1, \cdots, t\} \setminus \{n\}$ with $d \notin L(q_j)$. We may suppose without loss of generality that $j \in \{1, \cdots, n-1\}$, since, for any extension of ψ' to an L-coloring of $G^0_{R^\dagger}$, we may interchange the colors in p_1, p_2 to produce a new extension of ψ' to a proper L-coloring of $G^0_{R^1}$, as $L_{\psi'}(p_1) = L_{\psi'}(p_2)$. Now let $\ell \in S'$ be the lone color of $L(q_n) \setminus \{\psi(p_1), d\}$, where $\ell \notin L_{\psi'}(p_i)$ for each $i \in \{1, 2\}$. Since $(G^*, v^*, p_1p_2, L^T_{\psi'})$ satisfies Statement 2c) of Theorem 4.1.3, there is a pair of colors in $\{\psi(p_1), \psi(p_2), d\}$ lying in $\bigcap_{i=1}^{n-1} L(q_i)$. Since $d \notin L(q_j)$, we have $\{\psi(p_1), \psi(p_2)\} \subseteq L(q_i)$ for each $i = 1, \cdots, n-1$.

Claim 4.2.17.

1) $L_{\psi'}(v^*) = \{\psi(p_2), d\}$*; AND*

2) $\mathcal{Z}_{H_1}(\psi(q_0), \psi(p_2), \bullet) = \mathcal{Z}_{H_1}(\psi(q_0), d, \bullet) = S'$*; AND*

3) $\{\psi(p_1), \psi(p_2), d\} = L(q_{n-1})$*; AND*

4) H_2 is not a triangle; AND

5) $\mathcal{Z}_{H_1}(\psi(q_0), \psi(p_1), \bullet) = \{d\}$.

Proof: Since $d \notin L(q_j)$, we have $|\mathcal{Z}_{H_1}(\psi(q_0), d, \bullet)| \geq 2$. Likewise, since $\psi(p_2) \notin L(q_n)$, we have $|\mathcal{Z}_{H_1}(\psi(q_0), \psi(p_2), \bullet)| \geq 2$ and $|\mathcal{Z}_{H_2}(\bullet, \psi(p_2), \psi(q_{t+1}))| \geq 2$.

Since the crown $(G^*, v^*, p_1p_2, L^T_{\psi'})$ satisfies Statement 2c) of Theorem 4.1.3, we have $|L_{\psi'}(v^*)| = 2$ and $L_{\psi'}(v^*) \subseteq \{\psi(p_1), \psi(p_2), d\}$. Note that $L_{\psi'}(v^*) \neq \{\psi(p_1), \psi(p_2)\}$, since ψ is a proper L-coloring of $G^1_{R^1}$. Thus, we have $d \in L_{\psi'}(v^*)$. Now, suppose toward a contradiction that $\psi(p_2) \notin L_{\psi'}(v^*)$. Now, since $|\mathcal{Z}_{H_1}(\psi(p_0), d, \bullet)| \geq 2$ and $|\mathcal{Z}_{H_2}(\bullet, \psi(p_2), \psi(q_{t+1}))| \geq 2$, we get that $[d, \psi(p_2), \psi']$ is a pointer for G^1_Q, contradicting Claim 4.2.13. This proves Fact 1.

Now, we have $\mathcal{Z}_{H_1}(\psi(q_0), d, \bullet) = L(q_n) \setminus \{d\} = S'$. Suppose toward a contradiction that $d \in \mathcal{Z}_{H_1}(\psi(q_0), \psi(p_2), \bullet)$. Then $\mathcal{Z}_{H_1}(\psi(q_0), d, \bullet) \cup \mathcal{Z}_{H_1}(\psi(q_0), \psi(p_2), \bullet) = L(q_n)$. But then $[\{d, \psi(p_2)\}, \psi(p_1), \psi']$ is a pointer for G^1_Q, contradicting Claim 4.2.13. Thus, $d \notin \mathcal{Z}_{H_1}(\psi(q_0), \psi(p_2), \bullet)$ so we have $\mathcal{Z}_{H_1}(\psi(q_0), \psi(p_2), \bullet) = L(q_n) \setminus \{d\} = S'$. This proves Fact 2.

Now we prove Fact 3. Since $\{\psi(p_1), \psi(p_2)\} \subseteq L(q_{n-1})$, it just remains to show that $d \in L(q_{n-1})$. Suppose toward a contradiction that $d \notin L(q_{n-1})$. In that case, the L-coloring $(\psi(q_0), \psi(p_2), d)$ of $q_0p_1q_n$ extends to an L-coloring of H_1, and so $d \in \mathcal{Z}_{H_1}(\psi(q_0), \psi(p_2), \bullet)$ contradicting Fact 2. Thus, we have $L(q_{n-1}) = \{\psi(p_1), \psi(p_2), d\}$, as desired. This completes the proof of Fact 3.

Now we prove Fact 4. Suppose toward a contradiction that H_2 is a triangle. Then we have $n = t$. Now we again consider the coloring ϕ of G^1_Q with $L_\phi(p_1) = L_\phi(p_2) = \{a, b, r\}$ and $L_\phi(v^*) = \{a, b\}$. Recall that S' is a set of two colors in $\{a, b, r\}$. Furthermore, note that $\{b, r\} \not\subseteq L(q_n)$, or else, since $\{b, r\} \subseteq \bigcap_{i=1}^{n-1} L(q_i)$ by assumption, we are back to Case 2.1, which we have already ruled out. Thus, $S' \neq \{b, r\}$, so we have either $S' = \{a, b\}$ or $S' = \{a, r\}$. Thus, consider the following cases:

Case 1: $S' = \{a, b\}$

In this case, since $\{\psi(p_1), \psi(p_2)\} \subseteq L(q_{n-1})$, at least one of a, b does not lie in $L(q_{n-1})$. Thus, we have $|\mathcal{Z}_{H_1}(\phi(q_0), a, \bullet) \cup \mathcal{Z}_{H_1}(\phi(q_0), b, \bullet)| \geq 2$. Furthermore, since $\{b, r\} \not\subseteq L(q_n)$, we have $r \notin L(q_n)$ and so $|\mathcal{Z}_{H_2}(\bullet, r, \phi(q_{n+1}))| \geq 2$. Thus, since $L_\phi(v^*) = \{a, b\}$, we get that $[\{a, b\}, r, \phi]$ is a pointer for G^1_Q, contradicting Observation **??**, so we have ruled out the possibility that $S' = \{a, b\}$.

Case 2: $S' = \{a, r\}$.

In this case, since $\{b, r\} \not\subseteq L(q_n)$, we have $b \notin L(q_n)$ and thus $|\mathcal{Z}_{H_1}(\phi(q_0), b, \bullet)| \geq 2$ and $|\mathcal{Z}_{H_2}(\bullet, b, \phi(q_{n+1}))| \geq 2$. Furthermore, since H_2 is a triangle and $\phi(q_{n+1}) \notin \{a, b, r\}$, we have $\{a, r\} \subseteq \mathcal{Z}_{H_2}(\bullet, b, \phi(q_{n+1}))$. Thus, we get $\mathcal{Z}_{H_1}(r) = \{\psi(p_1)\}$, or else $[r, b, \phi]$ is a pointer for G^1_Q, contradicting Claim 4.2.13. Since $\mathcal{Z}_{H_1}(\phi(q_0), r, \bullet) = \{\psi(p_1)\}$, we have $r \in L(q_{n-1})$ and thus $L(q_{n-1}) = \{\psi(p_1), \psi(p_2), r\}$ by Fact 3. Thus, we have $d = r$, since $\{\psi(p_1), \psi(p_2)\} \cap S' = \varnothing$. But then, by Fact 4 of Claim 4.2.16, we have $r \notin L(q_j)$, and thus $\mathcal{Z}_{H_1}(\phi(q_0), r, \bullet)| \geq 2$, contradicting the fact that $\mathcal{Z}_{H_1}(\phi(q_0), r, \bullet) = \{\psi(p_1)\}$. So we have ruled out the possibility that $S' = \{a, r\}$, completing the proof of Fact 4.

Now we prove Fact 5. Firstly, we have $|\mathcal{Z}_{H_1}(\psi(q_0), \psi(p_1), \bullet)| = 1$, or else $\mathcal{Z}_{H_1}(\psi(q_0), \psi(p_1), \bullet) \cap \mathcal{Z}_{H_2}(\bullet, \psi(p_2), q_0) \neq \varnothing$, and then $[\psi(p_1), \psi(p_2), \psi']$ is a pointer for G^1_Q, contradiction Claim 4.2.13. Now suppose toward a contradiction that $\mathcal{Z}_{H_1}(\psi(q_0), \psi(p_1), \bullet) \neq \{d\}$. Thus, since $|\mathcal{Z}_{H_1}(\psi(q_0), \psi(p_1), \bullet)| = 1$, we have $\mathcal{Z}_{H_1}(\psi(q_0), \psi(p_1), \bullet) =$

$\{\ell\}$. Furthermore, we have $\mathcal{Z}_{H_2}(\bullet, \psi(p_2), \psi(q_{t+1}) = \{\psi(p_1), d\}$, or else, again, we have $\mathcal{Z}_{H_1}(\psi(q_0), \psi(p_1), \bullet) \cap \mathcal{Z}_{H_2}(\bullet, \psi(p_2), \psi(q_{t+1})) \neq \varnothing$, and thus $[\psi(p_1), \psi(p_2), \psi']$ is a pointer for G_Q^1, contradicting Claim 4.2.13.

Since $\psi(p_1) \notin L_\psi(v^*)$, we have $\ell \notin \mathcal{Z}_{H_2}(\bullet, \psi(p_2), \psi(q_{t+1})) \cup \mathcal{Z}_{H_2}(\bullet, d, \psi(q_{t+1}))$, or else $[\psi(p_1), \{\psi(p_2), d\}, \psi']$ is a pointer for G_Q^1,contradicting Claim 4.2.13. Thus, we have $\mathcal{Z}_{H_2}(\bullet, \psi(p_2), \psi(q_{t+1})) = \{d, \psi_1(p_1)\}$ and $\mathcal{Z}_{H_2}(\bullet, d, \psi(q_{t+1})) = \{\psi(p_1)\}$. Now, since H_2 is not a triangle, q_{n+1} is an internal vertex of $H_2 \setminus \{p_2\}$, and since $|\mathcal{Z}_{H_2}(\bullet, d, \psi(q_{t+1}))| = 1$, we have $d \in L(q_{n+1})$. Furthermore, we have $\ell \in L(q_{n+1})$, or else $\ell \in \mathcal{Z}_{H_2}(\bullet, \psi(p_2), \psi(q_{t+1}))$, contradicting the fact that $\mathcal{Z}_{H_2}(\bullet, \psi(p_2), \psi(q_{t+1})) = \{\psi(p_1), d\}$. Thus, $L(q_{n+1}) = \{\ell, d, \psi(p_1)\}$ and $\mathcal{Z}_{H_2}(\bullet, \psi(p_2), \psi(q_{t+1})) = L(q_n)$, again contradicting the fact that $\ell \notin \mathcal{Z}_{H_2}(\bullet, \psi(p_2), \psi(q_{t+1}))$. This proves Fact 5. ■

By Fact 5 of Claim 4.2.17, we have $\ell \notin \mathcal{Z}_{H_1}(\psi(q_0), \psi(p_1), \bullet)$. Since $L(q_{n-1}) = \{\psi(p_1), \psi(p_2), d\}$ by Fact 3 of Claim 4.2.17, we have $\ell \notin L(q_{n-1})$. But then, the L-coloring $(\psi(q_0), \psi(p_1), \ell)$ of $q_0p_1q_n$ extends to an L-coloring of H_1, contradicting the fact that $\ell \notin \mathcal{Z}_{H_1}(\psi(q_0), \psi(p_1), \bullet)$. This rules out Subcase 2.2. Thus, having ruled out each subcase of Case 2, we conclude that our assumption that $(G^*, v^*, p_1p_2, L^T_{\psi'})$ satisfies Statement 2c) of Theorem 4.1.3 is false. Thus, our original assumption that there exists a bad path in $\mathcal{K}^4(C, \mathcal{T})$ is false. This completes the proof of Lemma 4.2.1. □

4.3 3-Chords Incident to an Internal Vertex of the Precolored Path

In this section and the next one we complete the proof of Theorem 4 by analyzing 3-chords of an open ring of a critical mosaic in which precisely one endpoint is an internal vertex of the precolored path. In Chapter 3 we showed that, given an open ring C in a critical mosaic, with precolored path $\mathbf{P}$, there is no 3-chord of C with both endpoints in $C \setminus \mathring{\mathbf{P}}$. We now deal with the case where one endpoint of the 3-chord lies in $\mathring{\mathbf{P}}$. We begin with the following simple observation:

Observation 4.3.1. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let $C \in \mathcal{C}$ be an open $\mathcal{T}$-ring. Let $\mathbf{P} := p_1 \cdots p_m$, and let $Q := x_1x_2x_3x_4$ be a 3-chord of C with precisely one endpoint in $V(\mathring{\mathbf{P}})$. Then the following hold.*

1) *$x_1 \in \{p_2, p_3, p_{m-1}, p_m\}$; AND*
2) *If x_1, x_4 have no common neighbor in G_Q^{small}, and Q is an induced subgraph of G_Q^{small}, then, for any $v \in V(G_Q^{\text{small}}) \setminus Q)$, $G[N(v) \cap V(Q)]$ is a subpath of Q of length at most one. An analogous statement holds for G_Q^{small}.*

Proof. 1) is just an immediate consequence of 4) of Theorem 2.2.4. Now let v^* be a vertex of $V(G_Q^{\text{small}} \setminus Q)$ with three neighbors in Q. If $G[N(v^*) \cap V(Q)]$ is not a subpath of Q, then, without loss of generality, we have $N(v^*) \cap V(Q) = \{x_1, x_2, x_4\}$. But since G is short-separation-free and Q is an induced subgraph of G_Q^{small}, we have $x_3 \in N(v^*)$ by our triangulation conditions, contradicting our assumption. Thus, $G[N(v^*) \cap V(Q)]$ is a subpath of Q. On the other hand, if v^* has less than three neighbors on Q and $G[N(v^*) \cap V(Q)]$ is not a subpath of Q, then, without loss of generaity, we let $N(v^*) \cap V(Q) = \{x_1, x_3\}$, since $N(x_1) \cap N(x_4) \cap V(G_Q^{\text{small}}) = \varnothing$. Since Q is an induced subgraph of G_Q^{small} and G is short-separation-free, we have $x_2 \in N(v^*)$ as well by our triangulation conditions, contradicting our assumption. An identical argument holds for G_Q^{large}. □

The remaindmer of Section 4.3 consists of two facts, which are Proposition 4.3.2 and Proposition 4.3.4.

Proposition 4.3.2. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let $C \in \mathcal{C}$ be an open $\mathcal{T}$-ring, and let $\mathbf{P} := \mathbf{P}_{\mathcal{T}}(C)$. Let $\mathbf{P} := p_1 \cdots p_m$, and let $Q := x_1x_2x_3x_4$ be a 3-chord of C with $x_1 \in V(\mathring{\mathbf{P}})$ and $x_4 \in V(C \setminus \mathring{\mathbf{P}})$ (possibly x_4 is an endpoint of P). If the following conditions hold, then $V(G_Q^{\text{large}} \cup P)$ is L-colorable.*

1) *$|V(G_Q^{\text{small}}) \setminus V(C \cup Q)| > 4$; AND*
2) *x_1, x_2, x_3 do not have a common neighbor in $G_Q^{\text{large}} \setminus Q$; AND*
3) *Q is an induced subpath of G; AND*
4) *$N(x_3) \cap V(\mathbf{P} - x_4) = \varnothing$ and neither endpoint of P lies in $N(x_2)$.*

Proof. Applying Observation 4.3.1, without loss of generality, let $x_1 \in \{p_2, p_3\}$. We construct a tessellation with fewer vertices than G which contains G_Q^{large} as a subgraph. We have to be somewhat careful to avoid violating our distance conditions and to avoid creating a vertex with three or more neighbors on the precolored path. The main technical obstacle to creating a smaller tessellation is the possibility that the edge p_1x_4 is present in G. Consider the following cases:

Case 1: $p_2 \in N(x_2)$

In this case, we have $x_1 \in \{p_2, p_3\}$. Let $Q^* := p_2x_2x_3x_4$. Since $x_1 \in \{p_2, p_3\}$ we have $G_{Q^*}^{\text{small}} \subseteq G_Q^{\text{small}}$ (possibly $Q^* = Q$). Furthermore, since $G_Q^{\text{small}} \setminus G_{Q^*}^{\text{small}} \subseteq \{p_2\}$, we have $|V(G_{Q^*}^{\text{small}}) \setminus V(C \cup Q^*)| > 4$, so let $s \in V(G_{Q^*}^{\text{small}}) \setminus V(C \cup Q^*)$. Note that $x_4 \neq p_1$, or else the 4-cycle $p_1p_2x_2x_3$ separates s from G_Q^{large}. Since Q is an induced subgraph of G and C is an induced cycle of G, Q^* is also an induced subgraph of G. W now break Case 1 into two subcases.

Subcase 1.1 x_2, x_3, x_4 do not have a common neighbor in $G_{Q^*}^{\text{large}} \setminus Q^*$

In this case, by Observation 4.3.1, for each $v \in V(G_{Q^*}^{\text{large}} \setminus Q^*)$, $G[N(v) \cap V(Q*)]$ is a subpath of Q^* of length at most one. Let $G^\dagger$ be a graph obtained from $G_{Q^*}^{\text{large}} + p_2p_1$ by first adding to $G_{Q^*}^{\text{large}} + p_2p_1$ the edge p_1x_4, and then adding a lone vertex v^* adjacent to each vertex in the 5-cycle $p_2x_2x_3x_4p_1$. Let $C^\dagger := (C \cap G_Q^{\text{large}}) + p_2p_1x_4$ and let $C_*^\dagger$ be the outer face of $G^\dagger$. Let $L^\dagger$ be a list-assignment for $G^\dagger$ where $L^\dagger(v^*)$ is an arbitrary 5-list and otherwise $L^\dagger = L$.

Since Q^* is an induced subgraph of G, and $G[N(v) \cap V(Q*)]$ is a subpath of Q^* of length at most one for each $v \in V(G_{Q^*}^{\text{large}})$, it follows that $G^\dagger$ is short-separation-free. Thus, Thus, $\mathcal{T}^\dagger := (G^\dagger, (\mathcal{C} \setminus \{C\}) \cup \{C^\dagger\}, L^\dagger, C_*^\dagger)$ is a tessellation, where $C^\dagger$ is an open ring which also has precolored path $\mathbf{P}$, so $\mathcal{T}^\dagger$ clearly satisfies M0) of Definition 2.1.6. Since $C^\dagger$ is an induced cycle of $G^\dagger$, and $N(v^*) \cap V(\mathbf{P}) = \{p_1, p_2\}$, $\mathcal{T}^\dagger$ also satisfies M1), and M2) is immediate.

If $\mathcal{T}^\dagger$ does not satisfy the distance conditions of Definition 2.1.6 then there exists a $C' \in \mathcal{C} \setminus \{C\}$ and an $H \subseteq C'$ such that $d_{G^\dagger}(H, v^*) < d_G(H, C \setminus \mathring{\mathbf{P}})$. Let R be a shortest (H, v^*)-path in $G^\dagger$. Then there is an $(H, \{p_2, x_4\})$-path in G of length $|E(R)| - 1$, so there is a is an $(H, C \setminus \mathring{\mathbf{P}})$-path of length at most $|E(R)|$, contradicting the distance conditions on $\mathcal{T}$. Thus, $(G^\dagger(\mathcal{C} \setminus \{C\}) \cup \{C^\dagger\}, L^\dagger, C_*^\dagger)$ satisfies the distance conditions of Definition 2.1.6 and is thus a mosaic. As indicated above, we have $|V(G_{Q^*}^{\text{small}}) \setminus V(C \cup Q^*)| > 1$, so we get $|V(G^\dagger)| < |V(G)|$. Thus, $G^\dagger$ is L-colorable by the minimality of $\mathcal{T}$, so let ϕ be an $L^\dagger$-coloring of $G^\dagger$ and let ϕ' be the restriction of ϕ to $G_{Q^*}^{\text{large}}$. If $p_1x_4 \notin E(G)$, then, since C is an induced subgraph of G, it follows that ϕ' extends to a proper L-coloring of $V(G_{Q^*}^{\text{large}}) \cup \{p_1\}$, since $N(x_3) \cap V(\mathbf{P} \setminus \{x_4\}) = \varnothing$. Likewise, if $p_1x_4 \in E(G)$, then, by construction of $G^\dagger$, we have $\phi(p_1) \neq \phi(x_4)$, and thus, again, ϕ' extends to a proper L-coloring of $V(G_{Q^*}^{\text{large}}) \cup \{p_1\}$. Thus, ϕ' extends to a proper L-coloring of $V(G_Q^{\text{large}}) \cup \{p_1, p_2\}$, so we are done in this case.

Subcase 1.2 x_2, x_3, x_4 have a common neighbor in $G_{Q^*}^{\text{large}} \setminus Q^*$

In this case, we let w be the common neighbor of x_2, x_3, x_4 in $G_{Q^*}^{\text{large}} \setminus Q^*$.

Claim 4.3.3. $N(w) \cap V(\mathbf{P}) = \varnothing$

Proof: Suppose there is a $p \in N(w) \cap V(\mathbf{P})$. By our conditions on Q, we have $x_1 \notin N(w)$, and Q separates w from $V(p_1\mathbf{P}x_1) \setminus \{x_1\}$, so $p \in (x_1\mathbf{P}p_m) \setminus \{x_1\}$. By Corollary 2.3.14, we have $|E(\mathbf{P})| = \lfloor \frac{2N_{\text{mo}}}{3} \rfloor$ and $p_m, p_{m-1} \notin N(w)$. Thus, by 4) of Theorem 2.2.4, we get $N(w) \cap V(\mathbf{P}) \subseteq \{p_1, p_2\}$, contradicting the fact that $p \in V(x_1\mathbf{P}p_m) \setminus \{x_1\}$. ■

Now we break Subcase 1.2 into two subcases.

Subcase 1.2.1 $p_3 \in N(x_2)$

In this case, let $Q^\dagger := p_3x_2wx_4$. Since $p_3 \notin N(w)$, $Q^\dagger$ is induced in G. Furthermore, $|V(G^{\text{small}}_{Q^\dagger}) \setminus V(C \cup Q^\dagger)| > |V(G^{\text{small}}_Q) \setminus V(C \cup Q)| > 4$, and $N(w) \cap V(\mathbf{P}) = \varnothing$.

Thus, if the three vertices p_3, x_2, w do not have a common neighbor in $G^{\text{large}}_{Q^\dagger}$, then, $Q^\dagger$ satisfies all the conditions of Proposition 4.3.2. In that case, we apply Subcase 1.1 with the role of Q replaced by $Q^\dagger$, since x_2, w, x_4 do not have a common neighbor in $G^{\text{large}}_{Q^\dagger} \setminus Q^\dagger$, or else G contains a copy of $K_{2,3}$. Thus, $V(G^{\text{large}}_{Q^\dagger}) \cup \{p_1, p_2\}$ admits an L-coloring ϕ, and ϕ extends to L-color x_3 as well, since x_3 only has three neighbors in dom(ϕ). Thus, in that case, $V(G^{\text{large}}_Q) \cup \{p_1, p_2\}$ is L-colorable, as desired.

Now suppose that the three vertices p_2, x_2, w have a common neighbor w' in $V(G^{\text{large}}_{Q^\dagger} \setminus Q^\dagger)$. Then G contains the 3-chord $Q^{\dagger\dagger} := p_3w'wx_4$ of C. Now we let $G^\dagger := G^{\text{large}}_{p_3x_2x_3x_4}$ and we let $L^\dagger$ be a list-assignment for $G^\dagger$ where $L^\dagger(x_2) = L(p_2)$ and $L^\dagger(x_3) = L(p_1)$. That is, since $p_4 \neq x_1$, our new precolored path is $\mathbf{P}'' := p_m \cdots p_3x_2x_3$. Let $C^\dagger := (G^\dagger \cap C) + p_3x_2x_3x_4$ and let $C^\dagger_*$ be the outer face of $G^\dagger$. Note that $\mathbf{P}''$ is an induced subgraph of $C^\dagger$, so $\mathbf{P}''$ is $L^\dagger$-colorable, since $\mathbf{P}$ is L-colorable. Thus, $\mathcal{T}^\dagger := (G^\dagger, (\mathcal{C} \setminus \{C\}) \cup \{C^\dagger\}, L^\dagger, C^\dagger_*)$ is a tessellation.

We claim now that $\mathcal{T}^\dagger$ is a mosaic. Firstly, if $\mathcal{T}^\dagger$ does not satisfy the distance conditions of Definition 2.1.6, then there is a $C' \in \mathcal{C} \setminus \{C\}$ and a subgraph H of C' such that $d(H, x_3) < d(H, C \setminus \mathring{\mathbf{P}})$. Since the 3-chord $p_3w'wx_4$ of C separates H from x_3, the vertex x_2 is an endpoint of $R \setminus \{x_3\}$, since the only other possible endpoint is w, which is adjacent to $x_4 \in V(C \setminus \mathring{\mathbf{P}})$. Thus, the endpoint of $R \setminus \{x_2, x_3\}$ adjacent to x_2 is among p_3, w', w, x_4, but each of these vertices of distance at most two from $C \setminus \mathring{\mathbf{P}}$, contradicting the fact that $d(H, x_3) < d(H, C \setminus \mathring{\mathbf{P}})$. Thus, $\mathcal{T}^\dagger$ does indeed satisfy the distance conditions of Definition 2.1.6. Furthermore, since $x_4 \neq p_1$, we have $N(w) \cap V(\mathbf{P}'') = \{x_2, x_3\}$, and thus $\mathcal{T}^\dagger$ also satisfies M1). M2) is immediate, and M0) is satisfied since $|E(\mathbf{P}'')| = |E(\mathbf{P})|$. Thus, $\mathcal{T}^\dagger$ admits an L-coloring ϕ. By construction of $L^\dagger$, we have $\phi(x_4) \neq L(p_1)$, since $L(p_1) = \{\phi(x_3)\}$. Thus, since Q^* is an induced subgraph of G and neither of x_2, x_3 is adjacent to p_1, ϕ extends to a proper L-coloring of $V(G^{\text{large}}_{Q^*} \cup P)$, so $V(G^{\text{large}}_Q)) \cup \{p_1\}$ is L-colorable, as desired.

Subcase 1.2.2 $p_3 \notin N(x_2)$

In this case, we have $Q = Q^*$. If x_1, x_2, w do not have a common neighbor in $G^{\text{large}}_{x_1x_2wx_4} \setminus \{x_1, x_2, w, x_4\}$, then, as above, the 3-chord $x_1x_wx_4$ satisfies all the conditions of Proposition 4.3.2, and since x_2, w, x_4 do not have a common neighbor in $G^{\text{large}}_{p_2x_2wx_4}$, we apply Subcase 1.1 with Q replaced by $x_1x_2wx_4$. Thus, the subgraph of G induced by $V(G^{\text{large}}_{x_1x_2wx_4} \cup \{p_1\}$ admits an L-coloring ϕ, and ϕ extends to L-color x_3 as well, since x_3 only has three neighbors in dom(ϕ). Thus, $V(G^{\text{large}}_Q) \cup \{p_1\}$ is L-colorable, as desired.

So now suppose that x_1, x_2, w have a common neighbor in $G^{\text{large}}_{x_1x_2wx_4} \setminus \{x_1, x_2, w, x_4\}$. Let w' be the unique common neighbor of x_1, x_2, w in $G^{\text{large}}_{x_1x_2wx_4} \setminus \{x_1, x_2, w, x_4\}$. Then G contains the 3-chord $R := x_1w'wx_4$ of C, and, since G is short-separation-free, we have $V(G^{\text{small}}_R \setminus \{w, w'\}) = V(G^{\text{small}}_Q)$. Let G' be the graph obtained from G^{large}_R by adding to G^{large}_R the edges x_1p_1 and p_1x_4. Then G' contains the 5-cycle cycle $D := x_1p_1x_4ww'$, and D is a cyclic facial subgraph of G'. Let U be the unique connected open region of $\mathbb{R}^2$ such that $\partial(U) = D$ and $U \cap V(G') = \varnothing$.

Let $G^\dagger$ be a graph obtained from G' by adding to U a 5-cycle $u_1u_2u_3u_4u_5$, where each vertex of $\{u_1, u_2, u_3, u_4, u_5\}$ is adjacent to an edge of D, and then adding a lone vertex u^* in U adjacent to all five vertices of $\{u_1, u_2, u_3, u_4, u_5\}$. Then $G^\dagger$ is still short-separation-free. Let $C^\dagger; = (C \cap G_Q^{\text{large}} + x_1p_1x_4$ and let $C_*^\dagger$ be the outer face of $G^\dagger$. Finally, let $L^\dagger$ be a list-assignment for $V(G^\dagger)$ where each vertex of $\{u_1, u_2, u_3, u_4, u_5, u^*\}$ is assigned an arbitrary 5-list, and otherwise $L^\dagger = L$. Then $\mathcal{T}^\dagger := (G^\dagger, (\mathcal{C} \setminus \{C\}) \cup \{C^\dagger\}, L^\dagger, C_*^\dagger)$ is a tessellation.

We claim that $\mathcal{T}^\dagger$ is a mosaic. Firstly, since $x_1 = p_2$, we get that $d_{G^\dagger}(w', p_1) = 2$ by our construction of $G^\dagger$. Furthermore, R separates each element of $\mathcal{C} \setminus \{C\}$ from p_1. Thus, since G contains a 2-path from w' to x_4, and w is adjacent to x_4, it follows that $\mathcal{T}^\dagger$ satisfies the distance conditions of Definition 2.1.6. M0) and M2) are immediate, and, by constriction of $G^\dagger$, $C^\dagger$ is an induced cycle of $G^\dagger$, and, for each $1 \leq i \leq 5$, $G^\dagger[N(u_i) \cap V(\mathbf{P})]$ is a path of length at most one. Thus, $\mathcal{T}^\dagger$ is indeed a mosaic. By assumption, we have $|V(G_Q^{\text{small}} \setminus V(C \cup Q)| > 4$. Thus, since $x_2, x_3 \in V(G_R^{\text{large}}) \setminus V(C \cup R)$ and $x_2, x_3 \in V(Q)$, we have $|V(G^\dagger)| < |V(G)|$, since G contains at least seven vertices outside of $V(G_R^{\text{large}}) \cup V(C)$. Thus, by the minimality of $\mathcal{T}$, $G^\dagger$ admits an L-coloring ϕ. Let ϕ' be the restriction of ϕ to $V(G_R^{\text{large}}) \cup \{p_1\}$. Then ϕ extends to color x_2, x_3 as well, since each of x_2, x_3 has only at most three neighbors in dom(ϕ') by our conditions on Q. Thus, we obtain a proper L-coloring of $V(G_Q^{\text{large}} \cup \mathbf{P})$, as desired. This completes Case 1 of Proposition 4.3.2.

Case 2: $p_2 \notin N(x_2)$

In this case, we have $x_1 = p_3$ and $N(x_2) \cap V(\mathbf{P}) = \{p_3\}$. We break this into two subcases.

Subcase 2.1 $p_1 \neq x_4$

In this case, Let $G^\dagger$ be a graph obtained from G_Q^{large} by adding to G_Q^{large} a vertex v^* adjacent to each of x_1, x_2, x_3 and a vertex v^{**} adjacent to each of v^*, x_3, x_4. Let $C^\dagger := (C \cap G_Q^{\text{large}}) + x_1v^*v^{**}x_4$ and let $\mathbf{P}' := p_m \cdots p_3v^*v^{**}$. Let $L^\dagger$ be a list-assignment for $V(G^\dagger)$, where $L^\dagger(v^*) = L(p_2)$ and $L^\dagger(v^{**}) = L(p_1)$. Since $x_1 = p_3$, we have $|E(\mathbf{P}')| = |E(\mathbf{P})|$. Let $C_*^\dagger$ be the outer face of $G^\dagger$ and let $\mathcal{T}^\dagger := (G^\dagger, (\mathcal{C} \setminus \{C\}) \cup \{C^\dagger), L^\dagger, C_*^\dagger)$.

Since Q is an induced subpath of G and x_1, x_2, x_3 do not have a common neighbor in G, $G^\dagger$ is short-separation-free, so $\mathcal{T}^\dagger$ is a tessellation. Furthermore, $C^\dagger$ is an open $\mathcal{T}^\dagger$-ring with precolored path $\mathbf{P}'$, and $\mathbf{P}'$ is $L^\dagger$-colorable, since $\mathbf{P}$ is L-colorable.

By assumption, $p_3 \notin N(x_3)$, and thus, by Observation 4.3.1, $G^\dagger[N(x_2) \cap V(\mathbf{P}')]$ consists of p_3v^* and $G^\dagger[N(x_3) \cap V(P')] = v^*v^{**}$, so M0)-M2) are satisfied. Finally, we have $d_{G^\dagger}(x_2, v^{**}) = 2$ and $d_{G^\dagger}(x_3, v^{**}) = 1$. On the other hand, in G, x_3 has a neighbor in $C \setminus \mathring{\mathbf{P}}$ and $x_2x_3x_4$ is a 2-path from x_2 to $C \setminus \mathring{\mathbf{P}}$, so, since $\mathcal{T}$ satisfies the distance conditions of Definition 2.1.6, $\mathcal{T}^\dagger$ does as well. Thus, $\mathcal{T}^\dagger$ is a mosaic, and, since $|V(G_Q^{\text{small}}) \setminus V(C \cup Q)| > 1$ by assumption, we have $|V(G^\dagger)| < |V(G)|$, so $G^\dagger$ is $L^\dagger$-colorable by the minimality of $\mathcal{T}$. Thus, $G^\dagger$ is $L^\dagger$-colorable, so let ϕ be an $L^\dagger$-coloring of $G^\dagger$ and let ϕ' be the restriction of ϕ to G_Q^{large}.

Since Q is an induced subgraph of G, ϕ' is a proper L-coloring of $V(G_Q^{\text{large}})$. If $p_1x_4 \notin E(G)$, then, since C is an induced subgraph of G, it follows from condition 3) that ϕ' extends to a proper L-coloring of $V(G_Q^{\text{large}}) \cup \{p_1, p_2\}$. On the other hand, if $p_1x_4 \in E(G)$, then, by construction of $L^\dagger$, we have $L(p_1) \neq \{\phi(x_4)\}$, since $\phi(x_4) \neq \phi(v^{**})$. Thus, again, ϕ' extends to a proper L-coloring of $V(G_Q^{\text{large}}) \cup \{p_1, p_2\}$.

Subcase 2.2 $x_4 = p_1$

In this case, the 5-cycle $D := p_3p_2p_1x_3x_2$ is a cyclic facial subgraph of G_Q^{small}. We break this into two subcases.

Subcase 2.2.1 x_2, x_3, x_4 do not have a common neighbor in $G_Q^{\text{large}} \setminus Q$

In this subcase, we let $G^\dagger$ be a graph obtained from G by deleting all the vertices of $G_Q^{\text{small}} \setminus D$ and replacing them with the edges x_2p_2, x_3p_2. Note that in this subcase, $G[N(v) \cap V(Q)]$ is a subpath of Q of length at most one for each $v \in V(G_Q^{\text{large}} \setminus Q)$, so $G^\dagger$ is short-separation-free, since Q is an induced subpath of G. Thus, $\mathcal{T}^\dagger = (G^\dagger, \mathcal{C}, L^\dagger, C_*)$ is a tessellation. In $G^\dagger$, we have $N(x_2) \cap V(\mathbf{P}) = \{p_3, p_2\}$ and $N(x_3) \cap V(P) = \{p_1, p_2\}$, so $\mathcal{T}^\dagger$ satisfies M1) of Definition 2.1.6. Since G contains the edge x_3p_1 and the 2-path $x_2x_3p_1$, $\mathcal{T}^\dagger$ also satisfies the distance conditions of Definition 2.1.6, and M0), M2) are immediate, so $\mathcal{T}^\dagger$ is a mosaic. Since $|V(G^\dagger)| < |V(G)|$, $G^\dagger$ admits an L-coloring, so $V(G_Q^{\text{large}} \cup \mathbf{P})$ admits an L-coloring.

Subcase 2.2.2 x_2, x_3, x_4 have a common neighbor in $G_Q^{\text{large}} \setminus Q$

In this case, let w be the unique common neighbor of x_2, x_3, x_4 in $G_Q^{\text{large}} \setminus Q$. Note that, since Q separates w from p_2, we have $N(w) \cap V(\mathbf{P} \setminus \{p_1\}) = \varnothing$. We now break Subcase 2.2.2 into two subcases.

Subcase 2.2.2.1 x_1, x_2, w do not have a common neighbor in $G_Q^{\text{large}} \setminus Q$

In this case, we simply let $G^\dagger = G_Q^{\text{large}} + p_3x_3$ and we let $C^\dagger := (C \cap G_Q^{\text{large}}) + p_3x_3p_1$. We claim that $G^\dagger$ is short-separation-free. To see this, note that, since Q is an induced subgraph of G, if $G^\dagger$ is not-short-separation-free, then there is a 2-path in $G_Q^{\text{large}} \setminus \{x_2\}$ with endpoints w, p_3. By assumption on Q, we have $p_3 \notin N(w)$, and thus, since G is short-separation-free, it follows from our triangulation conditions that the midpoint if this path is adjacent to x_2, contradicting the assumption of this subcase. Thus, $G^\dagger$ is indeed short-separation-free.

Let $C_*^\dagger$ be the outer face of $G^\dagger$ and let $L^\dagger$ be a list-assignment for $V(G^\dagger)$ where $L^\dagger(x_3) = L(p_2)$, and otherwise $L^\dagger = L$. Let $\mathbf{P}' := p_m\mathbf{P}p_3x_3p_1$. Since $\mathbf{P}$ is L-colorable, $\mathbf{P}'$ is $L^\dagger$-colorable. Thus, $\mathcal{T}^\dagger := (G^\dagger, (\mathcal{C} \setminus \{C\}) \cup \{C^\dagger\}, L^\dagger, C_*^\dagger)$ is a tessellation. We claim now that $\mathcal{T}^\dagger$ is a mosaic. In $G^\dagger$, we have $N(x_2) \cap V(\mathbf{P}') = \{x_3, p_1\}$. Thus, since $C^\dagger$ is an induced subgraph of $G^\dagger$, $\mathcal{T}^\dagger$ satisfies M1), and M2) is immediate. Since $|E(\mathbf{P}')| = |E(\mathbf{P})|$, we have M1) as well, so $\mathcal{T}^\dagger$ is indeed a mosaic. Since $|V(G^\dagger)| < |V(G)|$, $G^\dagger$ admits an $L^\dagger$-coloring ϕ by the minimality of $\mathcal{T}$. Let ϕ' be the restriction of ϕ to $G_{p_3x_2wp_1}^{\text{large}}$. Then ϕ' extends to an L-coloring ϕ'' of $V(G_Q^{\text{large}})$, since x_3 has precisely four neighbors among $\text{dom}(\phi')$. Since $\phi(p_1) \neq \phi(x_3)$ by constriction of $L^\dagger$, ϕ'' is a proper L-coloring of its domain, and since, in G, p_2 has no neighbors in $\{x_2, x_3\}$, ϕ'' extends to a proper L-coloring of $V(G_Q^{\text{large}} \cup \mathbf{P})$.

Subcase 2.2.2.2 x_1, x_2, w have a common neighbor in $G_Q^{\text{large}} \setminus Q$

In this case, let w' be the unique common neighbor of x_1, x_2, w in$G_Q^{\text{large}} \setminus Q$. Then G contains the 3-chord $R := x_1w'wx_4$ of C, and, since G is short-separation-free, we have $V(G_R^{\text{small}} \setminus \{w, w'\}) = V(G_Q^{\text{small}})$. Let G' be the graph obtained from G_R^{large} by adding to G_R^{large} the edges x_1p_2 and p_2x_4. Then G' contains the 5-cycle cycle $D := x_1p_2x_4ww'$, and D is a cyclic facial subgraph of G'. Let U be the unique connected open region of $\mathbb{R}^2$ such that $\partial(U) = D$ and $U \cap V(G') = \varnothing$.

Let $G^\dagger$ be a graph obtained from G' by adding to U a 5-cycle $u_1u_2u_3u_4u_5$, where each vertex of $\{u_1, u_2, u_3, u_4, u_5\}$ is adjacent to an edge of D, and then adding a lone vertex u^* in U adjacent to all five vertices of $\{u_1, u_2, u_3, u_4, u_5\}$. Then $G^\dagger$ is still short-separation-free. Let $C^\dagger; = (C \cap G_Q^{\text{large}} + x_1p_1x_4$ and let $C_*^\dagger$ be the outer face of $G^\dagger$. Finally, let $L^\dagger$ be a list-assignment for $V(G^\dagger)$ where each vertex of $\{u_1, u_2, u_3, u_4, u_5, u^*\}$ is assigned an arbitrary 5-list, and otherwise $L^\dagger = L$. Then $\mathcal{T}^\dagger := (G^\dagger, (\mathcal{C} \setminus \{C\}) \cup \{C^\dagger\}, L^\dagger, C_*^\dagger)$ is a tessellation.

We claim that $\mathcal{T}^\dagger$ is a mosaic. Firstly, we have $d_{G^\dagger}(w', p_1) = 2$ by our construction of $G^\dagger$. Thus, since G contains a 2-path from w' to x_4, and w is adjacent to x_4, it follows that $\mathcal{T}^\dagger$ satisfies the distance conditions of Definition 2.1.6. M0) and M2) are immediate, and, by constriction of $G^\dagger$, $C^\dagger$ is an induced cycle of $G^\dagger$, and, for each $1 \leq i \leq 5$, $G^\dagger[N(u_i) \cap V(P)]$ is a path of length at most one. Thus, $\mathcal{T}^\dagger$ is indeed a mosaic. By assumption, we have $|V(G_Q^{\text{small}} \setminus V(C \cup Q)| > 4$. Thus, since $x_2, x_3 \in V(G_R^{\text{large}}) \setminus V(C \cup R)$ and $x_2, x_3 \in V(Q)$, we have $|V(G^\dagger)| < |V(G)|$, since G contains at least seven vertices outside of $V(G_R^{\text{large}}) \cup V(C)$. Thus, by the minimality of $\mathcal{T}$, $G^\dagger$ admits an L-coloring ϕ. Let ϕ' be the restriction of ϕ to $V(G_R^{\text{large}}) \cup \{p_1, p_2\}$. Then ϕ extends to color x_2, x_3 as well, since each of x_2, x_3 has only at most three neighbors in dom(ϕ') by our conditions on Q. Thus, we obtain a proper L-coloring of $V(G_Q^{\text{large}} \cup \mathbf{P})$, as desired. This completes the proof of Proposition 4.3.2. □

We now prove the following, which is the second of two facts which make up Section 4.3.

Proposition 4.3.4. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let $C \in \mathcal{C}$ be an open $\mathcal{T}$-ring, and let $\mathbf{P} := \mathbf{P}_{\mathcal{T}}(C)$. Let $\mathbf{P} := p_1 \cdots p_m$, and let $Q := x_1x_2x_3x_4$ be a 3-chord of C with $x_1 \in V(\mathring{\mathbf{P}})$ and $x_4 \in V(C \setminus \mathring{\mathbf{P}})$. Suppose that $V(G_Q^{\text{small}} \setminus C) \neq \{x_2, x_3\}$ and $V(G_Q^{\text{small}}) \subseteq B_1(C)$. Letting $x_1 \in \{p_2, p_3\}$, we have $x_2 \in N(w)$ and $V(G_Q^{\text{small}}) \setminus V(C \cup Q)$ consists of a lone vertex adjacent to all vertices in the cycle $C \cap G^{\text{large}}_{p_2x_2x_3x_4}) + p_2x_2x_3x_4$. An analogous statement holds in the case where $x_1 \in \{p_{m-1}, p_{m-2}\}$.*

Proof. Let C^1 be the 1-necklace of G. Given a 3-chord $Q := x_1x_2x_3x_4$ of C, we say that Q is *defective* if the following hold.

1) Precisely one endpoint of Q lies in $\mathring{\mathbf{P}}$; *AND*
2) $V(G_Q^{\text{small}}) \subseteq B_1(C)$; *AND*
3) $|V(G_Q^{\text{small}} \setminus C)| > 3$.

We claim now that there is no defective 3-chord of C. Suppose toward a contradiction that there is a defective 3-chord Q of C, and, among all defective 3-chords of C, we choose Q so that $|V(G_Q^{\text{small}})|$ is minimized. Let $Q := x_1x_2x_3x_4$.

By Observation 4.3.1, suppose without loss of generality that $x_1 \in \{p_2, p_3\}$ and $G_Q^{\text{small}} \cap \mathbf{P} = p_1\mathbf{P}x_1$. Then we have $V(G_Q^{\text{small}}) = V(G_Q^{\text{small}} \cap C) \cup V(G_Q^{\text{small}} \cap C^1)$. Since Q is defective, the graph $G_Q^{\text{small}} \cap C^1$ is a path of length at least three with endpoints x_2, x_3, so let $G_Q^{\text{small}} \cap C^1 = w_0w_1 \cdots w_sw_{s+1}$ for some $s \geq 2$, where $w_0 = x_2$ and $w_{s+1} = x_3$. Then G_Q^{small} contains the cycle $D := w_0w_1 \cdots w_{s+1}$.

Claim 4.3.5. *$x_2p_1 \notin E(G)$ and $x_3p_1 \notin E(G)$*

<u>Proof:</u> Suppose that $x_2p_1 \in E(G)$. By M2), we have $x_1 = p_3$, and the 3-chord $Q^* := p_1x_2x_3x_4$ of C separates a vertex of $\{w_1, \cdots, w_s\}$ from $G_{Q^*}^{\text{large}} \setminus V(Q^*)$. Since $Q^* \in \mathcal{K}(C, \mathcal{T})$ and Q^* has an endpoint in $\{p_1, p_m\}$, this contradicts Theorem 3.0.2. Thus, $x_2p_1 \notin E(G)$. Now suppose toward a contradiction that $x_3p_1 \in E(G)$ and let $Q^{**} := x_1x_2x_3p_1$. Then each vertex of $\{w_1, \cdots w_s\}$ lies in $G_{Q^{**}}^{\text{small}}$. Again, since $Q^{**} \in \mathcal{K}(C, \mathcal{T})$ and one endpoint of Q^{**} lies in $\{p_1, p_m\}$, this contradicts Theorem 3.0.2. ■

Thus, we have $x_2p_1 \notin E(G)$. Since $x_1 \in \{p_2, p_3\}$. Since C^1 contains the unique common neighbor of p_1, p_2, there is a $j \in \{1, \cdots, s\}$ such that w_j is the unique common neighbor of p_1, p_2 in G. Since $p_3 \notin N(w_j)$ by M2), p_2 is an endpoint of the path $G[V(C) \cap N(w_j)]$, so we have $w_{j-1} \in N(p_2)$ by our triangulation conditions.

We note now that $N(w_j) \cap V(D) = \{w_{j-1}, w_j\}$. If this does not hold, then there is a chord of C^1 with w_j as an endpoint, so let $k \in \{0, \cdots, s+1\} \setminus \{j-1, j, j+1\}$ be such that w_k is the other endpoint of this chord. Suppose that w_k does not have a neighbor among $C \setminus \mathring{\mathbf{P}}$. In that case, we have $0 \leq k < j$, since each vertex of $w_jw_{j+1} \cdots w_{s+1}$ has a neighbor on $C \setminus \mathring{\mathbf{P}}$.

If $p_2 \in N(w_k)$, then the 3-cycle $w_jp_2w_k$ separates w_{j-1} from G_Q^{large}, contradicting short-separation-freeness. Likewise, if $p_3 \in N(w_k)$, then the 4-cycle $w_jp_2p_3w_k$ separates w_{j-1} from G_Q^{large}, contradicting short-separation-freeness. Thus, w_k has a neighbor in $C \setminus \mathring{\mathbf{P}}$, and since w_1 is adjacent to p_1, we contradict Theorem 3.0.2. Thus, we get

$N(w_j) \cap V(D) = \{w_{j-1}, w_j\}$, as desired. Let $C \cap G_Q^{\text{large}} = x_1 \mathbf{P} p_1 u_1 \cdots u_t$ for some $t \geq 0$ (possibly $t = 0$ and $p_1 = x_4$). We now have the following:

Claim 4.3.6. *$x_4 \neq p_1$, and, for each $m \in \{j+1, \cdots, s+1\}$, we have $N(w_m) \cap V(\mathbf{P}) = \varnothing$*

Proof: We first show that $N(x_{j+1}) \cap V(\mathbf{P}) = \varnothing$. We first note that $p_2 \notin N(w_{j+1})$, or else G contains a copy of K_4 on the vertices $\{p_2, w_{j-1}, w_j, w_{j+1}\}$. We claim now that we also have $p_3 \notin N(w_{j+1})$. Furthermore, $p_1 \notin N(w_{j+1})$, or else G contains a $K_{2,3}$ with bipartition $\{p_1, w_{j-1}\}, \{p_2, w_j, w_{j+1}\}$. Now suppose toward a contradiction that $p_3 \notin N(w_{j+1})$. Then G contains the 4-cycle $p_3 p_2 w_{j-1} w_{j+1}$, and since $p_2 \notin N(w_{j+1})$, we have $p_3 \in N(w_{j-1})$ by our triangulation conditions, so G contains a $K_{2,3}$ with bipartition $\{p_2, w_{j+1}\}, \{w_{j-1}, w_j, p_3\}$, contradicting short-separation-freeness. Thus, we have $N(v_{j+1}) \cap V(\mathbf{P}) = \varnothing$. Since $N(w_{j+1}) \cap V(\mathbf{P}) = \varnothing$, there exists an $r \in \{1, \cdots, t\}$ such that the graph $G[V(C) \cap N(w_j)]$ is a path $p_2 p_1 u_1 \cdots u_r$ by Theorem 3.0.2. In particular, since $t \geq 1$, we have $x_1 \neq p_4$.

Since $x_3 \in N(u_t)$, the cycle $w_0 \cdots w_{j+1} u_r \cdots u_t w_{s+1}$ separates each vertex of $\{w_{j+2}, \cdots, w_s\}$ from $\mathbf{P}$, so we have $N(w_m) \cap V(P) = \varnothing$ for each $m \in \{j+1, \cdots, s\}$. Since $w_{s+1} = x_3$, we just need to check that $N(x_3) \cap V(P) = \varnothing$.

Suppose that $x_3 x_1 \in E(G)$. By Theorem 3.0.2, we have $V(G^{\text{small}}_{x_1 x_3 x_4}) \subseteq V(C) \cup \{x_3\}$, so $x_1 x_3 \notin E(G_Q^{\text{large}})$. Thus, $x_1 x_3 \in E(G_Q^{\text{small}})$, and the triangle $x_1 x_2 x_3$ separates a vertex of $\{w_1, \cdots, w_s\}$ from G_Q^{large}), which is false. Thus, $x_1 \notin N(x_3)$. Now suppose that $p_2 \in N(x_3)$. Then $x_1 = p_3$, and, by Theorem 3.0.2, we have $V(G^{\text{small}}_{p_2 x_3 x_4}) \subseteq V(C) \cup \{x_3\}$, so the 4-cycle $x_1 p_2 x_3 x_2$ separates a vertex of $\{w_1, \cdots, w_s\}$ from G_Q^{large}, which is false. We just need to make sure that $p_1 \notin N(x_3)$. Then the cycle D lies in $G^{\text{small}}_{x_1 x_2 x_3 p_1}$, so $x_1 x_2 x_3 p_1$ is also a defective 3-chord of C. Since $x_4 \neq p_1$, this contradicts the minimality of Q. ■

By Claim 4.3.6, since each vertex of $\{w_0, \cdots, w_{s+1}\}$ is in $B_1(C, G)$, each vertex of $\{w_{j+1}, \cdots, w_{s+1}\}$ has a neighbor among $\{u_1, \cdots, u_t\}$.

Claim 4.3.7. *$N(w_0) \cap \{w_{j+2}, \cdots, w_s\} = \varnothing$ and, for each $a \in \{1, \cdots, j-1\}$, we have $N(w_a) \cap \{w_{j+2}, \cdots, w_{s+1}\} = \varnothing$.*

Proof: Suppose toward a contradiction that there is an $m \in \{j+2, \cdots, s\}$ with $x_2 \in N(w_m)$. Since w_m has a neighbor among $\{u_1, \cdots, u_t\}$, let $u \in \{u_1, \cdots, u_t\} \cap N(w_m)$ and let $Q^\dagger := x_1 x_2 w_m u$. Since $x_2 w_m$ is a chord of D, we have $|V(G^{\text{small}}_{Q^\dagger})| < |V(G^{\text{small}}_Q)|$, and since $w_j, w_{j+1} \in V(G^{\text{small}}_{Q^\dagger}) \setminus V(Q^\dagger)$, $Q^\dagger$ is also defective, so this contradicts the minimality of Q.

Now suppose toward a contradiction that there is an $a \in \{1, \cdots, j-1\}$ and an $m \in \{j+2, \cdots, s+1\}$ with $w_m \in N(w_{j-1})$. Since w_m has a neighbor among $\{u_1, \cdots, u_t\}$ and w_a has a neighbor among $\{p_2, p_3\}$, let $u \in \{u_1, \cdots, u_t\} \cap N(w_m)$ and let $p \in \{p_2, p_3\} \cap N(w_a)$. Let $Q^\dagger := p w_a w_m u$. Since $w_a w_m \neq x_2 x_3$, $w_a w_m$ is a chord of D, and we have $|V(G^{\text{small}}_{Q^\dagger})| < |V(G^{\text{small}}_Q)|$. Since $w_j, w_{j+1} \in V(G^{\text{small}}_{Q^\dagger}) \setminus V(Q^\dagger)$, $Q^\dagger$ is also defective, so this contradicts the minimality of Q. ■

Now we have the following:

Claim 4.3.8. *There is no chord of D with both endpoints in $w_{j+1} \cdots w_{s+1}$.*

Proof: Suppose toward a contradiction that there is a pair of indices $k, \ell \in \{j+1, \cdots, s+1\}$, where $\ell > k+1$ and $w_k w_\ell \in E(G)$. Since each of w_k, w_ℓ has a neighbor on $\{u_1, \cdots, u_t\}$, we let $u \in N(w_k) \cap \{u_1, \cdots, u_t\}$ and

$u' \in N(w_\ell) \cap \{u_1, \cdots, u_t\}$. Note that $u \neq u'$, or else G contains a triangle which separates w_{k+1} from G_Q^{large}. Now let $Q^\dagger := uw_K w_\ell u'$. Then $Q^\dagger \in \mathcal{K}(C, \mathcal{T})$ and $w_{k+1} \in V(G_{Q^\dagger}^{\text{small}} \setminus Q^\dagger)$. By Theorem 3.0.2, w_k is not a $\mathcal{C}$-shortcut, and since $w_k \neq w_{s+1}, w_0$, Q separates w_k from each element of $\mathcal{C} \setminus \{C\}$. By Claim 4.3.6, w_k has no neighbors in $V(\mathbf{P})$, and thus, since w_k is not a $\mathcal{C}$-shortcut, w_k is adjacent to x_2, contradicting Claim 4.3.7. ■

Combining Claim 4.3.8 with Claim 4.3.7, we have $w_{j+1} = w_{s+1} = x_3$, or else, since D is triangulated by chords in G_Q^{small}, there is either a chord of D with both endpoint in $w_{j+1} \cdots w_{s+1}$, or a chord of D with one endpoint in $\{w_{j+1}, \cdots, w_{s+1}\}$ and one endpoint in $\{w_0, \cdots, w_{j-1}\}$.

We claim now that $j = 2$. Firstly, we have $j > 1$, or else $x_2 = w_{j-1}$ and $V(G_Q^{\text{small}} \setminus C) = \{x_2, x_3, w_1\}$, contradicting the fact that Q is defective. Now suppose toward a contradiction that $j > 2$. Then $w_0 \cdots w_{j-1}$ is a path of length at least two, and since D is triangulated by chords in G_Q^{small}, it follows from Claim 4.3.7 that G_Q^{small} has a chord of D with both endpoints in $\{w_0, \cdots, w_{j-1}\}$. Let $0 \leq k < \ell \leq j-1$, where $w_k w_\ell \in E(G) \setminus E(D)$ and $\ell > k+1$. Since each of w_k, w_ℓ have a neighbor among p_2, p_3, there is a cycle of length at most four which separates w_{k+1} from G_Q^{large}, contradicting short-separation-freeness.

We conclude that $j = 2$ and $V(G_Q^{\text{small}} \setminus C) = \{x_2, x_3, w_1, w_2\}$. Furthermore, we have $x_1 = p_3$ and $w_1 \in N(x_1)$, or else G contains a $K_{2,3}$ with bipartition $\{x_2, w_1, w_2\}, \{p_2, x_3\}$. Note that Q is induced in G and $N(x_1) \cap N(x_4) = \varnothing$, or else there is a 2-chord of C with endpoints p_3, x_4, contradicting Theorem 3.0.2. Now we have the following:

Claim 4.3.9. $N(x_3) \cap V(C \cap G_Q^{\text{small}}) = \{x_4\}$.

Proof: By Claim 4.3.6, x_3 has no neighbors in $\mathbf{P}$, so if this does hold, then there is an $r \in \{1, \cdots, t-1\}$ with $u_r \in N(x_3)$. Then $V(G_{x_1x_2x_3u_r}^{\text{small}})| < |V(G_Q^{\text{small}})|$, and since $w_1, w_2 \in V(G_{x_1x_2x_3u_r}^{\text{small}}))$, $x_1x_2x_3u_r$ is defective, contradicting the minimality of Q. ■

Since $N(x_3) \cap V(C \cap G_Q^{\text{small}}) = \{x_4\}$, it follows from our triangulation conditions that w_2 is adjacent to each vertex of $\{p_1, u_1, \cdots, u_t\}$. Furthermore, G contains the 3-chord $Q^\dagger := x_1w_1w_2x_4$ of C. For each $i = 1, 2, 3$, let $\{q_i\} = L(p_i)$.

Claim 4.3.10. *Let $L^\dagger$ be a list-assignment for $G_{Q^\dagger}^{\text{small}}$ such that the following hold.*

1) $|L^\dagger(w_1)| = |L^\dagger(w_2)| = 1$*; AND*
2) $L^\dagger(w_1) \neq L^\dagger(w_2)$ *and* $L^\dagger(w_1) \neq q_3$*; AND*
3) $L^\dagger(v) = L(v)$ *for all* $v \in V(G_{Q^\dagger}^{\text{small}}) \setminus \{w_1, w_2\}$.

Then $G_{Q^\dagger}^{\text{large}}$ is $L^\dagger$-colorable. In particular, there exists an L-coloring ϕ of G_Q^{large} such that $\phi(x_1) \neq q_4$.

Proof: Let $C^\dagger := (C \cap G_{Q^\dagger}^{\text{large}}) + x_1w_1w_2x_4$ and let $C_*^\dagger$ be the outer face of $G_{Q^\dagger}^{\text{large}}$. Let $\mathbf{P}' := p_m \cdots p_3 w_1 w_2$ and let $\mathcal{T}^\dagger := (G_{Q^\dagger}^{\text{large}}, (\mathcal{C} \setminus \{C\}) \cup \{C^\dagger\}, L^\dagger, C_*^\dagger)$. By conditions 1) and 2), $\mathbf{P}'$ is $L^\dagger$-colorable, and $\mathcal{T}^\dagger$ is a tessellation, where $C^\dagger$ is an open $\mathcal{T}^\dagger$-ring with precolored path $\mathbf{P}'$. We claim that $\mathcal{T}^\dagger$ is a tessellation. Since $N(x_4) \cap V(\mathbf{P}) = \varnothing$ and $w_1x_4 \notin E(G)$, there is no chord of $C^\dagger$ with an endpoint in $\mathbf{P}'$. Furthermore, we have $N(x_2) \cap V(\mathbf{P}') = \{p_3, w_1\}$ and $N(x_3) \cap V(\mathbf{P}') = \{x_1, w_2\}$, so $\mathcal{T}^\dagger$ satisfies M1), and M0) holds since $|E(\mathbf{P}')| = |E(\mathbf{P})|$. M2) is immediate, so we just need to check that our distance conditions hold. If not, then there is a $C' \in \mathcal{C} \setminus \{C\}$ and a $H \subseteq C'$ such that $d(H, w_2) < d(H, C \setminus \mathring{\mathbf{P}})$. Let R be a shortest (H, w_2)-path. Then $V(R \cap Q)$ does not contain either of x_3, x_4, since each of these are of distance at most one from $C \setminus \mathring{\mathbf{P}}$, so $V(R \cap Q)$ contains a vertex of x_1, x_2, since Q separates H from w_2. But each of x_1, x_2 have distance two from w_2, and distance two from $C \setminus \mathring{\mathbf{P}}$, so we have a contradiction. Thus, $\mathcal{T}^\dagger$ is a mosaic, and since $p_1, p_2 \notin V(G_{Q^\dagger}^{\text{large}})$, $G_{Q^\dagger}^{\text{large}}$ admits an $L^\dagger$-coloring by the minimality of $\mathcal{T}$. ■

As a consequence of the above, we have the following:

Claim 4.3.11. $G_Q^{\text{small}} \cap C = p_3p_2p_1x_4$.

<u>Proof:</u> Suppose not. Then u_1 is an internal vertex of the path $p_1u_1\cdots u_t$. Let $a, b \in L(u_1) \setminus \{q_1\}$ and let d be a color of $L(w_2) \setminus \{q_1, q_2, a, b\}$ and let $f \in L(w_1) \setminus \{q_2, q_3, d\}$. By Claim 4.3.10, there is an L-coloring ϕ of $G_{Q^\dagger}^{\text{large}}$ with $\phi(w_1) = f$ and since $p_1x_4 \notin E(G)$, ϕ is a proper L-coloring of the subgraph of G induced by $V(G_{Q^\dagger}^{\text{large}})$. By our choice of d, f, ϕ extends to an L-coloring ϕ' of $V(G_{Q^\dagger}^{\text{large}}) \cup \{p_1, p_2\}$. Since at least one of q_1, d lies outside of $L(u_1)$, ϕ' extends to the broken wheel $G_{p_1w_2x_4}^{\text{small}}$, and thus G is L-colorable, which is false. ■

Now we have the following:

Claim 4.3.12. *For any L-coloring ϕ of G_Q^{large}, if $\phi(x_4) \neq q_1$, then $\phi(x_4) = \phi(x_2)$.*

<u>Proof:</u> Let $a, b \in L(w_1) \setminus \{q_1, q_2, q_3\}$. Let $L^\dagger$ be a list-assignment for $G_{Q^\dagger}^{\text{large}}$ with $L^\dagger(w_1) = a$, $L^\dagger(w_2) = q_1$, and otherwise $L^\dagger = L$. By Claim 4.3.10, $G_{Q^\dagger}^{\text{large}}$ admits an $L^\dagger$-coloring ϕ. Let ϕ' be the restriction of ϕ to G_Q^{large}. Note that $\phi(x_4) \neq q_1$ so ϕ' extends to a proper L-coloring ϕ'' of the subgraph of G induced by $V(G_Q^{\text{large}})) \cup \{p_1, p_2\}$. Since G is not L-colorable, ϕ'' does not extend to L-color the edge w_1w_2. Since each of w_1, w_2 has precisely four neighbors in dom(ϕ'') and $a \in L_{\phi''}(w_1)$, we have $L_{\phi''}(w_1) = L_{\phi''}(w_2) = \{a\}$. The same argument shows that there is an L-coloring ψ'' of $V(G_Q^{\text{large}}) \cup \{p_1, p_2\}$ such that $L_{\psi''}(w_1) = L_{\psi''}(w_2) = \{b\}$, so $a, b \in L(w_2) \setminus \{q_1, q_2, q_3\}$.

Now let ζ be an arbitrary L-coloring of G_Q^{large}, where $\zeta(x_4) \neq q_1$. Since $\zeta(x_4) \neq q_1$, ζ extends to a proper L-coloring ζ' of $V(G_Q^{\text{large}}) \cup \{p_1, p_2\}$. Suppose toward a contradiction that $\zeta(x_2) \neq \zeta(x_4)$.

Since ζ' does not extend to an L-coloring of G, we have $|L_{\zeta'}(w_1)| = |L_{\zeta'}(w_2)| = 1$ and $L_{\zeta'}(w_1) = L_{\zeta'}(w_2)$. Let $L_{\zeta'}(w_1) = L_{\zeta'}(w_2) = \{c\}$. If $c \notin \{a, b\}$, then we have $\{a, b\} = \{\zeta(x_2), \zeta(x_3)\}$ and $\{a, b\} = \{\zeta(x_3), \zeta(x_4)\}$, so $\zeta(x_2) = \zeta(x_4)$, contradicting our assumption. Thus, we have $c \in \{a, b\}$ so suppose without loss of generality that $c = a$. Then b appears among $\{\zeta(x_2), \zeta(x_3)\}$ and among $\{\zeta(x_3), \zeta(x_4)\}$, so, since $\zeta(x_2) \neq \zeta(x_4)$, we have $\zeta(x_3) = b$, and there exist colors d_1, d_2 with $d_1 \neq d_2$, where $d_1 \in L(w_1) \setminus \{a, b, q_2, q_3\}$ and $d_2 \in L(w_1) \setminus \{a, b, q_1, q_2\}$. Since $d_1 \neq q_1$, let $L^\dagger$ be a list-assignment for $G_{Q^\dagger}^{\text{large}}$ where $L^\dagger(w_1) = \{d_1\}$ and $L^\dagger(w_2) = \{q_1\}$ and otherwise $L^\dagger = L$. By Claim 4.3.10, $G_{Q^\dagger}^{\text{large}}$ admits an $L^\dagger$-coloring Φ, so let Φ' be the restriction of Φ to G_Q^{large}. Since $\Phi(x_4) \neq q_1$, Φ' extends to a proper L-coloring Φ'' of $V(G_Q^{\text{large}}) \cup \{p_1, p_2\}$, and this coloring leaves over d_1 for w_1, but since $d_1 \notin \{a, b, d_2\}$ and at least one of the colors a, b, d_2 is not used among $\Phi(x_3), \Phi(x_4)$, there is a color left over in $L(w_2)$, so Φ'' extends to an L-coloring of G, which is false. ■

Now we have the following:

Claim 4.3.13. *Let $G' := G_Q^{\text{large}} + x_2x_4$ and $G'' := G_Q^{\text{large}} + x_1x_3$. Then neither G' nor G'' is short-separation-free.*

<u>Proof:</u> Suppose toward a contradiction that G' is short-separation-free. Let $d_1 \in L(x_4) \setminus \{q_1\}$ and let $d_2 \in L(x_2) \setminus \{q_3, d_1\}$. Let L' be a list-assignment for $V(G')$ where $L'(x_2) = \{d_2\}$, $L'(x_4) = \{d_1\}$, and otherwise $L' = L$. Let $C' := (C \cap G_Q^{\text{large}}) + x_1x_2x_4$ and let C'_* be the outer face of G'. Let $\mathcal{T}' := (G', (\mathcal{C} \setminus \{C\}) \cup \{C'\}, L', C'_*)$ and let $\mathbf{P}' := p_m \cdots p_3x_2x_4$. Note that, by Theorem 2.3.2, we have $V(C \setminus \mathbf{P}) \neq \{x_4\}$, and thus, since G' is short-separation-free, $\mathcal{T}'$ is a tessellation where C' is an open $\mathcal{T}'$-ring with precolored path $\mathbf{P}'$ (in particular, since C is induced in G and $V(C \setminus \mathbf{P}) \neq \{x_4\}$, $G'[V(\mathbf{P}')]$ is L'-colorable).

Since Q is induced in G and $N(x_1) \cap N(x_4) = \varnothing$, we have $N(x_3) \cap V(\mathbf{P}') = \{x_1, x_4\}$, and C' is induced in G', so $\mathcal{T}'$ satisfies condition M1) of Definition 2.1.6. Since $V(C' \setminus \mathring{\mathbf{P}}') \subseteq V(C \setminus \mathring{\mathbf{P}})$,we get that $\mathcal{T}'$ satisfies the distance conditions Definition 2.1.6. To see this, just note that if either of M3)-M4) is violated, then there is a path in G' from x_1 to $C' \setminus \mathring{\mathbf{P}}'$ of length less than two using the edge x_2x_4, which is false. Furthermore, we have $|E(\mathbf{P}')| = |E(\mathbf{P})|$, so M0) is satisfied as well. M2) is immediate, so $\mathcal{T}'$ is a mosaic.

Thus, since $|V(G')| < |V(G)|$, G' admits an L'-coloring, so G^{large} admits an L-coloring ϕ in which $\phi(x_4) \neq q_1$ and $\phi(x_2) \neq \phi(x_4)$, contradicting Claim 4.3.12. Now we show that G'' is not short-separation-free. Suppose toward a contradiction that G'' is short-separation-free. Since $|L(x_2) \setminus \{q_3\}| \geq 4$ and $|L(w_1) \setminus \{q_2, q_3\}| \geq 3$, we fix a color $d \in L(x_2) \setminus \{q_3\}$ such that $|L(w_1) \setminus \{q_2, q_3, d\}| \geq 3$, and we fix a color $d_2 \in L(x_4) \setminus \{q_1, d\}$. Let L'' be a list-assignment for $V(G'')$ where $L''(x_3) = \{d\}$, $L''(x_4) = \{d_2\}$, and otherwise $L'' = L$. Let $\mathbf{P}'' := p_m \cdots x_1x_3x_4$. Let $C'' := (C \cap G_Q^{\text{large}}) + x_1x_2x_4$ and let C''_* be the outer face of G''. Let $\mathcal{T}'' := (G'', (\mathcal{C} \setminus \{C\}) \cup \{C''\}, L'', C''_*)$. As above, we have $V(C'') \neq V(\mathbf{P}'')$ by Theorem 2.3.2, and since G'' is short-separation-free, $\mathcal{T}''$ is a tessellation, where C'' is an open $\mathcal{T}''$-ring with precolored path $\mathbf{P}''$.

Crucially, since $N(x_2) \cap V(\mathbf{P}) = \{x_1\}$, we have $N(x_2) \cap V(P'') = \{x_1, x_4\}$. Thus, since Q is induced in G, $N(x_1) \cap N(x_4) = \varnothing$, and C'' is induced in G'', $\mathcal{T}''$ satisfies M1), and the other conditions are immediate as in the case of $\mathcal{T}'$. In particular, the distance conditions hold since $V(C'' \setminus \mathring{\mathbf{P}}'') \subseteq V(C \setminus \mathring{\mathbf{P}})$ and there is no path of length less than two in G'' from x_1 to x_4 using the edge x_1x_3. Thus, since $|V(G'')| < |V(G)|$, G'' admits an L''-coloring ϕ, and ϕ extends to an L-coloring ϕ' of $V(G_Q^{\text{large}}) \cup \{p_1, p_2\}$. We have $|L_{\phi'}(w_2)| \geq 1$, and, by our choice of d, we have $|L_{\phi'}(w_1)| \geq 2$, so ϕ' extends to L-color the edge w_1w_2, contradicting the fact that G is not L-colorable. ■

Since Q is an induced subgraph of G, $N(x_1) \cap N(x_4) = \varnothing$, it follows from Claim 4.3.13 that $G_Q^{\text{large}} \setminus Q$ contains path R of of length either two or three, where R has endpoints x_2, x_4 and$R$ is otherwise disjoint from Q.

Claim 4.3.14. *x_2, x_3, x_4 do not have a common neighbor in $G_Q^{\text{large}} \setminus Q$.*

Proof: Suppose there is a common neighbor z of x_2, x_3, x_4 in $G_Q^{\text{large}} \setminus Q$. Let d be a color in $L(w_1) \setminus \{q_1, q_2, q_3\}$, and let $L^\dagger$ be a list-assignment for $G_{Q^\dagger}^{\text{large}}$, where $L^\dagger(w_1) = d$, $L^\dagger(w_2) = \{q_1\}$, and otherwise $L^\dagger = L$. By Claim 4.3.10, $G_{Q^\dagger}^{\text{large}}$ admits an $L^\dagger$-coloring ϕ. Let ϕ' be the restriction of ϕ to $G_Q^{\text{large}} \setminus \{x_3\}$. Then ϕ' is an L-coloring of its domain, and, since $\phi'(x_4) \neq q_1$, ϕ' extends to a proper L-coloring ϕ^* of $\text{dom}(\phi') \cup \{p_1, p_2\}$.

Now, ϕ^* does not extend to the triangle $w_1w_2x_3$, and since $N(x_3) \cap \text{dom}(\phi^*) = \{x_2, z, x_4\}$, each of x_3, w_1, w_2 has an L_{ϕ^*}-list of size precisely two. Thus ϕ uses two different colors on x_2, x_4. Since $|L_{\phi^*}(x_3)| = 2$, there is an L-coloring of G_Q^{large} using two different colors on x_4, p_1 and two different colors on x_2, x_4, contradicting Claim 4.3.12. ■

Since x_2, x_3, x_4 do not have a common neighbor in $G_Q^{\text{large}} \setminus \{x_3\}$, and $G_Q^{\text{large}} + x_2x_4$ is not short-separation-free, there is a path in $G_Q^{\text{large}} \setminus \{x_3\}$ with endpoints x_2, x_4 and length precisely three. Note that any such path is disjoint to Q except for its endpoints, since $N(x_1) \cap N(x_4) = \varnothing$.

Claim 4.3.15. *For any path $R := x_2zz'x_4$ in $G_Q^{\text{large}} \setminus \{x_3\}$ of length precisely three, each of z, z' is adjacent to x_3.*

Proof: Let G' be a graph obtained from G_Q^{large} by adding to G_Q^{large} a lone vertex w^* adjacent to each of x_1, x_2, x_3, x_4. Let $d \in L(x_4) \setminus \{q_1\}$ and let L' be a list-assignment for $V(G')$, where $L'(x_4) = \{d\}$, $L'(w^*)$ is a lone color distinct from q_3, d, and otherwise $L' = L$. Let $\mathbf{P}' := p_m \cdots p_3w^*x_4$ and let $C' := (C \cap G_Q^{\text{large}}) + x_1w^*x_4$. Let C'_* be the outer face of G' and let $\mathcal{T}' := (G', (\mathcal{C} \setminus \{C\}) \cup \{C'\}L', C'_*)$. Note that G' is short-separation-free, or else, since Q is induced in G and x_1, x_4 have no common neighbor in G_Q^{large}, the vertices x_2, x_4 have a common neighbor z in G_Q^{large}.

But then, since G is short-separation-free and Q is induced in G, it follows from our triangulation conditions that $x_3 \in N(z)$, contradicting our assumption. Thus, G' is indeed short-separation-free, and $\mathcal{T}'$ is a tessellation, where C' is an open $\mathcal{T}'$-ring with precolored path $\mathbf{P}'$, since $V(C \setminus \mathbf{P}) \neq \{x_4\}$ by Theorem 2.3.2. Since $V(C' \setminus \mathring{\mathbf{P}}') \subseteq V(C \setminus \mathring{\mathbf{P}})$, it is immediate that $\mathcal{T}'$ satisfies the distance conditions of Definition 2.1.6, and $|E(\mathbf{P}')| = |E(\mathbf{P})|$, so $\mathcal{T}'$ satisfies M0) as well. Since $N(x_2) \cap V(\mathbf{P}') = \{p_3, w^*\}$ and $N(x_3) \cap V(\mathbf{P}') = \{w^*, p_4\}$,a nd C' is induced in G', $\mathcal{T}'$ satisfies M1) as well, and M2) is immediate, so $\mathcal{T}'$ is a tessellation. Since $|V(G')| < |V(G)|$, G' admits an L'-coloring ϕ, and, by Claim 4.3.12, we have $\phi(x_2) = \phi(x_4)$.

Now, G contains the 4-chord $R^* := x_1x_2zz'x_4$ of C, so let ϕ' be the restriction of ϕ to $G^{\text{large}}_{R^*}$. Note that ϕ' extends to an L-coloring ϕ'' of $G^{\text{large}}_{R^*} \cup \{p_1, p_2\}$

Suppose toward a contradiction that at least one of z, z' is not adjacent to x_3. Then x_3 has an $L_{\phi''}$-list of size three, since dom$(\phi'') \cap N(x_3)$ has at most three vertices and ϕ'' uses the same color on x_2, x_4. Thus, ϕ'' extends to L-color the triangle $x_3w_1w_2$, since each of w_1, w_2 has an $L_{\phi''}$-list of size at least two, so let ϕ^* be the resulting extension of ϕ'' to dom$(\phi'') \cup \{w_1, w_2, x_3\}$. The cycle $F := x_2x_3x_4z'z'$ is a cyclic facial subgraph of $H := G^{\text{small}}_{R^*} \setminus (G^{\text{small}}_Q \setminus Q)$, where each vertex of F is precolored by ϕ^*. Since x_2, x_3, x_4 do not have a common neighbor in $G^{\text{large}}_Q \setminus Q$, there is no vertex of $H \setminus F$ adjacent to all five vertices of F, so ϕ^* extends to an L-coloring of H, and thus L-colors all of G, contradicting the fact that $\mathcal{T}$ is critical. ■

Now we have enough to finish showing that there is no defective 3-chord of C. Since there is a path in $G^{\text{large}}_Q \setminus V(Q)$ with endpoints x_2, x_4 and length precisely three, let $R := x_2zz'x_4$ be such a path. By Claim 4.3.15, R is the unique 3-chord of Q in G^{large}_Q with endpoints x_2, x_4, and each of z, z' is adjacent to x_3. Furthermore, by Claim 4.3.13, since $G^{\text{large}}_Q + x_1x_3$ is not short-separation-free, and Q is induced in G, there is a path in $G^{\text{large}}_Q \setminus \{x_2\}$ with endpoints x_1, x_3 and length either two or three. This path does not have length two, or else G has a copy of $K_{2,3}$, since w_1 is adjacent to each of x_1, x_2, x_3. Thus, this path has length three, so let $x_1uu'x_3$ be such a path. Then $u' \in \{z, z'\}$. By Claim 4.3.10, we fix an L-coloring ϕ of G^{large}_Q with $\phi(x_4) \neq q_1$. Now consider the following cases:

Case 1: $u' = z$

In this case let $R^\dagger := x_1uzz'x_4$. Since G is short-separation-free and $G^{\text{small}}_{R^\dagger}$ contains the 4-cycle x_1uzx_2, we have $G^{\text{small}}_{R^\dagger} \setminus G^{\text{small}}_Q = uzz'$. Let ϕ' be the restriction of ϕ to $V(G^{\text{small}}_{R^\dagger})$. Since $N(x_2) \cap \text{dom}(\phi') \subseteq \{x_1, u, z\}$, there is a color in $c \in L_{\phi'}(x_2)$ distinct from $\phi(x_4)$, and since dom$(\phi') \cap N(x_3) = \{z, z', x_4\}$, there is a color left over for x_3 in $L_{\phi'}(x_3) \setminus \{c\}$, so ϕ' extends to an L-coloring of G^{large}_Q which uses two different colors on p_1, x_4 and two different colors on x_2, x_4, contradicting Claim 4.3.12.

Case 2: $u' = z'$

In this case let $R^\dagger := x_1uz'x_4$. Let ϕ' be the restriction of ϕ to $V(G^{\text{small}}_{R^\dagger}$. If $z' \in n(x_2)$ then G contains a K_4 with vertices $\{x_2, x_3, z', z\}$, which is false, and if $u \in N(x_2)$, then G contains a $K_{2,3}$ with bipartition $\{u, z, x_3\}, \{x_2, z'\}$, which is false. Thus, since Q is induced in G, we have $N(x_2) \cap \text{dom}(\phi') = \{x_1\}$, so we simply choose a color $d \in L(x_2) \setminus \{q_3\}$ with $d \neq \phi(x_4)$. Since $N(x_3) \cap \text{dom}(\phi') = \{x_4\}$, there is an extension of ϕ' to dom$(\phi') \cup \{x_2, x_3\}$ in which x_2 is colored with d. Finally, the resulting L-coloring ϕ^* of dom$(\phi') \cup \{x_2, x_3\}$ extends to $G^{\text{small}}_{R^\dagger} \setminus G^{\text{small}}_Q$, since the 5-cycle $F := x_1uz'zx_2$ is properly colored by ϕ^*, and there is no vertex of $G^{\text{small}}_{R^\dagger} \setminus G^{\text{small}}_Q$ adjacent all five vertices of F, or else, since x_3 is adjacent to x_2, z, z', G contains a copy of $K_{2,3}$. Thus ϕ^* extends to L-coloring the rest of G^{large}_Q, so we have constructed an L-coloring of G^{large}_Q which uses two different colors on p_1, x_4 and two different colors on x_2, x_4, contradicting Claim 4.3.12.

We conclude that there is no defective 3-chord of C. Now let $Q := x_1x_2x_3x_4$ be any 3-chord of C, where $x_1 \in V(\mathring{\mathbf{P}})$

and $x_4 \in V(C \setminus \mathring{\mathbf{P}})$. Without loss of generality, let $x_1 \in \{p_2, p_3\}$ and suppose that $V(G_Q^{\text{small}} \setminus C) \neq \{x_2, x_3\}$ and $V(G_C^{\text{small}}) \subseteq B_1(C)$. Since Q is not defective and $V(G_Q^{\text{small}} \setminus C)$ is contained in $V(C^1)$, there is a lone vertex $w \in D_1(C)$ such that $G_Q^{\text{small}} \setminus C$ consists of the triangle x_2x_3w. Now, we have $p_1 \notin N(x_2)$, or else, since G is short-separation-free, the 3-chord $p_1x_2x_3x_4$ separates w from G_Q^{large}. Since one end of this 3-chord is an endpont of P, and the other does not lie in $\mathring{\mathbf{P}}$ this contradicts Theorem 3.0.2.

Claim 4.3.16. *Q is an induced subgraph of G*

Proof: If not, then since $|V(C)| > 3$ and C is induced in G, G contains one of the edges x_1x_3 or x_2x_4. If either of these edges lies in $E(G_Q^{\text{small}}) \setminus E(Q)$, then G contains a triangle separating w from $G_Q^{\text{large}} \setminus Q$, which is false. Thus, one of these edges lies in $E(G_Q^{\text{large}} \setminus E(Q)$, so there is a 2-chord of C separating w from $G_Q^{\text{large}} \setminus Q$, contradicting Theorem 3.0.2. ■

We claim now that $p_1 \notin N(x_3)$. Suppose that $p_1 \in N(x_3)$ (possibly $x_4 = p_1$). Then we have $x_1 = p_3$, or else the cycle $x_1p_1x_3x_2$ separates w from $G_Q^{\text{large}} \setminus Q$, contradicting short-separation-freeness. Furthermore, since w is the lone vertex of $G_Q^{\text{small}} \setminus (Q \cup C)$, we get that w is the lone vertex of $G^{\text{small}}_{x_1x_2x_3p_1} \setminus V(Q \cup C)$. Since Q and C are both induced in G and $p_1 \notin N(x_2)$, it follows from our triangulation conditions that w is adjacent to all five vertices of the cycle $x_1p_2p_1x_3x_2$, so w is adjacent to p_1, p_2, p_3, contradicting M1). Thus, $x_2, x_3 \notin N(p_1)$. Furthermore, by our triangulation conditions, since C and Q are induced in G, we have $w \in N(x_4)$ and $w \in N(x_1)$. If $x_1 = p_2$, then $G_Q^{\text{small}} + Q$ is a cycle where w is adjacent to each vertex of $G_Q^{\text{small}} + Q$, so we are done in that case. If $x_1 = p_3$, then, since C is induced in G and $N(x_2) \subseteq \{p_2, p_3\}$, we have $p_2 \in N(w)$ by our triangulation conditions and $N(x_2) \cap V(\mathbf{P}) = \{p_2, p_3\}$. Thus $G^{\text{small}}_{p_2x_2x_3x_4}$ is a wheel with central vertex w. This completes the proof of Proposition 4.3.4. □

4.4 Completing the Proof of Theorem 4.0.1

With Proposition 4.3.4 in hand, we prove the following, which is enough to complete the proof of Theorem 4.0.1 and thus complete Chapter 4. The result below is the lone result of Section 4.4.

Theorem 4.4.1. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let $C \in \mathcal{C}$ be an open $\mathcal{T}$-ring, and let $\mathbf{P} := \mathbf{P}_{\mathcal{T}}(C)$. Let $\mathbf{P} := p_1 \cdots p_m$, and let $Q := x_1x_2x_3x_4$ be a 3-chord of C with precisely one endpoint in $V(\mathring{\mathbf{P}})$. Then $V(G_Q^{\text{small}}) \subseteq B_1(C)$.*

Proof. Given a 3-chord $Q := x_1x_2x_3x_4$ of C, where $x_1 \in V(\mathring{\mathbf{P}})$ and $x_4 \in V(C \setminus \mathring{\mathbf{P}})$, we say that Q is *bad* if $V(G_Q^{\text{small}}) \not\subseteq B_1(C)$.

Claim 4.4.2. *For any bad 3-chord Q' of C, the following hold.*

1) *Q' is an induced subgraph of G; AND*
2) *x_1, x_4 do not have a common neighbor in $G_{Q'}^{\text{large}}$; AND*
3) *x_2 is not adjacent to either endpoint of $\mathbf{P}$, and x_3 is not adjacent to any vertex of $\mathbf{P}$, except possibly the lone vertex of $\{p_1, p_m\} \cap V(G_Q^{\text{small}})$; AND*
4) *$V(G_{Q'}^{\text{small}}) \setminus V(C \cup Q')| > 4$*

Proof: Let $S' := V(G^{\text{small}}_{Q'}) \setminus B_1(C, G)$. Since Q' is bad, we have $S' \neq \varnothing$. By Observation 4.3.1, suppose without loss of generality that $x_1 \in \{p_2, p_3\}$. Now suppose that Q' is not an induced subgraph of G. Then, since C is an induced subgraph of G, and neither of p_2, p_3 is an endpoint of $\mathbf{P}$ by Corollary 2.3.14, G contains one of the edges x_2x_4, x_1x_3. Thus, G contains a 2-chord of C with endpoints x_1, x_4. Thus, since x_1 is an internal vertex of C, we have $x_1 = p_2$ by 4) of Theorem 2.2.4. Consider the following cases:

Case 1: $x_2x_4 \in E(G)$

In this case, G contains the 2-chord $x_1x_2x_4$ of C. By Theorem 3.0.2, we have $V(G^{\text{small}}_{x_1x_2x_4}) \setminus V(C) = \{x_2\}$. If $x_2x_4 \in E(G^{\text{large}}_{Q'})$, then $G^{\text{small}}_{Q'} \subseteq G^{\text{small}}_{x_1x_2x_4}$, and thus, $x_3 \in V(G^{\text{small}}_{x_1x_2x_4}) \setminus V(C)$, which is false. On the other hand, if $x_2x_4 \in E(G^{\text{small}}_{Q'})$, then, since G is short-separation-free, we have $V(G^{\text{small}}_{Q} \setminus G^{\text{small}}_{x_1x_2x_4}) = \{x_3\}$, and thus $V(G) \setminus V(C) = \{x_2, x_3\}$, contradicting the fact that $S' \neq \varnothing$.

Case 2: $x_1x_3 \in E(G)$

In this case, G contains the 2-chord $x_1x_3x_4$ of C. By Theorem 3.0.2, we have $V(G^{\text{small}}_{x_1x_3x_4}) \setminus V(C) = \{x_3\}$. If $x_1x_3 \in E(G^{\text{large}}_{Q'})$, then $G^{\text{small}}_{Q} \subseteq G^{\text{small}}_{x_1x_3x_4}$, and thus, $x_2 \in V(G^{\text{small}}_{x_1x_3x_4}) \setminus V(C)$, which is false. On the other hand, if $x_1x_3 \in E(G^{\text{small}}_{Q})$, then, since G is short-separation-free, we have $V(G^{\text{small}}_{Q'} \setminus G^{\text{small}}_{x_1x_3x_4}) = \{x_3\}$, and thus $V(G) \setminus V(C) = \{x_2, x_3\}$, contradicting the fact that $S' \neq \varnothing$. We conclude that Q' is an induced subgraph of G, as desired.

Now suppose that x_1, x_4 have a common neighbor v^* in $G^{\text{large}}_{Q'}$. Since Q' is induced in G, we have $v^* \notin V(Q')$. Since $v^* \in V(G^{\text{large}}_{Q'})$, we have $S' \subseteq V(G^{\text{small}}_{x_1v^*x_4})$, contradicting Theorem 3.0.2. This proves 2). Now suppose toward a contradiction that x_2 has a neighbor among $\{p_1, p_m\}$. By M1), we have $p_1 \in N(x_2)$, since $|E(\mathbf{P})| > 3$ by Corollary 2.3.14. Likewise by M1), we have $x_1 = p_2$.

By Theorem 3.0.2, we have $S' \cap V(G^{1}_{p_1x_2x_3x_4}) = \varnothing$, and thus the triangle $p_1x_2x_1\mathbf{P}p_1$ separates S from $G^{\text{large}}_{Q'}$ contradicting short-separation-freeness. Now suppose there is a vertex p of $\mathbf{P}$ adjacent to x_3. By Corollary 2.3.14, we have $E(\mathbf{P})| = \lfloor \frac{2N_{\text{mo}}}{3} \rfloor$ and $x_3 \notin N(p_m)$. Thus, if $p \in V(G^{\text{large}}_{Q}) \setminus V(Q)$, then p is an internal vertex of P and we have $S \subseteq G^{\text{small}}_{px_3x_4}$, contradicting Theorem 3.0.2. Thus, we have $p \in V(C^{\text{small}}_{Q'} \cap \mathbf{P})$. Suppose toward a contradiction that $p \neq p_1$. Then, since Q' is an induced subgraph of G, we have $x_1 = p_3$ and $p' = p_2$. Since $S \cap V(G^{\text{small}}_{px_3x_4}$, the cycle $x_1x_2x_3p$ separates S from G^{large}_{Q}, contradicting short-separation-freeness.

Now we prove 4). Let $T := V(G^{\text{small}}_{Q'} \setminus V(C \cup Q)$ and suppose toward a contradiction that $|T| \leq 4$. Note that $S' \subseteq T$.

Since $S' \neq \varnothing$, let $s \in S'$ Since each vertex of T has degree at least five, and s has no neighbors among $V(C)$, it follows that $|T| = 4$ and s is adjacent to x_2, x_3 and each vertex of $T \setminus \{s\}$, so let $T = \{s, s_1, s_2, s_3\}$. Note that G^{small}_{Q} contains a 5-cycle with vertices $\{x_2, x_3, s_1, s_2, s_3\}$, or else, by our triangulation conditions, s ha a neighbor among $V(G^{\text{small}}_{Q}) \setminus (T \cup \{x_2, x_3\}$, contradicting the fact that $s \notin B_1(C, G)$. Thus, suppose without loss of generality that G^{small}_{Q} contains the 5-cycle $x_2s_1s_2s_3x_3x_2$. This is an induced cycle of C, or else, since s is adjacent to each vertex of $x_2s_1s_2s_3x_3x_2$, G contains a copy of K_4. Thus, since s_2, s_3 are not adjacent to x_2, and $T = \{s_1, s_2, s_3, s\}$, we then get from our triangulation conditions that s_1 is the unique common neighbor of x_1, x_2 in G^{small}_{Q}, and likewise, s_3 is the unique common neighbor of x_3, x_4 in G^{small}_{Q}. Thus, G contains the 4-chord $Q^* := x_1s_1s_2s_3x_4$ of C.

If $s_2x_1 \in E(G)$ then G contains a $K_{2,3}$ with bipartition $\{x_2, s_1, s_2\}, \{s, x_1\}$. Thus, we have $s_2x_1 \notin E(G)$, and the same argument shows that $s_2x_4 \notin E(G)$. Furthermore, we have $s_1x_4 \notin E(G)$, or else. G contains a $K_{2,3}$ with bipartition $\{s_1, s_3, x_3\}, \{s, x_4\}$, The same argument shows that $s_3x_1 \notin E(G)$.

Thus, Q^* is an induced path in G, and, by assumption $G^{\text{small}}_{Q^*} \setminus \{s_1, s_2, s_3\}$ consists of the path $C \cap G^{\text{small}}_{Q}$. Thus, since

C is an induced subgraph of G, it follows from our triangulation conditions that the three vertices s_1, s_2, s_3 have a common neighbor on $C \cap G_Q^{\text{small}}$, and thus G contains a copy of $K_{2,3}$, which is false. This completes the proof of Claim 4.4.2. ■

Suppose toward a contradiction that a bad 3-chord Q of C exists, and, among all such 3-chords of C, we choose Q so that $|V(G_Q^{\text{small}})|$ is minimized. By Observation 4.4.1, we have $x_1 \in \{p_2, p_3, p_{m-2}, p_{m-3}\}$, so suppose without loss of generality that $x_1 \in \{p_2, p_3\}$ and $G_Q^{\text{small}} \cap \mathbf{P} = p_1\mathbf{P}x_1$. Let $G_Q^{\text{small}} \cap C = x_1\mathbf{P}p_1u_1 \cdots u_t$, where $u_t = x_4$ (possibly, $t = 0$ and $x_4 = p_1$). Let $S := V(G_Q^{\text{small}}) \setminus B_1(C, G)$. We now have the following:

Claim 4.4.3. $N(x_3) \cap \{u_1, \cdots, u_t\} = \varnothing$*, and furthermore, if* $x_4 \neq p_1$ *then* $N(x_3) \cap V(\mathbf{P}) = \varnothing$*, and if* $x_4 = p_1$ *then* $N(x_3) \cap V(\mathbf{P}) = \{x_4\}$.

Proof: Suppose toward a contradiction that $N(x_3) \cap V(\mathbf{P}) \neq \varnothing$. Then x_3 has a neighbor $p \in V(p_1\mathbf{P}x_1)$. By Claim 4.4.2, we have $p = p_1$. Since $S \cap V(G^1_{x_4x_3p_1})$ by Theorem 3.0.2, we have $S \subseteq G^{\text{small}}_{x_1x_2x_3p_1}$. But then, since $|V(G^{\text{small}}_{x_1x_2x_3p_1})| < |V(G_Q^{\text{small}})|$, the 3-chord $x_1x_3x_3p_1$ of C contradicts the minimality of Q.

Now suppose there is an $i \in \{1, \cdots, t\}$ with $u_i \in N(x_3)$. Then G contains the 2-chord $Q' := x_4x_3u_i$ of C and the 3-chord $Q'' := u_ix_3x_2x_1$ of C. By Lemma 3.1.1, we have $S \cap V(G^{\text{small}}_{Q'}) = \varnothing$, so $S \subseteq V(G^{\text{small}}_{Q''})$. Since $x_4 \notin V(G^{\text{small}}_{Q''})$ and $u_i \notin V(\mathring{\mathbf{P}})$, this contradicts the minimality of Q. ■

We have a similar claim for x_2:

Claim 4.4.4. $N(x_2) \cap V(p_1\mathbf{P}x_1) = \{x_1\}$ *and* $N(x_2) \cap \{u_1, \cdots, u_t\} = \varnothing$

Proof: Suppose toward a contradiction that there is a $p \in V(p_1\mathbf{P}x_1) \setminus \{x_1\}$ with $p \in N(x_2)$. By Claim 4.4.2, we have $p \neq p_1$. Thus, we have $x_1 = p_3$ and $p = p_2$. We have $S \subseteq G^{\text{small}}_{p_2x_2x_3x_4})$, or else the triangle $p_2x_2p_3$ separates S from G_Q^{large}, contradicting short-separation-freeness. Since $S \subseteq G^{\text{small}}_{p_2x_2x_3x_3})$ and $V(G^{\text{small}}_{p_2x_2x_3x_3}) = V(G_Q^{\text{small}}) \setminus \{p_3\}$, we contradict the minimality of Q. Thus, we have $N(x_2) \cap V(p_1\mathbf{P}x_1) = \{x_1\}$, as desired. Now suppose there is an $i \in \{1, \cdots, t\}$ with $u_i \in N(x_2)$. Then G contains the 2-chord $u_ix_2x_1$ of C, and since $x_1 \in \{p_2, p_3\}$, we have $x_1 = p_2$, or else we contradict Theorem 3.0.2. Thus, G contains the 2-chord $p_2x_2u_i$ of C, and, by Theorem 3.0.2, we have $S \cap V(G^{\text{small}}_{p_2x_2u_i}) = \varnothing$ and $x_2p_1 \in E(G)$, contradicting the fact that $N(x_2) \cap V(p_1\mathbf{P}x_1) = \{x_1\}$. ■

Now let $D := u_1 \cdots u_tx_3x_2x_1\mathbf{P}p_1$. By Claim 4.4.2, Q is induced in G. Combining this with Claim 4.4.3 and Claim 4.4.4, together with the fact that C is induced in G, we get that D is an induced cycle of G_Q^{small}. We also have the following:

Claim 4.4.5. $V(G_Q^{\text{large}} \cup \mathbf{P})$ *is L-colorable.*

Proof: By Claim 4.4.2, we have $|V(G_Q^{\text{small}}) \setminus V(Q \cup C)| > 4$, so Q satisfies the first condition of Proposition 4.3.2. Since Q is induced in G, Q also satisfies condition 3) of Proposition 4.3.2. Finally, by Claim 4.4.3 and Claim 4.4.4, Q also satisfies condition 4) of Proposition 4.3.2. If there does not exist a common neighbor of x_1, x_2, x_3 in $G_Q^{\text{large}} \setminus Q$, then Q satisfies all the conditions of Proposition 4.3.2, so $V(G_Q^{\text{large}} \cup \mathbf{P})$ is L-colorable, as desired. So we are done in that case.

Now suppose there is a $w \in V(G_Q^{\text{large}} \setminus Q)$ adjacent to x_1, x_2, x_3. Note that $w \notin V(C)$, or else since C is an induced subgraph of G, w is the unique neighbor of x_1 on the path $C \cap G_Q^{\text{large}}$, and thus we have $x_1 = p_1$ and $w = p_2$, which is false as $x_1 \in \{p_2, p_3\}$.

Thus, $w \notin V(C)$, so $Q^* := x_1wx_3x_4$ is a 3-chord of C with the same endpoints as Q. Since G is short-separation-free, we have $G_Q^{\text{small}} = G_{Q^*}^{\text{small}} \setminus \{x_2\}$. We claim that $V(G_{Q^*}^{\text{large}} \cup \mathbf{P})$ is L-colorable. We just need to check that Q^* satisfies all four conditions of Proposition 4.3.2. Firstly, x_1, w, x_3 do not have a common neighbor in $G_{Q^*}^{\text{large}} \setminus Q^*$, or else G contains a copy of $K_{2,3}$, so condition 2) of Proposition 4.3.2 is satisfied. Since $S \subseteq G_Q^{\text{small}} \subseteq G_{Q^*}^{\text{small}}$, Q^* is also a bad 3-chord of C, and thus, by Claim 4.4.2, Q^* is an induced subgraph of G. By Claim 4.4.2 we have $|V(G_{Q^*}^{\text{small}})| > 4$. As shown above, we have $N(x_3) \cap V(\mathbf{P} \setminus \{x_4\}) = \varnothing$, and since Q separates w from p_1, we have $p_1 \notin N(w)$. By Claim 4.4.2, $p_m \notin N(w)$, so Q^* satisfies all four conditions of of Proposition 4.3.2. Thus, $V(G_{Q^*}^{\text{large}} \cup \mathbf{P})$ admits an L-coloring ϕ, and since x_2 has precisely three neighbors in $V(G_{Q^*}^{\text{large}} \cup \mathbf{P})$, ϕ extends to an L-coloring of $V(G_Q^{\text{large}} \cup \mathbf{P})$, as desired. ■

Applying Claim 4.4.5, we fix an L-coloring ϕ of $V(G_Q^{\text{large}} \cup \mathbf{P})$ for the remainder of the proof of Theorem 4.4.1.

Claim 4.4.6. $V(G_Q^{\text{small}} \cap C) \setminus V(\mathbf{P})| > 1$

Proof: We first rule out the possibility that $x_4 = p_1$. Suppose toward a contradiction that $x_4 = p_1$. Then $D = x_1\mathbf{P}p_1x_3x_2$, and $|V(D)| \leq 5$. Thus, $|V(D)| = 5$, or else there is a 4-cycle separating S from $G_Q^{\text{large}} \setminus Q$.

Since $\mathcal{T}$ is critical, ϕ does not extend to $G_Q^{\text{small}} \setminus D$, so $G_Q^{\text{small}} \setminus D$. Thus, since G is short-separation-free, G_Q^{small} consists of a lone vertex adjacent to all five vertices of D, contradicting 4) of Claim 4.4.2. Now suppose toward a contradiction that $x_4 = u_1$. Then $D = x_1\mathbf{P}p_1x_4x_3x_2$ and $|V(D)| \leq 6$. Thus, $5 \leq |V(D)| \leq 6$, or else there is a 4-cycle separating S from $G_Q^{\text{large}} \setminus Q$. But then, since ϕ does not extend to an L-coloring of $G_Q^{\text{small}} \setminus D$, we get that $G_Q^{\text{small}} \setminus D$ either consists of a lone vertex adjacent to all the vertices of D, or two vertices, each with at least four neighbors on D. In either case, we contradict 4) of Claim 4.4.2. ■

Now we have the following:

Claim 4.4.7. *$N(p_2) \cap N(x_4) = \varnothing$, and in particular, $N(x_1) \cap N(x_4) = \varnothing$. Furthermore, there is no vertex of $G_Q^{\text{small}} \setminus D$ adjacent to both x_4 and $p_1\mathbf{P}x_1$.*

Proof: Suppose toward a contradiction that there is a $v^* \in N(p_2) \cap N(x_4)$. Since C is an induced subgraph of G and $|V(C)| > 3$, we have $v^* \in V(G \setminus C)$. By Claim 4.4.2, we have $v^* \notin V(Q)$ and if $v^* \in V(G_Q^{\text{large}} \setminus Q)$, then we have $p_2 = x_1$ and we contradict 2) of Claim 4.4.2. Thus, we have $v^* \in V(G_Q^{\text{small}} \setminus Q)$. By Theorem 3.0.2, $G_{p_2v^*x_4}^{\text{small}}$ consists of a broken wheel with principal path $p_2v^*x_4$. By Claim 4.4.6, we have $t > 1$. Now let a, b be two colors in $L(u_1) \setminus L(p_1)$ and, since $|L(v^*)| = 5$, let d be a color of $L(v^*) \setminus (\{a, b\} \cup L(p_1) \cup L(p_2))$. Let $G' := G_{p_2v^*x_4}^{\text{large}}$.

Let $\mathbf{P}' := p_m \cdots p_2v^*$ and let L' be a list-assignment for $V(G')$ where $L'(v^*) = \{d\}$ and otherwise $L' = L$. Let $C' := (C \cap G')$ and let C'_* be the outer face of $G_{p_2v^*x_4}^{\text{large}}$. Since $N(v^*) \cap V(\mathbf{P}) \subseteq \{p_1, p_2\}$, $\mathbf{P}'$ is an induced subgraph of G'. Thus, $\mathbf{P}'$ is L'-colorable, and $\mathcal{T}' := (G', (\mathcal{C} \setminus \{C\}) \cup \{C'\}, L', C'_*)$ is a tessellation, where C' is an open $\mathcal{T}'$-ring with precolored path $\mathbf{P}'$.

If $\mathcal{T}'$ is a mosaic, then G' admits an L-coloring ψ by the minimality of $\mathcal{T}$. Since $d \neq L(p_1)$, ϕ extends to an L-coloring ψ' of $V(G') \cup \{p_1\}$. By construction of L', at least one of $\psi'(p_1), d$ lies outside of $L(u_1)$, so, since $t > 1$, ψ' extends to color the broken wheel $G_{p_2v^*x_4}^{\text{small}}$, and thus G is L-colorable, contradicting the fact that $\mathcal{T}$ is critical.

Thus, $\mathcal{T}'$ is not a tessellation. Note that, since $p_3 \notin N(v^*)$ and Q separates v^* from each element of $\mathcal{C} \setminus \{C\}$, $\mathcal{T}'$ satisfies the distance conditions of Definition 2.1.6. Since $|E(\mathbf{P}')| = |E(\mathbf{P})|$, the only condition that $\mathcal{T}'$ violates is M1), and, in particular, Since C' is induced in G', there is a lone vertex $w \in V(G' \setminus C')$ adjacent to each of p_2, p_3, v^*, and so $x_1 = p_3$. Note that $v^* \notin \{x_2, x_3\}$ by Claim 4.4.4 and Claim 4.4.3. Thus, $w \in V(G_Q^{\text{small}} \setminus D)$, and G contains the

3-chord $Q'' := x_1wv^*x_4$ of C. Let $f \in L(w)\setminus(\{d\}\cup L(p_2)\cup L(p_3))$ and let L'' be a list-assignment for $G^{\text{large}}_{Q''}$, where $L''(v^*) = \{d\}$, $L''(w) = \{f\}$, and otherwise $L'' = L$. Let $\mathbf{P}'' := p_m \cdots p_3wv^*$. Let $C'' := (C \cap G^{\text{large}}_{Q} + x_1wv^*x_4$ and let C''_* be the outer face of $G^{\text{large}}_{Q''}$. Finally, let $\mathcal{T}'' := (G^{\text{large}}_{Q''}, (\mathcal{C}\setminus\{C\})\cup\{C''\}, L'', C''_*)$. Then C'' is an open $\mathcal{T}''$-ring with precolored path $\mathbf{P}''$, and since P'' is induced in $G^{\text{large}}_{Q''}$, we get that $\mathbf{P}''$ is L''-colorable by our construction of L''. Thus, $\mathcal{T}''$ is a tessellation. We claim now that $\mathcal{T}''$ is a mosaic. As above with $\mathcal{T}'$, if $\mathcal{T}''$ is not a mosaic, then condition M1) is violated, and, in particular, there is a vertex z of $G^{\text{large}}_{Q''}\setminus C''$ adjacent to both x_1 and v^*. But then G contains a $K_{2,3}$ with bipartition $\{z, p_2, w\}, \{x_1, v^*\}$, contradicting short-separation-freeness.

Thus, $\mathcal{T}''$ is indeed a tessellation, and $|VG^{\text{large}}_{Q''})| < |V(G)|$, so $G^{\text{large}}_{Q''}$ admits an L''-coloring ψ''. By Claim 4.4.6, we have $x_4p_1 \notin E(G)$, so ψ'' is a proper L-coloring of the subgraph of G induced by $G^{\text{large}}_{Q''}$, and, by our choice of lists for v^*, w, ψ'' extends to an L-coloring $\psi^\dagger$ of $V(G^{\text{large}}_{Q''})\cup\{p_1, p_2\}$. Since $\psi''(v^*) = d$ and $t > 1$, $\psi^\dagger$ also extends to the broken wheel $G^{\text{small}}_{p_2v^*x_4}$. Thus, since $\psi^\dagger$ does not extend to an L-coloring of G, the precoloring of the 6-cycle $D^* := x_1x_2x_3x_4v^*w$ with $\psi^\dagger$ does not extend to $G^{\text{small}}_{Q}\setminus(V(D)\cup\{v^*, w\})$.

Let $W \subseteq \mathbb{R}^2$ be the unique open set such that $\partial(W) = D^*$ and $W\cap V(C) = \varnothing$. Since D^* is a 6-cycle and $\psi^\dagger$ does not extend to L-coloring $W\cap V(G)$, the graph $G\cap W$ is either a lone vertex, an edge, or a triangle by Theorem 1.3.5. In each case, each vertex of $V(G)\cap W$ is adjacent to a subpath of D^* of length at least two, so each vertex in $V(G)\cap W$ has a neighbor in $\{x_1, p_2, x_4\}$, contradicting the fact that $S\subseteq V(G)\cap W$. We conclude that $N(x_4)\cap N(p_2) = \varnothing$, as desired.

Note that since $N(p_2)\cap N(x_4) = \varnothing$, we have $N(x_1)\cap N(x_4) = \varnothing$, or else we have $x_1 = p_3$. Yet since C is induced and $|V(C)| \geq 3$, we get that p_3, x_4 have a common neighbor in $G\setminus C$, contradicting 4) of Theorem 2.2.4. Finally, suppose toward a contradiction that there is a $v^* \in V(G^{\text{small}}_{Q}\setminus D)$ adjacent to both x_4 and $p_1\mathbf{P}x_1$. Since $N(p_2)\cap N(x_4) = \varnothing$, and $N(x_1)\cap N(x_4) = \varnothing$, we have $p_1 \in N(v^*)$ and G^{small}_{Q} contains the 3-chord $p_1v^*x_4$ of C, where $p_1v^*x_4 \in \mathcal{K}(C, \mathcal{T})$. By Theorem 3.0.2, $G^{\text{small}}_{p_1v^*x_4}\setminus\{v^*\}$ is the path $p_1u_1\cdots u_t$, and $t > 1$ by Claim 4.4.6. Let a, b be two colors in $L(u_1)\setminus L(p_1)$, and let $G^\dagger := G^{\text{large}}_{p_1v^*x_4}$ and let $L^\dagger$ be a list-assignment for $G^\dagger$, where $L^\dagger(v^*) = L(v)\setminus\{a, b\}$, and otherwise $L^\dagger = L$. Let $C^\dagger := (G\cap G^{\text{large}}_{Q}) + x_1\mathbf{P}p_1v^*x_4$, and let $C^\dagger_*$ be the outer face of $G^\dagger$. Let $\mathcal{T}^\dagger := (G^\dagger, (\mathcal{C}\setminus\{C\})\cup\{C^\dagger\}, L^\dagger, C^\dagger_*)$. Then $\mathcal{T}^\dagger$ is a tessellation, where $C^\dagger$ is an open $\mathcal{T}^\dagger$-ring which also has precolored path $\mathbf{P}$. We claim now that $\mathcal{T}^\dagger$ is a mosaic.

Firstly, since $N(p_2)\cap N(x_4) = \varnothing$ and $N(x_1)\cap N(x_4) = \varnothing$, we have $p_2, p_3 \notin N(v^*)$, so $C^\dagger$ is an induced cycle of $G^\dagger$. Thus, since $C^\dagger$ has the same precolored path as C, $\mathcal{T}^\dagger$ satisfies M0) and M1), and M2) is immediate. Furthermore, since v^* has a neighbor in $C\setminus\mathring{\mathbf{P}}$ and Q separates v^* from each element of $\mathcal{C}\setminus\{C\}$, $\mathcal{T}^\dagger$ satisfies the distance conditions of Definition 2.1.6

Thus, $\mathcal{T}^\dagger$ is a mosaic, and since $|V(G^\dagger)| < |V(G)|$, $G^\dagger$ admits an $L^\dagger$-coloring ψ by the minimality of $\mathcal{T}$. By our choice of $L^\dagger$, either $\psi(p_1)\notin L(u_1)$ or $\psi(v^*)\notin L(u_1)$. In either case, the coloring ψ of the principal path $p_1v^*x_4$ of $G^{\text{small}}_{p_1v^*x_4}$ extends an L-coloring of the entire broken wheel, so ψ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. ■

As a consequence of the above, we have the following:

Claim 4.4.8. *x_3, p_1 have no common neighbor in G.*

Proof: Suppose toward a contradiction that x_3, p_1 have a common neighbor z. Since $x_4 \neq p_1$ and Q is induced in G, it follows from Claim 4.4.3 that $N(x_3)\cap V(D) = \{x_4\}$, so $z \notin V(C)$. Now, G contains the 3-chord $p_1zx_3x_4$ of C, and, by Theorem 3.0.2, we have $G^{\text{small}}_{p_1zx_3x_4} \setminus \{z, x_3\} = p_1u_1 \cdots u_t$. Since $N(x_3) \cap V(D) = \{x_4\}$, it follows from our triangulation conditions that $z \in N(x_4)$, contradicting Claim 4.4.7. ■

We also have the following:

Claim 4.4.9. *For any $v^* \in V(G^{\text{small}}_Q \setminus Q)$, $G[N(v^*) \cap V(Q)]$ is a subpath of Q of length at most one. Furthermore, for any $v^* \in V(G^{\text{small}}_Q \setminus D)$, if $x_2 \in N(v^*)$, then v^* has no neighbors on the path $p_1u_1 \cdots u_t$.*

Proof: By Claim 4.4.7, we have $N(x_1) \cap N(x_4) = \varnothing$. Thus, if there is a $v^* \in V(G^{\text{small}}_Q \setminus Q)$ such that $G[N(v^*) \cap V(Q)]$ is a subpath of Q of length at most one, then, since Q is an induced subgraph of G and it follows from Observation 4.3.1 that $G[N(v^*) \cap V(Q)]$ is a subpath of Q of length precisely two. Consider the following cases.

Case 1: $N(v^*) \cap V(Q) = \{x_1, x_2, x_3\}$

In this case, let $Q' := x_1v^*x_3x_4$. Since G is short-separation-free, we have $V(G^{\text{small}}_{Q'} \setminus G^{\text{small}}_Q) = \{x_2\}$. Since v^* has a neighbor in C, we have $v^* \notin S$, and since G is short-separation-free, we have $S \subseteq G^{\text{small}}_{Q'}$, contradicting the minimality of Q.

Case 2: $N(v^*) \cap V(Q) = \{x_2, x_3, x_4\}$

In this case, let $Q'' := x_1x_2v^*x_4$. Since G is short-separation-free, we have $V(G^{\text{small}}_{Q''} \setminus G^{\text{small}}_Q) = \{x_3\}$. Again, since v^* has a neighbor in C, we have $v^* \notin S$, and since G is short-separation-free, we have $S \subseteq G^{\text{small}}_{Q'}$, contradicting the minimality of Q.

Now suppose toward a contradiction there is a $v^* \in V(G^{\text{small}}_Q \setminus D)$ and a $u \in \{p_1, u_1, \cdots, u_t\}$ such that v^* is adjacent to both x_2 and u. Then G contains the 3-chord $Q' := x_1x_2v^*u$ of C and the 4-chord $Q'' := uv^*x_2x_3x_4$ of C. Note that Q'' lies in $\mathcal{K}(C, \mathcal{T})$, and that $u \neq u_t$, or else we contradicting the fact that $G[N(v^*) \cap V(Q)]$ is a subpath of Q of length at most one. Furthermore, since $u \in V(C \setminus \mathring{\mathbf{P}})$, we have $S \cap V(G^{\text{small}}_{Q'}) = \varnothing$, or else we contradict the minimality of Q. But then, since $G^{\text{small}}_{Q''} = G^1_{Q''}$, we have $S \subseteq V(G^1_{Q''})$. Since $v^*u \in E(G)$, we have $v^* \notin S$, so $S \subseteq V(G^1_{Q''}) \setminus V(Q'')$, contradicting Lemma 4.2.1. ■

The above claims have the following simple consequence, which we use repeatedly:

Claim 4.4.10. *Let $Q^\dagger$ be a proper generalized chord of C with endpoints p_1, x_4, where $Q^\dagger \subseteq G^{\text{small}}_Q$, and suppose that $x_2, x_3 \notin V(Q^\dagger)$. Then ϕ extends to an L-coloring of $G^{\text{small}}_{Q^\dagger}$.*

Proof: Let $G^{\text{small}}_{Q^\dagger} \setminus \{p_1, x_4\}$ and let F be the unique facial subgraph of G^* containing the path $u_1 \cdots u_{t-1}$ Since $t > 1$ by Claim 4.4.6, this is well defined. Suppose there is a vertex $u \in V(F) \setminus \{u_1, \cdots, u_{t-1}\}$ with at least three neighbors in dom(ϕ). Then $u \in V(Q^\dagger \setminus \{p_1x_4\})$. If u is adjacent to p_1, then $p_3 \notin N(u)$ by M1), and $x_2 \notin N(u)$ by Claim 4.4.9. Furthermore, $x_4 \notin N(u)$ by Claim 4.4.7. Since u has at least three neighbors on dom(ϕ), we have $N(u) \cap \text{dom}(\phi) = \{x_3, p_1, p_2\}$. But then G contains the 3-chord $p_1zx_3x_4$ of C, and since $N(x_3) \cap \{p_1, u_1, \cdots, u_t\} = \{u_t\}$, it follows from Theorem 3.0.2 and our triangulation conditions that $x_4 \in N(z)$, which is false.

Thus, every vertex of $F \setminus \{u_1, \cdots, u_{t-1}\}$ has an L_ϕ-list of size at least three. Furthermore, for each $u \in V(G^* \setminus F)$, we have $|L_\phi(u)| \geq 5$. If $t = 2$, then $|L_\phi(u_1)| \geq 1$, and $|L_\phi(u)| \geq 3$ for all $u \in V(F) \setminus \{u_1\}$, so G^* is L_ϕ-colorable by Theorem 0.2.3. If $t > 2$, then $|L_\phi(u_1)| \geq 2$, $|L_\phi(u_{t-1})| \geq 2$, and $|L_\phi(u)| \geq 3$ for all $u \in V(F) \setminus \{u_1, u_{t-1}\}$. Thus, by Theorem 1.3.4, G^* is L_ϕ-colorable. In either case, we are done. ■

Let U be the set of vertices in $V(G_Q^{\text{small}} \setminus D)$ with at least three neighbors among $V(Q \cup x_1\mathbf{P}p_1)$, and let $p' \in \{p_1, p_2\}$ be the lone neighbor of x_1 in $x_1\mathbf{P}p_1$. Now we have the following:

Claim 4.4.11. *There exist a $v \in V(G_Q^{\text{small}} \setminus D)$ such that $\{v\} = U$ and $N(v) \cap V(D) = \{x_1, x_2, p'\}$.*

Proof: Suppose toward a contradiction that $U = \varnothing$, and let F be the lone facial subgraph of $G_Q^{\text{small}} \setminus (V(Q) \cup V(x_1\mathbf{P}p_1))$ containing all vertices of $G_Q^{\text{small}} \setminus (V(Q) \cup V(x_1\mathbf{P}p_1))$ with $L_{\phi'}$-lists of size less than 3. Since C is an induced cycle of G and $(N(x_2) \cup N(x_3)) \cap \{u_1, \cdots, u_{t-1}\} = \varnothing$, we have the following: If $t = 1$, then $|L_{\phi'}(u_1)| \geq 1$ and $|L_{\phi'}(z)| \geq 3$ for all $z \in V(F) \setminus \{u_1\}$. On the other hand, if $t > 1$, then $|L_{\phi'}(u_1)| \geq 2$ and $|L_{\phi'}(u_{t-1})| \geq 2$, and $|L_{\phi'}(z)| \geq 3$ for all $z \in V(F) \setminus \{u_1, u_{t-1}\}$. In either case, ϕ' extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. Thus, $U \neq \varnothing$ so let $v \in U$. By M2), we have $G[N(v) \cap V(x_1\mathbf{P}p_1)]$ is a subpath of P of length at most one, and by Claim 4.4.9, we get that $G[N(v) \cap V(Q)]$ is a subpath of Q of length at most one, so it just suffices to check that $x_1 \in N(v)$. Then v is the unique vertex of G_Q^{small} adjacent to each of x_1, x_2, p'. Suppose toward a contradiction that $x_1 \notin N(v)$. By Claim 4.4.7, we have $x_4 \notin N(v)$. Consider the following cases:

Case 1: $x_2 \in N(v)$

In this case, since $x_1 \notin N(v)$ by assumption and $N(v)$ has nonempty intersection with $v(p_1\mathbf{P}x_1)$, it follows from Claim 4.4.9 that $x_1 = p_3$ and $N(v) \cap V(p_1\mathbf{P}x_1) = \{p_2\}$, so G contains the 4-cycle $x_1p_2v^*x_2$. But then, since $x_2p_2 \notin E(G)$ by Claim 4.4.4 and G is short-separation-free, we have $v^*x_1 \in E(G)$ by our triangulation conditions. contradicting our assumption.

Case 2: $x_2 \notin N(v)$

In this case, $N(v) \cap V(Q) = \{x_3\}$, and thus, since v has three neighbors on $V(Q \cup x_1\mathbf{P}p_1)$, we have $x_1 = p_3$ and $N(v) \cap V(Q \cup x_1\mathbf{P}p_1) = \{x_3, p_2, p_1\}$, contradicting Claim 4.4.8. This completes the proof of Claim 4.4.11. ■

This also implies the following:

Claim 4.4.12. $x_1 = p_3$.

Proof: Suppose not. Then we have $x_1 = p_2$, and, applying Claim 4.4.11, there is a $v^* \in V(G_Q^{\text{small}} \setminus D)$ adjacent to p_1, p_2, x_2, so G contains the 4-chord $Q^\dagger := p_1v^*x_3x_3x_4$ of C, and $Q^\dagger \in \mathcal{K}(C, \mathcal{T})$. Since G is short-separation-free, we have $V(G_Q^{\text{small}}) \setminus V(G_{Q^\dagger}^{\text{small}}) = \{p_2\}$, so $S \subseteq V(G_{Q^\dagger}^{\text{small}})$. Since $v^* \notin S$, this contradicts Lemma 4.2.1. ■

Let $U = \{v^*\}$ and let $Q^\dagger := p_2v^*x_2x_3x_4$. For any extension of ϕ to an L-coloring ϕ' of $\text{dom}(\phi) \cup A$, we let $T(A, \phi')$ be the set $\{z \in V(G_Q^{\text{small}}) \setminus \text{dom}(\phi') : |L_{\phi'}(z)| < 3\}$.

Claim 4.4.13. *Let $A \subseteq V(G_Q^{\text{small}}) \setminus (V(Q) \cup \{p_1, p_2\})$ and suppose that each vertex of A either lies in D or has a neighbor in $V(Q) \cup \{p_1, p_2\}$. Let $B \subseteq A$ and let ϕ' be an extension of ϕ to $\text{dom}(\phi) \cup B$. Suppose that B and ϕ' satisfy the following additional conditions.*

1) *each vertex of $A \setminus B$ is $L_{\phi'}$-inert; AND*
2) *For each $j \in \{u_1, \cdots, u_{t-1}\}$, if $u_j \in A$, then A either contains the path $p_1u_1 \cdots u_j$, or the path $u_j \cdots u_t$ (possibly both); AND*
3) *For each $u \in \{u_1, \cdots, u_{t-1}\} \setminus A$, $N(u) \cap B \subseteq V(D)$.*

Then $T(B, \phi') \setminus A \neq \varnothing$.

Proof: Let A, B, ϕ' be as above and suppose toward a contradiction that $T(B, \phi') \subseteq A$. Since each vertex of A either lies in D or has a neighbor in $V(Q) \cup \{p_1, p_2\}$, the graph $G^* := G_Q^{\text{small}} \setminus (A \cup V(Q) \cup \{p_1, p_2\})$ has a unique facial subgraph F containing all the neighbors of $\text{dom}(\phi')$. Thus, every vertex of $G^* \setminus F$ has an $L_{\phi'}$-list of size five. Since $T(B, \phi') \subseteq A$, each vertex of $F \setminus \{u_1, \cdots, u_{t-1}\}$ has an $L_{\phi'}$-list of size at least three. Furthermore, by our conditions on A, there exist indices $1 \leq i \leq j \leq t-1$ such that $u_1 \cdots u_{t-1} \setminus A$ consists of the path $u_i \cdots u_j$. By our conditions on A, each internal vertex of $u_i \cdots u_j$ has no neighbors in B and thus an $L_{\phi'}$-list of size at least three. Again by our conditions on A, if $j > i$, then each of u_i, u_j has an $L_{\phi'}$-list of size at least two, and if $j = i$, then u_i has an $L_{\phi'}$-list of size at least one. In either case, G^* is $L_{\phi'}$-colorable, and since $A \setminus B$ is $L_{\phi'}$-inert, ϕ' extends to an L-coloring of all of G_Q^{small}, so G is L-colorable, which is false. ■

Now we have the following:

Claim 4.4.14. *Let ϕ' be any extension of ϕ to $\text{dom}(\phi) \cup \{v^*\}$. Then $1 \leq |T(v^*, \phi')| \leq 2$, and, for each $w \in T(v^*, \phi')$, either $N(w) \cap (V(Q) \cup \{p_1, p_2, v^*\}) = \{p_1, p_2, v^*\}$ or $N(w) \cap (V(Q) \cup \{p_1, p_2, v^*\}) = \{x_2, x_3 v^*\}$. Furthermore, if $N(w) \cap (V(Q) \cup \{p_1, p_2, v^*\}) = \{x_2, x_3 v^*\}$, then $N(w) \cap V(D) = \{x_2, x_3\}$.*

Proof: Let ϕ' be as above. By Claim 4.4.11, v^* has no neighbors in $V(D) \setminus \text{dom}(\phi)$. Thus, letting $A = B = \{v^*\}$, this choice of A, B, ϕ' satisfies the conditions of Claim 4.4.13. so we have $T(v^*, \phi') \neq \varnothing$. Thus, let $w \in T(\phi', A)$. By Claim 4.4.11, $w \notin T(\phi, \varnothing)$. Thus, since $N(p_3) \cap V(G_Q^{\text{small}}) = \{p_2, v^*, x_2\}$, w is adjacent to v^* and has precisely two neighbors among $\{p_1, p_2, x_2, x_3, x_4\}$. Consider the following cases:

Case 1: $p_1 \in N(w)$

In this case, G contains the 4-cycle $p_1 w v^* p_2$, and thus, since $p_1 \notin N(v^*)$, we have $p_2 \in N(w)$. Since w has precisely two neighbors in $\{p_1, p_2, x_2, x_3, x_4\}$, we have $N(w) \cap (V(Q) \cup \{p_1, p_2, v^*\}) = \{p_1, p_2, v^*\}$.

Case 2:$p_1 \notin N(w)$

In this case, we first claim that $p_2 \notin N(w)$. Suppose that $p_2 \in N(w)$. By Claim 4.4.7, we have $x_4 \notin N(w)$, so w has a neighbor among x_2, x_3. If $x_2 \in N(w)$, then G contains a $K_{2,3}$ with bipartition $\{p_3, v^*, w\}$, $\{p_2, x_2\}$, which is false. Thus, we have $N(w) \cap \{x_2, x_3\} = \{x_3\}$, and G contains the 4-cycle $v^* w x_3 x_2$. Since $x_3 \notin N(v^*)$, we have $x_2 \in N(w)$ by our triangulation conditions, contradicting the fact that $N(w) \cap \{x_2, x_3\} = \{x_3\}$.

Thus, $p_2 \notin N(w)$ as well, so w has precisely two neighbors among $\{x_2, x_3, x_4\}$. Thus, if $N(w) \cap \{x_2, x_3, x_4\} \neq \{x_2, x_3\}$, then, by Claim 4.4.9, we have $N(w) \cap \{x_2, x_3, x_4\} = \{x_3, x_4\}$, so G contains the 4-cycle $x_2 v^* w x_3$. Since $x_3 \notin N(v)$ we have $x_2 \in N(w)$ by our triangulation conditions, contradicting the fact that $N(w) \cap \{x_2, x_3, x_4\} = \{x_3, x_4\}$. Thus, we get $N(w) \cap V(\mathbf{P} \cup Q) = \{x_2, x_3\}$, as desired. Furthermore, since w is adjacent to x_2, we have $N(w) \cap V(D) = \{x_2, x_3\}$ by Claim 4.4.9. Finally, since G is short-separation-free, we have $1 \leq |T(v^*, \phi')| \leq 2$. This completes the proof of Claim 4.4.14. ■

Now we have the following critical claim:

Claim 4.4.15. *Suppose there is a vertex $z \in V(G_Q^{\text{small}} \setminus D)$ adjacent to p_1, p_2, v^*. Let u be the non-p_2 endpoint of $G[N(z) \cap V(C)]$. Then the following hold.*

1) *$u = p_1$. In particular, u, v^* have no common neighbor in G except for z; AND*

2) *z, x_4 have no common neighbor in G.*

Proof: Let $H := G[\{z\} \cup (N(z) \cap V(C)]$. Then H is a broken wheel with principal path p_2zu. Suppose toward a contradiction that $u \neq p_1$. Let $L(p_i) = \{q_i\}$ for each $i = 1, 2$. Let $H' := H \setminus \{p_2\}$. Since $u \neq p_1$, H' is a broken wheel with principal path p_1zu.

Applying Corollary 1.4.6, there exist two colors $c_1, c_2 \in L(u)$ such that $\mathcal{Z}_{H'}(q_1, \bullet, c_1) \cap \mathcal{Z}_{H'}(\phi(p_1), \bullet, c_2) \neq \varnothing$, so let $d \in \mathcal{Z}_{H'}(q_1, \bullet, c_1) \cap \mathcal{Z}_{H'}(q_1, \bullet, c_2)$. Let $d_1 \in L(v^*) \setminus (L(p_3) \cup L(p_2) \cup \{d\})$. Let $G^\dagger := G \setminus (H \setminus \{u, z\})$ and let $L^\dagger$ be a list-assignment for $V(G^\dagger)$, where $L^\dagger(u) := \{c_1, c_2, d\}$, $L^\dagger(z) := \{d\}$ and $L^\dagger(v^*) := \{d_1\}$, and finally, $L^\dagger(a) := L(a)$ for all $a \in V(G^\dagger) \setminus \{v^*, z, u\}$.

Let $\mathbf{P}^\dagger := p_m\mathbf{P}p_3v^*z$ and let $C^\dagger := (C \cap G_Q^{\text{large}}) + x_1v^*zu \cdots u_t$. Let $C_*^\dagger$ be the outer face of $G^\dagger$ and let $\mathcal{T}^\dagger := (G^\dagger, (\mathcal{C} \setminus \{C\}) \cup \{C^\dagger, C_*^\dagger, L^\dagger)$. Then $\mathcal{T}^\dagger$ is a tessellation, where $C^\dagger$ is an open $\mathcal{T}^\dagger$-ring with precolored path $\mathbf{P}^\dagger$. We claim now that $\mathcal{T}$ is a mosaic. Since $|E(\mathbf{P}^\dagger)| = |E(\mathbf{P})|$, M0) is satisfied. By Claim 4.4.4, v^* has no neighbors in $C^\dagger$ except for x_1, z, and furthermore, $x_2z \notin E(G)$, or else G. Thus, $C^\dagger$ is induced in $G^\dagger$, and furthermore, since $N(x_2) \cap V(\mathbf{P}) = \{p_3\}$, we have $N(x_2) \cap V(\mathbf{P}^\dagger) = \{p_3, v^*\}$, and any other vertex of $G_Q^{\text{small}} \setminus D$ is adjacent to at most $\{v^*, z\}$ among the vertices of $\mathbf{P}^\dagger$, so $\mathcal{T}^\dagger$ satisfies M1), and M2) is immediate.

To see that the distance conditions of Definition 2.1.6 hold, just note that, in G, each vertex of Q is of distance at most two from $C \setminus \mathring{\mathbf{P}}$, and z is of distance at least two from each vertex of Q. Thus, since $G^\dagger \subseteq G$ and $C^\dagger \setminus (V(\mathring{\mathbf{P}}) \cup \{v^*, z\}) \subseteq C \setminus \mathring{\mathbf{P}}$, $\mathcal{T}^\dagger$ satisfies the distance conditions of Definition 2.1.6 as well. Thus, $\mathcal{T}^\dagger$ is a mosaic, and since $|V(G^\dagger)| < |V(G)|$, $G^\dagger$ admits an $L^\dagger$-coloring ψ by the minimality of $\mathcal{T}$. Since $\psi(z) = d$, we have $\psi(u) \in \{c_1, c_2\}$, so ψ is an L-coloring of $V(G^\dagger)$. Furthermore, by our choice of c_1, c_2, d, ψ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical.

Thus, we have $N(z) \cap V(C) = \{p_1, p_2\}$, as desired. In particular, since $u = p_1$, we get that u, v^* have no common neighbor in G except for v^*, or else G contains a copy of $K_{2,3}$. This proves 1) of Claim 4.4.15.

Now we prove 2). Suppose toward a contradiction that z, x_4 have a common neighbor z'. We have $z' \notin V(\mathbf{P})$ by Claim 4.4.7, and since z has no neighbors in $u_1, \cdots, u_t$ by 1), we have $z' \in V(G_Q^{\text{small}} \setminus C)$. But then G contains the 3-chord $p_1zz'x_4$ of C, and, by Theorem 3.0.2, we have $V(G_{p_1zz'x_4}^{\text{small}}) \setminus \{z, z'\} = \{p_1, u_1, \cdots, u_t\}$. Since z has no neighbors in $u_1, \cdots, u_t$, it follows from our triangulation conditions that $p_1 \in N(z')$, which is false. This proves 2) of Claim 4.4.15. ■

With the above in hand, we prove the following:

Claim 4.4.16. *There exists a vertex $w \in N(v^*) \cap V(G_Q^{\text{small}} \setminus D)$ such that $N(w) \cap V(D) = \{x_2, x_3\}$.*

Proof: Suppose toward a contradiction that there is no vertex $w \in N(v^*) \cap V(G_Q^{\text{small}} \setminus D)$ satisfying the claim. By Claim 4.4.14, there is a vertex $z \in N(v^*) \cap V(G_Q^{\text{small}} \setminus D)$ such that $N(z) \cap (V(Q) \cup \{p_1, p_2\}) = \{p_1, p_2\}$.

Now set $A^* = B^* = \{z, v^*\}$ and let $\psi \in \Phi(\phi, A^*)$. Applying 1) of Claim 4.4.15, since z has no neighbors in $\{u_1, \cdots, u_t\}$, this choice of A^*, B^*, ψ satisfies the conditions of Claim 4.4.13, so there is a vertex $y \in V(G_Q^{\text{small}}) \setminus (V(D) \cup A)$ with at least three neighbors in dom(ψ). Since the path $x_2v^*zp_1$ separates y from $\{x_1, p_2\}$, y has at least three neighbors among $\{x_2, x_3, x_4, v^*, z, p_1\}$.

Subclaim 4.4.17. $p_1, z \notin N(y)$

Proof: Suppose that $p_1 \in N(y)$. In that case, by Claim 4.4.9, we have $x_2 \notin N(y)$ by Claim 4.4.9. Furthermore, we have $x_3 \notin N(y)$ by Claim 4.4.8, and $x_4 \notin N(y)$ by Claim 4.4.7, so dom$(\psi) \cap N(y) = \{v^*, z, p_1\}$, contradicting 1) of Claim 4.4.15.

Thus, we have $p_1 \notin N(y)$. Now suppose toward a contradiction that $z \in N(y)$. In that case, by 2) of Claim 4.4.15, we have $x_4 \notin N(y)$, so $N(y) \cap \text{dom}(\psi)$ consists of at least three vertices of $\{z, v^*, x_2, x_3\}$. By our triangulation conditions, since $zv^*x_2x_3$ is an induced subpath of G, $G[N(y) \cap \text{dom}(\psi)]$ is a subpath of $zv^*x_2x_3$ of length either two or three. If this subpath of $zv^*x_2x_3$ has x_3 as an endpoint, then $\{v^*, x_2, x_3\} \subseteq N(y)$, and, by Claim 4.4.9, we have $N(y) \cap V(D) = \{x_2, x_3\}$, contradicting our assumption. The only remaining possibility is that $N(y) \cap \text{dom}(\psi) = \{z, v^*, x_2\}$. Since $|L_\phi(v^*)| = 2$ and $|L_\phi(z)| \geq 3$, let $c \in L_\phi(z) \setminus L_\phi(v^*)$ and let ψ be an extension of ϕ to $\text{dom}(\phi) \cup \{z\}$ obtained by coloring z with c. Let $A := \{v^*, z\}$ and $B := \{z\}$. Note that v^* is L_ψ-inert by our choice of c.

Since z has no neighbors in $u_1, \cdots, u_t$, this choice of A, B, ψ satisfies the conditions of Claim 4.4.13, so there is a $z' \in V(G_Q^{\text{small}} \setminus D) \setminus \{v^*, z\}$ with at least three neighbors in $V(Q) \cup \{p_1, p_2, z\}$. Since $x_3 \notin N(y)$, we have $|L_\psi(y)| \geq 3$, so $z' \neq y$.

Now, the path $p_1zyx_2x_3x_4$ separates z' from each vertex of $\text{dom}(\psi) \setminus \{p_1, z, x_2, x_3, x_4\}$, so z' has at least three neighbors among $\{z, x_2, x_3, x_4\}$. Suppose that $x_4 \in N(z')$. Then, by 2) of Claim 4.4.15, we have $z \notin N(z')$, and, by Claim 4.4.7, we have $p_1 \notin N(z')$, so z' is adjacent to all of x_2, x_3, x_4, contradicting Claim 4.4.9.

Thus, $x_4 \notin N(z')$, so z' is adjacent to each vertex of $\{z, x_2, x_3\}$.But then G contains a $K_{2,3}$ with bipartition $\{v^*, y, z'\}$, $\{x_2, z\}$, which is false. ■

Since $p_1, z \notin N(y)$, we get that $N(y) \cap \text{dom}(\psi)$ consists of at least three vertices of $\{x_2, x_3, x_4, v^*\}$. By our triangulation conditions, the graph $G[N(y) \cap \{x_2, x_3, x_4, v^*\}$ is a subpath of $v^*x_2x_3x_4$, and since, by Claim 4.4.9, x_2, x_4 are not both adjacent to v^*, we get that $N(y) \cap \text{dom}(\psi) = \{v^*, x_2, x_3\}$. But then, by Claim 4.4.9, we have $N(y) \cap V(D) = \{x_2, x_3\}$, contradicting our assumption. This completes the proof of Claim 4.4.16. ■

We fix a vertex $w \in V(G_Q^{\text{small}} \setminus D)$ satisfying Claim 4.4.16. Now we have the following:

Claim 4.4.18. *There is a $w^* \in N(w)$ such that $N(w) \cap (V(Q) \cup \{p_1, p_2\}) = \{x_3, x_4\}$.*

Proof: Suppose toward a contradiction that no vertex satisfying the Claim 4.4.18 exists. Consider the following cases:

Case 1 of Claim 4.4.18: There exists a vertex of $G_Q^{\text{small}} \setminus D$ adjacent to each of p_1, p_2, v^*

In this case, let z be the unique vertex of $G_Q^{\text{small}} \setminus D$ adjacent to each of p_1, p_2, v^*. We fix a color $c \in L_\phi(z) \setminus L_\phi(v^*)$ and we let ψ be an extension of ϕ to $\{z, v^*, w\}$. Let $A = B = \{z, v^*, w\}$. Then this choice of A, B, ψ satisfies the conditions of Claim 4.4.13, since z has no neighbor in $u_1, \cdots, u_t$ by 1) of Claim 4.4.15. Thus, there exists a vertex $z' \in V(G_Q^{\text{small}} \setminus D) \setminus A$ with at least three neighbors in $\text{dom}(\psi)$, so z' has at least three neighbors among $\{p_1, z, v^*, w, x_3, x_4\}$.

Subclaim 4.4.19. $x_3 \notin N(z')$.

Proof: Suppose that $x_3 \in N(z')$. Then $v^* \notin N(z')$, or else G contains a copy of $K_{2,3}$ with bipartition $\{x_2, w, z'\}$, $\{x_3, v^*\}$. If $w \in N(z')$ as well, then $x_4 \notin N(z')$ by assumption, so $\{x_3, w, z\} \subseteq N(z')$. But then G contains the 4-cycle $v^*zz'w$, and since $v^* \notin N(z')$, we have $wz \in E(G)$ by our triangulation conditions, which is false. Thus, $w \notin N(z')$, so z' has adjacent to x_3 and at least two of $\{x_4, p_1, z\}$. By Claim 4.4.7, we have $\{x_4, p_1\} \not\subseteq N(z')$. By 2) of Claim 4.4.15, we have $\{x_4, z'\} \not\subseteq N(z)$. The only possibility left is that z' is adjacent to each of x_3, z, p_1, and thus G contains the 3-chord $p_1z'x_3x_4$ of C. By Theorem 3.0.2, since $N(x_3) \cap \{p_1, u_1, \cdots, u_t\} = \{u_t\}$, we get that $x_4 \in N(z')$, which is false. ■

Applying Subclaim 4.4.19, $x_3 \notin N(z')$, and z' has at least three neighbors among $\{p_1, z, w, v^*, x_4\}$.

Subclaim 4.4.20. $x_4 \notin N(z')$.

Proof: Suppose toward a contradiction that $x_4 \in N(z')$. Then $w \notin N(z')$, or else G contains the 4-cycle $x_3x_4z'w$, and thus $x_3z' \in E(G)$ by our triangulation conditions, which is false. Furthermore, $p_1 \notin N(x_4)$ by Claim 4.4.7, so z' is adjacent to each of v^*, z, x_4, contradicting 2) of Claim 4.4.15 . ■

Since $x_3, x_4 \notin N(z')$ by the two subclaims above, z' has at least three neighbors among $\{p_1, z, v^*, w\}$. Suppose that $p_1 \in N(z')$. Then $v^* \notin N(z')$, or else we contradict 1) of Claim 4.4.15, so $N(z') \cap \text{dom}(\psi) = \{w, z, p_1\}$. But then G contains the 4-cycle wv^*zz', and $v^* \in N(z')$ by our triangulation conditions, which is false.

Thus, $p_1 \notin N(z')$, so $N(z') \cap \text{dom}(\psi) = \{z, v^*, w\}$. Thus, since G is $K_{2,3}$-free, we conclude that, for any $\psi \in \Phi(\phi, A)$, we have $T(A, \psi) = \{z'\}$ and $N(z') \cap \text{dom}(\psi) = \{z, v^*, w\}$. Now, if $L_\phi(w) \cap L_\psi(z) \neq \varnothing$, then we choose a color $d \in L_\phi(w) \cap L_\phi(z)$, and since $|L_\phi(v^*) \setminus \{d\}| \geq 1$, there is an extension of ϕ to an L-coloring ψ of $\text{dom}(\phi) \cup \{w, v^*\}$ in which $\psi(w) = \psi(z) = d$. But then $|L_\psi(z)| \geq 3$, so $z' \notin T(A, \psi)$, a contradiction.

Thus, we have $L_\psi(w) \cap L_\psi(z) = \varnothing$, so $|L_\phi(w) \cup L_\phi(z)| \geq 6$. Since $|L(y)| = 5$ and $|L_\psi(v^*)| \geq 2$, there is an extension of ϕ to an L-coloring ψ of $\text{dom}(\phi) \cup \{w, v^*\}$ in which $|L_{\psi'}(z)| \geq 3$, contradicting the fact that $T(A, \psi) = \{z'\}$. This completes Case 1 of Claim 4.4.18.

Case 2 of Claim 4.4.18: There does not exist a vertex of $G_Q^{\text{small}} \setminus D$ adjacent to each of p_1, p_2, v^*

In this case, we first note the following:

Subclaim 4.4.21. *w and p_2 have no common neighbor in $G_Q^{\text{small}} \setminus D$ other than v^*.*

Proof: Suppose toward a contradiction that there is a $z \in V(G_Q^{\text{small}} \setminus D)$ other than v^* which is adjacent to each of p_2, w. Then G_Q^{small} contains the 5-cycle $K := x_1p_2zwx_2$, and v^* is adjacent to each of p_2, x_1, x_2, w. Since G is short-separation-free, we get from our triangulation conditions that v^* is also adjacent to w^*, i.e $G[V(K) \cup \{v^*\}]$ is a wheel with central vertex v^*. Now set $A := \{v^*, w\}$ and $B := \{w\}$. Since $|L_\phi(w)| \geq 3$ and $L_\phi(v^*)| = \{a, b\}$, let $c \in L_\phi(w) \setminus \{a, b\}$ and let ϕ' be an extension of ϕ to $\text{dom}(\phi) \cup \{w\}$ with $\phi'(w) = c$. Then z is $L_{\phi'}$-inert, and since w has no neighbors on the path $p_1u_1 \cdots u_t$, this choice of A, B, ϕ' satisfies the conditions of Claim 4.4.13, so there is a vertex $w^* \in V(G_Q^{\text{small}} \setminus (V(D) \cup \{v^*, w\})$ with at least three neighbors in $\text{dom}(\phi')$. Since the path p_2zwx_3 separates w^* from each of x_1, x_2, w^* has at least three neighbors among $\{p_1, p_2, w, x_3, x_4\}$. Since $w^* \neq v^*$, we have by Claim 4.4.11 that $w^*w \in E(G)$ and w^* has precisely two neighbors among $\{p_1, p_2, x_3, x_4\}$.

If $x_4 \in N(w^*)$, then, by Claim 4.4.7, we have $p_1, p_2 \notin N(w^*)$ so $N(w^*) \cap (V(Q) \cup \{p_1, p_2\}) = \{x_3, x_4\}$, contradicting our assumption. Thus, $x_4 \notin N(w^*)$. If $p_2 \in N(x_3)$, then G contains a $K_{2,3}$ with bipartition $\{v^*, z, w^*\}$, $\{w, p_2\}$, so we get that $N(w^*) \cap \{p_1, p_2, x_3, x_4\} = \{p_1, x_3\}$, and G contains the 3-chord $Q^\dagger := p_1w^*x_3x_4$ of C. By Theorem 3.0.2, we have $V(G_{Q^\dagger}^{\text{small}}) = \{p_1, u_1, \cdots, u_t\} \cup \{x_3, w^*\}$. Since x_3 has no neighbors on $p_1, \cdots, u_{t-1}$ by Claim 4.4.3, we have $x_4 \in N(w^*)$ by our triangulation conditions, which is false. ■

Now we return to Case 2 of Claim 4.4.18. Let $\phi'' \in \Phi(\phi, \{v^*, w\})$. Setting $A = B = \{v^*, w\}$, the choice of $A, B\phi''$ satisfies the conditions of Clam 4.4.13, so there is a vertex $w^* \in V(G_Q^{\text{small}}) \setminus (V(D) \cup A)$ with at least three neighbors in $\text{dom}(\phi'')$. We claim now that $N(w^*) \cap (V(Q) \cup \{p_1, p_2, v^*\}) = \{x_3, x_4\}$.

Note that w^* has at most two neighbors among $\text{dom}(\phi) \cup \{v^*\}$, or else we contradict Claim 4.4.14. Thus, w^* is adjacent to w and has precisely two neighbors among $V(Q) \cup \{p_1, p_2, v^*\}$. Furthermore, by Subclaim 4.4.21, we have $p_2 \notin N(w^*)$. Since $p_2v^*wx_3$ separates p_2, p_3 from w^*, w^* has precisely two neighbors among $\{p_1, v^*, x_3, x_4\}$.

Suppose toward a contradiction that $v^* \in N(w^*)$. In that case, we have $p_1 \notin N(w^*)$, or else G contains the 4-cycle $p_1 w^* v^* p_2$, and since $p_1 \notin N(v^*)$, we have $p_2 \in N(w^*)$ by our triangulation conditions, which is false. Furthermore, if $x_3 \in N(w^*)$, then G contains a $K_{2,3}$ with bipartition $\{x_2, w, w^*\}$, $\{v^*, x_3\}$, contradicting short-separation-freeness. Thus, we have $N(W^*) \cap \{p_1, v^*, x_3, x_4\} = \{v^*, x_4\}$, so G contains the 4-cycle $wx_3x_4w^*$. Since $x_4 \notin N(w)$, we have $x_3 \in N(w^*)$ by our triangulation conditions, so we have a contradiction.

Thus, our assumption that $v^* \in N(w^*)$ is false, so $N(w^*) \cap (\{p_1, p_2\} \cup V(Q))$ consists of precisely two vertices of $\{p_1, x_3, x_4\}$. By Claim 4.4.7, this set of two vertices is either $\{p_1, x_3\}$ or $\{x_3, x_4\}$ .Suppose that $N(w^*) \cap (\{p_1, p_2\} \cup V(Q)) = \{p_1, x_3\}$. Then G contains the 3-chord $Q' := p_1 w^* x_3 x_4$ of C. By Theorem 3.0.2, we have $G^{\text{small}}_{Q'} = \{p_1, u_1, \cdots, u_t\} \cup \{x_3, w^*\}$. Since $u_{t-1} \notin N(x_3)$ by Claim 4.4.3, and $x_4 \notin N(w)$, we have $x_4 \in N(w^*)$ by our triangulation conditions, contradicting our assumption that $N(w^*) \cap (\{p_1, p_2\} \cup V(Q)) = \{p_1, x_3\}$. We conclude that $N(w^*) \cap (\{p_1, p_2, v^*\} \cup V(Q)) = \{x_3, x_4\}$, as desired. This completes the proof of Claim 4.4.18. ■

Now let w^* be as in Claim 4.4.18 above. Then G contains the 4-chord $p_2 v^* w w^* x_4$ of C, which separates each vertex of p_3, x_2, x_3 from $G^{\text{small}}_Q \setminus (V(Q) \cup \{v*, w, w^*\})$. Let $H^\dagger$ be the subgraph of G induced by $\{w^*\} \cup (N(w^*) \cap V(C))$. Since $N(w^*) \cap V(P) = \varnothing$, there exists a $u^\dagger \in \{u_1, \cdots, u_{t-1}, u_t\}$ such that either $u^\dagger = u_t$ and $H^\dagger$ is the edge $w^* u^\dagger$, or $u^\dagger \in \{u_1, \cdots, u_{t-1}\}$ and $H^\dagger$ is a broken wheel with principal path $u_t w^* u^\dagger$.

Claim 4.4.22.

1) w^* has no common neighbor with either of p_1, p_2 in $G^{\text{small}}_Q \setminus D$; AND

2) $u^\dagger \neq u_1$; AND

3) $u^\dagger, p_2$ have no common neighbor in $G^{\text{small}}_Q \setminus D$; AND

4) w and $u^\dagger$ have no common neighbor in $G^{\text{small}}_Q \setminus D$.

Proof: Suppose toward a contradiction that there is a $z \in V(G^{\text{small}}_Q \setminus D)$ adjacent to each of p_2, w^*. Since $v^* \notin N(w^*)$, we have $z \neq v^*$, and G contains the 3-chord $R := p_2 z w^* x_4$ of C. Note that ϕ extends to an L-coloring of $\text{dom}(\phi) \cup V(G^{\text{small}}_R)$ by Claim 4.4.10.

Thus, let ψ be an extension of ϕ to an L-coloring of $\text{dom}(\phi) \cup V(G^{\text{small}}_R)$. Let $K := p_2 v^* w w^* z p_2$. Since $p_2 v^* w w^*$ is a chordless path, ψ extends to color w, v^*, so let $\psi' \in \Phi(\psi, \{w, v^*\})$. Let $W \subseteq \mathbb{R}^2$ be the unique open region such that $\partial(W) = K$ and $W \cap V(C) = \varnothing$. Since ψ L-colors $G \setminus W$ and ψ does not extend to an L-coloring of G, it follows from Theorem 1.3.5 that $W \cap V(G)$ consists of a lone vertex z' adjacent to all five vertices of K, and G contains the 3-chord $R^\dagger := x_4 w^* z' p_2$ with $R \subseteq G^{\text{small}}_{R^\dagger}$.

By the minimality of Q, we have $V(G^{\text{small}}_{R^\dagger}) \subseteq B_1(C)$, and since $z \in V(G^{\text{small}}_{R^\dagger}) \setminus V(C \cup R^\dagger)$, it follows from Proposition 4.3.4 that $G^{\text{small}}_{R^\dagger} \setminus C$ consists of the triangle $w^* z z'$, and $G^{\text{small}}_R \setminus \{w^*, z\} = p_2 u_1 \cdots u_t$. Since C is an induced cycle in G, it follows from Claim 4.4.7 that $u^\dagger \neq u_t$ and $N(z) \cap V(C) = \{p_2, p_1, u_1, \cdots, u^\dagger\}$, contradicting Claim 4.4.7.

Thus, our assumption that w^* and p_2 have a neighbor in $G^{\text{small}}_Q \setminus D$, is false. Now suppose toward a contradiction that w^*, p_1 have a common neighbor z in $V(G^{\text{small}}_Q \setminus D)$. Since $v^* \notin N(w^*)$, we have $z \neq v^*$, and G contains the 3-chord $R := p_1 z w^* x_4$ of C. Note that ϕ extends to an L-coloring of $\text{dom}(\phi) \cup V(G^{\text{small}}_R)$ by Claim 4.4.10.

Thus, let ψ be an extension of ϕ to an L-coloring of $\text{dom}(\phi) \cup V(G^{\text{small}}_R)$. Let $K := p_2 v^* w w^* z p_1 p_2$. Since $p_2 v^* w w^*$ is a chordless path, ψ extends to color w, v^*, so let $\psi' \in \Phi(\psi, \{w, v^*\})$. Let $W \subseteq \mathbb{R}^2$ be the unique open region such that $\partial(W) = K$ and $W \cap V(C) = \varnothing$. Since ψ L-colors $G \setminus W$ and ψ does not extend to an L-coloring of G, it follows from Theorem 1.3.5 that $|V(G) \cap W| \leq 3$, and each vertex of $V(G) \cap W$ is adjacent to a subpath of K of length at least two. Note that no vertex of W is adjacent to each of p_1, w^*, or else, if such a $y \in W \cap V(G)$ exists, then the 3-chord $p_1 y w^* x_4$ of C separates z from $G_Q^{\text{large}} \setminus Q$, contradicting Theorem 3.0.2. Thus, we have $|V(G) \cap W| > 1$, so consider the following cases:

Case 1: $|V(G) \cap W| = 2$

In this case, $G \cap W$ consists of an edge yy' in which each endpoint is adjacent to a subpath of K of length precisely three. Furthermore, as shown above, p_2, w^* have no common neighbor in $G_Q^{\text{small}} \setminus D$. Thus, we have $N(y) \cap V(K) = \{v^*, p_2, p_1, z\}$ and $N(y') \cap V(K) = \{z, w^*, w, v^*\}$. In that case, $G \setminus \text{dom}(\psi)$ consists of the graph in Figure 4.4.1, with lower bounds on the sizes of the L_ψ-lists labelled in red.

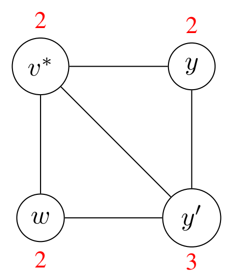

Figure 4.4.1: Case 1 of Fact 1

Thus, ψ extends to an L-coloring of G, contradicting the fact that G is not L-colorable.

Case 2: $|V(G) \cap W| = 3$

In this case, again applyin the fact that p_1, w^* have no common neighbor in W, the graph $G \cap W$ consists of a triangle $y_1 y_2 y_3$ such that $G[N(y_1) \cap V(K)] = \{p_2, p_1, z\}$, $G[N(y_2) \cap V(K)] = \{z, v^*, w\}$, and $G[N(y_3) \cap V(K)] = \{w, v^*, p_2\}$. Then $G \setminus \text{dom}(\psi)$ consists of the graph in Figure 4.4.2, with lower bounds on the L_ψ-lists labelled in red.

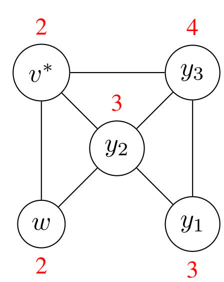

Figure 4.4.2: Case 2 of Fact 1

Thus, ψ extends to an L-coloring of G, contradicting the fact that G is not L-colorable. Thus, our assumption that w^* and p_1 have a neighbor in $G_Q^{\text{small}} \setminus D$, is false. This proves 1) of Claim 4.4.22.

Now we prove 2). Suppose that $u^\dagger = u_1$. Then G contains the 6-cycle $K := p_1 u_1 w^* w v^* p_2$. Let $W \subseteq \mathbb{R}^2$ be the unique open set such that $\partial(W) = K$ and $W \cap V(C) = \varnothing$. Let ψ be an extension of phi to $\text{dom}(\phi) \cup V(H^\dagger) \cup \{w, v^*\}$. To see that such a ψ exists, just note that, by Proposition 1.4.5, there are at least two colors c_1, c_2 in $\mathcal{Z}_{H^\dagger}(\psi(x_4), \bullet, c_i) \neq \varnothing$ for each $i = 1, 2$, so we choose $c_i \neq \phi(p_1)$ and let $d \in \mathcal{Z}_{H^\dagger}(\phi(x_4), \bullet, c_i)$. The resulting extension of ϕ to $V(H^\dagger)$ extends to the edge wv^*, so such a ψ does indeed exist, and ψ is an L-coloring of $G \setminus W$.

Since ψ does not extend to L-color G, it follows from Theorem 1.3.5 that $|V(G) \cap W| \leq 3$. By 1), p_2, w^* have no common neighbor in W, so $|V(G) \cap W| > 1$. Consider the following cases:

Case 1: $|V(G) \cap W| = 2$

In this case, since p_2, w^* have no common neighbor in W, and p_1, w^* have no common neighbor in W, we get that $G \cap W$ consists of an edge yy', where $G[N(y) \cap V(K)] = \{v^*, w, w^*, u_1\}$ and $G[N(y') \cap V(K)] = \{u_1, p_1, p_2, v^*\}$. But then the vertex y' contradicts 1) of Claim 4.4.15.

Case 2: $|V(G) \cap W| = 3$

In this case, again since p_1, w^* have no common neighbor in W, $G \cap W$ consists of a triangle $y_1y_2y_3$, where $G[N(y_1) \cap V(K)] = \{p_2, p_1, u_1\}$, $G[N(y_2) \cap V(K)] = \{u_1, w^*, w\}$, and $G[N(y_3) \cap V(K)] = \{w, v^*, p_2\}$. Now let ψ^* be the restriction of ψ to dom$(\psi) \setminus \{w, v^*\}$. Then $G \setminus$ dom(ψ^*) consists of the graph in Figure 4.4.3, where the lower bounds on the sizes of the L_{ψ^*}-lists are labelled in red.

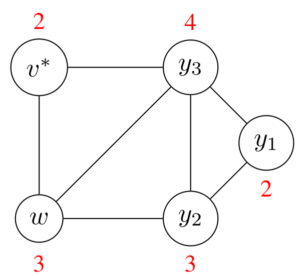

Figure 4.4.3: Case 2 of Fact 2

Thus, ψ^* extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. This completes the proof of 2) of Claim 4.4.22.

Now we prove 3). Suppose toward a contradiction that $u^\dagger$ and p_2 have a common neighbor z in $G_Q^{\text{small}} \setminus D$. By Claim 4.4.7, we have $u^\dagger \neq x_4$. Furthermore, G contains the 3-chord $p_2zu^\dagger$ of G. By Theorem 3.0.2, we have $V(G^{\text{small}}_{p_2zu^\dagger}) = \{z\} \cup \{p_2, p_1, u_1, \cdots, u^\dagger\}$, and z is adjacent to each of $p_2, p_1, u_1, \cdots, u^\dagger$. Since $u^\dagger \neq p_1$, this contradicts 1) of Claim 4.4.15. This proves 3).

Now we prove 4) of Claim 4.4.22. Suppose toward a contradiction that $w, u^\dagger$ have a common neighbor y in $G_Q^{\text{small}} \setminus D$. In that case, we have $u^\dagger \neq u_t$, or else G contains a $K_{2,3}$ with bipartition $\{x_3, w^*, y\}, \{u_t, w\}$, so $u^\dagger \in \{u_1, \cdots, u_{t-1}\}$ and $H^\dagger$ is a broken wheel. Furthermore, since G contains the 4-cycle $wyu^\dagger w^*$ and $u^\dagger \notin N(w)$, we have $w^* \in N(y)$ as well. By 1), no vertex of $\{p_1, p_2\}$ lies in $N(y)$.

By Corollary 1.4.6, there is a color $d \in L_\phi(u^\dagger)$ such that, for some pair of colors $c_1, c_2 \in L_\phi(w^*)$, we have $d \in \mathcal{Z}_{H^\dagger}(\phi(x_4), c_1, \bullet) \cap \mathcal{Z}_H(\phi(x_4), c_2, \bullet)$. This is permissible since, by 2), we have $u^\dagger \neq u_1$, so $|L_\phi(u^\dagger)| = 3$. We now extend ϕ to an L-coloring ϕ' of dom$(\phi) \cup \{u^\dagger, w, v^*\}$ by first coloring $u^\dagger$ with d and coloring w with a color in $L_\phi(w) \setminus \{c_1, c_2\}$, which is permissible as $|L_\phi(w)| \geq 3$. Finally, there is a color left over for v^* since $|L_\phi(v^*)| \geq 2$ and $u \notin N(v^*)$.

We now set $A = V(H^\dagger) \cup \{w, v^*\}$ and $B = \{v^*, w, u^\dagger\}$. We note that the choice A, B, ϕ' satisfies the conditions of Claim 4.4.13. To see this, just note that $V(H^\dagger) \setminus \{x_4, u^\dagger\}$ is $L_{\phi'}$-inert, since, after coloring y, there is at least one color of c_1, c_2 left over for w^*. Since the choice A, B, ϕ' satisfies the conditions of Claim 4.4.13, there exists a vertex z' with at least three neighbors in dom(ϕ').

Subclaim 4.4.23. *$N(y) \cap \mathrm{dom}(\phi') = \{w, u^\dagger\}$. In particular, $z' \neq y$.*

Proof: Firstly, by 1), we have $p_1, p_2 \notin N(y)$. Since the path $p_1p_2v^*ww^*u^\dagger$ separates y from $\mathrm{dom}(\phi') \setminus \{p_1, p_2, v^*, w, w^*, u^\dagger\}$, we have $N(y) \cap \mathrm{dom}(\phi) \subseteq \{w, u^\dagger, v^*\}$. Suppose toward a contradiction that $v^* \in N(y)$. Then G contains the 3-chord $Q^\dagger := p_2v^*yu^\dagger$ of C. By the minimality of Q, we have $V(G^{\text{small}}_{Q^\dagger}) \subseteq B_1(C)$. If $V(G^{\text{small}}_{Q^\dagger}) \setminus \{v^*, y\} = \{p_2, p_1, u_1, \cdots, u^\dagger\}$, then, since $p_1 \notin N(v^*)$, we have $p_1 \in N(y)$, which is false. Thus, by Proposition 4.3.4, there is a lone vertex q adjacent to every vertex in the cycle $p_1p_1u_1 \cdots u^\dagger yv^*$. Since $u^\dagger \neq p_1$, the vertex q contradicts 1) of Claim 4.4.15. Thus, $v^* \notin N(y)$, so we are done. ∎

Since $z' \neq y$, z' has at most one neighbor among $w, u^\dagger$, or else G contains a copy of $K_{2,3}$. If $u^\dagger \in N(z')$, then, by 3), we have $p_2 \notin N(z')$, so $N(z') \cap \{p_1, p_2, v^*, w, u\} = \{p_1, v^*, u^\dagger\}$. In that case, G contains the 4-cycle $p_1p_2v^*z'$. Thus, by our triangulation conditions, since $p_1 \notin N(v^*)$, we have $p_2 \in N(z')$, which is false. Thus, $u^\dagger \notin N(z')$, so $N(z') \cap \mathrm{dom}(\phi')$ consists of at least three vertices of $\{p_1, p_2, v^*, w\}$. Since $p_1p_2v^*w$ is a chordless subpath of G, it follows from our triangulation conditions that $G[N(z') \cap \mathrm{dom}(\phi')]$ is a subpath of $p_1p_2v^*w$ of length either two or three, and, in particular, since G is $K_{2,3}$-free, z' is the unique vertex of $T(B, \phi') \setminus A$. Consider the following cases:

Case 1: $\{p_1, p_2, v^*\} \subseteq N(z')$

In this case, we let $A^* := V(H^\dagger \setminus \{x_4\}) \cup \{w, v^*, z\}$ and $B^* := \{u^\dagger, w, v^*, z\}$. By 1) of 1) of Claim 4.4.15, z' has no neighbors in $u_1, u_2, \cdots, u_t$, so, for any $\psi \in \Phi(\phi', z')$, the choice A^*, B^*, ψ satisfies the conditions of Claim 4.4.13.

Subclaim 4.4.24. *There exists a vertex $q \in V(G^{\text{small}}_Q \setminus D) \setminus A^*$ such that the following hold:*

1) For any extension of ϕ' to an L-coloring $\psi \in \Phi(\phi', z')$, we have $T(B^*, \psi) \setminus A^* = \{q\}$; AND

2) $N(q) \cap B^* = \{z', v^*, w\}$; AND

3) q has no neighbors among $u_1, \cdots, u_t$.

Proof: Let $\psi \in \Phi(\phi', z')$. By Claim 4.4.13, we have $T(B^*, \psi) \setminus A^* \neq \varnothing$, so let $q \in T(B^*, \psi) \setminus A^*$. Since the path $p_1z'v^*wyu^\dagger$ separates q from $\mathrm{dom}(\psi) \setminus \{p_1, z', v^*, w, u^\dagger\}$, we have $N(q) \cap \mathrm{dom}(\psi) \subseteq \{p_1, z', v^*, w, u^\dagger\}$. Since z' is the unique vertex of $T(B, \phi') \setminus A$, q' is adjacent to z' and q has precisely two neighbors among $\{p_1, v^*, w, u^\dagger\}$. We claim now that $u^\dagger \notin N(q)$.

Suppose toward a contradiction that $u^\dagger \in N(q)$. In that case, G contains the 3-chord $p_1z'qu^\dagger$ of C, and, by Theorem 3.0.2, we have $G^{\text{small}}_{p_1z'qu^\dagger} \setminus \{z', q\} = p_1u_1 \cdots u^\dagger$. Since z' has no neighbors among $u_1, \cdots, u^\dagger$, we have $p_1 \in N(q)$ by our triangulation conditions as well, and $N(q) \cap V(C) = \{p_1, u_1, \cdots, u^\dagger\}$. Thus, G contains the broken wheel H^{mid} which has principal path $p_1qu^\dagger$, where $H^{\text{mid}} \setminus \{q\} = p_1u_1 \cdots u^\dagger$. Now, G contains the 6-cycle $K := ww^*u^\dagger qz'v^*$. Let $W \subseteq \mathbb{R}^2$ be the unique open region such that $\partial(W) = K$ and $W \cap V(C) = \varnothing$. Note that $y \in W$. Now let Ψ be an extension of ϕ to an L-coloring of $G \setminus W$. Consider the following cases.

Case 1 of Subclaim 4.4.24: $yq \in E(G)$

In this case, G contains the 5-cycle $K' := v^*z'qyw$. Let $W' \subseteq \mathbb{R}^2$ be the unique open region such that $\partial(W') = K'$ and $W' \subseteq W$. Extending Ψ to y, we have an L-coloring of $G \setminus W'$. Since G is not L-colorable, it follows from Theorem 1.3.5 that there is a lone vertex q' adjacent to all five vertices of K'. Now we uncolor w, v^*, z' (that is, restrict Ψ to an L-coloring Ψ' of $\mathrm{dom}(\Psi) \setminus \{v^*, w, z'\}$). Then $G \setminus \mathrm{dom}(\Psi')$ consists of the graph showin in Figure 4.4.4, with lower bounds on the $L_{\Psi'}$-lists shown in red.

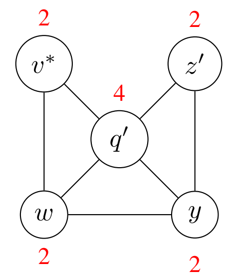

Figure 4.4.4: Case 1 of Subclaim 4.4.24

Thus, Ψ' extends to L-color G, contradicting the fact that $\mathcal{T}$ is critical.

Case 2 of Subclaim 4.4.24: $yq \notin E(G)$

In this case, we let $K' := v^*z'qu^\dagger uw$. Then $K' \subseteq \text{Cl}(W)$ and K' is a chordless cycle. Let $W' \subseteq \mathbb{R}^2$ be the unique open region such that $\partial(W') = K'$ and $W' \subseteq W$. Note that $w, y, u^\dagger$ have no common neighbor in W', since they are adjacent to w^* and G is $K_{2,3}$-free. Extending Ψ to y, we have an L-coloring of $G \setminus W'$. Thus, since $|V(G) \cap W| \leq 3$, we have $|V(G) \cap W'| \leq 2$, and each vertex of $V(G) \cap W'$ is adjacent to a subpath of K of length at least three. But then, since $w, y, u^\dagger$ have no common neighbor in W', there is a vertex $q' \in V(G) \cap W'$ adjacent to each of $u^\dagger, q, z'$, and thus G contains $K_{2,3}$ with bipartition $\{u^\dagger, q, z'\}$, $\{q', p_1\}$, contradicting short-separation-freeness.

Thus, $u^\dagger \notin N(q)$, as desired, so q' has precisely two neighbors among $\{p_1, v^*, w\}$. Since $z' \in N(q)$ and $p_1z'v^*w$ is a chordless subpath of G, it follows from our triangulation conditions that $G[N(q) \cap B^*]$ is a subpath of $p_1z'v^*w$ of length precisely two and which contains z'. If $p_1 \in N(z')$, then we contradict 1) of Claim 4.4.15, so $G[N(q) \cap B^*] = \{z', v^*, w\}$.

Thus, since G is $K_{2,3}$-free, q is the unique vertex of G such that $T(B^*, \psi) \setminus A^*) = \{q\}$ for each $\psi \in \Phi(\phi', z')$. Now suppose toward a contradiction that q has a neighbor in $\{u_1, \cdots, u_t\}$ and let $i \in \{1, \cdots, t\}$ be the minimal index such that $u_i \in N(q)$. Then G contains the 3-chord $p_1z'qu_i$ of C, and since $qp_1 \notin E(G)$, q has no neighbors in $\{p_1, u_1, \cdots, u_{i-1}\}$. By Theorem 3.0.2, we have $G^{\text{small}}_{p_1z'qu_i} \setminus \{q, z'\} = p_1u_1 \cdots u_i$, so $u_i \in N(z')$ by our triangulation conditions, which is false. ■

Let q be as in Subclaim 4.4.24. Note that $w \notin N(z')$, or else G contains a K_4 on the vertices $\{v^*, w, z', q\}$. We also have the following:

Subclaim 4.4.25. $L_\phi(v^*) \subseteq L_\phi(w)$

Proof: Suppose there is a color $d \in L_\phi(v^*) \setminus L_\phi(w)$. Let $\phi^* \in \Phi(\phi, \{v^*, z'\})$ be the extension of ϕ obtained by coloring v^* with d any choosing any remaining color for z'. Let $A'' = B'' = \{v^*, z'\}$. Since this choice of A'', B'', ϕ^* satisfies the conditions of Claim 4.4.13, we have $T(B'', \phi^*) \setminus A'' \neq \varnothing$, so there is a vertex $q^* \in V(G^{\text{small}}_Q \setminus D) \setminus A''$ with at least three neighbors in $\text{dom}(\phi^*)$.

Thus, by our choice of d, since $w \notin N(z')$, we have $w \notin T(B'', \phi^*)$. Likewise, since each of $q, y, u^\dagger$ has at most two neighbors in $\text{dom}(\phi^*)$, we have $q^* \notin \{q, w, y, u^\dagger\}$. Yet then the path $p_1z'qwyu^\dagger$ separates q^* from $\text{dom}(\phi^*) \setminus \{p_1, z'\}$, so $|N(q^*) \cap \text{dom}(\phi^*)| \leq 2$, a contradiction. ■

Now we return to the main proof of Case 1 of 4) of Claim 4.4.22. Let $L_\phi(v^*) = \{a, b\}$ and let $c \in L_\phi(z') \setminus \{a, b\}$. Since q has no neighbors in $V(Q) \cup \{p_1, p_2\}$, we have $|L_\phi(q)| = 5$, so, applying Subclaim 4.4.25, there is a color $f \in L_\phi(q) \setminus \{a, b, c\}$ such that $|L_\phi(w) \setminus \{f\}| \geq 3$. Let $\psi \in \Phi(\phi, \{z', q\})$ be obtained by L-coloring the edge $z'q$

with (c, f). Let $A^\dagger := \{v^*, z', q\}$ and $B^\dagger := \{z', q\}$. By Subclaim 4.4.24, q has no neighbors among $u_1, \cdots, u_t$, so this choice of $A^\dagger, B^\dagger, \psi$ satisfies the conditions of Claim 4.4.13, so there exists a $q' \in V(G) \setminus A^\dagger$ with three neighbors among $\text{dom}(\psi)$. Since the path separates $p'_z qwyu^\dagger$ separates q' from $\text{dom}(\psi) \setminus \{p_1, z', q\}$, q' is adjacent to all of p_1, z, q. Since G is $K_{2,3}$-free, we have $T(B^\dagger, \psi) \setminus A^\dagger = \{q'\}$.

Now we simply uncolor z'. That is, we let ψ' be the restriction of ψ to $\text{dom}(\phi) \cup \{q\}$. Then $|L_{\psi'}(q')| \geq 3$, so $T(\{q\}, \psi') \setminus \{v^*, z, q\} = \varnothing$. Yet, by our choice of $\psi'(q)$, the set $\{v^*, z\}$ is $L_{\psi'}$-inert, since G contains the 6-cycle $x_1 p_2 p_1 q' q v^*$, so we contradict Claim 4.4.13. This completes Case 1 of 4) of Claim 4.4.22.

Case 2: $\{p_1 p_2, v^*\} \not\subseteq N(z')$

In this case, $G[N(z') \cap B]$ is the path $p_2 v^* w$, and G contains a 6-wheel with central vertex v^*, where $N(v^*) = \{x_1, x_2, w, z', p_2\}$. As in the previous case, we have the following:

Subclaim 4.4.26. $L_\phi(v^*) \subseteq L_\phi(w)$.

Proof: Suppose there is a color $d \in L_\phi(v^*) \setminus L_\phi(w)$. Let $\phi^* \in \Phi(\phi, z')$ be the extension of ϕ obtained by coloring v^* with d. Let $A'' = B'' = \{v^*\}$. Since this choice of A'', B'', ϕ^* satisfies the conditions of Claim 4.4.13, we have $T(B'', \phi^*) \setminus A'' \neq \varnothing$, so there is a vertex $q \in V(G_Q^{\text{small}} \setminus D) \setminus A''$ with at least three neighbors in $\text{dom}(\phi^*)$. By our choice of d, we have $q \neq w$, and since $|L_{\phi^*}(z')| \geq 3$, we have $q \neq z'$. Yet then the path $p_2 z' wyu^\dagger$ separates q from every vertex of $\text{dom}(\phi^*) \setminus \{p_1, p_2\}$, contradicting the fact that $|L_{\phi^*}(q)| < 3$. ■

Since $p_1 \notin N(z')$, we have $|L_\phi(z')| \geq 4$, so, applying Subclaim 4.4.26, there is a color $f \in L_\phi(z')$ such that $|L_\phi(v^*) \setminus \{f\}| \geq 2$ and $|L_\phi(w) \setminus \{f\}| \geq 3$. Now set $A^\dagger := \{v^*, z'\}$ and $B^\dagger := \{z'\}$. Let $\phi^\dagger$ be the extension of ϕ to z' obtained by coloring z' with f. We claim now that this choice of $A^\dagger, B^\dagger, \phi^\dagger$ satisfies the conditions of Claim 4.4.13. If z' has a neighbor $u \in \{u_1, \cdots, u_t\}$, then G contains the 2-chord $p_2 z' u$ of C, and thus, by Theorem 3.0.2, we have $p_1 \in N(z')$, which is false. Furthermore, by our choice of f, v^* is $L_{\phi^\dagger}$-inert, so our choice of $A^\dagger, B^\dagger, \phi^\dagger$ does indeed satisfy the conditions of Claim 4.4.13. Thus, there is a vertex $q \in T(B^\dagger, \phi^\dagger) \setminus A^\dagger$. Since the path $p_2 z' wyu^\dagger$ separates q from every vertex of $\text{dom}(\psi^\dagger) \setminus \{p_1, p_2, z'\}$, and G is $K_{2,3}$-free, this vertex q is unique, and $N(q) \cap \text{dom}(\psi^\dagger) = \{p_1, p_2, z'\}$.

Now we repeat the process by adding q' to $A^\dagger$ and extending $\phi^\dagger$ to a L-coloring $\phi^{\dagger\dagger}$ of $\text{dom}(\phi) \cup B^\dagger \cup \{q\}$. By 1) of Claim 4.4.15, q has no neighbors in $u_1, \cdots, u_t$, so this choice of $A^\dagger \cup \{q\}, B^\dagger, \phi^\dagger$ again satisfies the conditions ofClaim 4.4.15. Thus, there is a vertex $q' \in V(G_Q^{\text{small}} \setminus D) \setminus (A^\dagger \cup \{q\})$ with at least three neighbors in $\text{dom}(\psi^{\dagger\dagger})$.

Subclaim 4.4.27. $q' \notin \{w, u^\dagger\}$.

Proof: Suppose that $q' = w$. In that case, by our choice of $\phi^\dagger(z')$, we have $q \in N(w)$, or else $|L_{\phi^{\dagger\dagger}}(w)| \geq 3$. But then G contains a $K_{2,3}$ with bipartition $\{p_2, z', w\}$, $\{v^*, q\}$, contradicting short-separation-freeness. Thus, $q' \neq w$. If $q' = u^\dagger$, then, by 3), we have $z', q \notin N(q')$, and thus $N(q') \cap \text{dom}(\psi^{\dagger\dagger}) \subseteq \{p_1\}$. In that case, since $|L_{\psi^{\dagger\dagger}}(q')| < 3$, we have $p_1 \in N(u^\dagger)$, and since C is induced in G, this contradicts 2). ■

Since $q' \notin \{w, u^\dagger\}$, the path $p_1 q z' w w^* u^\dagger$ separates q' from $\text{dom}(\psi^{\dagger\dagger}) \setminus \{p_1, q, z'\}$. But then q' is adjacent to all three of p_1, q, z', so G contains a $K_{2,3}$ with bipartition $\{p_2, q, q'\}$, $\{z', p_1\}$. This completes the proof of Claim 4.4.22. ■

Claim 4.4.28. *There is a $z \in V(G_Q^{\text{small}} \setminus D)$ adjacent to each of v^*, p_2, p_1*

Proof: Suppose that no such vertex exists. Let $A = B = V(H^\dagger \setminus \{x_4\}) \cup \{v^*, w\}$ and let ψ be an extension of ϕ to $\text{dom}(\phi) \cup A$. Then this choice of A, B, ϕ' satisfies the conditions of Claim 4.4.13, so there is a vertex $z \in V(G_Q^{\text{small}} \setminus D) \setminus A$ with at least three neighbors in $\text{dom}(\psi)$. Since the path $p_2 v^* w w^* u^\dagger$ separates z from each vertex in $A \setminus \{u^\dagger, w^*, w, v^*, p_2, p_1\}$, z has at least three neighbors among $\{u^\dagger, w^*, w, v^*, p_2, p_1\}$.

Subclaim 4.4.29. $u^\dagger \notin N(z)$.

Proof: Suppose that $u^\dagger \in N(z)$. By 3) of Claim 4.4.22, we have $p_2 \notin N(z)$. Furthermore, by 4) of Claim 4.4.22, we have $w \notin N(z)$, so z has at least two neighbors among $\{w^*, v^*, p_1\}$. If z is adjacent to each of v^*, p_1, then $p_2 \in N(z)$ by our triangulation conditions, which is false. If z is adjacent to each of v^*, w^*, then $w \in N(z)$ by our triangulation conditions, which is false. The only remaining possibility is that $N(z) \cap \{w^*, v^*, p_1\} = \{w^*, p_1\}$, contradicting 1) of Claim 4.4.22. ■

Thus, we have $u^\dagger \notin N(z)$. Thus, z has at least three neighbors among $\{w^*, w, v^*, p_2, p_1\}$. If $p_1 \in N(z)$, then $w^* \notin N(z)$ by 1) of Claim 4.4.22, and $v^* \notin N(z)$ or else $p_2 \in N(z)$ by our triangulation conditions, contradicting our assumption. Thus, in that case, z is adjacent to each of p_1, p_2, w, so G contains the 4-cycle p_2zwv^*. Again by our triangulation conditions, $z \in N(v^*)$, contradicting our assumption. We conclude that $p_1 \notin N(z)$, so z has at least three neighbors among $\{w^*, w, v^*, p_2\}$. By 1) of Claim 4.4.22, $N(z) \cap \{w^*, w, v^*, p_2\}$ is either $\{p_2, v^*, w\}$ or $\{v^*, w, w^*\}$. In either case, since G is $K_{2,3}$-free, z is the unique vertex such that $T(A, \psi) = \{z\}$ for any $\psi \in \Phi(\phi, A)$.

Case 1: $w^* \in N(z)$

In this case, $N(z) \cap \{w^*, w, v^*, p_2\} = \{v^*, w, w^*\}$, and G contains a 6-wheel with central vertex w, where w is adjacent to each vertex of $v^*zw^*x_3x_2$. Now we set $B' := A \setminus \{w\}$. Since $|L_\phi(w^*)| \geq 3$, $L_\phi(w)| \geq 3$, and $|L_\phi(v^*)| \geq 2$, we let $c_1 \in L\phi(*v^*)$ and $c_2 \in L_\phi(w^*)$, where $L_\phi(w) \setminus \{c_1, c_2\}| \geq 2$. Since $u^\dagger \neq u_1$ and $v^* \notin N(u^\dagger)$, there is a $\psi \in \text{dom}(\phi, B')$, where $\psi(v^*) = c_1$ and $\psi(w^*) = c_2$. Since $B' \subseteq A$ and $|L_\psi(z)| \geq 3$, we have $T(B', \psi) \setminus A = \varnothing$. Since w is L_ψ-inert, this contradicts Claim 4.4.13.

Case 2: $w^* \notin N(z)$

In this case, $N(z) \cap \{w^*, w, v^*, p_2\} = \{v^*, w, w^*\}$, and G contains a 6-wheel with central vertex v^*, where w is adjacent to each vertex of $x_1p_2zwx_2$. Let $B' := A \setminus \{v^*\}$. Since $|L_\phi(w)| \geq 3$ and $|L_\phi(v^*)| = 2$, let $c \in L_\phi(w) \setminus L_\phi(v^*)$. As above, since $u^\dagger \neq u_1$, there is a $\psi \in \text{dom}(\phi, B')$, where $\psi(w) = c$. We have $|L_\psi(z)| \geq 3$, or else $p_1 \in N(z)$, which is false. Thus, since $B' \subseteq A$, we have $T(B', \psi) \setminus A = \varnothing$. Since v^* is L_ψ-inert by our choice of c, this contradicts Claim 4.4.13. ■

Now we are ready to finish the proof of Theorem 4.4.1. Let z be as in Claim 4.4.28. Let $R := p_1zv^*ww^*u^\dagger$ and let $A := V(R \setminus \{p_1\}) \cup V(H^\dagger \setminus \{x_4\})$. For any $\psi \in \Phi(\phi, A)$, we have $T(A, \psi) \neq \varnothing$ by Claim 4.4.13, and, for each $y \in T(A, \psi)$, we have $N(y) \cap \text{dom}(\psi) \subseteq V(R)$, since the path R separates y from $\text{dom}(\psi) \setminus R$. Let $R' := zv^*ww^*$. We break the remainder of Theorem 4.4.1 into two cases:

Case 1 of Theorem 4.4.1: There is no $y \in V(G_Q^{\text{small}} \setminus D)$ with $|N(y) \cap V(R)| \geq 3$ adjacent to each of $p_1, u^\dagger$

In this case, we first note the following:

Claim 4.4.30. *For any $\psi \in \Phi(\phi, zv^*ww^*)$, ψ extends to L-color* $\text{dom}(\psi) \cup V(H^\dagger)$.

Proof: This is trivial if $H^\dagger$ is just an edge, since $\text{dom}(\psi) \cup V(H^\dagger)$. is already colored. If $H^\dagger$ is a broken wheel, then we simply choose a color $d \in \mathcal{Z}_{H^\dagger}(\phi(x_4), \psi(w^*), \bullet)$. Possibly $d = \phi(p_1)$. This is permissible as $u^\dagger \neq u_1$ by 2) of Claim 4.4.22. In either case, ψ extends to L-color $\text{dom}(\psi) \cup V(H^\dagger)$. ■

Since ϕ extends to an L-coloring ψ of $\text{dom}(\phi) \cup A$, and $T(A, \psi) \neq \varnothing$ by Claim 4.4.13, there is a $y \in V(G_Q^{\text{small}} \setminus D) \setminus A$ with three neighbors in R. We claim now that y is the unique vertex of $y \in V(G_Q^{\text{small}} \setminus D) \setminus A$ with three neighbors in

R, and that y is not adjacent to either of $p_1, u^\dagger$.

Suppose that y is adjacent to p_1. By 1) of Claim 4.4.15, y is not adjacent to v^*, and, by our assumption, y is not adjacent to $u^\dagger$. If y is adjacent to w^*, then G contains the 3-chord $u^\dagger w^* y p_1$ of C. But then, by Theorem 3.0.2, $G^{\text{small}}_{p_1 y w^* u^\dagger} \setminus \{y, w^*\} = p_1 u_1 \cdots u^\dagger$, and since w^* has no neighbors among $\{p_1, u_1, \cdots, u^\dagger\} \setminus \{u^\dagger\}$, it follows from our triangulation conditions y is adjacent to $u^\dagger$ as well, contradicting our assumption. Thus, y is not adjacent to w^* either, so $N(y) \cap V(R) = \{u, z, w\}$. But then G contains the 4-cycle wv^*zy, and $v^* \in N(y)$ by our triangulation conditions, which is false. We conclude that $y \notin N(p_1)$, as desired.

Now suppose that y is adjacent to $u^\dagger$. A similar argument to the one above rules out the possibility. By 4) of Claim 4.4.22, y is not adjacent to w, and, by our assumption, y is not adjacent to p_1. If y is adjacent to z, then G contains the 3-chord $u^\dagger y z p_1$ of C. Since this 3-chord of C lies in $\mathcal{K}(C, \mathcal{T})$ and has one endpoint in P, it follows from Theorem 3.0.2 that $G^{\text{small}}_{u^\dagger y z p_1} \setminus \{y, z\} = p_1 u_1 \cdots u^\dagger$, and y is adjacent to $u^\dagger$, which is false. Thus, y is not adjacent to z either, so $N(y) \cap V(R) = \{u^\dagger, w^*, v^*\}$. But then G contains the 4-cycle $v^* w w^* y$, and $w \in N(y)$ by our triangulation conditions, which is false. Thus, $y \notin N(u^\dagger)$. We conclude that y is not adjacent to either of $p_1, u^\dagger$. By our triangulation conditions, $G[N(y) \cap V(R)]$ is a subpath of ${}'R$ of length either two or three, and, in particular, y is the unique vertex of G such that, for any $\psi \in \Phi(\phi, A)$, we have $T(A, \psi) = \{y\}$.

Claim 4.4.31. *There exists a partial L_ϕ-coloring ψ of R' such that $L_\psi(y)| \geq 3$ and $V(R') \setminus \text{dom}(\phi \cup \psi)$ is $L_{\phi \cup \psi}$-inert.*

Proof: It is easy to check that this holds in the case where $G[N(y) \cap V(R)]$ is a subpath of R' of length two, since $|L_\phi(v^*)| \geq 2$ and each vertex of $R' \setminus \{v^*\}$ has an L_ϕ-list of size at least three. Now suppose that y is adjacent to all four vertices of R', and suppose that no partial L_ϕ-coloring of R' satisfying the claim exists. We fix a $c \in L_\phi(z) \setminus L_\phi(v^*)$, since $|L_\phi(v^*)| = 2$. Furthermore, G contains the 7-cycle $K := x_1 x_2 x_3 v^* y z p_2$, and, letting $W \subseteq \mathbb{R}^2$ be the unique open set such that $\partial(W) = K$ and $W \cap V(C) = \varnothing$, we get that $G \cap W$ consists of the edge wv^*.

We claim now that $L_\phi(w) = L_\phi(w^*)$ and $L_\phi(v^*) \subseteq L_\phi(w)$. Let $d \in L_\phi(w^*)$ and let ψ be the L_ϕ-coloring of $\{z, w^*\}$ where $\psi(z) = c$ and $\psi(w^*) = d$. We have $|L_\psi(y)| \geq 3$, and since ψ does not satisfy Claim 4.4.31, there is an extension of $\phi \cup \psi$ to an L-coloring f of $G \setminus (V(G) \cap W)$, such that f does not extend to L-color the edge $v^* w$. By our choice of c, we have $|L_f(v^*)| = |L_f(w)| = 1$ and $L_f(v^*) = L_f(w)$. We conclude that $d \in L_\phi(w)$ and $|L_\phi(w)| = 3$, and $L_\phi(w) \setminus \{d\} = L_\phi(v^*)$. Since this holds for each $d \in L_\phi(w^*)$, we have $L_\phi(w) = L_\phi(w^*)$ and $L_\phi(v^*) \subseteq L_\phi(w)$.

Now, we simply let ψ be an L_ϕ-coloring of $\{v^*, w^*, z\}$ in which the same color is used on v^*, w^*. Then $|L_\psi(y)| \geq 3$, and, w is $L_{\phi \cup \psi}$-inert, since $N(w) = \{v^*, y, w^*, x_3, x_2\}$ and v^*, w^* use the same color. But then ψ satisfies the conditions of Claim 4.4.31, contradicting our assumption. ■

Combining Claim 4.4.31 with Claim 4.4.30, there is an extension of ϕ to a partial L-coloring ϕ^* of $\text{dom}(\phi) \cup A$ such that $|L_{\phi^*}(y)| \geq 3$ and each vertex of $A \setminus \text{dom}(\phi^*)$ is L_{ϕ^*}-inert. Let $B := \text{dom}(\phi^*)$, and, as above, let ψ be an extension of ϕ^* to all of $\text{dom}(\phi) \cup A$. Since $T(B, \phi^*) \setminus A \subseteq T(A, \phi^{**}) = \{y\}$, we have $T(B, \phi^*) \setminus A = \varnothing$, contradicting Claim 4.4.13.

Case 2 of Theorem 4.4.1: There exists a $y \in V(G^{\text{small}}_Q \setminus D) \setminus R$ with $|N(y) \cap V(R)| \geq 3$, where y is adjacent to each of $p_1, u^\dagger$

In this case, $v^* \notin N(y)$ by 1) of Claim 4.4.15, and $w^* \notin N(y)$ by 1) of Claim 4.4.22. Furthermore, y is not adjacent to w, or else G contains the 4-cycle wv^*zy, and so $v^* \in N(y)$ by our triangulation conditions, which is false. Thus, $N(y) \cap A = \{p_1, u^\dagger, z\}$, and G contains a broken wheel H^{mid} with principal path $u^\dagger y p_1$, where

$H^{\text{mid}} \setminus \{y\} = p_1u_1 \cdots u^\dagger$. Now, G contains the 6-cycle $K'' := wv^*zyu^*w^*$. Let $W'' \subseteq \mathbb{R}^2$ be the unique open set such that $\partial(W'') = K''$ and $V(C) \cap W'' = \varnothing$. Let $c \in L_\phi(z) \setminus L_\phi(v^*)$.

By Proposition 1.4.5, there is a $d' \in L(y) \setminus \{c, \phi(p_1)\}$ such that $|\mathcal{Z}_{H^{\text{mid}}}(\phi(p_1), d', \bullet)| \geq 2$. Likewise, there is a $d^* \in \mathcal{Z}_{H^\dagger}(\phi(x_4), d^*, \bullet) \geq 2$. Since $|L(u^\dagger)| = 3$, there is an extension of ϕ to an L-coloring ψ of $G \setminus W''$ in which $\psi(z) = c$, $\psi(y) = d'$, and $\psi(w^*) = d^*$. Possibly $d' = d^*$. This is permissible as $w^*y \notin E(G)$. Note that there is no vertex of $V(G) \cap W''$ adjacent to all three of $w, w^*, u^\dagger$ by 4) of Claim 4.4.22. Furthermore there is no vertex $y' \in V(G) \cap W'$ adjacent to all three of $z, y, u^\dagger$, or else G contains the 3-chord $p_1zy'u^\dagger$ of C which separates y from G_Q^{large}. Since this 3-chord of C lies in $\mathcal{K}(C, \mathcal{T})$ and has p_1 as an endpoint, this contradicts Theorem 3.0.2. Thus, it follows from 1.3.5 that $G \cap W''$ consists of a triangle $y_1y_2y_3$, where $N(y_1) \cap V(K'') = \{y, u^\dagger, w^*\}$, and $N(y_2) \cap V(K'') = \{y, z, v^*\}$, and $N(y_3) \cap V(K'') = \{v^*, w, w^*\}$. Thus, $G \setminus \text{dom}(\psi)$ consists of diagram in Figure 4.4.5, with the lower bounds on the sizes of the L_ψ-lists labelled in red.

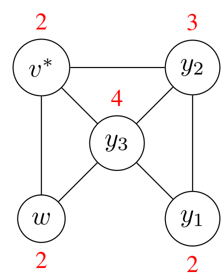

Figure 4.4.5: Main Case 2

We have $|L_\psi(v^*)| \geq 2$ by our choice of c, and the above graph is L_ψ-colorable. Note that this is not necessarily true if $|L_\psi(v^*)| = 1$, since in that case, the diagram above possibly reduces to a triangle in which all three vertices have the same 2-list. Since the graph above is L_ψ-colorable, ϕ extends to an L-coloring of G, which is false. This completes the proof of Theorem 4.4.1. $\square$

With Theorem 4.4.1 in hand, we can finally finish the proof of the main theorem of Chapter 4, i.e Theorem 4.0.1. Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let $C \in \mathcal{C}$ be an open ring. Let $\mathbf{P} = p_1 \cdots p_m$, and let uv be a chord of C^1 where u has a neighbor in $V(\mathring{\mathbf{P}})$ and v has a neighbor in $V(C \setminus \mathring{\mathbf{P}})$. Let $w \in N(u) \cap V(\mathring{\mathbf{P}})$ and $w' \in N(v) \cap V(C \setminus \mathring{\mathbf{P}})$. By Observation 4.3.1, we suppose without loss of generality that $w \in \{p_2, p_3\}$. Let $R := wuvw'$.

Combining Theorem 4.4.1 and Proposition 4.3.4, we get that $|V(G_R^{\text{small}}) \setminus V(C \cup R)| = 1$, and if $w = p_2$, then G_R^{small} is a wheel whose central vertex is the lone vertex of $V(G_R^{\text{small}}) \setminus V(C \cup Q)$. If $w = p_3$, then $G_R^{\text{small}} \setminus \{p_3\}$ is a wheel whose central vertex is the lone vertex of $V(G_R^{\text{small}}) \setminus V(C \cup R)$. In either case, the lone central vertex of this wheel is the lone vertex of $D_1(C)$ adjacent to p_1, p_2, and $N(w) \cap V(\mathbf{P})$ is either $\{p_2\}$ or $\{p_2, p_3\}$.

Combining this Theorem 3.0.2, we get that, for any 3-chord R' of C with at least one endpoint in $C \setminus \mathring{\mathbf{P}}$, R' does not separate two vertices of $V(C \cup C^1)$. Thus, by Lemma 4.2.1, together with our triangulation conditions, G contains a cycle C^2 such that, letting $G = G' \cup G''$ be the natural C^2-partition of G, where $C \subseteq G'$, we have $C^2 \cap C^1 = \mathbf{P}^1$ and $V(G') = V(C \cup C^1 \cup C^2)$, and furthermore, $V(C^2 \setminus \mathbf{P}^1) = D_2(C \setminus \mathbf{P}) \setminus V(C^1)$. This completes the proof of Theorem 4.0.1. In Chapter 5 and Chapter 6 we apply the results of Chapters 3 and 4 describing the structure of a critical mosaic near each open ring to delete vertices on and near the open rings.

Chapter 5

Deleting Vertices of Distance One from Open Rings of Critical Mosaics

In this chapter we apply our boundary analysis results for open rings from Chapters 3 and 4 to color and delete a strip of the 1-necklace of an open ring near the precolored path. The main result of this chapter is somewhat technical because it requires very careful coloring and deleting of vertices of this open ring and the 1-neckalce of this open ring to avoid creating any lists of size two on the remaining vertices of the open ring.

5.1 Preliminaries

Applying the structural results from Chapters 3 and 4, we first have the following.

Observation 5.1.1. *Let $\mathcal{T}$ be a critical mosaic and let C be an open $\mathcal{T}$-ring. Let p be an endpoint of $\mathbf{P}_{\mathcal{T}}(C)$, and let $\mathbf{P} := p_1 \cdots p_m$, and $p = p_1$. Let $C := p_m \cdots p_1 u_1 \cdots u_n$ for some $n \geq 1$. Let $C^1 := x_1 \cdots x_r$ be the 1-necklace of C, where x_1 is the unique common neighbor of p_1 in C^1. Then there exist indices m_2, m_3 with $1 < m_2 < m_3 < r$ and indices t_1, t_2, t_3 with $1 \leq t_1 < t_2 < t_3 \leq n$ such that the following hold.*

1) *$N(x_1) \cap V(C \setminus \mathbf{P}) = \{u_1, \cdots, u_{t_1}\}$; AND*
2) *$N(x_{m_2}) \cap V(C) = \{u_{t_1}, u_{t_1+1}, \cdots, u_{t_2}\}$; AND*
3) *For each internal vertex x of the path $x_1 x_2 \cdots x_{m_2}$, $N(x) \cap V(C) = \{u_{t_1}\}$; AND*
4) *$N(x_{m_3}) \cap V(C) = \{u_{t_2}, u_{t_2+1}, \cdots, u_{t_3}\}$; AND*
5) *For each internal vertex x of the path $x_{m_2} x_{m_2+1} \cdots x_{m_3}$, $N(x) \cap V(C) = \{u_{t_2}\}$.*

Proof. Since x_1 has at least two neighbors on C, $G[N(x_1) \cap C]$ is a broken wheel with principal vertex x_1. Since $G[N(x) \cap V(\mathbf{P})]$ is either p_1 or p_2, and C is an induced subgraph of G, there is a $t_1 \in \{1, \cdots, n\}$ such that $G[N(x) \cap V(C \setminus \mathbf{P})] = u_1 \cdots u_{t_1}$. Note that $t_1 < n$, or else we contradict 1) of Theorem 2.3.2. Thus, let $m_2 \in \{2, \cdots, r\}$, where x_{m_2} is the unique common neighbor of u_{t_1}, u_{t_1+1} in C^1, and, for each index $1 < j < m_2$, $N(x_j) \cap V(C) = \{u_{t_1}\}$. Note that $N(x_{m_2}) \cap V(\mathbf{P}) = \varnothing$, or else, letting $p \in V(\mathbf{P}) \cap N(x_{m_2})$, the path $p_1 x_1 u_{t_1} x_{m_2} p$ is a C-band, contradicting 1) of Theorem 2.3.2.

Thus, $G[N(x_{m_2}) \cap V(C)] = u_{t_1} u_{t_1+1} \cdots u_{t_2}$ for some $t_2 \in \{t_1 + 1, \cdots, n\}$. Furthermore, $t_2 < n$, or else the path $p_1 x_1 u_{t_1} x_{m_2} u_n p_m$ is a C-band, contradicting 1) of Theorem 2.3.2. Thus, there is an $m_3 \in \{m_2 + 1, \cdots, r\}$ such that x_{m_3} is the unique common neighbor of u_{t_2}, u_{t_2+1} in C^1, and, for each $m_2 < j < m_3$, we have $N(x_j) \cap V(C) =$

$\{u_{t_2}\}$. Finally, we have $N(x_{m_3}) \cap V(\mathbf{P}) = \varnothing$, or else, letting $p \in N(x_{m_3}) \cap V(\mathbf{P})$, the path $p_1 x_1 u_{t_1} x_{m_2} u_{t_2} x_{m_3} p$ is a short C-band. Since $\frac{N_{\text{mo}}}{4} > 7$, this contradicts 1) of Theorem 2.3.2. Thus, there is a $t_3 \in \{t_2+1, \cdots, n\}$ such that $G[N(x_{m_3}) \cap V(C)] = u_{t_2} \cdots u_{t_3}$, so we are done. $\square$

In this chapter, we show how partially color the path $x_1 \cdots x_{m_3}$ in such a way that the path $u_1 \cdots u_{t_3 - 1}$ can be removed. Given Observation 5.1.1, it is natural to introduce the following definition:

Definition 5.1.2. Let $\mathcal{T}$ be a critical mosaic and let C be an open $\mathcal{T}$-ring. Let p be an endpoint of $\mathbf{P}$, where $\mathbf{P} := p_1 \cdots p_m$ and $p = p_1$. Let $C := p_m \cdots p_1 u_1 \cdots u_n$ for some $n \geq 1$. Let $C^1 := x_1 \cdots x_r$ be the 1-necklace of C, where x_1 is the unique common neighbor of p_1 in C^1. Let $m_2, m_3 \in \{1, \cdots, r\}$ and $t_1, t_2, t_3 \in \{1, \cdots, n\}$ be as in Observation 5.1.1. Then we let Π_p^0 denote the path $u_1 \cdots u_{t_3-1}$ and we let Π_p^1 denote the path $x_1 \cdots x_{m_3}$. The vertex x_{m_2} is called the *overlap point* of Π_p^1.

Observation 5.1.3. *Let $\mathcal{T}$ be a critical mosaic and let C be an open $\mathcal{T}$-ring. Let p be an endpoint of $\mathbf{P}_{\mathcal{T}}(C)$, and let C^1 be the 1-necklace of C. Let $\Pi_p^1 = x_1 \cdots x_{m_2} \cdots x_{m_3}$, where x_{m_2} is the overlap point of Π_p^1. Then either Π_p^1 is an induced subgraph of G or Π_p^1 has precisely one chord, which is $x_{m_2-1}x_{m_2+1}$. Furthermore, $G \setminus B_1(C)$ contains a path $z_1 \cdots z_\ell$ such that the following hold.*

1) *$\{z_1, \cdots, z_\ell\} \cap B_1(C) = V(\Pi_p^1)$; AND*
2) *For each $i \in \{1, \cdots, \ell\}$ $G[N(z) \cap V(\Pi_p^1)]$ is a subpath of Π_p^1 of length at most two, and furthermore, if $G[N(z) \cap V(\Pi_p^1)]$ is a subpath of Π_p^1 of length precisely two, then $G[N(z) \cap V(\Pi_p^1)] = x_{m_2-1}x_{m_2}x_{m_2+1}$, and Π_p^1 is an induced subgraph of G.*

Proof. Firstly, since G is short-separation-free, there is no chord of Π_p^1 with both endpoints in $x_1 \cdots x_{m_2}$, since each vertex of $\{x_1, \cdots, x_{m_2}\}$ is adjacent to u_{t_1}. Likewise, there is no chord of Π_p^1 with both endpoints in $x_{m_2} \cdots x_{m_3}$, since each vertex of $\{x_{m2}, \cdots, x_{m_3}\}$ is adjacent to u_2. Thus, by Theorem 3.0.2, either Π_p^1 is an induced subpath of G, or there is precisely one chord of Π_p^1, which is $x_{m_2-1}x_{m_2+1}$. Now let $z \in V(G) \setminus B_1(C)$, where z has at least two neighbors in Π_p^1. Let x_i, x_j be the endpoints of $\Pi_p^1 \cap G[N(z)]$, where $1 \leq i < j \leq m_3$. We claim now that if $j \neq i+1$, then $i = m_2 - 1$ and $j = m_2 + 1$. Suppose that $j \neq i + 1$.

If each of x_i, x_j lie in $\{x_1, \cdots, x_{m_2}\}$, then, since $j > i+1$, $u_{t_1} x_i z x_j$ is a separating cycle in G, contradicting the fact that G is short-separation-free. Likewise, at most one of x_i, x_j lies in $\{x_{m_2}, \cdots, x_{m_3}\}$, so $x_i \in \{x_1, \cdots, x_{m_2-1}\}$ and $x_j \in \{x_{m_2-1}, \cdots, x_{m_2+1}\}$, so G contains the 4-chord $u_{t_1} x_i z x_j u_{t_2}$ of C, and $u_{t_1} x_i z x_j u_{t_2} \in \mathcal{K}(C, \mathcal{T})$. Thus, by Theorem 4.0.1, since Π_p^1 is an induced subgraph of G, except possibly for the chord $x_{m_2-1}x_{m_2+1}$, z is adjacent to each vertex of $\{x_i, x_{i+1}, \cdots, x_j\} \setminus \{x_{m_2}\}$, so we have $i = m_2 - 1$ and $j = m_2 + 1$, as desired. $\square$

Given Observation 5.1.3, it is natural to introduce the following definition:

Definition 5.1.4. Let $\mathcal{T}$ be a critical mosaic and let C be an open $\mathcal{T}$-ring. Let p be an endpoint of $\mathbf{P}_{\mathcal{T}}(C)$. We denote the path $z_1 \cdots z_\ell$ from Observation 5.1.3 as Π_p^2.

We use the notation Π_p^0, Π_p^1, and Π_p^2 for our analysis of a critical mosaic near each open ring throughout Chapters 5 and 6. This notation is always used in a context in which we have fixed a critical mosaic and an open ring of the critical mosaic, and the letter Π is not used in any other context. The diagram at the end of this section shows the three paths $\Pi_p^0, \Pi_p^1, \Pi_p^2$ and is a useful reference point. To state the main result of Chapter 5, we first introduce the following definitions, which makes precise the idea of puncturing an open ring near the precolored path.

Definition 5.1.5. Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let C be an open $\mathcal{T}$-ring. Let p, p^* be the endpoints of $\mathbf{P}$ and let ϕ be the unique L-coloring of $V(\mathbf{P})$. A *C-wedge* is a pair (H, ψ), where H is a subgraph of $V(\Pi_p^0 \cup \Pi_{p^*}^0) \cup V(\Pi_p^1 \cup \Pi_{p^*}^1)$ and ψ is a partial L_ϕ-coloring of $V(H)$, such that the following hold.

1) For each $q \in \{p, p^*\}$, the following hold.
   a) $H \cap \Pi_q^0$ is a terminal subpath of Π_q^0 containing the lone endpoint of Π_q^0 adjacent to q*; AND*
   b) $H \cap \Pi_q^1$ is a terminal subpath of Π_q^1 containing the lone endpoint of Π_q^1 adjacent to q, and this path either consists of a lone vertex, or ends in the overlap point of Π_q^1, or consists of all of Π_q^1.
2) $V(H)$ is $(L, \phi \cup \psi)$-inert*; AND*
3) Each vertex of $D_1(H \cup \mathbf{P}) \setminus \mathbf{P}^1$ has an $L_{\phi \cup \psi}$-list of size at least three*; AND*
4) Each vertex of $\mathbf{P}^1 \setminus H$ has an $L_{\phi \cup \psi}$-list of size at least two.

Our main result for Chapter 5 is the following.

Theorem 5.1.6. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let C be an open $\mathcal{T}$-ring. Then there exists a C-wedge.*

To prove this, we first introduce the following notation and terminology and then prove a simple lemma about broken wheels. The purpose of the lemma below is to allow us to to delete vertices in the ball of distance one from an open ring without leaving nearby vertices on the ring with lists of size less than three.

Definition 5.1.7. Let H be a broken wheel with principal path $P := p_1p_2p_3$ and let L be a list-assignment for H.

1) We denote by $\mathcal{M}_L^P(H, p_1p_2)$ the set of L-colorings ϕ of the edge p_1p_2 satisfying the following conditions.
   a) $|L_\phi(p_3)| \geq 3$*; AND*
   b) For any $c \in L_\phi(p_3)$, the L-coloring $(\phi(p_1), \phi(p_2), c)$ of $p_1p_2p_3$ extends to an L-coloring of H.
2) We say that the edge p_1p_2 is an *L-shield* for H if there exist two distinct elements ψ_1, ψ_2 of $\mathcal{M}_L^P(H, p_1p_2)$ which satisfy one of the following.
   a) ψ_1, ψ_2 use the same color on the principal vertex p_2*; OR*
   b) There exist $a, b \in L(p_1) \cap L(p_2)$ such that $a = \psi_1(p_1) = \psi_2(p_2)$ and $b = \psi_1(p_2) = \psi_2(p_1)$, i.e ψ_1, ψ_2 are obtained from each other by interchanging colors on p_1p_2.

If the principal path P is clear from the context then we drop the superscript P from the notation $\mathcal{M}_L^P(H, p_1p_2)$.

Lemma 5.1.8. *Let H be a broken wheel with principal path $P := p_1p_2p_3$, and let L be a list-assignment for H such that $|L(p_2)| \geq 5$ and $|L(x)| = 3$ for all $x \in V(H) \setminus \{p_2\}$. Then the following hold.*

1) If H is not a triangle, then each edge of P is an L-shield for H; AND

2) If H is a triangle and p_1p_2 is not an L-shield for H, then $|L(p_1) \cap L(p_3)| \geq 2$.

Proof. For the proof of this lemma, given a $d \in L(p_1)$ and $d' \in L(p_2)$, the ordered pair (d, d') denotes an L-coloring of p_1p_2 using d, d' on the respective vertices p_1, p_2.We first prove 1). The two sides are symmetric so it suffices to show that p_1p_2 is an L-shield for H. Since H is not a triangle, let $H - p_2 = p_1v_1 \cdots v_\ell p_3$ for some $\ell \geq 1$. Since $L(p_2)| \geq 5$ and $|L(p_3)| = 3$, we fix two colors $a, b \in L(p_2) \setminus L(p_3)$. Suppose toward a contradiction that p_1p_2 is not an L-shield for H.

Claim 5.1.9. $\{a, b\} \subseteq L(v_j)$ *for each* $j = 1, \cdots, \ell$.

Proof: Suppose not, and suppose without loss of generality that there is a $j \in \{1, \cdots, \ell\}$ with $a \notin L(v_j)$. Since $|L(p_1)| = 3$, there exist $c, c' \in L(p_1) \setminus \{a\}$. By Proposition 1.4.4, each of (c, a) and (c', a) lies in $\mathcal{M}_L(H, p_1p_2)$, contradicting our assumption. ■

Claim 5.1.10. ℓ *is even.*

Proof: Suppose ℓ is odd. Since $|L(p_1) \setminus \{a\}| \geq 3$ and p_1p_2 is not an L-shield for H, there is a $c \in L(p_1) \setminus \{a\}$ with $(c, a) \notin \mathcal{M}_L(H, p_1p_2)$. Thus, there is an L-coloring σ_{ca} of $p_1p_2p_3$, where σ_{ca} uses c, a on p_1, p_2 respectively and does not extend to an L-coloring of H. By Claim 5.1.9, we have $b \in L(v_i)$ for each $i = 1, \cdots, \ell$. Since $\sigma_{ca}(p_3) \notin \{a, b\}$ and ℓ is odd, we extend σ_{ca} to an L-coloring of H by coloring each of $v_1, v_3, \cdots, v_\ell$ with b, which leaves a color for each of $v_2, v_4, \cdots, v_{\ell-1}$, contradicting our assumption that σ_{ca} does not extend to an L-coloring of H. ■

Claim 5.1.11. $\{a, b\} \subseteq L(p_1)$.

Proof: Suppose not, and suppose without loss of generality that $a \notin L(p_1)$. Since $|L(p_1) \setminus \{a\}| \geq 3$, it follows from our assumption on H that there are two distinct colors $c, c' \in L(p_1) \setminus \{a\}$ such that neither (c, a) nor (c', a) lies in $\mathcal{M}_L(H, p_1p_2)$. Thus, there exsit two L-colorings σ, σ' of $p_1p_2p_3$, neither of which extends to an L-coloring of H, such that σ uses c, a on the respective vertices p_1, p_2, and σ' uses c', a on the respective vertices p_1, p_2. By Proposition 1.4.4, $\sigma(p_3) \in L(v_\ell)$ and $\sigma'(p_3) \in L(v_\ell)$. By Claim 5.1.9, we have $|L(v_\ell) \setminus \{a, b\}| = 1$. Since $a, b \notin L(p_3)$, we have $\sigma(p_3) = \sigma'(p_3) = r$ for some color $r \in L(p_3) \setminus \{a, b\}$. By Observation 1.4.2, the L-coloring (a, r) of p_2p_3 extends to an L-coloring of H using of of c, c' on p_1, so we have a contradiction. ■

By assumption, at most one of $(a, b), (b, a)$ lies in $\mathcal{M}_L(H, p_1p_2)$,. Suppose without loss of generality that $(a, b) \notin \mathcal{M}_L(H, p_1p_2)$. Thus, there is an L-coloring σ_{ab} of $p_1p_2p_3$, using a, b on the respective vertices p_1, p_2, where σ_{ab} does not extend to an L-coloring of H. Since ℓ is even and $a \notin L(p_3)$, we now color each of $v_2, v_4, \cdots, v_\ell$ with a, which leaves a color for each of $v_1, v_3, \cdots, v_{\ell-1}$, so σ_{ab} extends to an L-coloring of H, which is false. This proves 1).

Now suppose that H is a triangle and that p_1p_2 is not an L-shield for H. Suppose toward a contradiction that $|L(p_1) \cap L(p_3)| \leq 1$ and let $a, b \in L(p_1) \setminus L(p_3)$. If $a, b \in L(p_2)$, then $(a, b), (b, a) \in \mathcal{M}_L(H, p_1p_2)$, so p_1p_2 is al L-shield for H, contradicting our assumption. Thus, there exists an $f \in L(p_2) \setminus (L(p_3) \cup \{a, b\})$, so $(a, f), (b, f) \in \mathcal{M}_L(H, p_1p_2)$, contradicting our assumption. □

For the remainder of Chapter 5, in order to avoid repetition, we fix a critical mosaic $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ and an open $\mathcal{T}$-ring C. As above, let $\mathbf{P} := \mathbf{P}_{\mathcal{T}}(C)$ and $\mathbf{P}^1 := \mathbf{P}^1_{\mathcal{T}}(C)$, and let ϕ be the unique L-coloring of $\mathbf{P}$. To prove Theorem 5.1.6, we first note that it is sufficient to restrict ourselves to one side of the precolored path. To do this, we first introduce the following terminology.

Definition 5.1.12. Let p, p^* be the endpoints of $\mathbf{P}$. Given a $q \in \{p, p^*\}$, a subgraph H_q of $G[V(\Pi^0_q \cup \Pi^1_q)]$, and an extension of ϕ to a partial L-coloring ψ_q of $V(\mathbf{P} \cup V(H_q)$, we call the pair (H_q, ψ_a) a (C, q)-*wedge* if H_q is a subgraph of $G[V(\Pi^0_q \cup \Pi^1_q)]$, ψ_q is partial L_ϕ-coloring of $V(H_q)$, and the following hold.

1a) $H \cap \Pi^0_q$ is a terminal subpath of Π^0_q containing the lone endpoint of Π^0_q adjacent to *q; AND*

1b) $H \cap \Pi^1_q$ is a terminal subpath of Π^1_q containing the lone endpoint of Π^1_q adjacent to q, and this path either consists of a lone vertex, or ends in the overlap point of R^1_q, or consists of all of Π^1_q*; AND*

2) $V(H_q)$ is $(L, \phi \cup \psi_q)$-inert in G*; AND*

3) Each vertex of $D_1(H_q) \setminus \mathbf{P}^1$ has an $L_{\phi\cup\psi_q}$-list of size at least three*; AND*

4) Each vertex of $\mathbf{P}^1 \setminus H_{p'}$ has an $L_{\phi\cup\psi_q}$-list of size at least two.

With the terminology above in hand, we now have the following simple observation.

Claim 5.1.13. *Let p, p^* be the endpoints of $\mathbf{P}$. If there exists a (C, p)-wedge (H_p, ψ_p) and a p^*-wedge ψ_{p^*}, then $(H_p \cup H_{p^*}, \psi_p \cup \psi_{p^*})$ is a wedge.*

Proof: Firstly, each vertex of $\Pi_p^0 \cup \Pi_p^1$ is of distance at most six from p. Likewise, each vertex of $\Pi_{p'}^0 \cup \Pi_{p'}^1$ is of distance at most from p' six. Since $\frac{N_{\text{mo}}}{4} > 6 + 6 + 2$, it follows from 1) of Theorem 2.3.2 that G contains no path of length at most two with one endpoint in $V(\Pi_p^0 \cup \Pi_p^1)$ and one endpoint in $V(\Pi_{p'}^0 \cup \Pi_{p'}^1)$. It immediately follows that $(H_p \cup H_{p^*}, \psi_p \cup \psi_{p^*})$ is a C-wedge. ■

Thus, we now fix an endpoint p of $\mathbf{P}$. The remainder Chapter 5 consists of the proof of the following result, which is sufficient to prove Theorem 5.1.6.

Theorem 5.1.14. *There exists a (C, p)-wedge.*

In Figure 5.1.1, we have a diagram in which the indices m_2, m_3, t_1, t_2, t_3 corresponding to Observation 5.1.1, where this diagram shows the paths $\Pi_p^0 + \{pu_1, u_{t_3-1}u_{t_3}\}, \Pi_p^1, \Pi_p^2$ and the edges between the these three paths. The three paths are on the respective levels 0, 1, 2 of the drawing, as indicated on the right. The graph below is not necessarily an induced subgraph of G, since, possibly, there are edges of $G \setminus E(\Pi_p^2)$ with both endpoints in Π_p^2, but the diagram does show all the edges of the subgraph of G induced by the paths $pu_1 \cdots u_{t_3}$ and $x_1 \cdots x_{m_2} \cdots x_{m_3}$. It is not necessarily the case that $x_{m_2-1}, x_{m_2}, x_{m_2+1}$ have a common neighbor, as shown in the diagram, but this vertex of Π_p^2, if it exists, is the only vertex of Π_p^2 whose neighborhood on Π_p^1 is not a subpath of Π_p^1 of length at most one. Note that Figure 5.1.1 also shows a neighbor of x_1 in Π_p^2, but there is also the possibility that this vertex does not have any neighbors outside of $V(C \cup C^1)$, since, by Theorem 4.0.1, there is possibly a 3-chord of C with one endpoint in x_2, where the other endpoint is the lone vertex of $\mathbf{P}$ adjacent to p.

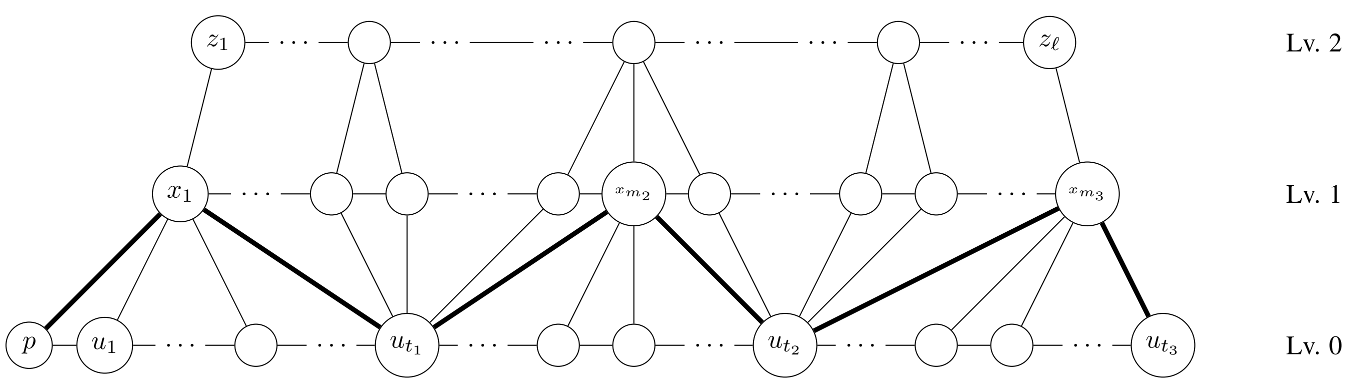

Figure 5.1.1: Vertices and Edges Near the Precolored Path

Let H_1 be the broken wheel with principal path $px_1u_{t_1}$, where $H_1 - x_1 = pu_1 \cdots u_{t_1}$. Likewise, let H_2 be the broken wheel with principal path $u_{t_1}x_{m_2}u_{t_2}$, where $H_2 = x_{m_2} = u_{t_1} \cdots u_{t_2}$. Finally, let H_3 be the broken wheel with principal path $u_{t_3}x_{m_3}u_{t_3}$, where $H_3 - x_{m_3} = u_{t_2} \cdots u_{t_3}$. The three principal paths of the respective broken wheels are indicated by the thick edges of the diagram. Note that p is not necessarily the only neighbor of x_1 on $\mathbf{P}$, since, possibly x_1 is adjacent to a subpath of $\mathbf{P}$ of length one. In any case, we have $|L_\phi(x_1)| \geq 3$ by M1).

Let $Q_{\text{left}} := x_1 \cdots x_{m_2}$ and let $Q_{\text{right}} := x_{m_2} \cdots x_{m_3}$. Each of Q_{left} and Q_{right} is an induced subpath of G, each vertex of which, except for x_1, has an L_ϕ-list of size at least five. Furthermore, there is no chord of Π^1_p except possibly $x_{m_2-1}x_{m_2+1}$. Thus, we immediately have the following simple observation.

Proposition 5.1.15. *For any $T \subseteq L(u_{t_1})$ of size at most two and $T' \subseteq L(u_{t_2})$ of size at most two, the following holds: Any L_ϕ-coloring of $\{x_1, x_{m_2}\}$ extends to an L-coloring of $V(Q_{\text{left}})$ in which each internal vertex of Q_{left} is colored by a color not in T. Likewise, any L_ϕ-coloring of $\{x_{m_2}, x_{m_3}\}$ extends to an L-coloring of $V(Q_{\text{right}})$ in which each internal vertex of Q_{right} is colored by a color not in T'.*

Next, we have the following simple fact.

Proposition 5.1.16. *If H_1 is not a triangle and there does not exist a (C,p)-wedge, then there does not exist an $s \in L_\phi(x_1)$ such that $\mathcal{Z}_{H_1, L^p_\phi}(\phi(p), s, \bullet) = L(u_{t_1})$.*

Proof. Suppose toward a contradiction that there is such an s. Let σ be the L_ϕ-coloring of $\{x_1\}$ where $\sigma(x_1) = s$. Since H_1 is not a triangle, we have $H_1 \setminus \{p, u_{t_1}\} \neq \varnothing$, and the pair $(H_1 \setminus \{p, u_{t_1}\}, \sigma)$ is a (C,p)-wedge, contradicting our assumption. □

We break the remainder of the proof of Theorem 5.1.14 into two parts, which are the remainder of Chapter 5.

5.2 Dealing with 3-Chords of C Near the Precolored Path

This section consists of the following lone result.

Lemma 5.2.1. *If there is a 3-chord of C which separates p from an element of $\mathcal{C} \setminus \{C\}$, then there exists a (C,p)-wedge.*

Proof. Let pp_2p_3 be the unique terminal subpath of $\mathbf{P}$ of length two which has p as an endpoint, and let x^* be the lone neighbor of x_1 on the path $C^1 - x_2$. Suppose there is a 3-chord of C separating p from each element of $\mathcal{C} \setminus \{C\}$. By Theorem 4.0.1, this 3-chord is unique, and its lone internal edge is x^*x_2. Furthermore, $x^* \in N(p_2)$, x_1 is the central vertex of a wheel, and $N(x_1)$ consists of all the vertices in the cycle $x^*p_2p_1u_1 \cdots u_{t_1}x_2$. Furthermore, since G is short-separation-free, H_1 is not a triangle. Suppose toward a contradiction that there does not exist a (C,p)-wedge.

Definition 5.2.2. Let Skip(H_1) be the set of partial L_ϕ-colorings ψ of the triangle $x_1x_2u_{t_1}$ such that $x_2, u_{t_1} \in \text{dom}(\psi)$ and one of the following holds.

1) $x_1 \notin \text{dom}(\psi)$ and $|\mathcal{Z}_{H_1, L^p_\phi}(\phi(p), \bullet, \psi(u_{t_1})) \setminus \{\psi(x_2)\}| \geq 2$*; OR*
2) $x_1 \in \text{dom}(\psi)$, $\psi(x_1) \in \mathcal{Z}_{L^p_\phi}(\phi(p), \bullet, \psi(u_{t_1}))$, and $L_{\phi\cup\psi}(x^*)| \geq 2$.

Claim 5.2.3. *For each L_ϕ-coloring ψ of $x_1u_{t_1}$ with $\psi(u_{t_1}) \in \mathcal{Z}_{H_1, L^p_\phi}(\phi(p), \psi(x_1), \bullet)$, there is an extension of ψ to an L_ϕ-coloring ψ^* of $x_1u_{t_1}x_2$ with $\psi^* \in$* Skip(H_1).

Proof: This is just an immediate consequence of the fact that $|L_{\phi\cup\psi}(x^*)| \geq 2$ and $|L_{\phi\cup\psi}(x_2)| \geq 3$. ■

With the above in hand, it is natural to introduce the following definition.

Definition 5.2.4. Given a $k \geq 1$, a *k-bouquet* is a set of k elements of Skip(H_1) which all use the same color on u_{t_1} and k distinct colors on x_2. The color used on u_{t_1} is called the *stem* of the k-bouquet.

For any $\psi \in$ Skip(H_1), $V(H_1)$ is $(L, \psi \cup \phi)$-inert in G, because the only uncolored vertex of $N(x_1) \setminus V(H_1)$ is x^*.

Claim 5.2.5. *There is an $r^{\downarrow} \in L(u_{t_1})$ with $|\mathcal{Z}_{H_1, L_\phi^p}(\phi(p), \bullet, r^{\downarrow})| \geq 2$ and a 2-bouquet $\{\psi_0, \psi_1\}$ using $r^{\downarrow}$ on u_{t_1} such that one of the following holds:*

Bq1) *There exists $\psi_2, \psi_3 \in$ Skip(H_1) such that $\{\psi_0, \psi_1, \psi_2, \psi_3\}$ is a 4-bouquet; OR*

Bq2) *For some $q^{\downarrow} \in L(u_{t_1}) \setminus \{r^{\downarrow}\}$, there exist $\psi_2, \psi_3 \in$ Skip(H_1) such that $\{\psi_2, \psi_3\}$ is a 2-bouquet using $q^{\downarrow}$ on u_{t_1}, and $|\{\psi_0(x_2), \psi_1(x_2)\} \cap \{\psi_2(x_2), \psi_3(x_2)\}| \leq 1$; OR*

Bq3) *There exist $s_0, s_1 \notin \{r^{\downarrow}, \psi_0(x_2), \psi_1(x_2)\}$ such that $\{s_0, s_1, r^{\downarrow}\} \subseteq L_\phi(x_1) \cap L_\phi(x^*) \cap Lu_{t_1})$ and furthermore, $\{s_0, s_1\} = \mathcal{Z}_{H_1, L_\phi^p}(\phi(p), r^{\downarrow}, \bullet)$ and $r^{\downarrow} \in \mathcal{Z}_{H_1, L_\phi^p}(\phi(p), s_j, \bullet))$ for each $j = 0, 1$.*

<u>Proof:</u> By Corollary 1.4.6, there is an $r^{\downarrow} \in L_\phi(u_{t_1})$ and a pair of colors $s_0, s_1 \in L_\phi(x_1)$ such that $s_0, s_1 \in \mathcal{Z}_{H_1, L_\phi^p}(\phi(p), \bullet, r^{\downarrow})$. Since $|L_\phi(x_2)| \geq 5$, there is a pair of L_ϕ-colorings ψ_0, ψ_1 of $u_{t_1}x_2$, each of which uses $r^{\downarrow}$ on u_{t_1} and a color of $L_\phi(x_2) \setminus \{r^{\downarrow}, s_0, s_1\}$ on x_2. For each $j = 0, 1$, $V(H_1)$ is $(L, \phi \cup \psi_j)$-inert in G, and since x_1 is uncolored, we have $\psi_0, \psi_1 \in$ Skip(H_1). Thus, $\{\psi_0, \psi_1\}$ is a 2-bouquet.

For the remainder of the proof of Claim 5.2.5, an ordered triple always denotes an L_ϕ-coloring of the triangle $x_1u_{t_1}x_2$, where the first, second, and third coordinates are the colors used on the respective vertices x_1, u_{t_1}, x_2. Since H_1 is not a triangle, we have $L_\phi(u_{t_1}) = L(u_{t_1})$. Now consider the following cases.

Case 1: Either $\{s_0, s_1\} \not\subseteq L_\phi(x^*)$ or $|L_\phi(x^*)| > 3$

In this case, there is a $j \in \{0, 1\}$ such that $|L_\phi(x^*) \setminus \{s_j\}| \geq 3$, say $j = 0$ without loss of generality.

Subcase 1.1 $s_0 \in L_\phi(x_2)$

In this case, since $|L_\phi(x_2)| \geq 5$, let $f \in L_\phi(x_2) \setminus \{s_0, r^{\downarrow}, \psi_0(x_2), \psi_1(x_2)\}$. Letting $\psi_2 := (s_0, r^{\downarrow}, f)$ and $\psi_3 := (s_1, r^{\downarrow}, s_0)$, each of ψ_2, ψ_3 lies in Skip(H_1), and $\{\psi_0, \psi_1, \psi_2, \psi_3\}$ is a 4-bouquet, so our choice of $r^{\downarrow}$ satisfies Bq1).

Subcase 1.2 $s_0 \notin L_\phi(x_2)$

In this case, there exist distinct $f, f' \in L_\phi(x_2) \setminus \{s_0, r^{\downarrow}, \psi_0(x_2), \psi_1(x_2)\}$. Letting $\psi_2 := (s_0, r^{\downarrow}, f)$ and $\psi_3 := (s_0, r^{\downarrow}, f')$, each of ψ_2, ψ_3 lies in Skip(H_1), and $\{\psi_0, \psi_1, \psi_2, \psi_3\}$ is a 4-bouquet, so our choice of $r^{\downarrow}$ satisfies Bq1).

Case 2: $\{s_0, s_1\} \subseteq L_\phi(x^*)$ and $|L_\phi(x^*)| = 3$

In this case, since $|L_\phi(x_1)| \geq 3$, let $s_2 \in L_\phi(x_1) \setminus \{s_0, s_1\}$ and consider the following subcases.

Subcase 2.1. There is a $q^{\downarrow} \in L(u_{t_1}) \setminus \{r^{\downarrow}\}$ and a $j \in \{0, 1\}$ such that $q^{\downarrow} \in \mathcal{Z}_{H_1, L_\phi^p}(\phi(p), s_j, \bullet) \cap \mathcal{Z}_{H_1, L_\phi^p}(\phi(p), s_2, \bullet)$

In this case, since $|L(x_2) \setminus \{\psi_0, \psi_1, q^{\downarrow}, s_2\}| \geq 1$ and at most one of $\{\psi_0(x_2), \psi_1(x_2)\}$ is equal to s_2, there is a pair of L_ϕ-colorings ψ_2, ψ_3 of $u_{t_1}x_2$, each of which uses $q^{\downarrow}$ on u_{t_1}, such that $|\{\psi_0(x_2), \psi_1(x_2)\} \cap \{\psi_2(x_2), \psi_3(x_2)\}| \leq 1$ and $s_2, s_j \notin \{\psi_2(x_2), \psi_3(x_2)\}$. Thus, we have $\psi_2, \psi_3 \in$ Skip(H_1), so $\{\psi_2, \psi_3\}$ is a 2-bouquet, and Bq2) is satisfied.

Subcase 2.2 For all $q^{\downarrow} \in L(u_{t_1}) \setminus \{r^{\downarrow}\}$ and $j \in \{0, 1\}$, we have $q^{\downarrow} \notin \mathcal{Z}_{H_1, L_\phi^p}(\phi(p), s_j, \bullet) \cap \mathcal{Z}_{H_1, L_\phi^p}(\phi(p), s_2, \bullet)$

In this case, consider the following subcases:

Subcase 2.2.1 For each $j = 0, 1$, $\mathcal{Z}_{H_1, L_\phi^p}(\phi(p), s_j, \bullet) = \{r^{\downarrow}\}$

Since $|\mathcal{Z}_{H_1,L^p_\phi}(\phi(p), s_j, \bullet)| = 1$ for each $j = 0, 1$, we have $s_0, s_1 \in L(u_{t_1})$ by Proposition 1.4.4, so $L(u_{t_1}) \setminus \{r^\downarrow\} = \{s_0, s_1\}$, and, by Proposition 1.4.5, $\{s_0, s_1\} \subseteq \mathcal{Z}_{H_1,L^p_\phi}(\phi(p), s_2, \bullet)$. Since H_1 is not a triangle, we have $\phi(p), s_0, s_1 \in L(u_1)$, so $s_2 \notin L(u_1)$. If $r^\downarrow \neq s_2$, then $\mathcal{Z}_{H_1,L^p_\phi}(\phi(p), s_2, \bullet) = L(u_{t_1})$, contradicting Proposition 5.1.16, so we have $r^\downarrow = s_2$ and $L(u_{t_1}) = \{s_0, s_1, s_2\}$. If $s_2 \in L_\phi(x^*)$, then Bq3) is satisfied, so we are done in that case.

Now suppose that $s_2 \notin L_\phi(x^*)$. In that case, let $f, f' \in L(x_2) \setminus \{\psi_0(x_2), s_0, s_2\}$ and set $\psi_2 := (s_2, s_0, f)$ and $\psi_3 := (s_2, s_0, f')$. We then have $\psi_2, \psi_3 \in \text{Skip}(H_1)$ and $|\{\psi_0(x_2, \psi_1(x_2) \cap \{\psi_2(x_2), \psi_3(x_2)\}| \leq 1$. Since $\{\psi_2, \psi_3\}$ is a 2-bouquet using s_0 on u_{t_1}, Bq2) is satisfied.

Case 2.2.2 For some $j \in \{0, 1\}$, $\mathcal{Z}_{H_1,L^p_\phi}(\phi(p), s_j, \bullet) \neq \{r^\downarrow\}$

In this case, let $L(u_{t_1}) = \{r^\downarrow, q_0, q_1\}$ and suppose without loss of generality that $q_1 \in \mathcal{Z}_{H_1,L^p_\phi}(\phi(p), s_0, \bullet)$. Since $r^\downarrow \in \mathcal{Z}_{H_1,L^p_\phi}(\phi(p), s_0, \bullet)$ as well, it follows from the assumption of Subcase 2.2 that $\mathcal{Z}_{H_1,L^p_\phi}(\phi(p), s_2, \bullet) = \{q_0\}$.

Suppose first that $q_1 \in \mathcal{Z}_{H_1,L^p_\phi}(\phi(p), s_0, \bullet) \cap \mathcal{Z}_{H_1,L^p_\phi}(\phi(p), s_1, \bullet)$. In that case, since $q_1 \neq r^\downarrow$ and $|L_\phi(x_2)| \geq 5$, there exists a pair of colors $f, f' \in L(x_2) \setminus \{s_0, s_1, q_1\}$ such that $\{f, f'\} \neq \{\psi_0(x_2), \psi_1(x_2)\}$. Thus, each of the L_ϕ-colorings (q_1, f) and (q_1, f') of $u_{t_1}x_{m_2}$ lies in Skip(H_1), and Bq2) is satisfied

Now suppose $q_1 \notin \mathcal{Z}_{H_1,L^p_\phi}(\phi(p), s_0, \bullet) \cap \mathcal{Z}_{H_1,L^p_\phi}(\phi(p), s_1, \bullet)$. Thus, we have $\mathcal{Z}_{H_1,L^p_\phi}(\phi(p), s_1, \bullet) = \{r^\downarrow\}$. Since H_1 is not a triangle, we have $L(u_1) = \{\phi(p), s_1, s_2\}$ by Proposition 1.4.4, and $s_0 \notin L(u_1)$. Thus, we have $q_0 = s_0$. But now, by 1) of Proposition 1.4.7, we have $q_0 = s_0 = s_1$, which is false. This completes the proof of Claim 5.2.5. ■

The claim above has the following useful consequence.

Claim 5.2.6. *Either there is a 4-bouquet or there are two 2-bouquets using different colors on u_{t_1}.*

Proof: Let $r^\downarrow, \psi_0, \psi_1$ be as in the statement of Claim 5.2.5. If either of Bq1) orf Bq2) hold, then we are done. If not, then Bq3) holds, so let $L(u_{t_1}) = \{r^\downarrow, s_0, s_1\}$. Then, for each $i, j \in \{0, 1\}$, the L_ϕ-coloring $(s_j, \psi_i(x_2))$ of $u_{t_1}x_2$ lies in Skip(H), since $L_\phi(x^*) \setminus \{\psi_i(x_2)\}| \geq 3$ and $s_j \neq \psi_i(x_2)$, so we are done. ■

Definition 5.2.7. Let Skip$^{\text{aug}}(H_1)$ be the set of partial L_ϕ-colorings ψ' of $V(Q_{\text{left}}) \cup \{u_{t_1}\}$ such that $V(Q_{\text{left}} - x_1) \cup \{u_{t_1}\} \subseteq \text{dom}(\psi')$ and ψ' restricts to an element of Skip(H_1). Given an integer k, an *augmented k-bouquet* is a set of elements of Skip$^{\text{aug}}(H_1)$ all using the same color on u_{t_1}.

Note that if $m_2 = 2$, then Skip$^{\text{aug}}(H_1) = \text{Skip}(H_1)$ and an augmented k-bouquet is just a k-bouqet. By Claim 5.2.5, there exists at least one 2-bouquet, and, since Q_{left} is an induced subpath of G, we immediately have the following:

Claim 5.2.8. *If $m_2 > 2$, then there exists an augmented 4-bouquet. In particular, given a 2-bouquet $\{\psi_0, \psi_1\}$, there exists an augmented 2-bouquet whose elements restrict to $\{\psi_0, \psi_1\}$, and, if $m_2 > 2$, then there exists an augmented 4-bouquet whose elements restrict to $\{\psi_0, \psi_1\}$.*

Now we have the following simple observation.

Claim 5.2.9. *For any $\psi \in$ Skip(H_1) and any $c \in L_{\phi\cup\psi}(x_{m_2})$, we have $\mathcal{Z}_{H_2,L}(\psi(u_{t_1}), c, \bullet) \neq L(u_{t_2})$.*

Proof: Suppose there is a $c \in L_{\phi\cup\psi}(x_{m_2})$ such that $\mathcal{Z}_{H_2,L}(\psi(u_{t_1}), c, \bullet) = L(u_{t_2})$. By Proposition 5.1.15, there is an extension of ψ to to an L_ϕ-coloring σ of dom$(\psi) \cup \{x_3, \cdots, x_{m_2}\}$ using c on x_{m_2}. Let J be the subgraph of G

induced by $V(H_1 \cup H_2 \cup Q_{\text{left}}) \setminus \{p, u_{t_2}\}$. Since $\mathcal{Z}_{H_2,L}(\psi(u_{t_1}), c, \bullet) = L(u_{t_2})$ and no vertex of Π_p^2 has more than two neighbors in Q_{right}, (J, σ) is a (C, p)-wedge, contradicting our assumption. ■

Now we have the following:

Claim 5.2.10. *Either $x_{m_2-1}x_{m_2+1} \in E(G)$ or there is a $z \in V(\Pi_p^2)$ adjacent to each of $x_{m_2-1}, x_{m_2}, x_{m_2+1}$.*

Proof: Suppose that neither of these hold. Thus, Π_p^1 is an induced subgraphof G and each vertex of Π_p^2 has at most two neighbors in Π_p^1. Let $r^\downarrow \in L_\phi(x_1)$ and let $\psi_0, \psi_1 \in \text{Skip}(H_1)$, where $r^\downarrow, \psi_0, \psi_1$ are as in the statement of Claim 5.2.5. Now we have the following.

Subclaim 5.2.11. *$u_{t_2}x_{m_3}$ is an L-shield for H_3.*

Proof: Suppose that $u_{t_2}x_{m_3}$ is not an L-shield for H_3. By Lemma 5.1.8, H_3 is a triangle and $|L(u_{t_2}) \cap L(u_{t_3})| \geq 2$. We first show that, for each $\psi \in \text{Skip}^{\text{aug}}(H_1)$, we have $\mathcal{Z}_{H_2,L}(\psi(u_{t_1}, \psi(x_{m_2}), \bullet) \subseteq L(u_{t_3})$. Suppose towards a contradiction that there is a $\psi \in \text{Skip}^{\text{aug}}(H_1)$ for which this does not hold, and let $d \in \mathcal{Z}_{H_2,L}(\psi(u_{t_1}), \psi(x_{m_2}), \bullet)$ with $d \notin L(u_{t_3})$. Let τ be the extension of ψ obtained by coloring u_{t_2} with d. Since $|L_{\phi\cup\tau}(u_{t_3})| = 3$ and $|L_{\phi\cup\tau}(x^*)| \geq 2$, the pair $(G[V(\Pi_p^0 \cup \Pi_p^1)], \tau)$ is a (C, p)-wedge, contradicting our assumption. Now consider the following cases:

Case 1: There exists a $\psi \in \text{Skip}^{\text{aug}}(H_1)$ such that $\psi(x_2) \notin L(u_{t_2})$

Since $|L_{\phi\cup\psi}(x_{m_3})| \geq 4$, we fix an $r^* \in L_{\phi\cup\psi}(x_{m_3}) \setminus L(u_{t_3})$. As shown above, since $r^* \notin L(u_{t_3})$, we have $r^* \notin \mathcal{Z}_{H_2,L}(\psi(u_{t_1}), \psi(x_2), \bullet)$. Since Π_p^1 is an induced subgraph of G, there is an extension of ψ to an L_ϕ-coloring ψ' of $\text{dom}(\psi) \cup V(\Pi_p^1)$ such that $\psi'(x_{m_3}) = r^*$. Since $r^* \neq \psi(x)$, this is true even if Q_{right} is an edge. By assumption $(G[V(\Pi_p^0 \cup \Pi_p^1)], \psi')$ is not a (C, p)-wedge, so the inertness condition is violated. That is, ψ' extends to an L-coloring τ of $\text{dom}(\phi \cup \psi') \cup \{u_{t_3}\}$ such that τ does not extend to L-color the path $u_{t_1+1} \cdots u_{t_3-1}$. Since H_3 is a triangle and $r^* \notin \mathcal{Z}_{H_2,L}(\psi(u_{t_1}), \psi(x_{m_2}), \bullet)$, we have $\mathcal{Z}_{H_2,L}(\psi(u_{t_1}), \psi(x_{m_2}), \bullet) = \{\tau(u_{t_3})\}$. By Proposition 1.4.4, since $\psi(x_{m_2}) \notin L(u_{t_2})$, we have $|\mathcal{Z}_{H_2,L}(\psi(u_{t_1}), \psi(x_{m_2}), \bullet)| > 1$, a contradiction.

Case 2: For all $\psi \in \text{Skip}^{\text{aug}}(H_1)$, we have $\psi(x_2) \in L(u_{t_2})$

In this case, since $|L(u_{t_2})| = 3$, there does not exist an augment 4-bouquet. Since $\{\psi_0, \psi_1\}$ is a 2-bouquet, it follows from Claim 5.2.8 that $m_2 = 2$ and $\text{Skip}^{\text{aug}}(H_1) = \text{Skip}(H_1)$. Now we apply Claim 5.2.6. There exists a 2-bouquet $\{\psi_2, \psi_3\}$ and a $q^\downarrow \in L_\phi(u_{t_1}) \setminus \{r^\downarrow\}$ such that ψ_2, ψ_3 use $q^\downarrow$ on u_{t_1}. We now fix an $r^* \in L(x_{m_3}) \setminus (L(u_{t_2}) \cup L(u_{t_3})$. By the assumption of Case 2, we have $r^* \neq \psi(x_2)$ for each $\psi \in \text{Skip}(H_1)$. Thus, for each $j = 0, 1, 2, 3$, there is an extension of ψ_j to an L_ϕ-coloring ψ'_j of of $\text{dom}(\psi_j) \cup V(\Pi_p^1)$ such that $\psi'_j(x_{m_3}) = r^*$. Since $r^* \neq \psi_j(x_2)$, this is true even if Q_{right} is an edge.

For each $j = 0, 1, 2, 3$, since $(G[V(\Pi_p^0 \cup \Pi_p^1)], \psi'_j)$ is not a (C, p)-wedge, the inertness condition is violated, so there is an extension of $\phi \cup \psi'_j$ to an L-coloring τ_j of $\text{dom}(\phi \cup \psi_j) \cup \{u_{t_3}\}$ such that τ_j does not extend to L-color the path $u_{t_1+1} \cdots u_{t_3-1}$. For each $j = 0, 1, 2$, since H_3 is a triangle and $r^* \notin \mathcal{Z}_{H_2,L}(\psi_j(u_{t_1}), \psi_j(x_2), \bullet)$, we have $\mathcal{Z}_{H_2,L_p^\phi}) = \{\tau_j(u_{t_3})\}$.

Now, since $\{\psi_0(x_2), \psi_1(x_2)\} \cup \{\psi_2(x_2), \psi_3(x_2)\} \subseteq L(u_{t_2})$, we suppose without loss of generality that $\psi_2(x_2) = \psi_0(x_2) = c$ for some color $c \in L(x_2)$. Let $c' \in L(u_{t_3}) \setminus \{\tau_0(u_{t_3}), \tau_2(u_{t_3})\}$. By Observation 1.4.2, the L-coloring (c, c') of $x_2u_{t_2}$ extends to L-coloring H_2 using one of $r^\downarrow, q^\downarrow$ on u_{t_1}, contradicting the fact that $\mathcal{Z}_{H_)2,L}(\psi_j(u_{t_1}), c, \bullet) = \{\tau_j(u_{t_3})\}$ for each $j = 0, 2$. ■

Since $u_{t_2}x_{m_3}$ is an L-shield for H_3, there exist two elements σ_0, σ_1 of $\mathcal{M}_L(H_3, u_{t_2}x_{m_3})$ such that either $\{\sigma_0(u_{t_2}), \sigma_0(x_{m_3})\} = \{\sigma_1(u_{t_2}), \sigma_1(x_{m_3})\}$ or σ_0, σ_1 use the same color on x_{m_3}. Now we have the following simple observation.

Subclaim 5.2.12. *For each $i \in \{0,1\}$ and $\psi \in \text{Skip}^{\text{aug}}(H_1)$, at least one of the following holds.*

1) $\sigma_i(u_{t_2}) \notin \mathcal{Z}_{H_2,L}(\psi(u_{t_1}), \psi(x_{m_2}), \bullet)$; OR

2) Q_{right} is an edge and $\psi(x_{m_2}) = \sigma_i(x_{m_3})$.

Proof: If there exist an $i \in \{0,1\}$ and a $\psi \in \text{Skip}^{\text{aug}}(H_1)$ for which this does not hold, then the union $\psi \cup \sigma_i$ is a proper L_ϕ-coloring of its domain which extends to an L_ϕ-coloring ψ^* of dom$(\psi \cup \sigma_i) \cup V(H_2 \cup Q_{\text{right}})$, and $(G[V(\Pi_p^0 \cup \Pi_p^1)], \psi^*)$ is a (C,p)-wedge, contradicting our assumption. ■

Let $S_{\text{right}} := \{\sigma_0(u_{t_2}), \sigma_1(u_{t_2})\}$. We now have the following.

Subclaim 5.2.13. *σ_0, σ_1 use the same color on x_{m_3}*

Proof: Since σ_0, σ_1 do not use the same color on x_{m_3}, there is a pair of colors $a, b \in L(u_{t_2}) \cap L(x_{m_3})$ such that $\{\sigma_i(u_{t_2}), \sigma_i(x_{m_3})\} = \{a,b\}$ for each $i = 0,1$, so suppose that $\sigma_0(u_{t_2}) = a$ and $\sigma_1(u_{t_2}) = a$.

If there is a $\psi \in \text{Skip}^{\text{aug}}(H_1)$ such that either Q_{right} is not an edge or $\psi(x_{m_2}) \notin \{a,b\}$, then, by Observation 1.4.2, we have $\{a,b\} \cap \mathcal{Z}_{H_2,L}(\psi(u_{t_1}), \psi(x_{m_2}), \bullet) \neq \varnothing$, contradicting Subclaim 5.2.12. Thus, Q_{right} is an edge, and $\psi(x_{m_2}) \in \{a,b\}$ for each $\psi \in \text{Skip}^{\text{aug}}(H_1)$. In particular, there does not exist an augmented 4-bouquet, and, by Claim 5.2.8, $m_2 = 2$ and $\text{Skip}^{\text{aug}}(H_1) = \text{Skip}(H_1)$.

Now we apply Claim 5.2.5. Since $\psi(x_2) \in \{a,b\}$ for each $\psi \in \text{Skip}(H_1)$, neither Bq1) nor Bq2) is satisfied, so $r^\downarrow, \psi_0, \psi_1$ satisfy Bq3). Let $s_0, s_1 \in L(x_1)$ be as in Bq3) of Claim 5.2.5. Since $L(x_2)| \geq 5$, there are distinct colors $f_0, f_1 \in L(x_2)$ such that, for each $i = 0,1$, $f_i \notin L(x_2) \setminus \{a, b, s_i, r^\downarrow\}$. Possibly $f_i = s_{1-i}$ for each $i = 0,1$. We now note the following simple observation

$$\text{There exists an } i \in \{0,1\} \text{ such that } s_i, r^\downarrow \in \mathcal{Z}_{H_2,L}(\bullet, f_i, \sigma_i(u_{t_2})). \tag{$\star$}$$

Suppose that $(\star)$ does not hold. Since $\{a,b\} = \{\psi_0(x_2), \psi_1(x_2)\}$, we have $\{a,b\} \cap \{r^\downarrow, s_0, s_1\} = \varnothing$. Since $|\{s_i, r^\downarrow\} \cap \mathcal{Z}_{H_2,L}(\bullet, f_i, \sigma_i(u_{t_2}))| \leq 1$ for each $i = 0,1$, H_2 is not a triangle, and, by Proposition 1.4.4, we have $\{a,b\} \cup \{f_0, f_1\} \subseteq L(u_{t_2 - 1})$, contradicting the fact that $|L(u_{t_2-1})| = 3$. Thus, $(\star)$ holds, so suppose without loss of generality that $s_0, r^\downarrow \in \mathcal{Z}_{H_2,L}(\bullet, f_0, a)$ and let σ_0^* by an extension of σ_0 to an L_ϕ-coloring of $\{u_{t_2}, x_{m_3}, x_2\}$ obtained by coloring x_2 with f_0.

By assumption, the pair $(G[V(\Pi_p^0 \cup \Pi_p^1)], \sigma_0^*)$ is not a (C,p)-wedge, so the inertness condition is violated. Thus, there is an extension of $\phi \cup \sigma_0^*$ to an L-coloring τ of dom$(\phi \cup \sigma_0^*) \cup \{x^*, u_{t_3}\}$ which does not extend to L-color $V(H_1 \cup \Pi_p^0)$. Only four neighbors of x_1 are colored, so $|L_\tau(x_1)| \geq 1$. Since σ_0^* restricts to an element of $\mathcal{M}_L(H_3, u_{t_2}x_{m_3})$, it follows that, for each $q \in L_\tau(x_1)$, we have $\mathcal{Z}_{H_1, L_\phi^p}(\phi(p), q, \bullet) \cap \{s_0, r^\downarrow\} = \varnothing$. Thus, for each $q \in L_\tau(x_1)$, we have $\mathcal{Z}_{H_1, L_\phi^p}(\phi(p), q, \bullet) = \{s_1\}$, and, by Observation 1.4.2, $q \in \{s_0, r^\downarrow\}$, yet, since Bq3) is satisfied, we have $r^\downarrow \in \mathcal{Z}_{H_1, L_\phi^p}(\phi(p), s_0, \bullet)$ and $s_0 \in \mathcal{Z}_{H_1, L_\phi^p}(\phi(p), r^\downarrow, \bullet)$, a contradiction. ■

Applying Subclaim 5.2.13, there is a color $q \in L(x_{m_3})$ such that $\sigma_i(x_{m_3}) = q$ for each $i = 0,1$.

Subclaim 5.2.14. $m_2 = 2$ *and* $\text{Skip}^{\text{aug}}(H_1) = \text{Skip}(H_1)$. *Furthermore, for any $\psi \in \text{Skip}(H_1)$, we have $\psi(x_2) \in S_{\text{right}} \cup \{q\}$.*

Proof: Let $\psi \in \text{Skip}^{\text{aug}}(H)$. If either Q_{right} is not an edge or $\psi(x_{m_2}) \neq q$, then, by Subclaim 5.2.12, we have $\mathcal{Z}_{H_2,L}(\psi(u_{t_1}), \psi(x_{m_2}), \bullet) \cap S_{\text{right}} = \varnothing$ and thus, by Observation 1.4.2, $\psi(x_{m_2}) \in S_{\text{right}}$. Thus, we have

$\psi(x_{m_2}) \in S_{\text{right}} \cup \{q\}$ for each $\psi \in$ Skip, and there is no augmented 4-bouquet. Thus, by Claim 5.2.8, we have $m_2 = 2$. ■

Applying Subclaim 5.2.14, there does not exist a 4-bouquet. By Claim 5.2.6, there is a 2-bouquet $\{\psi_2, \psi_3\}$ using a color other than $r^{\downarrow}$ on u_{t_1}. At least one of $\psi_0(x_2), \psi_1(x_2)$ is distinct from q. Likewise, at least one of $\psi_2(x_2), \psi_3(x_2)$ is distinct from q. Suppose without loss of generality that $\psi_0(x_2), \psi_2(x_2) \neq q$. For each $i = 0, 2$, we then have $\mathcal{Z}_{H_2,L}(\psi_j(u_{t_1}), \psi_j(x_2), \bullet) \cap S_{\text{right}} = \varnothing$ by Subclaim 5.2.12. Thus, there is a lone color $c \in L(u_{t_2}) \setminus S_{\text{right}}$ such that $\mathcal{Z}_{H_2,L} = (\psi_j(u_{t_1}), \psi_j(x_2), \bullet) = \{c\}$ for each $j = 0, 2$. Since $\psi_0(u_{t_1}) \neq \psi_2(u_{t_1})$, it follows from 2) of Proposition 1.4.7 that $\psi_0(x_2) = \psi_2(u_{t_1})$ and $\psi_2(x_2) = \psi_0(u_{t_1})$. Thus, $\{\psi_0(x_2), \psi_2(x_2)\} = S_{\text{right}}$, and $S_{\text{right}} \subseteq L(u_{t_1})$.

Since $\mathcal{Z}_{H_2,L}(\psi_j(u_{t_1}), \psi_j(x_2), \bullet) = \{c\}$ for each $j = 0, 2$, H_2 is not a triangle, and, by Proposition 1.4.4, $S_{\text{right}} \subseteq L(u_{t_1}) \cap L(u_{t_2-1})$.

The trick now is to leave x_1 uncolored. Since $|L(x_2)| \geq 5$, we let ζ_0, ζ_1 be two L-coloring of $\{x_2\}$ with $\zeta_0(x_2), \zeta_1(x_2) \notin L(u_{t_2})$. Let J be the subgraph of G induced by $V(H_1 \cup H_2 \cup Q_{\text{left}}) \setminus \{p, u_{t_2}\}$. For each $k = 0, 1$, (J, ζ_k) is not a (C, p)-wedge, and since x_1 is uncolored and $\zeta_k(x_2) \notin L(u_{t_2})$, we have $|L_{\phi \cup \zeta_k}(x^*)| \geq 2$ and $|L_{\phi \cup \zeta_k}(u_{t_2})| = 2$, so the inertness condition is violated. That is, there is an extension of $\phi \cup \zeta_k$ to an L-coloring τ_k of $\text{dom}(\phi \cup \zeta_k) \cup \{x^*, u_{t_2}\}$ such that τ_k does not extend to L-color the pair of broken wheels $H_1 \cup H_2$. For each $k = 0, 1$, since x_1 has four colored neighbors, we have $|L_{\tau_k}(x_1)| \geq 1$, so we immediately get the following.

Subclaim 5.2.15. *$k \in \{0, 1\}$ and $d \in L_{\tau_k}(x_1)$, we have $\mathcal{Z}_{H_1, L^p_\phi}(\phi(p), d, \bullet) \cap \mathcal{Z}_{H_2,L}(\bullet, \zeta_k(x_2), \tau_k(u_{t_2})) = \varnothing$.*

Since $S_{\text{right}} \subseteq L(u_{t_1}) \cap L(u_{t_2})$, there is a $k \in \{0, 1\}$ such that $\zeta_k(x_2) \notin L(u_{t_1})$, say $k = 0$ without loss of generality. If $\zeta_0(x_2) \notin L(u_{t_1+1}) \cap L(u_{t_2-1})$, then, by Proposition 1.4.4, $\mathcal{Z}_{H_2,L}(\bullet, \zeta_0(x_2), \tau_0(x_2)) = L(u_{t_1})$, contradicting Subclaim 5.2.15. Thus, $L(u_{t_1+1}) = L(u_{t_2-1}) = S_{\text{right}} \cup \{\zeta_0(x_2)\}$ and $\zeta_1(x_2) \in L(u_{t_1})$, so $L(u_{t_1}) = S_{\text{right}} \cup \{\zeta_1(x_2)\}$.

Let $f \in L(x_2) \setminus (S_{\text{right}} \cup \{q, \zeta_1(x_2)\})$. Since Π^1_p is an induced subgraph of G, we let ψ^* be an L_ϕ-coloring of $\Pi^1_p - x_1$ in which $\psi^*(x_2) = f$ and $\psi^*(x_{m_3}) = q$, and furthermore, each internal vertex of $\Pi^1_p - x_1$ use a color not lying in S_{right}. Since $f \neq q$, such a ψ^* exists even if Q_{right} is an edge.

Consider the pair $(G[V(\Pi^0_p \cup \Pi^1_p)], \psi^*)$. Since q is used by an element of $\mathcal{M}_L(H_3, u_{t_2} x_{m_3})$, on x_{m_3}, we have $q \notin L(u_{t_3})$, and since x_1 is uncolored, we have $|L_{\phi \cup \psi^*}(x^*)| \geq 2$ and $|L_{\phi \cup \psi^*}(u_{t_3})| \geq 3$. By assumption, $(G[V(\Pi^0_p \cup \Pi^1_p)], \psi^*)$ is not a (C, p)-wedge, so the inertness condition is violated. That is, $\phi \cup \psi^*$ extends to an L-coloring $\psi^\dagger$ of $\text{dom}(\phi \cup \psi^*) \cup \{x^*, u_{t_3}\}$ which does not extend to L-color $V(H_1 \cup \Pi^0_p)$. Since $|L_{\psi^\dagger}(x_1)| \geq 1$, let $d \in L_{\psi^\dagger}(x_1)$. Since $L(u_{t_1}) = S_{\text{right}} \cup \{\zeta_1(x_2)\}$, we have $f \notin L(u_{t_1})$, so $\psi^\dagger$ extends to an L-coloring $\psi^{\dagger\dagger}$ of $\text{dom}(\psi^\dagger) \cup V(H_1)$ using d on x_1. Now, since $f \notin S_{\text{right}}$, we have $\mathcal{Z}(\psi^{\dagger\dagger}(u_{t_1}), \psi^{\dagger\dagger}(x_2), \bullet) \cap S_{\text{right}} \neq \varnothing$ by Observation 1.4.2. Since each coloring of $u_{t_2} x_{m_3}$ using q on x_{m_3} and a color of S_{right} on u_{t_2} lies in $\mathcal{M}_L(H_3, u_{t_2} x_{m_3})$, it follows that $\psi^{\dagger\dagger}$ extends to L-color $V(H_2 \cup H_3)$, so $\psi^\dagger$ extends to L-color $V(H_1 \cup \Pi^0_p)$, a contradiction. This completes the proof of Claim 5.2.10. ■

Since $N(x_1) \setminus V(C) = \{x_2, x^*\}$, it follows from Claim 5.2.10 that $m_2 > 2$. We now have the following useful fact.

Claim 5.2.16. *If there exist two distinct colors of $L(u_{t_1})$ which are the ties of 2-bouquets, then $u_{t_1} x_{m_2}$ is not an L-shield for H_2.*

Proof: Suppose that $u_{t_1} x_{m_2}$ is an L-shield for H_2 and suppose toward a contradiction that there are two colors $r^{\downarrow}, q^{\downarrow}$ of $L(u_{t_1})$ which are the both the ties of 2-bouquet. Thus, there is a pair of elements $\mathcal{M}_L(H_2, u_{t_1} x_{m_2})$ using different colors on u_{t_1}, and since $|L(u_{t_1})| = 3$, there is an element of $\mathcal{M}_L(H_2, u_{t_1} x_{m_2})$ using one of $r^{\downarrow}, q^{\downarrow}$ on u_{t_1}. Since each of $r^{\downarrow}, q^{\downarrow}$ is the stem of a 2-bouquet, it follows that there is a $\psi \in \text{Skip}(H_1)$ and a $\zeta \in \mathcal{M}_L(H_2, u_{t_1} x_{m_2})$ such that $\psi \cup \zeta$

is a proper L_ϕ-coloring of its domain, and thus $\psi\cup\zeta$ extends to an L_ϕ-coloring ζ^* of $\text{dom}(\psi\cup\zeta)\cup\{x_3,\cdots,x_{m_2-1}\}$. By definition, $\mathcal{Z}_{H_2,L}(\zeta^*(u_{t_1}),\zeta^*(x_{m_2}),\bullet)=L(u_{t_2})$ contradicting Claim 5.2.9. ■

We now deal with the case where there is a chord of Π_p^1.

Claim 5.2.17. *Π_p^1 is an induced subgraph of G.*

Proof: Suppose not. We then have $x_{m_2-1}x_{m_2+1}\in E(G)$. Then $N(x_{m_2})=\{x_{m_2-1},x_{m_2+1}\}\cup V(H_2-x_{m_2})$. Since G is short-separation-free, H_2 is not a triangle. Let $r^\downarrow,\psi_0,\psi_1$ be as in Claim 5.2.5, where $r^\downarrow\in L(u_{t_1})$ and $\{\psi_0,\psi_1\}$ is a 2-bouquet using $r^\downarrow$ on u_{t_1}.

Subclaim 5.2.18. *$u_{t_2}x_{m_3}$ is an L-shield for H_3.*

Proof: Suppose not. By Lemma 5.1.8, H_3 is a triangle and $L(u_{t_2})\cap L(u_{t_3})|\geq 2$, so let $q\in L(x_{m_3})\setminus(L(u_{t_1})\cup L(u_{t_2}))$. By Proposition 1.4.5, there is an $s\in L(x_2)\setminus\{q,r^\downarrow\}$ such that $|\mathcal{Z}_{H_2,L}(r^\downarrow,s,\bullet)|\geq 2$. Let S_{right} be a set of two colors in $\mathcal{Z}_{H_2,L}(r^\downarrow,s,\bullet)$. Consider the following cases.

Case 1: Either $\{x_{m_2},x_{m_3}\}\neq\{x_3,x_4\}$ or $\{\psi_0(x_2),\psi_1(x_2)\}\neq\{s,q\}$

In this case, there is an $i\in\{0,1\}$ and an extension of ψ_i to an L_ϕ-coloring σ of $\text{dom}(\phi_i)\cup V(\Pi_p^1-x_1)$ such that σ uses s,q on the respective vertices x_{m_2},x_{m_3} and does not use a color of S_{right} on any internal vertex of Q_{right}. By assumption, $(G[V(\Pi_p^0\cup\Pi_p^1)],\sigma)$ is not a (C,p)-wedge, and since no vertex of Π_p^2 is adjacent to more than two vertices of Π_p^1, the inertness conditions is violated. Since H_3 is a triangle and σ restricts to an element of $\text{Skip}(H_1)$, it follows that $\phi\cup\sigma$ extends to an L-coloring τ of $\text{dom}(\phi\cup\sigma)\cup V(H_1)\cup\{u_{t_3}\}$ such that τ does not extend to L-color H_2. Thus, we have $L_\tau(u_{t_2})\cap S_{\text{right}}=\varnothing$. But since $|L(u_{t_2})|=3$ and $q\notin L(u_{t_2})$, we have $S_{\text{right}}\cap L_\tau(u_{t_2})\neq\varnothing$, a contradiction.

Case 2: $\{x_{m_2},x_{m_3}\}=\{x_3,x_4\}$ and $\{\psi_0(x_2),\psi_1(x_2)=\{s,q\}$

In this case, $\Pi_p^0=x_1x_2x_3x_4$, $m_2=3$, $m_3=4$, and the lone chord of Π_p^1 is x_2x_4. If $q\notin L(x_2)$, then $|L(x_{m_2})\setminus\{s,q,r^\downarrow\}|\geq 3$ and, by Proposition 1.4.5, there is an $s'\in L(x_{m_2})\setminus\{s,q,r^\downarrow\}$ with $|\mathcal{Z}_{H_2,L}(r^\downarrow,s',\bullet)|\geq 2$, so we are back to Case 1 with s replaced by s'.

Now suppose that $q\in L(x_{m_2})$. Since $\{s,q\}=\{\psi_0(x_2),\psi_1(x_2)\}$, we have $q\neq r^\downarrow$ and there iss precisely one $i\in\{0,1\}$ such that $q\in L_{\phi\cup\psi_i}(x_{m_2})$. By Claim 5.2.9, we have $\mathcal{Z}_{H_2,L}(r^\dagger,q,\bullet)\neq L(u_{t_2})$. Since $q\notin L(u_{t_2})$ and H_1 is not a triangle, it follows from Proposition 1.4.4 that $r^\dagger,q\in L(u_{t_1+1})$. Thus, there is an $s'\in L(x_2)\setminus\{r^\downarrow,q,s\}$ with $s'\notin L(u_{t_1+1})$, so, again applying Proposition 1.4.4, we have $|\mathcal{Z}_{H_2,L}(r^\downarrow,s',\bullet)|\geq 2$, and since $s'\notin\{\psi_0(x_2),\psi_1(x_2)\}$, we are back to Case 1 with s replaced by s'. ■

Now we have the following:

Subclaim 5.2.19. *Let $\psi\in\text{Skip}(H_1)$ and $\sigma\in\mathcal{M}_L(H_3,u_{t_2}x_{m_3})$ and suppose there is an $s\in\mathcal{Z}_{H_2,L}(\psi(u_{t_1}),\bullet,\sigma(u_{t_2})\cap L_{\phi\cup\psi}(x_{m_2})\cap L_{\phi\cup\sigma}(x_{m_2}))$. Then $\{x_{m_2},x_{m_3}\}=\{x_3,x_4\}$ and $\psi(x_{m_2-1})=\sigma(x_{m_2+1})$.*

Proof: Suppose that at least one one of these does not hold. Then $\psi\cup\sigma$ is a proper L_ϕ-coloring its domain which extends to an L_ϕ-coloring σ^* of $\text{dom}(\psi\cup\sigma)\cup V(\Pi_p^1-x_1)$ such that $\sigma^*(x_2)=s$. Since σ^* restricts to an element of $\text{Skip}(H_1)$ and to an element of $\mathcal{M}_L(H_3,u_{t_2}x_{m_3})$, the inertness condition of Definition 5.1.12 are satisfied, and $(G[V(\Pi_p^0\cup\Pi_p^1)],\sigma^*)$ is a (C,p)-wedge, contradicting our assumption. ■

Since $u_{t_2}x_{m_3}$ is an L-shield for H_3, let σ_0,σ_1 be a pair of elements of $\mathcal{M}_L(H_3,u_{t_2}x_{m_3})$ such that either σ_0,σ_1 use the same color on x_{m_3} or $\{\sigma_0(u_{t_2}),\sigma_0(x_{m_3})\}=\{\sigma_1(u_{t_2}),\sigma_1(x_{m_3})\}$.

Subclaim 5.2.20. *σ_0, σ_1 use the same color on x_{m_3}*

Proof: Suppose not. Then there is a pair of colors $\{a, b\}$ such that $\{\sigma_i(x_{m_2}), \sigma_i(u_{t_2})\} = \{a, b\}$ for each $i = 0, 1$. Consider the following cases:

Case 1: $\{\psi_0(x_2), \psi_1(x_2)\} \neq \{a, b\}$

In this case, suppose without loss of generality that $\psi_0(x_2) \notin \{a, b\}$. Since $|L_\phi(x_2)| \geq 5$, there is an $s \in L_\phi(x_2) \setminus \{r^\downarrow, \psi_0(x_2), a, b\}$. By Observation 1.4.2, $\mathcal{Z}_{H_2,L}(r^\downarrow, s, \bullet) \cap \{a, b\} \neq \varnothing$ so there is an $i \in \{0, 1\}$ such that $s \in \mathcal{Z}_{H_2,L}(r^\downarrow, \bullet, \sigma_i(u_{t_2}))$ and such that $s \in L_{\phi \cup \sigma_i}(x_{m_2})$. But since $s \in L_{\phi \cup \psi_0}(x_{m_2})$ as well, we contradict Subclaim 5.2.19.

Case 2: $\{\psi_0(x_2), \psi_1(x_2)\} = \{a, b\}$

In this case, we just choose an $s \in L_\phi(x_2) \setminus \{r^\downarrow, a, b\}$. As above, since $\mathcal{Z}_{H_2,L}(r^\downarrow, s, \bullet) \cap \{a, b\} \neq \varnothing$ there is an $i \in \{0, 1\}$ such that $s \in \mathcal{Z}_{H_2,L}(r^\downarrow, \bullet, \sigma_i(u_{t_2}))$ and such that $s \in L_{\phi \cup \sigma_i}(x_{m_2})$. There is precisely one $j \in \{0, 1\}$ such that $\psi_j(x_2) \neq \sigma_i(x_{m_3})$. But since $s \notin \{r^\downarrow, \psi_0(x_2), \psi_1(x_2)\}$, we have $s \in L_{\phi \cup \psi_j}(x_{m_2})$, contradicting Subclaim 5.2.19. ■

Applying Subclaim 5.2.20, let $q \in L(x_{m_3})$, where each of σ_0, σ_1 uses q on x_{m_3}. Let $\{\sigma_0(u_{t_2}), \sigma_1(u_{t_2})\} = \{a, b\}$ and let c be the lone color of $L(u_{t_2}) \setminus \{a, b\}$.

Subclaim 5.2.21. *For any $\psi \in \text{Skip}(H_1)$ with $\psi(x_2) \neq q$, we have $L(x_{m_2}) = \{\psi(u_{t_1}), \psi(x_2), q, a, b\}$, and furthermore, $\mathcal{Z}_{H_2,L}(\psi(u_{t_1}), a, \bullet) = \mathcal{Z}_{H_2,L}(\psi(u_{t_1}), b, \bullet) = \{c\}$.*

Proof: If there is an $s \in L(x_2) \setminus \{\psi(u_{t_1}), \psi(x_2), q, a, b\}$, then, by Observation 1.4.2, $\mathcal{Z}_{H_2,L}(\psi(u_{t_1}), s, \bullet) \cap \{a, b\} \neq \varnothing$, so there exists a $j \in \{0, 1\}$ with $s \in \mathcal{Z}_{H_2,L}(\psi(u_{t_1}), \bullet, \sigma_j(u_{t_2}))$. Furthermore, $s \in L_{\phi \cup \psi}(x_{m_2})$ and, since $s \notin \{q, a, b\}$, we have $s \in L_{\phi \cup \sigma_j}(x_{m_2})$, so we contradict Subclaim 5.2.19. We conclude that $L(x_2) = \{\psi(u_{t_1}), \psi(x_{m_2}), q, a, b\}$, and, again applying Subclaim 5.2.19, $\mathcal{Z}_{H_2,L}(\psi(u_{t_1}), a, \bullet) = \mathcal{Z}_{H_2,L}(\psi(u_{t_1}), b, \bullet) = \{c\}$. ■

Subclaim 5.2.21 immediately implies that $L(x_{m_2}) \setminus \{q, a, b\}| = 2$ and that, for any $\psi \in \text{Skip}(H_1)$, either $\psi(x_2) = q$ or $\psi(x_2) \in L(x_{m_2}) \setminus \{q, a, b\}$. Thus, there does not exist a 4-bouquet.

Now, applying Claim 5.2.6, there is a 2-bouquet $\{\psi_0', \psi_1'\}$ using a color $q^\downarrow \in L(u_{t_1}) \setminus \{r^\downarrow\}$ on u_{t_1}. At least one of $\{\psi_0(x_2), \psi_1(x_2)\}$ is distinct from q, and likewise for $\{\psi_0', \psi_1'\}$, so suppose without loss of generality that $q \neq \psi_0(x_{m_2}), \psi_0'(x_{m_2})$. By Subclaim 5.2.21, we have $\{a, b\} \cap \{r^\downarrow, q^\downarrow\} = \varnothing$, and furthermore, $\mathcal{Z}_{H_2,L}(r^\downarrow, a, \bullet) = \mathcal{Z}_{H_2,L}(q^\downarrow, a, \bullet) = \{c\}$. By Observation 1.4.2, the L-coloring (a, b) of $x_{m_2}u_{t_2}$ extends to an L-coloring of H_2 using one of $r^\downarrow, q^\downarrow$ on u_{t_1}, so we have a contradiction. This completes the proof of Claim 5.2.17. ■

Applying Claim 5.2.10 and Claim 5.2.17, there is a vertex $z \in V(\Pi_p^2)$ adjacent to each of $x_{m_2-1}, x_{m_2}, x_{m_2+1}$. We now have the following simple observation.

Claim 5.2.22. *Let r be the stem of a 2-bouquet. If H_2 is not a triangle, then there is at most one color $c \in L(x_2) \setminus \{r\}$ such that $\mathcal{Z}_{H_2,L}(r, c, \bullet)| = 1$.*

Proof: Suppose toward a contradiction that there exist two colors $c_0, c_1 \in L(x_{m_2}) \setminus \{r\}$ such that $|\mathcal{Z}_{H_2,L}(r, c_i, \bullet)| = 1$ for each $i = 0, 1$ Since H_2 is not a triangle, it follows from Proposition 1.4.4 that $c_0, c_1 \in L(u_{t_1+1}) \cap L(u_{t_2})$ and that $r \in L(u_{t_1})$. Since $L(u_{t_1}) \cap L(u_{t_2})| \geq 2$, there is an $r^* \in L(x_{m_2}) \setminus (L(u_{t_1+1}) \cup L(u_{t_2}))$. Since $r \in L(u_{t_1+1})$, we have $r^* \neq r$. Since there is a 2-bouquet using r on u_{t_1}, there is an element of $\text{Skip}^{\text{aug}}(H_1)$ using r, r^* on the respective vertices u_{t_1}, x_{m_2}. Since $r^* \notin L(u_{t_1+1})$, we have $\mathcal{Z}(r, r^*, \bullet) = L(u_{t_2})$, contradicting Claim 5.2.9. ■

As above, we first deal with the case where $u_{t_2}x_{m_3}$ is not an L-shield for H_3.

Claim 5.2.23. *$u_{t_2}x_{m_3}$ is an L-shield for H_3.*

Proof: Suppose not. By Lemma 5.1.8, H_3 is a triangle and $|L(u_{t_2}) \cap L(u_{t_3})| \geq 2$. Thus, we fix a color $r^* \in L(x_{m_3}) \setminus (L(u_{t_2}) \cup L(u_{t_3}))$. Since each vertex of Q_{right} has an L_ϕ-list of size at least five, there is an L_ϕ-coloring σ of $Q_{\text{right}} - x_{m_2}$ such that $\sigma(x_{m_3}) = r^*$ and σ does not use a color of $L(u_{t_2})$ on any vertex of $Q_{\text{right}} - x_{m_2}$. Possibly Q_{right} is an edge and this is just a coloring of a lone vertex.

Subclaim 5.2.24. *Let $\psi \in \text{Skip}(H_1)$ and let ψ' be an extension of ψ to an L_ϕ-coloring of $\text{dom}(\psi) \cup \{x_2, \cdots, x_{m_2-1}\}$. Then the following holds:*

1. *There is at most one $c \in L(x_{m_2}) \setminus \{\sigma(x_{m_2+1}), \psi'(u_{t_1}), \psi'(x_{m_2-1})\}$ such that $|\mathcal{Z}_{H_2,L}(\psi'(u_{t_1}), c, \bullet)| \geq 2$.*
2. *$\psi'(u_{t_1}) \in L(u_{t_1+1})$ and $|L_{\psi' \cup \sigma}(x_{m_2}) \cap L(u_{t_1+1})| \geq 1$*

Proof: We first prove 1). Suppose toward a contradiction there are two colors $c_0, c_1 \in L(x_{m_2}) \setminus \{\sigma(x_{m_2+1}), \psi'(u_{t_1}), \psi'(x_{m_2-1})\}$ such that this holds. Since Π_p^1 is an induced subgraph of G, $\psi' \cup \sigma$ is a proper L_ϕ-coloring of its domain. By assumption, $(G[V(\Pi_p^0 \cup \Pi_p^1)], \psi' \cup \sigma)$ is not a (C, p)-wedge. Since x_{m_2} is uncolored and $r^* \notin L(u_{t_3})$, the inertness condition is violated. Since H_3 is a triangle and ψ' restricts to an element of $\text{Skip}(H_1)$, there is an extension of $\phi \cup \psi' \cup \sigma$ to an L-coloring τ of $\text{dom}(\phi \cup \psi' \cup \sigma) \cup \{z, u_{t_3}\}$ such that τ does not extend to L-color H_2. At least one of c_0, c_1 is distinct from $\tau(z)$, so suppose without loss of generality that $c_0 \in L_\tau(x_{m_2})$. Since $|\mathcal{Z}_{H_2,L}(\psi'(u_{t_1}), c_0, \bullet)| \geq 2$, we have $|\mathcal{Z}_{H_2,L}(\psi'(u_{t_1}), c_0, \bullet \setminus \{\tau(u_{t_3})\}| \geq 1$, so τ extends to L-color H_2, a contradiction.

Since $|L_{\psi' \cup \sigma}(x_{m_2})| \geq 2$, it follows from 1) that there is a $c \in L_{\psi \cup \sigma}(x_{m_2})$ with $|\mathcal{Z}_{H_2,L}(\psi'(u_{t_2}), c, \bullet)| < 2$. Thus, by 2) of Proposition 1.4.4, $\psi'(u_{t_1}), c \in L(u_{t_1+1})$. This proves 2). ■

Now let $r^\downarrow, \psi_0, \psi_1$ be as in Claim 5.2.5.

Subclaim 5.2.25. *$m_2 = 3$ and $r^\downarrow$ is not the stem of a 4-bouquet.*

Proof: If at least one of these does not hold, then, since $L(x_{m_2}) \setminus \{r^\downarrow, \sigma(x_{m_2+1})\}| \geq 3$, there is a $\psi \in \text{Skip}(H_1)$ using $r^\downarrow$ on u_{t_1} and a $\psi' \in \Phi_{L_\phi}(\psi, \{x_2, \cdots, x_{m_2-1}\})$ such that $|L_{\psi' \cup \sigma}(x_{m_2})| \geq 3$. Thus, by Claim 5.2.22, there are two colors c_0, c_1 such that $|\mathcal{Z}_{H_2,L}(r^\downarrow, c_i, \bullet)| > 1$ for each $i = 0, 1$, contradicting 1) of Subclaim 5.2.24. ■

Since $r^\downarrow$ is not the stem of a 4-bouquet, It follows from Claim 5.2.6 that there is a $q^\downarrow \in L(u_{t_1}) \setminus \{r^\downarrow\}$ which is also the stem of a 2-bouquet. By Claim 5.2.16, $u_{t_1}x_{m_2}$ is not an L-shield for H_2. Recalling that $m_2 = 3$, we now we have the following.

Subclaim 5.2.26. *For each $\psi \in \text{Skip}(H_1)$, we have $|L_{\psi \cup \sigma}(x_3)| = 2$.*

Proof: Suppose that this does not hold. Thus, here is a $\psi \in \text{Skip}(H_1)$ such that $|L_{\psi \cup \sigma}(x_3)| \geq 3$. By assumption, the pair $(G[V(\Pi_p^0 \cup \Pi_p^1)], \psi \cup \sigma)$ is not a (C, p)-wedge, so the inertness condition is violated. Since $\psi \cup \sigma$ restricts to an element of $\text{Skip}(H_1)$, there is an extension of $\phi \cup \psi \cup \sigma$ to an L-coloring τ of $\text{dom}(\phi \cup \psi \cup \sigma) \cup V(H_1) \cup \{z, u_{t_3}\}$ such that τ does not extend to L-color the triangle H_2. Since σ does not use any color in $L(u_{t_2})$, we have

$|L_\tau(u_{t_2})| \geq 1$, so τ extends to L-color u_{t_2} as well, and, since $|L_\tau(x_{m_2})| \geq 2$ there is a color left over for x_{m_2}, contradicting our assumption. ■

Since $|L(x_2)| \geq 5$ and $|L_\sigma(x_3)| \geq 4$, we now fix a $c \in L(x_2)$ with $|L_\sigma(x_3) \setminus \{c\}| \geq 4$. Since $u_{t_1}x_{m_2}$ is not an L-shield for H_2, it follows from Lemma 5.1.8 that H_2 is a triangle.

By Subclaim 5.2.26, we have $\{r^{\downarrow}, q^{\downarrow}\} \subseteq L_\sigma(x_3)$, so $c \notin \{r^{\downarrow}, q^{\downarrow}\}$. Again by Subclaim 5.2.26, no element of $\text{Skip}(H_1)$ uses c on x_2. The trick now is to leave u_{t_1} uncolored. Let σ^* be an extension of σ to $V(Q_{\text{right}} - x_3) \cup \{x_2\}$ obtained by coloring x_2 with c.

Since x_3 is uncolored, we have $|L_{\phi\cup\sigma^*}(z)| \geq 3$, and since x_1 is uncolored, we have $|L_{\phi\cup\sigma^*}(x^*)| \geq 2$. By assumption, $(G[V(\Pi_p^0 \cup \Pi_p^1)], \phi \cup \sigma^*)$ is not a (C, p)-wedge, so the inertness condition is violated. That is, there is an extension of $\phi \cup \sigma^*$ to an L-coloring τ of $\text{dom}(\phi \cup \sigma^*) \cup \{x^*, z, u_{t_3}\}$ such that τ does not extend to L-color $H_1 \cup H_2$.

If τ extends to an L-coloring τ^* of $\text{dom}(\tau) \cup V(H_1)$, then, by our choice of c, we have $|L_{\tau^*}(x_{m_2})| \geq 2$. Since H_2 is a triangle and σ uses no color of $L(u_{t_2})$, we have $|L_{\tau^*}(u_{t_2})| \geq 1$ and thus τ^* extends to the edge $x_{m_2}u_{t_2}$, contradicting our assumption. We conclude that τ does not extend to an L-coloring of $\text{dom}(\tau) \cup V(H_1)$. Since $|L_\tau(x_1)| \geq 1$, it follows from Theorem 0.2.3 that $c \in L(u_{t_1})$ and there is a $d \in L_\tau(x_1)$ such that $\mathcal{Z}_{L_\phi^p}(\phi(p), d, \bullet) = \{c\}$. By Claim 5.2.3, there is an element of $\text{Skip}(H_1)$ using c, d on the respective vertices u_{t_1}, x_1, and since $|L(x_3) \setminus \{c, \sigma(x_4)\}| \geq 4$, this contradicts Subclaim 5.2.26. This completes the proof of Claim 5.2.23. ■

Now we have the following.

Claim 5.2.27. *There does not exist a pair of elements of $\mathcal{M}_L(H_3, u_{t_2}x_{m_3})$ using the same color on x_{m_3}.*

Proof: Suppose toward a contradiction that there is a $q \in L(x_{m_3})$ and a pair of elements $\sigma_0, \sigma_1 \in \mathcal{M}_L(H_3, u_{t_2}x_{m_3})$ such that $\sigma_0(x_{m_3}) = \sigma_1(x_{m_3}) = q$. Let $S_{\text{right}} := \{\sigma_0(u_{t_2}), \sigma_1(u_{t_2})\}$. Since each vertex of Q_{right} has an L_ϕ-list of size at least five, we fix an L_ϕ-coloring σ of $Q_{\text{right}} - x_{m_2}$ such that $\sigma(x_{m_3}) = q$ and σ uses no color of S_{right}. Possibly $m_3 = m_2 + 1$ and σ is just a coloring of a lone vertex.

We also fix a $c \in L(x_{m_2-1})$ such that $L(x_{m_2}) \setminus \{c, \sigma(x_{m_2+1})\}| \geq 4$. Since Π_p^1 is an induced subgraph of G, there is an extension of σ to an L_ϕ-coloring σ^* of $V(\Pi_p^1) \setminus \{x_1, x_{m_2}\}$ such that $\sigma^*(x_{m_2-1}) = c$ and no vertex of $\{x_2, \cdots, x_{m_2-2}\}$ is colored by a color of $L(u_{t_1})$. Possibly $c \in L(u_{t_1})$. Let f be the lone color of $L(u_{t_2}) \setminus S_{\text{right}}$,

Subclaim 5.2.28. *H_2 is not a triangle and $S_{\text{right}} \subseteq L(u_{t_1+1})$.*

Proof: Consider the following cases:

Case 1: There does not exist a exist a $d \in L_{\phi\cup\sigma^*}(x_1)$ such that $\mathcal{Z}_{H_1, L_\phi^p}(\phi(p), d, \bullet) = \{c\}$

By assumption, the pair $(G[V(\Pi_p^0 \cup \Pi_p^1), \sigma^*)$ is not a (C, p)-wedge. Since x_1, x_{m_2} are uncolored, $|L_{\phi\cup\sigma^*}(x^*)| \geq 2$ and $|L_{\phi\cup\sigma^*}(z)| \geq 3$, so the inertness condition is violated. Thus, there is an extension of $\phi \cup \sigma^*$ to an L-coloring τ of $\text{dom}(\phi \cup \sigma^*) \cup \{x^*, z, u_{t_3}\}$ such that τ does not extend to L-coloring $(VH_1 \cup H_2 \cup H_3)$. Since $|L_\tau(x_1)| \geq 1$, there is a $d \in L_\tau(x_1)$, and, by assumption, $\mathcal{Z}_{H_1, L_\phi^p}(\phi(p), d, \bullet) \neq \{c\}$, so τ extends to an L-coloring τ' of $\text{dom}(\tau) \cup V(H_1)$, since c is the only color used by τ on a vertex of Q_{left} which possibly lies in u_{t_1}.

By our choice of c, we have $|L_{\tau'}(x_{m_2})| \geq 2$. Since τ' does not extend to L-color $H_2 \cup H_3$, it follows that, for each $d \in L_{\tau'}(x_{m_2})$, we have $\mathcal{Z}(\tau', d, \bullet) \cap S_{\text{right}} = \varnothing$. By Observation 1.4.2, we thus have $L_{\tau'}(x_{m_2}) = S_{\text{right}}$.

Recalling that f is the lone color of $L(u_{t_2}) \setminus S_{\text{right}}$, we have $\mathcal{Z}_{H_2,L}(\tau'(u_{t_1}), s, \bullet) = \{f\}$ for each $s \in S_{\text{right}}$, so H_2 is not a triangle, and, by Proposition 1.4.4, $S_{\text{right}} \subseteq L(u_{t_1+1})$, as desired.

Case 2: There exists a $d \in L_{\phi \cup \sigma^*}(x_1)$ such that $\mathcal{Z}_{H_1, L^p_\phi}(\phi(p), d, \bullet) = \{c\}$

By Claim 5.2.3, there is $\psi \in \text{Skip}(H_1)$ which colors x_1, u_{t_1} with with the respective colors d, c, and, since Π^1_p is an induced subgraph of G, $\sigma \cup \psi$ is a proper L_ϕ-coloring of its domain which extends to an L_ϕ-coloring $\sigma^\dagger$ of $\text{dom}(\phi \cup \sigma) \cup V(Q_{\text{left}} \setminus \{x_1, x_{m_2}\})$.

By assumption, the pair $(G[V(\Pi^0_p \cup \Pi^1_p], \sigma^\dagger)$ is not a (C, p)-wedge, and since $|L_{\phi \cup \sigma^\dagger}(x^*)| \geq 2$ and x_{m_2} is uncolored, the inertness condition is violated, so there is an extension of $\phi \cup \sigma^\dagger$ to an L-coloring $\tau^\dagger$ of $\text{dom}(\phi \cup \sigma^\dagger) \cup \{z, u_{t_3}\}$ such that $\tau^\dagger$ does not extend to L-color $H_2 \cup H_3$. Since u_{t_1} is colored with c, we have $|L_{\tau^\dagger}(x_{m_2})| \geq 2$. Since $\tau^\dagger$ does not extend to L-color $H_2 \cup H_3$, we have $\mathcal{Z}_{H_2,L}(c, c', \bullet) \cap S_{\text{right}} = \varnothing$ for each $c' \in L_{\tau^\dagger}(x_{m_2})$, and thus, as above, it follows from Observation 1.4.4 that $L_{\tau^\dagger}(x_{m_2}) = S_{\text{right}}$, and, for each $s \in S_{\text{right}}$, $\mathcal{Z}_{H_2,L}(c, s, \bullet) = \{f\}$. Thus, H_2 is not a triangle, and, by Proposition 1.4.4, $S_{\text{right}} \subseteq L(u_{t_1+1})$, as desired. ■

By Claim 5.2.8, there is an augmented 4-bouquet, since $m_2 > 2$. Thus, there is a $\psi^* \in \text{Skip}^{\text{aug}}(H_1)$ with $\psi^*(x_{m_2}) \notin L(u_{t_2})$. By Subclaim 5.2.28, H_2 is not a triangle. Since $S_{\text{right}} \subseteq L(u_{t_1+1})$, at least one of $\psi^*(x_{m_2}), \psi^*(u_{t_1})$ does not lie in $L(u_{t_1+1})$. Since $\psi^*(x_{m_2}) \notin L(u_{t_2})$ and H_2 is not a triangle, it follows from Proposition 1.4.4 that $\mathcal{Z}_{H_2,L}(\psi^*(u_{t_1}), \psi^*(x_{m_2}), \bullet) = L(u_{t_2})$, contradicting Claim 5.2.9. This completes the proof of Claim 5.2.27. ■

By Claim 5.2.23, $u_{t_2}x_{m_3}$ is an L-shield for H_2, and, by Claim 5.2.27, there exists a pair of elements $\sigma_0, \sigma_1 \in \mathcal{M}_L(H_3, u_{t_2}x_{m_3})$ and a pair of colors $a, b \in L(x_{m_3})$ such that $\{\sigma_0(u_{t_2}), \sigma_0(x_{m_3})\} = \{\sigma_1(u_{t_2}), \sigma_1(x_{m_3})\} = \{a, b\}$. $\sigma_0(u_{t_2}) = a$ and $\sigma_1(u_{t_2}) = b$.

Claim 5.2.29. *Q_{right} is an edge.*

Proof: Suppose not. Thus, we have $m_3 > m_2 + 1$. By Claim 5.2.5, $\text{Skip}(H_1) \neq \varnothing$ so we fix a $\psi \in \text{Skip}(H_1)$ and an extension of ψ to an L_ϕ-coloring ψ' of $\text{dom}(\psi) \cup \{x_2, \cdots, x_{m_2-1}\}$. Let d_0, d_1, d_2 be three colors of $L(x_{m_2-1}) \setminus \{a, b\}$. For each $i = 0, 1$ and $j = 0, 1, 2$, there is an extension of σ_i to an L-coloring σ_{ij} of $\{u_{t_2}\} \cup V(Q_{\text{right}} - x_{m_2})$ using d_j on x_{m_2+1}. Consider the following cases:

Case 1: Either $\psi'(u_{t_1}) \notin \{a, b\}$ or H_2 is not a triangle

In this case, for each $i = 0, 1$ and $j = 0, 1, 2$, the union $\psi' \cup \sigma_{ij}$ is a proper L_ϕ-coloring of its domain. By assumption, for each such i, j, the pair $(G[V(\Pi^0_p \cup \Pi^1_p), \psi' \cup \sigma_{ij})$ is not a (C, p)-wedge, so the inertness condition is violated and, since $\psi' \in \text{Skip}(H_1)$, there is an extension of $\phi \cup \psi' \cup \sigma_{ij}$ to an L-coloring τ_{ij} of $\text{dom}(\phi \cup \psi' \cup \sigma_{ij}) \cup V(H_1) \cup \{z, u_{t_3}\}$ such that τ_{ij} does not extend to L-color $V(H_2 \cup H_3)$. Thus, for each $i = 0, 1$ and $j = 0, 1, 2$, we have $L_{\tau_{ij}}(x_{m_2}) \cap \mathcal{Z}(\psi'(u_{t_1}), \bullet, \sigma_i(u_{t_2})) = \varnothing$.

Subclaim 5.2.30. *For each $j = 0, 1, 2$, $L_{\tau_{0j}}(x_{m_2}) \cap L_{\tau_{1j}}(x_{m_2}) = \varnothing$.*

Proof: Let $j \in \{0, 1, 2\}$ and suppose there is a $d \in L_{\tau_{0j}}(x_{m_2}) \cap L_{\tau_{1j}}(x_{m_2})$. Then $d \notin \{a, b\}$, and, by Observation 1.4.2, d either lies in $\mathcal{Z}_{H_2,L}(\psi'(u_{t_1}, \bullet, a)$ or $\mathcal{Z}_{H_2,L}(\psi'(u_{t_1}), \bullet, b)$, which is false. ■

Now we not the following:

Subclaim 5.2.31. $\{d_0, d_1, d_2\} \not\subseteq L(x_{m_2}) \setminus \{\psi'(u_{t_1}), \psi'(x_{m_2})\}$.

Proof: Suppose toward a contradiction that $\{d_0, d_1, d_2\} \not\subseteq L(x_{m_2}) \setminus \{\psi'(u_{t_1}), \psi'(x_{m_2})\}$. Since $d_0, d_1, d_2 \notin \{a, b\}$, it follows that for each $i \in \{0, 1\}$ and $j \in \{0, 1, 2\}$, we have $L_{\tau_{ij}}(x) \cap (\{d_0, d_1, d_2\} \setminus \{d_j\})| \geq 1$.

Thus, suppose without loss of generality that $d_1 \in L_{\tau_{00}}(x_{m_2})$. By Subclaim 5.2.30, we have $d_1 \notin L_{\tau_{10}}(x_{m_2})$, so $d_2 \in L_{\tau_{10}}(x_{m_2})$. We then have $a \notin \mathcal{Z}_{H_2,L}(\psi'(u_{t_1}), d_1, \bullet)$ and $b \notin \mathcal{Z}_{H_2,L}(\psi'(u_{t_1}), d_2, \bullet)$. By assumption, $\psi'(u_{t_1}) \neq a, b$, so H_2 is not a triangle, and, by Proposition 1.4.4, we have $\{a,b\} \cup \{d_1, d_2\} \subseteq L(u_{t_2-1})$, contradicting the fact that $|L(u_{t_2-1})| = 3$. ■

Applying Subclaim 5.2.31, suppose without loss of generality that $d_0 \notin L(x_{m_2}) \setminus \{\psi'(u_{t_1}), \psi'(x_{m_2-1})\}$. Since $d_0 \notin \{a,b\}$ it follows that, for each $i = 0,1$, we have $L_{\tau_{i0}}(x_{m_2}) \neq \varnothing$. By Subclaim 5.2.30, there exist distinct colors c_0, c_1 such that, for each $i = 0,1$, $c_i \in L_{\tau_{i0}}(x_{m_2})$ and $\sigma_i(u_{t_2}) \notin \mathcal{Z}(\psi'(u_{t_1}), c_i, \bullet)$. Since $c_0 \neq a$ and $c_1 \neq b$, H_2 is not a triangle, and, by Proposition 1.4.4, $\{c_0, c_1\} \cup \{a,b\} \subseteq L(u_{t_2-1})$. Since $|L(u_{t_2-1})| = 3$, we have either $c_0 = b$ or $c_1 = a$, so suppose without loss of generality that $c_0 = b$. Thus, we have $L_{\tau_{00}}(x_{m_2}) = \{b\}$ and $L(x_{m_2}) = \{\tau_{00}(z), \psi'(u_{t_1}), \psi'(x_{m_2}), b, d_0\}$, contradicting the fact that $d_0 \notin L(x_{m_2}) \setminus \{\psi'(u_{t_1}), \psi'(x_{m_2-1})\}$.

Case 2: H_2 is a triangle and $\psi'(u_{t_1}) \in \{a,b\}$

In this case, suppose without loss of generality that $\psi'(u_{t_1}) = a$. Then, for each $j = 0,1,2$, $\psi' \cup \sigma_{1j}$ is a proper L_ϕ-coloring of its domain. Since $|L(x_{m_2}) \setminus \{a, b, \psi'(x_{m_2-1})\}| \geq 2$, there is a $j \in \{0,1,2\}$ with $|L(x_{m_2}) \setminus \{a, b, \psi'(x_{m_2-1}), d_j\}| \geq 2$, say $j = 0$. Since H_2 is a triangle, $(G[V(\Pi_p^0 \cup \Pi_p^1)], \psi' \cup \sigma_{10})$ satisfies the inertness condition and is thus a (C,p)-wedge, contradicting our assumption. This completes the proof of Claim 5.2.29. ■

Claim 5.2.32. *For any $\psi \in \text{Skip}(H_1)$ and any extension of ψ to an L_ϕ-coloring ψ' of* $\text{dom}(\psi) \cup \{x_2, \cdots, x_{m_2-1}\}$*, we have $\psi'(u_{t_1}), \psi'(x_{m_2-1}) \in L(x_{m_2}) \setminus \{a,b\}$. For each $i = 0,1$, we have $|L_{\psi' \cup \sigma_i'}(x_{m_2+1})| \geq 3$,*

<u>Proof:</u> Suppose there is a $\psi \in \text{Skip}(H_1)$ for which this does not hold and let ψ' be an extension of ψ to an L_ϕ-coloring ψ' of $\text{dom}(\psi) \cup \{x_2, \cdots, x_{m_2-1}\}$ with $\{\psi'(u_{t_1}), \psi'(x_{m_2-1})\} \not\subseteq L(x_{m_2}) \setminus \{a,b\}$. Consider the following cases:

Case 1: Either $\psi'(u_{t_1}) \notin \{a,b\}$ or H_2 is not a triangle

In this case, for each $i = 0,1$, the union $\psi' \cup \sigma_i$ is a proper L_ϕ-coloring of its domain. For each $i = 0,1$, since $(G[V(\Pi_p^0 \cup \Pi_p^1)], \psi' \cup \sigma_i)$ is not a (C,p)-wedge, and x_{m_2} is uncolored, the inertness condition is violated, and since $\psi' \cup \sigma_i$ restricts to an element of $\text{Skip}(H_1)$, there is an extension of $\phi \cup \psi' \cup \sigma$ to an L-coloring τ_i of $\text{dom}(\phi \cup \psi' \cup \sigma_i) \cup V(H_1) \cup \{z, u_{t_3}\}$ such that τ_i does not extend to L-color $V(H_2 \cup H_3)$.

For each $i = 0,1$, since $\{\psi'(u_{t_1}), \psi'(x_{m_2})\} \not\subseteq L(x_{m_2}) \setminus \{a,b\}$, we have $|L_{\tau_i}(x_{m_2})| \geq 1$. Since $\sigma_i \in \mathcal{M}_L(H_3, u_{t_2}x_{m_3})$, it follows that, for each $d \in L_{\tau_i}(x_{m_2})$, we have $\sigma_i(u_{t_2}) \notin \mathcal{Z}_{H_2,L}(\psi'(u_{t_1}, d, \bullet)$. Since $a, b \notin L_{\tau_0}(x_{m_2}) \cup L_{\tau_1}(x_{m_2})$, we have $L_{\tau_0}(x_{m_2}) \cap L_{\tau_1}(x_{m_2}) = \varnothing$. To see this, suppose there is a $d \in L_{\tau_0}(x_{m_2}) \cap L_{\tau_1}(x_{m_2})$. By Observation 1.4.2, one of a, b lies in $\mathcal{Z}_{H_2,L}(\psi'(u_{t_1}), d, \bullet)$, which is false. Thus, let c_0, c_1 be distinct colors of $L(x_{m_2}) \setminus \{a,b\}$ with $c_i \in L_{\tau_i}(x_{m_2})$ for each $i = 0,1$. If H_2 is a triangle, then, by assumption, we have $\psi'(u_{t_1}) \notin \{a,b\}$, and, for each $i = 0,1$, we have $\{a,b\} \subseteq \mathcal{Z}_{H_2,L}(\psi'(u_{t_1}), c_i, \bullet)$, which is false. Thus, H_2 is not a triangle. But then, by Proposition 1.4.4, we have $\{a,b\} \cup \{c_0, c_1\} \subseteq L(u_{t_2-1})$, contradicting the fact that $|L(u_{t_2-1})| = 3$.

Case 2: $\psi'(u_{t_1}) \in \{a,b\}$ and H_2 is a triangle

In this case, suppose without loss of generality that $\psi'(u_{t_1}) = a$. Then $\psi' \cup \sigma_1$ is a proper L_ϕ-coloring of its domain. Since $(G[V(\Pi_p^0 \cup \Pi_p^1)], \psi' \cup \sigma_1)$ is not a (C,p)-wedge, there is an extension of $\phi \cup \psi' \cup \sigma_1$ to an L-coloring τ of $\text{dom}(\phi \cup \psi' \cup \sigma_1) \cup V(H_1) \cup \{z, u_{t_3}\}$ which does not extend to L-color H_2. As H_2 is a triangle, we have $L_\tau(x_{m_2}) = \varnothing$. Since Q_{right} is an edge and x_{m_3}, u_{t_1} are colored with the same color, we have $|L_\tau(x_{m_2})| \geq 1$, a contradiction. ■

Now we let $r^\downarrow, \psi_0, \psi_1$ be as in Claim 5.2.5. It immediately follows from Claim 5.2.32 that $|L(x_{m_2}) \setminus \{a,b\}| = 3$ and $r^\downarrow \in L(x_{m_2}) \setminus \{a,b\}$.

Claim 5.2.33. *$m_2 = 3$ and $r^{\downarrow}$ is not the stem of a 4-bouquet.*

Proof: If either $r^{\downarrow}$ is the stem of a 4-bouquet or $m_2 > 3$, then, since $|L(x_{m_2}) \setminus \{a, b, r^{\downarrow}\}| = 2$, there is a $\psi \in \text{Skip}(H_1)$ and a $\psi' \in \Phi_{L_\phi}(\psi, \{x_2, \cdots, x_{m_2-1})$ such that $\psi'(x_{m_2-1}) \notin L(x_{m_2}) \setminus \{a, b\}$, contradicting Claim 5.2.32. ■

Since $r^{\downarrow}$ is not the stem of a 4-bouquet, it follows from Claim 5.2.6 that there is $q^{\downarrow} \in L(u_{t_1}) \setminus \{r^{\downarrow}\}$ which is the stem of a 2-bouquet. By Claim 5.2.16, $u_{t_1}x_{m_2}$ is not an L-shield and thus, By Lemma 5.1.8 H_2 is a triangle and $|L(u_{t_1}) \cap L(u_{t_2})| \geq 2$. By Claim 5.2.32, we have $r^{\downarrow}, q^{\downarrow} \notin \{a, b\}$, so there is a $d \in \{r^{\downarrow}, q^{\downarrow}\}$ with $d \notin L(u_{t_2})$, and, by Claim 5.2.8, since $m_2 > 2$, there is an augmented 4-bouquet using d on u_{t_1}. Thus, there is a $\psi^* \in \text{Skip}^{\text{aug}}(H_1)$ such that $\psi, \psi \notin L(u_{t_2})$, and since H_2 is a triangle we have $\mathcal{Z}_{H_2,L}(\psi^*(u_{t_1}), \psi^*(x_{m_2}), \bullet) = L(u_{t_2})$, contradicting Claim 5.2.9. This completes the proof of Lemma 5.2.1. □

5.3 Completing the Proof of Theorem 5.1.6

With Lemma 5.2.1 in hand, in order to finish the proof of Theorem 5.1.6, we deal with the case where there is no 3-chord of C which separates p from each ring of $\mathcal{C} \setminus \{C\}$. Note that, if no such 3-chord of C exists, then condition 4) of Definition 5.1.12 is automatically satisfied, so any pair which fails to satisfy the conditions of Definition 5.1.12 violates one of 1)-3). Section 5.3 consists of the following lone result.

Lemma 5.3.1. *If there does not exist a 3-chord of C which separates p from an element of $\mathcal{C} \setminus \{C\}$, then there exists a (C, p)-wedge.*

Proof. Suppose toward a contradiction that there does not exist a (C, p)-wedge. We now have the following.

Claim 5.3.2. *If Q_{left} is an edge then at most one of H_1, H_2 is a triangle.*

Proof: Suppose toward a contradiction that each of H_1, H_2 is a triangle. Thus, we have $t_1 = 1$ and $t_2 = 2$. Since Q_{left} is an edge, we have $m_2 = 2$. Furthermore, since H_2 is a triangle and G is short-separation-free, we have $x_1x_3 \notin E(G)$, so Π_p^1 is an induced subgraph of G. Let S_{left} be a set of two colors in $L(u_1) \setminus \{\phi(p_1)\}$. Since $|L_\phi(x_1)| \geq 3$, we also fix a $c^* \in L_\phi(x_1) \setminus S_{\text{left}}$.

Subclaim 5.3.3. *If $\mathcal{M}_L(H_3, u_2x_{m_3}) \neq \varnothing$, then Q_{right} is an edge and there exists a $z \in V(\Pi_p^2)$ adjacent to all three of x_1, x_2, x_3.*

Proof: Let $\sigma \in \mathcal{M}_L(H_3, u_2x_{m_3})$ and suppose toward a contradiction that here does not exist a $z \in V(\Pi_p^2)$ adjacent to all three of x_1, x_2, x_3. Now we simply choose a color $s \in S_{\text{left}} \setminus \{\sigma(u_2)\}$. Since $|L_\phi(x_2)| \geq 5$, there is a color left in $L_\phi(x_2) \setminus \{s, c^*, \sigma(u_2), \sigma(x_{m_3})$. Thus, since Π_P^1 is an induced subgraph of G, there is an extension of σ to an L_ϕ-coloring τ of $V(\Pi_p^0) \cup \{u_1, u_2\}$. Since there is no $z \in V(\Pi_p^2)$ adjacent to all three of x_1, x_2, x_3, the pair $(G[V(\Pi_p^0 \cup \Pi_p^1)], \tau)$ is a (C, p)-wedge, contradicting our assumption.

Now suppose toward a contradiction that Q_{right} is not an edge. Thus, $m_2 < m_3 - 1$ and $|L_{\phi\cup\sigma}(x_{m_2+1})| \geq 3$ (possibly $m_2 = m_3 - 2$). Since $|L_\phi(x)| \geq 3$ and Π_p^1 is an induced subgraph of G, there is an extension of σ to an L_ϕ-coloring σ^* of $V(Q_{\text{right}} - x_2) \cup \{u_1, u_2\}$ such that $|L_{\phi\cup\sigma^*}(x_2)| \geq 2$, so $\{x_2\}$ is $(L, \phi \cup \sigma^*)$-inert in G. Since $N(x_2) = \{x_1, x_3, u_1, u_2, z\}$, the pair $(G[V(\Pi_p^0 \cup \Pi_p^1)], \sigma^*)$ is a (C, p)-wedge, contradicting our assumption. ■

Now we have the following.

Subclaim 5.3.4. *$u_2x_{m_3}$ is not an L-shield for H_3.*

Proof: Suppose toward a contradiction that $u_2x_{m_3}$ is an L-shield for H_3. By Subclaim 5.3.3, Q_{right} is an edge and there exists a $z \in V(\Pi_p^2)$ adjacent to all three of x_1, x_2, x_3, so $N(x_2) = \{x_1, x_3, u_1, u_2, z\}$. Since each of Q_{left} and Q_{right} is an edge $x_{m_3} = x_3$ and $\Pi_p^1 = x_1x_2x_3$. Consider the following cases:

Case 1: There exists a $\sigma \in \mathcal{M}_L(H_3, u_2x_3)$ such that either $\sigma(u_2) \notin S_{\text{left}}$ or $\{\sigma(u_2), \sigma(x_3)\} \cap L_\phi(x_1) \neq \varnothing$

In this case, there is a $\sigma \in \mathcal{M}_L(H_3, u_2x_3)$ and a $c \in L_\phi(x_1)$ such that either $\sigma(u_2) \notin S_{\text{left}}$ or $c \in \{\sigma(u_2), \sigma(x_3)\}$. Since each vertex of Q_{right} has an L_ϕ-list of size at least five, there is an extension of σ to an L_ϕ-coloring τ of $V(Q_{\text{right}} - x_2) \cup \{x_1, u_2\}$ such that $\tau(x_1) = c$. By our choice of σ, the set $\{u_1, x_2\}$ is $(L, \phi \cup \tau)$-inert in G, so the pair $(G[V(\Pi_p^0 \cup \Pi_p^1)], \tau)$ is a (C, p)-wedge, contradicting our assumption.

Case 2: For each $\sigma \in \mathcal{M}_L(H_3, u_2x_3)$, $\sigma(u_2) \in S_{\text{left}}$ and $\{\sigma(u_2), \sigma(x_3)\} \cap L_\phi(x_1) = \varnothing$

In this case, since u_2x_3 is an L-shield for H_3, there are distinct $\psi_1, \psi_2 \in \mathcal{M}_L(H_3, u_2x_3)$ such that $S_{\text{left}} = \{\psi_1(u_2), \psi_2(u_2)\}$ and $S_{\text{left}} \cap L_\phi(x_1) = \varnothing$. As $|L_\phi(x_1)| \geq 3$, there is a $c \in L_\phi(x_1) \setminus \{\psi_2(u_2)\}$ such that $|L_\phi(x_2) \setminus \{\psi_1(x_3), \psi_1(u_2), \psi_2(u_2), c\}| \geq 2$. Thus, there is an extension of ψ_1 to an L_ϕ-coloring ψ_1^* of $\{x_1, x_3, u_1, u_2\}$ with $L_{\phi \cup \psi_1^*}(x_2)| \geq 2$, so $(G[V(\Pi_p^0 \cup \Pi_p^1)], \psi_1^*)$ is a (C, p)-wedge, contradicting our assumption. ■

Since $u_2x_{m_3}$ is not an L-shield for H_3, it follows from Lemma 5.1.8 that H_3 is a triangle and $|L(u_2) \cap L(u_3)| \geq 2$, so all three of H_1, H_2, H_3 are triangles and $\Pi_p^0 = u_1u_2$. Since $|L(u_2) \cap L(u_3)| \geq 2$, there is an $r \in L(x_{m_3}) \setminus (L(u_2) \cup L(u_3))$. Since each vertex of Q_{right} has an L-list of size at least five, there is an L-coloring σ_{right} of Q_{right} (which is also an L_ϕ-coloring of Q_{right}) in which $\sigma_{\text{right}}(x_{m_3}) = r$ and every vertex of Q_{right} is colored with a color not in $L(u_3)$.

Subclaim 5.3.5. *There exists a $z \in V(\Pi_p^2)$ adjacent to all three of x_1, x_2, x_3*

Proof: Suppose not. Since Π_p^1 is an induced subpath of G, we have $|L_{\phi \cup \sigma_{\text{right}}}(x_1)| \geq 2$ and $|L_{\phi \cup \sigma_{\text{right}}}(u_1)| \geq 1$, so σ_{right} extends to an L_ϕ-coloring τ of $V(\Pi_p^0) \cup \{u_1\}$. By our choice of r, we have $|L_{\phi \cup \tau}(u_3)| = 3$, and by our construction of τ, we have $|L_{\phi \cup \tau}(u_2)| \geq 2$, so $\{u_2\}$ is $(L, \phi \cup \tau)$-inert in G. Since no vertex of Π_p^2 is adjacent to x_1, x_2, x_3, the pair $(G[V(\Pi_p^0 \cup \Pi_p^1)], \tau)$ is a (C, p)-wedge, contradicting our assumption. ■

Now we have enough to finish the proof of Claim 5.3.2. Let σ'_{right} be the restriction of σ_{right} to $x_3 \cdots x_{m_3}$ and let z be the lone vertex of Π_p^2 adjacent to x_1, x_2, x_3. As above, since Π_p^1 is an induced subgraph of G, σ'_{right} extends to an L_ϕ-coloring τ of $V(Q_{\text{right}} - x_3) \cup \{x_1, u_1\}$. By our construction of τ, we have $|L_{\phi \cup \tau}(u_2)| \geq 2$. Since $N(x_2) = \{x_1, x_3, u_1, u_2, z\}$, the set $\{x_2, u_2\}$ is $(L, \phi \cup \tau)$-inert in G, so the pair $(G[V(\Pi_p^0 \cup \Pi_p^1)], \tau)$ is a (C, p)-wedge, contradicting our assumption. This completes the proof of Claim 5.3.2. ■

Claim 5.3.6. *$u_{t_1}x_{m_2}$ is an L-shield for H_2.*

Proof: Suppose toward a contradiction that $u_{t_1}x_{m_2}$ is an L-shield for H_2. By Lemma 5.1.8, H_2 is a triangle, so $t_2 = t_1 + 1$. Let J be the subgraph of G induced by $V(H_1 \cup H_2 \cup Q_{\text{left}}) \setminus \{p, u_{t_2}\}$.

Subclaim 5.3.7. *$L(u_{t_1}) \neq L(u_{t_2})$.*

Proof: Suppose that $L(u_{t_1}) = L(u_{t_2})$. Since each of u_{t_1}, u_{t_2} has an L-list of size three, there are two colors $r, s \in L(x_{m_2}) \setminus (L(u_{t_1}) \cup L(u_{t_2}))$. By Proposition 1.4.5, there is a $c \in L_\phi(x_1)$ such that $\mathcal{Z}_{H_1, L_\phi^p}(\phi(p), c, \bullet)| \geq 2$. Let S_{left} be a set of two colors in $\mathcal{Z}_{H_1, L_\phi^p}(\phi(p), c, \bullet)$. Since each vertex of $x_2 \cdots x_{m_2}$ has an L_ϕ-list of size at least five and Q_{left} is an induced subpath of G, there is an L_ϕ-coloring σ_{left} of $x_1 \cdots x_{m_2}$ such that $\sigma_{\text{left}}(x_1) = c$, $\sigma_{\text{right}}(x_{m_2}) \in \{r, s\}$, and no internal vertex of $x_1 \cdots x_{m_2}$ is colored with a color of S_{left}. Since at least one of r, s

is distinct from c, this is true even if Q_{left} is an edge. Now consider the pair $(J, \sigma_{\text{left}})$. By assumption, this is not a (C,p)-wedge. By our choice of $\sigma_{\text{left}}(x_{m_2})$, we have $|L_{\phi\cup\sigma_{\text{left}}}(u_{t_2})| = 3$, so the inertness condition is violated. That is, there is an extension of $\phi\cup\sigma_{\text{left}}$ of an L-coloring τ of dom$(\phi\cup\sigma_{\text{left}})\cup\{u_{t_2}\}$ such that τ does not extend to L-color the path $u_1\cdots u_{t_1}$. Yet since $c\notin S_{\text{left}}$ and $\sigma_{\text{left}}(x_{m_2})\notin L(u_{t_1})$, it follows that $L_\tau(u_{t_1})$ contains a color of S_{left}, so τ extends to L-color the path $u_1\cdots u_{t_1}$, a contradiction. ■

Since $L(u_{t_1})\neq L(u_{t_2})$, we have $|L(u_{t_1})\cap L(u_{t_2})| = 2$ by 2) of Lemma 5.1.8. Thus, there is an $r\in L(x_{m_2})\setminus(L(u_{t_1})\cup L(u_{t_2}))$.

Subclaim 5.3.8. *All three of the following hold.*

1) Q_{left} *is an edge; AND*

2) $r\in L_\phi(x_1)$*; AND*

3) *For each* $s\in L_\phi(x_1)\setminus\{r\}$, $|\mathcal{Z}_{H_1,L^p_\phi}(\phi(p),s,\bullet)| = 1$.

Proof: Suppose at least one of these does not hold. Since $|L_\phi(x_1)|\geq 3$, it follows from Proposition 1.4.5 that there is a $c\in L_\phi$ such that $|\mathcal{Z}_{H_1,L^p_\phi}(\phi(p),c,\bullet)|\geq 2$ and such that either $c\neq r$ or $x_1x_{m_2}\notin E(G)$. Let S_{left} be a set of two colors in $\mathcal{Z}_{H_1,L^p_\phi}(\phi(p),c,\bullet)$. By Claim 5.1.15, there is an L_ϕ-coloring σ_{left} of Q_{left} using c, r on the respective vertices x_1, x_{m_2}, where each internal vertex of Q_{left} is colored by a color not in S_{left}. By assumption, (J,σ_{left}) is not a (C,p)-wedge. Since $|L_{\phi\cup\sigma_{\text{left}}}(u_{t_2})| = 3$, the inertness condition is violated, so there is an extension of $\phi\cup\sigma_{\text{left}}$ to an L-coloring τ of dom$(\phi\cup\sigma_{\text{left}})\cup\{u_{t_2}\}$ which does not extend to L-color $u_1\cdots u_{t_1}$. Since $r\notin L(u_{t_1})$, we have $S_{\text{left}}\cap L_\tau(u_{t_1})\neq\varnothing$, so τ extends to L-color $u_1\cdots u_{t_1}$, a contradiction. ■

Since $|L(u_{t_1})\cap L(u_{t_2}| = 2$, let d be the lone color of $L(u_{t_1})\setminus L(u_{t_2})$.

Subclaim 5.3.9. *For each* $s\in L_\phi(x_1)\setminus\{r\}$, $d\notin\mathcal{Z}_{H_1,L^p_\phi}(\phi(p),s,\bullet)$.

Proof: Suppose there is an $s\in L_\phi(x_1)\setminus\{r\}$ such that $d\in\mathcal{Z}_{H_1,L^p_\phi}(\phi(p),s,\bullet)$ and let ψ be the L_ϕ-coloring of x_1x_2 where $\psi(x_1) = s$ and $\psi(x_2) = r$. Then $|L_{\phi\cup\psi}(u_{t_2})| = 3$, and since $d\notin\{r,s\}$ and $d\notin L(u_{t_2})$, the pair (J,ψ) is a (C,p)-wedge, contradicting our assumption. ■

Now let $s_1, s_2\in L_\phi(x_1)\setminus\{r\}$. By Subclaim 5.3.9, we have $d\notin\mathcal{Z}_{H_1,L^p_\phi}(\phi(p),s_j,\bullet)$ for each $j = 1,2$. By Claim 5.3.2, H_1 is not a triangle, as Q_{left} is an edge and H_2 is a triangle. Thus, by Proposition 1.4.4, we have $L(u_{t_1-1}) = \{s_1,s_2,d\}$, so $r\notin L(u_{t_1-1})$. Since $r\notin L(u_{t_1})$, it also follows from Proposition 1.4.4 that $\mathcal{Z}_{H_1,L^p_\phi}(\phi(p),r,\bullet) = L(u_{t_1})$. Since $|L(x_2)|\geq 5$ and $m_2 = 2$, let $r^*\in L(x_2)\setminus(L(u_{t_2})\cup\{r^*\})$ and let ψ^* be the L_ϕ-coloring of the edge x_1x_2 using r, r^* on the respective vertices x_1, x_2. Since $\mathcal{Z}_{H_1,L^p_\phi}(\phi(p),r,\bullet) = L(u_{t_1})$, the pair (J,ψ^*) is a (C,p)-wedge, contradicting our assumption. This completes the proof of Claim 5.3.6. ■

Claim 5.3.10. *Both of the following hold.*

1) H_1 *is not a triangle, and* Q_{left} *is an edge (i.e* $m_2 = 2$*);*

2) *There exist two elements* ψ_1,ψ_2 *of* $\mathcal{M}_L(H_2, u_{t_1}x_{m_2})$ *and a color* $r\in L(x_{m_2})$ *such that the following hold:*

   a) $\psi_1(x_{m_2}) = \psi_2(x_{m_2}) = r$ *and* $L(u_{t_1}) = L_\phi(x_1) = \{r,\psi_1(u_{t_1}),\psi_2(u_{t_1})\}$*; AND*

   b) $\mathcal{Z}_{H_1,L^p_\phi}(\phi(p_1),\psi_1(u_{t_1},\bullet) = \mathcal{Z}(\phi(p_1),\psi_2(u_{t_1},\bullet) = \{r\}$ *and* $\{\psi_1(u_{t_1}),\psi_2(u_{t_1})\}\subseteq Z_{H_1,L^p_\phi}(\phi(p),r,\bullet)$.

Proof: We first set J to be the subgraph of G induced by $V(H_1\cup H_2\cup Q_{\text{left}})\setminus\{p,u_{t_2}\}$. By Claim 5.3.6, $u_{t_1}x_{m_2}$ is an L-shield for H_2. Thus, there exist two elements of $\mathcal{M}_L(H_2,u_{t_1}x_{m_2})$ which use different colors on u_{t_1}. Suppose toward a contradiction that H_1 is a triangle and let $\psi \in \mathcal{M}_L(H_2, u_{t_1}x_{m_2})$ with $\psi(u_{t_1}) \neq \phi(p_1)$. By Theorem 0.2.3, ψ extends to an L_ϕ-coloring ψ^* of $V(Q_{\text{left}}) \cup \{u_{t_1}\}$, and, by our choice of ψ, the pair (J, ψ^*) is a (C, p)-wedge, contradicting our assumption. Thus, H_1 is not a triangle.

Now we show that there are two elements of $\mathcal{M}_L(H_2, u_{t_1}x_{m_2})$ using the same color on x_{m_2}. Suppose not. Since $u_{t_1}x_{m_2}$ is an L-shield for H_2, there exist two elements ψ_1, ψ_2 of $\mathcal{M}_L(H_2, u_{t_1}x_{m_2})$ and two colors $a, b \in L(u_{t_1}) \cap L(x_{m_2})$, where $\psi_1(u_{t_1}) = \psi_2(x_{m_2}) = a$ and $\psi_1(x_{m_2}) = \psi_2(u_{t_2}) = b$. AS $|L_\phi(x)| \geq 3$, there is an $s \in L_\phi(x) \setminus \{a, b\}$. By Observation 1.4.2, at least one of a, b lies in $\mathcal{Z}_{H_1, L^p_\phi}(\phi(p), s, \bullet)$, so suppose without loss of generality that $a \in \mathcal{Z}_{H_1, L^p_\phi}(\phi(p), s, \bullet)$. Since $s \notin \{a, b\}$, there is an extension of ψ_1 to an L_ϕ-coloring ψ^* of $\{x_1, x_{m_2}, u_{t_1}\}$ using s on x_1. Since $s \notin \{a, b\}$, this is true even if Q_{left} is an edge, so (J, ψ^*) is a (C, p)-wedge, contradicting our assumption.

Thus, there exist two elements ψ_1, ψ_2 of $\mathcal{M}_L(H_1,)$ and an $r \in L_\phi(x_{m_2})$ with $\psi_1(x_{m_2}) = \psi_2(x_{m_2}) = r$ and $\psi_1(u_{t_1}) \neq \psi_2(x_{m_2})$. If there exists an $i \in \{1, 2\}$ such that either $\mathcal{Z}_{H_1, L^p_\phi}(\phi(p), \bullet, \psi_i(u_{t_1})) \not\subseteq \{r\}$ or Q_{left} has length greater than one, then there is an extension of ψ_i to an L_ϕ-coloring ψ_i^* of $V(Q_{\text{left}}) \cup \{u_{t_1}\}$ such that $\psi_i^*(x_1) \in \mathcal{Z}_{H_1, L^p_\phi}(\phi(p), \bullet, \psi_i(u_{t_1}))$. But then the pair (J, ψ_i^*) is a (C, p)-wedge, contradicting our assumption. Thus, $m_2 = 2$ and, for each $j = 1, 2$, we have $\mathcal{Z}_{H_1, L^p_\phi}(\phi(p), \bullet, \psi_j(u_{t_1})) \subseteq \{r\}$.

For each $s \in L_\phi(x) \setminus \{r\}$, since $\mathcal{Z}_{H_1, L^p_\phi}(\phi(p), s, \bullet) \neq \varnothing$ by Theorem 0.2.3, we have $\psi_1(u_{t_1}), \psi_2(u_{t_1}) \notin \mathcal{Z}_{H_1, L^p_\phi}(\phi(p), s, \bullet)$. Thus, there is a lone color $c \in L(u_{t_1})$ such that $L(u_{t_1}) = \{c, \psi_1(u_{t_1}), \psi_2(u_{t_1})\}$ and, for each $s \in L_\phi(x_1) \setminus \{r\}$, we have $\mathcal{Z}_{H_1, L^p_\phi}(\phi(p), s, \bullet) = \{c\}$. By Proposition 1.4.5, we have $|L_\phi(x_1) \setminus \{r\}| = 2$ and $L_\phi(x_1) \setminus \{r\} = \{\psi_1(u_{t_1}), \psi_2(u_{t_1})\}$. Thus, $r \in L_\phi(x_1)$, and, again by Proposition 1.4.5, we have $\{\psi_1(u_{t_1}), \psi_2(u_{t_1})\} \subseteq \mathcal{Z}_{H_1, L^p_\phi}(\phi(p), r, \bullet)$.

To finish, we just need to show that $r = c$. Suppose toward a contradiction that $r \neq c$. By Proposition 5.1.16, we have $\mathcal{Z}_{H_1, L^p_\phi}(\phi(p), r, \bullet) \neq L(u_{t_1})$. Since $\{\psi_1(u_{t_1}), \psi_2(u_{t_1})\} \subseteq \mathcal{Z}_{H_1, L^p_\phi}(\phi(p), r, \bullet)$, we have $c \notin \mathcal{Z}_{H_1, L^p_\phi}(\phi(p), r, \bullet)$. Since H_1 is not a triangle and $\mathcal{Z}_{H_1, L^p_\phi}(\phi(p), s, \bullet) = \{c\}$ for each $s \in \{\psi_1(u_{t_1}), \psi_2(u_{t_1})\}$, it follows from Proposition 1.4.4 that $\psi_1(u_{t_1}), \psi_2(u_{t_1})\} \subseteq L(u_{t_1 - 1})$. Since $c \notin \mathcal{Z}_{H_1, L^p_\phi}(\phi(p), r, \bullet)$ and $c \neq r$, we have $c \in L(u_{t_1 - 1})$. Since $|L(u_{t-1})| = 3$, we have $L(u_{t-1}) = \{c, \psi_1(u_{t_1}), \psi_2(u_{t_1})\}$, so $r \notin L(u_{t_1 - 1})$. Thus, by Proposition 1.4.4, we have $c \in \mathcal{Z}_{H_1, L^p_\phi}(\phi(p), r, \bullet)$, a contradiction. This completes the proof of Claim 5.3.10. ∎

Claim 5.3.11. *If $u_{t_2}x_{m_3}$ is an L-shield for H_3, then there exists an $r^* \in L(x_{m_3})$ and two distinct elements ψ_1, ψ_2 of $\mathcal{M}_L(H_3, u_{t_2}x_{m_2})$ using r^* on x_{m_3}.*

Proof: Suppose toward a contradiction that there is no such color in $L(x_{m_3})$. By definition, there exist $a, b \in L(u_{t_2}) \cap L(x_{m_3})$ and two distinct elements of $\mathcal{M}_L(H_3, u_{t_2}x_{m_3})$ such that $\psi_1(u_{t_2}) = \psi_2(x_{m_3}) = a$ and $\psi_1(x_{m_3}) = \psi_2(u_{t_2}) = b$. Applying Claim 5.3.10, we have $m_2 = 2$, and we fix a color $r \in L(x_2) \cap L_\phi(x_1) \cap L(u_{t_1})$ such that there are two distinct elements of $\mathcal{M}_L(H_2, u_{t_1}x_2)$ using r on x_2. Since $\{a, b\} \subseteq L(u_{t_2})$, we have $r \notin \{a, b\}$ by definition of $\mathcal{M}_L(H_2, u_{t_1}x_2)$. Again by Claim 5.3.10, there are $s_0, s_1 \in L_\phi(x_1) \cap L(u_{t_1})$ such that $\mathcal{Z}_{H_1, L}(\phi(p), s_j, \bullet) = \{r\}$ for each $j = 0, 1$. Since $|L(x_2)| \geq 5$, we fix two colors $f_0, f_1 \in L(x_2) \setminus \{r, a, b\}$. By Observation 1.4.2, we have $\mathcal{Z}_{H_2, L}(r, f_j, \bullet) \cap \{a, b\} \neq \varnothing$ for each $j = 0, 1$.

> **Subclaim 5.3.12.** *There is an L_ϕ-coloring $\sigma^\dagger$ of $V(\Pi^1_p \cup H_1 \cup H_2)$ such that the restriction of $\sigma^\dagger$ to $u_{t_2}x_{m_3}$ is one of ψ_1, ψ_2.*
>
> Proof: It suffices to show that there is an L_ϕ-coloring of $\{x_1, x_2, u_{t_2}, x_{m_3}\}$ using $\{a, b\}$ on $\{u_{t_2}, x_{m_3}\}$ and using one of $\{s_0, s_1\}$ on x_1 and one of $\{f_0, f_1\}$ on x_2. Since $r, f_0, f_1 \notin \{a, b\}$, the only nontrivial case is the case where $m_3 = 3$ and Π^1_p is not an induced subgraph of G. In that case, we have $x_1x_3 \in E(G)$. Possibly $\{s_0, s_1\} = \{a, b\}$, but such a $\sigma^\dagger$ exists in any case, since $\{f_0, f_1\} \cap \{a, b\} = \varnothing$ and $u_{t_3} \notin N(x_1)$. ∎

Now we return to the proof of Claim 5.3.11. We note that there is a vertex of Π_p^2 adjacent to x_1, x_2, x_3.To see this, suppose not. Let $\sigma^\dagger$ be an L_ϕ-coloring of $V(\Pi_p^1 \cup H_1 \cup H_2)$ as in Subclaim 5.3.12. As no vertex of Π_p^2 has more than two neighbors on Π_p^1, the pair $(G[V(\Pi_p^0 \cup \Pi_p^1)], \sigma^\dagger)$ is a (C, p)-wedge, contradicting our assumption. Thus, let z be the lone vertex of Π_p^2 adjacent to each of x_1, x_2, x_3. Since G is short-separation-free, we have $x_1x_3 \notin E(G)$, so Π_p^1 is an induced subgraph of G. The trick now is to leave u_{t_1}, x_2 uncolored. Let $\psi_a^\star$ be an L_ϕ-coloring of $V(\Pi_p^1 - x_2) \cup V(H_1) \cup \{u_{t_2}\}$, where $\psi^\star(x_1) = r$, $\psi^\star(u_{t_2}) = a$, $\psi^\star(x_{m_3}) = b$, and $\psi^\star(x_3) \notin \{f_0, f_1\}$. We define an L_ϕ-coloring $\psi_b^\star$ of $V(\Pi_p^1 - x_2) \cup V(H_1) \cup \{u_{t_2}\}$ analogously, with the roles of a, b interchanged.

By assumption, the pair $(G[V(\Pi_p^0 \cup \Pi_p^1)], \psi_a^\star)$ is not a (C, p)-wedge, and since x_2 is uncolored, the inertness conditions is violated. Since the restriction of $\psi_a^\star$ to the edge $u_{t_2}x_{m_3}$ is an element of $\mathcal{M}_L(H_3, u_{t_2}x_{m_3})$, it follows that there is an extension of $\phi \cup \psi_a^\star$ to an L-coloring τ_a of $\mathrm{dom}(\phi \cup \psi_a^\star) \cup V(H_3) \cup \{z\}$ which does not extend to L-color $H_2 \setminus \{u_{t_1}, u_{t_2}\}$. Likewise, there is an extension of $\phi \cup \psi_b^\star$ to an L-coloring τ_b of $\mathrm{dom}(\phi \cup \psi_a^\star) \cup V(H_3) \cup \{z\}$ such that τ_b does not extend to L-color $H_2 \setminus \{u_{t_1}, u_{t_2}\}$.

Subclaim 5.3.13. *$L_{\tau_a}(z) \subseteq \{s_0, s_1\}$, and $L_{\tau_b}(z) \subseteq \{s_0, s_1\}$.*

Proof: Suppose not, and suppose without loss of generality that there is a $c \in L_\tau(z) \setminus \{s_0, s_1\}$. By Observation 1.4.2, we have $\mathcal{Z}_{H_2, L}(\bullet, c, a) \cap \{s_0, s_1\} \neq \varnothing$, and since $\{s_0, s_1\} = \mathcal{Z}_{H_1, L_\phi^p}(\phi(p), r, \bullet)$, τ_a extends to L-color H_2, contradicting our assumption. ■

Since u_{t_1} is not colored by τ_a, we have $|L_{\tau_a}(z)| \geq 1$, so, by Subclaim 5.3.13, suppose without loss of generality that $s_0 \in L_{\tau_a}(z)$. Since $\{s_0, s_1\} = \mathcal{Z}_{H_1, L_\phi^p}(\phi(p), r, \bullet)$ and τ_a does not extend to L-color H_2, we have $\mathcal{Z}_{H_2, L}(\bullet, s_0, a) = \{r\}$. By Observation 1.4.2, since $s_1 \notin \mathcal{Z}_{H_2, L}(\bullet, s_0, a)$, we have $b \in \mathcal{Z}(s_1, s_0, \bullet)$. As τ_b does not extend to L-color H, we have $s_0 \notin L_{\tau_b}(z)$. Since $|L_{\tau_b}(z)| \geq 1$, it follows from Subclaim 5.3.13 that $s_1 \in L_{\tau_b}(z)$, so $\mathcal{Z}_{H_2, L}(\bullet, s_0, a) = \mathcal{Z}_{H_2, L}(\bullet, s_1, b) = \{r\}$, contradicting 2) of Proposition 1.4.7. This completes the proof of Claim 5.3.11. ■

Claim 5.3.14. *If $u_{t_2}x_{m_3}$ is an L-shield for H_3, then there exists a $z \in V(\Pi_p^2)$ adjacent to each of $x_{m_2-1}, x_{m_2}, x_{m_2+1}$.*

Proof: Suppose toward a contradiction that no such $z \in V(\Pi_p^2)$ exists. Applying Claim 5.3.10, we first fix a color $c \in L_\phi(x_1)$ such that $|\mathcal{Z}_{H_1, L_\phi^p}(\phi(p), c, \bullet)| \geq 2$. Let S_{left} be a set of two colors in $\mathcal{Z}_{H_1, L_\phi^p}(\phi(p), c, \bullet)$. Since $u_{t_2}x_{m_3}$ is an L-shield for H_3, it follows from Claim 5.3.11 that exist an $r \in L(x_{m_3}) \setminus L(u_{t_3})$ and two distinct elements of ψ_1, ψ_2 of $\mathcal{M}_L(H_3, u_{t_2}x_{m_2})$ which use r on x_{m_3} and different colors on u_{t_2}. and let $S_{\text{right}} := \{\psi_1(u_{t_2}), \psi_2(u_{t_1})\}$.

Subclaim 5.3.15. *All of the following hold.*

1) $x_{m_2-1}x_{m_2+1} \in E(G)$; AND

2) $m_2 = 2 = m_3 - 1$ (i.e Π_p^1 has length two); AND

3) $c = r$.

Proof: Suppose that at least one of these does not hold. Since $|L(x_{m_2})| \geq 5$, we fix an $f \in L(x_{m_2}) \setminus (\{c, r\} \cup S_{\text{left}})$. Suppose now that at at least one of conditions 1)-3) above does not hold. Since Π_p^1 has no chords in G, except possibly $x_{m_2-1}x_{m_2+1}$, and each internal vertex of Π_p^1 has an L_ϕ-list of size at least five, there is an L_ϕ-coloring σ of $V(\Pi_p^1)$ in which x_1, x_{m_2}, x_{m_3} are colored with respective colors c, f, r, and furthermore, each internal vertex of Q_{left} is colored with a color not in S_{left}, and each internal vertex of Q_{right} is colored with a color not in S_{right}.

Consider the pair $(G[V(\Pi_p^0 \cup \Pi_p^1)], \sigma)$. By assumption, this pair is not a (C, p)-wedge. By the assumption of the subclaim, there is no vertex of Π_p^2 adjacent to more than two vertices of Π_p^1, so the only condition which is violated is the inertness condition. It follows that there is an extension of $\phi \cup \sigma$ to an L-coloring τ of the path

$p_1x_1\Pi^1_px_{m_3}u_{t_3}$ such that τ does not extend to $u_1\cdots u_{t_3-1}$. Since $|S_{\text{right}}\setminus\{f\}|\geq 1$, it follows from the definition of S_{right} that τ extends to an L-coloring τ^* of $\text{dom}(\tau)\cup V(H_3)$ using a color of S_{right} on u_{t_2}. Since $f\notin S_{\text{left}}$, it follows from Observation 1.4.2 that the L-coloring $(f,\tau^*(u_{t_2}))$ of $x_{m_2}u_{t_2}$ extends to an L-coloring of H_2 using a color of S_{left} on u_{t_1}, and thus τ^* extends to L-color $u_1\cdots u_{t_3-1}$, contradicting our choice of τ. We conclude that conditions 1)-3) above all hold, as desired. ■

Since $x_{m_2-1}x_{m_2+1}\in E(G)$ and G is short-separation-free, H_2 is not a triangle. By Lemma 5.1.8, $u_{t_2}x_{m_2}$ is also an L-shield for H_2. Since $|L(u_{t_2})|=3$, there is a $\sigma\in\mathcal{M}_L(H_2,u_{t_2}x_{m_2})$ such that $\sigma(u_{t_2})\in S_{\text{left}}$. By Subclaim 5.3.15, we have $r\in L_\phi(x_1)$, since $c=r$, so it follows from Claim 5.3.10 that $r\in\ L(u_{t_1})$. Since $\sigma\in\mathcal{M}_L(H_2,u_{t_2}x_{m_2})$, we thus have $\sigma(x_{m_2})\neq r$. Since $|L_\phi(x_1)|\geq 3$, let τ be an L_ϕ-coloring of $V(\Pi^1_p)$ in which x_2 is colored with $\sigma(x_2)$, x_3 is colored with r, and x_1 is colored with a color of $L_\phi(x_1)\setminus\{r,\sigma(x_2)\}$. Since $\sigma(x_{m_2})\neq r$, τ is a proper L_ϕ-coloring of its domain. Since $\sigma(u_{t_2})\in S_{\text{left}}$), any extension of $\tau\cup\phi$ to an L-coloring of $\text{dom}(\tau\cup\phi)\cup\{u_{t_3}\}$ also extends to L-color the path $u_{t_1}\cdots u_{t_3-1}$, so the pair $(G[V(\Pi^0_p\cup\Pi^1_p)],\tau)$ is a (C,p)-wedge, contradicting our assumption. ■

Claim 5.3.16. *$u_{t_3}x_{m_2}$ is not an L-shield for H_3. In particular, H_3 is a triangle and $|L(u_{t_2})\cap L(u_{t_3})|\geq 2$.*

Proof: Suppose that $u_{t_3}x_{m_2}$ is an L-shield for H_3. By Claim 5.3.14, there is a $z\in V(\Pi^2_p)$ adjacent to each of $x_{m_2-1},x_{m_2},x_{m_2+1}$, so $N(x_{m_2})=\{z,x_{m_2-1},x_{m_2+1}\}\cup\{u_{t_1},\cdots,u_{t_2}\}$ and Π^1_p is an induced subgraph of G. By Claim 5.3.11, there exists an $r\in L(x_{m_3})$ and two distinct elements ψ_1,ψ_2 of $\mathcal{M}_L(H_3,u_{t_2}x_{m_2})$ using r on x_{m_3}. Let $T_{\text{right}}:=\{\psi_1(u_{t_2}),\psi_2(u_{t_2})\}$. By Claim 5.1.15, since $r\notin T_{\text{right}}$, there is an L-coloring σ_{right} of $x_{m_2+1}\cdots x_{m_3}$ such that $\sigma_{\text{right}}(x_{m_3})=r$ and each vertex of $x_{m_2+1}\cdots x_{m_3}$ is colored by a color not lying in T_{right}. Let $r':=\sigma_{\text{right}}(x_{m_2+1})$. Possibly $m_2+1=m_3$ and $r'=r$. Since $|L_{\sigma_{\text{right}}}(x_{m_2})|\geq 4$, there is an $S\subseteq L_{\sigma_{\text{right}}}(x_{m_2})\setminus T_{\text{right}}$ with $|S|=2$.

By Claim 5.3.10, there exist distinct $c_1,c_2\in L_\phi(x_1)$ and an $r^*\in L(u_{t_1})$ such that $\mathcal{Z}_{H_1,L^p_\phi}(\phi(p),c_j,\bullet)=\{r^*\}$ for each $j=1,2$. Thus, it immediately follows from Proposition 1.4.5 that there is a $c\in L_\phi(x_1)$ such that $\mathcal{Z}_{H_1,L^p_\phi}(\phi(p),c,\bullet)\not\subseteq S$.

Subclaim 5.3.17. $m_2=2$.

Proof: Suppose that $m_2>2$. As indicated above, there is a $c\in L_\phi(x_1)$ with $\mathcal{Z}_{H_1,L^p_\phi}(\phi(p),c,\bullet)\not\subseteq S$. Let $c^*\in\mathcal{Z}_{H_1,L^p_\phi}(\phi(p),c,\bullet)\setminus S$. Since $m_2>2$, we have $L_\phi(x_{m_2-1})=L(x_{m_2-1})$, and since each vertex of $x_2\cdots x_{m_2-1}$ has an L_ϕ-list of size at least five, there is an L_ϕ-coloring σ_{left} of $\{u_{t_1}\}\cup V(Q_{\text{left}}-x_{m_2})$ such that $\sigma_{\text{left}}(x_1)=c$, $\sigma(u_{t_1})=c^*$, and each vertex of $x_2\cdots x_{m_2-1}$ is colored with a color not lying in S. Since Π^0_p is an induced subgraph of G, $\sigma_{\text{left}}\cup\sigma_{\text{right}}$ is an induced subgraph of G. By assumption, $(G[V(\Pi^0_p\cup\Pi^1_p)],\sigma_{\text{left}}\cup\sigma_{\text{right}})$ is not a (C,p)-wedge, so there exists an extension of $\phi\cup\sigma_{\text{left}}\cup\sigma_{\text{right}}$ to an L-coloring τ of $\text{dom}(\phi\cup\sigma_{\text{left}}\cup\sigma_{\text{right}})\cup\{z,u_{t_3}\}$ such that τ does not extend to L-color $\{x_{m_2}\}\cup\{u_{t_1+1}\cdots u_{t_3-1}\}$. Yet by our construction of σ_{left}, we have $L_\tau(x_{m_2})\setminus S\neq\varnothing$, so it follows from Observation 1.4.2 that τ extends to an L-coloring of H_2 using a color of T_{right} on u_{t_2}. Thus, τ extends to L-color $\{x_{m_2}\}\cup\{u_{t_1+1}\cdots u_{t_3-1}\}$, contradicting our assumption. ■

For each $c\in L_\phi(x_1)$, let σ^c be the extension of σ_{right} to an L_ϕ-coloring of $\{x_1\}\cup\{x_{m_2+1},\cdots,x_{m_3}\}$ obtained by coloring x_1 with c. Since Π^1_p is an induced subgraph of G, each of these is a proper L_ϕ-coloring of its domain. By assumption, for each $c\in L_\phi(x_1)$, the pair $(G[V(\Pi^0_p\cup\Pi^1_p)],\sigma^c)$ is not a (C,p)-wedge, so there is an extension of $\phi\cup\sigma^c$ to an L-coloring τ^c of $\text{dom}(\phi\cup\sigma^c)\cup\{z,u_{t_3}\}$ which does not extend to L-color $\{x_{m_2}\}\cup V(\Pi^0_p)$.

Subclaim 5.3.18. *For each $c\in L_\phi(x_1)$, $L_{\tau^c}(x_2)=T_{\text{right}}$.*

Proof: Suppose there is a contradiction that there is a $c\in L_\phi(x_1)$ with $L_{\tau^c}(x_2)\neq T_{\text{right}}$. Since $|L_{\tau^c}(x_2)|\geq 2$, there is a $d\in L_{tau^c}(x_2)\setminus T_{\text{right}}$. By Theorem 0.2.3, $\mathcal{Z}_{L^p_\phi,H_1}(\phi(p_1),c,\bullet)\neq\varnothing$, so it follows from Observation

1.4.2 that τ^c extends to an L-coloring of dom$(\tau^c) \cup V(H_1 \cup H_2)$ using d on x_2 a color of T_{right} on u_{t_2}. By definition of T_{right}, it follows that τ^c extends to an L-coloring of Π_p^0, contradicting our assumption. ■

It follows from Subclaim 5.3.18 that $|L(x_2)| = 5$ and $L_\phi(x_1) \cup \{r\}$ is a subset of $L_\phi(x_2) \setminus T_{\text{right}}$ of size four, but it also follows from Subclaim 5.3.18 that $|L_\phi(x_2) \setminus T_{\text{right}}| = 3$, a contradiction. ■

Applying Claim 5.3.16, we fix a color $r^* \in L(x_{m_3}) \setminus (L(u_{t_2}) \cup L(u_{t_3}))$. By Claim 5.3.10, Q_{left} is an edge. Since Q_{right} is an induced subpath of G and each vertex of Q_{right} has an L_ϕ-list of size at least five, there is an L-coloring σ of $Q_{\text{right}} - x_2$ such that $\sigma(x_{m_3}) = r^*$ and each vertex of of $x_3 \cdots x_{m_3}$ is colored with a color outside of $L(u_{t_2})$. Applying Claim 5.3.10, we fix a color $r \in L_\phi(x_1) \cap L(x_2)$ such that the following hold.

1) There are two elements of $\mathcal{M}_L(H_2, u_{t_1}x_{m_2})$ using r on x_{m_2}; *AND*
2) $\mathcal{Z}_{H_1, L_\phi^p}(\phi(p), r, \bullet) = L(u_{t_1}) \setminus \{r\} = L_\phi(x_1) \setminus \{r\}$.

Let $T_{\text{left}} := \mathcal{Z}_{H_1, L_\phi^p}(\phi(p), r, \bullet)$. We now have the following:

Claim 5.3.19. *There is a $z \in V(\Pi_p^2)$ adjacent to all three of $x_{m_2-1}, x_{m_2}, x_{m_2+1}$.*

Proof: Suppose not. Consider the following cases:

Case 1: Either Π_p^1 is induced or $\sigma(x_3) \neq r$.

In this case, since $L_\phi(x_2) \setminus (T_{\text{left}} \cup \{r, \sigma(x_3)\})| \geq 1$, there is an extension of σ to an L_ϕ-coloring σ' of $V(\Pi_p^1)$ with $\sigma'(x_1) = r$ and $\sigma'(x_2) \notin T_{\text{left}}$. By construction of σ', we have $|L_{\phi\cup\sigma'}(u_{t_3})| = 3$. By assumption, the pair $(G[V(\Pi_p^0 \cup \Pi_p^1)], \sigma')$ is not a (C,p)-wedge, so the inertness condition is violated, i.e there is an extension of $\phi \cup \sigma'$ to an L-coloring τ of dom$(\phi \cup \sigma') \cup \{u_{t_3}\}$ which τ does not extend to L-color the path Π_p^0. Since H_3 is a triangle and $|L_{\phi\cup\sigma'}(u_{t_2})| \geq 2$, there is a color left in $L_\tau(u_{t_2})$. Since $\sigma'(x_2) \notin T_{\text{left}}$, it follows from Observation 1.4.2 that τ extends to L-color $u_{t_1} \cdots u_{t_3}$ using a color of T_{left} on u_{t_1}, so τ extends to L-color the path Π_p^0, a contradiction.

Case 2: Π_p^1 is not induced and $\sigma(x_3) = r$

In this case, x_1x_3 is the lone chord of of Π_p^1 and $N(x_2) = \{x_1, x_3\} \cup \{u_{t_1}, \cdots, u_{t_2}\}$. Since $T_{\text{left}} \subseteq L_\phi(x_1) \setminus \{r\}$ and each vertex of Q_{right} has an L_ϕ-list of size at least five, there is an extension of σ to an L_ϕ-coloring σ' of $V(Q_{\text{right}} - x_2) \cup V(H_1)$ which colors x_1 with a color of T_{left} and colors u_{t_1} with r. By assumption, the pair $(G[V(\Pi_p^0 \cup \Pi_p^1)], \sigma')$ is not a (C,p)-wedge, so the inertness condition is violated. That is, there exists an extension of σ' to an L-coloring τ of dom$(\phi \cup \sigma') \cup \{u_{t_3}\}$ such that τ does not extend to L-color $\{x_2\} \cup \{u_{t_1+1}, \cdots, u_{t_3-1}\}$. Now, since each of u_{t_1}, x_3 is colored with r, we have $|L_\tau(x_{m_2})| \geq 3$. By our construction of σ', we have $|L_{\phi\cup\sigma'}(u_{t_2})| = 3$, so $L_\tau(u_{t_2})| \geq 2$, and there is a $d \in L_\tau(x_{m_2})$ such that $|L_\tau(u_{t_2}) \setminus \{d\}| \geq 2$. Thus, applying Observation 1.4.2 to the edge $u_{t_1}x_{m_2}$, together with the fact that H_3 is a triangle, τ extends to L-color $\{x_2\} \cup \{u_{t_1+1}, \cdots, u_{t_3-1}\}$, a contradiction. ■

Applying Claim 5.3.19, since $m_2 = 2$, let z be the lone vertex of Π_p^2 adjacent to each of x_1, x_2, x_3. Since G is short-separation-free, we have $x_1x_3 \notin E(G)$, so Π_p^1 is an induced subgraph of G. Let $T_{\text{left}} = \{s_0, s_1\}$.

Claim 5.3.20. *The following hold.*

1) *$T_{\text{left}} \subseteq L(x_2)$; AND*
2) *$\sigma(x_3) \in L(x_{m_2}) \setminus (\{r\} \cup T_{\text{left}})$; AND*
3) *There exists a $d \in L(u_{t_2})$ such that $L(u_{t_2}) = T_{\text{left}} \cup \{d\}$ and, for each $j = 0, 1$, $\mathcal{Z}_{H_2, L}(s_j, s_{1-j}, \bullet) = \{d\}$.*

Proof: As Π_p^1 is an induced subgraph of G, σ extends to an L_ϕ-coloring $\sigma^\dagger$ of $V(\Pi_p^1 - x_2)$ with $\sigma^\dagger(x_1) = r$. By assumption, $(G[V(\Pi_p^0 \cup Pi_p^1)], \sigma^\dagger)$ is not a (C,p)-wedge. Since x_2 is uncolored, the inertness condition is violated and there is an extension of $\phi \cup \sigma^\dagger$ to an L-coloring τ of dom$(\phi \cup \sigma^\dagger) \cup \{z, u_{t_3}\}$ which does not extend to L-color $\{x_2\} \cup V(\Pi_p^0)$. Sin $T_{\text{left}} = \mathcal{Z}_{L_\phi^p, H_1}(\phi(p), r, \bullet)$, there is no L_τ-coloring of H_2 using one of $\{s_0, s_1\}$ on u_{t_1}. By our construction of σ, we have $|L_\tau(u_{t_2})| \geq 2$. Since $|L_\tau(x_2)| \geq 2$, and there is no L_τ-coloring of H_2 using a color of T_{left} on u_{t_1}, it follows from Observation 1.4.2 that $L_\tau(x_2) = L_\tau(u_{t_2}) = T_{\text{left}}$. Furthermore, for each $j = 0, 1$, the lone color of $\mathcal{Z}_{H_2,L}(s_j, s_{1-j}, \bullet)$ is $\tau(u_{t_3})$, so each of 1)-3) hold. ■

Applying Claim 5.3.20, let $L(u_{t_2}) = \{s_0, s_1, d\}$ for some color d. Recall that r has been chosen so that there is an element of $\mathcal{M}_L(H_2, u_{t_1}x_2)$ using r on x_2, so, by definition, $r \notin L(u_{t_2})$ and $r \neq d$. Ssince $\mathcal{Z}_{H_2,L}(s_j, s_{1-j}, \bullet) = \{d\}$ for each $j = 0, 1$, it follows from Proposition 1.4.4 that $\{s_0, s_1\} \subseteq L(u_k)$ for each $k = t_1 + 1, \cdots, t_2 - 1$.

Claim 5.3.21. *For each $j = 0, 1$, $|\mathcal{Z}_{H_2,L}(r, s_j, \bullet)| = 1$.*

Proof: Suppose there is a $j \in \{0, 1\}$ for which this does not hold, say $j = 1$ without loss of generality. Recall that $\mathcal{Z}_{H_1, L_\phi^p}(\phi(p), s_0, \bullet) = \{r\}$. Since Π_p^1 is an induced subgraph of G, σ extends to an L_ϕ-coloring $\sigma^\star$ of $V(H_1) \cup V(Q_{\text{right}} - x_2)$ using s_0, r on the respective vertices x_1, u_{t_1}. By assumption, $(G[V(\Pi_p^0 \cup \Pi_p^1), \sigma^\star)$ is not a (C,p)-wedge. Since x_2 is uncolored and $|L_{\phi \cup \sigma^\star}(u_{t_3})| = 3$, the inertness condition is violated, so there is an extension of $\phi \cup \sigma^\star$ to an L-coloring τ of dom$(\phi \cup \sigma^\star) \cup \{z, u_{t_3}\}$ which does not extend to L-color $V(H_2 \setminus \{u_{t_1}\})$. Since $|L_\tau(x_2)| \geq 1$, let $f \in L_\tau(x_2)$. We then have $\mathcal{Z}_{H_2,L}(r, f, \bullet) \cap L_\tau(u_{t_2}) = \varnothing$. By our construction of σ, $|L_{\phi \cup \sigma^\star}(u_{t_2})| = 3$, so $|L_\tau(u_{t_2})| \geq 2$. By assumption, $\mathcal{Z}_{H_2,L}(r, s_1, \bullet)| > 1$, so $\mathcal{Z}_{H_2,L}(r, s_1, \bullet) \cap L_\tau(u_{t_2}) \neq \varnothing$ and $f \neq s_1$. Since $\sigma^\star(x_1) = s_0$, we have $f \in L(x_{m_2}) \setminus \{s_0, s_1, r\}$. Since $|\mathcal{Z}_{H_2,L}(r, f, \bullet)| = 1$, we have $f = d$ by Proposition 1.4.4. If H_2 is a triangle, then $s_0, s_1 \in \mathcal{Z}_{H_2,L}(r, f, \bullet)$, which is false, so H_2 is not a triangle. But then, again by Proposition 1.4.4, we have $r, d \in L(u_{t_1+1})$. Since $|L(u_{t_1+1})| = 3$ and $\{s_0, s_1\} \subseteq L(u_{t_1+1})$, we have a contradiction. ■

For each $k = u_{t_1}, \cdots, u_{t_2-1}$, let H_2^k be the broken wheel with principal path $u_k x_2 u_{t_2}$, where $H_2^k - x_2 = u_k \cdots u_{t_2}$.

Claim 5.3.22. *For each $k = t_1, \cdots, t_2 - 1$, the following hold.*

1) *$L(u_k) = \{r, s_0, s_1\}$; AND*
2) *For each $j = 0, 1$ and $f \in L(u_k) \setminus \{s_{1-j}\}$, we have $|\mathcal{Z}_{H_2^k, L}(f, s_{1-j}, \bullet)| = 1$.*

Proof: We show this by induction on k. If $k = u_{t_1}$ then $H_2^k = H_2$. We have $L(u_{t_1}) = \{r, s_0, s_1\}$ and, by Claim 5.3.21, we have $|\mathcal{Z}_{H_2,L}(r, s_{1-j}, \bullet)| = 1$ for each $j = 0, 1$. By Claim 5.3.20, we have $\mathcal{Z}_{H_2,L}(s_j, s_{1-j}, \bullet) = \{d\}$ for each $j = 0, 1$. This completes the base case. If H_2 is a triangle, then we are done, so suppose now that H_2 is not a triangle, let $k \in \{u_{t_1+1}, \cdots, u_{t_2-1}\}$, and suppose that 1) and 2) above hold for $k - 1$. For each $j = 0, 1$ and $f \in L(u_{k-1})$, we have $|\mathcal{Z}_{H_2^{k-1}, L}(f, s_{1-j}, \bullet)| = 1$. Since H_2^{k-1} is not a triangle, it follows from Proposition 1.4.4 that $r \in L(u_k)$, so $L(u_k) = \{r, s_0, s_1\}$ and k satisfies 1). Suppose there is a $j \in \{0, 1\}$ and an $f \in \{r, s_j\}$ such that $|\mathcal{Z}_{H_2^k, L}(f, s_{1-j}, \bullet)| \geq 2$. Letting $f^* \in L(u_{k-1}) \setminus \{f, s_{1-j}\}$, we then have $\mathcal{Z}_{H_2^{k-1}, L}(f^*, s_{1-j}, \bullet)| \geq 2$, contradicting our induction hypothesis. ■

Let $k = u_{t_2} - 1$. By Claim 5.3.22, we have $r \in L(u_k)$ and $|\mathcal{Z}_{H_2^k, L}(r, s_j, \bullet)| = 1$ for each $j = 0, 1$, so $r \in L(u_{t_2})$, which is false. This completes the proof of Lemma 5.3.1 and Theorem 5.1.14 and thus completes the proof of Theorem 5.1.6. □

Chapter 6

Deleting Vertices Near the Open Rings of Critical Mosaics

In this chapter, we build on the work of Chapter 5 to carefully cut away part of an open ring in a critical mosaic near the precolored path. We begin with the following natural definition analogous to Definition 3.3.8..

Definition 6.0.1. Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let C be an open $\mathcal{T}$-ring. Let C^2 be the unique cycle of G specified in Theorem 4.0.1. We call C^2 the *2-necklace* of C.

When we cut away part of an open ring in a critical mosaic it is easier to analyze proper k-chords of the 2-necklace of an open ring, rather than proper k-chords of the specified open ring, for small values of k. We first introduce the following natural definition.

Definition 6.0.2. Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let C be an open $\mathcal{T}$-ring. Let C^2 be the 2-necklace of C. We define a subgraph $\hat{G}$ of G which we call the *large side of* C^2 as follows. We set $\hat{G}$ to be Int(C^2) if C is the outer face of G, and otherwise set $\hat{G}$ to be Ext(C^2). We call the graph $G \setminus (\hat{G} \setminus C^2)$ the *small side* of C^2.

We now have the following simple observation.

Observation 6.0.3. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let C be an open $\mathcal{T}$-ring. Let C^2 be the 2-necklace of C and let $\hat{G}$ be the large side of C^2. Let $k < \frac{N_{\text{mo}}}{4} - 4$ and let Q be a k-chord of C^2 in $\hat{G}$. Let $\hat{G} = G_0 \cup G_1$ be the natural Q-partition of $\hat{G}$. Then there exists an $i \in \{0, 1\}$ such that $C' \subseteq G_i$ for all $C' \in \mathcal{C} \setminus \{C\}$.*

Proof. Let u, u' be the endpoints of Q. If there is a k-chord Q' of C such that Q is a subpath of Q', then the desired result follows from Corollary 2.3.8. Now suppose that no such k-chord of C exists, and suppose toward a contradiction that there exist $C^0, C^1 \in \mathcal{C} \setminus \{C\}$ such that $C_i \subseteq G_i$ for each $i = 0, 1$. Thus, it follows that either Q is a cycle (i.e not a proper generalized chord) or Q is a proper generalized chord whose endpoints have a common neighbor in C^1. In either case, there is a cycle D which separates C^0 from C^1, where $Q \subseteq D$ and $|V(D) \setminus V(Q)| \leq 1$. Since $\text{Rk}(C) = 2N$ and $|V(D)| \leq \frac{N_{\text{mo}}}{4}$, it follows from Corollary 2.1.30 that $d(D, V(C \setminus \mathring{\mathbf{P}}) > 2N_{\text{mo}} - \frac{3N_{\text{mo}}}{8}$. Yet each vertex of Q is of distance at most 2 from $V(C)$ and thus, since $|E(\mathbf{P})| \leq \frac{2N_{\text{mo}}}{3}$, each endpoint of Q is of distance at most $\frac{N_{\text{mo}}}{3} + 2$ from $C \setminus \mathring{\mathbf{P}}$, a contradiction. $\square$

Given the observation above, it is natural to introduce the following notation.

Definition 6.0.4. Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let C be an open $\mathcal{T}$-ring. Let C^2 be the 2-necklace of C Let $\hat{G}$ be the large side of C^2. Let $k < \frac{N_{\text{mo}}}{4} - 4$ and let Q be a k-chord of C^2 (not necessarily proper). We let $\hat{G} = \hat{G}_Q^{\text{small}} \cup \hat{G}_Q^{\text{large}}$ be the natural Q-partition of $\hat{G}$, where, for each $C' \in \mathcal{C} \setminus \{C\}$, we have $C' \subseteq \hat{G}_Q^{\text{large}}$.

It is clear that the partition defined above respects the orientation defined by the subpath of the 2-necklace consisting of the neighbors of the precolored path, which is made precise by the following observation.

Observation 6.0.5. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let C be an open $\mathcal{T}$-ring. Let C^2 be the 2-necklace of C and let $\hat{G}$ be the large side of C^2. Let $k < \frac{N_{\text{mo}}}{4} - 4$ and let Q be a k-chord of C^2 in $\hat{G}$, where neither endpoint of Q lies in $\mathring{\mathbf{P}}^1$. Then $\mathbf{P}^1 \subseteq \hat{G}_Q^{\text{large}}$.*

Proof. If this does not hold, then either there is a cycle D of length at most $k+2$ with $d(C, D) \leq 2$, where D separates C from an element of $\mathcal{C} \setminus \{C\}$, or there is a $Q' \in \mathcal{K}(C, \mathcal{T})$ of length at most $k + 4$ such that Q' separates $\mathbf{P}$ from an element of $\mathcal{C} \setminus \{C\}$. In the first case, we contradict Corollary 2.1.30, and in the second case, we contradict 3) of Theorem 2.2.4. $\square$

We require some more setup before we state our main result for Chapter 6. Given an open ring C in a critical mosaic, we introduce the following very natural way to associate to a vertex z which is close to the 2-necklace of C a "span" of z which is determined by the neighbors of z and the vertices of C^2 of distance two from z.

Definition 6.0.6. Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let C be an open $\mathcal{T}$-ring. Let C^1 be the 1-necklace of C and let C^2 be the 2-necklace of C. Let $\hat{G}$ be the large side of C^2. Given a $z \in D_2(C^2, \hat{G})$, we associate to z a subgraph Span(z) of $\hat{G}$ in the following way.

1) If there exists a proper 4-chord P of C^2 in $\hat{G}$ whose midpoint in C, then we set Span(z) to be the unique proper 4-chord P of C^2 which minimizes the quantity $|V(\hat{G}_P^{\text{small}})|$.
2) If no such proper 4-chord of C^1 exists, then we define Span(z) in the following way.
    a) If $N(z) \cap D_1(C^2)$ consists of a lone vertex v, and $|N(v) \cap V(C^2)| = 1$, then we set Span(z) to be the unique 2-path with z as an endpoint and the other endpoint in C^2.
    b) If $N(z) \cap D_2(C)$ consists of a lone vertex v, and $|N(v) \cap V(C^2)| > 1$, then we set Span(z) to be the claw on the vertices $\{v, z, x, x'\}$, where Span(z) has central vertex z and xvx' is the unique 2-chord of C^1 with central vertex v which maximizes the quantity $|V(\hat{G}_{xvx'}^{\text{small}})|$.
    c) If $|N(z) \cap D_2(C)| > 1$, then, since G is $K_{2,3}$-free, there exist vertices v, v', x such that $N(z) \cap D_1(C^2) = \{v, v'\}$ and $N(v) \cap V(C^2) = N(v') \cap V(C^2) = \{x\}$, and set Span$(z)$ to be the 4-cycle $zvxv'$.

Thus, for each $z \in D_2(C^2, \hat{G})$, Span(z) is either a 4-path, a 4-cycle, a claw, or a 2-path. There is a natural way to associate to each such z a subpath of C^2 in the following way.

Definition 6.0.7. Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let C be an open $\mathcal{T}$-ring. Let C^2 be the 2-necklace of C and let $\hat{G}$ be the large side of C^2. Given a $z \in D_2(C^2, \hat{G})$, we let Pin(z) be the unique subpath of C^2 such that the following hold.

1) If Span(z) is either a 2-path or a 4-cycle, then Pin(z) is just the singleton path Span$(z) \cap C^2$*; AND*
2) The endpoints of Pin(z) are the vertices of Span$(z) \cap C^2$ and, in $\hat{G}$, Span(z) separates the edges of Pin(z) from all the elements of $\mathcal{C} \setminus \{C\}$.

We require one more definition and then we state our main result for Chapter 6.

Definition 6.0.8. Let $\mathcal{T} := (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let $C \in \mathcal{C}$ be an open $\mathcal{T}$-ring, let C^2 be the 2-necklace of C, and let $\hat{G}$ be the large side of C^2. Let ϕ be the unique L-coloring of $V(\mathbf{P})$. Given a $z \in D_2(C^2, \hat{G})$, a (C, z)-

opener is a pair $[K,\psi]$, where K is a connected subgraph of G which we call the *underlying graph* of $[K,\psi]$, and ψ is an extension of ϕ to an partial L-coloring of $V(K)$ such that the following hold.

1) $\mathbf{P}\subseteq K$ and, for each $v\in V(K)\cap B_1(C)$, we have $d(v,\mathbf{P})\leq 6$*; AND*
2) $V(K\setminus\text{dom}(\psi))$ is L_ψ-inert and, for each $u\in D_1(K)$, $|L_\psi(u)|\geq 3$*; AND*
3) There is at most one vertex of $(\text{dom}(\psi)\cap D_1(C^2,\hat{G}))\setminus\text{Sh}_4(C^2,\hat{G})$ which does not lie in $\text{Span}(z)$*; AND*
4) For any $v\in V(H)\cap\text{Sh}_4(C^2,\hat{G})$, either $v\in\text{Sh}_3(C^2,\hat{G})$, or $\text{Span}(z)$ is a 4-chord of C^2 which, in $\hat{G}$ separates v from every element of $\mathcal{C}\setminus\{C\}$*; AND*
5) $K\cap C^2$ is a subpath P of C^2 such that the following hold.
   a) Each of $\mathbf{P}^1$ and $\text{Pin}(z)$ is a subpath of P, and $V(K)\subseteq\text{Sh}_4(P,C^2,\hat{G})\cup V(C)\cup B_1(C^2)\cup\{z\}$. We call the unique $(\text{Pin}(z),\mathbf{P}^1)$-subpath of P the *head* of $[K,\psi]$*; AND*
   b) If $\text{Pin}(z)$ is not a terminal subpath of P, then each vertex of P has distance at most 8 from $\mathbf{P}$*; AND*
   c) If $\text{Pin}(z)\cap\mathbf{P}^1=\varnothing$, then every vertex of P outside of the head of $[K,\psi]$ has distance at most 8 from $\mathbf{P}$*; AND*
   d) If $\text{Pin}(z)\cap\mathbf{P}^1\neq\varnothing$, then every vertex of $P\setminus\text{Sh}_4(P,C^2,\hat{G})$ has distance at most 14 from $\mathbf{P}$*; AND*
   e) For all $v\in V(K)\cap V(C^1\setminus\mathbf{P}^1)$, all of the neighbors of v on C^2 lie in P.

When we construct a smaller counterexample from a critical mosaic by deleting a path between the outer face and another ring, we need to be careful in the case where the outer face is an open ring, because two internal rings are possibly both close to the outer face but still far from each other. This is not the case if the outer face is a closed ring, since closed rings in a mosaic are of bounded length, but if the outer face is an open ring, then we want to ensure that, in a small ball around the outer face, we have some control over how far our deletion set is from the precolored path of the outer face, otherwise the new tessellation possibly has an internal ring which is too close to the new outer face to satisfy the distance conditions of Definition 2.1.6. This is the reason for the somewhat technical conditions in 5) of Definition 6.0.8. Our main theorem for Chapter 6 is the following theorem. In order to deal with the case where the outer face is an open ring, we need to specify a direction along which we cut open the graph on the 2-necklace of the ouer face. This is the reason we need to prove 2) of the theorem below.

Theorem 6.0.9. *Let $\mathcal{T}:=(G,\mathcal{C},L,C_*)$ be a critical mosaic, let $C\in\mathcal{C}$ be an open $\mathcal{T}$-ring, and let C^2 be the 2-necklace of C. Let $\hat{G}$ be the large side of C^2 and let p,p' be distinct endpoints of $\mathbf{P}^1$. Then, for any $z\in D_2(C^2,\hat{G})\setminus\text{Sh}_4(C^2,\hat{G})$, the following hold.*

1) There exists a (Cz)-opener; AND
2) If $\text{Pin}(z)\cap\mathbf{P}^1=\varnothing$ and there is no $(\text{Pin}(z),p)$-path of length at most 16 on the small side of C^2, then there exists a (C,z)-opener whose head has p' as an endpoint.

6.1 Deleting a C-Wedge

For the remainder of this chapter, in order to avoid repetition, we fix the following data.

1) We let $\mathcal{T}:=(G,\mathcal{C},L,C_*)$ be a critical mosaic, ww let $C\in\mathcal{C}$ be an open ring, and we let C^1 be the 1-necklace of C and C^2 be the 2-necklace of C; *AND*

2) We let $\mathbf{P} := \mathbf{P}_{\mathcal{T}}(C)$ and $\mathbf{P}^1 := \mathbf{P}^1_{\mathcal{T}}(C)$; *AND*

3) We let ϕ be the unique L-coloring of $V(C)$; *AND*

4) We set $\hat{G}$ to be the large side of C^2 and, applying Theorem 5.1.6, we fix a C-wedge (H, ψ) and let $\varphi := \phi \cup \psi$

All of this fixed information is in the background of the remainder of Chapter 6 so that the statements of the intermediate results and definitions that we need for the proof of Theorem 6.0.9 do not become too long and unwieldy.

In order to prove Theorem 6.0.9 we perform a partial coloring and deletion similar to that of Section 1.7. An overview of this idea is as follows. It follows from Theorem 4.0.1 that C^2 is a facial subgraph of $\hat{G}$ in which most of the vertices have L_φ-lists of size five, except for the vertices of C^2 on a short subpath of C^2 which contains $\mathbf{P}^1$. We want an analogue to Theorem 1.7.5 for a subpath of C^2. One complication is that C^2 is a facial subgraph of $\hat{G}$ but not a facial subgraph of $G \setminus H$. However, most of the vertices of $C^1 \setminus (\mathbf{P}^1 \cup H)$ also have L_φ-lists of size five, and, in any case, we no longer need to deal with the remaining vertices of C, because we have cut away all the vertices of C with neighbors in C^2. The two propositions below make this precise by describing the graph obtained from G by deleting H.

Proposition 6.1.1. *$C^1 \setminus (H \cup \mathbf{P}^1)$ is a path and furthermore, there is a unique subpath Ω^1 of the subgraph of G induced by $C^1 \setminus (H \cup \mathbf{P}^1)$ such that Ω^1 satisfies all of the following.*

1) Every vertex of Ω^1 has an L_φ-list of size at least five, except for the endpoints of Ω^1, and each endpoint of Ω^1 has an L_φ-list of size at least three; AND

2) Ω^1 is an induced subgraph of G and every vertex of $C^1 \setminus (H \cup \mathbf{P}^1)$ with a neighbor in $G \setminus B_1(C)$ lies in $V(\Omega^1)$; AND

3) For every $v \in V(C^2 \setminus \mathbf{P}^1)$, if v has a neighbor in $V(G \setminus \hat{G}) \setminus V(H)$, then the subgraph of G induced by $N(v) \cap (V(G \setminus \hat{G}) \setminus V(H))$ is a subpath of Ω^1; AND

4) $|E(\Omega^1)| \geq 13$.

Proof. Recalling Theorem 3.0.2, every chord of the path $C^1 \setminus \mathbf{P}^1$ has endpoints which are also the endpoints of a subpath of $C^1 \setminus \mathbf{P}^1$ of length precisely two. Since G is short-separation-free, there is a unique subpath Ω of $G[V(C^1 \setminus \mathbf{P}^1)$ such that, for every $v \in V(G) \setminus B_1(C)$, if v has a neighbor in $C^1 \setminus \mathbf{P}$, then $N(v) \cap B_1(C) \subseteq V(\Omega)$, i.e Ω is the unique path obtained from C^1 by replacing all the 2-paths in C^1 whose endpoints are also the endpoints of a chord of C^1 with the corresponding chord of C^1. It follows from 1) of Theorem 2.3.2 the endpoints of Ω are not adjacent.

Note that the endpoints of $C^1 \setminus \mathbf{P}^1$ are also the endpoint of Ω, and, since endpoints of Ω are not adjacent in G, Ω is an induced subgraph of G. Let p, p' be the endpoints of $\mathbf{P}$. Recalling Definitions 5.1.2 and 5.1.5, $C^1 \setminus (\mathbf{P} \cup H)$ is a path, since $H \cap C^1$ consists of two disjoint connected components, where one of these components is a terminal subpath of Π^1_p and the other component is a terminal subpath of $\Pi^1_{p'}$. Let $\Omega^1 := \Omega \setminus H$. Note that Ω^1 is a subpath of Ω, and the only vertices of Ω^1 with neighbors in H are the endpoints of Ω^1. By Definition of $H, \psi)$, each endpoint of Ω^1 has an L_φ-list of size at least three, each each other vertex of Ω^1 has no neighbors in $\mathbf{P}$ and no neighbors in H, and thus has an L_φ-list of size at least five. Finally, it follows from Theorem 4.0.1 that, for every $v \in V(C^2 \setminus \mathbf{P}^1)$, if v has a neighbor in $V(G \setminus \hat{G}) \setminus V(H)$, then the subgraph of G induced by $N(v) \cap (V(G \setminus \hat{G}) \setminus V(H))$ is a subpath of Ω^1.

Let q, q' be the endpoints of Ω^1 and let p, p' be the endpoints of $\mathbf{P}$. Without loss of generality, let q have a neighbor $v \in V(H \cap \Pi^1_p)$ and let q' have a neighbor $v' \in V(H \cap \Pi^1_{p'})$. Now, $G[V(H) \cap V(\Pi^0_p \cup \Pi^1_p)]$ contains a $(v, \mathbf{P})$-path which has p as an endpoint and has length at most five, and likewise, $G[V(H) \cap V(\Pi^0_{p'} \cup \Pi^1_{p'})]$ contains a $(v', \mathbf{P})$-path which has p' as an endpoint and has length at most five, where these two paths are disjoint. Thus, if $|E(\Omega^1)| < 13$, then, recalling Definition 2.3.1, there exists a C-band of length at most $6+6+12$. Since $N_{\text{mo}} \geq 96$, this contradicts 1) of Theorem 2.3.2. $\square$

The second of our two propositions describes the lists of C^2 that result after we color and delete dom(φ) from $V(G)$.

Proposition 6.1.2. *$C^2 \cap H = \varnothing$ and there is a unique path $\Gamma^2 \subseteq C^2$ such that all of the following hold.*

1) *$E(C^2 \setminus \mathring{\Gamma}^2) \geq 13$; AND*
2) *$\mathbf{P}^1 \subseteq \Gamma^2$ and every vertex of Γ^2 is of distance at most 6 from $V(\mathbf{P})$ in G; AND*
3) *Every vertex of $C^2 \setminus \Gamma^2$ has an L_φ-list of size at least five; AND*
4) *Each endpoint of $\mathbf{P}^1$ is an internal vertex of Γ^2, and every vertex of Γ^2 has an L_φ-list of size at least three, except possible the endpoints of $\mathbf{P}^1$, which have L_φ-lists of size at least two; AND*
5) *The only vertices of Γ^2 with a neighbor in Ω^1 are the endpoints of Γ^2. Conversely, every vertex of $C^2 \setminus \mathring{\Gamma}^2$ has a neighbor in Ω^1.*

Proof. Firstly, we have $C^2 \cap H = \varnothing$ since $C^2 \cap (C \cup C^1) = \mathbf{P}^1$ and $H \cap \mathbf{P}^1 = \varnothing$. Let p, p' be the endpoints of $\mathbf{P}$. Recalling the notation of Definition 5.1.4, each of Π^2_p and $\Pi^2_{p'}$ is a terminal subpath of $C^2 \setminus \mathbf{P}^1$ and every vertex of $C^2 \setminus \mathbf{P}^1$ with a neighbor in H lies in $\Pi^2 \cup \Pi^2_{p'}$. By Definition 5.1.5, the set of vertices of Π^2_p with a neighbor in H form a subpath of π^2_p of Π^2_p which is a nonempty terminal subpath of $C^2 \setminus \mathbf{P}^1$. Likewise, set of vertices of Π^2_p with a neighbor in H form a subpath of π^2_p of Π^2_p, where $\pi^2_{p'}$ is a terminal subpath of $C^2 \setminus \mathbf{P}^1$ containing the other terminal vertex of $C^2 \setminus \mathbf{P}^1$.

We now set Γ^2 to be the subpath of of C^2 consisting of all the vertices of $V(\mathbf{P}^1 \cup \pi^2_p \cup \pi^2_{p'})$. Since each vertex of Π^2_p has distance at most 6 from p and each vertex of $\Pi^2_{p'}$ has distance at most 6 from p', Condition 2) is satisfied. If $|E(C^2 \setminus \mathring{\Gamma}^2| < 13$, then, as in the proof of Proposition 6.1.1, there exists a C-band of length at most $12+6+6$, contradicting 1) of Theorem 2.3.2, so Condition 3) is satisfied.

By Definition 5.1.5, each of $\pi^2_p, \pi^2_{p'}$ is nonempty, so each endpoint of $\mathbf{P}^1$ is an internal vertex of Γ^2. Let q, q' be the endpoints of $\mathbf{P}^1$. By Definition 5.1.5, there is no chord of C^1 with one endpoint in H and one endpoint in $\mathbf{P}^1 \setminus \{q, q'\}$, so each vertex of $\mathbf{P}^1 \setminus \{q, q'\}$ has an L_φ-list of size at least three, and, again by Definition 5.1.5, each of q, q' has an L_φ-list of size at least two. Thus, Condition 4) is satisfied. By our choice of paths $\pi^2_p, \pi^2_{p'}$, the only vertices of Γ^2 with a neighbor in Ω^1 are the respective endpoints of $\pi^2_p, \pi^2_{p'}$ which are also endpoints of Γ^2. Conversely, each vertex of $C^2 \setminus \Gamma^2$ is one endpoint of a 2-path whose other endpoint lies in $C^1 \setminus \mathbf{P}$ and whose midpoint lies in Ω^1. $\square$

We also have the following simple osbervation, which states that, for sufficiently small values of k, if we have a k-chord of C^2 in $\hat{G}$ where neither endpoint is an internal vertex of $\mathbf{P}^1$, then the "small" side of a k-chord of C^2 in $\hat{G}$ (as specified in Definition 6.0.4) does not separate the elements of $\mathcal{C} \setminus \{C\}$ from $\mathbf{P}^1$.

Proposition 6.1.3. *For any integer $1 \leq k \leq \frac{N_{\text{mo}}}{4} - 4$, any subpath Q of $C^2 \setminus \mathring{\mathbf{P}}^1$, and any k-chord R of C^2 with both endpoints in Q, we have $Q \subseteq \hat{G}_R^{\text{small}}$. In particular, Q is (k, L_φ)-short in $(C^2, \hat{G})$.*

Proof. Let $\hat{G} = \hat{G}_0 \cup \hat{G}_1$ be the natural R-partition of $\hat{G}$. If R is not a proper k-chord of $\hat{G}$ (i.e R is a cycle) then we are immediately done by Corollary 2.1.30. Now suppose that R is a proper k-chord of $\hat{G}$ and suppose without loss of generality that $\hat{G}_0 \cap Q$ has one connected component, and $\hat{G}_1 \cap Q$ has two connected components. In the notation of

Definition 6.0.4, we just need to check that $\hat{G}_0 = \hat{G}_R^{\text{small}}$. Firstly, since $\hat{G}_0 \cap Q$ is a subpath of Q, and both endpoints of R lie in Q, we have $\hat{G}_0 \cap C^2 = \hat{G} \cap Q$, so $\hat{G}_0 \cap C^2 \subseteq Q$ and $\mathbf{P}^1 \subseteq \hat{G}_1$. Suppose toward a contradiction that $\hat{G}_0 \neq \hat{G}_R^{\text{small}}$. Thus, we have $\hat{G}_0 = \hat{G}_R^{\text{large}}$.

Claim 6.1.4. *There is a a proper $k+4$-chord R' of C such that $R \subseteq R'$ and both endpoints of R' lie in $C \setminus \mathring{\mathbf{P}}$.*

Proof: Note that each vertex of $C^2 \setminus \mathbf{P}^1$ has a neighbor in $C^1 \setminus \mathbf{P}^1$, and each vertex of $C^1 \setminus \mathbf{P}^1$ has a neighbor in $C \setminus \mathring{\mathbf{P}}$. Thus, if no such proper $k+4$-chord of C exists, then G contains a cycle D with $R \subseteq D$ and $|V(D)| \leq k+4$, where, in G, D separates $\hat{G}_0$ from $\mathbf{P}$, and since D intersects with C on at most a lone vertex, D separates each element of $\mathcal{C} \setminus \{C\}$ from C. But since C is an open $\mathcal{T}$-ring, we have $\text{Rk}(\mathcal{T}|C) = 2N_{\text{mo}}$, and since $d(D, C) \leq 1$ and $|V(D) \leq$, we contradict Corollary 2.1.30. ■

Let R' be as in Claim 6.1.4. As neither endpoint of R' is an internal vertex of $\mathbf{P}$, we have $R' \in \mathcal{K}(C, \mathcal{T})$. Now, by 3) of Theorem 2.2.4, $\mathbf{P} \subseteq G_{R'}^{\text{large}}$ and thus $\mathbf{P}^1 \subseteq \hat{G}_R^{\text{large}}$. Since $\mathbf{P}^1 \subseteq \hat{G}_1$, we have $\hat{G}_0 = \hat{G}_R^{\text{small}}$, contradicting our assumption. □

In certain cases, we also deal with the special case of a 2-chord of a subpath of C^2 whose midpoint also lies in C^2. An identical argument to the one above shows the following simple observation.

Proposition 6.1.5. *Let Q be a subpath of C^2 of length at least one and let q, q' be the endpoints of Q. Let $v \in V(C^2 \setminus Q)$ and suppose that $\hat{G}$ contains each of vq and $v'q$ as chords of C^2. Let D be the cycle $Q + vqv'$ and let K be the subgraph of G consisting of all the edges and vertices in the unique closed region bounded by D which contains no edges of $E(C^2) \setminus E(Q)$. Then K contains no elements of $\mathcal{C} \setminus \{C\}$.*

In view of the above, it is natural to introduce the following definition.

Definition 6.1.6. Let Q be a subpath of C^2. A *Q-fulcrum* is a vertex $v \in V(Q)$ which is both a Q-hinge and satisfies the additional property that, if v is an internal vetex of Q, then there is no $w \in V(C^2)$ such that $\hat{G}$ contains no pair of chords which both have w as an endpoint and whose non-w-endpoints lie in different connected components of $Q - v$.

6.2 Extending $\text{Span}(z)$ for Vertices of Distance Two from C^2

Our approach to the proof of Theorem 6.0.9 is as follows. Given a $z \in V(\hat{G}) \cap D_2(C^2)$, we color and delete a subpath of C^2 which contains $\text{Span}(z) \cap V(C^2)$ and which contains all the vertices of C^2 with L_φ-lists of size less than five, where this subpath satisfies Condition 5) of Definition 6.0.8. We want the path we construct to contain every chord of C^2 in $\hat{G}$ with one endpoint in Γ^2 and the other endpoint in $C^2 \setminus \Gamma^2$, so we make the following definition.

Definition 6.2.1. The *chord-closure* of Γ^2 is the unique minimal subpath Γ^{2c} of C^2 such that $\Gamma^2 \subseteq \Gamma^{2c}$ and there is no chord of C^2 in $\hat{G}$ with one endpoint in $\Gamma^2 \setminus \mathring{\mathbf{P}}^1$ and the other endpoint in $C^2 \setminus \Gamma^{2c}$.

We check that this indeed a well-defined subpath of C^2.

Proposition 6.2.2. *Γ^{2c} is a proper subpath of C^2 and, in particular, $|E(C^2 \setminus \mathring{\Gamma}^2)| \geq 10$.*

Proof. Suppose that either Γ^{2c} is not a proper subpath of C^2 or it is a proper subpath of C^2 such that $|E(C^2 \setminus \mathring{\Gamma}^2)| < 9$. For each $v \in V(\Gamma^2)$, there is precisely one endpoint p of $\mathbf{P}$ such that the small side of G contains a $(v, \mathbf{P})$-path of length at most six whose $\mathbf{P}$-endpoint is p. Since each vertex of $C^2 \setminus \mathbf{P}^1$ has distance two from $C \setminus \mathbf{P}$, it follows from our assumption that one of the following holds.

1) There exists a C-band of length at most $9 + 7 + 7$*; OR*

2) There is a chord of C^2 with one endpoint in $\Gamma^2 \setminus \mathring{\mathbf{P}}^1$ and one endpoint in $C^2 \setminus \Gamma^2$ such that, in $\hat{G}$, this chord separates $\mathbf{P}^1$ from an element of $\mathcal{C} \setminus \{C\}$.

In the first case, we contradict 1) of Theorem 6.0.8. In the second case, we contradict Observation 6.0.5. $\square$

Analogous to Proposition 6.1.3, we have the following

Proposition 6.2.3. *For any $1 \leq k \leq \frac{N_{\text{mo}}}{4} - 8$, any subpath Q of C^2 with both endpoints in Γ^{2c}, and any k-chord R of C^2 with both endpoints in Q, we have $Q \subseteq \hat{G}^{\text{small}}_R$. In particular, Q is (k, L_φ)-short in $(C^2, \hat{G})$.*

Proof. Let $\hat{G} = \hat{G}_0 \cup \hat{G}_1$ be the natural R-partition of $\hat{G}$. If R is not a proper k-chord of $\hat{G}$ (i.e R is a cycle) then we are immediately done by Corollary 2.1.30. Now suppose that R is a proper k-chord of $\hat{G}$ and suppose without loss of generality that $\hat{G}_0 \cap Q$ has one connected component, and $\hat{G}_1 \cap Q$ has two connected components. As above, we just need to check that $\hat{G}_0 = \hat{G}^{\text{small}}_R$. Firstly, since $\hat{G}_0 \cap Q$ is a subpath of Q, and both endpoints of R lie in Q, we have $\hat{G}_0 \cap C^2 = \hat{G} \cap Q$, so $\hat{G}_0 \cap C^2 \subseteq Q$. Suppose toward a contradiction that $\hat{G}_0 \neq \hat{G}^{\text{small}}_R$. Thus, we have $\hat{G}_0 = \hat{G}^{\text{large}}_R$. At least one endpoint of R lies in $\mathring{\mathbf{P}}^1$, or else we contradict Proposition 6.1.3. By Observation 6.0.5, there is no chord of C^2 which separates an element of $\mathcal{C} \setminus \{C\}$ from $\mathbf{P}^1$, and since at least one endpoint of R lies in $\mathring{P}^1$, it follows that at least one of the following holds.

1) There is a C-band of length at most $k + 1 + 7$*; OR*

2) Both endpoints of R lie in $\mathring{\mathbf{P}}^1$, the endpoints of R have a common neighbor in C, and there is a a cycle of length at most $k + 2$ which separates C from an element of $\mathcal{C} \setminus \{C\}$.

In the first case, we contradict 1) of Theorem 2.3.2, and in the second case, we contradict Corollary 2.1.30. $\square$

Combining Propositions 6.1.3 and 6.2.3, we immediately have the following.

Proposition 6.2.4. *Let $z \in D_2(C^2, \hat{G})$. Then the following hold.*

1) If $\text{Span}(z) \cap C^2 \subseteq \Gamma^{2c}$, then $\text{Pin}(z)$ is a subpath of Γ^{2c}; AND

2) If $\text{Span}(z) \cap \Gamma^{2c} = \varnothing$, then $\text{Pin}(z)$ is a subpath of $C^2 \setminus \Gamma^{2c}$.

Given a $z \in D_2(C^2, \hat{G}) \setminus \text{Sh}_4(C^2, \hat{G})$, when we delete a subpath of C^2 which contains $\text{Span}(z) \cap C^2$ and contains Γ^{2c}, we need to make sure that our path does not wind sufficiently far around C^2 that we create unwanted interactions between its endpoints, so we introduce the following definition.

Definition 6.2.5. Let $z \in D_2(C^2, \hat{G})) \setminus \text{Sh}_4(C^2, \hat{G})$ with $\text{Span}(z) \cap \Gamma^{2c} = \varnothing$. Let p, p' be the two endpoints of Γ^{2c} and let P be the unique subpath of C^2 such that $\text{Pin}(z)$ is a terminal subpath of P, $\Gamma^{2c} \subseteq P$, and p' is the non-$\text{Pin}(z)$ endpoint of P. We say that p is a *good z-direction* if the following hold.

1) There is no $(p', \text{Span}(z) \cap C^2)$ path on the small side of C^2 which has length less than three*; AND*

2) For any $1 \leq k \leq 3$ and any proper k-chord R of C^2 in $\hat{G}$, if both endpoints of R lie in P, then $\hat{G}^{\text{small}}_R \cap C^2$ is a subpath of P.

Given a $z \in D_2(C^2, \hat{G}) \setminus \text{Sh}_4(C^2, \hat{G})$ with $\text{Span}(z) \cap \Gamma^{2c} = \varnothing$, it is possible that both endpoints of Γ^{2c} are good z-directions, but in any case, there is at least one choice of good z-direction.

Proposition 6.2.6. *Let $z \in D_2(C^2, \hat{G}) \setminus \mathrm{Sh}_4(C^2, \hat{G})$ with $\mathrm{Span}(z) \cap \Gamma^{2c} = \varnothing$. Then at least one endpoint of Γ^{2c} is a good z-direction.*

Proof. Let p_0, p_1 be the endpoints of Γ^{2c}. For each $i = 0, 1$, there is a uniquely specified endpoint q_i of $\mathbf{P}$ such that $q_0 \neq q_1$ and such that, for each $i = 0, 1$, $G \setminus (\hat{G} \setminus C^2)$ contains a $(p_i, \mathbf{P})$-path of length at most six which has q_i as an endpoint. For each $i = 0, 1$, let P_i be the unique subpath of C^2 such that $\Gamma^{2c} \subseteq P_i$ and such that $\mathrm{Pin}(z)$ is a terminal subpath of P_i, where p_{1-i} is the other terminal vertex of P_i. Furthermore, let v_0, v_1 be the vertices of $C^2 \cap \mathrm{Span}(z)$ (possibly is a 2-path or a 4-cycle and $v_0 = v_1$), where v_i is the unique non-Γ^{2c}-endpoint of P_i for each $i = 0, 1$. Note that p_0, v_1, v_0, p_1 is the ordering of these vertices on the path $C^2 \setminus \mathring{\Gamma}^{2c}$.

Suppose now that, for each $i = 0, 1$, there is a $(p_i, \{v_0, v_1\})$-path Q_i on the small side of C^2 which has length at most two. Note that each of Q_0, Q_1 is disjoint to $V(C)$. For each $i = 0, 1$, there is a (p_i, q_i)-path on the small side of C^2 which has length at most seven and is disjoint to $\mathbf{P}$ except for its $\mathbf{P}$-endpoint. Note that since $z \notin \mathrm{Sh}_4(C^2, \hat{G})$, there is no chord of C^2 with both endpoints in Γ^{2c} which, in $\hat{G}$, separates z from an element of $\mathcal{C} \setminus \{C\}$. Thus, since $\mathrm{Span}(z)$ contains (v_0, v_1)-path of length at most four, there exists a C-band with endpoints q_0, q_1, where this C-band has length at most $7 + 7 + 2 + 2 + 4$. Since $22 < \frac{N_{\mathrm{mo}}}{4}$, this contradicts 1) of Theorem 2.3.2.

Thus, suppose without loss of generality that there is no $(p_1, \{v_0, v_1\})$-path of length less than three on the small side of C^2. If p_0 also satisfies Condition 2) of Definition 6.2.5, then we are done, so suppose now that there exists a $1 \leq k \leq 3$ and a proper k-chord R_0 of C^2 in $\hat{G}$ such that $\hat{G}^{\mathrm{small}}_{R_0} \cap P_0$ is not a subpath of P_0. Since $\mathrm{Pin}(z)$ is a terminal subpath of P_0 and $z \in D_2(C^2, \hat{G}) \setminus \mathrm{Sh}_4(C^2, \hat{G})$, it follows that v_0 is an endpoint of R and $\hat{G}^{\mathrm{large}} \cap P_0$ is a terminal subpath of P_0 with v_0 as en endpoint.

Now we switch to the other side. We claim that p_1 is a good z-direction. We first check that p_1 satisfies Condition 1) of Definition 6.2.5. Suppose toward a contradiction that there is a $(p_0, \{v_0, v_1\})$-path of length at most two on the small side of C^2. As indicated above, Since $\mathrm{Span}(z)$ contains a (v_0, v_1)-path of length at most four and v_0 is an endpoint of R_0, it follows from Proposition 6.2.4 that one of the following holds.

1) There is a C-band of length at most $2 + 4 + 3 + 7 + 7$*; OR*

2) There is a separating cycle D, where $d(D, C) \leq 1$, $|E(D)| \leq 2 + 4 + 3 + 7 + 7$, and D separates C from an element of $\mathcal{C} \setminus \{C\}$.

In the first case, since $\frac{96}{4} = 24$, this contradicts 1) of Theorem 2.3.2. In this second case, we contradict Corollary 2.1.30, so now we just need to check that p_1 also satisfies Condition 2). Suppose not. Then there is a proper generalized chord R_1 of C^2 in $\hat{G}$ which has length at most three, where $\hat{G}^{\mathrm{small}}_{R_1} \cap P_1$ is not a subpath of P_1. Since $\mathrm{Pin}(z)$ is a terminal subpath of P_1 and $z \in D_2(C^2, \hat{G}) \setminus \mathrm{Sh}_4(C^2, \hat{G})$, it follows that v_1 is an endpoint of R_1 and $\hat{G}^{\mathrm{large}} \cap P_1$ is a terminal subpath of P_1 with v_1 as en endpoint. Thus, one of the following holds:

1. $R_0 \cap R_1 = \varnothing$ and there is a C-band of length at most $3 + 4 + 3 + 7 + 7$*; OR*
2. There is a separating cycle D, where $d(D, C) \leq 1$, $|E(D)| \leq 3 + 4 + 3 + 7 + 7$, and D separates C from an element of $\mathcal{C} \setminus \{C\}$.

In the first case, since $\frac{96}{4} = 24$, this contradicts 1) of Theorem 2.3.2. In this second case, we contradict Corollary 2.1.30. $\square$

We now describe the subpath of C^2 which we delete when we construct a (C, z)-opener for a $z \in D_2(C^2, \hat{G}) \setminus \mathrm{Sh}_4(C^2, \hat{G})$.

Definition 6.2.7. Given a $z \in D_2(C^2, \hat{G}) \setminus \mathrm{Sh}_4(C^2, \hat{G})$ and a subpath P of C^2 with $\Gamma^{2c} \subseteq P$, we say that P is a *z-bend* if $\mathrm{Pin}(z)$ is also a subpath of P and P is specified in the following way.

1) If $\mathrm{Span}(z) \cap C^2 \subseteq \Gamma^{2c}$, then $P := \Gamma^{2c}$.

2) If $\mathrm{Span}(z) \cap \Gamma^{2c} = \varnothing$, then $\mathrm{Span}(z) \cap C^2$ is a terminal subpath of P, and the other endpoint of P is the unique endpoint of Γ^{2c} which does not lie in $\mathrm{Span}(z)$, and furthermore, this endpoint of Γ is a good z-direction.

3) If $\mathrm{Span}(z) \cap C^2 \not\subseteq \Gamma^{2c}$ and $\mathrm{Span}(z) \cap \Gamma^{2cl} \neq \varnothing$, then P is the unique subpath of C^2 which has $\mathrm{Pin}(z)$ as a terminal subpath and whose unique non-$\mathrm{Pin}(z)$-endpoint is the lone endpoint of Γ^{2c} which does not lie in $\mathrm{Pin}(z)$.

In Cases 1) or 3) above, P is uniquely specified, and, in Case 2), there are possibly two z-bends. By Proposition 6.2.6, there is at least one z-bend in Case 2) above, so in any case, for any $z \in D_2(C^2, \hat{G}) \setminus \mathrm{Sh}_4(C^2, \hat{G})$, there exists a z-bend. The purpose of introducing Definition 6.2.7 is that, given a $z \in D_2(C^2, \hat{G}) \setminus \mathrm{Sh}_4(C^2, \hat{G})$, when we construct a (C, z)-opener, the subpath of C^2 which we delete is a z-bend. We now have the following simple result, which takes up the remainder of this section.

Proposition 6.2.8. *Let $z \in D_2(C^2, \hat{G}) \setminus \mathrm{Sh}_4(C^2, \hat{G})$ and let P be a z-bend. Then the following hold.*

A) P is a proper subpath of C^2 and $|E(C^2 \setminus \mathring{P})| \geq 2$; AND

B) For any $v \in D_1(C^2)$ on the small side of C^2, the graph $G[N(v)] \cap P$ is a subpath of P; AND

C) For any integer $1 \leq k \leq 3$, and any k-chord R of C^2 with both endpoints in P, the graph $C^2 \cap \hat{G}_R^{\text{small}}$ has one connected component and the gaph $C^2 \cap \hat{G}_R^{\text{large}}$ has two connected components.

Proof. We break this into three cases.

Case 1: $\mathrm{Span}(z) \subseteq \Gamma^{2c}$

In this case, it follows from Proposition 6.2.2 that A) is satisfied and it follows from Proposition 6.2.3 that C) is satisfied. Suppose that there is a $v \in D_1(C^2)$ on the small side of C^2 which violates condition B). Now, $G[N(v) \cap C^2$ is a subpath of C^2 so the endpoints of Γ^{2c} have a common neighbor on the small side of C^2, and thus G contains a C-band of length at most $7 + 7 + 2$, contradicting 1) of Theorem 2.3.2.

Case 2: $\mathrm{Span}(z) \cap \Gamma^{2c} = \varnothing$

In this case, it follows from Proposition 6.1.3 that Condition C) is satisfied and it follows from Definition 6.2.5 that A) is satisfied. Suppose there is vertex v on the small side a violating condition B). Now, $G[N(v) \cap C^2$ is a subpath of C^2, and thus v is adjacent to each endpoint of P, contradicting the fact that the unique endpoint of Γ^{2c} which is also an endpoint of P is a good z-direction.

Case 3: $\mathrm{Span}(z) \cap \Gamma^{2c} \neq \varnothing$ and $\mathrm{Span}(z) \not\subseteq \Gamma^{2c}$

In this case, $\mathrm{Pin}(z)$ is a subpath of C^2 with one endpoint in $C^2 \setminus \Gamma^{2c}$ and one endpoint in Γ^{2c}. We first check conditions A) and B). For any $v \in D_1(C^2)$ on the small side of G, the graph is a subpath of C^2, so if one of A), B) does not hold, then there is a path on the small side of C^2 which has length at most two and whose endpoints are the endpoints of P. Now, the lone endpoint of $\mathrm{Pin}(z)$ which is not an endpoint of P lies in Γ^2, and since $\mathrm{Span}(z)$ contains a path of length at most four between the endpoints of $\mathrm{Pin}(z)$, it follows that G contains a C-band of length at most $7 + 7 + 2 + 4$, contradicting 1) of Theorem 2.3.2.

Now we check C). Let R be a proper generalized chord of C^2 of length at most three, where R has both endpoints in P. Since $z \notin \mathrm{Sh}_4(C^2, \hat{G})$, it follows that R intersects with $\mathrm{Pin}(z)$ on at most one vertex, and if this vertex exists, then it is the unique Γ^{2c}-endpoint of $\mathrm{Pin}(z)$. Thus, both endpoints of R lie in Γ^{2c}, so it follows from Proposition 6.2.3 that Condition C) is indeed satisfied. $\square$

6.3 Channel Colorings

In this section, we prove an analogue of Theorem 1.7.5 for a subpath of C^2. This requires a slightly different approach because, in the context of Theorem 1.7.5, we are analyzing a subpath of a facial cycle in a planar graph, but the vertices of $C^2 \setminus (H \cup \mathbf{P}^1)$ have some neighbors in $\hat{G} \setminus C^2$ and neighbors in $C^1 \setminus H$.

Definition 6.3.1. Given a subpath Q of C^2, we introduce the following notation.

1) A partial L_φ-coloring σ of $V(Q)$ is called a *channel* of Q if the following hold.

   a) The endpoints of Q lie in $\mathrm{dom}(\sigma)$*; AND*

   b) For any $1 \leq k \leq 2$ and any 2-chord R of C^2 in $\hat{G}$, if the endpoints of R lie in $V(Q)$, then $V(\hat{G}^{\text{small}}_R) \setminus V(R)$ is $(L, \varphi \cup \sigma)$-inert in G*; AND*

   c) Every vertex of Ω^1 has an $L_{\varphi \cup \sigma}$-list of size at least three; *AND*

   d) Every vertex of $D_1(C^2, \hat{G}) \setminus \mathrm{Sh}_2(Q, C^2, \hat{G})$ has an $L_{\varphi \cup \sigma}$-list of size at least three*; AND*

   e) For every $x \in V(C^2 \setminus Q)$, if $v \notin V(\Gamma^2)$, then $|L_{\varphi \cup \sigma}(x)| \geq 3$, and, if $x \in V(\Gamma^2)$, then $|L_{\varphi \cup \sigma}(x)| \geq |L_\varphi(x)| - 2$.

2) For any vertex $v \in V(\Omega^1)$, subset $S \subseteq L_\varphi(v)$, and channel σ of Q, we say that σ is (v, S)*-avoiding* if $S \subseteq L_{\varphi \cup \sigma}(v)$.

Our main result for this section is the following.

Theorem 6.3.2. *Let $z \in D_2(C^2, \hat{G})$ and let P_z be a z-bend of C^2. For any subpath Q of P_z, there is a channel of Q.*

We break the proof of Theorem 6.3.2 into several lemmas. We first introduce the following definitions.

Definition 6.3.3. Let A be a proper generalized chord of C^2 in $\hat{G}$ with $1 \leq |E(A)| \leq 2$ and let D be the cycle $(\hat{G}^{\text{small}}_A \cap C^2) + A$.

1) We say that A is an *atom* if D is an induced subgraph of G and $\hat{G}^{\text{small}}_A \cap C^2$ is a path of length at least two

2) We say that an atom is *irreducible* if, for any 2-chord A' of D in $\hat{G}^{\text{small}}_A$ with both endpoints in C^2, letting w be the midpoint of A', the graph $G[N(w) \cap V(C^2)]$ is a subpath of Q.

Anaalogous to the above, we have the following.

Definition 6.3.4. Let Q be a supath of C^2 with $|E(Q)| \geq 1$ and let q, q' be the endpoints of Q.

1) A vertex $v \in V(C^2 \setminus Q)$ is called a *Q-prism* if $\hat{G}$ contains both of vq, vq' as chords of C^2*; AND*

2) We say that Q is a *rainbow* if $|E(Q)| \geq 2$ and there is a Q-prism v such that $Q + qvq'$ is an induced cycle*; AND*

3) We say that Q is an *irreducible rainbow* if Q is a rainbow and, for any $w \in V(\hat{G} \setminus C^2)$ with a neighbor in Q, the graph $G[N(w) \cap V(C^2)]$ is a subpath of Q.

Our first lemma in the proof of Theorem 6.3.2 is the following somewhat technical result.

Lemma 6.3.5. *Let Q be a subpath of C^2 satisfying one of the following two conditions.*

1) *Q is an irreducible rainbow; OR*
2) *There exists an irreducible atom A such that $Q = C^2 \cap \hat{G}_A^{\text{small}}$.*

Let q, q' be the endpoints of Q and suppose that $V(Q \setminus \{q\}) \subseteq V(C^2 \setminus \Gamma^2)$. Let v be the unique vertex of Ω^1 which is adjacent to both endpoints of the terminal edge of Q with q as an endpoint. Let $c \in L_\varphi(q)$ and let $\{d_0, d_1\}$ be a set of two colors of $L_\varphi(q')$ (possibly $c \in \{d_0, d_1\}$). Let $S \subseteq L_\varphi(v) \setminus \{c\}$ with $|S| = 3$. Then, for some $i \in \{0, 1\}$, there is a (v, S)-avoiding channel σ of Q such that σ uses c, d_i on the respective vertices q, q'.

Proof. We prove that this holds if there exists an irreducible atom A such that $Q = C^2 \cap \hat{G}_A^{\text{small}}$. An identical argument works for the case where Q is an irreducible rainbow.

Firstly, by 5) of Proposition 6.1.2, every vertex of $C^2 \setminus \mathring{\Gamma}^2$ has a neighbor in Ω^1, so the endpoints of the terminal edge of Q containing q do indeed have a unique common nieghbor in Ω^1. Given a partial L_φ-coloring σ of $V(Q)$, if we want to check that σ is a channel of Q, then we just ned to check conditions 1a)-c) of Definition 6.3.1, i.e, since $Q = C^2 \cap \hat{G}_A^{\text{small}}$, it follows that σ automatically satisfies 1d-e).

Suppose toward a contradiction that there is no (v, S)-avoiding channel of Q which uses c on q and uses one of d_0, d_1 on q'. Let $K := \hat{G}_A^{\text{small}}$ and let D be the cycle $A + (C^2 \cap K)$. Note that D is a cyclic facial subgraph of K. By definition of an atom, D is an induced subgraph of G and P has length at least two. Thus, it follows from our triangulation conditions that $V(K \setminus D) \neq \varnothing$. If P has length precisely two, then D is a separating cycle of length at most four, contradicting the fact that $\mathcal{T}$ is a tessellation. Thus, P has length at least three. Furthermore, since $V(Q \setminus \{q\}) \subseteq V(C^2 \setminus \Gamma^2)$, every vertex of $Q \setminus \{q\}$ has an L_φ-list of size at least five and every neighbor of Q in Ω^1, except possibly v, has an L_φ-list of size at least five. Let $Q := q_1 \cdots q_r$, where $q_1 = q$ and $q_r = q'$. We now let $P \subseteq Q$ be the path $Q \cap G[N(v) \cap V(C^2)]$. Note that P is a terminal subpath of Q and $q_1 q_2 \in E(P)$. Let $P := q_1 \cdots q_\ell$. Since G is short-separation-free and $V(K \setminus D) \neq \varnothing$, we have $\ell < r$, or else G contains a cycle of length at most four which separates $K \setminus D$ from all the elements of $\mathcal{C} \setminus \{C\}$. Since v is adjacent to q_1, q_2, we have $2 \leq \ell < r$.

Claim 6.3.6. *There is an L_φ-coloring σ of P such that the following hold.*

1) *$\sigma(q_1) = c$ and $S \subseteq L_{\varphi \cup \sigma}(v)$; AND*
2) *For each $z \in V(\Omega^1) \cup V(K \setminus D)$, $|L_{\varphi \cup \sigma}(z)| \geq 3$.*

Proof: Since P is an induced path in G and each vertex of $q_2 \cdots q_\ell$ has an L_φ-list of size at least five, there is an L_φ-coloring σ of $V(P)$ such that $\sigma(q_1) = c$ and no vertex of P is colored with a color of S. Since every vertex of P is adjacent to v and G is short-separation-free, there is no other vertex of G with more than two neighbors on P, so, for each $z \in V(\Omega^1) \cup V(K \setminus D)$, we have $|L_{\varphi \cup \sigma}(z)| \geq 3$. ■

We now fix an L_φ-coloring σ of $V(P)$ satisfying Claim 6.3.6.

Claim 6.3.7. $q_1 q_r \notin E(G)$.

Proof: Suppose toward a contradiction that $q_1 q_r \in E(G)$. By definition of an atom, we then have $A = q_1 q_r$. At least one of d_0, d_1 is distinct from c so suppose without loss of generality that $d_0 \neq c$. Since D is an induced subgraph of G and each vertex of $q_2 \cdots q_{r-1}$ has an L_φ-list of size at least five, it follows from Proposition 1.2.3 that σ extends to an L_φ-coloring σ' of $q_1 \cdots q_{r-2}$ such that every vertex of $V(\Omega^1) \cup V(K \setminus D)$ has an $L_{\varphi \cup \sigma'}$-list of size at least three. Note that $v \notin N(q_{r-1})$ or else $q_1 v q_{r-1} q_r$ is a 4-cycle which separates each vertex of $K \setminus D$ from each element of $\mathcal{C} \setminus \{C\}$, contradicting the fact that $\mathcal{T}$ is a tessellation.

Subclaim 6.3.8. *There is a vertex z of $N(q_{r-1}) \cap V(\Omega^1)$ such that $|N(z) \cap V(Q)| > 2$.*

Proof: Since D is an induced cycle in G, σ' extends to an L_φ-coloring τ of $V(D)$ such that $\tau(q_r) = d_0$. Let $B_0 := \{q_{r-1}, q_r\}$. By our choice of σ', every vertex of $K \setminus D$ has an $L^{B_0}_{\varphi \cup \tau}$-list of size at least three, so, retaining the precolored edge $q_{r-1}q_r$, it follows from Theorem 0.2.3 that $\varphi \cup \tau$ extends to L-color K. Since $q_{r-1}, q_r \notin N(v)$ we have $S \subseteq L_{\varphi \cup \tau}(v)$. Thus, by assumption, τ is not a channel of Q, so there is a vertex of Ω^1 with an $|L_{\varphi \cup \tau}(z)| < 3$, and we have $|N(z) \cap V(Q)| > 2$. By our choice of σ', this vertex z has a neighbor in B. Since $G[N(z) \cap V(Q)]$ is a subpath of Q of length at least two, it follows that, if $q_r \in N(z)$, then $q_{r-1} \in N(z)$ as well, so $q_{r-1} \in N(z)$ in any case. ■

Appyling Subclaim 6.3.8, let z be a vertex of $N(q_{r-1}) \cap V(\Omega^1)$ such that $G[N(z) \cap V(Q)]$ is a subpath of Q of length at least two. Since $q_{r-1}q_r$ is a terminal edge of Q, z is the only vertex of $V(\Omega^1) \cap N(q_{r-1})$ which is adjacent to a subpath of Q of length at least two. Since $q_{r-1} \notin N(v)$, we have $z \neq v$. Now consider the following cases:

Case 1: q_{r-1} is an internal vertex of $G[N(z) \cap V(Q)]$

In this case, $G[N(q) \cap V(Q)]$ contains $q_{r-2}q_{r-1}q_r$ as a subpath. Let $m \in \{1, 2, \cdots, r-2\}$ be the unique index such that $G[N(z) \cap V(Q)] = q_m \cdots q_r$. Let τ be an L_φ-coloring of $V(q_1 Q q_m) \cup \{q_r\}$ obtained from σ' by first restricting σ' to $\{q_1, \cdots, q_m\}$ and then coloring q_r with d_0. Now, we have $|L_{\varphi \cup \tau}(z)| \geq 3$, since $N(z) \cap \text{dom}(\tau) = \{q_m, q_r\}$. Furthermore, since $z \neq v$, we have $P \subseteq q_1 \cdots q_m$, so $S \subseteq L_{\varphi \cup \tau}(v)$. By assumption, τ is not a (v, S)-avoiding channel of Q, so the inertness condition is violated. Thus, there is an extension of $\varphi \cup \tau$ to an L-coloring ζ of $\text{dom}(\varphi \cup \tau) \cup \{z\}$ such that ζ does not extend to L-color K. Now we simply leave the edge $q_r z$ precolored. Let $B_1 := \{q_r, z\}$. By our choice of σ', each vertex of $K \setminus D$ has an $L^{B_1}_\zeta$-list of size at least three, and since $q_m \cdots q_{r-1}$ is an induced subgraph of G, each vertex of $q_{m+1} \cdots q_{r-1}$ has an $L^{B_1}_\zeta$-list of size at least three. Thus, by Theorem 0.2.3, ζ extends to L-color K, contradicting our assumption.

Case 2: q_{r-1} is not internal vertex of $G[N(z) \cap V(Q)]$

In this case, $G[N(z) \cap V(Q)]$ is a path containing $q_{r-3}q_{r-2}q_{r-1}$ as a terminal subpath. As above, let $m \in \{1, 2, \cdots, r-3\}$ be the unique index such that $G[N(z) \cap V(Q)] = q_m \cdots q_{r-1}$. Let τ be an L_φ-coloring of $V(q_1 Q q_m) \cup \{q_{r-1}, q_r\}$ obtained from σ' by first restricting σ' to $\{q_1, \cdots, q_m\}$ and then coloring the edge $q_{r-1}q_r$, where q_r is colored with d_0. As above, we have $|L_{\varphi \cup \tau}(z)| \geq 3$, since $N(z) \cap \text{dom}(\tau) = \{q_m, q_{r-1}\}$. Furthermore, since $z \neq v$, we have $P \subseteq q_1 \cdots q_m$, so $S \subseteq L_{\varphi \cup \tau}(v)$. assumption, τ is not a (v, S)-avoiding channel of Q, so the inertness condition is violated. Thus, there is an extension of $phi \cup \psi \cup \tau$ to an L-coloring ζ of $\text{dom}(\varphi \cup \tau) \cup \{z\}$ such that ζ does not extend to L-color K. This time, we retain the edge $q_{r-1}q_r$. Let $B_2 := \{q_{r-1}, q_r\}$. By our choice of σ', each vertex of $K \setminus D$ has an $L^{B_2}_\zeta$-list of size at least three, and since $q_m \cdots q_{r-1}$ is an induced subgraph of G, each vertex of $q_{m+1} \cdots q_{r-2}$ has an $L^{B_2}_\zeta$-list of size at least three. By Theorem 0.2.3, ζ extends to L-color K, contradicting our assumption. ■

Since $q_1 q_r \notin E(G)$, A is a 2-chord of C^2, so let w be the midpoint of A.

Claim 6.3.9. *There is an L_φ-coloring τ of $V(Q)$ such that the following hold.*

1) $\tau(q_1) = c$ *and* $\tau(q_r) \in \{d_0, d_1\}$*; AND*
2) $S \subseteq L_{\varphi \cup \tau}(v)$*; AND*

3) For any $z \in V(K \setminus D) \cup V(\Omega^1)$, if $|L_{\varphi\cup\tau}(z)| < 3$, then $z \in N(q_r)$ and $|L_{\varphi\cup\tau}(z) \cup \{\tau(q_r)\}| = 3$.

4) $V(K \setminus D)$ is $(L, \varphi \cup \tau)$-inert in G.

Proof: Since each vertex of $q_2 \cdots q_{r-1}$ has an L_φ-list of size at least five, it now follows from Proposition 1.2.3 that σ extends to an L_φ-coloring σ^* of $q_1 \cdots q_{r-1}$ such that every vertex of $V(\Omega^1) \cup V(K \setminus D)$ has an $L_{\varphi\cup\sigma^*}$-list of size at least three. By Claim 6.3.7, $q_1q_r \notin E(G)$, so $L_{\varphi\cup\sigma^*}(q_r) \cap \{d_0, d_1\} \neq \varnothing$. Thus, σ^* extends to an L_φ-coloring τ of $V(Q)$. Since $q_r \notin V(P)$, we have $S \subseteq L_{\varphi\cup\tau}(v)$, and, by our choice of σ^*, it follows that, for any $z \in V(K \setminus D) \cup V(\Omega^1)$, if $|L_{\varphi\cup\tau}(z)| < 3$, then $z \in N(q_r)$ and $|L_{\varphi\cup\tau}(z) \cup \{\tau(q_r)\}| = 3$.

To finish, we just need to check that $V(K \setminus D)$ is $(L, \varphi \cup \tau)$-inert in G. Let τ^* be an extension of τ to an L-coloring of dom$(\tau) \cup \{w\}$. Let $B_3 := \{w, q_r\}$. It follows from 3) that every vertex of $K \setminus D$ has an $L^{B_3}_{\tau^*}$-list of size at least three, so, retaining the precolored edge wq_r, it follows from Theorem 0.2.3 that τ^* extends to L-color K. Thus, $V(K \setminus D)$ is indeed $(L, \varphi \cup \tau)$-inert in G. ■

Let τ be as in Claim 6.3.9. By assumption, τ is not a (v, S)-avoiding channel of Q, so there is is a $z \in V(\Omega^1) \setminus \{v\}$ with $|L_{\varphi\cup\tau}(z)| < 3$. By 3) of Claim 6.3.9, we have $z \in N(q_r)$ and $G[N(z) \cap V(Q)]$ is a terminal subpath of Q of length at least two, and z ius unique. Let $m \in \{1, \cdots, r-2\}$ be the unique index such that $G[N(z) \cap V(Q)] = q_m \cdots q_r$. Let τ^* be the restriction of τ to dom$(\tau) \setminus \{q_{m+1}, \cdots, q_{r-1}\}$. Now, τ^* is also not a (v, S)-avoiding Q-channel, and since $N(z) \cap \text{dom}(\tau^*) = \{q_m, q_r\}$, the inertness condition is violated. Thus, there is an extension of $\varphi \cup \tau^*$ to an L-coloring ζ of dom$(\varphi \cup \tau^*) \cup \{z\}$ such that ζ does not extend to L-color K. As above, we retain the edge wq_r. Let $B := \{w, q_r\}$. It follows from 3) of Claim 6.3.9 that each vertex of $K \setminus D$ has an L^B_ζ-list of size at least three. Since Q is an induced path in G, each vertex of $\{q_{m+1}, \cdots, q_{r-1}$ has an L^B_ζ-list of size at least three. Thus, by Theorem 0.2.3, K is L^B_ζ-colorable, so ζ extends to L-color K, which is false. This completes the proof of Lemma 6.3.5. □

We now have the following by a straightforward induction argument. The lemma below, in combination with the work of Section 1.7, is sufficient to prove Theorem 6.3.2.

Lemma 6.3.10. *Let Q be a subpath of C^2 of length at least one, let q, q' be the endpoints of Q and suppose that $V(Q \setminus \{q\}) \subseteq V(C^2 \setminus \Gamma^2)$. Let e be the unique terminal edge of Q containing q, and let v be the unique vertex of Ω^1 adjacent to both endpoints of e. Let $c \in L_\varphi(q)$ and let let $S \subseteq L_\varphi(v) \setminus \{c\}$ with $|S| = 3$. Then the following hold.*

1) There is a (v, S)-avoiding channel of Q which uses c on q; AND

2) If there is an atom whose endpoints are q, q' then, for any $T \subseteq L_\varphi(q')$ of size two, there is a (v, S)-avoiding channel of Q which uses c on q and uses a color of T on q'.

a) There is an atom whose endpoints are q, q'; OR

b) Q is a rainbow.

We briefly describe how to apply the result of Theorem 6.3.2 to prove Theorem 6.0.9. Given a $z \in D_2(C^2)$, it follows from the work of Section 6.2 that a z-bend satisfies the distance conditions specified in Definition 6.0.8, and given a channel coloring σ of a z-bend P_z, we extend σ to L-coloring dom$(\sigma) \cup V(\tilde{G}^{\text{small}}_{\text{Span}(z)})$ and combine this with the work of Section 1.6 to produce a (C, z)-opener.

Chapter 7

An Internal 2-List Lemma

In this short chapter, we prove a general result which strengthens Theorem 1.7.5 before returning to the context of critical mosaics in Chapter 8. The idea is that, given a short-separation-free planar graph G and a cyclic facial subgraph C of G with a list-assignment L, we can obtain an analogue to Theorem 1.7.5 for a subpath P of C which has has an internal vertex with a list of size two, as long as some additional properties are satisfied by any 2-chord of C which separates this lone 2-list from the "large" side of the graph, where the meaning of large is made precise below. Our main result for this chapter is the following.

Theorem 7.0.1. *Let G be a short-separation-free graph and let C be an induced cyclic facial subgraph of G, Let P be a subpath of C of length at least two, let $u_\star \in V(P)$, and let $P_\star$ be a subpath of P. Let p, p' be the endpoints of P and let q, q' be the endpoints of $P_\star$, where the (not necessarily distinct) vertices of $\{p, p', q, q'\}$ have the order p', q', q, p on the path P. Suppose that that following conditions hold.*

1) *$L(u_\star)| \geq 2$ and $u_\star \in V(P_\star)$; AND*
2) *P is $(2, L)$-short and every vertex of $P - u_\star$ has an L-list of size at least three; AND*
3) *if $|V(P_\star)| > 1$, then $u_\star \notin \{q, q'\}$ and there is a vertex $w \in D_1(C)$ such that $G[N(w) \cap V(P)] = P_\star$ and such that any 2-chord of C with both endpoints in P which separates $u^\star$ from an edge of $E(C) \setminus E(P)$ has midpoint w and endpoints in $P_\star$.*

Then both of the following hold.

A) *$\mathrm{Link}_L(P) \neq \varnothing$; AND*
B) *If there is a $v \in V(pPq) \setminus \{u_\star\}$ such that $|L(v)| \geq 4$ and v is a P-hinge of C, then there exist two elements ψ_1, ψ_2 of $\mathrm{Link}_L(P)$ which use different colors on p and both restrict to the same partial L-coloring of $q'Pp'$.*

The reason we need this result is that, when we delete vertices on the 1-necklace of a closd ring in a critical mosaic, we use the results of Sections 1.4, 1.6, and 1.7, but we have the added complication that, after we delete the vertices of a closed ring C in a critical mosaic, there is possibly a lone 2-list left in the 1-necklace of C. This is due to Definition 2.1.3.

7.1 Broken Wheels with 2-Lists

This section consists of the following intermediate result, which we need in order to prove Theorem 7.0.1.

Theorem 7.1.1. *Let H be a broken wheel with principal path $P := p_1p_2p_3$, and let $u_\star \in V(H \setminus P)$. Let L be a list-assignment for H such that $|L(u_\star)| \geq 2$, and, for each $v \in V(H) \setminus \{u_\star, p_2\}$, $|L(v)| \geq 3$. Then the following hold.*

1) *There exists a pair of colors $(c, d) \in L(p_1) \times L(p_3)$ such that, for any L-coloring ϕ of $V(P)$ using c, d on p_1, p_3 respectively, ϕ extends to an L-coloring of H; AND*

2) *If $|L(p_3)| \geq 4$, then there exists a a color $c \in L(p_1)$ and two distinct colors $d_0, d_1 \in L(p_3)$ such that, for each $i = 0, 1$ and any L-coloring ϕ of $V(P)$ using c, d_i on p_1, p_3 respectively, ϕ extends to an L-coloring of H.*

This is a variant of Theorem 1.5.5 in the case where one of the vertices of the outer face not lying in the specified 2-path has a 2-list, but the vertices of the specified 2-path are not precolored. Unlike Theorem 1.5.5, we need to restrict ourselves to broken wheels in this case. The counterexample in Figure 7.1.1 illustrates why we restrict the structure of the graph in this way, as the analogue to Theorem 1.5.5 in the general case is false. In the graph in Figure 7.1.1, it is not possible to color only the endpoints of $p_1p_2p_3$ in such a way as to prevent the existence of a proper coloring of $p_1p_2p_3$ which uses the color a on p_2.

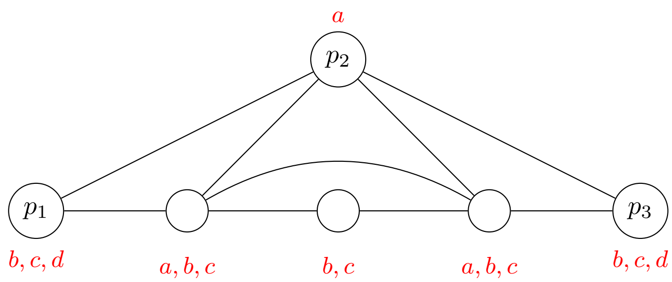

Figure 7.1.1: Theorem 1.5.5 does not hold if an internal 2-list is permitted

We now prove 1) of Theorem 7.1.1.

Proof. Let H be a vertex-minimal counterexample to 1) of Theorem 7.1.1, and let P, u, L be as in the statement of the theorem, where $P := p_1p_2p_3$. Let $H \setminus \{p_2\} = p_1u_1 \cdots u_tp_3$ for some $t \geq 1$. Let $n \in \{1, \cdots, t\}$, where $u = u_n$. Note that $|L(u_n)| = 2$, or else we contradict Theorem 1.5.5. Since H is a counterexample and $p_1p_3 \notin E(H)$, it follows that, for each pair $(c, d) \in L(p_1) \times L(p_3)$, there is an L-coloring σ^{cd} of P which uses c, d on p_1, p_3 respectively and which does not extend to an L-coloring of H. For each $q \in L(p_2)$, let $S_q := \{(c, d) \in L(p_1) \times L(p_3) : \sigma^{cd}(p_2) = q\}$.

For each $i = 1, \cdots, t$, let H_i^{left} be the subgraph of H induced by $\{p_1, p_2\} \cup \{u_1, \cdots, u_i\}$ and let $P_i^{\text{left}} := p_1p_2u_i$ be the principal path of H_i^{left}. Likewise, let H_i^{right} be the subgraph of H induced by $\{p_2, p_3\} \cup \{u_i, \cdots, u_t\}$ and let $P_i^{\text{right}} := u_ip_2p_3$ be the principal path of H_i^{left}.

Claim 7.1.2. *For each color $r \in L(u_n)$, $r \in L(p_2)$ and $S_r \neq \varnothing$.*

<u>Proof:</u> Let $r \in L(u_n)$ and suppose that this does not hold. Let $P' := p_1p_2u_n$ and $P'' := u_np_2p_3$. Let H' be the subgraph of H induced by $V(P') \cup \{u_2, \cdots, u_{n-1}\}$, and let H'' be the subgraph of H induced by $V(P'') \cup \{u_{n+1}, \cdots, u_t\}$. By Theorem 1.5.5, there exists a $c \in L(p_1)$ such that any L-coloring of P_n^{left} using c, r on p_1, u_n respectively extends to an L-coloring of H_n^{left}, and $c \neq r$ if $u_1 = u_n$. Likewise, there exists a $d \in L(p_3)$ such that any L-coloring of P_n^{right} using r, d on u_np_2 respectively extends to an L-coloring of H_n^{right} and $d \neq r$ if $u_n = u_t$. Since $\sigma^{cd}(p_2) \neq r$, it follows that $(c, \sigma^{cd}(p_2), r)$ is a proper L-coloring of $V(P_n^{\text{left}})$ and $(r, \sigma^{cd}(p_2), d)$ is a proper L-coloring of $V(P_n^{\text{right}})$, so σ^{cd} extends to an L-coloring of H, which is false. ■

We now fix two colors r, s such that $L(u_n) = \{r, s\}$. We now have the following:

Claim 7.1.3. $n \notin \{1, t\}$.

Proof: Suppose toward a contradiction that u_n is an endpoint of $H \setminus P$ and, without loss of generality, let $n = 1$. Since $|L(p_1)| \geq 3$ and $|L(u_1)| = 2$, there is a color $c \in L(p_1)$ such that $c \notin L(u_1)$. If $u_1 = u_n = u_t$, then, since $L(p_3)| \geq 3$, we choose a color $d \in L(p_3)$ such that $d \notin L(u_1)$. Then any L-coloring of P using c, d on p_1, p_3 respectively extends to H, since there is a color left over for u_1, contradicting the fact that H is a minimal counterexample. Thus, we have $t > 1$. For each $d \in L(p_3)$, the path $u_1 \cdots u_t$ is not $L_{\sigma_{cd}}$-colorable, and each internal vertex of $u_1 \cdots u_t$ has an $L_{\sigma_{cd}}$-list of size at least two. Since $t > 1$, it follows from our choice of c that, for each $d \in L(p_3)$, we have $|L_{\sigma^{cd}}(u_1)| \geq 1$ and $|L_{\sigma^{cd}}(u_t)| \geq 1$.

Thus, for each $d \in L(p_3)$, we have $|L_{\sigma^{cd}}(u_1)| = |L_{\sigma^{cd}}(u_t)| = 1$, and each internal vertex of $u_1 \cdots u_t$ has an $L_{\sigma^{cd}}$-list of size precisely two. Since $|L(u_t)| = 3$, we conclude that $L(p_3) = L(u_t)$, and, for each $d \in L(p_3)$ and $j = 1, \cdots, t$, $\sigma^{cd}(p_2)$ lies in $L(u_j)$. Applying Claim 7.1.2, we have $L(u_1) = \{\sigma^{cd}(p_2) : d \in L(p_3)\}$. In particular, we have $\sigma^{cr}(p_2) = s$ and $\sigma^{cs}(p_2) = r$. Since $c \notin L(u_1)$, we have $c \notin \{r, s\}$. Let $q \in L(p_3) \setminus \{r, s\}$. Since $\sigma^{cq}(p_2) \in \{r, s\}$, suppose without loss of generality that $\sigma^{cq}(p_2) = r$.

Let L' be a list-assignment for $u_1 \cdots u_t p_3$ obtained by deleting r from the L-list of each vertex in $u_1 \cdots u_t p_3$. Then each vertex of $u_2 \cdots u_t p_3$ has an L'-list of size at least two, and $L'(p_3) = \{s, q\}$. Thus, there is an L'-coloring of $u_1 \cdots u_t p_3$ which uses s on u_1, and thus one of σ^{cs}, σ^{cq} extends to an L-coloring of H, which is false. ■

Now we have the following:

Claim 7.1.4. *For any $(c, d) \in L(p_1) \times L(p_3)$, if $\sigma^{cd}(p_2) \notin \{r, s\}$, then $\sigma^{cd}(p_2) \in L(u_i)$ for each $i = 1, \cdots, n-1, n+1, \cdots, t$, and furthermore, either $L(u_{n-1}) = \{r, s, \sigma^{cd}(p_2)\}$, or $L(u_{n+1}) = \{r, s, \sigma^{cd}(p_2)\}$.*

Proof: Let $(c, d) \in L(p_1) \times L(p_3)$ and let $\sigma^{cd}(p_2) := q$ for some $q \notin \{r, s\}$. By Observation 1.4.2, since $q \notin L(u_n)$, we have $\mathcal{Z}_{H_n^{\text{left}}}(c, q, \bullet) \neq \varnothing$ and $\mathcal{Z}_{H_n^{\text{right}}}(\bullet, q, d) \neq \varnothing$. Note that $\mathcal{Z}_{H_n^{\text{left}}}(c, q, \bullet) \cap \mathcal{Z}_{H_n^{\text{right}}}(\bullet, q, d) = \varnothing$, or else σ^{cd} extends to an L-coloring of H, which is false. Thus, we have $|\mathcal{Z}_{H_n^{\text{left}}}(c, q, \bullet)| = |\mathcal{Z}_{H_n^{\text{right}}}(\bullet, q, d)| = 1$, so suppose without loss of generality that $\mathcal{Z}_{H_n^{\text{left}}}(c, q, \bullet) = \{r\}$ and $\mathcal{Z}_{H_n^{\text{right}}}(\bullet, q, d) = \{s\}$. Thus, by 2) of Proposition 1.4.4, we have $q \in L(u_i)$ for each $i = 1, \cdots, n-1$ and each $i = n+1, \cdots, t$, and since $s \notin \mathcal{Z}_{H_n^{\text{left}}}(c, q, \bullet)$, we have $s \in L(u_{n-1})$, and likewise, since $r \notin \mathcal{Z}_{H_n^{\text{right}}}(\bullet, q, d)$, we have $r \in L(u_{n+1})$.

To finish, we need to show that either $r \in L(u_{n-1})$ or $s \in L(u_{n+1})$. Suppose neither of these hold. Applying Claim 7.1.2, there exists a pair $(c_1, d_1) \in S_r$. Since $\sigma^{c_1 d_1}$ does not extend to an L-coloring of H, we have $\mathcal{Z}_{H_n^{\text{left}}}(c_1, r, \bullet) \cap \mathcal{Z}_{H_n^{\text{right}}}(\bullet, r, d_1) = \varnothing$. As $r \notin L(u_{n-1})$, it follows from 1) of Proposition 1.4.4 that $s \in \mathcal{Z}_{H_n^{\text{left}}}(c_1, r, \bullet)$. Since $s \notin L(u_{n+1})$, we have $s \in \mathcal{Z}_{H_n^{\text{right}}}(\bullet, r, d_1)$, contradicting the fact that $\mathcal{Z}_{H_n^{\text{left}}}(c_1, r, \bullet) \cap \mathcal{Z}_{H_n^{\text{right}}}(\bullet, r, d_1) = \varnothing$. ■

We can show that there exists such a pair of colors:

Claim 7.1.5. *There exist $c_* \in L(p_1)$ and $d_* \in L(p_3)$ such that $\sigma^{c_* d_*}(p_2) \notin \{r, s\}$.*

Proof: By Claim 7.1.2, we have $n \neq 1, t$. Thus, let $q \in L(u_{n-1})$ and $q' \in L(u_{n+1})$ with $q, q' \notin \{r, s\}$. By Theorem 1.5.5, there exists a $c \in L(p_1)$ such that any L-coloring of P_{n-1}^{left} using c, q on p_1, u_{n-1} respectively extends to an L-coloring of H_{n-1}^{left}, and $c \neq q$ if $n = 2$. Likewise, there exists a $d \in L(p_3)$ such that any L-coloring of P_{n+1}^{right} using q', d on u_{n+1}, p_3 respectively extends to an L-coloring of H_{n+1}^{right}, and $q' \neq d$ if $n + 1 = t$. Suppose that $\sigma^{cd}(p_2) \notin \{q, q'\}$. Then (c, σ^{cd}, q) is a proper L-coloring of $p_1 p_2 u_{n-1}$, and $(q', \sigma^{cd}(p_2), d)$ is a proper L-coloring of $u_{n+1} p_2 p_1$. By our choice of c, d, the coloring σ_{cd} extends to an L-coloring of $H - u_n$ using q, q' on u_{n-1}, u_{n+1} respectively. Since one of r, s is left over for u_n, σ^{cd} extend to an L-coloring of H, which is false. Thus, we have $\sigma^{cd}(p_2) \in \{q, q'\}$. ■

Applying Claim 7.1.5, we fix a pair $(c_*, d_*) \in L(p_1) \times L(p_3)$ such that $\sigma^{c_* d_*}(p_2) \notin \{r, s\}$. Let $q := \sigma^{c_* d_*}(p_2)$. By Claim 7.1.4, we have $q \in L(u_i)$ for each $i = 1, \cdots, n-1, n+1, \cdots, t$.

Claim 7.1.6. *For any pair of colors $(c, d) \in L(p_1) \times L(p_3)$, we have $\sigma^{cd}(p_2) \in \{q, r, s\}$.*

Proof: Suppose there is a $(c, d) \in L(p_1) \times L(p_3)$ with $\sigma^{cd}(p_2) \notin \{r, s, q\}$. By Claim 7.1.4, we have $\sigma^{cd}(p_2) \in L(u_{n-1}) \cap L(u_{n+1})$ and we have either $L(u_{n-1}) = \{r, s, q\}$ or $L(u_{n+1}) = \{r, s, q\}$, a contradiction. ■'

Now we have the following:

Claim 7.1.7. *For each $c \in L(p_1)$, there is at most one $d \in L(p_3)$ such that $(c, d) \in S_q$. Likewise, for each $d \in L(p_3)$, there is at most one $c \in L(p_1)$ such that $(c, d) \in S_q$.*

Proof: Suppose that this does not hold, and, without loss of generality, suppose that there exists a $c \in L(p_1)$ such that, for some distinct $d, d' \in L(p_3)$, we have $d, d' \in S_q$. Since $q \notin L(u_n)$ and $|L(u_n)| = 2$, it follows from Observation 1.4.2 that there is at $b \in \mathcal{Z}_{H_n^{\text{left}}}(c, q, \bullet)$. By Claim 7.1.3, $n \neq t$, and thus each of (b, q, d), (b, q, d') is a proper L-coloring of $u_n p_2 p_3$. Applying Observation 1.4.2 again, one of these two L-colorings extends to an L-coloring of H_n^{right}, so one of $\sigma^{cd}, \sigma^{cd'}$ extends to an L-coloring of H, contradicting our assumption. ■

Now we have the following:

Claim 7.1.8. *$L(p_1) = L(u_1)$ and $L(p_3) = L(u_t)$.*

Proof: Suppose that this does not hold, and, without loss of generality, suppose that $L(p_1) \neq L(u_1)$. Thus, there exists a $q' \in L(p_1)$ with $q' \notin L(u_1)$.

> **Subclaim 7.1.9.** *For each $d \in L(p_3)$, we have $\sigma^{q'd}(p_2) \in \{r, s\}$.*
>
> Proof: Suppose this does not hold. Then, by Claim 7.1.6, there exists a $d \in L(p_3)$ such that $(q', d) \in S_q$. Since $q' \notin L(u_1)$, and $u_1 \neq u_n$, we have $\mathcal{Z}_{H_n^{\text{left}}}(q', q, \bullet) = L(u_n) = \{r, s\}$ by Proposition 1.4.4. Again by Observation 1.4.2, we have $\mathcal{Z}_{H_n^{\text{right}}}(\bullet, q, d) \neq \varnothing$, since $q \notin \{r, s\}$. But then $\mathcal{Z}_{H_n^{\text{left}}}(q', q, \bullet) \cap \mathcal{Z}_{H_n^{\text{right}}}(\bullet, q, d) \neq \varnothing$, so $\sigma^{q'd}$ extends to an L-coloring of H, which is false. ■

Since $|L(p_3)| \geq 3$, it follows from Subclaim 7.1.9 that there exist $d, d' \in L(p_3)$ such that $\sigma^{q'd}(p_2) = \sigma^{q'd'}(p_2)$, say without loss of generality that $\sigma^{q'd}(p_2) = \sigma^{q'd'}(p_2) = r$. By Observation 1.4.2, the L-coloring of the edge $u_n p_2$ with (s, r) extends to an L-coloring ψ of H_n^{right} using one of d, d' on p_3. Suppose without loss of generality that $\psi(p_3) = d$. Since $q' \notin L(u_1)$, it follows from Proposition 1.4.4 that the coloring (q', r, s) of $p_1 p_2 u_n$ extends to an L-coloring ϕ of H_n^{left}. But then $\phi \cup \psi$ is an extension of $\sigma^{q'd}$ to an L-coloring of H, which is false. ■

By Claim 7.1.3, $n \notin \{1, t\}$. Since $q \in L(u_i)$ for each $i \in \{1, \cdots, n-1, n+1, \cdots, t\}$, it follows from Claim 7.1.8 that $q \in L(p_1) \cap L(p_3)$.

Claim 7.1.10. *$L(p_1) = L(p_3) = \{q, r, s\}$.*

Proof: Suppose not. Then, without loss of generality, suppose that $L(p_1) \neq \{q, r, s\}$. By Claim 7.1.8, we have $L(u_1) \neq \{q, r, s\}$. Since $q \in L(u_1)$ and $|L(u_1)| = 3$, one of r, s does not lie in $L(u_1)$, so suppose without loss of generality that $r \notin L(u_1)$.

Subclaim 7.1.11. *For any $c \in L(p_1)$, there is at most one one $d \in L(p_3)$ such that $(c, d) \in S_r$.*

Proof: Let $c \in L(p_1)$ and suppose toward a contradiction that there exist distinct $d, d' \in L(p_3)$ such that (c, d) and $(c, d') \in S_r$. By Claim 7.1.3, we have $n \neq 1$. Since $r \notin L(u_1)$, it follows from Proposition 1.4.4 that $s \in \mathcal{Z}_{H_n^{\text{left}}}(c, r, \bullet)$. Consider the two L-colorings (s, r, d), (s, r, d') of $u_n p_2 p_3$. By Claim 7.1.3, $n \neq t$, so each of (s, r, d), (s, r, d') is a proper L-coloring of $u_n p_2 p_3$, and, by Observation 1.4.2, one of these extends to an L-coloring of H_n^{right}. Thus, one of $\sigma^{cd}, \sigma^{cd'}$ extends to an L-coloring of H, which is false. ■

Now note the following:

Subclaim 7.1.12. $s \notin L(p_1)$.

Proof: Suppose that $s \in L(p_1)$. By Claim 7.1.6, we have $\sigma^{sd}(p_2) \in \{r, q\}$ for each $d \in L(p_3)$. By Subclaim 7.1.11, there is at most one $d \in L(p_3)$ such that $\sigma^{sd}(p_2) = r$, and, by Claim 7.1.7, there is at most one $d \in L(p_3)$ such that $\sigma^{sd}(p_2) = q$. Since $|L(p_3)| \geq 3$, we have a contradiction. ■

Since $s \notin L(p_1)$, it follows from Claim 7.1.8 that $s \notin L(u_1)$. Thus, we have $\{r, s\} \cap L(u_1) = \varnothing$. Recall that $q \in L(p_1)$. By Claim 7.1.6, we have $\sigma^{qd}(p_2) \in \{r, s\}$ for each $d \in L(p_3)$. By Subclaim 7.1.11, since $|L(p_3)| \geq 3$, there exist two distinct colors $d, d' \in L(p_3)$ such that $\sigma^{qd}(p_2) = \sigma^{qd'}(p_2) = s$.

Since $n \neq t$, each of (r, s, d), (r, s, d') is a proper L-coloring of $u_n p_2 p_3$, and, by Observation1.4.2, one of these extends to an L-coloring of H_n^{right}, so suppose without loss of generality that $d \in \mathcal{Z}_{H_n^{\text{right}}}(r, s, \bullet)$. Since $s \notin L(u_1)$ and $n \neq 1$, it follows from Proposition 1.4.4 that the L-coloring (q, s, r) of $p_1 p_2 u_n$ extends to an L-coloring of H_n^{left}, so σ^{qd} extend to an L-coloring of H, contradicting our assumption. ■

We now have enough to finish the proof of 1) of Theorem 7.1.1.

Claim 7.1.13. $\sigma^{rr}(p_2) = s$ *and* $\sigma^{ss}(p_2) = r$.

Proof: Suppose that one of these does not hold, and suppose without loss of generality that $\sigma^{rr}(p_2) \neq s$. By Claim 7.1.6, we have $\sigma^{rr}(p_2) = q$, and we also have $\sigma^{rs}(p_2) = q$, so we contradict Claim 7.1.7. ■

Applying Claim 7.1.6, we have $\sigma^{qq}(p_2) \in \{r, s\}$, so suppose without loss of generality that $\sigma^{qq}(p_2) = r$. By Observation 1.4.2, $\mathcal{Z}_{H_n^{\text{left}}}(\bullet, r, s)$ contains one of q, s and $\mathcal{Z}_{H_n^{\text{right}}}(s, r, \bullet)$ also contains one of q, s. If s lies in both of these lists, then, since $\sigma^{ss} = r$, the coloring σ^{ss} extends to an L-coloring of H, contradicting our assumption. Thus, suppose without loss of generality that $\mathcal{Z}_{H_n^{\text{left}}}(\bullet, r, s) = \{q\}$. If $q \in \mathcal{Z}_{H_n^{\text{right}}}(s, r, \bullet)$, then, since $\sigma^{qq}(p_2) = r$, it follows that σ^{qq} extends to an L-coloring of H, contradicting our assumption. Thus, $\mathcal{Z}_{H_n^{\text{right}}}(s, r, \bullet) = \{s\}$. Applying Claim 7.1.6 again, we have $\sigma^{qs}(p_2) = r$. Since $\mathcal{Z}_{H_n^{\text{left}}}(\bullet, r, s) = \{q\}$ and $\mathcal{Z}_{H_n^{\text{right}}}(s, r, \bullet) = \{s\}$, it follows that σ^{qs} extends to an L-coloring of H, contradicting our assumption. This completes the proof of 1) of Theorem 7.1.1. □

2) of Theorem 7.1.1 deals with the case where one of the two endpoints of the principal path has a 4-list. We now prove 2), which we restate with the lemma below.

Lemma 7.1.14. *Let H be a broken wheel with principal path $P := p_1 p_2 p_3$, and let $u_\star \in V(H \setminus P)$. Let L be a list-assignment for H such that the following hold.*

1) $|L(u_\star)| \geq 2$; AND

2) $|L(p_3)| \geq 4$, and, for each $v \in V(H) \setminus \{u_\star, p_2, p_3\}$, $|L(v)| \geq 3$.

Then there exists a a color $c \in L(p_1)$ and two distinct colors $d_0, d_1 \in L(p_3)$ such that, for each $i = 0, 1$ and any L-coloring ϕ of $V(P)$ using c, d_i on p_1, p_3 respectively, ϕ extends to an L-coloring of H.

Proof. Let H be a counterexample to the lemma. By removing colors from some of the lists if necessary, we suppose that $|L(v)| = 3$ for all $v \in V(H) \setminus \{u_\star, p_2, p_3\}$, $|L(p_3)| = 4$, and $|L(u_\star)| = 2$. Let $S \subseteq L(p_1) \times L(p_3)$ be the set of pairs (c, d) such that any L-coloring of P using c, d on p_1, p_3 respectively extends to an L-coloring of H. Let $H - p_2 = p_1 u_1 \cdots u_t p_3$ for some $t \geq 1$ and let $m \in \{1, \cdots, t\}$ with $u_m = u_\star$. Let H^{left} be the broken wheel with principal path $P^{\text{left}} := p_1 p_2 u_m$, where $H^{\text{left}} - p_2 = p_1 u_1 \cdots u_m$. Likewise, let H^{right} be the broken wheel with principal path $P^{\text{right}} := u_m p_2 p_3$, where $H^{\text{right}} - p_2 = u_m \cdots u_t p_3$. Since $|L(u_t)| = 3$ and $|L(p_3)| = 4$, let $L(p_3) = \{d_0, d_1, d_2, d_3\}$, where $d_3 \notin L(u_t)$.

Since H is a counterexample, there are is at most one pair in S whose second coordinate is d_3, so let $c_0, c_1 \in L(p_1)$ be distinct colors with $(c_0, d_3), (c_1, d_3) \notin S$. Thus, for each $i = 0, 1$, there is an L-coloring σ_{i3} of $V(P)$ using c_i, d_3 on the respective colors p_1, p_3, where σ_{i3} does not extend to an L-coloring of H.

Claim 7.1.15. *For each $i = 0, 1$, we have $\mathcal{Z}_{H^{\text{left}}}(c_i, \sigma_{i3}(p_2), \bullet) = \varnothing$ and $\sigma_{i3}(p_2) = d_i$. Furthermore, $L(u_m) = \{\sigma_{03}(p_2), \sigma_{13}(p_2)\}$.*

<u>Proof:</u> Suppose there is an L-coloring ϕ of H^{left} using $c_i, \sigma_{i3}(p_2)$ on the respective vertices p_1, p_2. Since $d_3 \notin L(u_t)$, it follows from Proposition 1.4.4 that the L-coloring $(\phi(u_m), \sigma_{i3}(p_2), d_3)$ of P^{right} extends to an L-coloring of H^{right} (this is true even if H^{right} is a triangle, since $d_3 \notin L(u_t)$). This contradicts our assumption that σ_{i3} does not extend to an L-coloring of H. Thus, we indeed have $\mathcal{Z}_{H^{\text{left}}}(c_i, \sigma_{i3}(p_2), \bullet) = \varnothing$ for each $i = 0, 1$. ■

For each $i = 0, 1$, let $r_i := \sigma_{i3}(p_3)$.

Claim 7.1.16. *For each $i = 0, 1$, $c_{1-i} = r_i$.*

<u>Proof:</u> Let h be a color distinct from r_0, r_1 and let L' be a list-assignment for $V(H^{\text{left}})$ where $L'(u_m) = \{r_0, r_1, h\}$ and otherwise $L' = L$. By Theorem 0.2.3, we have $\mathcal{Z}_{H^{\text{left}}, L'}(c_i, \sigma_{i3}(p_2), \bullet) \neq \varnothing$ for each $i = 0, 1$. Thus, by Claim 7.1.15, we have $\mathcal{Z}_{H^{\text{leftt}}, L'}(c_i, \sigma_{i3}(p_2), \bullet) = \{h\}$ for each $i = 0, 1$. Since $c_0 \neq c_1$, it follows from 2) of Proposition 1.4.7 that, for each $i = 0, 1$, we have $c_{1-i} = \sigma_{i2}(p_2)$ and thus $c_{1-i} = r_i$. ■

Combining Claim 7.1.15 and Claim 7.1.16, we have $\{c_0, c_1\} \subseteq L(p_2)$ and $L(u_m) = \{c_0, c_1\}$. Since $|L(p_1)| = 3$, let f be the lone color of $L(p_1) \setminus \{c_0, c_1\}$.

Claim 7.1.17. $\mathcal{Z}_{H^{\text{left}}}(\bullet, c_1, c_0) = \mathcal{Z}_{H^{\text{left}}}(\bullet, c_0, c_1) = \{f\}$ *and furthermore,* $\{c_0, c_1\} \subseteq L(u_1)$.

<u>Proof:</u> Suppose that there is an $i \in \{0, 1\}$ such that $\mathcal{Z}_{H^{\text{left}}}(\bullet, c_i, c_{1-i}) \neq \{f\}$, say $i = 0$ without loss of generality. Since $L(p_1) = \{c_0, c_1, f\}$, it follows from Theorem 0.2.3 that $c_1 \in \mathcal{Z}_{H^{\text{left}}}(\bullet, c_0, c_1)$. By Claim 7.1.16, we have $r_1 = c_0$. Yet we also have $r_1 = \sigma_{13}(p_2)$ and, by Claim 7.1.15, $\mathcal{Z}_{H^{\text{left}}}(c_1, r_1, \bullet) = \varnothing$, so we have a contradiction.We now check that $\{c_0, c_1\} \subseteq L(u_1)$. If H^{left} is not a triangle, then this immediately follows from Proposition 1.4.4, since $|\mathcal{Z}_{H^{\text{left}}}(\bullet, c_i, c_{1-i})| = 1$ for each $i = 0, 1$. If H^{left} is a triangle and there is an $i \in \{0, 1\}$ with $c_i \notin L(u_1)$, then $m = 1$ and $\mathcal{Z}_{H^{\text{left}}}(c_i, \sigma_{i3}(p_2), \bullet) \neq \varnothing$, contradicting Claim 7.1.15. ■

We now have the following:

Claim 7.1.18. $(f, d_3) \in S$

Proof: Suppose that $(f, d_3) \notin S$. Thus, there is an L-coloring τ of $p_1p_2p_3$ using f, d_3 on the respective vertices p_1, p_3, where τ does not extend to an L-coloring of H. At least one of r_0, r_1 is distinct from $\tau(p_2)$, suppose without loss of generality that $\tau(p_2) \neq r_0$. Note that $\tau(p_2) \neq f$, since $f = \tau(p_1)$. Since $d_3 \notin L(u_t)$ and $L(u_m) = \{r_0, r_1\}$, it follows from Proposition 1.4.4 that $r_0 \in \mathcal{Z}_{H^{\text{right}}}(\bullet, \tau(p_2), d_3)$. This is true even if H^{right} is a triangle, since $d_3, \tau(p_2) \neq r_0$. Thus, we have $f \notin \mathcal{Z}_{H^{\text{left}}}(\bullet, \tau(p_2), r_0)$, or else τ extends to an L-coloring of H. Since $r_0 = c_1$ and $f \in L(p_1) \setminus \{c_0, c_1\}$, we have $\tau(p_2), r_0 \neq f$, so H^{left} is not a triangle.

Since $r_0 = c_1$, $r_1 = c_0$, and $f \notin \mathcal{Z}_{H^{\text{left}}}(\bullet, \tau(p_2), r_0)$, we have $\tau(p_2) \neq r_1$, or else we contradict Caim 7.1.17. Thus, $\tau(p_2) \notin \{c_0, c_1, f\}$. By Proposition 1.4.4, since H^{left} is not a triangle, we have $f, \tau(p_2) \in L(u_1)$. By Claim 7.1.17, $c_0, c_1 \in L(u_1)$. Thus, $L(u_1)$ contains the two disjoint sets $\{f, \tau(p_2)\}$, $\{c_0, c_1\}$, which is false as$|L(u_1)| = 3$. ■

Since $(f, d_3) \in S$, it follows that $(f, d_0), (f, d_1), (f, d_2) \notin S$, or else we contradict our assumption that H is a counterexample. Thus, for each $i = 0, 1, 2$ there is an L-coloring τ_i of P using f, d_i on the respective vertices p_1, p_3, where τ_i does not extend to an L-coloring of H.

Claim 7.1.19. *For each $k = 0, 1, 2$, $\tau_k(p_2) \in \{c_0, c_1\}$.*

Proof: Let $k \in \{0, 1, 2\}$ and suppose that $\tau_k(p_2) \notin \{c_0, c_1\}$.. Since $L(u_m) = \{c_0, c_1\}$, it follows from Observation 1.4.2 that $\mathcal{Z}_{H^{\text{right}}}(\bullet, \tau_k(p_2), d_k) \neq \varnothing$, so let $j \in \{0, 1\}$ with $c_j \in \mathcal{Z}_{H^{\text{right}}}(\bullet, \tau_k(p_2), d_k)$. By Claim 7.1.17, we have $\{c_0, c_1\} \subseteq L(u_1)$. Since $|\{f, \tau_3(p_2)\}| = 2$ and $\{f, \tau_k(p_2)\} \cap \{c_0, c_1\} = \varnothing$, one of $f, \tau_k(p_2)$ does not lie in u_1, as $|L(u_1)| = 3$. By Proposition 1.4.4, $f \in \mathcal{Z}_{H^{\text{left}}}(\bullet, \tau_k(p_2), c_j)$, and τ_k extends to an L-coloring of H, which is false. ■

Since $\{\tau_0(p_2), \tau_1(p_2), \tau_2(p_2)\} \subseteq \{c_0, c_1\}$, there exist $j, k \in \{0, 1, 2\}$ and a $c \in \{c_0, c_1\}$ and such that $\tau_j(p_2) = \tau_k(p_2) = c$, say $c = c_0$ without loss of generality. Thus, we have $d_j, d_k \neq c_0$. By Claim 7.1.17, we have $f \in \mathcal{Z}_{H^{\text{left}}}(\bullet, c_0, c_1)$, and, by Observation 1.4.2, the L-coloring (c_0, c_1) of p_2u_m extends to an L-coloring of H^{right} using one of d_j, d_k on p_3. Thus, one of τ_j, τ_k extends to an L-coloring of H, which is false. This completes the proof of Lemma 7.1.14 and thus completes the proof of Theorem 7.1.1. □

7.2 Completing the Proof of Theorem 7.0.1

This short section consists of the proof of Theorem 7.0.1, which we do not restate as the statement is somewhat lengthy. Let $G, C, P, P_\star, u_\star$ be as in the statement of Theorem 7.0.1. Let p, p' be the endpoints of P and let q, q' be the endpoints of $P_\star$, where the (not necessarily distinct) vertices of $\{p, p', q, q'\}$ have the order p', q', q, p on the path P. The following easy observation is an immediate consequence of the assumption that C is induced in G.

Recall that, by 1) of Theorem 1.7.4 we have the following: For any subpath R of C, letting x, x' be the endpoints of R, and letting $y \in V(R)$ be an R-hinge, we get that, for any $\psi \in \text{Link}_L(xRy)$ and $\psi' \in \text{Link}_L(yRx')$, if $\psi(y) = \psi'(y)$, the union $\psi \cup \psi'$ lies in $\text{Link}_L(R)$. By Condition 3) of Theorem 7.0.1, each of q, q' is a P-hinge. Combining this with Theorem 1.7.5, we immediately have the following by taking appropriate unions.

Claim 7.2.1. *Let $v \in V(pPq)$ be a P-hinge with $v \neq u_\star$ and suppose there exist two elements $\psi_1, \psi_2 \in \text{Link}_L(vPq')$ with $\psi_1(v) \neq \psi_2(v)$. Then there exist two elements τ_1, τ_2 of $\text{Link}_L(P)$ such that $\tau_1(p) \neq \tau_2(p)$ and such that, for each $i \in \{1, 2\}$, the restriction of τ_i to $\text{dom}(\tau_i) \cap V(vPp')$ is one of ψ_1, ψ_2.*

Now we return to the proof of Theorem 7.0.1. We break this into two cases.

Case 1: There is no $w \in D_1(C)$ such that $G[N(w) \cap V(P)]$ is a subpath of P with $u_\star$ as an internal vertex

In this case, $u_\star$ is a P-hinge and $P_\star = u_\star = q = q'$. By Theorem 1.7.5, there is an element ψ of $\text{Link}_L(u_\star Pp')$, since every vertex of $u_\star Pp'$ has an L-list of size at least three, except possibly the endpoint $u_\star$. We first prove A). Applying 1) of Theorem 1.7.5, there exists a $\psi' \in \text{Link}_L(pPu_\star)$ such that $\psi'(u_\star) = \psi(u_\star)$. By 1) of Theorem 1.7.4, the union $\psi \cup \psi'$ lies in $\text{Link}_L(P)$. This proves Statement A) in this case. Now we prove B).

Let $v \in V(pPq) \setminus \{u_\star\}$ with $|L(v)| \geq 4$ and suppose that v is a P-hinge. By Theorem 1.7.5, there exist two elements ψ_1, ψ_2 of $\text{Link}_{L_\phi}(u_\star Pv^\dagger, C^1, \tilde{G})$ which use $\psi(u_\star)$ on $u_\star$ and which color $v^\dagger$ with two different colors. Since $u_\star \not\in T^{\text{int}}$, it follows from 1) of Theorem 1.7.4 that each of $\psi \cup \psi_1$ and $\psi \cup \psi_2$ lies in $\text{Link}_{L_\phi}(p'Pv^\dagger, C^1, \tilde{G})$. Combining this with Claim 7.2.1, we prove Statement B). Thus, Theorem 7.0.1 holds in this case.

Case 2: There exists a $w \in D_2(C)$ such that $G[N(w) \cap V(P)]$ is a subpath of P with $u_\star$ as an internal vertex.

In this case, applying Condition 3) of Theorem 7.0.1, let $w \in D_1(C)$ be the unique vertex such that $P_\star = G[N(w) \cap V(C)] = qPq'$, where $u_\star$ is an internal vertex of qPq'.

Claim 7.2.2. $\text{Link}_L(qPp') \neq \varnothing$.

Proof: By 1) of Theorem 7.1.1, there is an element ψ of $\text{Link}_L(P_\star)$ obtained by coloring q, q'. By Theorem 1.7.5, there is an element ψ' of $\text{Link}_L(q'Pp')$ which uses $\psi(q')$ on q'. Since q' is a P-hinge, it follows from 1) of Theorem 1.7.4 that the union $\psi' \cup \psi$ is an element of $\text{Link}_L(qPp')$. ■

We first prove Statement A) of Theorem 7.0.1. Applying Claim 7.2.2, there is a $\psi^* \in \text{Link}_L(qPp')$. Applying 1) of Theorem 1.7.5, there exists an element σ of $\text{Link}_L(pQq)$ with $\sigma(q) = \psi^*(q)$. Since q is a P-hinge, it follows from 1) of Theorem 1.7.4 that the union $\psi^* \cup \sigma$ lies in $\text{Link}_L(P)$. This proves A). Now we prove B). Let $v \in V(pPq)$, where v is a P-hinge and $|L(v)| \geq 4$. Since $u_\star$ is an internal vertex of $P_\star$, we have $v \neq u_\star$. We now break the proof of B) into two cases

Subcase 2.1 $v \neq q$

In this case, as above, we fix a $\psi^* \in \text{Link}_L(qPp')$ by applying Claim 7.2.2. Again applying Theorem 1.7.5, there exist two elements σ_1, σ_2 of $\text{Link}_L(vPq)$ which both color q with $\psi^*(q)$ and use different colors on v. Since v is a P-hinge, it follows from 1) of Theorem 1.7.4 that the union $\psi^* \cup \sigma_i$ lies in $\text{Link}_L(vPp')$ for each $i = 1, 2$. Combining this with Claim 7.2.1, we prove B) in this case.

Subcase 2.2 $v = q$

In this case, since $|L(q)| \geq 4$, it follows from 2) of Theorem 7.1.1 that there exist two L-colorings ψ_1, ψ_2 of $\{q, q'\}$ which use the same color on q' and different colors on q, where $\psi_1, \psi_2 \in \text{Link}_L(P_\star)$. Let $c = \psi_1(q') = \psi_2(q')$. As above, by Theorem 1.7.5, there is a $\psi^* \in \text{Link}_L(q'Pp')$ with $\psi^*(q') = c$. Applying 1) of Theorem 1.7.4, each of the unions $\psi^* \cup \psi_1$, $\psi^* \cup \psi_2$ lies in $\text{Link}_L(pPq')$. Combining the above with Claim 7.2.1, we complete the proof of B). This completes the proof of Theorem 7.0.1.

Chapter 8

Boundary Analysis for Closed Rings

In this chapter, we prove an analogue of Theorem 3.0.2 for closed rings. In order to state the main result of Chapter 8, we begin with the following observation.

Observation 8.0.1. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let C be a closed $\mathcal{T}$-ring. Then there is a unqiue cycle $C^1 \subseteq G$ such that $V(C^1) = B_1(C, G)$. Furthermore, letting $G = G_0 \cup G_1$ be the natural C-partition of G, where $C \subseteq G_0$, we have $E(G_0) = E(C) \cup E(C^1) \cup E(C, C^1)$.*

Proof. By Corollary 2.2.10, C is a chordless cycle in G. Furthermore, there does not exist an $x \in D_1(C, G)$ such that x is adjacent to each vertex of C, or else, since C is a facial subgraph of G and G is short-separation-free, we have $V(G) = V(C) \cup \{x\}$, contradicting Corollary 2.2.29. Since C is an L-predictable subgraph of G, $G[N(x) \cap V(C)]$ is a proper subpath of C for each $x \in D_1(C, G)$. Since C is a chordless cycle, it follows from our triangulation conditions that G contains a cycle C^1 with $V(C^1) = D_1(C, G)$, and C^1 separates C from $G \setminus B_1(C, G)$. □

Given a closed ring $C \in \mathcal{C}$, we call the ring C^1 above the *1-necklace* of C. Note that this is analogous to the 1-necklace of an open ring of $\mathcal{T}$ from Theorem 3.0.2. When we delete vertices near a closed ring $C \in \mathcal{C}$, it is easier to analyze proper k-chords of C^1 in $G \setminus C$ for small values of k, rather than proper k-chords of C for small values of k.

Observation 8.0.2. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let C be a closed $\mathcal{T}$-ring. Let C^1 be the 1-necklace of C and let $k < \frac{N_{\text{mo}}}{3} - 2$ and let Q be a proper k-chord of C^1. Let $G_0 \cup G_1$ be the natural (C^1, Q)-partition of $G \setminus C$. Then there exists an $i \in \{0, 1\}$ such that $C' \subseteq G_i$ for all $C' \in \mathcal{C} \setminus \{C\}$.*

Proof. Let u, u' be the endpoints of Q. If there exist $v \in N(u) \cap V(C)$ and $v' \in N(u') \cap V(C)$ with $v \neq v'$, then the claim follows from 1) of Theorem 2.2.4. If no such pair v, v' exists, then there exists a lone vertex $v \in V(C)$ such that $N(u) \cap V(C) = N(u') \cap V(C) = \{v\}$, and then the claim follows from 2) of Theorem 2.2.4. □

Given the result of Observation 8.0.2, it is natural to introduce the following notation analogous to Definition 2.3.9 and Definition 6.0.4.

Definition 8.0.3. Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let C be a closed $\mathcal{T}$-ring. We set $\tilde{G} := G \setminus C$. Let C^1 be the 1-necklace of C and let $k < \frac{N_{\text{mo}}}{3} - 2$ and let Q be a proper k-chord of C^1. We then let $\tilde{G} = \tilde{G}^{\text{small}}_Q \cup \tilde{G}^{\text{large}}_Q$ denote the natural (C^1, Q)-partition of $\tilde{G}$, where, for each $C' \in \mathcal{C} \setminus \{C\}$, we have $C' \subseteq \tilde{G}^{\text{large}}_Q$.

In this chapter, we analyze the structure of G near the C^1 to obtain a result analogous to the results for open rings from Chapters 3 and 4. This analysis is simpler and shorter than that of Chapters 3 and 4. The main result of Chapter 8 is the following, which is an analogue of Theorem 3.0.2.

Theorem 8.0.4. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let $C \in \mathcal{C}$ be a closed ring, let $\tilde{G} := G \setminus C$, and let C^1 be the 1-necklace of C. Then C^1 is an induced subgraph of G, and, for each 2-chord xwy of C^1 in $\tilde{G}$, the graph $\tilde{G}^{\text{small}}_{xwy}$ is a broken wheel with principal path xwy.*

8.1 3-Lists on the 1-Necklace of a Closed Ring

We begin with the following:

Lemma 8.1.1. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let $C \in \mathcal{C}$ be a closed ring. Let Q be a 3-chord of C and, suppose that $V(C^1 \cap G^{\text{small}}_Q) \not\subseteq V(Q)$. Then G^{large}_Q is L-colorable.*

Proof. Given a 3-chord Q of C, we say that Q is *bad* if $V(C^1 \cap G^{\text{small}}_Q) \not\subseteq V(Q)$, but G^{large}_Q is L-colorable. Suppose toward a contradiction that there exists a bad 3-chord Q of C, and, among all bad 3-chords of C, we choose Q so that $|V(G^{\text{large}}_Q)|$ is minimized. Let $Q := pxyp'$ and let $P := C \cap G^{\text{small}}_Q$ Note that $d(p, p') \geq 2$, or else G contains a 4-cycle separating an internal vertex of $C^1 \cap G^{\text{large}}_Q$ from $G^{\text{small}}_Q \setminus Q$.

Claim 8.1.2. *There is no chord of Q in G^{large}_Q, except possibly that $pp' \in E(C)$.*

<u>Proof:</u> Suppose toward a contradiction that there exists such a chord of Q. Since C is induced, G^{large}_Q contains either the edge py or the edge $p'x$. Suppose without loss of generality that $p'x \in E(G^{\text{large}}_Q)$. Since C is L-predictable and an induced subgraph of G, and x is adjacent to each of p, p', x is adjacent to each vertex of $C \cap G^{\text{large}}_Q$. But then the triangle $p'xy$ separates an element of $\mathcal{C} \setminus \{C\}$ from C, contradicting short-separation-freeness. ■

Let $G^\dagger$ be a graph obtained from G by deleting all the vertices of $G^{\text{small}}_{pxyp'} \setminus \{p, x, y, p'\}$ and replacing them with a single vertex p^{in} adjacent to each of x, y, p, p'.

Claim 8.1.3. *$G^\dagger$ is short-separation-free.*

<u>Proof:</u> If $G^\dagger$ is not short-separation-free, then G^{large}_Q either contains a chord of Q which is not an edge of C, or a 2-chord of Q whose endpoints are either p, y or p', x. In the former case we contradict Claim 8.1.2, so there exists a $v \in V(G^{\text{large}}_Q \setminus Q)$ such that $V(Q) \cap N(v)$ contains at least one of $\{p', x\}$ or $\{p, y\}$. Suppose without loss of generality that $\{p, y\} \subseteq N(v)$. If $v \in V(C)$, then, since C is an induced cycle in G we have $pv \in E(C)$, and, since C is L-predictable, y is adjacent to each vertex of $(C \cap G^{\text{large}}_Q) \setminus \{p\}$. But then the 4-cycle $pvyx$ separates an element of $\mathcal{C} \setminus \{C\}$ from p', contradicting the fact that $\mathcal{T}$ is a tessellation.

Since $v \notin V(C)$, $Q^* := pvyp'$ is a 3-chord of C. Since G is short-separation-free, we have $G^{\text{small}}_{Q^*} \setminus \{v\} = G^{\text{small}}_Q$ and $G^{\text{large}}_Q \setminus \{x\} = G^{\text{large}}_{Q^*}$. Thus, we have $V(C^1 \cap G^{\text{small}}_{Q^*}) \not\subseteq V(Q^*)$, since $C^1 \cap G^{\text{small}}_{Q^*}$ contains an internal vertex of $C^1 \cap G^{\text{small}}_Q$, and we have $|V(G^{\text{large}}_{Q^*})| < |V(G^{\text{large}}_Q)|$. By the minimality of $|V(G^{\text{large}}_Q)|$, it follows that $G^{\text{large}}_{Q^*}$ admits an L-coloring ψ, and $|L_\psi(x)| \geq 2$, so ψ extends to an L-coloring of G^{large}_Q, contradicting our assumption. ■

Let $C^\dagger$ be the cycle obtained from C by replacing P with $pp^{\text{in}}p'$. Let L' be a list-assignment for $V(G^\dagger)$ where $L'(p^{\text{in}})$ is a lone color not lying in $L(p) \cup L(x) \cup L(y) \cup L(p')$, and otherwise $L' = L$. Let $C^\dagger_*$ be the outer face of $G^\dagger$ and let $\mathcal{T}^\dagger := (G^\dagger, (\mathcal{C} \setminus \{C\}) \cup \{C^\dagger\}, L', C^\dagger_*)$. By Claim 8.1.3, $G^\dagger$ is short-separation-free. Since $C^\dagger$ is L'-colorable by our choice of L', $\mathcal{T}^\dagger$ is a tessellation, where C' is a closed $\mathcal{T}^\dagger$-ring.

We claim now that $\mathcal{T}^\dagger$ is a mosaic. Since $|V(P)| \geq 3$, we have $|V(C^\dagger)| \leq |V(C)|$, so M0) is satisfied, and M1) is immediate. Since C is induced in G, $C^\dagger$ is induced in $G^\dagger$, and each vertex of $D_1(C^\dagger, G^\dagger)$ still satisfies the property that its neighborhood in $C^\dagger$ consists of a subpath of $C^\dagger$. Thus, by our choice of $L'(p^{in})$, $C^\dagger$ is an L'-predictable facial subgraph of $\mathcal{T}^\dagger$, so M2) is satisfied. Finally, for any $C' \in \mathcal{C} \setminus \{C\}$, there is no shortest $(w_{\mathcal{T}}(C'), C)$-path in G whose C-endpoint is an internal vertex of P. Since $|V(C^\dagger)| \leq |V(C)|$, the rank of C has not increased, so $\mathcal{T}^\dagger$ also satisfies the distance conditions of Definition 2.1.6.

Thus, $\mathcal{T}^\dagger$ is a mosaic, as desired. By assumption, we have $V(C^1 \cap G_Q^{\text{small}}) \not\subseteq V(Q)$, so $|V(G^\dagger)| < |V(G)|$, so $G^\dagger$ admits an L'-coloring ψ by the minimality of $\mathcal{T}$. The restriction of ψ to G_Q^{large} is an L-coloring of G_Q^{large}. Thus, our assumption that Q is bad is false. $\square$

We now rule out some of the chords of the 1-necklace of a closed ring in a critical mosaic.

Lemma 8.1.4. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let $C \in \mathcal{C}$ be a closed ring, and let C^1 be the 1-necklace of C. Let ϕ be the unique L-coloring of $V(C)$. Let P be a subpath of C^1 and suppose that each internal vertex of P has an L_ϕ-list of size at least three. Then P is an induced subpath of G.*

Proof. Let $\tilde{G} := G \setminus C$, and suppose toward a contradiction that the claimed result does not hold. Then there is a chord xy of C^1 such that each internal vertex of $C^1 \cap \tilde{G}_{xy}^{\text{small}}$ has an L_ϕ-list of size at least three. Let $P^1 := \tilde{G}_{xy}^{\text{small}} \cap C^1$. That is, P^1 is a subpath of C^1 of length at least two, with endpoints x, y. Since each vertex of P^1 has a neighbor in C consisting of a subpath of C, let P^0 be the subpath of C such that $V(P_0) = D_1(P^1, C)$. For any $q, q' \in V(P_0)$ with $q \in N(x)$ and $q' \in N(y)$, q, q' are of distance at least two apart, or else G contains a 4-cycle separating an internal vertex of P^0 from $\tilde{G}_{xy}^{\text{small}}$. Let P_0^* be the subpath of P_0 intersecting $N(x)$ and $N(y)$ only on its endpoints. Note that $|V(P_0^*)| \geq 3$. Let p, p' be the endpoints of P_*^0, where $p \in N(x)$ and $p' \in N(y)$, and let $Q := pxyp'$.

Claim 8.1.5. *Q is an induced subpath of G_Q^{small}.*

<u>Proof:</u> If this does not hold, then, since $d(p, p') \geq 2$, G_Q^{small} contains one of the edges $p'x, py$, so suppose without loss of generality that $p'x \in E(G)$. Then the triangle $p'xy$ separates an internal vertex of P^1 from G_Q^{large}, contradicting short-separation-freeness. ■

Since $|V(P^1)| \geq 3$, it follows from Lemma 8.1.1 that there is an L-coloring ψ of G_Q^{large}. Since Q is an induced subpath of G_Q^{small}, ψ is an L-coloring of the subgraph of G induced by G_Q^{large}.

Since each neighbor of $\{x, y\}$ in C lies in $\text{dom}(\psi)$, the union $\psi \cup \phi$ is a proper L-coloring of $V(G_Q^{\text{large}} \cup P^0)$. Now, $G_Q^{\text{small}} \setminus P_*^0$ contains a cyclic facial subgraph $F := P^1 + xy$. By assumption, each vertex of $F \setminus \{x, y\}$ has an $L_{\psi \cup \phi}^{xy}$-list of size at least three, and furthermore, each vertex of $(G_Q^{\text{small}} \setminus P_*^0) \setminus F$ has an $L_{\psi \cup \phi}^{xy}$-list of size five. Thus, by Theorem 0.2.3, $\psi \cup \phi$ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. $\square$

We now have the following intermediate result which is analogous to Lemma 8.1.1.

Lemma 8.1.6. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let $C \in \mathcal{C}$ be a closed ring. Let Q be a 4-chord of C and, suppose that $|V(G_Q^{\text{small}} \setminus Q)| > 3$. Then G_Q^{large} is L-colorable.*

Proof. Given a 4-chord Q of C, we say that Q is *bad* if $|V(G_Q^{\text{small}} \setminus Q)| > 3$ but G_Q^{large} is L-colorable. Suppose toward a contradiction that there exists a bad 4-chord Q of C, and, among all bad 4-chords of C, we choose Q so that $|V(G_Q^{\text{large}})|$ is minimized. Let $Q := pxwyp'$, let $P^0 := C \cap G_Q^{\text{small}}$ and $P^1 := C^1 \cap G_Q^{\text{small}}$. Note that $d(p, p') \geq 2$, or else G contains a 4-cycle separating an internal vertex of $C^1 \cap G_Q^{\text{large}}$ from $G_Q^{\text{small}} \setminus Q$. Thus, we have $|V(P^0)| \geq 3$.

Claim 8.1.7. *There is no chord of Q in G_Q^{small}. Furthermore, if there is an edge $e \in E(G_Q^{\text{large}}$ which is a chord of Q, then $e = pp'$ and $e \in E(C')$.*

<u>Proof:</u> Suppose toward a contradiction that there is a chord e of Q in G_Q^{small}. Since C is an induced cycle of G and $|V(P^0)| \geq 2$, we have $e \neq pp'$. Furthermore, we have $e \notin \{wp, wp'\}$, since G_Q^{small} contains an (x, y)-path which is disjoint to Q except for its endpoints.

Thus, we have $e \in \{xy, py, p'x\}$. If $e = xy$, then G contains the 3-chord $Q^* := pxyp'$, and, since G is short-separation-free, we have $G_Q^{\text{large}} = G_{Q^*}^{\text{large}} - e$. In particular, $P_0 \subseteq G_{Q^*}^{\text{small}}$, and since $|V(P_0)| \geq 3$, it follows from Lemma 8.1.1 that $G_{Q^*}^{\text{large}}$ admits an L-coloring ψ. Since $G_Q^{\text{large}} = G_{Q^*}^{\text{large}} - e$, ψ is also an L-coloring of G_Q^{large}, contradicting our assumption.

We conclude that $xy \notin V(G_Q^{\text{small}})$, so $e \in \{py, p'x\}$. Since $xy \notin V(G_Q^{\text{small}})$, we have $|V(P_1)| \geq 3$. Suppose without loss of generality that $e = p'x$. Then the 4-cycle $xwyp'$ separates an internal vertex of P_1 from $G_Q^{\text{large}} \setminus Q$, contradicting short-separation-freeness. We conclude that there is no chord of Q in G_Q^{small}, as desired. Now suppose that there is a chord e of Q in G_Q^{large}. If $e = pp'$ then, since C is an induced cycle of G, we have $e \in E(C)$, and we are done in that case, so suppose toward a contradiction that $e \neq pp'$.

Suppose first that $e \in \{p'x, py\}$, and, without loss of generality, let $e = p'x$. Since G is L-predictable and C is induced in G, x is adjacent to each vertex of $C \setminus \mathring{P}^9$, so the 4-cycle $xwyp'$ separates an element of $\mathcal{C} \setminus \{C\}$ from each vertex of $G_Q^{\text{small}} \setminus Q$, contradicting short-separation-freeness.

Thus, we have $e \in \{xy, py, p'y\}$. In particular, the endpoints of e are of distance precisely two apart on Q. Let $e = qq'$ and let q^* be the unique vertex of Q such that q, q^*, q' are consecutive on Q. Since G is short-separation-free, G contains a 3-chord Q^* of C with the same endpoints as Q, where $G_{Q^*}^{\text{large}} = G_Q^{\text{large}} \setminus \{q^*\}$ and $G_{Q^*}^{\text{small}} = G_Q^{\text{small}} + qq'$. Thus, $P_1 \subseteq G_{Q^*}^{\text{large}}$, and since $|V(P_1)| \geq 3$, it follows from Lemma 8.1.1 that $G_{Q^*}^{\text{small}}$ admits an L-coloring ψ. Since $q^* \in \{x, w, y\}$, we have $|L_\psi(q^*)| \geq 3$, and ψ extends to an L-coloring of G_Q^{large}, contradicting our assumption. ■

Now we have the following:

Claim 8.1.8. *For any two vertices $q, q' \in V(Q)$ which are of distance precisely two apart in Q, q, q' do not have a common neighbor in $V(G_Q^{\text{large}}) \setminus V(Q)$.*

<u>Proof:</u> Suppose toward a contradiction that q, q' have a common neighbor w^* in $G_Q^{\text{large}} \setminus Q$, and let q'' be the unique common neighbor of q, q' on the path Q. Then G contains the 4-cycle $qw^*q'q''$. By Claim 8.1.7, $qq' \notin E(G)$, so we have $w^* \in N(q'')$ by our triangulation conditions.

We claim now that $w^* \notin V(C)$. Suppose that $w^* \in V(C)$. If $\{x, y\} \subseteq N(w^*)$, then, since C is L-predictable, x is adjacent to each vertex of the subpath of $C \cap G_Q^{\text{large}}$ with endpoints p, w^*, and y is adjacent to each vertex of the subpath of $C \cap G_Q^{\text{large}}$ with endpoints p', w^*. But then the 4-cycle $xwyw^*$ separates an element of $\mathcal{C} \setminus \{C\}$ from p, p', contradicting the fact that $\mathcal{T}$ is a tessellation. Thus, at least one of x, y lies outside of $N(w^*)$, so suppose without loss of generality that $y \notin N(w^*)$. Thus, we have $qq''q' = pxw$, and G contains the 3-chord $R := w^*wyp'$ of C. Since G is short-separation-free, we have $G_R^{\text{large}} = G_Q^{\text{large}} \setminus \{x, p\}$. Since $V(P^1) \subseteq V(G_R^{\text{small}})$, it follows from Lemma 8.1.1 that G_R^{large} admits an L-coloring ψ. Since w^* is precolored and wp is an edge of C, ψ extends to an L-color dom$(\psi) \cup \{p\}$, and the resulting extension leaves a color for x, since $|L_\psi(x)| \geq 3$. Thus, ψ extends to an L-coloring of G_Q^{large}, contradicting our assumption. We conclude that $w^* \notin V(C)$.

Since $w^* \notin V(C)$, G contains a 4-chord Q^* of C obtained from Q by replacing $qq''q'$ with qw^*q'. Since G is short-separation-free, we have $G_Q^{\text{small}} = G_{Q^*}^{\text{small}} \setminus \{w^*\}$ and $G_Q^{\text{large}} \setminus \{q''\} = G_{Q^*}^{\text{large}}$. Thus, we have $|V(G_{Q^*}^{\text{large}} \setminus Q^*)| > 3$ as well, and, by the minimality of Q, $G_{Q^*}^{\text{large}}$ admits an L-coloring ψ. By Claim 8.1.7, $N(q'') \cap V(Q) = \{q, q'\}$, so $|L_\psi(w)| \geq 2$. Thus, ψ extends to an L-coloring of G_Q^{large}, contradicting our assumption. ■

With the above in hand, we prove the following:

Claim 8.1.9. $|V(P^0)| = 3$

Proof: Suppose toward a contradiction that $|V(P^0)| \neq 3$. Since p, p' are of distance at least two apart, we have $|V(P^0)| > 3$. Now let $G^\dagger$ be a graph obtained from G in the following way: We first delete all the vertices of $G_Q^{\text{small}} \setminus (Q \cup C)$, and the contract P^0 to a path $pqq'p'$ of length three with endpoints p, p', deleting any loops. Finally, we add the edges xq, yq', and we add a vertex w^* adjacent to all five vertices of the cycle $wqq'yw$.

Note that $G_Q^{\text{large}} \subseteq G^\dagger$, and $G^\dagger \setminus G_Q^{\text{large}}$ consists of the triangle $qq'w^*$. Let $C^\dagger$ be the cycle obtained from C by the contraction of P^0 to $pqq'q'$.

We claim now that $G^\dagger$ is short-separation-free. Let $H \cup G_Q^{\text{large}}$ be the natural Q-partition of $G^\dagger$. Each of H and G_Q^{large} is short-separation-free, so if $G^\dagger$ is not short-separation-free, then there is either a chord of Q in G_Q^{large} which is not an edge of C, or a 2-chord of Q in G_Q^{large} whose endpoints are of distance precisely two apart on Q. In the former case, we contradict Claim 8.1.7, and in the latter case, we contradict Claim 8.1.8.

Thus, $G^\dagger$ is indeed short-separation-free. Let c, c' be colors where $c \neq c'$, $c \notin L(p) \cup L(x)$, and $c' \notin L(p') \cup L(y)$. Let L' be a list-assignment for $V(G^\dagger)$ where $L'(q) = \{c\}$, $L'(q') = \{c'\}$, $L'(w^*)$ is an arbitrary 5-list, and otherwise $L' = L$. Let $C_*^\dagger$ be the outer face of $G^\dagger$. By construction of $G^\dagger$ and L', each face of $G^\dagger$, except those among $(\mathcal{C} \setminus \{C\}) \cup \{C^\dagger\}$, is a triangle, and $V(C^\dagger)$ is L'-colorable. Thus, since $G^\dagger$ is short-separation-free, $\mathcal{T}^\dagger := (G^\dagger, (\mathcal{C} \setminus \{C^\dagger\}), L', C_*^\dagger)$ is a tessellation. We claim now that $\mathcal{T}^\dagger$ is a mosaic.

Since $|V(P_0)| > 3$, we have $|V(C^\dagger)| \leq |V(C)|$, so M0) is satisfied, and M1) is immediate. Since C is induced in G, $C^\dagger$ is induced in $G^\dagger$, and, by our construction of $G^\dagger$, ech vertex of $D_1(C^\dagger, G^\dagger)$ has a neighborhood in $C^\dagger$ consisting of a subpath of $C^\dagger$.

By our choice of colors c, c', $C^\dagger$ is L'-predictable, so M2) is satisfied as well. We just need to check that the distance conditions of Definition 2.1.6 hold. Since $|V(C^\dagger)| \leq |V(C)|$, the rank of C has not increased, and since $d_G(w, C) \leq 2$, it follows that, for any $C' \in \mathcal{C} \setminus \{C\}$, we have $d_{G^\dagger}(w_{\mathcal{T}}(C'), C^\dagger) \geq d_G(w_{\mathcal{T}}(C'), C)$. Since $w_{\mathcal{T}^\dagger}(C') = w_{\mathcal{T}}(C)$, $\mathcal{T}^\dagger$ satisfies the desired distance conditions. Thus, $\mathcal{T}^\dagger$ is a mosaic.

Since $G^\dagger \setminus G_Q^{\text{large}} = qq'w^*$ and $|V(G_Q^{\text{small}} \setminus Q)| > 3$, we have $|V(G^\dagger)| < |V(G)|$. By the minimality of $\mathcal{T}$, $G^\dagger$ admits an L'-coloring ψ, and ψ restricts to an L-coloring of G_Q^{large}, contradicting our assumption that Q is bad. ■

Since $|V(P^0)| = 3$, let p^{in} be the lone internal vertex of P^0. Then $P^0 := pp^{in}p'$. We now construct a smaller mosaic than $\mathcal{T}$ in the following way. Let G' be a graph obtained from G by first deleting all the vertices of $G_Q^{\text{small}} \setminus (Q \cup C)$ and replacing them with an edge w^*w^{**}, where $N(w^*) \cap V(C \cup Q) = \{p^{in}, x, p, w\}$ and $N(w^{**}) \cap V(C \cup Q) = \{p^{in}, w, y, p'\}$.

We claim now that G' is short-separation-free. Let $G' = H \cup G_Q^{\text{large}}$ be the natural Q'-partition of G'. Note that G' does not contain a separating cycle of length at most four containing both of p, p', or else, since C is an induced cycle of G', there is a 2-chord pup' of Q with $u \in V(G_Q^{\text{large}} \setminus Q)$, where pup' is not a subpath of C. But then, $pup'p^{in}$ is also a separating cycle in G, contradicting the fact that $\mathcal{T}$ is a tessellation. Since each of H and G_Q^{large} is short-separation-free, if G' is not short-separation-free, then there is either a chord of Q in G_Q^{large} which is not an edge of C, or a 2-chord of Q in G_Q^{large} whose endpoints are of distance precisely two apart on Q, or a 2-chord of Q. In the former case, we contradict Claim 8.1.7, and in the latter case, we contradict Claim 8.1.8. Thus, G' is short-separation-free.

Let L' be a list-assignment for $V(G')$ where each of $L'(w^*)$ and $L'(w^{**})$ is an arbitrary 5-list, and otherwise $L' = L$. Then $\mathcal{T}' := (G', \mathcal{C}, L', C_*)$ is a tessellation. M0) and M1) are immediate, and each of w^*, w^{**} has a neighborhood on C consisting of a path of length precisely one, so, since C is L-predictable in G, it is also L'-predictable in G'. Thus, M2) is satisfied as well. Finally, since $d_G(w, C) \leq 2$, it follows that, for any $C' \in \mathcal{C} \setminus \{C\}$, we have $d_{G'}(w_{\mathcal{T}}(C'), C) \geq d_G(w_{\mathcal{T}'}(C'), C)$. Since $w_{\mathcal{T}'}(C') = w_{\mathcal{T}}(C)$, $\mathcal{T}'$ satisfies the distance conditions of Definition 2.1.6. Thus, $\mathcal{T}'$ is a tessellation. Since $|V(G^{\text{small}} \setminus Q)| > 3$, we have $|V(G')| < |V(G)|$, so, by the minimality of $\mathcal{T}$, G' admits an L'-coloring ψ, and ψ restricts to an L-coloring of G_Q^{large}, contradicting our assumption that Q is bad. □

With the above in hand, we prove an analogue of Lemma 8.1.4 for 2-chords of the 1-necklace:

Lemma 8.1.10. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let $C \in \mathcal{C}$ be a closed ring, and let C^1 be the 1-necklace of C. Let $\tilde{G} := G \setminus C$ and let ϕ be the unique L-coloring of $V(C)$. Let P be a subpath of C^1 with $|V(P)| \geq 2$ and suppose that each internal vertex of P has an L_ϕ-list of size at least three. Let x, y be the endpoints of P and suppose there is a vertex $w \in D_1(C^1, \tilde{G})$ adjacent to each of x, y. Then $V(\tilde{G}_{xwy}^{\text{small}}) = V(P) \cup \{w\}$ and $\tilde{G}_{xwy}^{\text{small}}$ is a broken wheel with principal path xwy.*

Proof. For any 2-chord xwy of C^1 with $w \in D_1(C^1, G \setminus C)$, we say that wxy is *defective* if $|V(\tilde{G}_{xwy}^{\text{small}} \setminus C^1)| > 1$. Suppose toward a contradiction that there exists a defective 2-chord xwy of C^1, and, among all defective 2-chords, we choose wxy so that $|V(\tilde{G}_{xwy}^{\text{small}})|$ is minimized. Let $P^1 := \tilde{G}_{xwy}^{\text{small}} \cap C^1$. By Lemma 8.1.4, P^1 is an induced path in G.

Claim 8.1.11. *For each $v \in V(\mathring{P}^1)$, $w \notin N(v)$.*

<u>Proof:</u> Suppose toward a contradiction that there is a $v \in V(\mathring{P}^1)$ with $w \in N(v)$. Then $\tilde{G}$ contains the 2-chords xwv and vwy of C^1. Furthermore, $|\tilde{G}_{xwv}^{\text{small}}| < |V(\tilde{G}_{xwy}^{\text{small}})|$ and $|\tilde{G}_{vwy}^{\text{small}}| < |V(\tilde{G}_{xwy}^{\text{small}})|$. By the minimality of wxy, we have $\{w\} = V(\tilde{G}_{vwy}^{\text{small}} \setminus C^1) = V(\tilde{G}_{xwv}^{\text{small}} \setminus C^1)$, so $\{w\} = V(\tilde{G}_{xwy}^{\text{small}} \setminus C^1)$, contradicting the fact that xwy is defective. ■

Let P^0 be the unique subpath of C such that $V(P) = V(C) \cap D_1(P^1, G)$, and let P_*^0 be the subpath of P^0 intersecting $N(x) \cup N(y)$ only on its endpoints. Let p, p' be the endpoints of P_*^0, where $p \in N(x)$ and $p' \in N(y)$. Note that $|V(P_*^0)| > 1$, or else, since G is short-separation-free, $\tilde{G}_{xwy}^{\text{small}}$ consists of the triangle xwy, contradicting the fact that xwy is defective. Thus, $Q := pxwyp'$ is a proper 4-chord of C.

Let $S := V(\tilde{G}_{xwy}^{\text{small}}) \setminus V(C^1) \cup \{w\})$. Since xwy is defective, we have $S \neq \varnothing$. Furthermore, we have $|V(\mathring{P}^1)| \geq 2$, or else xP_1yw is a cycle of length at most four separating a vertex of S from $G_Q^{\text{large}} \setminus Q$.

Claim 8.1.12. *If e is a chord of Q in G_Q^{small}, then $e = pp'$ and $e \in E(C)$.*

<u>Proof:</u> Suppose there is a chord e of Q in G_Q^{small}. If $e = pp'$ then $e \in E(C)$, since C is an induced subgraph of G, so we are done. Now suppose toward a contradiction that $e \neq pp'$. Since $|V(\mathring{P}^1)| \geq 2$ and P^1 is an induced subpath of G_Q^{small}, we have $e \neq xy$. Since $w \in D_1(C^1, G \setminus C)$, we have $e \notin \{pw, p'w\}$, so $e \in \{p'x, py\}$. Suppose without loss of generality that $e = p'x$. Since C is an induced subgraph of G and C is L-predictable, x is adjacent to each vertex of P^0, so the 4-cycle $xwyp'$ separates a vertex of S from $G_Q^{\text{large}} \setminus Q$, contradicting short-separation-freeness. ■

We now have the following:

Claim 8.1.13. *For any L-coloring ψ of G_Q^{large}, $\psi \cup \phi$ is a proper L-coloring of $V(G_Q^{\text{large}} \cup P^0)$.*

Proof: Firstly, we note that ψ is an L-coloring of the subgraph of G induced by $V(G_Q^{\text{large}})$. To see this, just note that, by Claim 8.1.12, if e is a chord of Q in G_Q^{small}, then $e = pp'$, and the endpoints of e are precolored in L. Thus, ψ is indeed a proper L-coloring of the subgraph of G induced by $V(G_Q^{\text{large}}$. Furthermore, since each vertex of $(N(x) \cup N(y)) \cap V(C)$ lies in dom(ψ), the union $\psi \cup \phi$ is indeed a proper L-coloring of the subgraph of G induced by $V(G_Q^{\text{large}} \cup P^0)$. ■

We now establish the following.

Claim 8.1.14.

1) *If $|S| = 1$ then $|V(P_*^0)| \geq 3$; AND*
2) *G_Q^{large} admits an L-coloring.*

Proof: Let $|S| = 1$ and suppose toward a contradiction that $|V((P_*^0)| \leq 2$. Then P_*^0 consists of the edge pp'. Let $R := pxP^1yp'$. Since C^1 separates C from $G \setminus C^1$ and $P^0 = pp'$, we have $V(G_R^{\text{small}}) = V(R)$. Since P^1 is an induced subpath of G_Q^{small} and pp' is the only chord of Q in G_Q^{small}, it follows from our triangulation conditions that there is a vertex $q \in V(R) \setminus \{x, y\}$ such that q is adjacent to both vertices of pp'.

Since $V(\mathring{P}^1)| \geq 2$, there is a vertex $q' \in V(\mathring{P}^1)$ with $q' \neq q$. Suppose without loss of generality that q' lies in the subpath yRq of R. Then the edge qp' separates q' from p. Since $q' \in V(C^1)$, q' has a neighbor among p, p', so $N(q') \cap V(C^1) = \{p'\}$.

By Claim 8.1.12, Q has no chord in G_Q^{small} except for pp', and, by Claim 8.1.11, w has no neighbor in $P^1 \setminus \{x, y\}$. Thus, since P^1 is an induced subpath of G_Q^{small}, the cycle wyP^1x is an induced subgraph of G_Q^{small}. Since $|S| = 1$, it then follows from our triangulation conditions that S is a lone vertex u adjacent to every vertex of the cycle wyP^1x. Yet since each of q, q', y are adjacent to p', G contains a $K_{2,3}$ with bipartition $\{u, p'\}$, $\{q, q', y\}$, contradicting the fact that $\mathcal{T}$ is a tessellation. This proves 1). Now suppose toward a contradiction that G_Q^{large} is not L-colorable. By Lemma 8.1.6, $|V(G_Q^{\text{small}} \setminus Q)| \leq 3$. Since $S \neq \varnothing$, $|V(\mathring{P}^1)| \geq 2$, and $p \neq p'$, it follows that $|S| = 1$, $|V(\mathring{P}^1)| = 2$, and P_*^0 is the edge pp'. This contradicts 1). ■

Applying Claim 8.1.14, let ψ be an L-coloring of G_Q^{large}. By Claim 8.1.11, the union $\psi \cup \phi$ is a proper L-coloring of $V(G_Q^{\text{large}} \cup P^0)$. Let $P^1 := u_0 \cdots u_t$, where $u_0 = x$ and $u_t = y$.

Claim 8.1.15.

1) *$|S| = 1$, and $\tilde{G}_{xwy}^{\text{small}}$ consists of a wheel whose central vertex is adjacent to each vertex of $V(P^1) \cup \{w\}$; AND*
2) *For any L-coloring ψ' of G_Q^{large}, we have $|L_{\psi'}(u_\star)| = 2$.*

Proof: We first show that there exists a vertex $u_\star \in S$ adjacent to each of x, w, y. Suppose toward a contradiction that no such vertex exists. Now, $\tilde{G}_{xwy}^{\text{small}}$ contains a cyclic facial subgraph F such that $V(F) = \{w\} \cup V(P^1)$. Thus, $\tilde{G}_{xwy}^{\text{small}} \setminus \{x, w, y\}$ contains a facial subgraph F' such that $V(F') = V(\mathring{P}^1) \cup (D_1(\text{dom}(\phi \cup \psi), G) \cap S)$. By Claim 8.1.11, w has no neighbors among $w_1, \cdots, w_t$. Thus, since P^1 is an induced graph of G, we have $N(u) \cap \text{dom}(\psi \cup \phi) \subseteq V(C)$ for each $u \in \{u_2, \cdots, u_{t-1}\}$, and thus, by assumption, $|L_{\psi \cup \phi}(u_i)| \geq 3$ for each $i \in \{2, \cdots, t-2\}$. Since $t \geq 3$, we have $|L_{\psi \cup \phi}(u_1)| \geq 2$ and $|L_{\psi \cup \phi}(u_{t-1})| \geq 2$. Since no vertex of S is adjacent to all three of x, w, y, we have $|L_{\psi \cup \phi}(v)| \geq 3$ for all $v \in V(F') \setminus \{u_1, \cdots, u_t\}$, and each vertex of $(\tilde{G}_{xwy}^{\text{small}} \setminus \{x, w, y\}) \setminus F'$ has an $L_{\psi \cup \phi}$-list of size five. Thus, by Theorem 1.3.4, $\psi \cup \phi$ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical.

Thus, there is indeed a vertex $u_\star \in S$ adjacent to all three of x, w, y, so $G \setminus C$ contains the 3-chord $xu_\star y$ of C^1. By the minimality of xwy, we get that $xu_\star y$ is not defective, so $V(\tilde{G}^{\text{small}}_{xu_\star y} = V(P^1) \cup \{u_\star\}$. Since P^1 is an induced path in G, it follows from our triangulation conditions that $u_\star$ is adjacent to each vertex of $u_0, \cdots, u_t$. Thus, since G is short-separation-free, $\tilde{G}^{\text{small}}_{xwy}$ consists of a a wheel with central vertex $u_\star$ adjacent to every vertex of the cycle $u_0 \cdots u_t w$, and $S = \{u_\star\}$. This proves 1).

If ψ' is an L-coloring of G^{large}_Q and $|L_\psi(u_\star)| > 2$, then $\psi' \cup \phi$ is a proper L-coloring of $V(G^{\text{large}}_Q \cup P^0)$ by Claim 8.1.13, and each vertex of the broken wheel $\tilde{G}^{\text{small}}_{xu_\star y}$ has an $L_{\psi' \cup \phi}$-list of size at least three, except for u_1, u_{t-1}, which ave $L_{\psi' \cup \phi}$-lists of size at least two. Thus, by Theorem 1.3.4, $\psi' \cup \phi$ extends to the broken wheel $\tilde{G}^{\text{small}}_{xu_\star y}$, contradicting the fact that $\mathcal{T}$ is critical. This proves 2). ■

As in Claim 8.1.15, let $S = \{u_\star\}$. Since $\text{dom}(\psi \cup \phi) \cap N(u_\star) = \{x, w, y\}$, there are two colors $r, s \in L_{\psi \cup \phi}(u_\star)$. Since $\psi \cup \phi$ does not extend to an L-color the broken wheel $\tilde{G}^{\text{small}}_{xu_\star y}$, we immediately have the following by Proposition 1.4.4.

Claim 8.1.16.

1) $|L_{\psi \cup \phi}(u_1)| = L_{\psi \cup \phi}(u_{t-1})| = 2$ *and* $|L_{\psi \cup \phi}(u_i)| = 3$ *for each* $i \in \{2, \cdots, t-2\}$*; AND*

2) $r, s \in L(u_i)$ *for each* $i = 1, \cdots, t-1$*. In particular,* $L(u_\star) = \{r, s, \psi(x), \psi(w), \psi(y)\}$.

Now we have the following.

Claim 8.1.17.

1) *If* e *is a chord of* Q *in* G^{large}_Q*, then* $e = pp'$ *and* $e \in E(C)$.

2) *There is no vertex* $v \in V(G^{\text{large}}_Q \setminus Q)$ *such that* v *is adjacent to two vertices of* Q *which are of distance precisely two apart on* Q.

Proof: Firstly, since x, w, y are adjacent to $u_\star$, we have $xy \notin E(G^{\text{large}}_Q)$, and there is no vertex of $G^{\text{large}}_Q \setminus Q$ adjacent to x, y, or else G contains either a copy of K_4 or $K_{2,3}$, contradicting short-separation-freeness. Likewise, since p, x, w are adjacent to u_1, we have $wp \notin E(G^{\text{large}}_Q)$, and there is no vertex of $G^{\text{large}}_Q \setminus Q$ adjacent to w, p. Since p', y, w are adjacent to u_{t-1}, we have $wp' \notin E(G^{\text{large}}_Q)$, and there is no vertex of $G^{\text{large}}_Q \setminus Q$ adjacent to w, p'. The above proves 2), and shows that there is a chord e of Q in G^{large}_Q, then $e \in \{pp', xp', yp\}$. We just need to show that any such e is an edge of C. Suppose not. Then, since C is an induced cycle of G, G^{large}_Q contains one of the edges xp', yp, say xp' without loss of generality. Since C is L-predictable, x is adjacent to each vertex of $C \cap G^{\text{large}}_Q$, and the 4-cycle $xwyp'$ separates an element of $\mathcal{C} \setminus \{C\}$ from p, contradicting the fact that $\mathcal{T}$ is a tessellation. ■

Now let G' be a graph obtained from G by first deleting the vertices of $G^{\text{small}}_Q \setminus Q$ and replacing them with a triangle $qq'p^{in}$, where p^{in} is adjacent top each of p, p', q is adjacent to w, x, p and q' is adjacent to w, y, p' (alternatively phrased, we delete S and then contract P^0_* to a path of length two and P^1 to a path of length three). Let C' be the cycle obtained from C by replacing P^0_* with $pp^{in}p'$. Now, each facial subgraph of G', except possibly those among $(\mathcal{C} \setminus \{C\}) \cup \{C'\}$, is bounded by a triangle.

Note that G' is short-separation-free, or else G contains either a chord of Q which is not an edge of C, or a 2-chord of Q whose endpoints are of distance precisely two apart on Q. In either case, we contradict Claim 8.1.17. Now let c, d be distinct colors where $c, d \notin L(p) \cup L(p') \cup \{r, s, \psi(x), \psi(w), \psi(y)\}$. Let L' be a list-assignment for $V(G')$ defined as follows:

$$L'(v) := \begin{cases} \{c\} \text{ if } v = p^{in} \\ \{\phi(p), \psi(x), \psi(w), c, d\} \text{ if } v = q \\ \{\phi(p'), \psi(y), \psi(w), c, d\} \text{if } v = q' \\ L(v) \text{ if } v \in V(G_Q^{\text{large}}) \end{cases}$$

Let C'_* be the outer face of G'. By our choice of c, $V(C')$ is L'-colorable, and, since G' is short-separation-free, the tuple $\mathcal{T}' := (G', (\mathcal{C} \setminus \{C\}) \cup \{C'\}, L', C'_*)$ is a tessellation. We claim now that $\mathcal{T}'$ is a mosaic.

Since $|S| = 1$, we have $|V(P^0)| \geq 3$ by 1) of Claim 8.1.14. Thus, $|V(C')| \leq |V(C)|$, so M0) is satisfied, and M1) is immediate. By construction of G', C' is an induced subgraph of G', since C is an induced subgraph of G, and, for each $v \in D_1(C', G')$, the neighborhood of v on C' is a subpath of C'. Thus, by our choice of $L'(p^{in})$, since C is L-predictable, C' is also L'-predictable.

For any $C'' \in \mathcal{C} \setminus \{C\}$, Q separates C'' from each vertex of P^0_*, and, by definition of P^0_*, x, y have no neighbors in $V(\mathring{P}^0_*)$. Since $d_G(w, C) = 2$ and $pxu_\star yp'$ separates w from each vertex of P^0_*, there is no shortest $(w_\mathcal{T}(C''), C)$-path in G whose C-endpoint is an internal vertex of P^0_*. Thus, since $\mathcal{T}$ satisfies the distance conditions of Definition 2.1.6, and the rank of C has not increased, $\mathcal{T}'$ also satisfies the distance conditions of Definition 2.1.6.

We conclude that $\mathcal{T}'$ is a mosaic. Since $|V(P^0)| \geq 3$, $|V(P^1)| \geq 4$, and $S \neq \varnothing$, we have $|V(G')| < |V(G)|$. Thus, by the minimality of $\mathcal{T}$, G' admits an L'-coloring σ. Let σ^* be the restriction of σ to $V(G_Q^{\text{large}})$. By our construction of L', σ^* is an L-coloring of G_Q^{large}, and, by Claim 8.1.13, the union $\sigma^* \cup \phi$ is a proper L-coloring of $V(G_Q^{\text{large}} \cup P^0)$. By 2) of Claim 8.1.15, we have $|L_{\sigma^*}(u_\star)| = 2$, and, since $\sigma^* \cup \phi$ does not extend to an L-coloring of G, we have the following.

1) $\sigma^*(x) \in L_\phi(u_1)$ and $\sigma^*(y) \in L_\phi(u_{t-1})$; *AND*
2) $|L_{\sigma^*}(u_\star) \subseteq L_\phi(u_i)$ for each $i = 1, \cdots t-1$.

Recall that, by Claim 8.1.16, we have $|L_\phi(u_1) = \{r, s, \psi(x)\}$ and $L(u_\star) = \{r, s, \psi(x), \psi(w), \psi(y)\}$.

Claim 8.1.18. *Either* $\sigma(x) \neq \psi(x)$ *or* $\sigma(y) \neq \psi(y)$.

Proof: Suppose toward a contradiction that $\sigma(x) = \psi(x)$ and $\sigma(y) = \psi(y)$. If $\sigma(w) \neq \psi(w)$, then, since $|L_{\sigma*}(u_\star)| = 2$ and $L(u_\star) = \{r, s, \psi(x), \psi(w), \psi(y)\}$, the color $\psi(w)$ lies in $L_{\sigma^*}(u_\star)$. Since $L_\phi(u_1) = \{r, s, \psi(x)\}$, we have $\psi(w) not \in L_\phi(u_1)$, so, coloring $u_\star$ with $\psi(w)$, the union $\sigma^* \cup \phi$ extends to a proper L-coloring of the broken wheel $G_Q^{\text{small}} \setminus C$. Thus, G is L-colorable, contradicting the fact that $\mathcal{T}$ is critical. It follows that $\sigma(w) = \psi(w)$, so σ and ψ restrict to the same L-coloring of the path xwy. Yet, by our construction of L', σ uses the color d on each of q, q', contradicting the fact that σ is a proper L'-coloring of $V(G')$. ■

Applying Claim 8.1.18, suppose without loss of generality that $\sigma(x) \neq \psi(x)$. Since $\sigma(x) \in L_\phi(u_1)$, we have $\sigma^*(x) \in \{r, s\}$. Suppose without loss of generality that $\sigma^*(x) = r$. Thus, since $L_{\sigma^*}(u_\star) \subseteq L_\phi(u_1)$, we have $L_{\sigma^*}(u_\star) = \{s, \psi(x)\}$. Furthermore, by Claim 8.1.16, we have $L_\phi(u_{t-1}) = \{r, s, \psi(y)\}$, and these colors are distinct. Since $\psi(x) \neq \psi(y)$, there is a color of $L_{\sigma^*}(u_\star)$ not lying in $L_\phi(u_{t-1})$, so $\sigma^* \cup \phi$ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. □

To complete the proof of Theorem 8.0.4, we need analogues to Lemmas 8.1.4 and 8.1.10 in which we deal with vertices on the 1-necklace with lists of size less than three after we delete the precolored cycle. We obtain these lemmas and complete the proof of Theorem 8.0.4 in the remaining sections of Chapter 8.

8.2 Ruling Out the Remaining Chords

This section consists of the following lone result.

Lemma 8.2.1. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let $C \in \mathcal{C}$ be a closed ring, and let C^1 be the 1-necklace of C. Then C^1 is an induced cycle of G.*

Proof. Given a 3-chord Q of C, we say that Q is *defective* if $V(G_Q^{\text{small}} \cap C^1) \neq V(Q \cap C^1)$. Suppose toward a contradiction that C^1 is not an induced subgraph of G. Then G contains a defective 3-chord Q of C. Among all defective 3-chords of C, we choose Q so that the quantity $|V(G_Q^{\text{large}})|$ is minimized. Let $Q := xuu'y$, let $C_\dagger^0$ be the cycle $C \cap G_Q^{\text{large}} + Q$, and let $C_\dagger^1$ be the cycle $(C^1 \cap G_Q^{\text{large}}) + uu'$. Likewise, let $D_\dagger^0$ be the cycle $(C \cap G_Q^{\text{small}}) + Q$ and let $D_\dagger^1$ be the cycle $(C^1 \cap G_Q^{\text{small}}) + uu'$.

Let ϕ be the unique L-coloring of $V(C)$. By Lemma 8.1.4, there is an internal vertex $u_\star$ of the path $C^1 \cap G_Q^{\text{small}}$ such that $|L(u_\star)| < 2$. Since C is L-predictable and an induced subgraph of G, we have $|L(u_\star)| = 2$, and $u_\star$ is the unique vertex of C^1 with an L_ϕ-list of size less than three.

Claim 8.2.2. *$N(u) \cap V(C_\dagger^0) = \{x\}$ and $N(u') \cap V(C_\dagger^0) = \{y\}$. In particular, $C_\dagger^0$ is induced in G.*

<u>Proof:</u> Since each of $G[N(u) \cap V(C)]$ and $G[N(u') \cap V(C)]$ is a subpath of C, it is immediately from the minimality of Q that $N(u) \cap V(C_\dagger^0) = \{x\}$ and $N(u') \cap V(C_\dagger^0) = \{y\}$. Since C is an induced subgraph of G, it follows that $C_\dagger^0$ is also an induced subgraph of G. ■

We also have the following easy facts:

Claim 8.2.3. *$u_\star$ has at least one neighbor in $D_\dagger^0 \setminus (N(u) \cup N(u'))$. Furthermore, at least one of u, u' is not the endpoint of a chord of $D_\dagger^1$.*

<u>Proof:</u> Suppose that $u_\star$ has no neighbors in $D_\dagger^0 \setminus (N(y) \cup N(u'))$. since each of $G[Nu) \cap V(C)]$ and $G[N(u') \cap V(C)]$ is a subpath of C, u and u' have a common neighbor in $C \cap G_Q^{\text{small}}$, and G contains a 4-cycle which separates $u_\star$ from $G_Q^{\text{large}} \setminus Q$, contradicting the fact that $\mathcal{T}$ is a tessellation.

By Lemma 8.1.4, any chord of $(C \cap G_Q^{\text{small}}) + uu'$ with u as an endpoint separates u' from $u_\star$. Likewise, any chord of $(C \cap G_Q^{\text{small}}) + uu'$ with u' as an endpoint separates u from $u_\star$. Thus, if there is a chord of $(C \cap G_Q^{\text{small}}) + uu'$ with u as an endpoint, then there is no chord of $(C \cap G_Q^{\text{small}}) + uu'$ with u' as an endpoint, and vice-versa. ■

We now have the following:

Claim 8.2.4. *$C_\dagger^1$ is an induced subgraph of G. Furthermore, for each $w \in V(G_Q^{\text{large}}) \setminus V(C \cup C^1)$, the graph $G[N(w) \cap V(C_\dagger^1)]$ is a subpath of $C_\dagger^1$.*

<u>Proof:</u> Suppose toward a contradiction that G has a chord ww' of $C_\dagger^1$. Then ww' is also a chord of C^1, and, in $G \setminus C$, ww' separates $u_\star$ from each element of $\mathcal{C} \setminus \{C\}$, or else we contradict Lemma 8.1.4. Each of w, w' has a neighbor in $C \cap G_Q^{\text{large}}$, since each lies in C^1 and Q separates w, w' from each internal vertex of the path $D_\dagger^0 - uu'$. Thus, let $z, z' \in V(C \cap G_Q^{\text{large}}$ with $z \in N(w)$ and $z' \in N(w')$, and let $Q' := zww'z'$. Then $u_\star \in V(G_{Q'}^{\text{small}})$ and $G_Q^{\text{small}} \subseteq G_{Q'}^{\text{small}}$. Note that $ww' \neq uu'$, since uu' is not a chord of $C_\dagger^1$. Thus, we have $|V(G_Q^{\text{small}})| < |V(G_{Q'}^{\text{small}})|$ and $|V(G_Q^{\text{large}})| > |V(G_{Q'}^{\text{large}})|$, contradicting the minimality of Q.

Now let $w \in V(G_Q^{\text{large}}) \setminus V(C \cup C^1)$. If w is adjacent to at most one of u, u', then it immediately follows from Lemma 8.1.10 that $G[N(w) \cap V(C^1_\dagger)]$ is a subpath of C^1 and also a subpath of $C^1_\dagger$. Now suppose that w is adjacent to each of u, u'. Applying Lemma 8.1.10, the graph $C^1 \cap G[N(w)]$ consists of two disjoint subpaths of $C^1 \cap G_Q^{\text{large}}$, where one of these paths has u as an endpoint and the other has u' as an endpoint. Since $C^1_\dagger$ is an induced subgraph of G, the graph $G[N(w) \cap V(C^1_\dagger)]$ is a subpath of $C^1_\dagger$ containing the edge uu'. ■

We now note the following:

Claim 8.2.5. *For each $v \in \{u, u'\}$ there exist two elements $\psi_1, \psi_2 \in \Phi(\phi, G_Q^{\text{small}})$, where $\psi_1(v) \neq \psi_2(v)$.*

Proof: We first note that $G_Q^{\text{small}} \setminus C$ has a facial cycle $D^1_\dagger$ which contains every vertex of $G_Q^{\text{small}} \setminus C$ with an L_ϕ-list of size less than five. Furthermore, $u_\star$ is the lone vertex of this cycle with an L_ϕ-list of size less than three, and $|L_\phi(u_\star)| = 2$. Thus, it immediately follows from Theorem 1.3.4 that there exist $\psi_1, \psi_2 \in \Phi(\phi, G_Q^{\text{small}})$ with $\psi_1(v) \neq \psi_2(v)$. ■

We now have the following:

Claim 8.2.6. *If $u_\star$ is the only internal vertex of the path $D^1_\dagger - uu'$ with more than one neighbor in C, then $|N(u) \cap V(C)| > 1$ and $|N(u') \cap V(C)| > 1$.*

Proof: Suppose toward a contradiction that at least one of u, u' is adjacent to precisely one vertex of C, and thus suppose without loss of generality that $N(u) \cap V(C) = \{x\}$. Let G' be a graph obtained from G by deleting every vertex of $G_Q^{\text{small}} \setminus Q$ and replacing them with a lone vertex $v^\star$, where $v^\star$ is adjacent to each vertex of Q. Let C' be the cycle $(G_Q^{\text{large}} \cap C) + xv^\star y$. Now let a be a color in $L_\phi(u_\star)$ and let L' be a list-assignment for $V(G')$ where $L'(v^\star) = \{a\}$ and otherwise $L' = L$.

> **Subclaim 8.2.7.** *G' is short-separation-free, and furthermore, C' is an L'-predictable facial subgraph of G'.*
>
> Proof: Suppose toward a contradiction that G' is not short-separation-free. Since $C^1_\dagger$ is an induced subgraph of G', there is a vertex w of $G_Q^{\text{large}} \setminus Q$ with at least three neighbors in Q. Since Q is an induced subgraph of G and G is short-separation-free, it follows from our triangulation conditions that $G[N(w) \cap V(Q)]$ is a subpath of Q of length at least two, so suppose without loss of generality that w is adjacent to each of x, u, u'. Thus, $w \in V(C^1_\dagger) \setminus \{u, u'\}$. By Claim 8.2.4, $C^1_\dagger$ is an induced subgraph of G, so $C^1_\dagger = uu'w$, and G contains a triangle which separates C from each element of $\mathcal{C} \setminus \{C\}$, contradicting the fact that $\mathcal{T}$ is a tessellation. Thus, G' is indeed short-separation-free.
>
> Now we check that C' is L'-predictable. Let ϕ' be the unique L'-coloring of $V(C')$. We have $|L_\phi(u) \setminus \{a\}| \geq 3$, since $|N(u) \cap V(C)| = 1$. For each $w \in V(G' \setminus C')$, if $w \neq u, u'$, then $|L_{\phi'}(w) = L_\phi(w)$ and thus $|L_{\phi'}(w)| \geq 3$. Since $|L_{\phi'}(u)| \geq 3$ and $|L_{\phi'}(u')| \geq 2$, C' is indeed an L'-predictable facial subgraph of G'. ■

Let $\mathcal{T}' := (G', (\mathcal{C} \setminus \{C\}) \cup \{C'\}, L', C'_*)$. By our choice of $L'(v^\star)$, $V(C')$ is L'-colorable, and since G' is short-separation-free, $\mathcal{T}'$ is a tessellation in which C' is a closed ring. By Claim 8.2.3, $C \cap G_Q^{\text{small}}$ is a path of length at least two, so $|V(C')| \leq |V(C)|$. We claim now that $\mathcal{T}'$ is a mosaic. Since $|V(C')| \leq |V(C)|$, M0) is satisfied, and M1) is trivial. By Subclaim 8.2.7, C' is an L'-predictable facial subgraph of G', so M2) is satisfied as well.

Now, for each $C'' \in \mathcal{C} \setminus \{C\}$, there is no shortest $(w_{\mathcal{T}}(C''), C)$-path in G whose C-endpoint lies in $V(C \cap G_Q^{\text{small}} \setminus \{x, y\}$, or else, since Q separates C'' from $G_Q^{\text{small}} \setminus Q$, one of u, u' has a neighbor in $V(C^0_\dagger \setminus Q)$, contradicting Claim 8.2.2. Since $|V(C')| \leq |V(C)|$, we have $\text{Rk}(\mathcal{T}'|C') \leq \text{Rk}(\mathcal{T}|C)$. Since $w_{\mathcal{T}'}(C'') = w_{\mathcal{T}}(C'')$ for each $C'' \in \mathcal{C} \setminus \{C\}$, and $\mathcal{T}$ satisfies the distance conditions of Definition 2.1.6, it follows so $\mathcal{T}'$ is also satisfies the distance conditions of Definition 2.1.6.

We conclude that $\mathcal{T}'$ is a mosaic. Since $|V(G')| < |V(G)|$, G' admits an L'-coloring ψ. Let ψ' be the restriction of ψ to $G' \setminus \{v^\star\}$. Note that ψ' is an L-coloring of G_Q^{large}. We claim now that ψ' extends to an L-coloring of G. By Claim 8.2.2, $D_\dagger^0$ is an induced subgraph of G, so the union $\psi' \cup \phi$ is a proper L-coloring of the subgraph of G induced by $C \cup G_Q^{\text{large}}$. The graph $G_Q^{\text{small}} \setminus V(C \cup Q)$ has a face F which contains every vertex of $G_Q^{\text{small}} \setminus V(C \cup Q)$ with an $L_{\psi' \cup \phi}$ list of size less than five. If $G_Q^{\text{small}} \setminus V(C \cup Q)$ is $L_{\psi' \cup \phi}$-colorable, then ψ' extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. Thus, $G_Q^{\text{small}} \setminus V(C \cup Q)$ is not $L_{\psi' \cup \phi}$-colorable. Now consider the following cases:

Case 1: $u_\star$ is adjacent to at least one of u, u'

Suppose without loss of generality that $u_\star$ is adjacent to u. Since there is no chord of $D_\dagger^1$ with $u_\star$ as an endpoint, it follows that $u_\star u$ is an edge of $D_\dagger^1$, and since any chord of $D_\dagger^1$ separates $u_\star$ from at least one of u, u', there is no chord of $D_\dagger^1$ with u' is an endpoint.

Suppose first that both u, u' are adjacent to $u_\star$. In that case, $D_\dagger^1$ consists of the triangle $uu'u_\star$, and since G is short-separation-free, we have $G_Q^{\text{small}} \setminus C = uu'u_\star$. Since $\psi(v^\star) = a$, we have $\{\psi'(u), \psi'(u')\} \neq L_\phi(v^\star)$, so $\psi' \cup \phi$ extends to L-color the triangle $uu'u_\star$. Thus, $G_Q^{\text{small}} \setminus V(C \cup Q)$ is $L_{\psi' \cup \phi}$-colorable, which is false. The only remaining possibility in Case 1 is that u s adjacent to $u_\star$ and u' is not. Thus, we have $|L_{\psi' \cup \phi}(u_\star)| \geq 1$. Since there is no chord of $D_\dagger^1$ with u' is an endpoint, and $u_\star$ is the only internal vertex of the path $D_\dagger^1 - uu'$ with more than one neighbor in C, it follows that each vertex of $F \setminus \{u_\star\}$ has an $L_{\psi' \cup \phi}$-list of size at least three. Thus, by Theorem 0.2.3, $G_Q^{\text{small}} \setminus V(C \cup Q)$ is $L_{\psi' \cup \phi}$-colorable, which is false.

Case 2: $u_\star$ is adjacent to neither u nor u'

In this case, we have $|L_{\psi' \cup \phi}(u_\star)| \geq 2$. Since each internal vertex of $D_\dagger^1 - uu'$, except for $u_\star$, has an L_ϕ-list of size at least four, and u, u' have at most one common neighbor in $D_\dagger^1$, it follows that there is a vertex $v \in V(F) \setminus \{u_\star\}$ such that $|L_{\psi' \cup \phi}(v)| \geq 2$, and, for each $w \in V(F) \setminus \{v, u_\star\}$, $|L_{\psi' \cup \phi}(w)| \geq 3$. Thus, by Theorem 1.3.4, $G_Q^{\text{small}} \setminus V(C \cup Q)$ is $L_{\psi' \cup \phi}$-colorable, which is false. ■

We now have the following:

Claim 8.2.8. *Either* $|N(u) \cap V(C)| > 1$ *or* $|N(u') \cap V(C)| > 1$.

<ins>Proof:</ins> Suppose toward a contradiction that $N(u) \cap V(C) = \{x\}$ and $N(u') \cap V(C) = \{y\}$. Since at least one of u, u' is not the endpoint of a chord of $D_\dagger^1$, suppose without loss of generality that u is not the endpoint of any chord of $D_\dagger^1$. Let p be the unique neighbor of u on the path $C_\dagger^1 - uu'$ and let q be the unique neighbor of u on the path $D_\dagger^1 - uu'$. Since $p \neq u_\star$, we have $|L_\phi(p)| \geq 3$. Since $N(u) \cap V(C) = \{x\}$, we have $|L_\phi(u)| = 4$. Thus, there exists a $c \in L_\phi(u)$ such that $|L_\phi(p) \setminus \{c\}| \geq 3$,

Subclaim 8.2.9. *There exists a* $\psi \in \Phi(\phi, G_Q^{\text{small}})$ *such that* $\psi(u) = c$.

<ins>Proof:</ins> Let ϕ' be the extension of ϕ to $V(C) \cup \{u\}$ obtained by coloring u with c. Since $N(u') \cap V(C)| = 1$, we have $|L_\phi(u')| \geq 4$, and thus $|L_{\phi'}(u')| \geq 3$. Let F be the lone facial subgraph of $G_Q^{\text{small}} \setminus (V(C) \cup \{u\})$ containing all the vertices of $G_Q^{\text{small}} \setminus (V(C) \cup \{u\})$ with $L_{\phi'}$-lists of size less than five. Since there is no chord of $D_\dagger^1$ with u as an endpoint, each vertex of $F \setminus \{u_\star, q\}$ has an $L_{\phi'}$-list of size at least three. If $u_\star \neq q$, then each of $u_\star, q$ has an $L_{\phi'}$-list of size at least two, and thus ϕ' extends to an L-coloring of G_Q^{small} by Theorem 1.3.4. If $u_\star = q$, then

$|L_{\phi'}(u_\star)| \geq 1$, and thus, by Theorem 0.2.3, ϕ' extends to an L-coloring of G_Q^{small}. In any case, there exists such a ψ in $\Phi(\phi, G_Q^{\text{small}})$. ■

Now we let $\psi \in \Phi(\phi, G_Q^{\text{small}})$ with $\psi(u) = c$. Let L' be a list-assignment for $V(G_Q^{\text{large}})$ where $L'(v) = \{\psi(v)\}$ for each $v \in V(C_\dagger^0)$, and otherwise $L' = L$. Let C'_* be the outer face of G_Q^{large} and let $\mathcal{T}' := (G_Q^{\text{large}}, L', C'_*)$. Note that $\mathcal{T}'$ is a tessellation in which $C_\dagger^0$ is a closed $\mathcal{T}'$-ring. We claim now that $\mathcal{T}'$ is a mosaic. Firstly, by Claim 8.2.6, there is a vertex $w \in V(D_\dagger^1) \setminus \{u, u', u_\star\}$ with at least two neighbors in C. Since $G[N(w) \cap V(C)]$ and $G[N(u_\star) \cap V(C)]$ are paths which intersect at most on a common endpoint, and $G[N(u_\star) \cap V(C)]$ has length at least two, it follows that there $D_\dagger^0 \setminus \{x, y\}$ is a path of length at least two, so we have $|V(C')| < |V(C)|$.

Since $|V(C')| < |V(C)|$, $\mathcal{T}'$ satisfies M0), and M1) is trivially satisfied. Furthermore, the rank of C' has dropped by at least one (that is, $\text{Rk}(\mathcal{T}'|C') < \text{Rk}(\mathcal{T}|C)$, and thus, since $\mathcal{T}$ satisfies the distance conditions of Definition 2.1.6, $\mathcal{T}'$ does as well. Thus, we just need to check that C' is an L'-predictable facial subgraph of G'.

Since C is L-predictable in G and $u_\star \notin V(G_Q^{\text{large}})$, each neighbor of C' has an L_ψ-list of size at least three, except for the neighbors of u, u' on the cycle $C_\dagger^1$. By our choice of ψ, we have $|L_\psi(p)| \geq 3$. Let p' be the unique neighbor of u' on the path $C_\dagger^1 - uu'$. Note that $p \neq p'$, or else G contains a triangle which separates each element of $\mathcal{C} \setminus \{C\}$ from C, contradicting the fact that $\mathcal{T}$ is a tessellation. Furthermore, since $|L_\phi(p')| \geq 3$, we have $|L_\psi(p')| \geq 2$.

Let $v \in V(G_Q^{\text{large}} \setminus C')$. If $u, u' \notin N(v)$, then, since C is L-predictable in G, the graph $G[N(v) \cap V(C')]$ is a subpath of C'. If at least one of $u, u' \in N(v)$, then, since $C_\dagger^1$ is an induced subgraph of G, and $G[N(v) \cap V(C)]$ is a subpath of C, it again follows that $G[N(v) \cap V(C')]$ is a subpath of C'. Thus, C' is an L'-predictable facial subgraph of G_Q^{large}, and $\mathcal{T}'$ is indeed a mosaic. Since $|V(G_Q^{\text{large}})| < |V(G)|$, it follows from the minimality of $\mathcal{T}$ that G_Q^{large} admits an L'-coloring ψ'. Thus, $\psi \cup \psi'$ is an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. ■

We now have the following:

Claim 8.2.10. *Let $v \in \{u, u'\}$. If each vertex of $C_\dagger^1 - v$ has at least two neighbors in C, then $|V(C_\dagger^1)| < |V(C)|$.*

Proof: Suppose toward a contradiction that $|V(C_\dagger^1)| \geq |V(C)|$. Note that $C_\dagger^1 - v$ is a subpath of C^1. Let $R := G[D_1(C_\dagger^1 - v) \cap V(C)]$. Since C is L-predictable in G, and each vertex of $C_\dagger^1 - v$ has at least two neighbors in C, it follows that R is a subpath of C of length at least $|E(C_\dagger^1 - v)|$. Since $|E(C_\dagger^1)| = |E(C_\dagger^1 - v)| + 2$, it follows that $|E(C_\dagger^1)| \leq |E(R)| + 2$ and thus $|V(C)| \leq |E(R)| + 2$. Thus, we have $|V(C)| \leq |V(R)| + 1$.

By Claim 8.2.3, there is a neighbor p of $u_\star$ in $D_\dagger^0 \setminus (N(u) \cup N(u'))$, and since $D_\dagger^0 \setminus (N(u) \cup N(u'))$ is vertex-disjoint to R, p is the lone vertex of $C \setminus R$. If $|N(v) \cap V(C)| > 1$, then there is a neighbor of v lying in $C \setminus R$, and since $p \notin N(v)$, we contradict the fact that $V(C \setminus R) = \{p\}$. Thus, we have $|N(v) \cap V(C)| = 1$, and furthermore, since $\{p\} = V(C \setminus R)$, no internal vertex of the path $D_\dagger^1 - uu'$ other than $u_\star$, has more than one neighbor on C. Since $|N(v) \cap V(C)| = 1$, we contradict Claim 8.2.6. ■

We have an analogous result for a proper subpath of $C_\dagger^1 - uu'$, where we construct a new cycle in G by adjoining a path in C to a path $C_\dagger^1$:

Claim 8.2.11. *Let $v, v^* \in \{u, u'\}$ with $v \neq v^*$ and let x^* be the unique vertex of $N(v^*) \cap \{x, y\}$. Let P be a subpath of $C_\dagger^1 - uu'$ with v as an endpoint, where $V(P) \not\subseteq V(C_\dagger^1 - v^*)$. Let q be an endpoint of P where $q = v$ if $|V(P)| = 1$ and otherwise q is the other endpoint of P. Let z be the unique vertex of $N(q) \cap V(C)$ which is closest to x^* on the path $C_\dagger^0 - uu'$ and let P_0 be the unique subpath of $C_\dagger^0 - uu'$ with z, x^* as an endpoints. Finally, let C' be the cycle $vPqzP_0x^*v^*$. Then the following hold.*

1) C' is an induced subgraph of G, and, for each $w \in V(G) \setminus B_1(C')$, $G[N(w) \cap V(C')]$ is a subpath of C'; AND

2) If each vertex of $P - v$ has at least two neighbors in C, then $|V(C')| < |V(C)|$.

Proof: Suppose without loss of generality that $v = u$, so $v^* = u'$ and $x^* = y$. Since $V(P) \not\subseteq V(C^1_\dagger - u')$, there is a unique vertex p of $C^1_\dagger \setminus (V(P) \cup \{u'\})$ which is adjacent to q. We claim now that C' is an induced subgraph of C. To see this, we first note that $C^1_\dagger$ is an induced subgraph of G by Claim 8.2.4, and P_0 is an induced subgraph of G, since C is an induced subgraph of G'. Since C is L-predictable in G, and $q, p \neq u'$, there is no edge of G with one endpoint in P_0 and the other endpoint in $C' \setminus P_0$, except for yu' and qz. Thus C' is indeed an induced subgraph of G.

Now let $w \in V(G) \setminus B_1(C')$, if w has a neighbor in C', then $G'[N(w) \cap V(C')]$ is a subpath of C'. To see this, note that $G[N(w) \cap V(C)]$ is a subpath of C (possibly empty) since C is L-predictable in G, and, by Claim 8.2.4, $G'[N(w) \cap V(C^1_\dagger)]$ is a subpath of $C^1_\dagger$. Since $C^1_\dagger$ is an induced subgraph of G, $G'[N(w) \cap V(C')]$ is a subpath of C'.

Now suppose that each vertex of $P - v$ has at least two neighbors in C, and suppose toward a contradiction that $|V(C')| \geq |V(C)|$. Note that we have the disjoint union $V(C') = \{u\} \cup V(P) \cup V(P_0)$. Since C is an L-predictable and induced subgraph of G, the graph $G[V(C) \cap D_1(P)]$ is a subpath of C. Since each vertex of $P - v$ is adjacent to a subpath of C of length at least one, it follows that $|V(P)| \leq |D_1(P) \cap V(C)|$. Since $G[D_1(P) \cap V(C)]$ is a subpath of C which intersects with P_0 precisely on the point w, and $V(C') = \{u\} \cup V(P) \cup V(P_0)$, it follows that $|V(C')| \leq 1 + (|V(P_0)| + |D_1(P) \cap V(C)|) - 1$. Since $|V(C')| \geq |V(C)|$, we have $|V(C)| \leq (|V(P_0)| + |D_1(P) \cap V(C)|)$.

Now, $C \setminus (V(P_0) \cup D_1(P))$ is a subpath of $G^{\text{small}}_Q \cap C$. Since $|V(C)| \leq (|V(P_0)| + |D_1(P) \cap V(C)|)$ and the sum on the right counts the vertex w precisely twice, it follows that the path $C \setminus (V(P_0) \cup D_1(P))$ consists of at most one vertex. By Observation 8.2.3, this path consists of precisely one vertex, this vertex does not lie in $N(u')$, and $u_\star$ is the only vertex of $D^1_\dagger \setminus \{u, u'\}$ with more than one neighbor in C. Thus, since $N(u') \cap V(C^0_\dagger) = \{y\}$, we have $N(u') \cap V(C) = \{y\}$, contradicting Claim 8.2.6. ■

We now define a set $S_u \subseteq L_\phi(u)$ and a set $S_{u'} \subseteq L_\phi(u')$, where S_u is the set of colors used on u by elements of $\Phi(\phi G^{\text{small}}_Q)$, and $S_{u'}$ is the set of colors used on u' by elements of $\Phi(\phi G^{\text{small}}_Q)$. By Claim 8.2.5, we have $|S_u| \geq 2$ and $|S_{u'}| \geq 2$. We now define a subpath P_u of $C^1_\dagger - uu'$ in the following way: P_u is the unique maximal subpath of $C^1_\dagger - uu'$ such that u is an endpoint of P_u and P satisfies the property that, for each $v \in V(P_u - u)$, $S_u \subseteq L_\phi(v)$ and $|L_\phi(v)| = 3$. Likewise, we define a subpath $P'_{u'}$ of $C^1_\dagger - uu'$ in the following way: $P'_{u'}$ is the unique maximal subpath of $C^1_\dagger - uu'$ such that u' is an endpoint of $P'_{u'}$ and, for each $v \in V(P'_{u'} - u')$, $S_{u'} \subseteq L_\phi(v)$ and $|L_\phi(v)| = 3$. Note that each vertex of $P_u - u$ is adjacent to at least two vetices of C, and likewise for $P_{u'} - u'$.

Applying Claim 8.2.4, we define two subsets $\text{Ob}(u)$ and $\text{Ob}(u')$ of $V(G^{\text{large}}_Q) \setminus V(C \cup C^1)$ in the following way: Let $\text{Ob}(u)$ be the set of vertices $v \in V(G^{\text{large}}_Q) \setminus V(C \cup C^1)$ such that $G[N(v) \cap V(C^1)]$ is a subpath of $C^1_\dagger$ of length at least two with u as an endpoint. We define $\text{Ob}(u')$ analogously. Note that each of $\text{Ob}(u)$ and $\text{Ob}(u')$ has size at most two. Furthermore, at most one of these sets have size precisely two. To see this, suppose that $|\text{Ob}(u')| = 2$. Then there s a vertex $v \in V(G^{\text{large}}_Q) \setminus V(C \cup C^1)$ such that $G[N(v) \cap V(C^1)]$ is a subpath of C^1 with u as an internal vertex, and thus $\text{Ob}(u) = \varnothing$.

Claim 8.2.12. *If $|\text{Ob}(u')| \leq 1$, then $V(C^1_\dagger - u') \not\subseteq V(P_u)$. Likewise, if $\text{Ob}(u)| \leq 1$, then $V(C^1_\dagger - u) \not\subseteq V(P'_{u'})$.*

Proof: As the two claims are symmetric, suppose without loss of generality that $|\text{Ob}(u')| \leq 1$ and suppose toward a contradiction that $V(C^1_\dagger - u') \subseteq V(P_u)$. Let p' be the unique vertex of $C^1_\dagger - u$ which is adjacent to u'

Subclaim 8.2.13. *There exists a $\sigma \in \Phi(\phi, G_Q^{\text{small}} \cup C_Q^1)$ and a vertex $w \in V(G_Q^{\text{large}}) \setminus V(C \cup C^1)$ such that $|L_\sigma(w)| \geq 2$, and, for each $v \in V(G_Q^{\text{large}}) \setminus V(C \cup C^1)$, if $v \neq w$, then $|L_\sigma(v)| \geq 3$.*

Proof: We break this into two cases:

Case 1: $S_u \cap S_{u'} \neq \varnothing$

In this case, there exists a $\psi \in \Phi(\phi, G_Q^{\text{small}})$ such that ψ extends to an L-coloring of dom$(\psi) \cup V(C_\dagger^1 - p')$ obtained by 2-coloring of the path $C_\dagger^1 - p'$ with the colors of $\{\psi(u), \psi(u')\}$. Since $p' \neq u_\star$, we have $|L_\phi(p')| \geq 3$, so there is a color left over for p'. This is permissible as $C_\dagger^1$ is an induced subgraph of G. Let σ be the resulting extension of ψ to dom$(\psi) \cup V(C_\dagger^1)$. We claim that σ satisfies the desired properties. If there is a vertex $w \in V(G_Q^{\text{large}}) \setminus B_1(C)$ such that p' is an internal vertex of $G[N(w) \cap V(C_\dagger^1)$, then $|L_\sigma(w)| \geq 2$ and w is the lone vertex of $G_Q^{\text{large}} \setminus B_1(C)$ with an L_σ-list of size less than three, since, for any other vertex $v \in V(G_Q^{\text{large}}) \setminus B_1(C)$, the colors used by σ among the neighbors of v all lie in $\{\psi(u), \psi(u')\}$. On the other hand, if no such vertex exists, then, for any $w \in V(G_Q^{\text{large}}) \setminus B_1(C)$, if $|L_\sigma(w)| < 3$, we have $w \in$ Ob(u') and $|L_\sigma(w)| = 2$, so, again, we are done.

Case 2: $S_u \cap S_{u'} \neq \varnothing$

In this case, we simply choose a $\psi \in \Phi(\phi, G_Q^{\text{small}})$ and extend ϕ to an L-coloring of $C_\dagger^1$ by 2-coloring the path $C_\dagger^1 - u'$ using colors from S_u. Let σ be the resulting L-coloring of $V(G_Q^{\text{small}}) \cup V(C \cup C^1)$. For each $v \in V(G_Q^{\text{large}}) \setminus B_1(C)$, if $v \notin$ Ob(u'), then $|L_\sigma(v)| \geq 3$, and, if $v \in$ Ob(u'), then $|L_\sigma(v)| \geq 2$. ■

Let σ be as in the statement of Subclaim 8.2.13. We let $G' := G_Q^{\text{large}} \setminus C$ and let L' be a list-assignment for $V(G')$ defined as follows. For each $v \in V(C_\dagger^1)$, we set $L'(v) = \{\sigma(v)\}$, and, for each $v \in V(G' \setminus C_\dagger^1)$, we set $L'(v) = L(v)$. Let C'_* be the outer face of G' and let $\mathcal{T}' := (G', (\mathcal{C} \setminus \{C\}) \cup \{C_\dagger^1\}, L', C'_*)$. Note that $\mathcal{T}'$ is a tessellation in which C' is a closed ring. We claim now that $\mathcal{T}'$ is a mosaic.

Subclaim 8.2.14. $V(C_1^\dagger)| < |V(C)|$.

Proof: If $|N(u) \cap V(C)| > 1$, then, since $V(C_\dagger^1 - u') \subseteq V(P_u)$, it follows from the definition of P_u each vertex of $C_\dagger^1 - u'$ is adjacent to at least two vertices of C, so, by Claim 8.2.10, we have $|V(C_\dagger^1)| < |V(C)|$. Now suppose that $|N(u) \cap V(C)| = 1$. By Claim 8.2.8, we have $|N(u') \cap V(C)| > 1$. Since each vertex of $C_\dagger^1 - u'$ lies in P, it follows that each vertex of $C_\dagger^1 - u$ is adjacent to at least two vertices of C, so, again applying Claim 8.2.8, we have $|V(C_\dagger^1)| < |V(C)|$. ■

Since $|V(C_\dagger^1)| < |V(C)|$, it immediately follows that $\mathcal{T}'$ satisfies M0), and M1) is trivial. Combining Subclaim 8.2.13 with Claim 8.2.4, we immediately get that $C_\dagger^1$ is an L'-predictable facial subgraph of G', so M2) is satisfied as well. Since $|V(C_\dagger^1)| < |V(C)|$, we have Rk$(\mathcal{T}'|C_\dagger^1) <$ Rk$(C|\mathcal{T})$. Since $V(C_\dagger^1) \subseteq B_1(C, G)$ and $\mathcal{T}$ satisfies the distance conditions of Definition 2.1.6, it immediately follows that $\mathcal{T}'$ does as well. Thus, $\mathcal{T}'$ is indeed a mosaic. Since $|V(G')| < |V(G)|$, it follows from the minimality of $\mathcal{T}$ that G' admits an L'-coloring, so σ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. ■

We now have the following:

Claim 8.2.15. *Let $v, v^* \in \{u, u'\}$ with $|\text{Ob}(v^*)| \leq 1$ and $v \neq v^*$. Then the following hold.*

1) *There is a vertex $w \in V(G_Q^{\text{large}} \setminus B_1(C))$ which is adjacent to v^* and has at least two neighbors in P_v; AND*
2) *The path $C_\dagger^1 \setminus P_v$ consists of precisely one edge v^*p, where $|N(p) \cap V(C)| \geq 2$.*

Proof: Suppose without loss of generality that $v^* = u'$ and $v = u$, and suppose toward a contradiction that at least one of the two conditions is not satisfied. Applying Claim 8.2.12, we have $V(P) \not\subseteq V(C^1_\dagger - u')$. Let $P_u := u_0u_1 \cdots u_t$, where $u_0 = u$. Let p be the unique vertex which does not lie in P_u and is adjacent to u_t on the path $C^1_\dagger - uu'$. Since $V(P_u) \not\subseteq V(C^1_\dagger - u')$, the subpath of $C^1_\dagger - uu'$ with endpoints u_t, u' has length at least two. Furthermore, there is at most one vertex $w \in V(G^{\text{large}}_Q \setminus B_1(C))$ such that $w \in N(u')$ and w is adjacent to at least two vertices of P_u. If such a w exists and the path $G[N(w) \cap V(C^1_\dagger)]$ has u' as an internal vertex, then this is immediate, since u' has degree three in $G \setminus C$, and if u' is an endpoint of $G[N(w) \cap V(C^1_\dagger)]$, then it follows from our assumption that w is the only vertex of $G^{\text{large}}_Q \setminus B_1(C)$ with at least two neighbors in P_u.

Subclaim 8.2.16. *There is an L-coloring $\sigma \in \Phi(\phi, G^{\text{small}}_Q \cup P_u)$ such that the following hold.*

1) *σ 2-colors the elements of $V(P_u)$ with colors of S_u; AND*
2) *If there exists a $w \in N(u')$ with at least two neighbors in P, then $|L_\sigma(p)| \geq 3$.*

Proof: By definition of P_u, there exists a 2-coloring of the path $u_0 \cdots u_t$ with two colors from S_u, where either the color used on u_t does not lie in p, or $|L_\phi(p)| > 3$. Since $C^1_\dagger$ is a chordless cycle, we have $|L_\sigma(p)| \geq 2$. Furthermore, if there exists a $w \in N(u')$ with at least two neighbors in P, then, by assumption, either p is not adjacent to u', or, if p is adjacent to u', then $|L_\phi(v^*)| = 4$. In either case, we have $|L_\sigma(p)| \geq 3$. ■

We now fix a $\sigma \in \Phi(\phi, G^{\text{small}}_Q \cup P)$ satisfying Subclaim 8.2.16. Since C is L-predictable in G, the graph $G[N(u_t) \cap V(C \cap G^{\text{large}}_Q)]$ is a subpath of $C \cap G^{\text{large}}_Q$. Let z be the vertex of $G[N(u_t) \cap V(C \cap G^{\text{large}}_Q)]$ which is closest to y on this path. Let P_0 be the unique subpath of $C \cap G^{\text{large}}_Q$ with endpoints z, y, and let C' be the cycle $u_0 \cdots u_t z P_0 y u' u_0$. By Claim 8.2.11, C' is an induced subgraph of G and $|V(C')| < |V(C)|$. Let L' be a list-assignment for $V(G')$ in which $V(C')$ is precolored by σ and otherwise $L' = L$. Let $\mathcal{T}' := (G', L', (\mathcal{C} \setminus \{C\}) \cup \{C'\}, C'_*)$. Note that $\mathcal{T}'$ is a tessellation in which C' is a closed ring.

Subclaim 8.2.17. *C' is an L'-predictable facial subgraph of G'.*

Proof: It immediately follows from Claim 8.2.11 that, for each $w \in V(G') \setminus B_1(C')$, $G'[N(w) \cap V(C')]$ is a subpath of C'. Since the path P_u is 2-colored by σ, it follows from Subclaim 8.2.16 that, if $|L_\sigma(p)| < 3$, then $|L_\sigma(p)| = 2$ and p is the lone vertex of $G' \setminus B_1(C')$ with an L_σ-list of size less than three, and furthermore, if $|L_\sigma(p)| \geq 3$, then any vertex of $G' \setminus B_1(C')$ with an L_σ-list of size less than three is adjacent to u' and to at least two vertices of P_u. In the latter case, by assumption, there is precisely one vertex of $G' \setminus B_1(C')$ with an L_σ-list of size less than three, and this vertex has an L_σ-list of size two. In either case, C' is L'-predictable in G'. ■

As $|V(C')| < |V(C)|$, it follows that $\mathcal{T}'$ satisfies M0), and that the rank of C' has dropped by at least one, i.e $\text{Rk}(\mathcal{T}'|C') < \text{Rk}(\mathcal{T}|C)$. Thus, since $\mathcal{T}$ satisfies the distance conditions of Definition 2.1.6, $\mathcal{T}'$ does as well., and M1) is trivially satisfied. Since C' is an L'-predictable facial subgraph of G', M2) is satisfied as well. We conclude that $\mathcal{T}'$ is a mosaic. Since $|V(G^{\text{small}}_Q \setminus Q)| > 0$, we have $|V(G')| < |V(G)|$. By the minimality of $\mathcal{T}$, G' admits an L'-coloring, and thus σ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. ■

We now have the following:

Claim 8.2.18. $S_u \cap S_{u'} = \varnothing$.

Proof: Suppose toward a contradiction that $S_u \cap S_{u'} \neq \varnothing$. Since at most one of $\text{Ob}(u)$, $\text{Ob}(u')$ has size two, suppose without loss of generality that $|\text{Ob}(u')| \leq 1$. By Claim 8.2.12, $V(C^1_\dagger - u') \not\subseteq V(P_u)$. Since $S_u \cap S_{u'} \neq \varnothing$, there is a $\psi \in \Phi(\phi, G^{\text{small}}_Q)$ which extends to an L-coloring σ of $\text{dom}(\psi) \cup V(P)$, where σ colors the vertices of the path $P_u + uu'$ with the two colors $\{\psi(u), \psi(u')\}$. This is permissible since $C^1_\dagger$ is an induced subgraph of G and $V(P + uu') \neq V(C^1_\dagger)$. By Claim 8.2.15, there is a vertex p of $C^1_\dagger$ adjacent to each endpoint of $P + uu'$, where $V(C^1_\dagger) = V(P + uu') \cup \{p\}$. Let z be the unique vertex of the path $N(u_t) \cap V(C)$ which is closest to y, and let P_0 be the subpath of $C^0_\dagger - uu'$ with z, y as endpoints. Since $|V(C^1_\dagger)| > 4$ and $P_u + uu'$ consists all but a lone vertex of $C^1_\dagger$, P_u has length at least two, so let q be the non-u endpoint of P_u, and let $C' := uP_uqzP_0yu'u$.

Let $G' := G^{\text{large}}_Q \setminus (C \setminus P_0)$ and let C'_* be the outer face of G'. Let L' be a list-assignment for $V(G')$ where $L'(v) = \{\sigma(v)\}$ for all $v \in V(C')$, and otherwise $L' = L$. Let $\mathcal{T}' := (G', \mathcal{C} \setminus \{C\}) \cup \{C'\}, L', C'_*)$. Note that $\mathcal{T}'$ is a tessellation in which C' is a closed ring precolored by σ. It immediately follows from Claim 8.2.11 that C' is an induced subgraph of G', and for each $w \in V(G' \setminus C')$, the graph $G'[N(w) \cap V(C')]$ is a subpath of C'. By definition of P, we have $|L_\sigma(q)| \geq 2$, since either $|L_\phi(q)| > 3$ or $\{\sigma(u), \sigma(u')\} \not\subseteq L_\phi(q)$. Furthermore, for any $w \in V(G' \setminus C')$, if $w \neq q$, then $w \notin B_1(C)$, and thus we have $|L_\sigma(w)| \geq 3$, since σ only uses two colors among the neighbors of w. We conclude that C' is an L'-predictable facial subgraph of G'.

We claim that $\mathcal{T}'$ is a mosaic. By Claim 8.2.11, we have $|V(C')| < |V(C)|$, so it immediately follows that $\mathcal{T}'$ satisfies M0), and M1) is trivially satisfied. Since C' is L'-predictable in G', M2) is satisfied as well. Since $|V(C')| < |V(C)|$, we have $\text{Rk}(\mathcal{T}'|C') < \text{Rk}(\mathcal{T}|C)$. Since $V(C') \subseteq B_1(C)$, it follows that $\mathcal{T}'$ also satisfies the distance conditions of Ddefinition 2.1.6. Thus, $\mathcal{T}'$ is a mosaic. Since $|V(G')| < |V(G)|$, it follows from the minimality of $\mathcal{T}$ that G' admits an L'-coloring, so σ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. ■

Since at most one of $\text{Ob}(u)$, $\text{Ob}(u')$ has size two, suppose now without loss of generality that $\text{Ob}(u')| \leq 1$. Let p be the unique vertex of $C^1_\dagger - u$ adjacent to u'. By Claim 8.2.15 $P_u = C^1_\dagger - \{u', p\}$. Note that $P_{u'}$ is a subpath of $u'p$, or else, since $|V(C^1_\dagger)| > 4$, there is a vertex $q \in V(P_u \cap P_{u'}) \setminus \{u, u'\}$, and thus $|L_\phi(q)| = 3$ and $S_u \cup S_{u'} \subseteq L_{phi}(q)$. Since $|S_u| \geq 2$ and $|S_{u'}| \geq 2$, this contradicts Claim 8.2.18.

Claim 8.2.19. $|\text{Ob}(u)| = 2$.

Proof: Suppose that $\text{Ob}(u)| < 2$. Applying Claim 8.2.15 again, every vertex of $C^1_\dagger$, except for the three vertices in the 2-path in $C^1_\dagger$ with u as a midpoint, lies in $P_{u'} - u'$. Since $|V(C^1_\dagger)| > 4$, this contradicts the fact that $P_{u'}$ is a subpath of $u'p$. Thus, $\text{Ob}(u)| = 2$. ■

Since $\text{Ob}(u)| = 2$, there exist two vertices $w, w^* \in V(G^{\text{large}}_Q) \setminus B_1(C)$ such that $G[N(w) \cap V(C^1_\dagger)]$ and $G[N(w^*) \cap V(C^1_\dagger)]$ are subpaths of $C^1_\dagger$, each of length at least two, which intersect precisely on u. Thus, precisely one of these two paths, say $G[N(w^*) \cap V(C^1_\dagger)]$ for the sake of definiteness, contains u' as an internal vertex. In particular, we have $\text{Ob}(u') = \varnothing$. Let $C^1_\dagger = u_0u_1 \cdots u_tpu'$, where $u_0 = 0$ and $u_0u_1 \cdots u_t = P_u$. Since w^* is the unique vertex of $G^{\text{large}}_Q \setminus B_1(C)$ adjacent to u', it follows from Claim 8.2.15 that w^* has at least two neighbors in P_u, so p is also an internal vertex of $G[N(w^*) \cap V(C^1_\dagger)]$.

Claim 8.2.20. $|V(C^1_\dagger)| < |V(C)|$.

Proof: Since p is an internal vertex of $G[N(w^*) \cap V(C^1_\dagger)]$, it follows that $|N(p) \cap V(C)| > 1$, or else G contains a copy of $K_{2,3}$, contradicting the fact that G is short-separation-free. Thus, by definition of P_u, each vertex of $C^1_\dagger - \{u, u'\}$ is adjacent to at least two vertices of C. By Claim 8.2.8, at least one of u, u' is also adjacent to more than one vertex of C, so it immediately follows from Claim 8.2.10 that $|V(C^1_\dagger)| < |V(C)|$. ■

Now we have enough to finish the proof of Lemma 8.2.1. We define a $\sigma \in \Phi(\phi, G^{\text{small}}_Q \setminus \{p\})$ in the following way. By definition of P_u, there is an L-coloring $\psi \in \Phi(\phi, G^{\text{small}}_Q)$ such that ψ admits an extension σ to $\text{dom}(\psi) \cup V(P)$, where $|L_\phi(p) \setminus \{\sigma(u_t)\}| \geq 3$ and σ colors P_u with two colors from S_u. Let L' be a list-assignment for $V(G_Q^{\text{large}} \setminus C)$ defined as follows. We set $L'(q)$ to be a lone color not lying in $\{\sigma(v) : v \in N(w) \cap \text{dom}(\sigma)\} \cup L(w^*)$. For each $v \in V(C_\dagger^1 - q)$, we set $L'(v) = \{\sigma(v)\}$, and, otherwise we set $L' = L$.

Let C'_* be the outer face of $G_Q^{\text{large}} \setminus C$ and let $\mathcal{T}' := (G_Q^{\text{large}} \setminus C, (\mathcal{C} \setminus \{C\}) \cup \{C_\dagger^1\}, L', C'_*)$. Note that $\mathcal{T}'$ is a tessellation, where $C_\dagger^1$ is L'-precolored. We claim that $\mathcal{T}'$ is a mosaic. Since $|V(C_\dagger^1)| < |V(C)|$, it immediately follows that $\mathcal{T}'$ satisfies M0), and M1) is trivially satisfied. Let σ' be the unique L'-coloring of $V(C_\dagger^1)$. By our choice of $L'(p)$, we have $|L_{\sigma'}(w^*)| \geq 2$. For each $v \in V(G') \setminus V(C_\dagger^1)$, we have $|L_\sigma(v)| \geq 3$, since only at most two colors are used by σ' among the neighbors of v. Applying Claim 8.2.4, it follows that $C_\dagger^1$ is an L'-predictable facial subgraph of $G_Q^{\text{large}} \setminus C$, so $\mathcal{T}'$ satisfies M2) as well. Since $|V(C_\dagger^1)| < |V(C)|$, we have $\text{Rk}(\mathcal{T}'|C_\dagger^1) < \text{Rk}(\mathcal{T}|C)$, and thus, since $\mathcal{T}$ satisfies the distance conditions of Definition 2.1.6, $\mathcal{T}'$ does as well.

Thus, $\mathcal{T}'$ is a mosaic. Since $|V(G_Q^{\text{large}} \setminus C)| < |V(G)|$, it follows from the minimality of $\mathcal{T}$ that $G_Q^{\text{large}} \setminus C$ admits an L'-coloring τ. Let τ' be the restriction of τ to $\text{dom}(\tau) \setminus \{q\}$. By definition of L', τ' is an L-coloring of its domain, and, by our choice of σ, the union $\tau' \cup \phi$ is an L-coloring of $G - \{p\}$. Furthermore, by our choice of precoloring σ for $C_\dagger^1 - p$, there is a color left over in $L_\phi(p) \setminus \{\tau(w^*), \tau(u'), \tau(u_t)\}$, so $\tau' \cup \phi$ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. This completes the proof of Lemma 8.2.1. $\square$

8.3 A Box Lemma for Pairs of 2-Paths

In Section 8.5, we complete the proof of Theorem 8.0.4 by dealing with the 4-chords of a closed ring in a critical mosaic which are not dealt with by Lemma 8.1.10. In order to prove the lone result of Section 8.5, we first prove two intermediate results, the first of which is the content of this section, and the second of which is the content of Section 8.4. The lone result of this section is a "box lemma" deals with the case of a pair of 2-chords of the 1-necklace of a closed ring a critical mosaic, where this pair of 2-chords encloses a region consisting only of 5-lists (i.e the 2-chords are two sides of a box which otherwise consists of edges of the 1-necklace, hence the name), and it is stated in purely general terms, i.e it is not a statement about critical mosaics. We begin with the following

Definition 8.3.1. Given a short-separation-free planar graph H, a tuple $\langle D, z, z^*, L\rangle$ is called an *H-box* if D is a cyclic facial subgraph of H, z, z^* are distinct vertices of D, L is a list-assignment for $V(H)$, and the following hold.

1) $zz^* \notin E(D)$ and there is no chord of D with z^* as an endpoint, except possibly zz^*; *AND*
2) There is no chord of D which separates z from z^**; AND*
3) $|L(v)| \geq 3$ for all $v \in V(D) \setminus \{z, z^*\}$ and $|L(v)| \geq 5$ for all $v \in \{z, z^*\} \cup V(H \setminus D)$.

We introduce one more definition and then state and prove the lone result of this section.

Definition 8.3.2. Let H be a short-separation-free planar graph and let $\langle D, z, z^*, L\rangle$ be an H-box. Let y, y' be the two neighbors of z on D, and let u, u' be the two neighbors of z^* on D (possibly $\{u, u'\} \cap \{y, y'\} \neq \varnothing$). A $\langle D, z, z^*, L\rangle$-*corner coloring* is an L-coloring σ of $\{u, u', y, y'\}$ such that, for any $c \in L(z^*) \setminus \{\sigma(u), \sigma(u')\}$, σ extends to an L-coloring of H using c on z^*.

Lemma 8.3.3. *(Box Lemma) Let H be a short-separation-free planar graph and let $\langle D, z, z^*, L\rangle$ be an H-box. Let y, y' be the two neighbors of z on D. Then any L-coloring of $\{y, y'\}$ extends to $\langle D, z, z^*, L\rangle$-corner coloring.*

Proof. Suppose there is a short-separation-free H and an H-box for which this does not hold, and choose H to be vertex-minimal with respect to this property By assumption, there is an H-box $\langle D, z, z^*, L\rangle$, such that, letting y, y' be the neighbors of z on D, there is an L-coloring ψ of $\{y, y'\}$ which does not extend to a $\langle D, z, z^*, L\rangle$-corner coloring. Let u, u' be the neighbors of z^* on D. Let P, P' be the two connected components of of $D - \{z, z^*\}$ and suppose without loss of generality that P has endpoints u, y and P' has endpoints u', y', as shown in Figure 8.3.1.

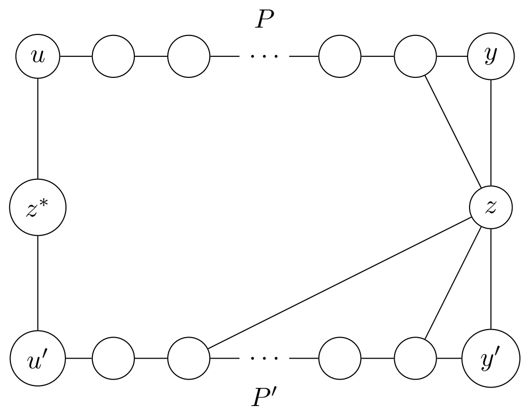

Figure 8.3.1: A box between P and P'

Claim 8.3.4. *There is no 2-chord of D which separates z from z^**

Proof: Let $\mathcal{P}$ be the set of 2-chords of D which separate z from z^*, and suppose toward a contradiction that $\mathcal{P} \neq \varnothing$. For each $Q \in \mathcal{P}$, we let $H = H_Q^{\text{left}} \cup H_Q^{\text{right}}$ be the natural Q-partition of H, where $z \in V(H_Q^{\text{right}})$ and $z^* \in V(H_Q^{\text{left}})$. Among all the elements of $\mathcal{P}$, we choose Q so that $|V(H_Q^{\text{right}})|$ is minimized. Precisely one endpoint of Q lies in P and the other endpoint lies in P', so let $Q := vwv'$, where $v \in V(P)$ and $v' \in V(P')$.

Let D^{right} be the cycle $wvPyzy'P'v'w$ and let D^{left} be the cycle $wvPuz^*u'P'v'w$. Now, since H has no chord of D which separates z from z^*, it follows that H_Q^{right} has no chord of D^{right} which separates z from w, and likewise, H_Q^{left} has no chord of D^{left} which separates z^* from w. Thus, $\langle D^{\text{left}}, w, z^*, L\rangle$ is an H_Q^{left}-box. By the minimality of Q, there is no chord of D^{right} in H_Q^{right} which has w as an endpoint, except possibly wz. Thus, $\langle D^{\text{right}}, z, w, L\rangle$ is an H_Q^{right}-box.

Since H is a minimal counterexample and $|V(H_Q^{\text{right}})| < |V(H)|$, ψ extends to a $\langle D^{\text{right}}, z, w, L\rangle$-corner coloring ψ^* of $\{v, v', y, y'\}$. Since $|V(H_Q^{\text{left}})| < |V(H)|$ and $\langle D^{\text{left}}, w, z^*, L\rangle$ is an H_Q^{left}-box, there is an $\langle D^{\text{left}}, w, z^*, L\rangle$-corner coloring σ^* of $\{u, u', v, v'\}$ which uses $\psi^*(v), \psi^*(v')$ on the respective vertices v, v'.

Let $\psi^\dagger$ be the extension of ψ to $\{u, u', y, y'\}$ obtained by coloring u, u' with the respective colors $\sigma^*(u), \sigma^*(u')$, and let $c \in L(z^*) \setminus \{\sigma^*(u), \sigma^*(u')\}$. Since σ^* is a $\langle D^{\text{left}}, w, z^*, L\rangle$-corner coloring, there is an extension of σ^* to an L-coloring τ of H_Q^{left} using c on z^*. Since ψ^* is a $\langle D^{\text{right}}, z, w, L\rangle$-corner coloring of $\{v, v', y, y'\}$ and $\tau(w) \in L(w) \setminus \{\psi^*(v), \psi^*(v')\}$, τ extends to an L-coloring of H_Q^{right} using $\psi(y), \psi(y')$ on the respective vertices y, y'. Thus, $\psi^\dagger$ is an extension of ψ to a $\langle D, z, z^*, L\rangle$-corner coloring, contradicting our choice of ψ. ■

Now consider the following cases:

Case 1: $zz^* \notin E(H)$

In this case, we apply the work of Section 1.7. Since each vertex of $P \cup P'$ has an L-list of size at least three, it follows from Theorem 1.7.5 that there is a $\sigma \in \text{Link}_L(P, D, H)$ and a $\sigma' \in \text{Link}_L(P', D, H)$, where σ uses $\psi(y)$

on y and σ' uses $\psi'(y')$ on y'. By definition, there is no chord of D with one endpoint in P and one endpoint in P', so $\sigma^\dagger := \sigma \cup \sigma'$ is proper L-coloring of its domain. Let ψ^* be the extension of ψ to an L-coloring of $\{y, y,', u, u'\}$ obtained by coloring u with $\sigma(u)$ and coloring u' with $\sigma(u')$. Note that we indeed have $u \in \text{dom}(\sigma)$ and $u' \in \text{dom}(\sigma')$ by definition, since u is an endpoint of P and u' is an endpoint of P'.

By definition of the sets $\text{Link}_L(P, D, H)$ and $\text{Link}_L(P, D, H)$, we have $|N(z) \cap \text{dom}(\sigma)| \leq 2$ and $|N(z) \cap \text{dom}(\sigma')| \leq 2$, so $L_{\sigma^\dagger}(z)| \geq 1$. By Claim 8.3.4, there is no vertex of $H \setminus D$ with one neighbor in P and one neighbor in P'. In the language of Definition 1.7.3, the vertex z is $D - z^*$-hinge for D, and since $|L_{\sigma^\dagger}(z)| \geq 1$, it follows that $\sigma^\dagger$ extends to an element τ of $\text{Link}_L(D - z^*, D, H)$.

Now, let $c \in L(z^*) \setminus \{\psi^*(u), \psi^*(u')\}$. Since $zz^* \notin E(D)$, there is no chord of D with z^* as an endpoint, so $c \in L_\tau(z^*)$. By 3) of Theorem 1.7.3, τ extends to an L-coloring of H using c on z^*, so ψ^* extends to an L-coloring of H using c on z^*, and ψ^* is an extension of ψ to a $\langle D, z, z^*, L \rangle$-corner coloring, contradicting our assumption. This rules out Case 1.

Case 2: $zz^* \in E(H)$

In this case, we apply the work of Section 1.6. Let $H = H^\uparrow \cup H^\downarrow$ be the natural zz^*-partition of H, where $P \subseteq H^\uparrow$ and $P' \subseteq H^\downarrow$. Let $D^\uparrow$ be the cycle $P + uz^*zy$ and let $D^\downarrow$ be the cycle $P' + u'z^*zy'$.

Claim 8.3.5. *There exists an L-coloring $\psi^\downarrow$ of $\{u', y'\}$ which uses $\psi(u')$ on u', such that any extension of $\psi^\downarrow$ to an L-coloring of $\{u', z^*, z, y'\}$ also extends to L-coloring $H^\downarrow$. Likewise, there exists an L-coloring $\psi^\uparrow$ of $\{u', y'\}$ which uses $\psi(u)$ on u, such that any extension of $\psi^\uparrow$ to an L-coloring of $\{u, z^*, z, y\}$ also extends to L-coloring $H^\uparrow$.*

<u>Proof:</u> These two statements are symmetric so it just suffices to prove that the first one holds. If $u' = y'$, then $D^\downarrow$ is a triangle and the claim follows immediately from Corollary 0.2.4. Now suppose that $u' \neq y'$. Thus, $u'z^*zy'$ is a proper subpath of $D^\downarrow$ of length three. By assumption, z^* has no neighbors in $D^\downarrow$ except for u', z. By Theorem 1.6.1, there is a $d \in L(u')$, where $d \neq \psi(y')$ if $u'y' \in E(H^\downarrow)$, such that any L-coloring of of $\{u', z^*, z, u'\}$ which uses $d, \psi(y')$ on the respective vertices u', y' also extends to L-color all of $D^\downarrow$, so we have our desired L-coloring of $\{u', y'\}$. ■

Let $\psi^\downarrow, \psi^\uparrow$ be as in Claim 8.3.5, the union $\psi^\downarrow \cup \psi^\uparrow$ is a $\langle D, z, z^*, L \rangle$-corner coloring and an extension of ψ, contradicting our assumption. This completes the proof of Lemma 8.3.3. □

8.4 An Improved Coloring Result for 4-Chords of Closed Rings

We use this lemma both in the remainder of Section 8.4 and in Section 8.5, The second of the two lemmas of this section is a coloring result for one side of a 4-chord of a closed ring in a a critical mosaic, where this lemma strengthens Lemma 8.1.6 under some additional conditions.

Lemma 8.4.1. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let $C \in \mathcal{C}$ be a closed ring, let C^1 be the 1-necklace of C, and let $\tilde{G} := G \setminus C$. Let ϕ be the unique L-coloring of $V(C)$, and let Q be a 4-chord of C such that the middle vertex of Q lies in $D_2(C)$ and there is an internal vertex of the path $C \cap G_Q^{\text{small}}$ with an L_ϕ-list of size two. Then, letting y, y' be the endpoints of $Q \setminus C$, then the following hold.*

A) $C^1 \cap G_Q^{\text{large}}$ is a path of length at least three and G_Q^{large} is an induced subgraph of G; AND

B) If $|V(C \cap G_Q^{\text{small}})| > 5$, then any L_ϕ-coloring of $\{y, y'\}$ extends to an L-coloring of the subgraph of G induced by $V(G_Q^{\text{large}})$.

Proof. We first prove A), which is the easier and shorter of the two results. Let $Q := xyzy'x'$, where $z \in D_2(C)$. Firstly, if $C^1 \cap G_Q^{\text{large}}$ has length less than three, then $(C^1 \cap G_Q^{\text{large}}) + yzy'$ is a cycle of length at most four which separates C from each element of $\mathcal{C} \setminus \{C\}$, contradicting the fact that $\mathcal{T}$ is a tessellation. Suppose now that G_Q^{large} is not an induced subgraph of G. Since $z \in D_2(C)$, G_Q^{small} does not contain the edges zx, zx', so $E(G_Q^{\text{small}})$ contains one of the edges $yy', yx', y'x$. By Lemma 8.2.1, C^1 is an induced subgraph of G, and, by assumption, the path $C^1 \cap G_Q^{\text{small}}$ has at least one internal vertex, so $yy' \notin E(G_Q^{\text{small}})$ and one of $yx', y'x$ lies in $E(G_Q^{\text{small}})$. Suppose without loss of generality that $yx' \in E(G_Q^{\text{small}})$. Thus, G contains the 4-cycle $yzy'x'$, and, by since $\mathcal{T}$ is a tessellation, it follows from our triangulation conditions that either zy' or yy' lies in $E(G_Q^{\text{small}})$, both of which have already been ruled out. This proves A) of Lemma 8.4.1. Now we prove B).

Definition 8.4.2. Given a 4-chord Q of C, we say that Q is *defective* if all of the following hold:

1) The middle vertex of Q lies in $D_2(C)$ and there is an internal vertex of the path $C \cap G_Q^{\text{small}}$ with an L_ϕ-list of size two*; AND*
2) $|V(C \cap G_Q^{\text{small}})| > 5$*; AND*
3) Letting y, y' be the endpoints of $Q \setminus C$, there exists an L_ϕ-coloring of $\{y, y'\}$ which does not extend to an L-coloring of $V(G_Q^{\text{large}})$.

It suffices to prove that there are no defective 4-chords of C. Suppose toward a contradiction that there is a defective 4-chord Q of C. Among all defective 4-chords of C, we choose Q so that $|V(G_Q^{\text{small}})$ is maximized. Let $Q := xyzy'x'$, let $P^0 := C \cap G_Q^{\text{large}}$, and let $P^1 := C^1 \cap G_Q^{\text{large}}$. Since Q is defective, let ψ be an L_ϕ-coloring of $\{y, y'\}$ which does not extend to an L-coloring of $V(G_Q^{\text{large}})$. By 1), G_Q^{large} is an induced subgraph of G, so ψ does not extend to an L-coloring of G_Q^{large}. By assumption, there is an internal vertex of the path $C^1 \cap G_Q^{\text{small}}$ with an L_ϕ-list of size two. Since C is induced in G and L-predictable, every vertex of P^1 has an L_ϕ-list of size at least three. Note that $P^1 + yzy'$ is a cyclic facial subgraph of $\tilde{G}_{yzy'}^{\text{large}}$.

Claim 8.4.3.

1) $yy' \notin E(G_Q^{\text{large}})$ *and* y, y' *have no common neighbor in* G_Q^{large} *other than z; AND*

2) $N(y) \cap V(P^0) = \{x\}$ *and* $N(y') \cap V(P^0) = \{x'\}$.

Proof: By Lemma 8.2.1, C^1 is induced in G, and, by A), $|E(P^1)| \geq 3$, so $yy' \notin E(G_Q^{\text{large}})$. Suppose toward a contradiction that there is a $w \in V(G_Q^{\text{large}})$ with $w \neq z$ such that w is adjacent to each of y, y'. Since G is short-separation-free and $yy' \notin E(G_Q^{\text{large}})$, it follows from our triangulation conditions that $wz \in E(G_Q^{\text{large}})$. If $w \in V(C^1)$, then, since C^1 is an induced subgraph of G, we have $P^1 = ywy'$, contradicting A). Thus, we have $w \in D_2(C)$, and, letting $Q' := xywy'x'$, $V(G_{Q'}^{\text{large}}) = V(G_Q^{\text{large}}) \cup \{w\}$, since G is short-separation-free, and $G_Q^{\text{small}} \subseteq G_{Q'}^{\text{small}}$. Thus, Q' also satisfies conditions 1) and 2) of Definition 8.4.2. By the maximality of $|V(G_Q^{\text{small}})|$, ψ extends to an L-coloring ψ^* of $V(G_{Q'}^{\text{large}})$, and since $|L_{\psi^*}(z)| \geq 2$, ψ^* extends to L-color G_Q^{large}, which is false. This proves 1).

Now we prove 2). Suppose one of the statements of 2) does not hold, and suppose without loss of generality that $N(y) \cap V(P^0) \neq \{x\}$. Let $x^* \in V(\mathring{P}_0) \cap N(y)$. Let $Q^* := x'yzyx^*$. Since C is L-predictable, we have $G_{Q^*}^{\text{large}} = V(G_Q^{\text{large}}) \cup V(x^*P^0x)$, and $G_Q^{\text{large}} \subseteq G_{Q^*}^{\text{large}}$. In particular, Q^* also satisfies conditions 1) and 2) of Definition 8.4.2 and, by the maximality of $|V(G_Q^{\text{small}})|$, $\phi \cup \psi$ extends to an an L_ϕ-coloring of $V(G_{Q^*}^{\text{large}})$. As every vertex of $G_Q^{\text{large}} \setminus G_{Q^*}^{\text{large}}$ is already precolored by ϕ, it follows that $\phi \cup \psi$ extends to an L_ϕ-coloring of $V(G_Q^{\text{large}})$, which is false. ■

Now we have the following:

Claim 8.4.4. *There is no 2-chord of the cycle $P^1 + yzy'$ in $\tilde{G}^{\text{large}}_{yzy'}$ which separates z from an element of $\mathcal{C} \setminus \{C\}$.*

Proof: Suppose toward a contradiction that there is a 2-chord R of $P^1 + yzy'$ which, in $\tilde{G}^{\text{large}}_{yzy'}$, separates z from an element of $\mathcal{C} \setminus \{C\}$. Among all such 2-chords of $P^1 + yzy'$, we choose R so that the quantity $|V(\tilde{G}^{\text{small}}_R)|$ is minimized. We have $\tilde{G}^{\text{small}}_R \supseteq \tilde{G}^{\text{small}}_{yzy'}$, and there is a 4-chord R^{aug} of C such that $R^{\text{aug}} \setminus C = R$ and $G^{\text{small}}_{R^{\text{aug}}} \supseteq G^{\text{small}}_Q$. Since R is a 2-chord of $P^1 + yzy'$, we have $|V(G^{\text{small}}_Q)| < |V(G^{\text{small}}_{R^{\text{aug}}})|$, and R^{aug} satisfies 1) and 2) of Definition 8.4.2. Thus, by the maximality of Q, R^{aug} violates 3) of Definition 8.4.2.

Let $R := uz^*u'$ and let $H := \tilde{G}^{\text{small}}_{uz^*u'} \cap \tilde{G}^{\text{small}}_{yzy'}$. Possibly one of u, u' lies in $\{y, y'\}$, but not both, or else we contradict 1) of Claim 8.4.3. Now, there is a unique cyclic facial subgraph F of H which contains the paths yzy' and uz^*u', and since $z \neq z^*$, $F - \{z, z^*\}$ consists of two disjoint paths P, P', where each of P, P' has one endpoint in $\{u, u'\}$ and the other in $\{y, y'\}$, so suppose without loss of generality that P has endpoints u, y and P' has endpoints u', y'.

Now we apply our box lemma. By the minimality of $|V(\tilde{G}^{\text{small}}_R)|$, z^* has no neighbors in $P \cup P'$, except for $\{u, u'\}$. By A), the path $C^1 \cap \tilde{G}^{\text{large}}_{uz^*u'}$ has at least one internal vertex, and, by assumption, the path $C^1 \cap \tilde{G}^{\text{small}}_{yzy'}$ has at least one internal vertex. Thus, since C^1 is an induced subgraph of G, there is no chord of F with one endpoint in P and the other endpoint in P'. Since all the vertices of $P \cup P'$ have L_ϕ-lists of size at least three, and all other vertices of H have L_ϕ-lists of size at least five, it follows that $\langle F, z, z^*, L_\phi \rangle$ is an H-box, and, by Lemma 8.3.3, ψ extends to a $\langle F, z, z^*, L_\phi \rangle$-corner coloring ψ^*, i.e ψ^* is an L_ϕ-coloring of $\{u, u', y, y'\}$ such that, for any $c \in L_\phi(z) \setminus \{\psi^*(u), \psi^*(u')\}$, ψ^* extends to an L_ϕ-coloring of H using c on z^*.

As indicated above, R^{aug} violates 3) of Definition 8.4.2, so there is an L-coloring τ of $V(G^{\text{large}}_{R^{\text{aug}}})$ using $\psi^*(u), \psi^*(u')$ on the respective vertices u, u'. By our choice of ψ^*, since $L_\phi(z^*) = L(z^*)$, it follows that τ extends to an L_ϕ-coloring of H using $\psi(y), \psi(y')$ on the respective vertices y, y', so ψ extends to an L-coloring of G^{large}_Q, contradicting our choice of ψ. ■

We now introduce the following notation.

Definition 8.4.5.

1) We set T to be a subset of $V(C^1 \cap G^{\text{large}}_Q)$, where $v \in T$ if and only if there is a $w \in D_2(C)$ such that $G[N(w) \cap V(P^1)]$ is a path with v as an internal vertex.
2) $G^\star$ is a graph obtained from G^{large}_Q by adding to G^{large}_Q a vertex $v^\dagger$ adjacent to all three of y, z, y'.
3) $C^\star$ is the cyclic facial subgraph $P^0 + xyv^\dagger y'x'$ of $G^\star$.
4) $C^\star_*$ is the outer face of $G^\star$.
5) $L^\star$ is a list-assignment for $V(G^\star)$, where $L^\star(y) = \{\psi(y)\}$, $L^\star(y') = \{\psi(y')\}$, and $L^\star(v^\dagger)$ is a lone color not lying in $\{\psi(y), \psi(y')\} \cup L(z)$. Otherwise, $L^\star = L$.

We now have the following.

Claim 8.4.6.

1) $(G^\star, (\mathcal{C} \setminus \{C\}) \cup \{C^\star\}, L^\star, C^\star_*)$ is a tessellation; AND

2) $|V(C^\star)| < |V(C)|$; AND

3) $C^\star$ is induced in $G^\star$ and, for any $w \in D_1(C^\star, G^\star)$, the graph $G^\star[N(w) \cap V(C^\star)]$ is a subpath of $C^\star$.

Proof: As shown above in A), P^1 is a path of length at least three, and since C^1 is an induced subgraph of G, it follows from Claim 8.4.3, y, y' have no common neighbor in G_Q^{large}. Thus, G^{aux} is short-separation-free, and $(G^\star, (\mathcal{C} \setminus \{C\}) \cup \{C^\star\}, L^\star, C^\star_*)$ is a tessellation in which $C^\star$ is a closed ring. This proves 1). By assumption, we have $|V(C \cap G_Q^{\text{small}})| > 5$, so $|V(C \setminus P^0)| > 3$ and $|V(C^\star)| < |V(C)|$. This proves 2).

Now we prove 3). By 2) of Claim 8.4.3, $N(y) \cap V(P^0) = \{x\}$ and $N(y') \cap V(P^0) = \{x'\}$. Since C is induced in G, it immediately follows that $C^\star$ is induced in $G^\star$. Now let $w \in D_1(C^\star, G^\star)$. We claim that $G[N(w) \cap V(C^\star)]$ is a subpath of $C^\star$.

If $N(w) \cap V(C^\star) \subseteq V(C)$, then we are done, since C is L-predictable and induced in G. Now suppose that $N(w) \cap V(C^\star \not\subseteq V(C)$. Thus, at least one of $y, y', v^\dagger$ is adjacent to w. If $v^\dagger \in N(w)$, then $w = z$ and $N(w) \cap V(C^\star) = \{y, v^\dagger, y'\}$, since $z \in D_2(C)$, so we are done in that case. Finally, suppose that $v^\dagger \notin N(w)$. By Claim 8.4.3, precisely one of y, y' is adjacent to w, so suppose without loss of generality that $N(w) \cap \{y, y', v^\dagger\} = \{y\}$. Thus, we have $w \in V(P^1) \setminus \{y, y'\}$. Since each of C and C^1 is an induced subgraph of G, and C is L-predictable in G, it follows that $G^\star[N(w) \cap V(C^\star)]$ is a subpath of $P^0 - \{y', z\}$ with y as an endpoint, so we are done. ■

Analogous to 3) of Claim 8.4.6, we have the following easy observation.

Claim 8.4.7. *Let $v_1 \cdots v_k$ be a subpath of P^1, where $v_1 = y$, $k > 1$, and $N(v_k) \cap V(C) \not\subseteq \{x\}$. Let x^* be the unique vertex of the path $G[N(x^*) \cap V(C)]$ which, on P^0, is farthest from x. Let D be the cycle obtained from $C^\star$ by replacing $V(xP^0x^*) \setminus \{x^*\}$ with $v_1 \cdots v_k x^*$. Then D is an induced subgraph of $G^\star \setminus (V(xP^0x^*) \setminus \{x^*\})$, and, for each $w \in V(G^\star) \setminus (V(xP^0x^*) \setminus \{x^*\})$ of distance one from D, the graph $G^\star[N(w) \cap V(D)]$ is a subpath of D.*

Proof: Since each of C, C^1 is induced in G and $v^\dagger$ has no neighbors in $P^1 \cup P^0$ except for y, y', it follows from our choice of x^* that D is induced in G. Let $w \in V(G^\star) \setminus (V(xP^0x^*) \setminus \{x^*\})$, where distance one from D. If $w \in V(C^1)$, then, since C is L-predictable in G and C^1 is induced in G, it follows that $G^\star[N(w) \cap V(D)]$ is either a subpath of $C \cap D$ or a subpath of D with v_k as an endpoint. Now suppose that $w \notin V(C^1)$. For any 2-chord uwu' of D with midpoint w, every vertex of $\tilde{G}^{\text{small}}_{uwu'}$ has an L_ϕ-list of size at least three, or else, in $G \setminus C$, uwu' separates z from an element of $\mathcal{C} \setminus \{C\}$, contradicting Claim 8.4.4. Thus, it follows from Lemma 8.1.10 that $G^\star[N(w) \cap V(D)]$ is a subpath of D. ■

We now have the following:

Claim 8.4.8. *Every internal vertex of the path P^1 has at least two neighbors in C.*

Proof: Suppose toward a contradiction that there is an internal vertex v of P^1 such that $|N(v) \cap V(C)| = 1$. Among all vertices of $\mathring{P}_1$ with precisely one neighbor on C, we choose v to be the one which is closest to y on the path P^1. Now let $\psi^\star$ be the unique $L^\star$-coloring of $V(C^\star)$.

Subclaim 8.4.9. $y, y' \notin N(v)$.

Proof: Suppose there is a $y^* \in \{y, y'\}$ such that $y^* \in N(v)$. Since C^1 is an induced subgraph of G, y^*v is a terminal edge of P^1. By A), P^1 has length at least three, and, since C^1 is an induced subgraph of G, we have $N(v) \cap \{y, y'\} = \{y^*\}$. Since $|N(v) \cap V(C)| = 1$ and $N(v) \cap \{y, y'\} = \{y^*\}$., we have $|L^\star(v)| \geq 3$. Since P^1 has at least three, there is a terminal vertex v' of $P^1 - \{y, y'\}$ with $v' \neq v$. Now, each vertex of $P^1 \setminus \{y, y'\}$, except for v', has an $L^\star_{\psi^\star}$-list of size at least three, and $|L^\star_{\psi^\star}(v')| \geq 2$. Furthermore, by our choice of color of

$L^\star(v^\dagger)$, we have $|L^\star_{\psi^\star}(z)| \geq 3$. Thus, applying 3) of Claim 8.4.6, $C^\star$ is an $L^\star$-predictable facial subgraph of $G^\star$. We claim now that $(G^\star, (\mathcal{C} \setminus \{C\}) \cup \{C^\star\}, L^\star, C^\star_*)$ is a mosaic.

By 1) of Claim 8.4.6, $\mathcal{T}^\star := (G^\star, (\mathcal{C} \setminus \{C\}) \cup \{C^\star\}, L^\star, C^\star_*)$ is a tessellation, and, since $C^\star$ is an $L^\star$-predictable facial subgraph of $G^\star$, this tessellation satisfies M2) of Definition 2.1.6. By 2) of Claim 8.4.6, $|V(C^\star)| < |V(C)|$, so M0) is satisfied, and M1) is trivially satisfied. Furthermore, since $\text{Rk}(\mathcal{T}^\star|C^\star) < \text{Rk}(\mathcal{T}|C)$, and $v^\dagger$ is separated from each element of $\mathcal{C} \setminus \{C\}$ by vertices of $B_2(C, G)$, it follows that the distance conditions of Definition 2.1.6 are also satisfied. Thus, $(G^\star, (\mathcal{C} \setminus \{C\}) \cup \{C^\star\}, L^\star, C^\star_*)$ is indeed a mosaic. Since $|V(G^{\text{small}} \setminus Q)| > 1$, we have $|V(G^\star)| < |V(G)|$. By the minimality of $\mathcal{T}$, it follows that $G^\star$ is $L^\star$-colorable, and thus $\psi \cup \phi$ extends to an L-coloring of G^{large}_Q, contradicting our choice of ψ. ■

Suppose toward a contradiction that there is a $v \in V(P^1) \setminus \{y, y'\}$ such that $|N(v) \cap V(C)| = 1$. By Subclaim 8.4.9, v is an internal vertex of $P^1 - \{y, y'\}$, and, since C^1 is an induced subgraph of G, $y, y' \notin N(v)$. By our choice of v, each internal vertex of the path yP^1v is adjacent to a subpath of C of length at least one. Let x^* be the unique neighbor of v in C^1. If $x^* = x$, then, for each $u \in V(vP^1y)$, we have $N(u) \cap V(C) = \{x\}$, and, by our choice of v, it follows that vy is a terminal edge of P^1, contradicting Subclaim 8.4.9. Thus, xP^0x^* is a path of length at least one.

Since $v \notin N(y)$, let $yP^1v = v_0 \cdots v_k$ for some $k \geq 2$, where $v_0 = y$ and $v_k = v$. Since $N(v) \cap V(C) = \{x^*\}$, x^* is a terminal vertex of $G[N(v_{k-1}) \cap V(C)]$. Since $N(y) \cap V(P^0) = \{x\}$, we get that x is a terminal vertex of $G[N(v_1) \cap V(C)]$. Now let C^{aux} be the cycle obtained from $C^\star$ by replacing xP^0x^* with $yv_1 \cdots v_{k-1}x^*$. Since each of $v_1, \cdots, v_{k-1}$ has at least two neighbors in C, we have $|V(C^{\text{aux}})| \leq |V(C^\star)|$. Let $G^{\text{aux}} := G^\star \setminus (V(xP^0x^*) \setminus \{x^*\})$

Since $|L_\phi(v_{k-1})| \geq 3$, it follows from Theorem 1.7.5 that there is a $\sigma \in \text{Link}_{L_\phi}(yP^1v_{k-1}, C^1, \tilde{G})$ using $\psi(y)$ on y. Note that $\psi^\star \cup \sigma$ is a proper $L^\star$-coloring of its domain in $G^\star$. Since $|N(v) \cap V(C)| = 1$, we have $v \notin T$, or else there are three consecutive vertices of P^1 with a common neighbor in $D_2(C)$ and a common neighbor in C, contradicting the fact that G is $K_{2,3}$-free. Thus, it follows from Lemma 8.1.10 that, in the notation of Section 1.7, we have $T \cap V(yP^1v_{k-1}) = \text{Sh}_{2,L_\phi}(yP^1v_{k-1}, C^1, \tilde{G})$

Let L^{aux} be a list-assignment for $V(G^{\text{aux}})$ defined as follows.

1) For each $u \in \text{dom}(\sigma)$, we set $L^{\text{aux}}(u) = \{\sigma(u)\}$, and, for each $u \in V(C^{\text{aux}}) \setminus (\text{dom}(\sigma) \cup T)$, we set $L^{\text{aux}}(u) = \{\psi^\star(u)\}$.
2) We set $\{L^{\text{aux}}(u) : u \in T \cap V(yP^1v_{k-1})\}$ to be a a collection of disjoint singletons, where, for each $u \in T \cap V(yP^1v_{k-1})$, the lone color of $L^{\text{aux}}(u)$ is disjoint from the L-lists of all the vertices in $B_2(C)$.
3) Otherwise, we set $L^{\text{aux}} = L^\star = L$.

By Lemma 8.1.10, we have $\text{Sh}_{2,L_\phi}(yP^1v_{k-1}, C^1, \tilde{G}) \subseteq T$, so the definition above yields a unique L^{aux}-coloring $\psi^{\star\star}$ of $V(C^{\text{aux}})$. Let C^{aux}_* be the outer face of G^{aux}. We claim now that $\mathcal{T}^{\text{aux}} := (G^{\text{aux}}, (\mathcal{C} \setminus \{C\}) \cup \{C^{\text{aux}}\}, L^{\text{aux}}, C^{\text{aux}}_*)$ is a mosaic. By Claim 8.4.6, $(G^\star, (\mathcal{C} \setminus \{C\}) \cup \{C^\star\}, L^\star, C^\star_*)$ is a tessellation, so $\mathcal{T}^{\text{aux}}$ is also a tessellation. Since $|L_\phi(v)| \geq 4$ and C^1 is an induced subgraph of G, we have $|L^{\text{aux}}_{\psi^{\star\star}}(v)| \geq 3$. Letting v' be the unique neighbor of y' on P^1, every vertex of $D_1(C^{\text{aux}}, G^{\text{aux}})$, except possibly v', has an $L^{\text{aux}}_{\psi^{\star\star}}$-list of size at least three, and v' has an $L^{\text{aux}}_{\psi^{\star\star}}$-list of size at least two. Combining this with Claim 8.4.7, it follows that C^{aux} is an L^{aux}-predictable facial subgraph of G^{aux}, so M2) is satisfied.

By 2) of Claim 8.4.6, we have $|V(C^\star)| < |V(C)|$. Since $|V(C^{\text{aux}})| \leq |V(C^\star)|$, $\mathcal{T}^{\text{aux}}$ satisfies M0) and $\text{Rk}(\mathcal{T}^{\text{aux}}|C^{\text{aux}}) < \text{Rk}(\mathcal{T}|C)$. Since $v^\dagger$ is separated from each element of $\mathcal{C} \setminus \{C\}$ by vertices of $B_2(C, G)$, and each vertex of $C^{\text{aux}} - v^\dagger$ lies in $B_1(C, G)$, it follows that the distance conditions of Definition 2.1.6 are also satisfied. M1) is trivial.

Thus, $\mathcal{T}^{\text{aux}}$ is indeed a mosaic. Since $|V(G^{\text{large}})| < |V(G)|$, ity follows from the minimality of $\mathcal{T}$ that G^{aux} admits an L^{aux}-coloring τ. Let τ^* be the restriction of τ to $V(G^{\text{aux}}) \setminus (T \cap V(yP^1v_{k-1}) \cup \{v^\dagger\})$. Then $\text{dom}(\tau^*) = V(G_Q^{\text{large}}) \setminus (T \cap V(yP^1v_{k-1})$ and τ^* is a proper L-coloring of its domain. Furthermore, since $\sigma \in \text{Link}_{L_\phi}(yP^1v_{k-1}, C^1, \tilde{G})$ and ϕ is the unique L-coloring of $V(C)$, it follows that τ^* extends to L-color the vertices of $T \cap V(yP^1v_{k-1})$, i.e τ^* extends to an L-coloring of G_Q^{large}, contradicting our assumption that Q is defective. ■

As a consequence of the above, we have the following:

Claim 8.4.10. $|V(C)| > |V(P^1 + yzy')| + 1$.

Proof: Since each internal vertex of P^1 has at least two neighbors in C, we have $|V(P^1)| \leq |V(P^0)| + 1$, and thus $|V(P^1 + yzy')| \leq |V(P^0)| + 2$. By assumption, $|V(C \cap G_Q^{\text{large}})| > 5$, so $|V(C)| > |V(P^0)| + 3$ Thus, we have $|V(C)| > |V(P^0)| + 3 > |V(P^1 + yzy')|$, so $|V(C)| > |V(P^1 + yzy')| + 1$. ■

We now introduce the following terminology. Given a $v \in V(P^1)$, we say that v is a *pivot vertex* if there is a $w \in D_2(C) \cap N(v)$ such that $G[N(w) \cap V(P^1)]$ is a subpath of P^1 of length at most two.

Claim 8.4.11. *Suppose there is at least one pivot vertex. Then there is a pivot vertex $v \in V(P^1)$, a $w \in D_2(C, G) \cap V(G_Q^{\text{large}})$ and an extension of ψ to an L_ϕ-coloring ψ^* of $(V(P^1) \setminus T) \cup \{v\}$ such that the following hold.*

1) $V(P^1) \setminus \text{dom}(\psi^*)$ is $(L, \phi \cup \psi^*)$-inert in G; AND

2) Every vertex of $D_2(C, G_Q^{\text{large}}) \setminus \{w\}$ has an $L_{\phi \cup \psi^*}$-list of size at least three; AND

3) $|L_{\phi \cup \psi^*}(w)| \geq 2$

Proof: As above, we apply the work of Section 1.7. Let v be a pivot vertex and consider the following cases:

Case 1: There is a $w \in D_2(C)$ such that $G[N(w) \cap V(C^1)]$ is a path of length two with midpoint v.

In this case, let $G[N(w) \cap V(C^1)] = uvu'$, where $u \in V(vP^1y)$ and $u' \in V(vP^1y')$.

By Theorem 1.7.5, there is a $\sigma_0 \in \text{Link}_{L_\phi}(yP^1u, C^1, \tilde{G})$ using $\psi(y)$ on y and a $\sigma_1 \in \text{Link}_{L_\phi}(u'P^1y', C^1, \tilde{G})$ using $\psi(y')$ on y'. Since $|L_\phi(v)| \geq 3$ and C^1 is an induced subgraph of G, $\sigma_0 \cup \sigma_1$ extends to a proper L_ϕ-coloring σ^* of $\text{dom}(\sigma_0 \cup \sigma_1) \cup \{v\}$, and $|L_{\phi \cup \sigma^*}(w)| \geq 2$. If there is a vertex w^* of $D_2(C, G_Q^{\text{large}}) \setminus \{w\}$ with an $L_{\phi \cup \sigma^*}$-list of size less than three, then, since $\sigma_0 \in \text{Link}_{L_\phi}(yP^1u, C^1, \tilde{G})$ and $\sigma_1 \in \text{Link}_{L_\phi}(u'P^1y', C^1, \tilde{G})$, it follows that w^* has a neighbor $p \in V(yP^1u)$ and a neighbor $p' \in V(u'P^1y')$. But then, by Claim 8.4.4, $\tilde{G}^{\text{small}}_{pw^*p'}$ contains the path pP^1p', and thus contains v, and, by Lemma 8.1.10, $\tilde{G}^{\text{small}}_{pw^*p'}$ is a broken wheel with principal path pw^*p', so $w^* = w$, which is false. So we are done in this case.

Case 2: There does not exist a $w \in D_2(C)$ such that $G[N(w) \cap V(C^1)]$ is a path of length two with midpoint v.

In this case, since v is a pivot vertex, there does not exist a $w \in D_2(C)$ such that v is an internal vertex of $G[N(w) \cap V(C)]$. It follows from our triangulation conditions that, for any $e \in E(P^1)$, there is a unique $w \in D_2(C)$ such that e is a subpath of $G[N(w) \cap V(C^1)]$ which contains e. Since v is a pivot vertex and P^1 has length at least three, it follows from Lemma 8.1.10 that there is an $e \in E(P^1)$ incident to v and a $w \in D_2(C)$ such that $G[N(w) \cap V(P^1)]$ is a path of length at most two which contain e and has v as an endpoint. If $G[N(w) \cap V(P^1)]$ has length precisely two, then its midpoint is also a pivot vertex and we are back to Case 1 with v replaced by the midpoint of $G[N(w) \cap V(P^1)]$, so suppose that this path has length one. Letting $e = vv'$, we have $G[N(w) \cap V(P^1)] = vv'$. Note that v' is also a pivot vertex. Possibly, v is an endpoint of P^1, but it is permissible to suppose that v is an internal vertex of P^1, because, if it is not, then we simply replace v with v', and, since P^1 has length at least three, v' is an internal vertex of P^1.

Thus, we suppose without loss ouf generality that v is an internal vertex of P^1, and we suppose further without loss of generality that v' lies in the subpath vP^1y' of P^1. Since v is an internal vertex of P^1, let v'' be the other neighbor of v on P^1. As above, it follows from Theorem 1.7.5 that there is a $\sigma_0 \in \text{Link}_{L_\phi}(yP^1v'')$ and a $\sigma_1 \in \text{Link}_{L_\phi}(v'P^1y')$. Since $|L_\phi(v)| \geq 3$, and since C^1 is an induced subgraph of G, $\sigma_0 \cup \sigma_1$ extends to a proper L_ϕ-coloring σ^* of $\text{dom}(\sigma_0 \cup \sigma_1) \cup \{v\}$. Possibly, there is a $w^* \in D_2(C)$ such that $G[N(w^*) \cap V(C^1)]$ is a subpath of $C^1 - v'$ which has length at least two and has v as an endpoint, and this vertex has an $L_{\phi \cup \sigma^*}$-list of size at least two, and every other vertex of $D_2(C, G_Q^{\text{large}})$ has an $L_{\phi \cup \psi^*}$-list of size at least three, so we are done. ■

With the above in hand, we have the following:

Claim 8.4.12. *There does not exist a pivot vertex.*

Proof: Suppose toward a contradiction that there is a pivot vertex. Then there is a pivot vertex v, a $w \in D_2(C) \cap V(G_Q^{\text{large}})$ and $\psi^* \in \Phi_{L_\phi}(\psi, (V(P^1) \setminus T) \cup \{v\})$ such that v, w, ψ^* satisfy Claim 8.4.11.

Recall that $G^\star$ is a graph obtained from G_Q^{large} by adding a lone vertex $v^\dagger$ adjacent to all three of y, z, y'. Let $H := G^\star \setminus C$ and let $D := P^1 + yv^\dagger y'$. Then D is a cyclic facial subgraph of H, and, since $|V(D)| = |V(P^1 + zyz')|$, it follows from Claim 8.4.10 that $|V(C)| > |V(D)| + 1$. Now we define a list-assignment L^* for H in the following way.

1) For each vertex of $\text{dom}(\psi^*)$, we set $L^*(u) = \{\psi^*(u)\}$.
2) We set $\{L^*(u) : u \in V(D) \setminus \text{dom}(\psi^*)\}$ to be a a collection of disjoint singletons, where, for each $u \in V(D) \setminus \text{dom}(\psi^*)$, the lone color of $L^*(u)$ is disjoint from the L-lists of all the vertices in $B_2(C)$.
3) Otherwise, we set $L^* = L$.

Now, let D_* be the outer face of H and consider the tuple $\mathcal{T}^* := (H, (\mathcal{C} \setminus \{C\}) \cup \{D\}, L^*, D_*)$. Note that $\mathcal{T}^*$ is a tessellation in which D is a closed ring. We claim that $\mathcal{T}^*$ is a mosaic. Since $|V(C)| > |V(D)| + 1$, M0) is trivially satisfied, and $\text{Rk}(\mathcal{T}|C) \geq \text{Rk}(\mathcal{T}^*|D)+2$. Since $v^\dagger$ is separated from each element of $\mathcal{C} \setminus \{C\}$ by vertices of $B_2(C, G)$, and each vertex of $D - \{v_\dagger\}$ lies in $B_1(C, G)$, it follows that $\mathcal{T}^*$ also satisfies the distance conditions of Definition 2.1.6. To finish, we just need to check that D is an L^*-predictable facial subgraph of H.

Letting τ be the unique L^*-coloring of $V(D)$, it follows from our construction of L^* that $L^*_\tau(w)| \geq 2$ and each vertex of H of distance 1 from D, other than w, has an L^*_τ-list of size at least three. Since C^1 is induced in G, D is induced in H. Combining Claim 8.4.4 with Lemma 8.1.10, it follows that, for every vertex $w' \in V(H)$ of distance 1 from D, the graph $H[N(w') \cap V(D)]$ is a subpath of D. Note that this is true even if z is the midpoint of this 2-chord.

Thus, D is an L^*-predictable facial subgraph of H, so $\mathcal{T}^*$ is a mosaic. Since $|V(H)| < |V(G)|$, H admits an L^*-coloring σ. Let σ' be the restriction of σ to $H \setminus (D \setminus \text{dom}(\psi^*))$. Then σ' is a proper L_ϕ-coloring of its domain. B our choice of ψ^*, $\sigma' \cup \phi$ extends to L-color the vertices of $V(P^1) \setminus \text{dom}(\psi^*)$, so G_Q^{large} is L-colorable, which is false. ■

We now define a cycle $C^\dagger$ of G as follows. We let $C^\dagger$ be the unique cycle of G which intersects with the cycle $P^1 + yzy'$ on precisely the vertices of $\{z\} \cup V(P^1 \setminus T)$, where, for each subpath R of P^1 of length at least two whose endpoints lie in $P^1 \setminus T$ and whose internal vertices lie in T, we replace $\mathring{R}$ with the unique 2-path whose endpoints are the endpoints of R and whose midpoint is the unique vertex of $D_2(C)$ adjacent to the endpoints of R.

Since there is no pivot vertex, no two vertices of $P^1 \setminus T$ are adjacent, so P^1 admits a partition into a collection of edge-disjoint paths $R_1, \cdots, R_k$ with $P^1 = R_1 \cup \cdots \cup R_k$ such that, for each $j = 1, \cdots, k$, the endpoints of R_i lie in $P^1 \setminus T$ and $V(\mathring{R}_j) \subseteq T$, and there is a unique vertex $w_j \in D_2(C)$ such that $R_j := G[N(w_j) \cap V(C^1)]$. Furthermore, each of the paths $R_1, \cdots, R_k$ has length at least three. For each $j = 1, \cdots, k$, let M_j be the unique 2-path whose midpoint is w_i and whose endpoints are the endpoints of R_j. Note that, for each $j = 1, \cdots, k$, we have $\frac{|E(R_j)|}{|E(M_j)|} \geq \frac{3}{2}$.

Now, $|V(C^\dagger)| = |E(C^\dagger)| = 2 + \sum_{j=1}^k |E(M_j)|$. On the other hand, $|V(P^1 + yzy')| = |E(P^1 + zyz')| = 2 + \sum_{j=1}^k |E(R_j)|$. Thus, $|V(P^1 + yzy')| \geq 2 + \sum_{j=1}^k \frac{3}{2}|E(M_j)|$, so $|V(P^1 + yzy')| \geq 2 + \frac{3}{2}(|V(C^\dagger)| - 2)$. By Claim 8.4.10, we thus have $|V(C)| > 3 + \frac{3}{2}(|V(C^\dagger)| - 2)$, so $|V(C)| > \frac{3}{2}|V(C^\dagger)|$. Since $y, y' \in V(C^\dagger \cap P^1)$, we have $d(C^\dagger, C) \leq 1$. Since $\text{Rk}(\mathcal{T}|C) = |V(C)|$ and $C^\dagger$ separates C from each element of $\mathcal{C} \setminus \{C\}$, this contradicts Corollary 2.1.30. Thus, our original assumption that Q is defective is false. This completes the proof of Lemma 8.4.1. $\square$

8.5 2-Chords of the 1-Necklace with a 2-List on the Small Side

This section consists of the lone result below, which, combined with Lemma 8.2.1, is enough to complete the proof of Theorem 8.0.4.

Lemma 8.5.1. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let $C \in \mathcal{C}$ be a closed ring, let C^1 be the 1-necklace of C, and let $\tilde{G} := G \setminus C$. Then, for each 2-chord xwy of C^1 in $\tilde{G}$, the graph $\tilde{G}^{\text{small}}_{xwy}$ is a broken wheel with principal path xwy.*

Proof. Given a 2-chord uzu' of C^1 in $\tilde{G}$, we say that uzu' is *bad* if $V(\tilde{G}^{\text{small}}_{uzu'}) \neq \{z\} \cup V(C^1 \cap \tilde{G}^{\text{small}}_{uzu'})$. By Lemma 8.2.1, C^1 is an induced cycle of G. It follows that, for any 2-chord uzu' of C^1 in $\tilde{G}$, if uzu' is not bad, then, by our triangulation conditions, $\tilde{G}^{\text{small}}_{uzu'}$ is a broken wheel with principal path uzu'. Thus, it suffices to prove that there are no bad 2-chords of C^1. Suppose toward a contradiction that there is bad 2-chord uzu' of C^1, where uzu' has been chosen to minimize the quantity $|V(\tilde{G}^{\text{small}}_{uzu'})|$ over all bad 2-chords of C^1. Let ϕ be the unique L-coloring of $V(C)$. By Lemma 8.1.10, there is an internal vertex $u_\star$ of the path $C^1 \cap \tilde{G}^{\text{small}}_{uzu'}$ with an L_ϕ-list of size less than three. Since C is L-predictable and an induced subgraph of G, $|L_\phi(u_\star)| = 2$ and every vertex of $C^1 - u_\star$ has an L_ϕ-list of size at least three. We now set Q_1^{large} to be the path $C^1 \cap \tilde{G}^{\text{large}}_{uzu'}$ and set Q_1^{small} to be the path $C^1 \cap \tilde{G}^{\text{small}}_{uzu'}$. Furthermore, we set F_1^{small} to be the cycle $Q_1^{\text{small}} + uzu'$.

Claim 8.5.2.

1) For any 2-chord yz^*y' of C^1 in $\tilde{G}$, if $\tilde{G}^{\text{small}}_{uzu'} \subseteq \tilde{G}^{\text{small}}_{yz^*y'}$, then $yy' \notin E(G)$. In particular $uu' \notin E(G)$; AND

2) F_1^{small} is induced in G.

Proof: Suppose that $yy' \in E(G)$. Since C^1 is an induced subgraph of G, and since $u_\star$ is an internal vertex of Q_1^{small}, we have $yy' \in E(C^1)$ and $yy' = Q_1^{\text{large}}$, so yz^*y' is a triangle separating an element of $\mathcal{C} \setminus \{C\}$ from $u_\star$, contradicting the fact that $\mathcal{T}$ is a tessellation. This proves 1). Now suppose that F_1^{small} is not induced in G. Since C^1 is an induced subgraph of G, it follows that $N(z) \cap V(C^1 \cap \tilde{G}^{\text{small}}_{uzu'}) \neq \{u, u'\}$. Thus, z has a neighbor u'' which is an internal vertex of the path $C^1 \cap \tilde{G}^{\text{small}}_{uzu'}$, and, by the minimality of uzu', neither uzu'' nor $u''zu'$ is a bad 2-chord of C^1, and since $\tilde{G}^{\text{small}}_{uzu'} = \tilde{G}^{\text{small}}_{uzu''} \cup \tilde{G}^{\text{small}}_{u''zu'}$, it follows that uzu' is also not bad, contradicting our assumption. ■

Applying the fact that C is induced in G and L-predictable, we now define the following subgraphs of G:

Definition 8.5.3.

1) For each vertex $v \in V(C^1)$, we let P_v be the path $G[N(v) \cap V(C)]$.
2) We set Q_0^{large} to be the unique subpath of $C \setminus \mathring{P}_{u_\star}$ which intersects with P_u on precisely an endpoint and intersects with $P_{u'}$ on precisely an endpoint,
3) We set Q_0^{small} to be the unique subpath of C consisting of the edges of $E(C) \setminus E(Q_0^{\text{large}})$.
4) We set Q_{0+}^{large} to be the path $G[V(Q_0^{\text{large}} \cup P_u \cup P_{u'})]$, and we set Q_{0-}^{small} to be the unique subpath of C consisting of the edges of $E(C) \setminus E(Q_{0+}^{\text{large}})$.
5) We set R to be the unique 4-chord of C whose endpoints are the endpoints of Q_0^{large} and whose middle two edges are uzu'. Likewise, we set R_+ to be the unique 4-chord of C whose endpoints are the endpoints of Q_{0+}^{large} and whose middle two edges are uzu'.

Since $|L_\phi(u_\star)| = 2$, the path $P_{u_\star}$ has length at least two, so Q_0^{large} is well-defined and Q_0^{small} is nonempty. Since Q_0^{large} intersects with each of $P_u, P_{u'}$ on an endpoint, Q_{0+}^{large} is a connected subgraph of G, and since $P_{u_\star}$ has length at least two, Q_{0+}^{large} is a subpath of $V(C)$ with $|V(Q_{0+}^{\text{large}})| < |V(C)|$, so all of the notation above is well-defined. We also have the following observation.

Claim 8.5.4.

1) *$P_u \cap P_{u'} = \varnothing$ and each of R, R_+ is a proper 4-chord of C; AND*
2) *Q_1^{small} has length at least three; AND*
3) *If u, u' have a common neighbor in G_Q^{small}, other than z, then $\tilde{G}_{uzu'}^{\text{small}}$ is a wheel with a central vertex adjacent to every vertex of F_1^{small}.*

Proof: Suppose that $P_u \cap P_{u'} \neq \varnothing$. Since C is an induced cycle and $P_{u_\star}$ has length at least two, P_u and $P_{u'}$ share a vertex v of Q_0^{large}, and $vuzu'$ is a 4-cycle which separates an element of $\mathcal{C} \setminus \{C\}$ from $u_\star$, contradicting the fact that $\mathcal{T}$ is a tessellation. Since $P_u \cap P_{u'} = \varnothing$, each of Q_0^{large} and Q_{0+}^{large} has length at least one, and so each of R, R_+ has distinct endpoints and is a proper 4-chord of C. If Q_1^{small} has length less than three then, $Q_1^{\text{small}} = uu_\star u'$, and since G is short-separation-free, $V(\tilde{G}_{uzu'}^{\text{small}}) = \{uz, u', u_\star\}$, contradicting the fact that uzu' is bad. This proves 1) and 2).

Now suppose that u, u' have a common neighbor w in G_Q^{small}, where $w \neq z$. Since $w \notin \{u, z, u'\}$, we have $w \notin V(C^1)$ by Claim 8.5.2, so $d(z^*, C) = 2$. By 1) of Claim 8.5.2, $uu' \notin E(G)$, so, since G is short-separation-free, it follows from our triangulation conditions that w is adjacent to each of u, z, u', and, by the minimality of uzu', the 2-chord uwu' of C^1 is not bad, so $V(\tilde{G}_{uwu'}^{\text{small}}) = \{w\} \cup V(C^1 \cap \tilde{G}_{uwu'}^{\text{small}})$. By Lemma 8.2.1, C^1 is induced in G, so it follows from our triangulation conditions that w is adjacent to each vertex of the path $C^1 \cap \tilde{G}_{uwu'}^{\text{small}}$, and, since G is short-separation-free, $\tilde{G}_{uzu'}^{\text{small}}$ is a wheel with central vertex adjacent w. ■

We now have the following:

Claim 8.5.5. *$V(\tilde{G}_{uzu'}^{\text{large}})$ is L_ϕ-colorable. Furthermore, if Q_1^{small} has length three, then $\tilde{G}_{uzu'}^{\text{small}}$ is a wheel.*

Proof: Let x_*, x'_* be the endpoints of R_+, where x_* is also an endpoint of P_u and x'_* is also an endpoint of $P_{u'}$. Suppose toward a contradiction that $\tilde{G}_{uzu'}^{\text{large}}$ is not L_ϕ-colorable. By A) of Lemma 8.4.1, $G_{R_+}^{\text{large}}$ is an induced subgraph of G, so any L-coloring of $G_{R_+}^{\text{large}}$ restricts to an L_ϕ-coloring of the subgraph of G induced by $V(\tilde{G}_{uzu'}^{\text{large}})$. Thus, $G_{R_+}^{\text{large}}$ is not L-colorable.

Subclaim 8.5.6. *All of the following hold.*

1) $|V(G^{\text{large}}_{R_+})| + 3 = |V(G)|$*; AND*

2) $u_\star$ *is adjacent to each endpoint of* R_+ *and* $P_{u_\star}$ *has length two; AND*

3) Q_1^{small} *is a path of length three.*

Proof: Firstly, if $|V(G^{\text{large}}_{R_+})| + 3 < |V(G)|$, then it immediately follows from Lemma 8.1.6 that $G^{\text{large}}_{R_+}$ is L-colorable, contradicting our assumption. Thus, we have $|V(G^{\text{large}}_{R_+})| \geq |V(G)| - 3$. By 2) of Claim 8.5.4, Q^1_{small} has length at least three, and since $P_{u_\star}$ has at least one internal vertex, we have $|V(G^{\text{small}}_{R_+} \setminus R^-+)| \geq 3$, and thus $|V(G^{\text{large}}_{R_+})| + 3 = |V(G)|$. Furthermore, since Q^1_{small} has at least two internal vertices, $P_{u_\star}$ is a path of length two whose endpoints are x_*, x'_*, and Q_1^{small} has precisely two internal vertices. ■

Appyling Subclaim 8.5.6, there is a vertex $v \in V(C^1)$ such that $Q_1^{\text{small}} - \{u, u'\} = vu_\star$, so suppose without loss of generality that $Q_1^{\text{small}} = uvu_\star u'$. Again by Subclaim 8.5.6, $u_\star$ is adjacent to all three vertices of $C \cap G^{\text{small}}_{R_+}$, and since C^1 is an induced subgraph of G, it follows that $N(v) \cap V(C) = \{x_*\}$.

Now let p be the lone internal vertex of $P_{u_\star}$) and let $C^\dagger$ be the cycle $(C \cap G^{\text{large}}_{R_+}) + x_* u_\star x'_*$. Note that $|V(C^\dagger)| = |V(C)|$ and $C^\dagger$ is a facial subgraph of $G - p$. Since $L_\phi(u_\star)| = 2$, let $L^\dagger$ be a list-assignment for $G - p$ in which $L^\dagger(u_\star)$ is a lone color of $L_\phi(u_\star)$, and otherwise $L^\dagger = L$. Let $C^\dagger_*$ be the outer face of $G - p$ and let $\mathcal{T}^\dagger := (G - p, (\mathcal{C} \setminus \{C\}) \cup \{C^\dagger\}, L^\dagger, C^\dagger_*)$. Then $\mathcal{T}^\dagger$ is a tessellation in which $C^\dagger$ is a closed ring. We claim that $\mathcal{T}^\dagger$ is a mosaic.

Firstly, since $|V(C^\dagger)| = |V(C)|$, we have $\text{Rk}(\mathcal{T}^\dagger | C^\dagger) = \text{Rk}(\mathcal{T} | C)$, and, by Claim 8.5.2, $u_\star \notin N(z)$, so $\mathcal{T}^\dagger$ satisfies the distance conditions of Definition 2.1.6. The only other nontrivial condition to check is that $C^\dagger$ is $L^\dagger$-predictable in $G - p$. Firstly, $C^\dagger$ is an induced subgraph of $G - p$ and, for every $w \in B_1(C^\dagger, G - p)$, the neighborhood of w in $C^\dagger$ is a subpath of $C^\dagger$. Let ψ be the unique $L^\dagger$-coloring of $C^\dagger$. Since v only has one neighbor in C, it has precisely two neighbors in $C^\dagger$, so every vertex of $B_1(C^\dagger, G - p)$ has an $L^\dagger_\psi$-list of size at least three, except possibly u'. Since $|L_\phi(u')| \geq 3$, we have $|L^\dagger_\psi(u')| \geq 2$, so $C^\dagger$ is indeed $L^\dagger$-predictable in $G - p$. Thus, $\mathcal{T}^\dagger$ is a mosaic, and since $|V(G - p)| < |V(G)|$, it follows from the minimality of $\mathcal{T}$ that $G - p$ is $L^\dagger$-colorable. Since $u_\star$ is the only neighbor of p in G which is not precolored, it follows that G is L-colorable, contradicting the fact that $\mathcal{T}$ is critical.

Thus, our assumption that $V(\tilde{G}^{\text{large}}_{uzu'})$ is not L-colorable is false. Let ψ be an L_ϕ-coloring of $V(\tilde{G}^{\text{large}}_{uzu'})$ and suppose Q_1^{small} has length three. By 2) of Claim 8.5.2, z is not adjacent to any internal vertex of Q_1^{small}. Since Q_1^{small} is induced in G, $Q_1^{\text{small}} - \{u, u'\}$ is an edge in which one endpoint has an $L_{\phi\cup\psi}$-list of size at least one and the other endpoint has an $L_{\phi\cup\psi}$-list of size at least two. Thus, $\phi \cup \psi$ extends to L-color $\text{dom}(\phi \cup \psi) \cup V(Q_1^{\text{small}})$, and, since G is not L-colorable, it follows from Theorem 1.3.5 that $\tilde{G}^{\text{small}}_{uzu'}$ is a wheel with a central vertex adjacent to all of F_1^{small}. ■

Claim 8.5.5 has the following useful consequence.

Claim 8.5.7. *For any vertex* $v \in V(Q_1^{\text{small}}) \setminus \{u, u'\}$*, if* v *is adjacent to either of* u, u'*, then* $|N(v) \cap V(C)| > 1$.

Proof: Suppose that this does not hold, and suppose without loss of generality that there is a $v \in V(Q_1^{\text{small}}) \setminus \{u, u'\}$ which is adjacent to u and has only one neighbor on C. Since C^1 is an induced subgraph of G, v is adjacent to u on the path Q_1^{small}, so let $Q_1^{\text{small}} = u_1 \cdots u_k$ for some $k \geq 2$, where $u_1 = u$, $u_k = u'$, and $u_2 = v$. Since $u_\star$ is an internal vertex of Q_1^{small}, we have $v \neq u_{k-1}$.

Applying Claim 8.5.5, we fix an L_ϕ-coloring ψ of $V(\tilde{G}^{\text{large}}_{uzu'})$. Now, $\tilde{G}^{\text{small}}_{uzu'} \setminus \{u, z, u'\}$ has a facial subgraph F containing all the vertices of $\tilde{G}^{\text{small}}_{uzu'} \setminus \{u, z, u'\}$ with $L_{\phi\cup\psi}$-lists of size less than five. By Claim 8.5.8, there is no common neighbor of u, z, u' in F, and, by Claim 8.5.2, z has no neighbors in F. Since C^1 is an induced subgraph of

F and $|L_\phi(v)| \geq 4$, it follows that every vertex of F, except possibly $u_\star, u_{k-1}$, has an $L_{\phi\cup\psi}$-lists of size at least three. If $u_\star \neq u_{k-1}$, then $u_\star$ has no neighbors in $\{u, z, u'\}$ and each of $u_\star, u_{k-1}$ has an $L_{\phi\cup\psi}$-list of size at least two. If $u_\star = u_{k-1}$, then $|L_{\phi\cup\psi}(u_{k-1})| \geq 1$. In either case, applying Theorem 1.3.4 or Theorem 0.2.3 respectively, ψ extends to an L_ϕ-coloring of $\tilde{G}^{\text{small}}_{uzu'}$, so G is L-colorable, contradicting the fact that $\mathcal{T}$ is critical. ■

Now we have the following simple observation:

Claim 8.5.8. *If there is a $v \in V(Q_1^{\text{small}}) \setminus \{u, u'\}$ such that $|N(v) \cap V(C)| = 1$, then u, u' have no common neighbor in $\tilde{G}^{\text{small}}_{uzu'}$, except for z.*

<ins>Proof:</ins> Suppose that u, u' have a common neighbor z^* in $V(\tilde{G}^{\text{small}}_{uzu'})$ with $z^* \neq z$. By 3) of Claim 8.5.4, $\tilde{G}^{\text{small}}_{uzu'}$ is a wheel with a central vertex adjacent to each vertex of $\mathrm{D}_1^{\text{small}}$. However, since there is an internal vertex of Q_1^{small} adjacent to only one vertex of C, and C is induced and L-predictable in G, there are three consecutive vertices of Q_1^{small} with a common neighbor in C, so G contains a copy of $K_{2,3}$, contradicting the fact that $\mathcal{T}$ is a tessellation. ■

We now make the following definition. An L_ϕ-coloring ψ of $\{u, u'\}$ is called *desirable* if every extension of ψ to an L_ϕ-coloring of uzu' also extends to L_ϕ-color all of $\tilde{G}^{\text{small}}_{uzu'}$. We now have the following key claim.

Claim 8.5.9. *There exists a desirable L_ϕ-coloring of $\{u, u'\}$.*

<ins>Proof:</ins> To prove this, we apply the work of Chapter 7. In order to use the main result of Chapter 7, we first prove the following easy observation:

> **Subclaim 8.5.10.** *There is a subpath $R_\star$ of Q_1^{small} such that each endpoint of $R_\star$ is a Q_1^{small}-hinge of F_1^{small} and such that precisely one of the following holds.*
>
> *1) $R_\star = u_\star$; OR*
>
> *2) There is a unique $w \in V(\tilde{G}^{\text{small}}_{uzu'}) \setminus V(D)$ such that $G[N(w) \cap V(Q_1^{\text{small}})]$ is a subpath of Q_1^{small} with $u_\star$ as an internal vertex, and any 2-chord of F_1^{small} which separates $u_\star$ from z has w as a midpoint.*
>
> <ins>Proof:</ins> If there is a 2-chord P of F_1^{small} in $\tilde{G}^{\text{small}}_{uzu'}$ which separates $u_\star$ from z, then, since uzu' is a minimal bad 2-chord of C^1, we have $V(\tilde{G}^{\text{small}}_P) = V(P) \cup V(C^1 \cap \tilde{G}^{\text{small}}_P)$, and, since C^1 is an induced subgraph of G, it follows from our triangulation conditions that $\tilde{G}^{\text{small}}_P$ is a broken wheel with principal path P. Thus, any such 2-chord of F_1^{small}, if it exists, has a unique midpoint w, and $G[N(w) \cap V(C^1)]$ is a subpath of F_1^{small} with $u_\star$ as an internal vertex. If no such 2-chord of F_1^{small} exists, then $u_\star$ is a Q_1^{small}-hinge of F_1^{small} by definition. ■

Let $R_\star$ be as in Sublaim 8.5.10. Since Q_1^{small} differs from D by only one vertex, it immediately follows from 3) of Theorem 1.7.3 that, for any $\sigma \in \text{Link}_{L_\phi}(Q_1^{\text{small}}, F_1^{\text{small}}, G_Q^{\text{small}})$, the restriction of σ to $\{u, u'\}$ is a desirable L_ϕ-coloring of $\{u, u'\}$. Each of u, u' is trivially a Q_1^{small}-hinge of F_1^{small}, because u, u' are the endpoints of Q_1^{small}. By Claim 8.5.2, F_1^{small} is induced in $\tilde{G}^{\text{small}}_{uuz'}$, so it follows from Sublaim 8.5.10 that all the conditions of Theorem 7.0.1 are satisfied, and it immediately follows from A) of Theorem 7.0.1 that $\text{Link}_{L_\phi}(Q_1^{\text{small}}, D, G_Q^{\text{small}}) \neq \varnothing$. Thus, there is at least one desirable L_ϕ-coloring of $\{u, u'\}$. ■

We now have the following:

Claim 8.5.11. *For any 4-chord M of C whose middle three vertices are u, z, u', we have $|V(G_M^{\text{small}} \cap C)| \leq 5$. In particular, we have $4 \leq |V(Q_{0-}^{\text{small}})| \leq |V(Q_0^{\text{small}})| \leq 5$.*

Proof: Suppose toward a contradiction that there is a 4-chord M of C whose middle three vertices are u, z, u', where $|V(G_M^{\text{small}} \cap C)| > 5$. Applying Claim 8.5.9, let ψ be a desirable L_ϕ-coloring of $\{u, u'\}$. By B) of Lemma 8.4.1, $\psi \cup \phi$ extends to an L-coloring ψ^* of $V(\tilde{G}_Q^{\text{large}})$, and, since ψ is desirable, ψ^* extends to L-color $\tilde{G}_{uzu'}^{\text{small}}$, so ψ^* extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. Thus, no such M exists.

Now, by 2) of Claim 8.5.4, Q_1^{small} has length at least three, so there is at least one vertex v of $Q_1^{\text{small}} \setminus \{u, u', u_\star\}$ with a neighbor in $\{u, u'\}$, and $P_v, P_{u_\star}$ intersect on at most a common endpoint. By Claim 8.5.7, $|V(P_v)| \geq 2$, and since $|V(P_{u_\star})| \geq 3$, there are at least two vertices of $C \setminus Q_{0+}^{\text{large}}$. We conclude that $4 \leq |V(Q_{0-}^{\text{small}})| \leq |V(Q_0^{\text{small}})|$. As shown above, we have $|V(Q_0^{\text{small}})| \leq 5$, since $Q_0^{\text{small}} = C \cap G_R^{\text{small}}$, so we are done. ■

Claim 8.5.11 has the following easy consequence.

Claim 8.5.12. *At least one of u, u' has precisely one neighbor in C.*

Proof: Suppose that each of u, u' has more than one neighbor in C. Then each of P_u and $P_{u'}$ is a path of length at least one, and the paths Q_0^{small} and Q_{0-}^{small} differ in length by at least two, contradicting Claim 8.5.11. ■

We now have the following:

Claim 8.5.13. *There is at least one internal vertex of the path $Q_1^{\text{small}} \setminus \{u, u'\}$ with more than one neighbor in C.*

Proof: Suppose not. By 2) of Claim 8.5.4, Q_1^{small} has length at least three and, by Claim 8.5.7, each endpoint of $Q_1^{\text{small}} - \{u, u'\}$ has more than one neighbor in C. Thus, it follows from our assumption that $u_\star$ is one of the endpoints of $Q_1^{\text{small}} \setminus \{u, u'\}$, and there is a $p \in V(Q_1^{\text{small}} \setminus \{u, u'\}$ such that $p \neq u_\star$, where p is the other endpoint of $Q_1^{\text{small}} \setminus \{u, u'\}$, and $p, u_\star$ are the only vertices of $Q_1^{\text{small}} - \{u, u'\}$ with more than one neighbor in C. Suppose without loss of generality that $uu_\star$ and pu' are the terminal edges of Q_1^{small}.

> **Subclaim 8.5.14.** $Q_1^{\text{small}} = uu_\star pu'$.
>
> Proof: By Claim 8.5.5, there is an L_ϕ-coloring ψ of $\tilde{G}_{uzu'}^{\text{large}}$. Now suppose toward a contradiction that Q^{small} has length strictly greater than three. Let p' be the lone neighbor of $u_\star$ on $Q_1^{\text{small}} - u$. Thus, $p' \neq p$, and, by assumption, we have $|N(p) \cap V(C)| = 1$. By Claim 8.5.8, the vertices u, u' do not have a common neighbor in $\tilde{G}_{uzu'}^{\text{small}}$ other than z. Since $|L_{\phi \cup \psi}(u_\star)| \geq 1$, ψ extends to an L_ϕ-coloring ψ^* of $\text{dom}(\psi) \cup \{u_\star\}$. We have $|L_\phi(p)| \geq 3$ and, by assumption, $|L_\phi(p')| \geq 4$. Let $H := \tilde{G}_{uzu'}^{\text{small}} \setminus \{u, z, u', u_\star\}$. Since F_1^{small} is an induced cycle in G, it follows from our triangulation conditions that $u, u_\star$ have a unique common neighbor $w \in D_2(C) \cap V(\tilde{G}_{uzu'}^{\text{small}})$.
>
> Since Q_1^{small} is an induced path in G and u, u' have no common neighbor in $\tilde{G}_{uzu'}^{\text{small}}$ other than z, it follows that H has a unique facial subgraph F such that every vertex of $H \setminus F$ has an $L_{\phi \cup \psi^*}$-list of size at least five and every vertex of $F \setminus \{w, p\}$ has an $L_{\phi \cup \psi^*}$-list of size at least three. Possibly w is as adjacent to z, but not to u', so each of p, w has an $L_{\phi \cup \psi^*}$-list of size at least two, and, by Theorem 1.3.4, H is $L_{\phi \cup \psi^*}$-colorable, so $\phi \cup \psi^*$ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. ■

Let $G^\dagger$ be the graph obtained from G by deleting from G all the internal vertices of Q_{0-}^{small}. Since $V(Q_{0-}^{\text{small}})| \geq 4$ by Claim 8.5.11, we have $|V(G^\dagger)| \leq |V(G)| - 2$. SiInce $L_\phi(p)| \geq 3$ and $|L_\phi(u_\star)| \geq 2$, let σ be an L_ϕ-coloring of $pu_\star$. Let $L^\dagger$ be a list-assignment for $V(G^\dagger)$, such that $L^\dagger(u_\star) = \{\sigma(u_\star)$ and $L^\dagger(p) = \{\sigma(p)\}$, and otherwise $L^\dagger = L$. Let x, x' be the endpoints of Q_{0-}^{small}, where x is an endpoint of P_u and x' is an endpoint of $P_{u'}$. Note that $G^\dagger$ contains the cyclic facial subgraph $C^\dagger := Q_{0+}^{\text{large}} + xu_\star px'$, and let $C_*^\dagger$ be the outer face of $G^\dagger$. By our construction of $L^\dagger$, we get that $V(C^\dagger)$ is $L^\dagger$-colorable, and since $G^\dagger \subseteq G$, we get that $G^\dagger$ is short-separation-free. Thus, the tuple $\mathcal{T}^\dagger := (G^\dagger, (\mathcal{C}\setminus\{C\})\cup\{C^\dagger, L^\dagger, C^\dagger_*)$ is a tessellation in which $C^\dagger$ is a closed ring. We claim now that $\mathcal{T}^\dagger$ is a mosaic.

Since Q_{0-}^{small} has at least two internal vertices, we have $|V(C^\dagger)| \leq |V(C)|$, so M0) is satisfied, and $\text{Rk}(\mathcal{T}^\dagger|C^\dagger) \leq \text{Rk}(\mathcal{T}|C)$. By 2) of Claim 8.5.2, z is not adjacent to either of $p, u_\star$. Since $xuzux'$ separates $pu_\star$ from all the elements of $\mathcal{C}\setminus\{C\}$, it follows that $\mathcal{T}^\dagger$ satisfies the distance conditions of Definition 2.1.6. M1) is trivially satisfied so the only thing left to check is that $C^\dagger$ is $L^\dagger$-predictable in $G^\dagger$. Since C is induced in G, $C^\dagger$ is induced in $G^\dagger$. Now, u us the unique common neighbor of $x, u_\star$ in $G^\dagger$ and u' is the unique common neighbor of x', p' in $G^\dagger$. Any other vertex of $D_1(C^\dagger, G^\dagger)$ with a neighbor in $\{p, u_\star\}$ is not adjacent to any vertex of C and is adjacent to one or both of $p, u_\star$. Since C is L-predictable in G, it follows that $C^\dagger$ satisfies the subpath condition of Definition 2.1.3.

Let $\phi^\dagger$ be the unique $L^\dagger$-coloring of $V(C^\dagger)$. As indicated above, any vertex of $D_1(C^\dagger, G^\dagger)\setminus\{u, u'\}$ with a neighbor in $\{u_\star, p\}$ has no other neighbors in $C^\dagger$ and thus has an $L^\dagger_{\phi^\dagger}$-list of size at least three. By Claim 8.5.12, at most one of u, u' has more than one neighbor in C, so every vertex of $D_1(C^\dagger, G^\dagger)$ has an $L^\dagger_{\phi^\dagger}$-list of size at least three, except possibly one of $\{u, u'\}$, which has an $L^\dagger_{\phi^\dagger}$-list of size at least two. Thus, $C^\dagger$ is indeed $L^\dagger$-predictable in $G^\dagger$ and $\mathcal{T}^\dagger$ is a mosaic. Since $|V(G^\dagger)| \leq |V(G)| - 2$, it follows from the minimality of $\mathcal{T}$ that $G^\dagger$ admits $L^\dagger$-coloring σ^*. As $G^\dagger$ contains all the neighbors of u, u' in C and $Q_1^{\text{small}}\setminus\{u, u'\}$ is precolored by σ, it follows that $\sigma^*\cup\phi$ is a proper L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. This completes the proof of Claim 8.5.13. ■

Combining Claim 8.5.13 with Claim 8.5.7, there are at least three vertices of $Q_1^{\text{small}}\setminus\{u, u'\}$ with more than one neighbor in C, and since $|E(P_{u_\star})| \geq 2$, it follows that $E(Q_{0-}^{\text{small}})| \geq 4$ and thus $|V(Q_{0-}^{\text{small}})| > 4$. By Claim 8.5.11, we have $|V(Q_{0-}^{\text{small}}| = |V(Q_0^{\text{small}}| = 5$, so each of u, u' has precisely one neighbor on C. Let p, p' be the endpoints of Q_1^{small}, where $pu, p'u'$ are the terminal edges of Q_1^{small}. As there are at least three vertices of $Q_1^{\text{small}}\setminus\{u, u'\}$ with more than one neighbor in C, let q be an internal vertex of $Q_1^{\text{small}}\setminus\{u, u'\}$ with more than one neighbor in C. Note that $u_\star \in \{p, p', q\}$ and every vertex of $Q_1^{\text{small}}\setminus\{p, p', q\}$ has precisely one neighbor in C. If at least one of these does not hold, then, since $|V(P_u)| = |V(P_{u'})| = 1$, it follows that $|E(Q_0^{\text{small}})| \geq 5$ and thus $|V(Q_0^{\text{small}})| \geq 6$, which is false.

Claim 8.5.15. *There is a terminal edge $Q_1^{\text{small}}\setminus\{u, u'\}$ which contains $u_\star$ and contains q (possibly $q = u_\star$).*

<u>Proof:</u> Suppose that this does not hold and, applying Claim 8.5.5, let ψ be an L_ϕ-coloring of $V(\tilde{G}^{\text{large}}_{uzu'})$. The assumption of Claim 8.5.15 implies that $Q^{\text{small}}_{uzu'} \neq upqp'u'$, so there is a vertex of $Q\setminus\{u, u'\}$ with precisely one neighbor in C. By Claim 8.5.8, u, u' have no common neighbor in $\tilde{G}^{\text{small}}_{uzu'}$ other than z. Since D_1^{small} is induced in G, we have $|L_{\phi\cup\psi^*}(u_\star)| \geq 1$, so ψ extends to an L-coloring ψ^* of $\text{dom}(\psi)\cup\{u_\star\}$. Let $H := \tilde{G}_1^{\text{small}}\setminus\{u, z, u', u_\star\}$. Then H has a unique facial subgraph F such that F contains $Q^{\text{small}}\setminus\{u, u', u_\star\}$ and every vertex of $H\setminus F$ has an $L_{\phi\cup\psi^*}$-list of size at least five. Consider the following cases:

Case 1: $u_\star = q$

In this case, $u_\star$ is an internal vertex of $Q_1^{\text{small}}\setminus\{u, u'\}$. Let v, v' be the two neighbors of q on Q_1^{small}. By assumption, $v, v' \notin \{p, p'\}$, so each of v, v' has precisely one neighbor on C. Each of v, v' has an $L_{\phi\cup\psi^*}$-list at least three, and each of p, p' has an $L_{\phi\cup\psi^*}$-list of size at least two. Since u, u' have no common neighbor in F, any remaining vertices of F also have $L_{\phi\cup\psi^*}$-lists of size at least three. By Theorem 1.3.4, H is $L_{\phi\cup\psi^*}$-colorable, so $\phi\cup\psi^*$ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical.

Case 2: $u_\star \neq q$

In this case, suppose without loss of generality that $u_\star = p$. By assumption, pq is not a terminal edge of $Q^{\text{small}}\setminus\{u, u'\}$, so there is a vertex $v \in V(Q_1^{\text{small}})\setminus\{p, p', q\}$ such that pv is a terminal edge of $Q_1^{\text{small}}\setminus\{u, u'\}$, and $|N(v)\cap V(C)| = 1$.

Since D_1^{small} is an induced subgraph of G, it follows that $|L_{\phi\cup\psi^*}(v)|\geq 3$ and $|L_{\phi\cup\psi^*}(p')|\geq 2$, and furthermore, for any vertex $w\in V(F)\setminus\{p'\}$ with an $L_{\phi\cup\psi^*}$-list of size less than three, we have $N(w)\cap\text{dom}(\psi^*)\subseteq\{u,z,u',u_\star\}$. As indicated above, any such w is adjacent to at most one of u,u', and since D_1^{small} is induced in G, it follows from our triangulation conditions that $N(w)\cap\text{dom}(\psi^*)\neq\{u',z,u_\star\}$, so $N(w)\cap\text{dom}(\psi^*)=\{z,u,u_\star\}$. Thus, such a w, if it exists, is unique and has an $L_{\phi\cup\psi^*}$-list of size two. By Theorem 1.3.4, $\phi\cup\psi^*$ extends to an L-coloring of H, and thus $\phi\cup\psi^*$ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. ■

Applying Claim 8.5.15, we suppose without loss of generality that q is adjacent to p (i.e pq is a terminal edge of $Q_1^{\text{small}}\setminus\{u,u'\}$) and $u_\star$ is an endpoint of pq.

Claim 8.5.16. $Q_1^{\text{small}}=upqp'u'$.

Proof: Suppose not. Let v be the lone neighbor of q on Q_1^{small} which is distinct from p. Since $v\neq q'$, we have $|N(v)\cap V(C)|=1$. By Claim 8.5.8, u,u' have no common neighbor in $\tilde{G}^{\text{small}}_{uzu'}$ except for z.

Subclaim 8.5.17. *The four vertices of $\{z,u,p,q\}$ have a unique common neighbor in $\tilde{G}^{\text{small}}_{uzu'}\setminus D_1^{\text{small}}$.*

Proof: Suppose not. Since D_1^{small} is induced in G, ψ extends to an L_ϕ-coloring $\psi^\dagger$ of $\text{dom}(\psi)\cup\{p,q\}$. Let $H:=\tilde{G}^{\text{small}}_{uzu'}\setminus\{u',z,u,p,q\}$. Note that H has a unique facial subgraph F which contains the path $Q^{\text{small}}\setminus\{u,u',p,q\}$, where each vertex of $H\setminus F$ has an $L_{\phi\cup\psi^\dagger}$-list of size at least five. We have $|L_{\phi\cup\psi^\dagger}(p')|\geq 2$ and $|L_{\phi\cup\psi^\dagger}(v)|\geq 3$.

Since u,u' have no common neighbor in F and D_1^{small} is induced in G, it follows that, for any other vertex w of $V(F)\setminus\{p'\}$ which has an $L_{\phi\cup\psi^\dagger}$-list of size less than three, w has at least three neighbors among $\{z,u,p,q\}$. By assumption, any such w is not adjacent to all four of these vertices, and, since D_1^{small} is induced in G, it follows from our triangulation conditions that $G[N(w)\cap\text{dom}(\psi^\dagger)$ is a subpath of $zuu_\star q$ of length precisely two, so any such w, if it exists, is unique, and has an $|L_{\phi\cup\psi^\dagger}$-list of size two. It now follows from Theorem 1.3.4 that H is $L_{\phi\cup\psi^\dagger}$-colorable, so $\phi\cup\psi^\dagger$ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. ■

Applying Subclaim 8.5.17, let w be the unique common neighbor of $\{z,u,p,q\}$ in $\tilde{G}^{\text{small}}_{uzu'}\setminus D_1^{\text{small}}$. Since every internal vertex of the path $qQ_1^{\text{small}}p'$ has precisely one neighbor in C, it follows that q,p' have a unique common neighbor $x^\dagger\in V(C)$, and $\tilde{G}^{\text{small}}_{uzu'}$ contains the 6-cycle $u'zupqx^\dagger p'u'$. Let $W\subseteq\mathbb{R}^2$ be the unique open region containing v. Since D_1^{small} is induced in G, it follows that $\phi\cup\psi$ extends to L-color $\text{dom}(\phi\cup\psi(\cup\psi(p,q,w,p'\}$ and since G is not L-colorable, it follows from Theorem 1.3.5 that $|V(G)\cap W|\leq 3$.

Subclaim 8.5.18. $\text{Cl}(W)$ *has no chord of $D^\dagger$.*

Proof: Suppose toward a contradiction that $\text{Cl}(W)$ contains a chord of $D^\dagger$. Since D_1^{small} is induced in G,it is easy to check that $\phi\cup\psi$ extends to an L-coloring σ of $\text{dom}(\phi\cup\psi(\cup\psi(p,q,w,p',v\}$, and furthermore, any chord of $D^\dagger$ chord has w as an endpont. Note that w is not adjacent to q', or else it follows from our triangulation conditions that w is also adjacent to u', which is false since u,u' have no common neighbor in $\tilde{G}^{\text{small}}_{uzu'}$ except for z.

Thus, wv is the unique chord of $D^\dagger$ in $\text{Cl}(W)$, or else, since G is short-separation-free, we have $W\cap V(G)=\{v\}$, and since $wq',wu'\notin E(G)\cap\text{Cl}(W)$, it then follows from our triangulation conditions that $zv\in E(G)\cap\text{Cl}(W)$, contradicting 2) of Claim8.5.2. We conclude that wz is the unique chord of $D^\dagger$ in W. Since σ does not extend to L-color G and $u'zwvq'$ is a 5-cycle, it follows from Theorem 1.3.5 that there is a vertex v' such that $V(G)\cap W=\{v,v'\}$ and v' is adjacent to all five vertices of $u'zwvq'$. Let ψ' be an extension of ψ to an L_ϕ-coloring of $\text{dom}(\psi)\cup\{p'\}$. Since $u_\star\in\{p,q\}$, consider the following cases:

Case 1: $q=u_\star$

In this case, we have $|L_{\phi\cup\psi'}(p)| \geq 2$. Coloring and deleting the vertices of $\text{dom}(\phi \cup \psi')$, we are left with the graph in Figure 8.5.1, with lower bounds on the sizes of the $L_{\phi\cup\psi'}$-lists.

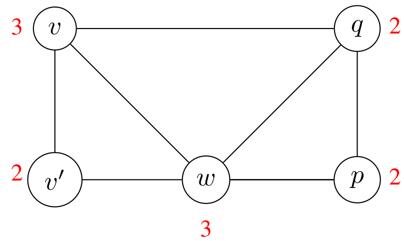

Figure 8.5.1: Case 1 of Subclaim 8.5.18

Case 2: $u_\star = p$

In this case, again deleting the domain of $\phi \cup \psi'$, we have the same graph as above except that $|L_{\phi\cup\psi'}(p)| \geq 1$ and $|L_{\phi\cup\psi'}(q)| \geq 3$, so we have the graph in Figure 8.5.2, with lower bounds on the sizes of the $L_{\phi\cup\psi'}$-lists.

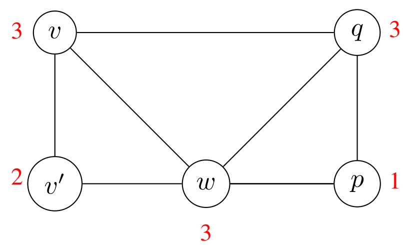

Figure 8.5.2: Case 2 of Subclaim 8.5.18

It is easy to check that the graph in Figure 8.5.2 colorable, so $\phi \cup \psi'$ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. This completes the proof of Subclaim 8.5.18. ■

Applying Subclaim 8.5.18, $\text{Cl}(W)$ has no chord of $D^\dagger$, and since v is an internal vertex of $qQ_1^{\text{small}}p'$, it follows from Theorem 1.3.5 that $(W \cap V(G)) \setminus V(qQ_1^{\text{small}}p')$ consists of precisely two vertices, or else all the vertices of $qq^{\text{small}}p'$ have a common neighbor in W and also have $x^\dagger$ as a common neighbor, contradicting the fact that is short-separation-free. Since $|V(G) \cap W| \leq 3$, it follows that v is the lone internal vertex of $qQ_1^{\text{small}}p'$ and $G \cap W$ consists of a triangle $ss'v$ for some $s, s' \in W \cap V(G)$. Let ψ^* be an extension of ψ to an L_ϕ-coloring of $\text{dom}(\psi) \cup \{p, q, w\}$. Then $G \setminus \text{dom}(\phi \cup \psi^*)$ consists of the graph in Figure 8.5.3, with lower bounds on the size of the $L_{\phi\cup\psi^*}$-lists of each vertex indicated in red. Note that each of s', v has an $L_{\phi\cup\psi^*}$-list of size at least three because p' has not been deleted.

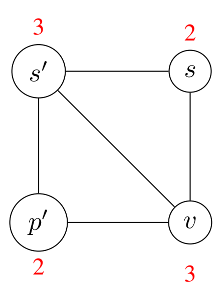

Figure 8.5.3: The last configuration in Claim 8.5.16

The graph in Figure 8.5.3 is $L_{\phi\cup\psi^*}$-colorable, so $\phi \cup \psi^*$ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. This completes the proof of Claim 8.5.16. ■

Applying Claim 8.5.16, we have $Q^{\text{small}} = upqp'u'$. Since $Q_1^{\text{small}} \setminus \{u, u'\}$ is a path in which $L_\phi(u_\star)| \geq 2$ and every other vertex has an L_ϕ-list of size at least three, there is an L_ϕ-coloring σ of Q_1^{small}. As in Claim 8.5.13, let $G^\dagger$ be the graph obtained from G by deleting all the internal vertices of Q_0^{small}. Since each of u, u' has precisely one neighbor on C, let $N(u) \cap V(C) = \{x\}$ and $N(u') \cap V(C) = \{x'\}$ for some $x, x' \in V(C)$.

Let $L^\dagger$ be a list-assignment for $V(G^\dagger)$ such that $L^\dagger(v) = \{\sigma(v)\}$ for each $v \in \{p, q, p'\}$ and otherwise $L^\dagger = L$. Note that $G^\dagger$ contains the cyclic facial subgraph $C^\dagger := Q_0^{\text{large}} + xpqp'x'$, and let $C_*^\dagger$ be the outer face of $G^\dagger$. By our construction of $L^\dagger$, we get that $V(C^\dagger)$ is $L^\dagger$-colorable, and since $G^\dagger \subseteq G$, we get that $G^\dagger$ is short-separation-free. Thus, the tuple $\mathcal{T}^\dagger := (G^\dagger, (\mathcal{C} \setminus \{C\}) \cup \{C^\dagger\}, L^\dagger, C_*^\dagger)$ is a tessellation in which $C^\dagger$ is a closed ring.

Claim 8.5.19. *$C^\dagger$ is an $L^\dagger$-predictable facial subgraph of $G^\dagger$.*

Proof: As C is induced in G, $C^\dagger$is induced in $G^\dagger$. Now, u us the unique common neighbor of $x, u_\star$ in $G^\dagger$ and u' is the unique common neighbor of x', p' in $G^\dagger$. Any other vertex of $D_1(C^\dagger, G^\dagger)$ with a neighbor in $\{p, q, p\}$ is not adjacent to any vertex of C. Since pqp' is an induced path in G, it follows from our triangulation conditions, that, for any $v \in D_1(C^\dagger, G^\dagger) \setminus \{u, u'\}$, if v has a neighbor in $C^\dagger \setminus C$, then $G^\dagger[N(v) \cap V(C^\dagger)]$ is a subpath of pqp'. Since C is L-predictable in G, it immediately follows that $C^\dagger$ satisfies the subpath condition of Definition 2.1.3.

Let $\phi^\dagger$ be the unique $L^\dagger$-coloring of $V(C^\dagger)$. As indicated above, any vertex of $D_1(C^\dagger, G^\dagger) \setminus \{u, u'\}$ with a neighbor in $\{p, q, p'\}$ has no other neighbors in $C^\dagger$. Furthermore, each of u, u' has precisely one neighbor in C and thus precisely two neighbors in $C^\dagger$. It follows that every vertex of $D_1(C^\dagger, G^\dagger)$ has an $L^\dagger_{\phi^\dagger}$-list of size at least three, unless that vertex is adjacent to all three of p, q, p'. Such a vertex, if it exists, as unique and has no other neighbors in $C^\dagger$ and thus has an $L^\dagger_{\phi^\dagger}$-list of size at least two. It follows that $C^\dagger$ is indeed $L^\dagger$-predictable in $G^\dagger$. ■

Since $|V(Q_0^{\text{small}})| = 5$, we have $|V(C^\dagger)| = |V(C)|$, so M0) is satisfied, and $\text{Rk}(\mathcal{T}^\dagger|C^\dagger) = \text{Rk}(\mathcal{T}|C)$. By 2) of Claim 8.5.2, z is not adjacent to any of $\{p, q, p'\}$, and since $xuzux'$ separates pqp' from all the elements of $\mathcal{C} \setminus \{C\}$, it follows that $\mathcal{T}^\dagger$ satisfies the distance conditions of Definition 2.1.6. M1) is trivially satisfied. It follows from Claim 8.5.19 that $\mathcal{T}^\dagger$ satisfies M2) as well and thus $\mathcal{T}^\dagger$ is indeed a mosaic. Since $|V(G^\dagger)| = |V(G)| - 3$, it follows from the minimality of $\mathcal{T}$ that $G^\dagger$ admits $L^\dagger$-coloring σ^*. Since $G^\dagger$ contains all the neighbors of u, u' in C and $Q_1^{\text{small}} \setminus \{u, u'\}$ is precolored by σ, it follows that $\sigma^* \cup \phi$ is a proper L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. We conclude that our original assumption that there exists a bad 2-chord of C^1 is false. This completes the proof of Lemma 8.5.1 and thus completes the proof of Theorem 8.0.4. □

Chapter 9

Corner Colorings

In Section 1.6, we proved a result for 3-paths of a facial cycle in a planar graph which showed that, under certain circumstances, we can find a coloring of the endpoints of the 3-path such that any extension of this precoloring to the entire 3-path also extends to the entire graph. Results of this form are very useful for the situation in which we want to delete the vertices on the small side of a 3-chord of a facial cycle in a critical mosaic while we are trying to precolor as few vertices as possible in order to avoid creating lists of size less than three. In this section and the next, we prove two variants of Theorem 1.6.1 in which, rather than only coloring the endpoints of the 3-path, we allow ourselves to precolor all but one internal vertex of the 3-path (i.e we leave a corner uncolored). The lone theorem which makes up the entiretiy of Chapter 9 is stated below. We use this theorem in Chapter 10.

Theorem 9.0.1. *Let H be a planar graph with facial cycle C, and let $P := p_1p_2p_3p_4$ be a subpath of C of length three. Let L be a list-assignment for H such that each vertex of $C \setminus P$ has an list of size at least three and each vertex of $H \setminus C$ has a list of size at least five. If either of the conditions below hold, then there exists an L-coloring ψ of $\{p_1, p_3, p_4\}$ such that, for any extension of ψ to an L-coloring ψ' of $V(P)$, ψ' extends to an L-coloring of H.*

1) *$|L(p_1)| \geq 2$ and $|L(p_4)| \geq 2$; OR*
2) *$\{p_1, p_4\}$ is L-colorable and there exists a vertex of $C \setminus P$ with a list of size at least four.*

Chapter 9 has two sections. In Section 9.1, we show that such a coloring always exists under the first condition, and in Section 9.2, we show that such a coloring always exists under the second condition.

9.1 Corner Colorings: Part I

This section consists of the following result.

Lemma 9.1.1. *Let H be a planar graph with facial cycle C, and let $P := p_1p_2p_3p_4$ be a subpath of C of length three. Let L be a list-assignment for H such that the following hold:*

1) *$|L(p_1)| \geq 2$ and $|L(p_4)| \geq 2$; AND*
2) *$|L(p_3)| \geq 4$ and, for each $v \in V(C \setminus P)$, $|L(v)| \geq 3$; AND*
3) *For each $v \in V(H \setminus C)$, $|L(v)| \geq 5$.*

Then there is an L-coloring ψ of $\{p_1, p_3, p_4\}$ such that any extension of ψ to an L-coloring of $V(P)$ also extends to an L-coloring of H.

Proof. Suppose that this does not hold and let H be a vertex-minimal counterexample to the claim. For convenience, we suppose, by applying an appropriate stereographic projection, that C is the outer face of H. By adding edges to H if necessary, we also suppose that every facial subgraph of H, except possibly C, is a triangle. By removing colors from the lists of $V(H)$ if necessary, we suppose forther that $|L(p_1)| = |L(p_4)| = 2$ and $|L(u)| = 3$ for each $u \in V(C \setminus P)$.

Since H is a counterexample, it follows that, for any proper L-coloring σ of $\{p_1, p_3, p_4\}$ there is an extension of σ to an L-coloring Ψ_σ of $V(P)$ such that Ψ_σ does not extend to an L-coloring of H. If $V(C) = V(P)$ then it follows from Corollary 0.2.4 that, for any L-coloring σ of $\{p_1, p_3, p_4\}$, Ψ_σ extends to an L-coloring of H, which is false. Thus, we have $|V(C)| > 4$, so let $C := p_4p_3p_2p_1u_1 \cdots u_t$ for some $t \geq 1$. As usual applying Theorem 0.2.3 and Corollary 0.2.4, we immediately have the following from the minimality of H.

Claim 9.1.2. *H is short-separation-free Any chord of C has an endpoint in $\{p_2, p_3\}$.*

We now fix two colors a_0, a_1 such that $L(p_1) = \{a_0, a_1\}$ and b_0, b_1 such that $L(p_4) = \{b_0, b_1\}$.

Claim 9.1.3. *Every chord of C has p_2 as an endpoint.*

Proof: We first rule out the possibility that $p_1p_3 \in E(H)$. Suppose toward a contradiction that $p_1p_3 \in E(H)$. Since H is short-separation-free, $H - p_2$ is bounded by outer cycle $C' := p_1u_1 \cdots u_tp_4p_3$. Since $|L(p_1)| \geq 2$ and $|L(p_4)| \geq 2$, it follows from Theorem 1.5.10 that there is a pair $(c, d) \in L(p_1) \times L(p_4)$ such that any L-coloring of $p_1p_3p_4$ coloring p_1, p_4 with c, d respectively extends to an L-coloring of $H - p_2$. Possibly $c = d$. This is permissible since $|V(C)| > 4$ and, by Claim 9.1.2, p_1p_4 is not a chord of C. Let σ be any L-coloring of $\{p_1, p_3, p_4\}$ using c, d on the respective vertices p_1, p_4. Then Ψ_σ extends to an L-coloring of H, which is false.

Thus, we have $p_1p_3 \notin E(H)$. Now suppose toward a contradiction that there is a chord of C which does not have p_2 as an endpoint. By Claim 9.1.2, there is a chord of C of the form p_3u_m for some $m \in \{1, \cdots, t\}$. Let m be the minimal index such that this holds. Let $H = K \cup K'$ be the natural p_3u_m-partition of H, where $p_1 \in V(K)$, and $p_4 \in V(K')$.

> **Subclaim 9.1.4.** $u_m \in N(p_2)$.
>
> Proof: Suppose toward a contradiction that $u_m \notin N(p_2)$. Let $P' := p_1p_2p_3u_m$. Note that K is bounded by outer face $C' := p_1p_2p_3u_m \cdots u_1$. By our choice of m, we have $N(p_3) \cap V(C') = \{p_2, u_m\}$, as we have already shown that $p_1p_3 \notin E(H)$. Since $|L(u_m)| = 3$, it follows from Theorem 1.6.1 that, for each $i = 0, 1$, there is a $d_i \in L(u_m)$ such that any L-coloring of $V(P')$ using a_i, d_i on the respective vertices p_1, p_4 extends to an L-coloring of K, where $a_i \neq d_i$ if $p_1u_m \in E(H')$ (possibly $d_0 = d_1$).
>
> Since $|L(p_3)| \geq 4$ and $|L(p_4)| = 2$, let $f \in L(p_3) \setminus (L(p_4) \cup \{d_0\})$. By our choice of f, it follows from Observation 1.4.2 that the L-coloring (d_0, f) of u_mp_3 extends to an L-coloring ϕ of K'. Now let σ be the L-coloring of $\{p_1, p_3, p_4\}$ using $a_0, f, \phi(p_4)$ on the respective vertices p_1, p_3, p_4. Since $u_m \notin N(p_2)$, the union $\phi \cup \sigma$ is a proper L-coloring of its domain in H, and, by our choice of d_0, it follows that Ψ_σ extends to an L-coloring of H, which is false. ■

Since $u_mp_2 \in E(H)$, let H' be the subgraph of H bounded by outer cycle $p_1p_2u_m \cdots u_1$. Note that H' and K' intersect precisely on u_m. Let $P_\ell := p_1p_2u_m$ and $P_r := u_mp_3p_4$.

Subclaim 9.1.5. *H' is a broken wheel with principal path P_ℓ.*

Proof: Suppose toward a contradiction that H' is not a broken wheel with principal path P_ℓ. By Claim 9.1.2, any chord of the outer face of H' has p_2 as an endpoint. By Theorem 1.5.3, there is an $i \in \{0,1\}$ such that any L-coloring of $V(P_\ell)$ using a_i on p_1 extends to an L-coloring of H'. Since $|L(p_3)| \geq 4$, let $c \in L(p_3) \setminus L(u_m)$ and let $j \in \{0,1\}$ with $b_j \neq c$. Let σ be an L-coloring of $\{p_1, p_3, p_4\}$ using a_i, c, b_j on the respective vertices p_1, p_3, p_4. Since $p_1p_3 \notin E(H)$, σ is a proper L-coloring of its domain. Since $c \notin L(u_m)$, it follows from Observation 1.4.2 that there is an extension of Ψ_σ to an L-coloring ϕ of $V(P) \cup V(K')$ such that $\phi(u_m) \neq \Psi_\sigma(p_2)$. By assumption, H' is not a triangle, so, by Claim 9.1.2, p_1u_m is not an edge of H'. Thus, the coloring $(a_i, \Psi_\sigma(p_2), \phi(p_2))$ is a proper L-coloring of $V(P_\ell)$. By our choice of a_i, this L-coloring of P_ℓ extends to an L-coloring of K', so Ψ_σ extends to an L-coloring of H, which is false. ■

We now make the following definition:

Definition 9.1.6. A *coloring matrix* for K' is a 2×2 array $(\phi^{ij} : 0 \leq i, j \leq 1)$ such that the following holds:

arabic*) For each $0 \leq i, j \leq 1$, ϕ^{ij} is a proper L-coloring of $\{p_1, p_3, p_4\}$ such that $\phi^{ij}(p_1) = a_i$; *AND*

arabic*) There exist $q_0, q_1 \in L(u_m)$ such that $q_0 \neq q_1$ and, for each $0 \leq i, j \leq 1$, $\mathcal{Z}_{H'}^{P_\ell}(a_i, \Psi_{\phi^{ij}}(p_2), \bullet) = \{q_j\}$.

Subclaim 9.1.7. *There does not exist a coloring matrix of K'.*

Proof: Suppose toward a contradiction that such an array $(\phi^{ij} : 0 \leq i, j \leq 1)$ exists. For each $0 \leq i, j \leq 1$, let $s_{ij} := \Psi_{\phi^{ij}}(p_2)$ and let q_0, q_1 be two distinct colors of $L(u_m)$ such that, for each $0 \leq i, j \leq 1$, $\mathcal{Z}_{H'}^{P_\ell}(a_i, s_{ij}, \bullet) = \{q_j\}$.

By Subclaim 9.1.5, H' is a broken wheel with principal path P_ℓ. For each $i \in \{0,1\}$, we have $s_{i0} \neq s_{i1}$, since $\mathcal{Z}_{H'}^{P_\ell}(a_i, s_{i0}\bullet) \neq \mathcal{Z}_{H'}^{P_\ell}(a_i, s_{i1}, \bullet)$. Thus, it immediately follows from 1) of Proposition 1.4.7 that $s_{i0} = q_1$ and $s_{i1} = q_0$ for each $i \in \{0,1\}$. Since each of the four colorings $\{\Psi_{\phi^{ij}} : 0 \leq i, j \leq 1\}$ is a proper L-coloring of its domain, it follows that $\{a_0, a_1\} \cap \{q_0, q_1\} = \varnothing$.

Now, if H' is a triangle, then, for each $0 \leq i, j \leq 1$, since $|\mathcal{Z}_{H'}^{P_\ell}(a_i, s_{ij}, \bullet)| = 1$, we have $a_i \in L(u_1)$ and $s_{ij} \in L(u_1)$. Likewise, if H' is not a triangle, then, for each pair $0 \leq i, j \leq 1$, since $\mathcal{Z}_{H'}^{P_\ell}(a_i, s_{ij}, \bullet)| = 1$, it immediately follows from Proposition 1.4.4 that $a_i \in L(u_1)$ and $s_{ij} \in L(u_1)$. Thus, in any case, we have $\{a_0, a_1\} \cup \{q_0, q_1\} \subseteq L(p_1)$. Since $\{a_0, a_1\} \cap \{q_0, q_1\} = \varnothing$. this contradicts the fact that $|L(u_1)| = 3$. ■

We now have the following:

Subclaim 9.1.8. *K' is a broken wheel with principal path P_r and $\{b_0, b_1\} \subseteq L(u_t)$.*

Proof: Suppose toward a contradiction that at least one of these does not hold. We claim now that there exists a $k \in \{0,1\}$ such that any L-coloring of $V(P_r)$ using b_k on p_4 extends to an L-coloring of K' and such that, if K' is a broken wheel with principal path P_r, then $b_k \notin L(u_t)$.

If K' is not a broken wheel with principal path P_r, then this immediately follows from Theorem 1.5.3, since, by Claim 9.1.2, there is no chord of the outer face of K' without p_3 as an endpoint. Now suppose that K' is a broken wheel with principal path P_r. By assumption, there is a $k \in \{0,1\}$ such that $b_k \notin L(u_t)$, and thus, by Proposition 1.4.4, any L-coloring of $V(P_r)$ using b_k on p_4 extends to an L-coloring of K' (possibly K' is a triangle and any proper L-coloring of $V(P_r)$ is also an L-coloring of K').

Let $k \in \{0,1\}$ be as above, and suppose without loss of generality that $k = 0$. Let q_0, q_1 be distinct colors in $L(p_3) \setminus \{b_0\}$. For each $i \in \{0,1\}$ and $j \in \{0,1\}$, let ϕ^{ij} be the L-coloring of $\{p_1, p_3, p_4\}$ obtained by coloring p_1, p_3, p_4 with the respective colors a_i, q_j, b_0. Since $p_1p_3 \notin E(H)$, each such ϕ^{ij} is a proper L-coloring of its domain. For each $0 \leq i, j \leq 1$, let $s_{ij} := \Psi_{\phi^{ij}}(p_2)$. We claim now that, for each such pair $0 \leq i, j \leq 1$, we have $\mathcal{Z}^{P_\ell}_{H'}(a_i, s_{ij}, \bullet) = \{q_j\}$.

Fix a pair $0 \leq i \leq 1$, $0 \leq j \leq 2$, and suppose toward a contradiction that $\mathcal{Z}^{P_\ell}_{H'}(a_i, s_{ij}, \bullet) \neq \{q_j\}$. Since $\mathcal{Z}^{P_\ell}_{H'}(a_i, s_{ij}, \bullet) \neq \varnothing$, let $q \in \mathcal{Z}^{P_\ell}_{H'}(a_i, s_{ij}, \bullet)$ with $q \neq q_j$. If K' is a triangle, then K' is a broken wheel with principal path P_r, and, by assumption, we have $b_0 \notin L(u_t)$. Since K' is a triangle, we have $u_m = u_t$ and $q \neq b_0$. Since we also have $q \neq q_j$ as well, $\Psi_{\phi^{ij}}$ extends to an L-coloring of H, which is false.

Thus, K' is not a triangle. Since u_mp_1 is not a chord of C, we have $u_mp_1 \notin E(K')$ and (q, q_j, b_0) is a proper L-coloring of the subgraph of H induced by $u_mp_3p_4$. By our choice of b_0, this coloring of P_r extends to an L-coloring of K', so $\Psi_{\phi^{ij}}$ extends to an L-coloring of H, which is false. We conclude that, for each $0 \leq i, j \leq 1$, we have $\mathcal{Z}^{P_\ell}_{H'}(a_i, s_{ij}, \bullet) = \{q_j\}$, as desired. Thus, $(\phi^{ij} : 0 \leq i, j \leq 1)$ is a coloring matrix for K', contradicting Subclaim 9.1.7. ■

We now have the following:

Subclaim 9.1.9. *K' is not a triangle.*

Proof: Suppose toward a contradiction that K' is a triangle. Thus, $u_m = u_t$. By Subclaim 9.1.8, we have $\{b_0, b_1\} \subseteq L(u_m)$, and there is a $c \in L(p_3) \setminus L(u_m)$ Since $p_1p_3 \notin E(H)$ and $|L(p_3)| \geq 4$, it follows that, for each pair $0 \leq i, j \leq 1$, there is an L-coloring ϕ^{ij} of $\{p_1, p_3, p_4\}$ using a_i, c, b_j on the respective vertices p_1, p_3, p_4. For each $0 \leq i, j \leq 1$, let $s_{ij} := \Psi_{\psi^{ij}}(p_2)$. Note that, for each $0 \leq i, j \leq 1$, we have $\mathcal{Z}^{P^\ell}_{H'}(a_i, s_{ij}, \bullet) = \{b_j\}$, or else, since $\mathcal{Z}^{P^\ell}_{H'}(a_i, s_{ij}, \bullet) \neq \varnothing$ and $c \notin L(u_m)$, $\Psi_{\phi^{ij}}$ extends to an L-coloring of H, which is false. Thus, $(\phi^{ij} : 0 \leq i, j \leq 1)$ is a coloring matrix for K', contradicting Subclaim 9.1.7. ■

We now have the following:

Subclaim 9.1.10. *There is a set $A \subseteq L(p_3)$ with $|A| = 3$ such that, for each $v \in \{u_{m+1}, \cdots, u_t\}$, $L(v) = A$.*

Proof: Suppose toward a contradiction that this does not hold. Since $|L(p_3)| \geq 4$, there is a pair of colors $f_0, f_1 \in L(p_3)$ such that, for each $i = 0, 1$, $f_i \notin \bigcap_{k=m+1}^t L(u_k)$. For each pair $0 \leq i, j \leq 1$, let ϕ^{ij} be an L-coloring of $\{p_1, p_3, p_4\}$ using a_i, f_j on the respective vertices p_1, p_3. Since $|L(p_4)| = 2$ and $p_1p_3 \notin E(H)$, there exists such a ϕ^{ij} for each pair $0 \leq i, j \leq 1$.

For each $0 \leq i, j \leq 1$, let $s_{ij} := \Phi_{\phi^{ij}}(p_2)$. We claim now that, for any $0 \leq i, j \leq 1$, we have $\mathcal{Z}^{P_\ell}_{H'}(a_i, s_{ij}, \bullet) = \{f_j\}$. Suppose toward a contradiction that there is a pair $0 \leq i, j \leq 1$ for which this does not hold. Since $\mathcal{Z}^{P_\ell}_{H'}(a_i, s_{ij}, \bullet) \neq \varnothing$, let $q \in \mathcal{Z}^{P_\ell}_{H'}(a_i, s_{ij}, \bullet)$ with $q \neq f_j$. By Subclaim 9.1.9, K' is not a triangle, so $(q, f_j, \phi^{ij}(p_4))$ is a proper L-coloring of $u_mp_3p_4$. By our choice of f_j, this L-coloring of P_r extends to an L-coloring of K', so $\Phi_{\phi^{ij}}$ extends to an L-coloring of H, which is false. Thus, we indeed have $\mathcal{Z}^{P_\ell}_{H'}(a_i, s_{ij}, \bullet) = \{f_j\}$ for all $0 \leq i, j \leq 1$, so $(\phi^{ij} : 0 \leq i, j \leq 1)$ is a coloring matrix for K', contradicting Subclaim 9.1.7. ■

Let $A \subseteq L(p_3)$ be as in Subclaim 9.1.10.

Subclaim 9.1.11. *Let $q \in L(p_3) \setminus A$ and let σ be an L-coloring of $V(P)$ with $\sigma(p_3) = q$. Then $\mathcal{Z}^{P_\ell}_{H'}(\sigma(p_1), \sigma(p_2), \bullet) = \{q\}$. In particular, $q \in L(u_m)$.*

Proof: Since $\mathcal{Z}^{P_\ell}_{H'}(\sigma(p_1), \sigma(p_2), \bullet) \neq \varnothing$, let $q^* \in \mathcal{Z}^{P_\ell}_{H'}(\sigma(p_1), \sigma(p_2), \bullet)$ and suppose that $q^* \neq q$. Since K' is not a triangle, $(q^*, q, \sigma(p_1))$ is a proper L-coloring of $u_mp_3p_4$, and since $q \notin A$, it follows from Proposition 1.4.4

that Ψ_σ extends to an L-coloring of H, which is false. Thus, we indeed have $\mathcal{Z}_{H'}^{P_\ell}(\sigma(p_1), \sigma(p_2), \bullet) = \{q\}$. ■

With the subclaims above in hand, we now have enough to finish the proof of Claim 9.1.3 by constructing a coloring matrix for K'. Since $|L(p_3)| \geq 4$ and $|L(u_m)| = 3$, let $q_0 \in L(p_3) \setminus A$ and let $q_1 \in L(p_3) \setminus L(u_m)$. By Subclaim 9.1.11, we have $q_0 \neq q_1$, since $q_1 \notin L(u_m)$. Since $\{b_0, b_1\} \subseteq A$, we have $q_0 \notin \{b_0, b_1\}$, so we fix a color $b \in \{b_0, b_1\} \setminus \{q_0, q_1\}$.

For each pair $0 \leq i, j \leq 1$, let ϕ^{ij} be an L-coloring of $\{p_1, p_3, p_4\}$ obtained by coloring the vertices p_1, p_3, p_4 with the respective colors a_i, q_j, b. Since $p_1p_3, p_1p_4 \notin E(H)$, it follows that, for each $0 \leq i, j \leq 1$, ϕ^{ij} is a proper L-coloring of its domain. For each $0 \leq i, j \leq 1$, let $s_{ij} := \Psi_{\phi^j}(p_2)$. It follows from Subclaim 9.1.11 that, for each $i \in \{0, 1\}$, we have $\mathcal{Z}_{H'}^{P_\ell}(a_i, s_{i0}, \bullet) = \{q_0\}$. We claim now that there exists a $c \in L(u_m) \setminus \{q_0\}$, such that, for each $i \in \{0, 1\}$, we have $\mathcal{Z}_{H'}^{P_\ell}(a_i, s_{i1}, \bullet) = \{c\}$.

Let $i \in \{0, 1\}$. Since $\mathcal{Z}_{H'}^{P_\ell}(a_i, s_{i1}, \bullet) \neq \varnothing$, let $c \in \mathcal{Z}_{H'}^{P_\ell}(a_i, s_{i1}, \bullet)$. Since $q_1 \notin L(u_m)$ and K' is not a triangle, (c, q_1, b) is a proper L-coloring of $u_mp_2p_3$, and since $\Psi_{\phi^{i1}}$ does not extend to an L-coloring of H, (c, q_1, b) does not extend to an L-coloring of K'. Now consider the following cases:

Case 1: $q_1 \notin \{b_0, b_1\}$

In this case, we have $c \in \{b_0, b_1\}$, or else we extend (c, q_1, b) to an L-coloring of K' by 2-coloring the path $u_{m+1} \cdots u_t$. If the path $K' - p_3$ has even length, then c is the lone color of $\{b_0, b_1\} \setminus \{b\}$, or else we extend $\Psi_{\phi^{ij}}$ to an L-coloring of H by 2-coloring $K' - p_3$ with $\{b_0, b_1\}$. Likewise, if the path $K' - p_3$ has odd length, then $c = b$, or else, again, we extend $\Psi_{\phi^{ij}}$ to an L-coloring of H by 2-coloring $K' - p_3$ with $\{b_0, b_1\}$. In any case, c is unique and independent of the choice of $i \in \{0, 1\}$, and $c \in \{b_0, b_1\}$, so $c \neq q_0$.

Case 2: $q_1 \in \{b_0, b_1\}$

In this case, let q' be the lone color of $A \setminus \{b_0, b_1\}$. Note that $c = q'$, or else we extend (c, q_1, b) to an L-coloring of K' by 2-coloring the path $u_{m+1} \cdots u_t$ with the colors of $\{b, q'\}$. Thus, c is unique and, since $c \in A$, we have $c \neq q_0$.

Since the case analysis above is independent of the choice of $i \in \{0, 1\}$, it follows that there exists a $c \in L(u_m) \setminus \{q_0\}$, such that, for each $i \in \{0, 1\}$, we have $\mathcal{Z}_{H'}^{P_\ell}(a_i, s_{i1}, \bullet) = \{c\}$. Thus, array $(\phi^{ij} : 0 \leq i, j \leq 1)$ is a coloring matrix for K', contradicting Subclaim 9.1.7. This completes the proof of Claim 9.1.3. ■

We now have the following:

Claim 9.1.12. *p_1, p_3 have no common neighbor in H, except for p_2.*

Proof: Suppose that p_1, p_3 have a common neighbor $w \in V(H - p_2)$. Since $|V(C)| > 4$, we have $w \notin V(C)$, or else there is a chord of C without p_2 as an endpoint, contradicting Claim 9.1.3. Since H is short-separation-free, $H - p_2$ is bounded by outer cycle $C' := p_1wp_3p_4u_t \cdots u_1$. By the minimality of H, there is an L-coloring σ of $\{p_1, p_3, p_4\}$ such that any extension of σ to an L-coloring of $\{p_1, w, p_3, p_4\}$ also extends to an L-coloring of $H - p_2$. Since $|L_{\Psi_\sigma}(w)| \geq 1$, it follows that Ψ_σ extends to an L-coloring of H, contradicting our assumption. ■

We now have the following:

Claim 9.1.13. *$p_2p_4 \notin E(H)$, and p_2, p_4 have no common neighbor in H, except for p_3.*

Proof: Suppose toward a contradiction that $p_2p_4 \in E(H)$. Since H is short-separation-free, $H - p_3$ is bounded by outer face $C' := p_1p_2p_4u_t \cdots u_1$. By Theorem 1.5.10, there is a pair $(c, d) \in L(p_1) \cap L(p_4)$, where $c \neq d$ if

$p_1p_4 \in E(H - p_3)$, such that any L-coloring of $p_1p_2p_4$ using c, d on the respective vertices p_1, p_4 extends to an L-coloring of H. Since $|L(p_3)| \geq 4$, let σ be an L-coloring of $\{p_1, p_3, p_4\}$ using c, d on the respective vertices p_1, p_4. Then Ψ_σ extends to an L-coloring of H, which is false. Thus, $p_2p_4 \notin E(H)$.

Now suppose toward a contradiction that p_2, p_4 have a common neighbor $w \in V(H - p_3)$. We first show that $w \notin V(C)$. Suppose that $w \in V(C)$. Then $w = u_t$, or else there is a chord of C without p_2 as an endpoint, contradicting Claim 9.1.3. Since $w = u_t$, H contains the 4-cycle $p_2u_tp_4p_3$, and since H is short-separation-free and $p_2p_4 \notin E(H)$, it follows from our triangulation conditions that $u_tp_3 \in E(H)$, which again contradicts Claim 9.1.3.

Thus, $w \notin V(C)$. Since H is short-separation-free, $H - p_3$ is bounded by outer cycle $C' := p_1p_2wp_4u_t \cdots u_1$. Since $|L(w)| \geq 5$, it follows from the minimality of H that there exist two L-colorings ψ_0, ψ_1 of $\{p_1, w, p_4\}$, where $\psi_0(w) \neq \psi_1(w)$, such that, for each $i = 0, 1$, any extension of ψ_i to an L-coloring of $p_1p_2wp_4$ also extends to an L-coloring of $H - p_3$. For each $i = 0, 1$, we let $c_i := \psi_i(w)$.

Subclaim 9.1.14. *For each $i \in \{0, 1\}$, $c_i \in L(p_2) \setminus \{\psi_i(p_1)\}$, and furthermore, letting τ_i be the L-coloring of $\{p_1, p_2, p_4\}$ obtained by coloring p_1, p_2, p_4 with the respective vertices $\psi_i(p_1), c_i, \psi_i(p_r)$, τ_i does not extend to an L-coloring of $H - p_3$.*

Proof: Since $|L(p_3)| \geq 4$, we define the following: For each $i \in \{0, 1\}$, let f_{i0}, f_{i1} be two distinct colors of $L(p_3) \setminus \{\psi_i(w), \psi_i(p_4)\}$. For each pair $0 \leq i, j \leq 1$, we let σ^{ij} be the L-coloring of p_1, p_3, p_4 with the respective colors $\psi_i(p_1), f_{ij}, \psi_i(p_4)$ (note that the four resulting colorings of p_1, p_3, p_4 are not necessarily distinct.)

We first note that, for each $0 \leq i, j \leq 1$, we have $\Psi_{\sigma^{ij}}(p_2) = c_i$. To see this, let $0 \leq i, j \leq 1$ and suppose toward a contradiction that $\Psi_{\sigma^{ij}}(p_2) \neq c_i$. Since $f_{ij} \neq c_i$, we then have $c_i \in L_{\Psi_{\sigma^{ij}}}(w) \setminus \{f_{ij}\}$, as ψ_i is a proper L-coloring of its domain in $H - p_3$ by assumption. Thus, $\Psi_{\sigma^{ij}}$ extends to an L-coloring of H, which is false. We conclude that $\Psi_{\sigma^{ij}}(p_2) = c_i$ for each pair $0 \leq i, j \leq 1$, so $c_i \in L(p_2) \setminus \{\psi_i(p_1)\}$.

For each $i = 0, 1$, let τ_i be as in the statement of the subclaim. Since $p_2p_4 \notin E(H)$, each of τ_0, τ_1 is a proper L-coloring of its domain. Suppose toward a contradiction that one of these extends to an L-coloring of $H - p_3$, and suppose without loss of generality that τ_0 extends to an L-coloring τ_0^* of $H - p_3$. Since one of f_{00}, f_{01} is left over in $L_{\tau_0^*}(p_3)$, it follows that one of $\Psi_{\sigma^{00}}, \Psi_{\sigma^{01}}$ extends to an L-coloring of H, which is false. ■

Let τ_0, τ_1 be as in the statement of Subclaim 9.1.14.

Subclaim 9.1.15. *There is a chord of C with p_2 as an endpoint. Furthermore, if m is the maximal index among $\{1 \leq j \leq t : u_j \in N(p_2)\}$, then $m < t$ and $u_m \in N(w)$.*

Proof: Suppose that this does not hold. By Claim 9.1.3, C is an induced cycle of H. Let F be the outer face of $H \setminus P$, and consider the list-assignment L_{τ_0} for $H \setminus P$. By Claim 9.1.12, we have $p_1 \notin N(w)$. Thus, since we have not colored p_3, we have $|L_{\tau_0}(w)| \geq 3$. Since H is short-separation-free, w is the unique common neighbor of p_1, p_3, so we have $|L_{\tau_0}(z)| \geq 3$ for all $z \in V(F) \setminus \{u_1, u_t\}$. If $t = 1$, then, since C is an induced subgraph of H, we have $|L(u_1)| \geq 1$, and, by Theorem 0.2.3, $H \setminus P$ is L_{τ_0}-colorable, contradicting Subclaim 9.1.14. If $t > 1$, then, again, since C is an induced subgraph of H, $|L_{\tau_0}(u_1)| \geq 2$ and $|L_{\tau_0}(u_t)| \geq 2$. Thus, follows from Theorem 1.3.4 that $H \setminus P$ is L_{τ_0}-colorable, contradicting Subclaim 9.1.14.

Since there is a chord of C with p_2 as an endpoint and $p_2p_4 \notin E(H)$, let m be the maximal index among $\{1$ $leqj \leq t : u_j \in N(p_2)\}$. Let $H - p_3 = H' \cup H''$, where $H' \cap H'' = p_2u_m$, $p_1 \in V(H')$, and $p_4 \in V(H'')$. Then the outer face of H'' is the cycle $u_mp_2w_4u_t \cdots u_{m+1}$, and since every chord of C in H has p_2 as an endpoint, this is an induced subgraph of H''. If $m = t$, then, in H, the 4-cycle $p_2p_3p_4u_t$ separates w from p_1, contradicting the fact that H is short-separation-free. Thus, $m < t$.

Let $i \in \{0, 1\}$. Then, by Theorem 0.2.3, we have $\mathcal{Z}_{H'}(\tau_i(p_1), c_i, \bullet) \neq \varnothing$, and since $m < t$ and the outer face of H'' is an induced subgraph of H'', it follows that there is an extension of τ_i to an L-coloring τ_i^* of $V(H') \cup \{w, p_4\}$. Since $u_m \notin N(w)$, we have $|L_{\tau_i^*}(w)| \geq 3$, so, applying the same argument as above, with the role of p_1 replaced by u_m, τ_i^* extends to an L-coloring of $H - p_3$, and thus τ_i extends to an L-coloring of $H - p_3$, contradicting Subclaim 9.1.14. ■

As above, let $m \in \{1, \cdots, t\}$ be the maximal index among $\{1 \leq j \leq t : u_j \in N(p_2)\}$, and let $H - p_3 = H' \cup H''$ be the natural p_2u_m-partition of $H - p_3$, where $p_1 \in V(H')$ and $p_4 \in V(H'')$. Since H is short-separation-free, $H'' - p_2$ is bounded by outer face $u_m w u_t u_{t-1} \cdots u_m$.

Subclaim 9.1.16. *$H'' - p_2$ is a broken wheel with principal path $u_m w p_4$, and $L(p_4) \subseteq L(u_t)$.*

Proof: Suppose that at least one of these conditions does not hold. Note that every chord of the outer face of $H'' - p_2$ has w as an endpoint, or else there is a chord of C which does not have p_2 as an endpoint, contradicting Claim 9.1.3. If $H'' - p_2$ is not a broken wheel with principal path $u_m w p_4$, it follows from Theorem 1.5.3 that there is a $b \in L(p_r)$ such that any L-coloring of $u_m w p_4$ using b on p_4 extends to an L-coloring of $H'' - p_2$. Likewise, if $H'' - p_2$ is a broken wheel with principal path $u_m w p_4$, but $L(p_4) \not\subseteq L(u_t)$, then it follows from Proposition 1.4.4 that there is a $b \in L(p_4)$ such that any L-coloring of $u_m w p_4$ using b on p_4 extends to an L-coloring of $H'' - p_2$. Thus, in any case, we fix such a $b \in Lp_4)$.

Let σ be an L-coloring of $\{p_1, p_3, p_4\}$ with $\sigma(p_4) = b$. Since $\mathcal{Z}_{H'}(\sigma(p_1), \Psi_\sigma(p_2), \bullet) \neq \varnothing$ and $u_m \notin N(p_4)$, Ψ_σ extends to an L-coloring Ψ^* of $V(H') \cup \{p_3, p_4\}$. Since $N(w) \cap \text{dom}(\Psi^*) = \{p_2, p_3, p_4, u_m\}$ and $|L(w)| \geq 5$, Ψ^* extends to L-color w as well, and, by our choice of b, the resulting L-coloring of the principal path $u_m w p_4$ extends to an L-coloring of H, so Ψ_σ extends to an L-coloring of H, which is false. ■

Recall that $L(p_4) = \{b_0, b_1\}$. Since $|L(u_t)| = 3$, let $L(u_t) = \{b_0, b_1, f\}$ for some color f. For each $i = 0, 1$, let τ_i^* be an extension of τ_i to an L-coloring of $V(H') \cup \{p_1\}$. As indicated above, such a τ_i^* exists $m < t$ and $u_m, p_2 \notin N(p_4)$.

Subclaim 9.1.17. *For each $i \in \{0, 1\}$, the following hold.*

1) $L_{\tau_i^*}(w) \subseteq \{b_0, b_1, f\}$ and $|L_{\tau_i^*}(w)| = 2$; AND

2) $L_{\tau_i^*}(w) \subseteq L(u_k)$ for each $k = m+1, \cdots, t$

Proof: Let $i \in \{0, 1\}$, and suppose without loss of generality that $i = 0$. We first prove 1). Suppose that at least one of these two conditions does not hold. Since $\tau_0^*(p_4) \in \{b_0, b_1\}$, it follows that there is a color $f^* \in L_{\tau_i^*}(w) \setminus \{b_0, b_1, f\}$. By Proposition 1.4.4, the L-coloring $(\tau_0^*(p_2), f^*, \tau_0^*(p_4))$ of $u_m w p_4$ extends to an L-coloring of $H'' - p_2$, so τ_0^* extends to an L-coloring of H, and thus τ_0 extends to an L-coloring of H, contradicting Subclaim 9.1.14. This proves 1). Likewise, since τ_0^* does not extend to an L-coloring of H, it follows from Proposition 1.4.4 that $L_{\tau_0^*}(w) \subseteq L(u_k)$ for each $k = m+1, \cdots, t$. ■

For each $i \in \{0, 1\}$, since τ_i^* does not extend to an L-coloring of H, we have $\tau_i^*(u_m) \notin \{b_0, b_1\}$, or else $L_{\tau_i^*}$ contains at least two colors not lying on $\{b_0, b_1\}$, contradicting Subclaim 9.1.17. Now let $i \in \{0, 1\}$ and suppose without loss of generality that $\tau_i^*(p_4) = b_0$. Applying Subclaim 9.1.17 we then have $L_{\tau_i^*}(w) = \{b_1, f\}$. As indicated above, we have $\{b_1, f\} \subseteq L(u_k)$ for each $k = m+1, \cdots, t$. Again applying Proposition 1.4.4, since $\tau_i^*(u_m) \notin L_{\tau_i^*}(w)$ and τ_i^* does not extend to an L-coloring of H, we have $L(u_{m+1}) = \{\tau_i^*(u_m), b_1, f\}$. Consider the following cases:

Case 1: $\tau_{1-i}^*(p_4) = b_0$

In this case, as with τ_i^*, we have $L_{\tau_{1-i}^*}(w) = \{b_1, f\}$ and $L(u_{m+1}) = \{\tau_{1-i}^*(u_m), b_1, f\}$. In particular, τ_{1-i}^* and τ_{1-i}^* use the same colors on u_m, p_4, and, by construction, they do not use the same color on p_2. Since each of $L_{\tau_0^*}(w)$ $L_{\tau_1^*}(w)$ has size two, it follows that $L_{\tau_0^*}(w) \neq L_{\tau_1^*}(w)$, contradicting the fact that $L_{\tau_0^*}(w) = L_{\tau_0^*}(w) = \{b_1, f\}$.

Case 2: $\tau_{1-i}^*(p_4) = b_1$

In this case, we have $L_{\tau_{1-i}^*}(w) = \{b_0, f\}$. Since $L_{\tau_i^*}(w) = \{b_1, f\}$, it follows from 2) of Subclaim 9.1.17 that $L(u_k) = \{b_0, b_1, f\}$ for all $k = m+1, \cdots, t$. Since $L(u_{m+1}) = \{\tau_i^*(u_m), b_1, f\}$, we have $\tau_i^*(u_m) = b_0$, so τ_i^* uses the same color on u_m, p_4. Thus, $|L_{\tau_i^*}(w)| \geq 3$, contradicting 1) of Subclaim 9.1.17. This completes the proof of Claim 9.1.13. ■

We now deal with any remaining chords of C.

Claim 9.1.18. *There exists a chord of C.*

Proof: Suppose that C is induced. If any three vertices of P have a common neighbor in $H \setminus C$ there is a vertex of $H \setminus C$ which is either a common neighbor of p_2, p_4, or a common neighbor of p_1, p_3, so we contradict either Claim 9.1.12 or Claim 9.1.13 Thus, no three vertices of P have a common neighbor in $H \setminus C$, and thus, by 1) of Proposition 1.5.1, for any L-coloring σ of $\{p_1, p_2, p_4\}$, Ψ_σ extends to an L-coloring of H, which is false. ■

By Claim 9.1.13, $p_2p_4 \notin E(H)$. By Claim 9.1.3, any chord of C has p_2 as an endpoint. Thus, let m be the maximal index in $\{1 \leq j \leq t : u_j \in N(p_2)\}$ and let $H = H_0 \cup H_1$ be the natural p_2u_m-partition of H, where $P_1 \in V(H_0)$, and $p_4 \in V(H_1)$. Then H_1 is bounded by outer cycle $C_1 := u_m p_2 p_3 p_4 u_t \cdots u_{m+1}$, and, by our choice of m, C_1 is an induced subgraph of H_1. Furthermore, C_1 contains the path $P_1 := u_m p_2 p_3 p_4$. As in Claim 9.1.13, $m \neq t$, or else, since H is short-separation-free and C_1 is an induced subgraph of H_1, we contradict our triangulation conditions.

Claim 9.1.19. *There exists a $w^\star \in V(H_1 \setminus C_1)$ such that $w^\star$ adjacent to each of u_m, p_2, p_3 and $w^\star$ is not adjacent to p_r. Furthermore, for any L-coloring σ of $\{p_1, p_3, p_4\}$ and any $d \in \mathcal{Z}_{H_0}(\sigma(p_1), \Psi_\sigma(p_2), \bullet)$, we have $|L(w^\star) \setminus \{d, \Psi_\sigma(p_2), \sigma(p_3)\}| = 2$.*

Proof: Let σ be an L-coloring of $\{p_1, p_3, p_4\}$. By Theorem 0.2.3, there is a $d \in \mathcal{Z}_{H_0}(\sigma(p_1), \Psi_\sigma(p_2), \bullet)$. Since C_1 is a chordless cycle and $m \neq t$, $(d, \Psi_\sigma(p_2), \sigma(p_3), \sigma(p_4))$ is a proper L-coloring of the path $u_m p_2 p_3 p_4$. Thus, there is an extension of Ψ_σ to an L-coloring Ψ^* of $V(H_0) \cup \{p_3, p_4\}$ such that $\Psi^*(u_m) = d$.

Since Ψ_σ does not extend to an L-coloring of H and C_1 is a chordless cycle in H_1, it follows from 1) of Proposition 1.5.1 that there is a $w^\star \in V(H_1 \setminus C_1)$ such that $|L_{\Psi^*}(w)| < 3$. By Claim 9.1.13, at most one of p_2, p_4 is adjacent to $w^\star$. Since $w^\star$ has at least three neighbors on the path $u_m p_2 p_3 p_4$ and H is short-separation-free, it follows from our triangulation conditions that $H[N(w^\star) \cap \{u_m, p_2, p_3, p_4\}]$ consists precisely of the path ump_2p_3. Thus, we conclude that $N(w^\star) \cap \text{dom}(\Psi^*) = \{u_m, p_2, p_3\}$ and $|L(w^\star) \setminus \{d, \Psi_\sigma(p_2), \sigma(p_3)\}| = 2$. The vertex $w^\star$ is the unique vertex of $H_1 \setminus C_1$ with at least three neighbors on $u_m p_2 p_3 p_4$ and is independent of our choice of σ. ■

We fix a vertex $w^\star$ as in Claim 9.1.19, and now note the following:

Claim 9.1.20. *For any L-coloring σ of $\{p_1, p_3, p_4\}$, $|\mathcal{Z}_{H_0}(\sigma(p_1), \Psi_\sigma(p_2), \bullet)| = 1$.*

Proof: Suppose there is a σ for which this does not hold. By Theorem 0.2.3, $\mathcal{Z}_{H_0}(\sigma(p_1), \Psi_\sigma(p_2), \bullet)$ is nonempty, so we have $|\mathcal{Z}_{H_0}(\sigma(p_1), \Psi_\sigma(p_2), \bullet)| > 2$. Let F be the outer face of $H_1 \setminus \{p_2, p_3, p_4\}$ and let $L^\dagger$ be a list-assignment for $H_1 \setminus \{p_2, p_3, p_4\}$, where $L^\dagger(u_m) = \mathcal{Z}_{H_0}(\sigma(p_1), \Psi_\sigma(p_2), \bullet)$ and otherwise $L^\dagger = L_{\Psi_\sigma}$.

Since $w^\star \notin N(p_4)$ and C_1 is an induced cycle of H_1, it follows that every vertex of F has an $L^\dagger$-list of size at least three, except for u_m, u_t. Since $m \neq t$, each of u_m, u_t has an $L^\dagger$-list of size at least two, so, by Theorem 1.3.4, $H_1 \setminus \{p_2, p_3, p_4\}$ admits an $L^\dagger$-coloring. Since u_m is not adjacent to either of p_3, p_4, it follows that Ψ_σ extends to an L-coloring Ψ^* of $V(H_1) \cup \{p_1\}$ which uses a color of $\mathcal{Z}_{H_0}(\sigma(p_1), \Psi_\sigma(p_2), \bullet)$ on u_m. Thus, Ψ_σ extends to an L-coloring of H, contradicting our assumption. ■

We now have the following:

Claim 9.1.21. *Let σ^0, σ^1 be two L-colorings of $\{p_1, p_3, p_4\}$ which differ only on the color used on p_1, where $\sigma^i(p_1) = a_i$ for each $i = 0, 1$. Then the following hold.*

1) $\mathcal{Z}_{H_0}(\sigma^0(p_1), \Psi_{\sigma^0}(p_2), \bullet) \neq \mathcal{Z}_{H_0}(\sigma^1)$; AND

2) $\Psi_{\sigma^0}(p_2) \neq \Psi_{\sigma^1}(p_2)$

Proof: For each $i = 0, 1$, let $s_i := \Psi_{\sigma^i}(p_2)$ and, Applying Claim 9.1.20, let c_i be the lone color of $\mathcal{Z}_{H_0}(a_i, s_i, \bullet)$. Let $b = \sigma^0(p_4) = \sigma^1(p_4)$ and $d = \sigma_0(p_3) = \sigma_1(p_3)$. We first prove the following intermediate result:

Subclaim 9.1.22. *If $c_0 = c_1$ then $s_0 = s_1$*

Proof: Let $c_0 = c_1 = c$ for some color c. Suppose toward a contradiction that $s_0 \neq s_1$. Let σ^* be the L-coloring of $\{u_m, p_3, p_4\}$ obtained by coloring u_m, p_3, p_4 with the respective colors c, d, b, and let F be the outer face of $H_1 \setminus P_1$. Since $p_4 \notin N(w^\star)$ and we have not colored p_2, we have $L_{\sigma^*}(w^\star)| \geq 3$. Since $w^\star$ is the unique common neighbor of u_m, p_3 outside of C_1, and C_1 is an induced subgraph of H_1, we have $|L_{\sigma^*}(u)| \geq 3$ for all $u \in V(F) \setminus \{u_{m+1}, u_t\}$. If $m+1 = t$, then $|L_{\sigma^*}(u_t)| = 1$ and thus, by Theorem 0.2.3, $H_1 \setminus P_1$ is L_{σ^*}-colorable. Likewise, if $m+1 < t$, then, by Theorem 1.3.4, $H_1 \setminus P_1$ is L_{σ^*}-colorable

Thus, in any case, σ^* extends to an L-coloring σ^{**} of $H_1 - p_2$. Since σ^{**} uses the colors c, d on the respective vertices u_m, p_3, and $s_0, s_1 \notin \{c, d\}$, it follows that one of s_0, s_1 is left over for p_2, as $\text{dom}(\sigma^{**}) \cap N(p_2) = \{w^\star, u_m, p_3\}$. Since $\{c\} = \mathcal{Z}_{H_0}(\sigma^i(p_1), \Psi_{\sigma^i}(p_2), \bullet)$ for each $i = 0, 1$, it follows that one of $\Psi_{\sigma^0}, \Psi_{\sigma^1}$ extends to an L-coloring of H, contradicting our assumption. ■

Now we prove 1). Suppose toward a contradiction that $c_0 = c_1 = c$ for some color c. Applying Subclaim 9.1.22, let $s_0 = s_1 = s$ for some color s. Thus, we have $s \notin \{a_0, a_1\}$, since each of σ^0, σ^1 is a proper L-coloring of its domain. Since $|L(u_m)| = 3$, let $s^* \in L(u_m) \setminus \{c, s\}$. It follows from Observation 1.4.2 that the L-coloring (s^*, s) of $u_m p_2$ extends to an L-coloring of H_0 using one of a_0, a_1 on p_1, contradicting the fact that $\mathcal{Z}_{H_0}(\sigma^i(p_1), \Psi_{\sigma^i}(p_2), \bullet) = \{c\}$ for each $i = 0, 1$. This proves 1).

Now we prove 2). Suppose toward a contradiction that $s_0 = s_1 = s$ for some s. Then $s \notin \{c_0, c_1\}$, and, by 1), $c_0 \neq c_1$. Let F be the outer face of $H_1 \setminus \{p_2, p_3, p_4\}$ and let σ' be the L-coloring of $p_2p_3p_4$ coloring p_2, p_3, p_4 with the respective colors s, d, b. Since $p_4 \notin N(w^\star)$, $|L_{\sigma'}(w^\star)| \geq 3$ and C_1 is an induced subgraph of H_1, every vertex of $F \setminus \{u_t, u_m\}$ has an $L_{\sigma'}$-list of size at least three and $|L_{\sigma'}(u_t)| \geq 2$. By Theorem 1.3.4, there is an $L_{\sigma'}$-coloring of $H_1 \setminus \{p_2, p_3, p_4\}$ using one of c_0, c_1 on u_m, so one of $\Psi_{\sigma^0}, \Psi_{\sigma^1}$ extends to an L-coloring of H, which is false. ■

For any L-coloring σ of the edge p_3p_4, we now define a set $S_\sigma \subseteq L(u_m)$ as follows. Let σ^0, σ^1 be the two extensions of σ to $\{p_1, p_3, p_4\}$, where $\sigma^i(p_1) = a_i$ for each $i = 0, 1$, and let $S_\sigma := \mathcal{Z}_{H_0}(a_0, \Psi_{\sigma^0}(p_2), \bullet) \cup \mathcal{Z}_{H_0}(a_1, \Psi_{\sigma^1}(p_2), \bullet)$. Combining Claim 9.1.20 with 1) of Claim 9.1.21, we have $|S_\sigma| = 2$ for any L-coloring σ of p_3p_4. Applying Proposition 1.4.4 and Claim 9.1.20, we immediately have the following.

Claim 9.1.23. *Let σ be an L-coloring of p_3p_4 and, for each $i = 0, 1$ let σ^i be the extension of σ to an L-coloring of $\{p_1, p_3, p_4\}$ in which p_1 is colored with a_i. Then, for each $k = 1, \cdots, m$, we have $\{\Psi_{\sigma^0}(p_2), \Psi_{\sigma^1}(p_2)\} \subseteq L(u_k)$, and furthermore, $\{a_0, a_1\} \subseteq L(u_1)$ and, if $m > 1$, then $S_\sigma \subseteq L(u_{m-1})$.*

We note now that the sets of the form S_σ are not constant as σ runs over all the L-colorings of p_3p_4.

Claim 9.1.24. *There does not exist a set $S \subseteq L(u_m)$ such that, for any L-coloring σ of p_3p_4, we have $S_\sigma = S$.*

Proof: Suppose toward a contradiction that such an S exists. Since $|L(p_3)| \geq 4$, there is a $d \in L(p_3)$ such that $|L(w^\star) \setminus (\{d\} \cup S)| \geq 3$. Since $|L(p_4)| = 2$, there is an L-coloring σ of p_3p_4 such that $\sigma(p_3) = d$. Let σ^0, σ^i be the two extensions of σ to $\{p_1, p_3, p_4\}$, where $\sigma^i(p_1) = a_i$ for each $i = 0, 1$. Since $S_\sigma = S$, there is an extension of σ to an L-coloring σ^* of $\{u_m, p_3, p_4\}$ such that $\sigma^*(u_m) \in S$ and $|L_{\sigma^*}(w^\star)| \geq 3$. But since σ^* uses a color of $\mathcal{Z}_{H_0}(a_0, \Psi_{\sigma^0}(p_2), \bullet) \cup \mathcal{Z}_{H_0}(a_1, \Psi_{\sigma^1}(p_2), \bullet)$ on u_m, we contradict Claim 9.1.19. ∎

We now fix an L-coloring σ of p_3p_4. For each $i = 0, 1$, let σ^i be the extension of σ to $\{p_1, p_3, p_4\}$ in which $\sigma^i(p_1) = a_i$. Furthermore, for each $i = 0, 1$, let $s_i := \Psi_{\sigma^i}(p_2)$, and let f_i be the lone color of $\mathcal{Z}_{H_0}(a_i, s_i, \bullet)$. Note that $S_\sigma = \{f_0, f_1\}$ and $f_0 \neq f_1$. By Claim 9.1.24, there is an L-coloring τ of p_3p_4 such that $S_\tau \neq S_\sigma$. Let τ^0, τ^1 be the two extension of τ to $\{p_1, p_3, p_4\}$, where. For each $i = 0, 1$, let $t_i := \Psi_{\tau^i}(p_2)$ and let g_i be the lone color of $\mathcal{Z}_{H_0}(a_i, t_i, \bullet)$. Note that $S_\tau = \{g_0, g_1\}$.

Claim 9.1.25. $\{t_0, t_1\} \neq \{s_0, s_1\}$.

Proof: Suppose toward a contradiction that $\{t_0, t_1\} = \{s_0, s_1\}$. By our choice of τ, we have $\{f_0, f_1\} \neq \{g_0, g_1\}$. If $s_0 = t_0$, then $s_1 = t_1$, and, for each $i = 0, 1$, we have $\mathcal{Z}_{H_0}(a_i, s_i, \bullet) = \mathcal{Z}_{H_0}(a_i, t_i, \bullet)$, contradicting the fact that $\{f_0, f_1\} \neq \{g_0, g_1\}$. Thus, we have $s_0 = t_1$ and $s_1 = t_0$. Now, since $s_0 \neq t_0$, it follows from 1) of Proposition 1.4.7 that $t_0 = f_0$ and $s_0 = g_0$. Likewise, $t_1 = f_1$ and $s_1 = g_1$. In particular, since $\{s_0, s_1\} = \{t_0, t_1\}$ we have $\{f_0, f_1\} = \{g_0, g_1\}$, contradicting our choice of τ. ∎

Since $\{t_0, t_1\} \neq \{s_0, s_1\}$ and $L(u_k)| = 3$ for each $k = 1, \cdots, m$, it follows from Claim 9.1.23 that $L(u_1) = \cdots = L(u_m)$ and $\{a_0, a_1\} \subseteq L(u_k)$ for each $k = 1, \cdots, m$. Let q be the lone color of $\bigcap_{k=1}^m L(u_k) \setminus \{a_0, a_1\})$. Since $S_\sigma \neq S_\tau$ we have $S_\sigma \cup S_\tau = L(u_m)$, so, by Claim 9.1.19, we have $L(u_m) \subseteq L(w^\star)$ and $|L(w^\star)| = 5$. For each $d \in L(p_3)$, there is an L-coloring of p_3p_4 using d on p_3, since $|L(p_4)| = 2$. Thus, again applying Claim 9.1.19, we have $L(p_3) \subseteq L(w^\star)$, so at least one of a_0, a_1 lies in $L(p_3)$. Suppose without loss of generality that $a_0 \in L(p_3)$.

Let ϕ be an L-coloring of p_3p_4 with $\phi(p_3) = a_0$. For each $i = 0, 1$, let ϕ^i be the extension of ϕ to $\{p_1, p_3, p_4\}$ obtained by coloring p_1 with a_i. If $\phi^i(u_m) = a_0$ for some $i \in \{0, 1\}$, then, since $u_mp_3 \notin E(H)$, we contradict Claim 9.1.19. Thus, we have $S_\phi = \{a_1, q\}$. For each $i = 0, 1$, let h_i be the lone color of $\mathcal{Z}_{H_0}(a_i, \Psi_{\phi^i}(p_2), \bullet)$. By Claim 9.1.23, we have $\{\Psi_{\phi^0}(p_2), \Psi_{\phi^1}(p_2)\} \subseteq \{a_0, a_1, q\}$, and, by Claim 9.1.21, $|\{\Psi_{\phi^0}(p_2), \Psi_{\phi^1}(p_2)\}| = 2$. Since Ψ_{ϕ^i} is a proper L-coloring of its domain for each $i = 0, 1$, we have $\Psi_{\phi^0}(p_2) = a_1$ and $\Psi_{\phi^1}(p_2) = q$. Since $|S_\phi| = 2$, we thus have $h_0 = a_1$ and $h_1 = q$. Now consider the following cases:

Case 1: m is odd

In this case, we extend the L-coloring (a_1, q) of p_1p_2 to an L-coloring of H_0 by coloring $u_1, u_3, \cdots, u_m$ with a_0, which leaves a color for each of $u_2, u_4, \cdots, u_{m-1}$, since each of these vertices has two neighbors using the same color. Thus, we have $a_0 \in \mathcal{Z}_{H_0}(a_1, \Psi_{\phi^1}(p_2), \bullet)$, which is false.

Case 2: m is even

In this case, we extend the L-coloring (a_0, a_1) of p_1p_2 to an L-coloring of H_1 by coloring $u_2, u_4, \cdots, u_m$ with a_0, which leaves a color for each of $u_1, u_3, \cdots, u_{m-1}$, as each of these vertices has two neighbors using the same color. Thus, we have $a_0 \in \mathcal{Z}_{H_0}(a_0, \Psi_{\phi^0}(p_2), \bullet)$, which is false. This completes the proof of Lemma 9.1.1. $\square$

9.2 Corner Colorings: Part II

In this section, we complete the proof of Theorem 9.0.1. We first prove the following simple lemma.

Lemma 9.2.1. *Let G be a planar graph, let C be a facial cycle of G, and let $P := p_1p_2p_3$ be a subpath of C. Suppose further that any chord of C is incident to p_2. Let L be a list-assignment for $V(G)$ such that each vertex of $C \setminus P$ has a list of size at least three and every vertex of $G \setminus C$ has a list of size at least five. Suppose further that there is a vertex of $C \setminus P$ with a list of size at least four. Then any L-coloring of $V(P)$ extends to an L-coloring of G.*

Proof. Suppose that this does not hold and let G be a vertex-minimal counterexample to the lemma. Thus, by assumption, there is an L-coloring ϕ of $V(P)$ which does not extend to an L-coloring of G. Let $\hat{u} \in V(C \setminus P)$, where $|L(\hat{u})| \geq 4$. For notational convenience, we suppose that C is the outer face of G. Applying Corollary 0.2.4, it immediately follows from the minimality of G that G is short-separation-free.

Claim 9.2.2. *There is no chord of C except possibly $\hat{u}p_2$.*

Proof: Suppose not. Since any chord of C has p_2 as an endpoint, G contains a chord of C of the form p_2u for some $u \in V(C \setminus P) \setminus \{\hat{u}\}$. Let $G = G_0 \cup G_1$ be the natural p_2u-partition of G, where $p_1 \in V(G_0)$, and $p_3 \in V(G_1)$. Suppose without loss of generality that $\hat{u} \in V(G_0) \setminus \{p_2, u\}$. Let C_0 be the outer face of G_0. By Theorem 0.2.3, the precoloring $(\phi(p_3), \phi(p_2))$ of the edge p_3p_2 extends to an L-coloring ψ of G_1. Since $\hat{u}$ is an internal vertex of the path $C_0 - p_2$, and every chord of C has p_2 as an endpoint, we have $p_1u \notin E(G)$, so $\phi \cup \psi$ of a proper L-coloring of its domain in G, even if $\phi(p_1) = \psi(u)$. Furthermore, every chord of C_0 has p_2 as an endpoint. Since $|V(G_0)| < |V(G)|$ and G_0 is also short-separation-free, the precoloring $(\phi(p_1), \phi(p_2), \psi(\hat{u}))$ of $p_1p_2\hat{u}$ extends to an L-coloring of G_0, so ϕ extends to an L-coloring of G, contradicting our assumption. ■

We now rule out the last remaining chord.

Claim 9.2.3. *There is no chord of C.*

Proof: Suppose C has a chord By Claim 9.2.2, $\hat{u}p$ is the lone chord of C. Let $G = G_0 \cup G_1$ be the natural $p_2\hat{u}$-partition of G, where $p_1 \in V(G_0)$ and $p_3 \in V(G_1)$. Let $P_0 := p_1p_2\hat{u}$ and let $P_1 := \hat{u}p_2p_3$. For each $i = 0, 1$, let C_i be the outer face of G_i.

If $V(C) = V(P) \cup \{\hat{u}\}$, then, since G is short-separation-free, it follows that G is a broken wheel with principal path $p_1p_2p_3$, and $G - p_2 = p_1\hat{u}p_3$. In that case, since $|L(\hat{u})| \geq 4$, ϕ extends to an L-coloring of G, contradicting our assumption. Thus, $V(P) \cup \{\hat{u}\}$ is a proper subset of $V(C)$, and there is an $i \in \{0, 1\}$ such that $V(C_i) \neq V(P_i)$, say $i = 0$ without loss of generality. Since there is no chord of C other than $p_2\hat{u}$, C_0 is an induced subgraph of G_0. Furthermore, as $V(C_0) \neq V(P_0)$, it follows that $p_1\hat{u} \notin E(G)$ and G_0 is not a broken wheel with principal path P_0.

Since $|L(\hat{u})| \geq 4$, it follows from Theorem 0.2.3 that the precoloring $(\phi(p_2), \phi(p_3))$ of the edge p_2p_3 extends to two distinct L-colorings of ψ, ψ^* of G_1 which use different colors on $\hat{u}$. Since $p_1\hat{u} \notin E(G)$, each of $(\phi(p_1), \phi(p_2), \psi(\hat{u}))$ and $(\phi(p_1), \phi(p_2), \psi^*(\hat{u}))$ is a proper L-coloring of $p_1p_2\hat{u}$. Since C_0 is an induced subgraph of G_0 and G_0 is not a broken wheel with principal path P_0, it follows from Theorem 1.5.3 that one of these extends to an L-coloring of G_0, so ϕ extends to an L-coloring of G, contradicting our assumption. ■

Since there is no chord of C and ϕ does not extend to an L-coloring of G, it follows from 1) of Proposition 1.5.1 that p_1, p_2, p_3 have a common neighbor w in $G \setminus C$. Let C' be the outer face of $G - p_2$. Since G is short-separation-free, we have $C' - p_2 = C - p_2$. Since there is no chord of C, every chord of C' in$G - p_2$ has w as an endpoint. Since $|L_\phi(w)| \geq 2$, let $c \in L_\phi(w)$, By the minimality of G, the L-coloring $(\phi(p_1), c, \phi(p_3))$ of p_1wp_3 extends to an L-coloring of $G - p_2$, and since $N(p_2) = \{p_1, w, p_3\}$, ϕ extends to an L-coloring of G, contradicting our assumption. □

We now state and prove the lone result which makes up the remainder of Section 9.2. The combination of the lemma below with Lemma 9.1.1 implies Theorem 9.0.1.

Lemma 9.2.4. *Let H be a planar graph with facial cycle C, and let $P := p_1p_2p_3p_4$ be a subpath of C of length three. Let $\hat{u}$ be a vertex of $C \setminus P$ and let L be a list-assignment for H such that the following hold.*

1) $\{p_1, p_4\}$ is L-colorable; AND

2) $|L(\hat{u})| \geq 4$ and $|L(p_3)| \geq 4$; AND

3) For each $v \in V(C) \setminus (V(P) \cup \hat{u})$, $|L(v)| \geq 3$; AND

4) For each $v \in V(H \setminus C)$, $|L(v)| \geq 5$.

Then there is an L-coloring ψ of $\{p_1, p_3, p_4\}$ such that any extension of ψ to an L-coloring of $V(P)$ also extends to an L-coloring of H.

Proof. Suppose that this does not hold and let H be a vertex-minimal counterexample to the lemma. For convenience, we suppose, by applying an appropriate stereographic projection, that C is the outer face of H. By adding edges to H if necessary, we also suppose that every facial subgraph of H, except possibly C, is a triangle.

Since $\{p_1, p_4\}$ is L-colorable, we fix an L-coloring σ of $\{p_1, p_4\}$. Note that $|L_\sigma(p_3)| \geq 2$, and, since H is a counterexample, it follows that, for each $c \in L_\sigma(p_3)$, there is an extension of σ to an L-coloring τ^c of $V(P)$ such that τ^c uses c on on p_3 and does not extend to an L-coloring of H. Thus, it follows from Corollary 0.2.4 that $|V(C)| > 4$, so let $C := p_4p_3p_2p_1u_1 \cdots u_t$ for some $t \geq 1$. By removing colors from $L(\hat{u})$ if necessary, we suppose that $|L(\hat{u})| = 4$. As usual applying Theorem 0.2.3 and Corollary 0.2.4, we immediately have the following from the minimality of H.

Claim 9.2.5. *H is short-separation-free and any chord of C has an endpoint in $\{p_2, p_3\}$*

We now have the following.

Claim 9.2.6. *$p_1p_3 \notin E(H)$ and p_1, p_3 have no common neighbor in $H \setminus C$. In particular, $|L_\sigma(p_3)| \geq 3$.*

<u>Proof:</u> Suppose that $p_1p_3 \in E(H)$ and let $c \in L_\sigma(u)$.. Since H is short-separation-free, it follows that $N(p_2) = \{p_1, p_3\}$, and $H - p_2$ has outer face $C' := p_1p_3p_4u_t \cdots u_1$. It follows from Claim 9.2.5 that any chord of C' has p_3 as an endpoint. Since $L(\hat{u})| \geq 4$, it follows from Lemma 9.2.1 that the coloring $(\sigma(p_1), c, \sigma(p_3))$ of $p_1p_3p_4$ extends to an L-coloring of $H - p_2$, and since $N(p_2) = \{p_1, p_3\}$, τ^c extends to an L-coloring of of H, contradicting our assumption. Since $p_1p_3 \notin E(H)$, we have $|L_\sigma(p_3)| \geq 3$.

Suppose that p_1, p_3 have a common neighbor $w \in V(H \setminus C)$. Since H is short-separation-free and $p_1p_3 \notin E(H)$, we have $wp_2 \in E(H)$ by our triangulation conditions, and $H - p_2$ has outer face $C' := p_1wp_3p_4u_t \cdots u_1$. Let $P' := p_1wp_3p_4$. Since $|V(H - p_2)| < |V(H)$, it follows from the minimality of H that there exists an extension of σ to an L-coloring ψ of $\{p_1, p_3, p_4\}$ such that any extension of ψ to an L-coloring of $V(P')$ also extends to an L-coloring of $H - p_2$. Let $c = \psi(p_3)$. Possibly $p_4 \in N(w)$, but in any case, since $|L(w) \setminus \{\sigma(p_1), \tau^c(p_2), c, \sigma(p_4)\}| \geq 1$, there is an extension of ψ to a proper L-coloring ψ^* of $V(P')$ such that $\psi^*(w) \neq \tau^c(p_2)$. Thus, ψ^* extends to an L-coloring of $H - p_2$, and since $N(p_2) = \{p_1, w, p_3\}$, τ^c extends to an L-coloring of H, contradicting our assumption. ■

We have an analogous result for the other side.

Claim 9.2.7. *$p_2p_4 \notin E(H)$ and p_2, p_4 have no common neighbor in $H \setminus C$.*

Proof: Suppose that $p_2p_4 \in E(H)$ and let $c \in L_\sigma(u)$.. Since H is short-separation-free, it follows that $N(p_3) = \{p_2, p_4\}$, and $H - p_3$ has outer face $C' := p_1p_2p_4u_t \cdots u_1$. It follows from Claim 9.2.5 that any chord of C' has p_2 as an endpoint. Since $L(\hat{u})| \geq 4$, it follows from Lemma 9.2.1 that the coloring $(\sigma(p_1), \tau^c(p_2), \sigma(p_4))$ of $p_1p_2p_4$ extends to an L-coloring of $H - p_3$, and since $N(p_3) = \{p_2, p_4\}$, τ^c extends to an L-coloring of of H, contradicting our assumption.

Now suppose toward a contradiction that p_2, p_4 have a common neighbor $w \in V(H \setminus C)$. Since H is short-separation-free and $p_2p_4 \notin E(H)$, it follows from our triangulation assumption that $p_3 \in N(w)$, and $H - p_3$ has outer face $C' := p_1p_2wp_4u_t \cdots u_1$. Let $P' := p_1p_2wp_4$. Since $|L(w)| \geq 5$, it follows from the minimality of H that there exist two distinct extensions ψ_0, ψ_1 of σ to L-colorings of $\{p_1, w, p_4\}$ such that, for each $i = 0, 1$, any extension of ψ_i to an L-coloring of $V(P')$ extends to an L-coloring of $H - p_3$. Since $|L_\sigma(p_3)| \geq 3$, there exists a $c \in L_\sigma(p_3) \setminus \{\psi_0(w), \psi_1(w)\}$. Since $\psi_0(w) \neq \psi_1(w)$, there exists an $i \in \{0, 1\}$ such that $\tau^c(p_3) \neq \psi_i(w)$. Thus, τ^c extends to an L-coloring of H, contradicting our assumption. ■

Claim 9.2.8. *p_2, p_3 have no common neighbor in C.*

Proof: Suppose toward a contradiction that p_2, p_3 have a common neighbor in C. By Claims 9.2.6 and 9.2.7 $p_1p_3, p_2p_4 \notin E(G)$. Thus, there exists an $s \in \{1, \cdots, t\}$ such that $u_s \in N(p_2) \cap N(p_3)$. Let H_0 be the subgraph of H bounded by outer cycle $C_0 := p_1p_2u_s \cdots u_1$ and let H_1 be the subgraph of H bounded by outer cycle $C_1 := u_s \cdots p_tp_4p_3$. Since H is short-separation-free, we have $H = (H_0 \cup H_1) + p_2p_3$. Let $P_0 := p_1p_2u_s$ and let $P_1 := u_sp_3p_4$. For each $c \in L_\sigma(p_3)$, since τ^c does not extend to an L-coloring of H, we have $\mathcal{Z}^{P_0}_{H_0}(\sigma(p_1), \tau^c(p_2), \bullet) \cap \mathcal{Z}^{P_1}_{H_1}(\bullet, c, \sigma(p_4)) = \varnothing$. By Claim 9.2.6, $|L_\sigma(p_3)| \geq 3$, so let $c_1, c_2, c_3 \in L_\sigma(p_3)$. Consider the following cases:

Case 1: $\hat{u} \in V(H_1) \setminus \{u_s\}$

In this case, by Claim 9.2.5, we have $u_sp_4 \notin E(H)$ so it follows from Lemma 9.2.1 that $\mathcal{Z}^{P_1}_{H_1}(\bullet, c, \sigma(p_4)) = L(u_s) \setminus \{c\}$ for each $c \in L_\sigma(p_3)$. Thus, for each $c \in L_\sigma(p_3)$, we have $\mathcal{Z}^{P_0}_{H_0}(\sigma(p_1), \tau^c(p_2), \bullet) = \{c\}$, since $\mathcal{Z}^{P_0}_{H_0}(\sigma(p_1), \tau^c(p_2), \bullet)$ is nonempty. It follows that $\tau^{c_1}(p_2), \tau^{c_2}(p_2), \tau^{c_3}(p_2)$ are three distinct colors, and, by Proposition 1.5.14, there is an $i \in \{1, 2, 3\}$ such that $|\mathcal{Z}^{P_0}_{H_0}(\sigma(p_1), \tau^{c_i}(p_2), \bullet)| \geq 2$, a contradiction.

Case 2: $\hat{u} \in V(H_0) \setminus \{u_s\}$

In this case, it follows from Claim 9.2.5 that $u_sp_1 \notin E(H)$ and thus, by Lemma 9.2.1, we have $\mathcal{Z}^{P_0}_{H_0}(\sigma(p_1), \tau^c(p_2), \bullet) = L(u_s) \setminus \{\tau^c(p_2)\}$ for each $c \in L_\sigma(p_3)$. Thus, for each $c \in L_\sigma(p_3)$, we have $\mathcal{Z}^{P_1}_{H_1}(\bullet, c, \sigma(p_4)) = \{\tau^c(p_2)\}$, since $\mathcal{Z}^{P_1}(\bullet, c, \sigma(p_4))$ is nonempty. Since $|L_\sigma(p_3)| \geq 3$, it follows from Proposition 1.5.14 that there is a $c \in L_\sigma(p_3)$ such that $|\mathcal{Z}^{P_1}_{H_1}(\bullet, c, \sigma(p_4))| \geq 2$, a contradiction.

Case 3: $\hat{u} = u_s$

In this case, since $|L(\hat{u})| = 4$, it follows from Theorem 0.2.3 that, for each $c \in L_\sigma(p_3)$ we have $|\mathcal{Z}_{H_0}^{P_0}(\sigma(p_1), \tau^c(p_2), \bullet)| \geq 2$. Furthermore, it follows from Proposition 1.5.14 that there is an $i \in \{1, 2, 3\}$ such that $|\mathcal{Z}_{H_1}^{P_1}(\bullet, c_i, \sigma(p_4))| \geq 3$. Thus, for some $i \in \{1, 2, 3\}$, we have $\mathcal{Z}_{H_0}^{P_0}(\sigma(p_1), \tau^c(p_2), \bullet) \cap \mathcal{Z}_{H_1}^{P_1}(\bullet, c, \sigma(p_4)) \neq \varnothing$, which is false. This completes the proof of Claim 9.2.8. ■

We now have the following.

Claim 9.2.9. *There are at least two distinct colors in* $\{\tau^c(p_2) : c \in L_\sigma(p_3)\}$.

Proof: Suppose not. Thus, there is a lone color d such that $\tau^c(p_2) = d$ for all $c \in L_\sigma(p_3)$. Let σ' be an extension of σ to an L-coloring of $\{p_1, p_2, p_4\}$ obtained by coloring p_2 with d. Note that $d \notin L_\sigma(p_3)$. By Claim 9.2.6, $|L_\sigma(p_3)| \geq 3$, so $|L_{\sigma'}(p_3)| \geq 3$. If σ' extends to an L-coloring of H, then there exists a $c \in L_{\sigma'}(p_3)$ such that τ^c extends to an L-coloring of H, which is false, so σ' does not extend to an L-coloring of H. Consider the following cases.

Case 1: There is no chord of C with p_2 as an endpoint.

In this case, by Claim 9.2.5, there is no chord of C with an endpoint in dom(σ'). By Claim 9.2.7, p_2, p_4 have no common neighbor in H. Thus, every vertex on the outer face of $H \setminus \text{dom}(\sigma')$ has an $L_{\sigma'}$-list of size at least three, except for the endpoints of $u_1 \cdots u_t$. Possibly $t = 1$ and $u_1 = \hat{u} = u_t$, but in any case, since $|L(\hat{u})| \geq 4$, each endpoint of $u_1 \cdots u_t$ has an $L_{\sigma'}$-list of size at least two, so, by Theorem 1.3.4, σ' extends to an L-coloring of H, which is false.

Case 2: There is a chord of C with p_2 as an endpoint.

Since $p_2p_4 \notin E(H)$, let m be the maximal index among $\{j \in \{1, \cdots, t\} : u_j \in N(p_2)\}$. Let $H = H_0 \cup H_1$ be the natural p_2u_m-partition of H, where $p_1 \in V(H_0)$. Since $p_2p_4 \notin E(H)$, it follows from Claim 9.2.5 and our choice of m that the outer face of H_1 has no chords with an endpoint in dom(ψ) (possibly there is a chord with p_3 as an endpoint).

Subcase 2.1 $m = t$

In this case, we have $\hat{u} \in \{u_1, \cdots, u_t\}$. If $\hat{u} \in \{u_1, \cdots, u_{t-1}\}$, then, by Lemma 9.2.4, we have $\mathcal{Z}_{H_0}(\sigma(p_1), d, \bullet) = L(u_t) \setminus \{d\}$. If $\hat{u} = u_t$, then it follows from Theorem 0.2.3 that $\mathcal{Z}_{H_0}(\sigma(p_1), d, \bullet)| \geq 2$. In any case, there exists an $f \in \mathcal{Z}_{H_0}(\sigma(p_1), d, \bullet)$ such that $f \neq \sigma(p_4)$. Thus, by Corollary 0.2.4, there is an extension of σ' to an L-coloring ψ of dom$(\sigma') \cup V(H_0)$ such that ψ also extends to L-color H_1, so σ' extends to an L-coloring of H, which is false.

Case 2.2 $m < t$

In this case, $u_mp_4 \notin E(H)$, so, since $\mathcal{Z}_{H_0}(\sigma(p_1), d, \bullet) \neq \varnothing$, it follows that σ' extends to an L-coloring ψ of dom$(\sigma') \cup V(H_0)$. By Claim 9.2.7, p_2, p_4 have no common neighbor in $H \setminus C$. Thus, every vertex on the outer face of $H \setminus (V(H_0) \cup \{p_4\})$ has an $L_{\sigma'}$-list of size at least three, except for the endpoints of $u_{m+1} \cdots u_t$. Applying Theorem 0.2.3 if $m + 1 = t$ and otherwise applying Theorem 1.3.4, it follows that ψ extends to an L-coloring of H so σ' extends to an L-coloring of H, which is false. This completes the proof of Claim 9.2.9. ■

Let $n \in \{1, \cdots, t\}$ where $\hat{u} = u_n$. We now have the following.

Claim 9.2.10. *There is no chord of C which separates $\hat{u}$ from p_2.*

Proof: Suppose toward a contradiction that there is a chord of C which separates $\hat{u}$ from p_2. By Claim 9.2.5, any such chord has p_3 as an endpoint, and, by Claim 9.2.6, $p_1p_3 \notin E(H)$, so there exists a chord of C of the form u_jp_3 for some $j \in \{1, \cdots, n-1\}$. Let m be the minimal index in $\{j \in \{1, \cdots, t\} : p_3 \in N(u_j)\}$. By assumption, such an m exists and $m \leq n-1$. Let $H := H^0 \cup H^1$ be the natural p_3u_m-partition of H, where $p_1 \in V(H_0)$. Let $P^0 := p_1p_2p_3u_m$ and $P^1 := u_mp_3p_4$.

Let C^0 be the outer face of H^0. Since $p_1p_3 \notin E(H)$, it follows from our choice of m that H_0 has no chord of C^0 with p_3 as an endpoint. Thus, by Theorem 1.6.1, there is a color $d \in L(u_m)$, where $d \neq \sigma(p_1)$ if $p_1u_m \in E(H^0)$, such that any L-coloring of $V(P^0)$ using $\sigma(p_1), d$ on the respective vertices p_1, u_m extends to an L-coloring of H^0. Possibly $d = \sigma(p_4)$, but, since $\hat{u} \in \{u_{m+1}, \cdots u_t\}$, it follows from Claim 9.2.5 that $u_mp_4 \notin E(H)$, so $(d, c, \sigma(p_4))$ is a proper L-coloring of $V(P^1)$. By Lemma 9.2.4, this L-coloring extends to an L-coloring of H^1. Possibly $d = \tau^c(p_2)$, but, by Claim 9.2.8, $p_2u_m \notin E(H)$, so $(\sigma(p_1), \tau^c(p_2), c, d)$ is a proper L-coloring of the subgraph of H induced by $p_1p_2p_3u_m$. Thus, by our choice of d, τ^c extends to an L-coloring of H^0, contradicting our assumption. ■

We have an analogous result for the other side.

Claim 9.2.11. *There is no chord of C which separates $\hat{u}$ from p_3.*

Proof: Suppose toward a contradiction that there is a chord of C which separates $\hat{u}$ from p_3. By Claim 9.2.5, any such chord has p_2 as an endpoint, and, by Claim 9.2.7, $p_2p_4 \notin E(H)$, so there exists a chord of C of the form u_jp_2 for some $j \in \{n+1, \cdots, t\}$. Let m be the maximal index in $\{j \in \{1 \cdots, t\} : p_3 \in N(u_j)\}$. By assumption, such an m exists and $m \geq n+1$. Let $H := H^0 \cup H^1$ be the natural p_3u_m-partition of H, where $p_1 \in V(H^0)$. Let $P^0 := p_1p_2p_3u_m$ and $P^1 := u_mp_3p_4$.

Let C^1 be the outer face of H^1. Since $p_2p_4 \notin E(H)$, it follows from our choice of m that H^1 has no chord of C^1 with p_2 as an endpoint. Thus, by Theorem 1.6.1, there is a color $d \in L(u_m)$, where $d \neq \sigma(p_4)$ if $p_4u_m \in E(H^1)$, such that any L-coloring of $V(P^1)$ using $\sigma(p_4), d$ on the respective vertices p_4, u_m extends to an L-coloring of H^1. By Claim 9.2.9, there exists a $c \in L_\sigma(p_3)$ such that $\tau^c(p_3) \neq d$. Thus, $(\sigma(p_1), \tau^c(p_2), d)$ is a proper L-coloring of the subgraph of H induced by $p_1p_2u_m$. By Claim 9.2.8, $u_mp_3 \notin E(H)$, so $(d, \tau^c(p_2), c, \sigma(p_4))$ is a proper L-coloring of the subgraph of H induced by $u_mp_2p_3p_4$. By Lemma 9.2.1, the coloring $(\sigma(p_1), \tau^c(p_2), d)$ of P^0 extends to an L-coloring of H_0, and, by our choice of d, the coloring of P^1 extends to an L-coloring of H^1, so τ^c extends to an L-coloring of H, contradicting our assumption. ■

We now have the following.

Claim 9.2.12. *Neither p_2 nor p_3 is adjacent to $\hat{u}$. In particular, $N(p_2) \cap V(C) \subseteq \{p_1, p_3\} \cup \{u_1, \cdots, u_{n-1}\}$ and $N(p_3) \cap V(C) \subseteq \{p_2, p_4\} \cup \{u_{n+1}, \cdots, u_t\}$.*

Proof: Suppose toward a contradiction that $\hat{u} \in N(p_3)$. Let H' be the subgraph of H bounded by outer cycle $C' := p_1p_2p_3\hat{u} \cdots u_1p_1$ and let H'' be the subgraph of H bounded by outer cycle $C'' := \hat{u} \cdots u_tp_4p_3$. Let $P'' := \hat{u}p_3p_4$. By Claim 9.2.6, $|L_\sigma(p_3) \geq 3$. Since $|L(\hat{u})| \geq 4$, it follows from Proposition 1.5.14 that there is a $c \in L_\sigma(p_3)$ such that $|\mathcal{Z}_L^{P''}(\bullet, c, \sigma(p_4)| \geq 3$. Let L^* be a list-assignment for $V(H'')$ defined as follows.

1) The vertices p_1, p_2, p_3 are precolored with the respective colors $\sigma(p_1), \tau^c(p_2), c$.
2) $L'(\hat{u}) = \mathcal{Z}_{H''}^{P''}(\bullet, c, \sigma(p_4))$.
3) Otherwise $L^* = L$.

Now we simply color and delete p_3. Let ψ be the lone L^*-coloring of $\{p_1, p_2, p_3\}$. By Claim 9.2.10, there is no chord of C' in H' with p_3 as an endpoint, so every vertex of the outer face of $H' - p_3$, other than p_1, p_2, has an $(L^*)^{p_1p_2}_{\psi}$-list of size at least three. Thus, by Theorem 0.2.3, ψ extends to an L^*-coloring of $V(H')$. Since $L^*(\hat{u}) = \mathcal{Z}^{P''}_{H''}(\bullet, c, \sigma(p_4))$, it follows that τ^c extends to an L-coloring of H, contradicting our assumption. Thus, we have $\hat{u} \notin N(p_3)$. Combining this with Claim 9.2.10, we get that $N(p_3) \cap V(C) \subseteq \{p_2, p_4\} \cup \{u_{n+1}, \cdots, u_t\}$.

Now we do the other side. Suppose toward a contradiction that $\hat{u} \in N(p_2)$. Let H^* be the subgraph of H bounded by outer cycle $C^* := p_1p_2\hat{u}\cdots u_t$ and let H^{**} be the subgraph of H bounded by outer cycle $C^{**} := \hat{u}\cdots u_tp_4p_3p_2$. Let $P^* := p_1p_2\hat{u}$. Since $|L(\hat{u})| \geq 4$, it follows from Theorem 0.2.3 that, for any $d \in L(p_2)$, we have $|\mathcal{Z}^{P^*}_{H^*}(\sigma(p_1), d, \bullet)| \geq 2$. By Claim 9.2.11, H^{**} has no chord of C^{**} with p_2 as an endpoint. Consider the following cases.

Case 1: There is no chord of C with p_3 as an endpoint

In this case, it follows from Claim 9.2.5 that C^{**} is an induced subgraph of H^{**}. Let $c \in L_\sigma(p_3)$ and let L^{**} be a list-assignment for $V(H^{**})$ defined as follows.

1) The vertices p_2, p_3, p_4 are precolored with the respective colors $\tau^c(p_2), c, \sigma(p_4)$.
2) $L^{**}(\hat{u}) = \{\tau^c(p_2)\} \cup \mathcal{Z}^{P^*}_{H^*}(\sigma(p_1), \tau^c(p_2), \bullet)$.
3) Otherwise $L^{**} = L$.

If H^{**} is L^{**}-colorable, then τ^c extends to an L-coloring of H, which is false, so H^{**} is not L^{**}-colorable. Note that $|L^{**}(\hat{u})| \geq 3$, since $|\mathcal{Z}^{P^*}_{H^*}(\sigma(p_1), \tau^c(p_2), \bullet)| \geq 2$ and $\tau^c(p_2) \notin \mathcal{Z}^{P^*}_{L}(\sigma(p_1), \tau^c(p_2), \bullet)$. By Claim 9.2.6, p_1, p_3 have no common neighbor in $H \setminus C$, and thus no common neighbor in $H^{**} \setminus C^{**}$. Thus, by 1) of Proposition 1.5.1, H^{**} is L^{**}-colorable, contradicting our assumption.

Case 2: There is a chord of C with p_3 as an endpoint.

In ths case, since $p_1p_3 \notin E(H)$, let $m^+ \in \{1, \cdots, n\}$ be the maximal index among $\{j \in \{1, \cdots, t\} : u_j \in N(p_3)\}$. As shown above, we have $m^+ \in \{n+1, \cdots, t\}$. Let $Q := u_{m^+}p_3p_2$ and let J be the subgraph of H bounded by outer cycle $u_{m^+}\cdots u_tp_4p_3$. By Proposition 1.5.14, since $|L_\sigma(p_3)| \geq 3$, there is a $c \in L_\sigma(p_3)$ such that $|\mathcal{Z}^{Q}_{J,L}(\bullet, c, \sigma(p_4))| \geq 2$. Let H^+ be the subgraph of H bounded by outer cycle $C^+ := p_2p_3u_{m^+}\cdots u_n$. Let L^+ be a list-assignment for $V(H^+)$ defined as follows.

1) p_2, p_3 are precolored with the respective colors $\tau^c(p_2), c$.
2) $L^+(\hat{u}) = \{\tau^c(p_2)\} \cup \mathcal{Z}^{P^*}_{H^*}(\sigma(p_1), \tau^c(p_2), \bullet)$ and $L^+(u_{m^+}) = \{c\} \cup \mathcal{Z}^{Q}_{J,L}(\bullet, c, \sigma(p_4))$.
3) Otherwise $L^+ = L$.

Note that $c \notin \mathcal{Z}^{Q}_{J}(\bullet, c, \sigma(p_4))$ and $\tau^c(p_2) \notin \mathcal{Z}^{P^*}_{H^*}(\sigma(p_1), \tau^c(p_2), \bullet)$. Since $|\mathcal{Z}^{P^*}_{H^*}(\sigma(p_1), \tau^c(p_2), \bullet)| \geq 2$ and $|\mathcal{Z}^{Q}_{J}(\bullet, c, \sigma(p_4))| \geq 2$, each of $\hat{u}, u_{m^+}$ has an L^+-list of size at least three. By Theorem 0.2.3, H^+ is L^+-colorable, so τ^c extends to an L-coloring of H, contradicting our assumption. ■

We now have the following.

Claim 9.2.13. *There is a chord of C with p_2 as an endpoint*

<u>Proof:</u> Suppose not, and consider the following cases.

Case 1: There is no chord of C with p_3 a an endpoint.

In this case, by Claim 9.2.8, C is an induced subgraph of G. Let $c \in L_\sigma(p_2)$. By Claim 9.2.6 p_1, p_3 have no common neighbor in $H \setminus C$, and, by Claim 9.2.7, p_2, p_4 have no common neighbor in $H \setminus C$. By 1) of Proposition 1.5.1, τ^c extends to an L-coloring of H, contradicting our assumption.

Case 2: There is a chord of C with p_3 as an endpoint.

Since $p_1p_3 \notin E(H)$, let $m \in \{1, \cdots, t\}$ be the minimal index among $\{j \in \{1, \cdots, t\} : p_j \in N(p_3)\}$. By Claim 9.2.12, $m \in \{n+1, \cdots t\}$. Let H' be the subgraph of H bounded by outer face $C' := p_1u_1 \cdots u_m p_3 p_2$ and let H'' be the subgraph of H bounded by outer face $C'' := u_m \cdots u_t p_4 p_3$. Let $P'' := u_m p_3 p_4$. Since $p_1p_3 \notin E(H)$, and there is no chord of C with p_2 as an endpoint, it follows from our choice of m that C' is an induced subgraph of H'. By Proposition 1.5.14, since $|L_\sigma(p_3)| \geq 3$, there is a $c \in L_\sigma(p_3)$ such that $|\mathcal{Z}^{P''}_{H''}(\bullet, c, \sigma(p_4))| \geq 2$. Let L' be a list-assignment for $V(H')$ defined as follows.

1) The vertices p_1, p_2, p_3 are precolored with the respective colors $\sigma(p_1), \tau^c(p_2), c$.
2) $L'(u_m) = \{c\} \cup \mathcal{Z}^{P''}_{H''}((\bullet, c, \sigma(p_4))$.
3) Otherwise $L' = L$.

Note that $|L'(u_m)| \geq 3$, since $c \notin \mathcal{Z}^{P''}_{H''}((\bullet, c, \sigma(p_4))$. If H' is L'-colorable, then there is an L-coloring of H' which uses a color of $\mathcal{Z}^{P''}_{H''}((\bullet, c, \sigma(p_4))$ on u_m, and thus τ^c extends, to an L-coloring of H, which is false. Thus H' is not L'-colorable. Since C' is an induced subgraph of H', it follows from 1) of Proposition 1.5.1 that p_1, p_2, p_3 have a common neighbor in $H' \setminus C'$ and thus a common neighbor in $H \setminus C$, contradicting Claim 9.2.6. ■

Applying Claim 9.2.13, since $p_2p_4 \notin E(H)$, let m^- be the maximal index in $\{j \in \{1, \cdots, t\} : p_2 \in N(u_j)\}$. We then have $m^- \in \{1, \cdots, n-1\}$. Let H^- be the subgraph of H bounded by outer cycle $p_1p_2u_{m^-} \cdots u_1$ and let $P^- := p_1p_2u_{m^-}$,

Claim 9.2.14. *There is no chord of C with p_3 as an endpoint.*

<u>Proof:</u> Suppose toward a contradiction that such a chord exists. By Claim 9.2.6, $p_1p_3 \notin E(H)$, so let m^+ be the minimal index in $\{j \in \{1, \cdots, t\} : p_3 \in N(u_j)\}$. By Claim 9.2.12, we have $m^+ \in \{n+1, \cdots, t\}$. Let H^{box} be the subgraph of H bounded by outer cycle $D := u_{m^-} \cdots u_{m^+} p_3 p_2$. Let $P^+ := u_{m^+} p_3 p_4$. Let H^+ be the subgraph of H bounded by outer cycle $u_{m^+} \cdots u_t p_4 p_3$. Applying Proposition 1.5.14, since $|L_\sigma(p_3)| \geq 3$, there is a $c \in L_\sigma(p_3)$ with $|\mathcal{Z}^{P^+}_{H^+}(\bullet, c, \sigma(p_4))| \geq 2$. Since $\mathcal{Z}^{P^-}_{H^-}(\sigma(p_1), \tau^c(p_2), \bullet) \neq \varnothing$, let $r \in \mathcal{Z}^{P^-}_{H^-}(\sigma(p_1), \tau^c(p_2), \bullet)$. Let L^* be a list-assignment for $V(H^{\text{box}})$ defined as follows.

1) u_{m^-}, p_2, p_3 are precolored with the respective colors $r, \tau^c(p_2), c$.
2) $L^*(u_{m^+}) = \{c\} \cup \mathcal{Z}^{P^+}_{H^+}(\bullet, c, \sigma(p_4))$.
3) Otherwise $L^* = L$.

Since $p_2p_4, p_1p_3 \notin E(H)$ and every chord of C has one of p_2, p_3 as an endpoint, it follows from our choice of m^-, m^+ that D is an induced subgraph of H^{box}. We now apply the work of Section 1.7. Let $Q := u_{m^-} \cdots u_{n-1}$. Q is a nonempty subpath of D since $m^- \leq n-1$.

Since every vertex of $H^{\text{box}} \setminus D$ has an L^*-list of size at least five, Q is $(2, L^*)$-short in (D, H). Applying i) of Theorem 1.7.5, there exists a $\psi \in \text{Link}_{L^*}(Q, D, H^{\text{box}})$ with $\psi(u_{m^-}) = r$. Let ϕ be the unique L^*-coloring of the edge p_2p_3. Since D is an induced subgraph of H^{box}, $\phi \cup \psi$ is a proper L^*-coloring of its domain. Again since D is an induced subgraph of G and $|L(u_n)| \geq 4$, we have $|L^*_\psi(u_n)| \geq 3$, and every vertex of $\{u_{n+1}, \cdots, u_{m^+}\}$ also has an L^*_ψ-list of size at least three. Thus, it immediately follows from 3a) of Theorem 1.7.4 that $\phi \cup \psi$ extends to an L^*-coloring φ of H^{box}. Thus, φ is an L-coloring of H^{box} which uses a color of $\mathcal{Z}^{P^-}_{H^-,L}(\sigma(p_1), \tau^c(p_2), \bullet)$ on u_{m^-} and a color of $\mathcal{Z}^{P^+}_{H^+,L}(\bullet, c, \sigma(p_4))$ on u_{m^+}, so it follows that τ^c extends to an L-coloring of H, which is false. ■

Let $H^\dagger$ be the subgraph of H bounded by outer cycle $D := u_{m^-} \cdots u_t p_4 p_3 p_2$. Since every chord of C has one of p_2, p_3 as an endpoint, it follows from Claim 9.2.14 and our choice of m^- that D is an induced subgraph of $H^\dagger$.

Claim 9.2.15. *There exist two distinct colors $c, c' \in L_\sigma(p_3)$ such that $\tau^c(p_2) = \tau^{c'}(p_2)$.*

Proof: Suppose not. Since $|L_\sigma(p_3)| \geq 3$, we have $|\{\tau^c(p_2) : c \in L_\sigma(p_3)\}| \geq 3$. Thus, by Proposition 1.5, there exists a $c \in L_\sigma(p_3)$ such that $|\mathcal{Z}^{P^-}_{H^-}(\sigma(p_1), \tau^c(p_2), \bullet)| \geq 2$. Let L' be a list-assignment for $V(H^\dagger)$ defined as follows.

1) p_2, p_3, p_4 are precolored with the respective colors $\tau^c(p_2), c, \sigma(p_4)$.
2) $L'(u_{m^-}) = \{c\} \cup \mathcal{Z}^{P^-}_{H^-}(\sigma(p_1), \tau^c(p_2), \bullet)$.
3) Otherwise $L' = L$.

Note that $|L'(u_{m^-})| \geq 3$. It follows from Claim 9.2.7 that there is no vertex of $H^\dagger \setminus D$ which is adjacent to each of p_2, p_4. Since D is an induced subgraph of H and $(\tau^c(p_2), c, \sigma(p_4))$ is a proper L'-coloring of $p_2p_3p_4$, it follows from 1) of Proposition 1.5.1 that $H^\dagger$ admits an L'-coloring ψ. Since $\psi(p_2) = c$, we have $\psi(u_{m^-}) \in \mathcal{Z}^{P^-}_{H^-}(\sigma(p_1), \tau^c(p_2), \bullet)$, so τ^c extends to an L-coloring of H, contradicting our assumption. ■

Now we have enough to finish the proof of Lemma 9.2.4. Let $c, c' \in L_\sigma(p_3)$ and let $d \in L_\sigma(p_3)$ such that $\tau^c(p_2) = \tau^{c'}(p_2) = d$. Let $r \in \mathcal{Z}^{P^-}_{H^-}(\sigma(p_1), d, \bullet)$. Let $L^\dagger$ be a list-assignment for $V(H^\dagger)$ defined as follows.

1. u_{m^-}, p_2, p_4 are precolored with the respective colors $r, d, \sigma(p_4)$.
2. $L^\dagger(p_3) = \{c_1, c_2, d\}$.
3. Otherwise $L^\dagger = L$.

Let Q be the subpath $u_{n+1} \cdots u_t p_4$ of D (possibly Q is just p_4). We again apply the work of Section 1.7. By i) of Theorem 1.7.5, there is a $\psi \in \text{Link}_{L^\dagger}(Q, D, H^\dagger)$. Let ϕ be the unique $L^\dagger$-coloring of $p_2p_3p_4$. Since D is an induced subgraph of $H^\dagger$, $\phi \cup \psi$ is a proper $L^\dagger$-coloring of its domain, and since $|L^\dagger(u_n)| \geq 4$, it follows that every vertex of $D \setminus (V(Q) \cup \{p_2, p_3\})$ has an $L^\dagger_\psi$-list of size at least three. Thus, by 3a) of Theorem 1.7.4, $\phi \cup \psi$ extends to an $L^\dagger$-coloring φ of $H^\dagger$. Since $\varphi(p_3) = d$, we have $\varphi(p_2) \in \{c, c'\}$ and since $\varphi(u_{m^-}) \in \mathcal{Z}^{P^-}_{H^-}(\sigma(p_1), d, \bullet)$, it follows that either τ^c or $\tau^{c'}$ extends to an L-coloring of H, contradicting our assumption. This completes the proof of Lemma 9.2.4. □

Chapter 10

Coils and Their Applications: Deleting Vertices Near the Closed Rings of Critical Mosaics

In this chapter, we prove an analogue of Theorem 6.0.9 for closed rings. In Chapter 11, which is the final chapter of the proof of Theorem 2.1.7, we then combine this result with Theorem 6.0.9 to construct a smaller counterexample from a critical mosaic. Theorem 6.0.9 is specific to the context of critical mosaics, but we state our analogous result for closed rings in more general terms. We begin with the following definition.

Definition 10.0.1. Given a short-separation-free planar graph G, a facial cycle C of G, and a list-assignment L-for $V(G)$, we say that C is an *L-coil* of G (or just an *L-coil*) if $V(C)$ is precolored by L, and, letting ϕ be the unique L-coloring of $V(C)$, the following hold.

Co1) C is an induced cycle and L-predictable facial subgraph of G*; AND*

Co2) For every $v \in B_2(C)$, every facial subgraph G containing v, except possibly C, is a triangle*; AND*

Co3) Every vertex of $V(G \setminus C) \cap B_2(C)$ has an L-list of size at least five, and furthermore, there is a cyclic facial subgraph $F^{<5}$ of G with $V(F^{<5}) \subseteq V(G) \setminus B_2(C)$, where all the vertices of $F^{<5}$ have L-lists of size less than five*; AND*

Co4) There is a unique cycle C^1 in G such that $V(C^1) = D_1(C)$, and C^1 satisfies the following:

   a) C^1 is chordless*; AND*

   b) For any $1 \leq k \leq 6$ and any k-chord P of C, letting $G = G_0 \cup G_1$ be the natural P-partition of G, there exists an $i \in \{0, 1\}$ such that every vertex of $V(G_i) \setminus V(C \cup P)$ has an L-list of size at least five*; AND*

   c) Either every vertex of C^1 has an L_ϕ-list of size at least three, or there is a vertex of C^1 with precisely one neighbor in C*; AND*

   d) If C^1 contains a vertex with an L_ϕ-list of size less than three, then, for every $w \in D_2(C)$, the graph $G[N(w) \cap V(C^1)]$ is a subpath of C^1*;*

The reason we introduce this definition is that we use the main result of Chapter 10 not only to complete the proof of Theorem 2.1.7, but also for part of the argument in Chapter 13 which involves an annulus consisting of two precolored cycles. That is, in this chapter we prove a result which holds for all short-separation-free G such that G contains a facial subgraph C which is an L-coil for some list-assignment L.

Note that Co4b) of Definition 10.0.1 is slightly weaker than the condition that C^1 is $(6, L)$-short, since, in Co4b) Definition 10.0.1, we do not require the vertices of $P \setminus C$ themselves to have L-lists of size at least five. When we apply the main result of Chapter 10 in the context of critical mosaics in Chapter 11, this distinction does not matter because of the distance conditions imposed on mosaics, but in Chapter 12 we apply the main result of Chapter 10 in the context of a graph with two precolored cycles which are possibly close together, and in that case, the distinction above does matter.

In order to state our lone main result for Chapter 10, we need several additional definitions. We first have the following simple observation, which is an immediate consequecne of short-separation-freeness.

Observation 10.0.2. *Let G be a short-separation-free planar graph with facial cycle C and list-assignment L, where C is an L-coil. Let $C^1 := G[D_1(C)]$, let $2 \leq k \leq 4$ and let R be a k-chord of C^1 in $G \setminus C$, where R separates two vertices of $G \setminus C$. Then R is a proper generalized chord of C^1. In particular, there is a $k+2$-chord R' of C in G such that $R \subseteq R'$.*

Note that Co3-Co4 of Definition 10.0.1, together with Observation 10.0.2, specify a unique small and large side of a k-chord of C^1 in $G \setminus C$ for $2 \leq k \leq 4$. Thus, analogous to Definition 8.0.3, we introduce the following natural notation.

Definition 10.0.3. Let G be a short-separation-free planar graph with facial cycle C and list-assignment L, where C is an L-coil. Let $C^1 := G[D_1(C)]$ and let ϕ be the unique L-coloring of $V(C)$. Letting $\tilde{G} = G \setminus C$, we introduce the following notation. For any $2 \leq k \leq 4$ and any k-chord R of C^1 in $\tilde{G}$, we set $\tilde{G}_R^{\text{small}}$ and $\tilde{G}_R^{\text{large}}$ to be the unique subgraphs of $\tilde{G}$ such that $\tilde{G}_R^{\text{small}} \cup \tilde{G}_R^{\text{large}} = \tilde{G}$ is the natural (C^1, R)-partition of $\tilde{G}$, where all of the vertices of $\tilde{G} \setminus (C^1 \cup R)$ with L-lists of size less than five lie in $V(\tilde{G}_R^{\text{large}})$. It follows from Co3 of Definition 10.0.1 that these two graphs are uniquely specified.

Analogous to Definition 6.0.6 from Chapter 6, we introduce the following natural definition.

Definition 10.0.4. Let G be a short-separation-free planar graph with facial cycle C and list-assignment L, where C is an L-coil. Let $C^1 := G[D_1(C)]$ and let ϕ be the unique L-coloring of $V(C)$. Given a $z \in D_2(C^1)$, we associate to z a subgraph Span(z) of $G \setminus C$ in the following way.

1) If there exists a proper 4-chord P of C^1 in $G \setminus C$ whose midpoint in C, then we set Span(z) to be the unique proper 4-chord P of C^1 in $G \setminus C$ which minimizes the quantity $|V(\tilde{G}_P^{\text{small}})|$.
2) If no such proper 4-chord of C^1 exists, then we define Span(z) in the following way:
   a) If $N(z) \cap D_2(C)$ consists of a lone vertex v, and $N(v) \cap V(C^1)| = 1$, then we set Span(z) to be the unique 2-path with z as an endpoint and the other endpoint in C^1.
   b) If $N(z) \cap D_2(C)$ consists of a lone vertex v, and $N(v) \cap V(C^1)| > 1$, then we set P to be the claw on the vertices $\{v, z, x, x'\}$, where P has central vertex z and xvx' is the unique 2-chord of C^1 with central vertex v which maximizes the quantity $|V(\tilde{G}_{xvx'}^{\text{small}})|$.
   c) If $|N(z) \cap D_2(C)| > 1$, then, since G is $K_{2,3}$-free, there exist vertices v, v', x such that $N(z) \cap D_2(C) = \{v, v'\}$ and $N(v) \cap V(C^1) = N(v') \cap V(C^1) = \{x\}$, and we set P to be the 4-cycle $zvxv'$.

Thus, for each $z \in D_2(C^1)$, Span(z) is either a 4-path, a 4-cycle, a claw, or a 2-path. To state oir main result for Chapter 10, which requires the following definition. This definition is a natural analogue of Definition 6.0.8 in the setting of L-coils (which, in particular, specializes to the setting of closed rings).

Definition 10.0.5. Let G be a short-separation-free graph with facial cycle C and list-assignment L, where C is an L-coil. Let $C^1 := G[D_1(C)]$, let ϕ be the unique L-coloring of $V(C)$. Given a $z \in D_2(C^1)$, a $z \in D_2(C^1) \setminus \text{Sh}_4(C^1, G \setminus C)$, a (C, z)-*opener* is a pair $[H, \psi]$, where ψ is an extension of ϕ to an partial L-coloring of G, and the following holds.

1) H is a connected subgraph of G and $\text{dom}(\psi) \subseteq V(H) \subseteq \text{Sh}_4(C^1, G \setminus C) \cup B_2(C) \cup \{z\}$*; AND*
2) $V(H) \setminus \text{dom}(\psi)$ is L_ψ-inert*; AND*
3) For each $u \in D_1(H)$, $|L_\psi(u)| \geq 3$*; AND*
4) There is at most one vertex of $(\text{dom}(\psi) \cap D_1(C^1, G \setminus C)) \setminus \text{Sh}_4(C^1, G \setminus C)$ which does not lie in $\text{Span}(z)$*; AND*
5) For any $v \in V(H) \cap \text{Sh}_4(C^1, G \setminus C)$, either $v \in \text{Sh}_3(C^1, G \setminus C)$ or $\text{Span}(z)$ is a proper 4-chord of C^1 which, in $G \setminus C$, separates v from every vertex of $G \setminus C$ with an L-list of size less than five.

We introduce one final definition and then we state the lone main result of Chapter 10.

Definition 10.0.6. Let G be a short-separation-free graph with facial cycle C and list-assignment L, where C is an L-coil. Let $C^1 := G[D_1(C)]$. Given a $z \in D_2(C^1)$, we say that z is a C-*pentagonal* vertex (or just *pentagonal* if the cycle C is clear from the context) if the following hold.

1) Every vertex of $B_2(z)$ has an L-list of size at least five*; AND*
2) For any $2 \leq k \leq 4$, there is no k-chord of C^1 in $G \setminus C$ which separates z from a vertex of $G \setminus C$ with an L-list of size less than five.

We are now ready to state the lone main result of Chapter 10. Analogous to Theorem 6.0.9, our lone main result for Chapter 10 is the following.

Theorem 10.0.7.

1) Let G be a short-separation-free graph with facial cycle C and list-assignment L, where C is an L-coil. Let $C^1 := G[D_1(C)]$. For any pentagonal $z \in D_2(C^1)$, there exists a (C, z)-opener; AND

2) Let $\mathcal{T} := (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let $C \in \mathcal{C}$ be a closed $\mathcal{T}$-ring. Let C^1 be the 1-necklace of C. Then C is an L-coil of G and furthermore, for any $z \in D_2(C^1) \setminus \text{Sh}_4(C^1, G \setminus C)$, there exists a (C, z)-opener.

We use 2) of Theorem 10.0.7 in Chapter 11, in combination with Theorem 6.0.9, to reduce a critical mosaic to a smaller counterexample by deleting a path between the outer face and an internal ring of a critical mosaic.

In Section 10.1, we show that closed rings of critical mosaics satisfy Definition 10.0.1, so for the remainder of Chapter 10 after Section 10.1, it just suffices to prove 1) of Theorem 10.0.7. In Section 10.2, we gather the preliminary facts we need in order to prove 1) of Theorem 10.0.7. In the remaining sections of Chapter 10, we prove a sequence of lemmas which we combine to prove 1) of Theorem 10.0.7. In each of these lemmas, we apply the work from Chapters 1, 7, 8, and 9 to produce our deletion set H by coloring and deleting a subpath of a specified cycle in a way which leaves some sets of vertices inert with respect to our coloring if those vertices are separated by our deletion set from the rings of $\mathcal{C} \setminus \{C\}$ by a 2- or 3-chord of a specified cycle. In Sections 10.2-10.3, we first introduce the machinery that we need in order to prove our sequence of results that make up the proof of Theorem 10.0.7. Theorem 10.0.7 is considerably more difficult than the analogous statement Theorem 6.0.9 for open rings.

10.1 Specializing to Closed Rings

The purpose of this short section is to prove the following lone result.

Lemma 10.1.1. *For any critical mosaic $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ and any closed ring $C \in \mathcal{C}$, C is an L-coil of G. In particular, if 1) of Theorem 10.0.7 holds, then 2) of Theorem 10.0.7 also holds.*

We begin with the following.

Proposition 10.1.2. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let C be a closed ring, let C^1 be the 1-necklace of C, and let ϕ be the unique L-coloring of $V(C)$. Then C^1 is $(4, L_\phi)$-short in $(C^1, G \setminus C)$, and, for any $w \in D_2(C)$, the graph $G[V(C^1) \cap N(w)]$ is a subpath of C^1.*

Proof. Let $\tilde{G} = G \setminus C$. For any $2 \leq k \leq 4$, any k-chord of C^1 in $\tilde{G}$ which separates two vertices of $\tilde{G}$ is a proper k-chord of C^1, since G is short-separation-free. Thus, it immediately follows from 2) of Theorem 2.2.4 that, for any k-chord R of C^1, one side of R contains all the elenents of $\mathcal{C} \setminus \{C\}$, so C^1 is $(4L_\phi)$-short in $(C^1, \tilde{G})$. The corresponding partition is specified in Definition 8.0.3. Note that, for any $u, u' \in N(w)$ with $u \neq u'$, the path $C^1 \cap \tilde{G}^{\text{large}}_{uwu'}$ has length greater than one, or else there is a cycle of length three which separates C from an element of $\mathcal{C} \setminus \{C\}$, contradicting short-separation-freeness. Thus, by Theorem 8.0.4, $G[N(w) \cap V(C^1)]$ is an subpath of C^1. $\square$

In order to prove Lemma 10.1.1, the only nontrivial thing left to check is that a closed ring of a critical mosaic satisfies property Co4c) of Definition 10.0.1.

Proposition 10.1.3. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let $C \in \mathcal{C}$ be a closed ring. Let C^1 be the 1-necklace of C. If there exists a vertex $v \in V(C^1)$ such that $|N(v) \cap V(C)| > 2$, then there also a exists a $v' \in V(C^1)$ such that $|N(v) \cap V(C)| = 1$.*

Proof. Suppose there is at least one vertex of C^1 with at least three neighbors in C. Since C is L-predictable and an induced subgraph of G, it follows that there is a vertex $u_\star \in V(C^1)$, where $|N(u_\star) \cap V(C)| > 2$, such that $|L_\phi(u_\star)| \geq 2$, and, for all $v \in V(C - u_\star)$, $|L_\phi(v)| \geq 3$.

Let $T^{<2} := \{v \in V(C^1) : |N(v) \cap V(C)| = 1\}$ and let T^{int} be the set of vertices in $v \in V(C^1)$ for which there exists a $w \in D_2(C)$ such that $G[N(w) \cap V(C^1)]$ is a subpath of C^1 with v as an internal vertex. Now suppose toward a contradiction that $T^{<2} = \varnothing$, i.e every vertex of C^1 has at least two neighbors in C. Thus, since $u_\star$ has at least three neighbors in C, it follows that $|V(C^1)| < |V(C)|$. Let $C^1 := v_1 \cdots v_k v_1$. Note that $k \geq 5$, as G is short-separation-free.

Claim 10.1.4. *There exists a partial L_ϕ-coloring ψ of $V(C^1)$ and a vertex $w \in B_2(C)$ such that the following hold.*

1) $|L_\psi(w)| \geq 2$, and, for all $w' \in B_2(C) \setminus \{w\}$, $|L_\psi(w')| \geq 3$; AND

2) $V(C^1) \setminus \text{dom}(\psi)$ is $L_{\phi \cup \psi}$-inert and a subset of T^{int}.

Proof: We first deal with the following easy case.

Subclaim 10.1.5. *Suppose there exists a $j \in \{1, \cdots, k\}$ such that, for any $w \in D_2(C) \cap N(v_j)$, w is adjacent to at most two vertices of C^1. Then there exists a partial L_ϕ-coloring ψ of $V(C^1)$ such that $V(C^1) \setminus \text{dom}(\psi)$ is $L_{\phi \cup \psi}$-inert and a subset of T^{int}, and furthermore, for all $w \in B_2(C)$, $|L_\psi(w)| \geq 3$.*

Proof: Let $j \in \{1, \cdots, k\}$ satisfy the condition above.

For any $w \in D_2(C) \cap N(v_j)$, $G[N(W) \cap V(C^1)]$ is a subpath of C^1 and, by assumption, w is adjacent to at most two vertices of C^1, so it follows that the path $C^1 - v_j$ is $(2, L_\phi)$-short in $(C^1, \tilde{G})$. Now consider the following cases:

Case 1: $v_j = u_\star$

In this case, every vertex of the path $C^1 - u_\star$ has an L_ϕ-list of size at least three. Since $|L_\phi(v_{j-1})| \geq 3$, let $c \in L_\phi(v_{j-1})$ with $|L_\phi(v_j) \setminus \{c\}| \geq 2$. By Theorem 1.7.5, there is a $\psi \in \text{Link}_{L_\phi}(C^1 - v_j, C^1, \tilde{G})$ such that $\psi(v_{j-1}) = c$. Since C^1 is an induced subgraph of $\tilde{G}$, ψ extends to a proper L_ϕ-coloring ψ^* of $V(C^1)$, and, by our assumption on v_j, it follows that, for each $w \in B_2(C)$, we have $|L_{\psi^*}(w)| \geq 3$. By definition of $\text{Link}_{L_\phi}(C^1 - v_j, C^1, \tilde{G})$, $V(C^1) \setminus \text{dom}(\psi^*)$ is $L_{\phi \cup \psi^*}$-inert in G and a subset of T^{int}.

Case 2: $v_j \neq u_\star$

In this case, since $u_\star$ is the only vertex of C^1 with an L_ϕ-list of size less than three, it again follows from Theorem 1.7.5 that there is a $\psi \in \text{Link}_{L_\phi}(C^1 - v_j, C^1, \tilde{G})$ (although we do not get to choose the color of v_{j-1} in this case). Since $|L_\phi(v_j)| \geq 3$ and C^1 is an induced subgraph of G, ψ extends to a proper L_ϕ-coloring ψ^* of $V(C^1)$, and, by our assumption on v_j, it follows that, for each $w \in B_2(C)$, we have $|L_{\psi^*}(w)| \geq 3$. By definition of $\text{Link}_{L_\phi}(C^1 - v_j, C^1, \tilde{G})$, $V(C^1) \setminus \text{dom}(\psi^*)$ is $L_{\phi \cup \psi^*}$-inert and a subset of T^{int}. ■

For the remainder of the proof of Claim 10.1.4, we suppose that, for each $j \in \{1, \cdots, k\}$, there is a $w \in B_2(C)$ such that $G[N(w) \cap V(C^1)]$ is a path of length two which contains v_j, since, if this does not hold, then, by Subclaim 10.1.5, we are done.

Subclaim 10.1.6. *There exists a $w \in B_2(C)$ such that $G[N(w) \cap V(C^1)]$ is a subpath of C^1 of length precisely two.*

Proof: Suppose toward a contradiction that no such w exists. Note that, by our assumption on $\{v_1, \cdots, v_j\}$, no two vertices of $C^1 \setminus T^{\text{int}}$ are consecutive in T^{int}. We now define a cycle C' as follows. We let C' be the unique cycle of G which intersects with C^1 on precisely the vertices of $C^1 \setminus T^{\text{int}}$, where, for each subpath P of C^1 with $V(\mathring{P}) \subseteq^{\text{int}}$ and $V(P) \setminus V(\mathring{P}) \subseteq V(C^1) \setminus T^{\text{int}}$, we replace P with the unique 2-path in G whose midpoint lies in $B_2(C)$ and whose endpoints are also the endpoints of P.

Since no two vertices of $C^1 \setminus T^{\text{int}}$ are adjacent in C^1, C^1 admits a partition $C^1 = P_1 \cdots P_r$, where $P_1, \cdots, P_r$ is a collection of edge-disjoint paths, and, for each $i = 1, \cdots, r$, the following hold.

i) The endpoints of P_i lie in $C^1 \setminus T^{\text{int}}$ and $V(\mathring{P}_i) \subseteq T^{\text{int}}$; AND

ii) There is a unique vertex $w_i \in D_2(C)$ such that $P_i := G[N(w_i) \cap V(C^1)]$; AND

iii) P_i, P_{i+1} intersect on a unique common endpoint of P_i, P_{i+1}, where the indices are read mod r.

Condition iii) holds since $r > 2$, or else there is a cycle of length four which separates C from an element of $\mathcal{C} \setminus \{C\}$, contradicting the fact that $\mathcal{T}$ is a tessellation. By assumption, each of the paths $P_1, \cdots, P_r$ has length at least three. For each $i = 1, \cdots, r$, let Q_i be the unique 2-path whose midpoint is w_i and whose endpoints are the endpoints of P_i, where $w_1, \cdots, w_r$ as as in ii) above. Note that, for each $i = 1, \cdots, r$, we have $\frac{|E(P_i)|}{|E(Q_i)|} \geq \frac{3}{2}$.

Since $|V(C^1)| = |E(C^1)| = \sum_{i=1}^r |E(P_i)|$ and $|V(C')| = |E(C')| = \sum_{i=1}^r |E(Q_i)|$, it follows that $|V(C_1)| \geq \lceil \frac{3}{2}|V(C')| \rceil$. Since C' has nonempty intersection with C_1, we have $d(C', C) = 1$. Since C' separates C from each element of $\mathcal{C} \setminus \{C\}$, it follows from Corollary 2.1.30 that $|V(C)| < 1 + \frac{3}{2}|V(C')|$. But since $|V(C_1)| < |V(C)|$, we have $|V(C)| > \lceil \frac{3}{2}|V(C')| \rceil$, so we get $\lceil \frac{3}{2}|V(C')| \rceil < |V(C)| < \frac{3}{2}|V(C')| + 1$, which is false. ■

Now we have enough to finish the proof of Claim 10.1.4. Applying Subclaim 10.1.6, let $w \in B_2(C)$, where $P := G[N(w) \cap V(C^1)]$ is a subpath of C^1 of length precisely two, and let $P := v_j v_{j+1} v_{j+2}$ for some $j \in \{1, \cdots, k\}$. Note that $v_{j+1} \in T^{\text{int}}$, and, since $G[N(w) \cap V(C^1)]$ has length precisely two, the path $C^1 - v_j$ is $(2, L_\phi)$-short in $(C^1, \tilde{G})$. Consider the following cases:

Case 1: $v_j = u_\star$

In this case, every vertex of the path $C^1 - u_\star$ has an L_ϕ-list of size at least three. Since $|L_\phi(v_{j-1})| \geq 3$, let $c \in L_\phi(v_{j-1})$ with $|L_\phi(v_j) \setminus \{c\}| \geq 2$. By Theorem 1.7.5, there is a $\psi \in \text{Link}_{L_\phi}(C^1 - v_j, C^1, \tilde{G})$ such that $\psi(v_{j-1}) = c$. Since C^1 is an induced subgraph of $\tilde{G}$, ψ extends to a proper L_ϕ-coloring ψ^* of $V(C^1)$, and, for each $w' \in B_2(C) \setminus \{w\}$, we have $|L_{\psi^*}(w')| \geq 3$. Furthermore, $|L_{\psi^*}(w)| \geq 2$, since P has length two, so our choice of ψ^*, w satisfies Claim 10.1.4.

Case 2: $v_j \neq u_\star$

In this case, by A) of Theorem 7.0.1, there is a $\psi \in \text{Link}_{L_\phi}(C^1 - v_j, C^1, \tilde{G})$ (although we no longer have control over the color $\psi(v_{j-1})$ in this case). Since $|L_\phi(v_j)| \geq 3$ and C^1 is an induced subgraph of G, ψ extends to a proper L_ϕ-coloring ψ^* of $V(C^1)$, and, for each $w' \in B_2(C) \setminus \{w\}$, we have $|L_{\psi^*}(w')| \geq 3$. As above, $|L_{\psi^*}(w)| \geq 2$, since P has length two, so our choice of ψ^*, w satisfies Claim 10.1.4. This completes the proof of Claim 10.1.4. ■

Now we have enough to finish the proof of Proposition 10.1.3. Let ϕ, w be as in Claim 10.1.4. We now define a list-assignment $L^\dagger$ for $V(G \setminus C)$ as follows. For each $x \in B_2(C)$, we set $L^\dagger(x) := L(x)$. For each $x \in \text{dom}(\psi)$, we set $L^\dagger(x) := \{\psi(x)\}$. Finally, for each $x \in V(C^1 \setminus \text{dom}(\phi))$, we set $L^\dagger(x)$ to consist of a lone color not lying in $\bigcup_{y \in N_G(x)} L(y)$, where these lone colors are chosen so that $V(C^1)$ is $L^\dagger$-colorable (we can just choose all of these singletons to be distinct colors). Now let $C_*^\dagger$ be the outer face of $\tilde{G}$ and consider the tuple $\mathcal{T}^\dagger := (\tilde{G}, (\mathcal{C} \setminus \{C\}) \cup \{C^1\}, L^\dagger, C_*^\dagger)$. Note that $\mathcal{T}^\dagger$ is a tessellation in which C^1 is a closed ring. We claim now that $\mathcal{T}^\dagger$ is a mosaic.

Since $|V(C^1)| < |V(C)|$ and $V(C^1) = B_1(C)$, it follows that $\mathcal{T}^\dagger$ satisfies the distance conditions of Definition 2.1.6, since $\text{Rk}(\mathcal{T}^\dagger | C^1) < \text{Rk}(\mathcal{T} | C)$, and it also immediately follows that $\mathcal{T}^\dagger$ satisfies M0).

Since each vertex of C^1 has an $L^\dagger$-list of size one, let σ be the unique $L^\dagger$-coloring of $V(C^1)$. By our choice of $L^\dagger$-lists for the vertices of $C^1 \setminus \text{dom}(\phi)$, we have $|L_\sigma(w)| \geq 2$, and, for each $w' \in B_2(C, G) \setminus \{w\}$, we have $|L_\sigma(w')| \geq 3$. By Theorem 8.0.4, each vertex of $w' \in B_2(C, G)$, the graph $G[N(w') \cap V(C^1)]$ is a subpath of C^1, and since C^1 is an induced subgraph of $G \setminus C$, it follows that C^1 is an $L^\dagger$-predictable facial subgraph of $G \setminus C$. Thus, $\mathcal{T}^\dagger$ also satisfies M2), and M1) is trivially satisfied. We conclude that $\mathcal{T}^\dagger$ is indeed a mosaic. Since $|V(G \setminus C)| < |V(G)|$, it follows from the minimality of $\mathcal{T}$ that there is an $L^\dagger$-coloring τ of $\tilde{G}$. Let τ^* be the restriction of τ to $V(\tilde{G}) \setminus V(C^1 \setminus \text{dom}(\psi))$. By our construction of ψ, the union $\phi \cup \tau^*$ is a proper L-coloring of $V(G) \setminus V(C^1 \setminus \text{dom}(\psi))$, and $\phi \cup \tau^*$ extends to an L-coloring of G, contradicting the fact that G is not L-colorable. This completes the proof of Proposition 10.1.3. □

Now we finish the proof of Lemma 10.1.1, which we restate below.

Lemma 10.1.1. *For any critical mosaic $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ and any closed ring $C \in \mathcal{C}$, C is an L-coil of G. In particular, if 1) of Theorem 10.0.7 holds, then 2) of Theorem 10.0.7 also holds.*

Proof. Let C^1 be the 1-necklace of C. We first check that C is an L-coil of G. Since C is an induced subgraph of G, it follows from Definition 2.1.6 that C satisfies Co1 of Definition 10.0.1. Since $\mathcal{T}$ is a tessellation, It immediately follows from Corollary 2.2.29 and the distance conditions of Definition 2.1.6 that Co2-Co3 hold as well. Now we just check that C^1 satisfies Co4. By Theorem 8.0.4, C^1 satisfies Co4a) and, by Proposition 10.1.2, C^1 satisfies Co4b) and

Co4d). By Proposition 10.1.3, C^1 satisfies Co4c), so C is indeed an L-coil of G. It also follows from the distance conditions on $\mathcal{T}$ that every vertex of $D_2(C^1) \setminus \text{Sh}_4(C^1, G \setminus C)$ is pentagonal. Thus, if 1) of Theorem 10.0.7 holds, then 2) of Theorem 10.0.7 also holds. $\square$

Since we have Lemma 10.1.1, the remainder of Chapter 10 deals entirely with 1) of Theorem 10.0.7, i.e all of the remaining work of Chapter 10 is exclusively in the context of the general structures defined in Definition 10.0.1.

10.2 Preliminaries to the Proof of 1) of Theorem 10.0.7

For the remainder of Chapter 10, we fix the following data.

1) A planar graph G with facial cycle C and list-assignment L, where C is an L-coil of G and $\tilde{G} := G \setminus C$.
2) An L-coloring ϕ of $V(C)$ (i.e the unique L-coloring of $V(C)$.
3) A cycle C^1, where $C^1 := G[D_1(C)]$.

We show in the remainder of Chapter 10 that, for every pentagonal vertex of $D_2(C^1)$, there exists a (C, z)-opener. We begin with the following definitions.

Definition 10.2.1.

1) Let $T^{<2} := \{v \in V(C^1) : |N(v) \cap V(C)| = 1\}$.
2) Let $S_\star$ be the set of vertices of C^1 with L_ϕ-lists of size less than three. Since C is L-predictable and induced in G, we have either $S_\star = \varnothing$ or * consists of a lone vertex with an L_ϕ-list of size two.
3) We define a subpath $S_\star^{\text{path}}$ of C^1 as follows.
   (a) If $S_\star = \varnothing$ then $S_\star^{\text{path}} = \varnothing$.
   (b) Otherwise, letting $u_\star$ be the lone vertex of $S_\star$, if $u_\star \notin \text{Sh}_{2,L_\phi}(C^1, \tilde{G})$, then we set $S_\star^{\text{path}} := u_\star$, and, if $u_\star \in \text{Sh}_{2,L_\phi}(C^1, \tilde{G})$, then we set $S_\star^{\text{path}} := G[N(w) \cap V(C^1)]$, where w is the unique element of $D_2(C)$ such that $u_\star$ is an internal vertex of $G[N(w) \cap V(C^1)]$.
4) Given a subpath Q of C^1, we say that Q is *divisible* if, for some $k \geq 2$, there is a proper k-chord R of C^1 in $\tilde{G}$ such that either $Q \subseteq C^1 \cap \tilde{G}_R^{\text{small}}$ or $Q \subseteq C^1 \cap \tilde{G}_R^{\text{large}}$.

Note that $S_\star^{\text{path}}$ is well-defined by the subpath condition Co4d) of Definition 10.0.1. We now have the following simple observation, which we use repeatedly.

Observation 10.2.2. *Let Q be a divisible subpath of C^1. Then Q is $(2, L_\phi)$-short in $(\tilde{G}, C^1)$.*

Proof. Suppose toward a contradiction that Q is not $(2, L_\phi)$-short in $(\tilde{G}, C^1)$). Since $G[N(w) \cap V(C^1)]$ is a subpath of C^1 for each $w \in D_2(C)$, it follows that there exists a $w \in D_2(C)$ such that $|N(w) \cap V(Q)| > 2$ and Q contains both endpoints of $G[N(w) \cap V(C^1)]$, but does not contain all of $G[N(w) \cap V(C^1)]$. That is, letting $P^w := G[N(w) \cap V(C^1)]$, Q contains an internal vertex of P^w but $Q \cap P^w$ is not connected.

By definition, for some $k \geq 2$, there is a proper k-chord R of C^1 in $\tilde{G}$ such that either $Q \subseteq C^1 \cap \tilde{G}_R^{\text{small}}$ or $Q \subseteq C^1 \cap \tilde{G}_R^{\text{large}}$. Consider the following cases.

Case 1: $w \in V(R)$

In this case, one endpoint of P^w lies in $\tilde{G}\setminus R$ and the other lies in $\tilde{G}\setminus R$, or else $P^w\cap Q$ is connected. But since both endpoints of P^w lie in Q,we contradict the fact that either $Q\subseteq\tilde{G}^{\text{small}}_R$ or $Q\subseteq\tilde{G}^{\text{large}}_R$.

Case 2: $w\notin V(R)$

In this case, suppose without loss of generality that $w\in V(\tilde{G}^{\text{large}}_R)\setminus V(R)$. Thus, P^w is a subpath of $C^1\cap\tilde{G}^{\text{large}}_R$ and intersects with $C^1\cap\tilde{G}^{\text{small}}_R$ at most on its endpoints. Since Q contains an internal vertex of P^w and both endpoints of P^w, we have $Q=P^w=\tilde{G}^{\text{small}}$, contradicting our assumption that $Q\cap P^w$ is not connected. $\square$

In view of the results of Sections 1.6 and 9.1-9.2, we introduce the following very natural definitions, since we frequently deal with 3-chords of C^1 in $\tilde{G}$.

Definition 10.2.3. Given a 3-chord R of C^1 in $\tilde{G}$, we have the following notation.

1) We set $\text{Base}(R)$ to be the set of L_ϕ-colorings ψ of the endpoints of R such that any extension of ψ to an L_ϕ-coloring of $V(R)$ extends to L_ϕ-color all of $\tilde{G}^{\text{small}}_R$.
2) For any $x\in V(\mathring{R})$, we set $\text{Corner}(R,x)$ to be the set of L_ϕ-colorings ψ of $V(R-x)$ such that any extension of ψ to an L_ϕ-coloring of $V(R)$ extends to L_ϕ-color all of $\tilde{G}^{\text{small}}_R$.

Unless otherwise specified, given a subpath Q of C^1, whenever we write $\text{Link}(Q)$ in the remainder of Chapter 10, we mean $\text{Link}_{L_\phi}(Q,C^1,\tilde{G})$, and likewise, whenever we write $\text{Sh}_2(Q)$, we mean $\text{Sh}_{2,L_\phi}(Q,C^1,\tilde{G})$. We supress these subscript and coordinates as they are clear from the context of the data that we fixed at the beginning of Section 10.2. Likewise, for any partial L_ϕ-coloring σ of $\tilde{G}$ and vertex set $A\subseteq V(\tilde{G})$, we always write $\Phi(\sigma,A)$ to mean $\Phi_{\tilde{G},L_\phi}(\sigma,A)$.

Proposition 10.2.4. *Suppose that $S_\star\neq\varnothing$ and let $u_\star$ be the lone vertex of $S_\star$. Let P be a divisible subpath of C^1 of length at least one, where $S^{\text{path}}_\star$ is a proper subpath of P. Let p,p' be the endpoints of P and let q,q' be the endpoints of $S^{\text{path}}_\star$, where the (not necessarily distinct) vertices of $\{p,p',q,q'\}$ have the order p',q',q,p on the path P. Then the following hold.*

1) $\text{Link}(P)\neq\varnothing$*; AND*
2) *If there is a vertex $v^\dagger\in V(qPp)\cap T^{<2}$, then there exist two elements ψ_1,ψ_2 of* $\text{Link}(P)$ *which use different colors on p and which both restrict to the same partial L_ϕ-coloring of $p'Pq'$.*

Proof. Since $S_\star\neq\varnothing$, it follows from Co4d) of Definition 10.0.1 that every vertex of $D^2(C)$. has a neighborhood in C^1 consisting of a subpath of C^1. Since G is $K_{2,3}$-free, it follows that, for any $w\in D_2(C)$, no vertex of $T^{<2}$ is an internal vertex of the path $G[N(w)\cap V(C^1)]$. In the language of Definition 1.7.3, any vertex of $V(P)\cap T^{<2}$ is a P-hinge of C, and since $u_\star\notin T^{<2}$, the proposition is an immediate consequence of Theorem 7.0.1. $\square$

The result above has the following compact corollary.

Corollary 10.2.5.

1) *For any divisible subpath Q of C^1,* $\text{Link}(Q)\neq\varnothing$*; AND*
2) *For any $x\in V(C^1)$, if C^1-x is a divisible subpath of C^1, then there exists a $\psi\in$* $\text{Link}(C^1-x)$ *such that* $|L_{\phi\cup\psi}(x)|\geq 1$
3) *If $S_\star\neq\varnothing$ and let $xx'\in E(C^1)\setminus E(S^{\text{path}}_\star)$, where C^1-xx' is a divisible subpath of C^1, then there is a $\sigma\in$* $\text{Link}(C^1-xx')$ *such that $\sigma(x)\neq\sigma(x')$ (i.e particular, σ is a proper L_ϕ-coloring of its domain in $\tilde{G}$)*

Proof. If $S_\star \cap V(\mathring{Q}) = \varnothing$, or there is a vertex $u_\star$ with $S_\star = \{u_\star\}$, where $S_\star^{\text{path}}$ intersects with Q on at most an endpoint, then we are done by Theorem 1.7.5. If $S_\star^{\text{path}} \subseteq Q$, then we are done by Proposition 10.2.4. The only remaining possibility is that there exist two vertices p, p' of $S_\star^{\text{path}}$, where $S_\star^{\text{path}} \cap Q = pQp'$, p is an endpoint of Q, p' is an internal vertex of Q and an endpoint of $S_\star^{\text{path}}$, and $u_\star$ is an internal vertex of pQp'. Let p^* be the non-p endpoint of Q. By Theorem 7.0.1, there is an element ψ of Link(pQp') obtained by coloring p, p'. By Theorem 1.7.5, there is a $\psi^* \in \text{Link}(p'Qp^*, C^1, \tilde{G})$ using $\psi(p')$ on p'. Since pQp' is a path of length at least two, there is a unique vertex $w \in B_2(C)$ such that $S_\star^{\text{path}} = G[N(w) \cap V(C^1)]$. Since $N(w) \cap V(Q) = V(pQp')$, and $p' \notin T^{\text{int}}$, we have $\psi \cup \psi' \in \text{Link}(pQp')$. This proves 1).

Now we prove 2). Suppose first that $x \notin S_\star$. By 1), there is a $\psi \in \text{Link}(C^1 - x,)$, and since C^1 is an induced subgraph of G and $L_\phi(x)| \geq 3$, we have $|L_{\phi\cup\psi}(x)| \geq 1$. Now suppose that $x \in S_\star$. Thus, $S_\star = \{x\}$ and $|L_\phi(x_\ell)| \geq 3$, so, there is a $c \in L_\phi(x_\ell)$ with $|L_\phi(x) \setminus \{c\}| \geq 2$. By Theorem 1.7.5, there is a $\psi \in \text{Link}_{L_\phi}(C^1 - x)$ with $\psi(x_\ell) = c$, so we again have $|L_{\phi\cup\psi}(x)| \geq 1$.

Now we prove 3). Since $S_\star \neq \varnothing$, it follows from Co4c) of Definition 10.0.1 that $T^{<2} \neq \varnothing$. Since xx' is not an edge of $S_\star^{\text{path}}$, then it immediately follows from Proposition 10.2.4 that there is a $\sigma \in \text{Link}(C^1 - xx')$ such that $\sigma(x) \neq \sigma(x')$, so σ is a proper L_ϕ-coloring of its domain in $\tilde{G}$. $\square$

We now return to setting up the proof of Theorem 10.0.7. For each $z \in D_2(C^1)$, we associate to z a partition of $\tilde{G}$ in the following way.

Definition 10.2.6. For each $z \in D_2(C^1)$, we let $\tilde{G}_z^{\text{small}}$ and $\tilde{G}_z^{\text{large}}$ be the unique subgraphs of $\tilde{G}$ such that $\tilde{G} = \tilde{G}_z^{\text{small}} \cup \tilde{G}_z^{\text{large}}$, where the following hold.

1) Each vertex of $\tilde{G} \setminus C^1$ with an L_ϕ-list of size less than five lies in $V(\tilde{G}_z^{\text{large}})$*; AND*
2) $\tilde{G}_z^{\text{small}} \cap \tilde{G}_z^{\text{large}} = \text{Span}(z)$*; AND*
3) If Span(z) is either a cycle or a proper 4-chord of C^1, then $\tilde{G}_z^{\text{small}} \cup \tilde{G}_z^{\text{large}}$ is the natural Span(z)-partition of $\tilde{G}$*; AND*
4) If Span(z) is a claw, then $\tilde{G}_z^{\text{small}} - z$ and $\tilde{G}_z^{\text{large}}$ are the two subgraphs of the natural partition of $\tilde{G}$ associated to the 2-chord Span$(z) \setminus \{z\}$ of $\tilde{G}$*; AND*
5) If Span(z) is a 2-path, then $\tilde{G}_z^{\text{small}} = \text{Span}(z)$ and $\tilde{G}_z^{\text{large}} = \tilde{G}$.

By Co3 of Definition 10.0.1, these two graphs are uniquely specified.

Note that, if Span(z) is a proper 4-chord P of $\tilde{G}$, then $\tilde{G}_z^{\text{small}} = \tilde{G}_P^{\text{small}}$ and $\tilde{G}_z^{\text{large}} = \tilde{G}_P^{\text{large}}$. If Span$(z)$ is a 2-path, then $\tilde{G}_z^{\text{small}} = \text{Span}(z)$ and $\tilde{G}_z^{\text{large}} = \tilde{G}$. If Span$(z)$ is a 4-cycle, the, since G is short-separation-free and $z \in D_2(C^1)$, it follows from our triangulation conditions that $\tilde{G}_z^{\text{small}}$ consists of the 4-cycle Span(z) and an edge between the two neighbors of z in $V(\text{Span}(z))$. Given the definitions above, there is a very natural way to associate to each $z \in D_2(C^1)$ a cycle obtained from C^1 by rerouting through a path in Span(z).

Definition 10.2.7. For each $z \in D_2(C^1)$, we associate to z a cycle C_z^1 in G in the following way. If Span(z) is either a 2-path or a 4-cycle, then we set $C_z^1 := C^1$. If Span(z) is a claw, then we set C_z^1 to be the cycle $(C^1 \cap \tilde{G}_z^{\text{large}}) + xwx'$, where xwx' is the 2-path Span$(z) \setminus \{z\}$. Finally, if Span(z) is a proper 4-chord of C^1, then we simply set C_z^1 to be the cycle $\tilde{G}_z^{\text{large}} + \text{Span}(z)$.

We now have the following.

Proposition 10.2.8. *For any pentagonal $z \in D_2(C^1)$, the following hold.*

1) *C^1_z is an induced subgraph of $\tilde{G}^{\text{small}}_z$; AND*
2) *No two vertices of $V(\text{Span}(z)) \cap D_2(C)$ have a common neighbor other than z in $\tilde{G}^1_z$; AND*
3) *For any $u \in N(z) \cap V(\tilde{G}^{\text{large}}_z \setminus \text{Span}(z))$, the set $N(u) \cap (V(C^1_z) \cup \text{Span}(z))$ consists of z and at most one vertex of $\text{Span}(z) \cap D_2(C)$.*

Proof. We first prove 1). This is trivial if $\text{Span}(z)$ is a 2-path or a 4-cycle, since in that case we have $C^1_z = C^1$ and so the proposition follows from Co4a) of Definition 10.0.1. Now suppose that $\text{Span}(z)$ is a claw, where $\text{Span}(z) - z$ is the 2-chord xwx' of C^1. We then have $N(w) \cap V(C^1) \subseteq V(\tilde{G}^{\text{small}}_z)$, or else we contradict the maximality of the 2-chord xwx', Thus, C^1_z is again an induced subgraph in this case, since $z \in D_2(C^1)$.

Now suppose that $\text{Span}(z)$ is a proper 4-chord $xyzy'x'$ of C^1. W first show that $x'y, xy' \notin E(\tilde{G}^{\text{large}}_z)$. Suppose this does not hold, and suppose without loss of generality that $x'y \in E(\tilde{G}^{\text{large}}_z)$. Then $\tilde{G}$ contains the 2-chord xyx' of C^1, and since G is short-separation-free, the 4-cycle $x'yzy'$ does not separate x from any vertex of $G \setminus B_4(C)$ with an L_ϕ-list of size less than five. Thus, we have $\tilde{G}^{\text{small}}_z \subseteq \tilde{G}^{\text{small}}_{xyx'}$, and $z \in V(\tilde{G}^{\text{small}}_{xyx'})$, contradicting the fact that z is pentagonal. Now suppose toward a contradiction that C^1_z is not an induced subgraph of $\tilde{G}^{\text{large}}_z$. Since C^1 is an induced subgraph of G, $x'y, xy' \notin E(\tilde{G}^{\text{large}}_z)$, and $z \in D_2(C)$, it follows that there exists an edge $e \in E(\tilde{G}^{\text{large}}_z)$ with one endpoint in $\{y, y'\}$ and the other endpoint in $V(C^1_z \setminus \text{Span}(z))$. Without loss of generality, let $e = yu$, and note that $u \in V(C^1) \setminus \{x, x'\}$.

Now, in $\tilde{G}^{\text{large}}_z$, the chord yu of C^1_z separates $zy'x'$ from no vertex of $G \setminus B_4(C)$ with an L_ϕ-list of size less than five, and, letting $P^* := uyzy'x'$, we have $\tilde{G}^{\text{small}}_z \subseteq \tilde{G}^{\text{small}}_{P^*}$. Since $u \notin V(\tilde{G}^{\text{small}}_z)$, we have $|V(\tilde{G}^{\text{small}}_z)| < |V(\tilde{G}^{\text{small}}_{P^*})|$, so we contradict the maximality of $\text{Span}(z)$. This proves 1).

Now we prove 2) and 3) together. We first show that 2) and 3) hold if $\text{Span}(z)$ is either a claw, a 4-cycle, or a 2-path. 2) is trivial if $\text{Span}(z)$ is either a claw of a 2-path, since there is only vertex of $\text{Span}(z) \cap D_2(C)$ in that case, and, if $\text{Span}(z)$ is a 4-cycle, then the claim immediately follows from the fact that G is $K_{2,3}$-free. Now let $u \in N(z) \cap V(\tilde{G}^{\text{large}}_z \setminus \text{Span}(z))$. If $\text{Span}(z)$ is a 2-path, a claw, or a 4-cycle, then, by definition, we have $N(u) \cap V(C^1) = \varnothing$, so, by 2), $N(u) \cap (V(C^1_z) \cup \text{Span}(z))$ consists of at most z and the lone vertex of $D_2(C) \cap V(\text{Span}(z))$. Thus, 2) and 3) hold in the case where $\text{Span}(z)$ is either a claw, a 4-cycle, or a 2-path.

Now we show that 2) and 3) hold in the case where $\text{Span}(z)$ is a proper 4-chord $xyzy'x'$ of C^1. We claim that, for any $u \in V(\tilde{G}^{\text{large}}_z \setminus C^1_z)$ with $u \in N(z)$, the set $N(u) \cap V(C^1_z)$ is either a subset of $\{z, y\}$ or a subset of $\{z, y'\}$. Let $u \in V(\tilde{G}^{\text{large}}_z - z)$ and suppose toward a contradiction that u is adjacent to each of y, y'. By 1), we have $u \notin V(C^1)$, and $xyuy'x'$ is a proper 4-chord of C^1. In $G \setminus C$, this 4-chord of C^1 separates z from a vertex of $G \setminus B_4(C)$ with an L_ϕ-list of size less than five, since G is short-separation-free. This contradicts the fact that z is pentagonal. Thus, u is adjacent to at most one of y, y'. Now let $u \in V(\tilde{G}^{\text{large}}_z \setminus C^1_z)$ with $u \in N(z)$. Suppose toward a contradiction that u has a neighbor $u' \in V(C^1_z) \setminus \{z, y, z'\}$. Note that $u' \in V(C^1)$. Consider the following cases:

Case 1: $u \in \{x, x'\}$

Suppose without loss of generality that $u = x$. Then, in $G \setminus C$, the 4-chord $P^* := xuzy'x'$ of C^1 separates y from a vertex of $G \setminus B_4(C)$ with an L_ϕ-list of size less than five, since G is short-separation-free. In particular, $\tilde{G}^{\text{small}}_z \subseteq \tilde{G}^{\text{small}}_{P^*} - u$, contradicting the maximality of $\tilde{G}^{\text{small}}_z$.

Case 2: $u \notin \{x, x'\}$

In this case, $\tilde{G}$ contains the proper 4-chords $Q := u'uzyx$ and $Q' := u'uzy'x'$ of C^1, and we have either $\tilde{G}_z^{\text{small}} \subseteq \tilde{G}_Q^{\text{small}}$ or $\tilde{G}_z^{\text{small}} \subseteq \tilde{G}_{Q'}^{\text{small}}$. Suppose without loss of generality that $\tilde{G}_z^{\text{small}} \subseteq \tilde{G}_Q^{\text{small}}$. Since $u', u \notin V(\tilde{G}_z^{\text{small}})$, we have $|V(\tilde{G}_z^{\text{small}}| < |V(\tilde{G}_Q^{\text{small}})|$, contradicting the maximality of $\tilde{G}_z^{\text{small}}$. Thus, u has no neighbors in $V(C_z^1) \setminus \{z, y, z'\}$. Since u is adjacent to at most one of y, y', $N(u) \cap V(C_z^1)$ is either a subset of $\{z, y\}$ or a subset of $\{z, y'\}$. $\square$

The final proposition we prove in this section is short but extremely useful.

Proposition 10.2.9. *Let $k \leq 2 \leq 3$ and let R be a proper k-chord of C^1. Let P be a subpath of C^1, where $C^1 \cap \tilde{G}_R^{\text{small}} \subseteq P$ and each endpoint of R is a P-hinge. Then, for any $\psi \in \text{Link}(P)$, any extension of ψ to an L_ϕ-coloring of* $\text{dom}(\psi) \cup V(R)$ *extends to L_ϕ-color all of $\tilde{G}_R^{\text{small}}$ as well.*

Proof. Let u, u' be the endpoints of R and let Q be the path $C^1 \cap \tilde{G}_R^{\text{small}}$. Since each of u, u' is a P-hinge, we have $u, u' \in \text{dom}(\psi)$, and ψ restricts to an element ψ' of $\text{Link}_{L_\phi}(Q, C^1, \tilde{G})$. Now, $\tilde{G}_R^{\text{small}}$ contains a cyclic facial subgraph $F := Q + R$, where each vertex of $\tilde{G}_R^{\text{small}} \setminus F$ has an L_ϕ-list of size at least five, so ψ' is also an element of $\text{Link}_{L_\phi}(Q, F, \tilde{G}_R^{\text{small}})$. The desired result now follows immediately from 3b) of Theorem 1.7.4. $\square$

10.3 Matchable Colors

In order to prove Theorem 10.0.7 in the most difficult and general case, which is there case where we deal witha $z \in D_2(C^1)$ such that $\text{Span}(z)$ is a proper 4-chord of C^1, (i.e $\text{Span}(z)$ has no degeneracies) we need some results about partial colorings of $\text{Span}(z)$ which extend to $\tilde{G}_z^{\text{small}}$. The purpose of Section 10.3 is to gather the results of this form that we need. To state the lone main result of Section 10.3, we first introduce the following terminology.

Definition 10.3.1. Let $k \geq 3$ and let P be a proper k-chord of C^1 in $\tilde{G}$, where $k \geq 3$. Let $xy, x'y'$ be the terminal edges of P, where $x, x' \in V(C^1)$. Let $c \in L_\phi(x')$ and let A be a subgraph of xy.

1) We say that c is (A, P)-*matchable* if there is at most on L_ϕ-coloring of $\{x'\} \cup V(A)$ which uses c on x' and does not extend to an L_ϕ-coloring of $V(\tilde{G}_P^{\text{small}})$*; AND*

2) We say that c is *highly* (A, P)-*matchable* if every L_ϕ-coloring of $\{x'\} \cup V(A)$ which uses c on x' extends to an L_ϕ-coloring of $V(\tilde{G}_P^{\text{small}})$.

Our lone result for Section 10.3 is the following.

Lemma 10.3.2. *Let P be a proper k-chord of C^1 in $\tilde{G}$, where $3 \leq k \leq 4$. Let $xy, x'y'$ be the terminal edges of P, where $x, x' \in V(C^1)$, and let $Q := C^1 \cap \tilde{G}_P^{\text{small}}$. Suppose that $\tilde{G}_P^{\text{large}}$ has no chord of P, except possibly xx'. Then the following holds.*

Pm1) *Suppose that $V(\mathring{Q}) \cap S_\star = \varnothing$, and suppose further that either x', y have no common neighbor in $\tilde{G}_P^{\text{small}} \setminus P$, or each of the sets $V(\tilde{Q}) \cap T^{<2} \neq \varnothing$ and $S_\star$ is nonempty. Then every color of $L_\phi(x')$. is highly (xy, P)-matchable.*

Pm2) *At most one color of $L_\phi(x')$ is not (x, P)-matchable; AND*

Pm3) *If $|L_\phi(x')| \geq 4$, then there is a color of $L_\phi(x')$ which is highly (x, P)-matchable; AND*

Pm4) *If $N(y) \cap S_\star \cap V(\mathring{Q}) = \varnothing$ and x', y have no common neighbor in $\tilde{G}_P^{\text{small}} \setminus P$, then there is a color of $L_\phi(x')$ which is highly (xy, P)-matchable.*

Pm5) *If $N(y) \cap S_\star \cap V(\mathring{Q}) = \varnothing$ and $V(\mathring{Q}) \cap T^{<2} \neq \varnothing$, then at most one color in $L_\phi(x')$ is not highly (xy, P)-matchable.*

Proof. In the proof of each of Pm1)-Pm5), whenever we have a partial L_ϕ-coloring of $V(P)$ whose domain includes x, x', and we want to show that this extends to L_ϕ-color $V(\tilde{G}_P^{\text{small}})$, it suffices to check that this partial coloring extends to an L_ϕ-coloring of $\tilde{G}_P^{\text{small}}$, since $\tilde{G}_P^{\text{large}}$ has no chord of P, except possibly xx'. We now fix the following definitions.

A) Let p be the unique vertex of $N(y) \cap V(Q)$ which, on Q, is closest to x' (possibly $p = x'$).

B) Let K be a subgraph of G, where $K = yp$ if $p = x$, and otherwise $K := \tilde{G}_{xyp}^{\text{small}}$.

C) Let v be the unique neighbor of x on Q and let v' be the unique neighbor of x' on Q.

D) Let D be the cyclic facial subgraph $Q + \text{Span}(z)$ of $\tilde{G}_P^{\text{small}}$.

We begin by proving Pm1). Suppose that P satisfies the conditions of Pm1). We now have the following.

Claim 10.3.3. *Any L_ϕ-coloring of $\{x, x', y\}$ extends to an L_ϕ-coloring of $\{x'\} \cup V(K)$.*

<u>Proof:</u> Note that, if K is not an edge, then, since every internal vertex of pQx' has an L_ϕ-list of size three, we have $\mathcal{Z}_K(\psi(u), \psi(y), \bullet) \neq \varnothing$.

Case 1: x', y have no common neighbor in $\mathring{Q}$

In this case, we have either $p = x$ or $px' \notin E(G_P^{\text{small}})$. If $p = x$, then K is an edge and we are done, so suppose that $p \neq x$. Thus, we have $px' \notin E(G_P^{\text{small}})$, and there is an extension of ψ to an L_ϕ-coloring ψ^* of $\{x'\} \cup V(K)$, even if $\mathcal{Z}_{K,L_\phi}(\psi(u), \psi(y), \bullet) = \{\psi(x')\}$.

Case 2: x', y have a common neighbor a $\mathring{Q}$

In this case, by assumption, we have $V(\mathring{Q}) \cap T^{<2} \neq \varnothing$ and $S_\star \neq \varnothing$, and, by Co4d) of Definition 10.0.1, $G[N(y) \cap V(C^1)]$ is a subpath of C^1. Since G is $K_{2,3}$-free, no internal vertex of the path $G[N(y) \cap V(C^1)]$ lies in $T^{<2}$, and since C^1 is an induced subgraph of G, it follows that $v \in T^{<2}$ and px' is the unique terminal edge of Q incident to x'. Since $|L_\phi(p)| \geq 4$, it follows from Theorem 0.2.3 that there is at least one color in $\mathcal{Z}_K(\psi(u), \psi(y), \bullet)$ other than $\psi(x')$, so again, ψ extends to an L_ϕ-coloring of $\{x'\} \cup V(K)$. ■

Suppose toward a contradiction that ψ does not extend to an L_ϕ-coloring of K. Let $H := \tilde{G}_P^{\text{small}} \setminus (K \setminus \{y, v\})$. By Claim 10.3.3, ψ extends to an L_ϕ-coloring ψ^* of $\{x'\} \cup V(K)$. If $p = x'$, then H contains the cyclic facial subgraph $x'Pyx'$ and $x \notin V(H)$, and, applying Theorem 0.2.3 to the edge yx', we get that that ψ^* extends to an L_ϕ-coloring of $V(\tilde{G}_P^{\text{small}})$, contradicting our assumption. Thus, we have $p \neq x'$.

Claim 10.3.4. *There is a vertex of $H \setminus (P \cup Q)$ is adjacent to all three of $\{x', p, y\}$.*

<u>Proof:</u> Suppose not. Let H^* be a planar embedding obtained by adding to H a vertex p^{new} adjacent to x', y, so that H^* has a facial cycle $D^{\text{new}} := x'p^{\text{new}}ypQx'$. Let $S := \{p, y, x'\}$ and let L^* be a list-assignment for H^* where $L^*(p^{\text{new}})$ is a lone color disjoint to all the L_ϕ-lists of the vertices of $V(H)$, and otherwise $L^* := L_{\phi \cup \psi^*}^S$. Note that all the vertices of $H^* \setminus D^{\text{new}}$ have L^*-lists of size at least five. Now, the path $pyp^{\text{new}}x'$ admits an $L_{\phi \cup \psi^*}^S$-precoloring φ which is an extension of ψ^*. Since Q is a chordless path and $N(y) \cap V(D^{\text{new}}) = \{p, p^{\text{new}}\}$, there is no chord of D^{new} with an endpoint in $pyp^{\text{new}}x'$. By assumption, no vertex of $H^* \setminus D^{\text{new}}$ is adjacent to all three of $\{x', p, y\}$. Thus, by 1) of Proposition 1.5.1, φ extends to an L^*-coloring of H^*, so ψ^* extends to an L_ϕ-coloring of H, which is false. ■

Let $w^\dagger \in V(H) \setminus V(P \cup Q)$ be adjacent to all three of x', p, y. We now have $V(\mathring{Q}) \cap T^{<2} \neq \varnothing$ by assumption, and, by Co4d) of Definition 10.0.1, every vertex of $D^2(C)$. has a neighborhood in C^1 consisting of a subpath of C^1. Since no internal vertex of $K - y$ and no internal vertex of the path $G[N(w^\dagger) \cap V(C^1)]$ lies in $T^{<2}$, we have

$v \in V(\mathring{Q}) \cap T^{<2}$, and G contains the broken wheel K' with principal path $vw^\dagger x'$, where $K' - w^\dagger = pQx'$. Since $|L_\phi(v)| \geq 4$, we have $|\mathcal{Z}_K(\psi(x), \psi(y), \bullet)| \geq 2$. Since $p \neq x$, we have $|L_{\phi\cup\psi}(w^\dagger)| \geq 3$, so there is a $c \in L_{\phi\cup\psi}(w^\dagger)$ with $|\mathcal{Z}_K(\psi(x), \psi(y), \bullet) \setminus \{c\}| \geq 2$. By Observation 1.4.2, $\mathcal{Z}_K(\psi(x), \psi(y), \bullet) \cap \mathcal{Z}_{K'}(\bullet, c, \psi(x')) \neq \varnothing$. Thus, ψ extends to an L_ϕ-coloring σ of $V(K) \cup V(K')$. Since every vertex of $G \setminus (K \cup K')$ has an L_ϕ-list of size at least five, the precoloring $(\sigma(y), \sigma(w^\dagger), \sigma(x'))$ of $yw^\dagger x'$ extends to L_ϕ-color $\tilde{G}^{\text{small}} \setminus (Q - x')$, so σ extends to an L_ϕ-coloring of $\tilde{G}_P^{\text{small}}$, contradicting our assumption. This proves Pm1).

Now we prove Pm2). If every internal vertex of Q has an L_ϕ-list of size at least three, then it immediate from 2) of Proposition 1.5.1 that any any L_ϕ-coloring of $\{x, x'\}$ extends to an L_ϕ-coloring of $\tilde{G}_P^{\text{small}}$, since all the vertices of $\mathring{P}$ have L_ϕ-lists of size at least five, so we are done in that case. Now suppose there is an internal vertex of Q with an L_ϕ-list of size less than three. Thus there is a $u_\star \in V(Q) \setminus \{x, x'\}$ with $S_\star = \{u_\star\}$.

Since every internal vertex of $x'Qu_\star$ has an L_ϕ-list of size at least three, it follows from ii) of Theorem 1.7.5 that, for all but at most one $c \in L_\phi(x')$, there is a $\psi \in \text{Link}_{L_\phi}(x'Qu_\star, C^1, \tilde{G})$ using c on x'. Thus, given a $\psi \in \text{Link}_{L_\phi}(x'Qu_\star, C^1, \tilde{G})$, it suffices to show that $\psi(x')$ is (x, P)-matchable.

Note that ψ is also an element of $\text{Link}_{L_\phi}(x'Qu_\star, D, \tilde{G}_P^{\text{small}})$, since every vertex of $\tilde{G}_P^{\text{small}} \setminus D$ has an L_ϕ-list of size at least five. Let $H := \tilde{G}_P^{\text{small}} \setminus (\text{dom}(\psi) \cup \text{Sh}_{2,L_\phi}(x'Qu_\star, D, \tilde{G}_P^{\text{small}}))$. Since $u_\star$ is an internal vertex of Q, three is a unique vertex q on the path $u_\star Qx$ which is adjacent to $u_\star$. Now, H has a unique facial subgraph F such that $V(D) \setminus V(x'Qu_\star) \subseteq V(F)$, where all the vertices of $H \setminus F$ have $L_{\phi\cup\psi}$-lists of size at least five. By 2) of Theorem 1.7.4, each vertex of $F \setminus D$ has a an list of size at least three, and each vertex of $\mathring{P}$ has an $L_{\phi\cup\psi}$-list of size at least three. Since C^1 is an induced subgraph of G and $u_\star \in \text{dom}(\psi)$, each vertex of $V(F) \setminus \{x, q\}$ has an $L_{\phi\cup\psi}$-list of size at least three, and $|L_{\phi\cup\psi}(q)| \geq 2$. By our inertness condition, any $L_{\phi\cup\psi}$-coloring of H extends to an $L_{\phi\cup\psi}$-coloring of $\tilde{G}_P^{\text{small}} \setminus \text{dom}(\psi)$. Let $c := \psi(x')$ and consider the following cases:

Case 1: $q \neq x$

In this case, for any L_ϕ-coloring σ of $\{x, x'\}$ such that $\sigma(x') = c$, the union $\psi \cup \sigma$ is a proper L_ϕ-coloring of its domain, since C^1 is an induced subgraph of G. By Theorem 1.3.4, there is at most one color of $L_{\phi\cup\psi}(x)$ which is not used by any $L_{\phi\cup\psi}$-coloring of H, so there is at most one L_ϕ-coloring of $\{x, x'\}$ which uses c on x' and does not extend to an L_ϕ-coloring of $\tilde{G}_P^{\text{small}}$, so we are done in this case.

Case 2: $q = x$

In this case, $u_\star x$ is a terminal edge of Q. We claim that any L_ϕ-coloring of $\{x, x'\}$ which uses c on x' and does not use $\psi(u_\star)$ on x extends to an L_ϕ-coloring of $\tilde{G}$. Then we are done. Let σ be an L_ϕ-coloring of $\{x, x'\}$ with $\sigma(x') = c$ and $\sigma(x) \neq \psi(u_\star)$. Then $\psi \cup \sigma$ is a proper coloring of its domain in G, as C^1 is an induced subgraph of G. Note that since $\psi \in \text{Link}_{L_\phi}(Q - x, D, \tilde{G}_P^{\text{small}})$ as well, since every vertex of $\tilde{G}^{\text{small}} \setminus D$ has an L_ϕ-list of size at least five. Since each internal vertex of P has an $L_{\phi\cup\psi}$-list of size at least three, it follows from 3a) of Theorem 1.7.4 that $\psi \cup \sigma$ extends to an L_ϕ-coloring of $\tilde{G}_P^{\text{small}}$, so we are done. This proves Pm2).

Before proving Pm3)-Pm5), we show the following.

Claim 10.3.5. *Let q be a vertex of $V(x'Qp) \setminus \{p\}$ and let q^* be the unique neighbor of q which, on Q, is closer to p. Let $\psi \in \text{Link}_{L_\phi}(x'Qq, C^1, \tilde{G})$. Let ψ' be an L_ϕ-coloring of $V(K)$ and suppose that $\psi \cup \psi'$ is a proper L_ϕ-coloring of its domain. Suppose that at least one of the following holds.*

1) $|L_{\phi\cup\psi}(q^*)| \geq 3$; OR

2) $S_\star \subseteq V(x'Qq)$; OR

3) q^* is the lone vertex of $S_\star$ and $|L_{\phi\cup\psi}(q^*)| \geq 2$.

Then $\psi \cup \psi'$ extends to an L_ϕ-coloring of $V(\tilde{G}_P^{\text{small}})$.

Proof: Let $K^\dagger := \tilde{G}_P^{\text{small}} \setminus (K \setminus \{y, p\})$ and let $D^\dagger$ be the facial cycle $x'y'zypQx'$ of $K^\dagger$. Now, ψ is also an element of $\text{Link}_{L_\phi}(x'Qq, D^\dagger, K^\dagger)$, since every vertex of $K^\dagger \setminus D^\dagger$ has an L_ϕ-list of size at least five. If $q^*p \in E(G)$, then q^*p is an edge of Q, and in that case, it follows from 3a) of Theorem 1.7.4 that the precoloring $(\psi'(y), \psi'(p))$ of the edge yp extends to an L_ϕ-coloring of $K^\dagger$ which is also an extension of ψ, so in that case, $\psi \cup \psi$ extends to an L_ϕ-coloring of $V(\tilde{G}_P^{\text{small}})$, and we are done. So now suppose that $q^*p \notin E(G)$.

Let $\sigma := \psi \cup \psi'$ and let $K^{\dagger\dagger} := K^\dagger \setminus \text{Sh}_{2,L_\phi}(x'Qq, C^1, \tilde{G})$. Note that $\text{dom}(\psi \cup \psi') \subseteq V(\tilde{G}_P^{\text{small}}) \setminus \text{Sh}_{2,L_\phi}(x'Qq, C^1, \tilde{G})$. We just need to show that $\psi \cup \psi'$ extends to an L_ϕ-coloring of $V(K^{\dagger\dagger})$, and then it follows from our inertness condition that $\psi \cup \psi'$ extends to an L_ϕ-coloring of $V(\tilde{G}_P^{\text{small}})$. Now, $K^{\dagger\dagger}$ has a facial subgraph F such that every vertex of $K^\dagger \setminus F$ has an $L_{\phi\cup\sigma}$-list of size at least five. Let p^* be the unique neighbor of p on $x'Qp$. Since $q^*p \notin E(G)$, we have $q^* \neq p^*$. Furthermore, each of y', z has an $L_{\phi\cup\sigma}$-list of size at least three. Consider the following cases.

Case 1: No vetex of q^*Qp^* lies in $S_\star$

In this case, since C^1 is an induced subgraph of G, it follows from our assumption that every vertex of $F \setminus \{p^*, q^*\}$ has an $L_{\phi\cup\sigma}$-list of size at least three. Since $p^* \neq q^*$ and $p^*, q^* \notin S_\star$, each of p^*, q^* has an $L_{\phi\cup\sigma}$-list of size at least two. By Theorem 1.3.4, σ extends to an L_ϕ-coloring of $K^{\dagger\dagger}$.

Case 2: There is a vertex of q^*Qp^* lies in $S_\star$

In this case, if $S_\star = \{q_*\}$, then it follows from our choice of ψ that each of p^*, q^* has an $L_{\phi\cup\sigma}$-list of size at least two, and, by Theorem 1.3.4, σ extends to an L_ϕ-coloring of $K^{\dagger\dagger}$. Now suppose that $S_\star \neq \{q_*\}$. Thus, since C^1 is an induced subgraph of G, it follows from our choice of ψ that every vertex of $F \setminus (S_\star \cup \{p_*\})$ has an $L_{\phi\cup\sigma}$-list of size at least three and each vertex of $S_\star \cup \{p_*\}$ has an $L_{\phi\cup\sigma}$-list of size at least two. By Theorem 1.3.4, σ extends to an L_ϕ-coloring of $K^{\dagger\dagger}$, so we are done. ■

Now we prove Pm3). Suppose that $|L_\phi(x')| \geq 4$. Thus, there is a color $c \in L_\phi(x')$ such that either $|L_\phi(v') \setminus \{c\}| \geq 3$ or both $v' \in S_\star$ and $c \notin L_\phi(v')$. Now, any L_ϕ-coloring of the singleton x' is a trivially an element of $\text{Link}_{L_\phi}(x', C^1, \tilde{G})$, so it follows from Claim 10.3.5 that c is (x, P)-matchable.

Now we prove Pm4). Suppose that $N(y) \cap S_\star \cap V(\mathring{Q}) = \varnothing$ and that x', y have no common neighbor in $\mathring{Q}$. Any choice of color for x' extend to an L_ϕ-coloring of $\{x, y, x'\}$. Thus, if $V(\mathring{Q}) \cap S_\star = \varnothing$, then we are done by Pm1), so now suppose there is a lone vertex $u_\star \in V(\mathring{Q})$ such that $\{u_\star\} = S_\star$. Since $u_\star \notin N(y)$, it follows that $u_\star$ is an internal vertex of $x'Qp$. Thus, there is a unique neighbor $v_\star$ of $u_\star$ on $x'Qu_\star$. Since every vertex of $x'Qv_\star$ has an L_ϕ-list of size at least three, it follows from i) of Theorem 1.7.5 that there is a $\psi \in \text{Link}_{L_\phi}(xQv_\star, C^1, \tilde{G})$ such that $\psi(v_\star) \notin L_\phi(u_\star)$. We claim now that $\psi(x')$ is (xy, P)-matchable. Let σ be an L_ϕ-cloring of $\{x, y, x'\}$ using $\psi(x')$ on x'.

By Theorem 0.2.3, the L_ϕ-coloring $(\sigma(x), \sigma(y))$ of xy extends to an L_ϕ-coloring σ^* of $V(K)$, since $u_\star \notin V(K)$. Since C^1 is an induced subgraph of G and $u_\star$ is an internal vertex of $x'Qp$, the union $\sigma^* \cup \psi$ is a proper L_ϕ-coloring of its domain, and an extension of σ. By Claim 10.3.5, $\sigma^* \cup \psi$ extends to an L_ϕ-coloring of $V(\tilde{G}_P^{\text{small}})$, so $\psi(x')$ is indeed (xy, P)-matchable. This proves Pm4).

Now we prove Pm5). Suppose that $N(y) \cap S_\star \cap V(\mathring{Q}) = \varnothing$ and let $\hat{u} \in V(\mathring{Q}) \cap T^{<2}$. If $S_\star \cap (V(\mathring{Q}) \setminus N(y)) = \varnothing$, then, by Pm1), every color in $L_\phi(x')$ is (xy, P)-matchable, so we are done in that case. Now suppose that $S_\star \cap$

$(V(Q) \setminus N(y)) \neq \varnothing$ and let $S_\star = \{u_\star\}$. Suppose toward a contradiction that there are at least two colors of $L_\phi(x')$ which are not (xy, P)-matchable.

Claim 10.3.6. *x' has no neighbors in K and $\hat{u} \notin V(x'Qu_\star)$.*

Proof: Since at least two colors of $L_\phi(x')$ are not (xy, P)-matchable, there is an L_ϕ-coloring σ of $\{x, y, x'\}$ such that $\sigma(x') = c$ and σ does not extend to an L_ϕ-coloring of $V(\tilde{G}_P^{\text{small}})$.

Suppose that x' has a neighbor in K. Firstly, if $x' \in N(y)$, then $Q = \tilde{G}_{x'yx}^{\text{small}} \cap C^1 = G[N(y) \cap V(C^1)]$ by Co4d) of Definition 10.0.1, since $S_\star \neq \varnothing$, and furthermore, and $\hat{u}$ is an internal vertex of the path $K - y$, which is false, since $\hat{u} \in T^{<2}$. Thus, we have $x' \notin N(y)$, and x' is adjacent to p. Since C^1 is an induced subgraph of G, $x'p$ is an edge of Q. Since $\hat{u}$ is not an internal vertex of the path $K - y$, we have $\hat{u} = p = v'$ and K is a broken wheel with principal path xyp. Since $|L_\phi(p)| \geq 4$, we have $\mathcal{Z}_K(\sigma(x), \sigma(y), \bullet) \setminus \{\sigma(y')\} \neq \varnothing$, so σ extends to an L_ϕ-coloring of $\tilde{G}^{\text{small}} \setminus (K \setminus \{y, p\})$. Now, $x'Pypx'$ is a cyclic facial subgraph of $\tilde{G}_P^{\text{small}} \setminus (K \setminus \{y, p\})$, and every vertex of $\tilde{G}_P^{\text{small}} \setminus (K \setminus \{y, p\})$, except for x', p, has an L_ϕ-list of size at least five, so the precoloring $(\sigma(x'), \sigma(p), \sigma(y))$ of the 2-path $x'py$ extends to L_ϕ-color $\tilde{G}_P^{\text{small}} \setminus (K \setminus \{y, p\})$. Thus σ extends to L_ϕ-color $\tilde{G}_P^{\text{small}}$, contradicting our assumption. We conclude that x' has no neighbors in K.

Since $S_\star \cap V(K) = \varnothing$ and x' has no neighbors in K, it follows that σ extends to an L_ϕ-coloring τ of $V(K) \cup \{x'\}$. Now suppose toward a contradiction that $\hat{u} \in V(x'Qu_\star)$. Let $u^\dagger$ be the unique neighbor of $\hat{u}$ on $x'Q\hat{u}$. Possibly $u^\dagger = x'$, but, in any case, by i) of Theorem 1.7.5, there is a $\psi \in \text{Link}(x'Qu^\dagger)$ with $\psi(x') = \sigma(x')$, since very vertex of $V(x'Qu^\dagger) \setminus \{x'\}$ has an L_ϕ-list of size at least three. Furthermore, we have $|L_{\phi \cup \psi}(\hat{u})| \geq 3$, since $\hat{u} \in T^{<2}$. Now, y has no neighbors in $\text{dom}(\psi)$, and since $u_\star \notin V(K)$, we get that $u_\star$ is an internal vertex of $u^\dagger Qp$. Since C^1 is an induced subgraph of G, the union $\psi \cup \tau$ is a proper L_ϕ-coloring of its domain. By Claim 10.3.5, $\psi \cup \tau$ extends to an L_ϕ-coloring of $V(\tilde{G}_P^{\text{small}})$, contradicting our choice of σ. ■

Now, by assumption, there exist two colors $c_0, c_1 \in L_\phi(x')$ such that neither c_0 nor c_1 is (xy, P)-matchable. By ii) of Theorem 1.7.5, there exists a $\psi \in \text{Link}_{L_\phi}(x'Qu_\star, C^1, \tilde{G})$ such that $\psi(x') \in \{c_0, c_1\}$. Suppose without loss of generality that $\psi(x') = c_0$. Since c_0 is not (xy, P)-matchable, there exists an L_ϕ-coloring σ of $\{x, y, x'\}$ with $\sigma(x') = c_0$, where σ does not extend to an L_ϕ-coloring of $V(\tilde{G}_P^{\text{small}})$.

Claim 10.3.7. *There exists an extension of σ to an L_ϕ-coloring σ^* of $V(K) \cup \{x'\}$.*

Proof: Firstly, by Claim 10.3.6, we have $x'y \notin E(G)$, and since $K - y$ is disjoint to $x'Qu_\star$, y has no neighbors in $\text{dom}(\psi)$. If p also has no neighbors in $\text{dom}(\psi)$, then we are immediately done, so suppose that p has a neighbor in $\text{dom}(\psi)$. Possibly $p = x$ and $xx' \in E(G)$, but then we are done since x', p are both precolored by σ. Since C^1 is an induced subgraph of G, the only remaining possibility is that $u_\star p$ is an edge of C^1. By Claim 10.3.6, $\hat{u} \notin V(x'Qu_\star)$ and since $\hat{u}$ is not an internal vertex of the path $K - y$, we have $\hat{u} = p$ (recall that, by Co4d) of Definition 10.0.1, K is a broken wheel with principal path xyp, as $S_\star \neq \varnothing$). Since $|L_\phi(\hat{u})| \geq 4$, we have $\mathcal{Z}_K(\sigma(x), \sigma(y), \bullet) \setminus \{\psi(u_\star)\} \neq \varnothing$, so we are done. ■

Letting σ^* be as in Claim 10.3.7, it now follows immediately from Claim 10.3.5 that $\sigma^* \cup \psi$ extends to an L_ϕ-coloring, contradicting our assumption that σ does not extend to an L_ϕ-coloring of $V(\tilde{G}_P^{\text{small}})$. This completes the proof of Lemma 10.3.2. □

10.4 Non-End-Repelling Vertices

That is, throughout the remainder of Chapter 10, we reserve the Lemma environment exclusively for statements of the form "if the following conditions hold, then there exists a (C, z)-opener". We use the Proposition environment for any other auxiliary facts we need to prove along the way.

In this section, we deal with the special case where we have a pentagonal $z \in D_2(C^1)$ such that Span(z) is part of a cycle of length at most six which separates C from a vertex of $G \setminus B_4(C)$ with an L_ϕ-list of size less than five. We begin with the following definition.

Definition 10.4.1. Let $z \in D_2(C^1)$ be a pentagonal vertex. We say that z is *end-repelling* if $|E(C^1 \cap \tilde{G}_z^{\text{large}})| > 1$, and, letting e, e' be the two edges of $C^1 \cap \tilde{G}_z^{\text{large}}$ which are incident to Span(z), there is no 2-chord of $G[V(C_z^1 \cup \text{Span}(z))]$ which separates both of e, e' from z.

The lemma below is the lone result of Section 10.4 and the first in the sequence of lemmas which make up the proof of Theorem 10.0.7.

Lemma 10.4.2. *Let $z \in D_2(C^1)$ be a pentagonal vertex and suppose that one of the following holds:*

1) $|E(C^1 \cap \tilde{G}_z^{\text{large}})| \leq 1$*; OR*
2) $|E(C^1 \cap \tilde{G}_z^{\text{large}})| > 1$ *and, letting e, e' be the two edges of C^1 which are incident to* Span(z)*, there exists a 2-chord of $G[V(C_z^1 \cup \text{Span}(z))]$ which separates both of e, e' from z.*

Then $|E(C^1 \cap \tilde{G}_z^{\text{small}})| > 1$ and there exists a (C, z)-opener.

Proof. We first show the following.

Claim 10.4.3. Span(z) *is a proper 4-chord of C^1.*

<u>Proof:</u> Suppose not. Thus, Span(z) is either a 2-path, a 4-cycle, or a claw. If Span(z) is a claw, then $|E(C^1 \cap \tilde{G}_z^{\text{large}})| > 1$, or else there is a triangle which separates C from an element of $\mathcal{C}$. Likewise, if Span(z) is eitehr a 4-cycle or a 2-path, then $|E(C^1 \cap \tilde{G}_z^{\text{large}})| > 1$, since G is a simple graph. Thus, in any case, $|E(C^1 \cap \tilde{G}_z^{\text{large}})| > 1$, so there exist two distinct edges e, e' of $C^1 \cap \tilde{G}_z^{\text{large}}$ which are incident to Span(z), and there is a 2-chord of C_z^1 which separates each of e, e' from z. Since G has no repeated edges, Span(z) is a claw, and G contains a 4-cycle which separates z from C, contradicting short-separation-freeness. ■

Since Span(z) is a proper 4-chord of C^1, the graph $C^1 \cap \tilde{G}_z^{\text{large}}$ is a subpath of C^1 (i.e not equal to all of C^1). S Let Span$(z) := xyzy'x'$. Now suppose toward a contradiction that there does not exist a (C, z)-opener. Let $Q^{\text{small}} := C^1 \cap \tilde{G}_z^{\text{small}}$ and $Q^{\text{large}} := C^1 \cap \tilde{G}_z^{\text{large}}$.

Claim 10.4.4. $|E(Q^{\text{small}})| > 1$.

<u>Proof:</u> Suppose not. Thus, we have $Q^{\text{small}} = xx'$, and since G is a simple graph, $|E(Q^{\text{large}})| > 1$. By assumption, there is a vertex $w \in V(\tilde{G}_z^{\text{large}} \setminus C_z^1)$ with at least one neighbor in $\{x, y\}$ and at least one neighbor in $\{x', y'\}$. By 2) of Proposition 10.2.8, $\{y, y'\} \not\subseteq N(w)$, so suppose without loss of generality that $x \in N(w)$. Since w is adjacent to at least one of x', y', G contains a 4-cycle which separates z from C, contradicting short-separation-freeness. ■

Let U be the set of vertices $w \in V(\tilde{G}_z^{\text{large}} \setminus C_z^1)$ such that $N(w)$ has nonempty intersection with each of $\{x, y\}$ and $\{x', y'\}$. By 3) of Proposition 10.2.8, we have $U \cap N(z) = \varnothing$.

Claim 10.4.5. *If $|E(Q^{\text{large}})| \leq 1$ then there is a vertex $w \in U$ with three neighbors on* $\text{Span}(z)$*, and $N(w) \cap V(\text{Span}(z))$ consists of $\{x, x'\}$ and precisely one of y, y'.*

Proof: In this case, we have $Q^{\text{large}} = xx'$. Let H be the subgraph of G induced by $V(C^1 \cup \tilde{G}_z^{\text{small}})$. Every vertex of C^1 has an L_ϕ-list of size at least three, except for possibly a lone vertex with a list of size precisely two, so, by Theorem 0.2.3, H admits an L_ϕ-coloring ψ. By assumption, the pair $[G[V(H \cup C)], \phi \cup \psi]$ is not a (C, z)-opener, so there exists a vertex $w \in V(\tilde{G}_z^{\text{large}} \setminus C_z^1)$ such that $|L_{\phi^*}(w)| < 3$. Since C_z^1 is the 5-cycle $xyzy'x'$, it follows that w has at least three neighbors in $\text{Span}(z)$. Since $z \notin N(w)$. Since w is adjacent to at most of y, y', we get that $\{x, x'\} \subseteq N(w)$ and precisely one of y, y' lies in $N(w)$. ■

We now have the following.

Claim 10.4.6. *$xy', y'x \notin E(G)$. Furthermore, there is a $w^\dagger \in U$ such that no vertex of $\tilde{G}_z^{\text{large}} \setminus (C_z^1 \cup \text{Sh}_2(Q^{\text{large}}))$, except possibly $w^\dagger$, has more than two neighbors in C_z^1.*

Proof: If $|E(Q^{\text{large}})| > 1$, then we have $U \neq \varnothing$ by assumption, and, if $|E(Q^{\text{large}})| \leq 1$, then $U \neq \varnothing$ by Claim 10.4.6, so we have $U \neq \varnothing$ in any case.

> **Subclaim 10.4.7.** *$xy', x'y \notin E(G)$, and no vertex of $V(\tilde{G} \setminus C_z^1) \setminus U$ has more than two neighbors in C_z^1.*
>
> Proof: Suppose toward a contradiction that $E(G)$ contains one of $xy', x'y$, say $xy' \in E(G)$ without loss of generality. Since $U \neq \varnothing$, let $w \in U$. By 1) of Proposition 10.2.8, we have $x'y \in E(\tilde{G}_z^{\text{small}})$, and thus $x'wxy'$ is a 4-cycle which separates C from a vertex of $G \setminus B_4(C)$ with an L_ϕ-list of size less than five, contradicting short-separation-freeness.
>
> Now suppose toward a contradiction that there is a $w \in V(\tilde{G}_z^{\text{large}} \setminus C_z^1)$ with more than two neighbors in C_z^1, where $w \notin U$. Since $V(\tilde{G}_{xw^\dagger x'}^{\text{small}}) = V(Q^{\text{small}}) \cup \{w^\dagger\}$, it follows that $N(w) \cap V(C_z^1) \subseteq \{x, y, z, y', x'\}$, so w has at least three neighbors in $\text{Span}(z)$. By 3) of Proposition 10.2.8, $z \notin N(w)$ and, by 2) of Proposition 10.2.8, $\{y, y'\} \not\subseteq N(w)$, so both of x, x' lie in $N(w)$, contradicting our assumption that $w \notin U$. ■

To finish the proof of Claim 10.4.6, it just suffices to show that at most one vertex of $U \setminus \text{Sh}_2(Q^{\text{large}})$ has more than two neighbors in C_z^1. Suppose toward a contradiction that there are two such vertices w, w^*. Note that neither of w, w^* lies in $N(x) \cap N(x')$, or else, if each of w, w^* is adjacent to both of x, x', then one of x, x' lies in $\text{Sh}_2(Q^{\text{large}})$, which is false.

Since $w \in U$ and $w \notin N(x) \cap N(x')$, suppose without loss of generality that $x, y' \in N(w)$. Thus, $\tilde{G}$ contains the 3-chord $R := xwy'x'$ of C^1. Consider the following cases.

Case 1: $w^* \in V(\tilde{G}_R^{\text{large}})$,

In this case, since z is pentagonal, we have $N(w^*) \cap V(C_z^1) \subseteq \{y', z, y, x\}$. Since $U \cap N(z) = \varnothing$ and w^* has at least three neighbors on C_z^1, it follows that w^* is adjacent to each of y, y', contradicting 2) of Proposition 10.2.8.

Case 2: $w^* \in V(\tilde{G}_R^{\text{small}})$,

In this case, since $w^* \in U$, $N(w^*)$ has nonempty intersection with each of $\{x', y'\}$ and $\{y\}$. Since $w^* \notin N(x) \cap N(x')$, it follows that w^* is adjacent to each of x, y', and we are back to Case 1 with the roles of w, w^* interchanged, so we are done. ■

Let $w^\dagger \in U$ be the vertex specified in Claim 10.4.6. Since no vertex of U lies in $N(y) \cap N(y')$, suppose without loss of generality that $y' \notin N(w^\dagger)$, and thus $x' \in N(w^\dagger)$. Since $U \cap N(z) = \varnothing$, we have $N(w^\dagger) \cap V(\text{Span}(z)) \subseteq \{x, y, x'\}$. By Claim 10.4.4, $|E(Q^{\text{small}})| > 1$, so every element of $\text{Link}(Q^{\text{large}})$ is a proper L_ϕ-coloring of its domain in $\tilde{G}$.

Claim 10.4.8. *For any $\sigma \in \text{Link}(Q^{\text{large}})$, the following hold.*

1) *For any extension of σ to an L_ϕ-coloring σ^* of $\text{dom}(\sigma) \cup \{y\}$, either $|L_{\phi \cup \sigma^*}(w^\dagger)| = 2$ or σ^* does not extend to an L_ϕ-coloring of $V(\tilde{G}_z^{\text{small}})$; AND*
2) *$\sigma(x')$ is not highly $(xy, \text{Span}(z))$-matchable; AND*
3) *If $y \notin N(w^\dagger)$, then $\sigma(x')$ is not highly $(x, \text{Span}(z))$-matchable.*

Proof: Since $w^\dagger \in U$, $w^\dagger$ has a neighbor in $\{x, y\}$, so, for some $2 \leq k \leq 3$, $\tilde{G}$ contains a k-chord R of C^1, where R is either $x'w^\dagger x$ or $x'w^\dagger yx$. Since z is pentagonal, we have $\tilde{G}_R^{\text{small}} \cap \tilde{G}_z^{\text{small}} \subseteq \{x, y, x'\}$. Now, by Proposition 10.2.9, $V(\tilde{G}_R^{\text{small}} - w^\dagger)$ is $(L, \phi \cup \sigma)$-inert in G.

We first prove 1). Let σ^* be an extension of σ to an L_ϕ-coloring of $\text{dom}(\sigma) \cup \{y\}$. Suppose that $|L_{\phi \cup \sigma^*}(w^\dagger)| \neq 2$. Since $z \notin V(C^1)$ and $|L_{\phi \cup \sigma}(w^\dagger)| \geq 3$, we have $|L_{\phi \cup \sigma^*})(w^\dagger)| > 2$. Now suppose toward a contradiction that σ^* extends to an L_ϕ-coloring τ of $V(\tilde{G}_z^{\text{small}})$. Since $N(w^\dagger) \cap V(\text{Span}(z)) \subseteq \{x, y, x'\}$, we have $|L_{\phi \cup \tau}(w^\dagger)| \geq 3$. Let H be the subgraph of G induced by $\text{dom}(\phi \cup \sigma^*) \cup V(\tilde{G}_z^{\text{small}}) \cup V(-w^\dagger)$. As indicated above, $V(\tilde{G}_R^{\text{small}} - w^\dagger)$ is $(L, \phi \cup \sigma)$-inert in G. By Claim 10.4.6, $w^\dagger$ is the only vertex of $\tilde{G}_z^{\text{large}} \setminus (C_z^1 \cup \text{Sh}_2(Q^{\text{large}}))$ with more than two neighbors in C_z^1, so $[H, \phi \cup \tau]$ is a (C, z)-opener, contradicting our assumption.

Now we prove 2) and 3) together. If one of these does not hold, then, letting $A := G[N(w^\dagger) \cap \{x, y\}]$, $\sigma(x')$ is highly $(A, \text{Span}(z))$-matchable. If $y \notin N(w^\dagger)$, then, by assumption, $\sigma(x')$ is highly $(x, \text{Span}(z))$-matchable, and since $V(\tilde{G}_R^{\text{small}}) \cap V(\tilde{G}_z^{\text{small}}) \subseteq \{x, y, x'\}$, σ extends to an L_ϕ-coloring τ of $\text{dom}(\sigma) \cup V(\tilde{G}_z^{\text{small}})$. Since $|L_{\phi \cup \sigma}(w^\dagger)| \geq 3$ and $y \notin N(w^\dagger)$, we have $|L_{\phi \cup \tau}(w^\dagger)| \geq 3$, contradicting 1). Thus, we have $y \in N(w^\dagger)$. By Claim 10.4.6, $x'y \notin E(G)$, and since $N(y) \cap V(C^1) \subseteq V(Q^{\text{small}})$, we have $|L_{\phi \cup \sigma}(y)| \geq 4$. Thus, σ extends to an L_ϕ-coloring σ^* of $\text{dom}(\sigma) \cup \{y\}$ such that $|L_{\phi \cup \sigma^*}(w^\dagger)| \geq 3$. Since $\sigma(x')$ is highly $(xy, \text{Span}(z))$-matchable, we again contradict 1). ■

By 1) of Corollary 10.2.5, we have $\text{Link}(Q^{\text{small}}) \neq \varnothing$, so, by 2) of Claim 10.4.8, we immediately have the following.

Claim 10.4.9. *At least one color of $L_\phi(x')$ is not highly $(xy, \text{Span}(z))$-matchable. Furthermore, if $y \notin N(w^\dagger)$, then at least one color of $L_\phi(x')$ is not highly $(x, \text{Span}(z))$-matchable.*

Now let $p \in V(Q^{\text{small}})$ be the neighbor of y, which, on the path Q^{small}, is closest to x'. Let K be a subgraph of G, where $K := xy$ if $p = x$, and otherwise $K := \tilde{G}_{xyp}^{\text{small}}$.

Claim 10.4.10. $N(w^\dagger) \cap V(\text{Span}(z)) = \{x, x'\}$.

Proof: Suppose not. Since $y', z \notin N(w^\dagger)$ and $w^\dagger \in U$, we have $y \in N(w^\dagger)$. We now note the following.

> **Subclaim 10.4.11.** *x', y have no common neighbor in $\tilde{G}_z^{\text{small}}$.*
>
> Proof: Suppose that x', y have a common neighbor in $\tilde{G}_z^{\text{small}}$. Possibly this common neighbor lies in $\text{Span}(z)$, but, in any case, since every chord of $\text{Span}(z)$, except possibly xx', lies in $\tilde{G}_z^{\text{small}}$, and x', y are both adjacent to $w^\dagger$, it follows that G contains a 4-cycle which separates C from a vertex of $G \setminus B_4(C)$ with an L_ϕ-list of size less than five, contradicting short-separation-freeness. ■

We now have the following.

Subclaim 10.4.12. $S_\star \cap V(\mathring{Q}^{\text{small}}) \neq \varnothing$.

Proof: Suppose toward a contradiction that $S_\star \cap V(\mathring{Q}^{\text{small}}) = \varnothing$. Thus, every internal vertex of Q^{small} has an L_ϕ-list of size at least three. By Subclaim 10.4.11, x', y have no common neighbor in $\tilde{G}_z^{\text{small}}$, and thus, by Pm1) of Lemma 10.3.2, every color of $L_\phi(x')$ is highly $(xy, \text{Span}(z))$-matchable, contradicting Claim 10.4.9. ■

Since $yx' \notin E(G)$, we get that, for each vertex $v \in N(y) \cap N(C^1)$, $\tilde{G}$ contains the 3-chord $R^v := vyw^\dagger x'$ of C^1 and the 4-chord $P^v := vyzy'x'$ of C^1. Note that $P^x = \text{Span}(z)$. Since z is pentagonal, the 5-cycle $x'y'zyw^\dagger$ separates C from a vertex of $G \setminus B_4(C)$ with an L_ϕ-list of size less than five. In particular, since $N(y) \cap V(C^1) \subseteq V(Q^{\text{small}})$, it follows that $V(\tilde{G}^{\text{small}}_{R^v} \cap \tilde{G}^{\text{small}}_{P^v}) \subseteq \{v, y, x'\}$ for each $v \in N(y) \cap V(C^1)$.

Subclaim 10.4.13. $S_\star \cap N(y) \neq \varnothing$.

Proof: Suppose toward a contradiction that $S_\star \cap N(y) = \varnothing$. By Subclaim 10.4.11 , x', y have no common neighbor in $\tilde{G}_z^{\text{small}}$, so it immediately follows from Pm4) of Lemma 10.3.2 that there is a color $c \in L_\phi(x')$ is highly $(xy, \text{Span}(z))$-matchable. Since $u_\star \notin V(Q^{\text{large}})$, it follows from i) of Theorem 1.7.5 that there is a $\sigma \in \text{Link}(Q^{\text{large}})$ using c on x', contradicting 2) of Claim 10.4.8. ■

Since $S_\star \neq \varnothing$, it follows from Co4d) of Definition 10.0.1 that K is either an edge or a broken wheel with principal path xyp. Applying Subclaim 10.4.13, there is a lone vertex $u_\star \in N(y)$ such that $S_\star = \{u_\star\}$. Since $u_\star \in V(\mathring{Q}^{\text{small}})$, K is a broken wheel with principal path xyp. Possibly $x \in N(w^\dagger)$, but the trick now is to leave x uncolored. By Subclaim 10.4.11, $px' \notin E(G)$, so it follows that $\tilde{G}^{\text{small}}_{R^{u_\star}}$ is an induced subgraph of G. Since $|L_\phi(u_\star)| = 2$, $|L_\phi(x')| \geq 3$, and $|L_\phi(y)| \geq 5$, it follows from 1) of Theorem 9.0.1 that there exists an L_ϕ-coloring ψ of $\{u_\star, y, x'\}$ such that any extension of ψ to an L_ϕ-coloring of $V(R^{u_\star})$ also extends to an L_ϕ-coloring of $\tilde{G}^{\text{small}}_{R^{u_\star}}$.

Since $u_\star \in N(y)$, every internal vertex of $u_\star Q^{\text{small}} x'$ has an L_ϕ-list of size at least three, and, by Subclaim 10.4.11, x', y have no common neighbor in $\tilde{G}_z^{\text{small}}$, so it follows from Pm1) of Lemma 10.3.2 that ψ extends to an L_ϕ-coloring ψ^* of $V(\tilde{G}^{\text{small}}_{P^{u_\star}})$. Let H be the subgraph of G induced by $\text{dom}(\phi \cup \psi^*) \cup V(\tilde{G}^{\text{small}}_{R^{u_\dagger}} - w^\dagger)$. By our choice of ψ^*, we get that $V(\tilde{G}^{\text{small}}_{R^{u_\star}})$ is $(L, \phi \cup \psi^*)$-inert in G. Since x is uncolored, we have $|L_{\phi \cup \psi^*}(w^\dagger)| \geq 3$. Since $S_\star \neq \varnothing$ and every vertex of $D_2(C)$ has a neighborhood on C^1 consisting precisely of a subpath of C^1, we have $\text{Sh}_2(Q^{\text{large}}) \subseteq V(Q^{\text{large}})$, and, by Claim 10.4.6, $w^\dagger$ is the only vertex of $G \setminus C^1_z$ with more than two neighbors in C^1_z. Thus, $[H, \phi \cup \psi^*]$ is a (C, z)-opener, contradicting our assumption. This completes the proof of Claim 10.4.10. ■

Note that, since z is pentagonal, the graphs $\tilde{G}^{\text{small}}_{xw^\dagger x'}$ and $\tilde{G}^{\text{small}}_z$ intersect precisely on x, x', and $Q^{\text{large}} = C^1 \cap \tilde{G}^{\text{small}}_{xw^\dagger x'}$. By Claim 10.4.9, there is a color of $L_\phi(x')$ which is not highly $(x, \text{Span}(z))$-matchable, since $y \notin N(w^\dagger)$.

If $S_\star \cap V(\mathring{Q}^{\text{small}}) = \varnothing$, then, by 2) of Proposition 1.5.1, any L_ϕ-coloring of $\{x, x'\}$ extends to an L_ϕ-coloring of $V(\tilde{G}_z^{\text{small}})$, so every color of $L_\phi(x')$ is highly $(x, \text{Span}(z))$-matchable. Thus, we have $S_\star \cap V(\mathring{Q}^{\text{small}}) \neq \varnothing$. Since $S_\star \cap V(\mathring{Q}^{\text{small}}) \neq \varnothing$, every vertex of Q^{large} has an L_ϕ-list of size at least three, so, by i) of Theorem 1.7.5, for every $d \in L_\phi(x')$, there is an element of $\text{Link}(Q^{\text{small}})$ using d on x'. Thus, no color of x' is highly $(x, \text{Span}(z))$-matchable.

Let $u_\star$ be the lone vertex of $S_\star$. By Co4c)-d) of Definition 10.0.1, there is a $\hat{u} \in T^{<2}$, and every vertex of $D_2(C)$ has a neighborhood on C^1 which consists precisely of a subpath of C^1. Since no internal vertex of $G[N(w^\dagger) \cap V(C^1)]$ lies in $T^{<2}$, we have $\hat{u} \in V(Q^{\text{small}})$. If $\hat{u} = x'$, then, since $|L_\phi(\hat{u})| \geq 4$, it follows from Pm3) of Lemma 10.3.2 that there is a highly $(x, \text{Span}(z))$-matchable color in $L_\phi(x')$, contradicting Claim 10.4.9. Thus, we have $\hat{u} \in V(Q^{\text{small}} - x')$. Consider the following cases:

Case 1: $\hat{u} = x$

By Pm2) of Lemma 10.3.2, there is an $(x, \text{Span}(z))$-matchable color $c \in L_\phi(x')$. Since every internal vertex of Q^{large} has an L_ϕ-list of size at least three, it follows from i) of Theorem 1.7.5 that there are two elements σ_0, σ_1 of $\text{Link}(Q^{\text{large}})$ which use c on x' and use different colors on x. Since c is $(x, \text{Span}(z))$-matchable, there exists an $i \in \{0,1\}$ such that σ_0 extends to an L_ϕ-coloring of $V(\tilde{G}_z^{\text{small}})$. Since $y \notin N(w^\dagger)$, this contradicts 1) of Claim 10.4.8.

Case 2: $\hat{u} \in V\mathring{Q}^{\text{small}})$

In this case, since no color of $L_\phi(x')$ is highly $(x, \text{Span}(z))$-matchable, it follows from Pm5) of Lemma 10.3.2 that $u_\star \in N(y)$. Let $P^\times := pyzy'x'$. Now, applying Corollary 10.2.5, we fix a $\sigma \in \text{Link}(pQ^{\text{small}}xQ^{\text{large}}x')$. By 2) of Theorem 1.7.4, each of y, z, y' has an $L_{\phi\cup\sigma}$-list of size at least three, so, by 2) of Proposition 1.5.1 applied to $\tilde{G}_{P^\times}^{\text{small}}$, we get that σ extends to an L_ϕ-coloring σ^* of $\text{dom}(\sigma) \cup V(\tilde{G}_{P^\times}^{\text{small}}$. Since σ restricts to an element of $\text{Link}(xQ^{\text{small}}p)$, the L_ϕ-coloring $(\sigma(x), \sigma^*(y), \sigma(p))$ of xyp extends to L_ϕ-color K as well, so σ extends to L_ϕ-color an L_ϕ-coloring τ of $\text{dom}(\sigma) \cup V(\tilde{G}_z^{\text{small}})$. Since $y \notin N(w^\dagger)$, we have $\text{dom}(\phi\cup\tau) \cap N(w^\dagger) = \{x, x'\}$ and $|L_{\phi\cup\tau}(w^\dagger)| = 3$, contradicting 1) of Claim 10.4.8. This completes the proof of Lemma 10.4.2. $\square$

10.5 Obstruction Vertices

When we construct a (C, z)-opener for a given a pentagonal $z \in D_2(C^1)$, the main obstacle is the presence of vertices of $\tilde{G}_z^{\text{large}} \setminus V(C^1 \cup \text{Span}(z))$ which have neighbors on C^1 and neighbors in $\text{Span}(z)$, so we introduce the following terminology.

Definition 10.5.1. Let $z \in D_2(C^1)$ be a pentagonal vertex and suppose that $|E(C^1 \cap \tilde{G}_z^{\text{large}})| > 1$. Let $e = xx_*$ be one of the two edges of $C^1 \cap \tilde{G}_z^{\text{large}}$ which is incident to $\text{Span}(z)$, where $x \in V(\text{Span}(z) \cap C^1)$ and $x_* \in V(C^1 \cap \tilde{G}_z^{\text{large}}) \setminus V(\text{Span}(z))$. We then have the following definitions.

1) An *e-obstruction* is a vertex $w \in V(\tilde{G}_z^{\text{large}}) \setminus V(C_z^1 \cup \text{Span}(z))$ such that the following hold.

    a) w is adjacent to at least one endpoint of the lone edge of $\text{Span}(z)$ incident to x*; AND*

    b) There is a 2-chord of $G[V(C_z^1 \cup \text{Span}(z))]$ with w as a midpoint, where, in $\tilde{G}_z^{\text{large}}$, this 2-chord separates z from the edge xx_*.

2) We denote the set of e-obstruction vertices by $\text{Ob}_z(e)$. We say that an e-obstruction w is *maximal* if there does not exist an e-obstruction w' and a 2-chord of $G[V(C_z^1 \cup \text{Span}(z))]$ which has w' as a midpoint and which separates w from z.

Proposition 10.5.2. *Let $z \in D_2(C^1)$ be a pentagonal, end-repelling vertex and let e be one of the two edges of $C^1 \cap \tilde{G}_z^{\text{large}}$ which is incident to* $\text{Span}(z)$. *Then* $\text{Ob}_z(e) \neq \varnothing$ *and there exists a unique* $w \in \text{Ob}_z(e)$ *and a 2-chord R_e of $G[V(C_z^1 \cup \text{Span}(z))]$ with w as a midpoint, where $\tilde{G}_z^{\text{large}}$ admits a partition $\tilde{G}_z^{\text{large}} = J_e^0 \cup J_e^1$, such that the following hold.*

1) *$J_e^0 \cap J_e^1$ is the natural R_e-partition of $\tilde{G}_z^{\text{large}}$, where $z \in V(J_e^0)$ and each vertex of $\tilde{G} \setminus C^1$ with an L_ϕ-list of size less than five lies in $V(J_e^0)$; AND*

2) *w is an e-obstruction and each e-obstruction lies in J_e^1; AND*

3) *$N(w) \cap V(C^1) \subseteq V(J_e^1)$*

Proof. Let $e = xx^*$, where $x \in V(\text{Span}(z))$, and let xy be the unique edge of $\text{Span}(z)$ incident to x. Note that, since C^1 is an induced subgraph of G, it follows that, for each endpoint e of $C^1 \cap \tilde{G}_z^{\text{large}}$, an e-obstruction always exists, since the endpoints of e have a common neighbor in $D_2(C)$. We break the remainder of the proof into two cases:

Case 1: There does not exist an element of $\text{Ob}_z(xx_*)$ adjacent to y

In this case, we let $\mathcal{P}$ be the set of 2-chords P of C^1 in $G \setminus C$ such that P has x as an endpoint and the midpoint of P lies in $\tilde{G}_z^{\text{large}}$. There is a unique element of $\mathcal{P}$ which maximizes the quantity $|V(\tilde{G}_P^{\text{small}})|$, and the midpoint of this element of $\mathcal{P}$ is the unique maximal obstruction vertex.

Case 2: There exists an element of $\text{Ob}_z(xx_*)$ adjacent to y

In this case, let $\mathcal{S}$ be the set of proper 3-chords P of C^1 in G_z^{large} in which xy is a terminal edge, the non-y endpoint of the middle edge of P is an e-obstruction, and the 3-chord P of C^1 separates xu from z. By assumption, $\mathcal{S} \neq \varnothing$. Note that each element of $\mathcal{S}$ is a 2-chord of $G[V(C_z^1 \cup \text{Span}(z))]$.

To show that the proposition holds in this case, it suffices to show that, for any $Q, Q' \in \mathcal{S}$, we have either $\tilde{G}_Q^{\text{small}} \subseteq \tilde{G}_{Q'}^{\text{small}}$ or $\tilde{G}_{Q'}^{\text{small}} \subseteq \tilde{G}_Q^{\text{small}}$. If this holds, then, letting Q be the unique element of $\mathcal{S}$ which maximizes $|V(\tilde{G}_Q^{\text{small}})|$, the choice $R_e = Q$ satisfies the proposition. If this total ordering of the elements of $\mathcal{S}$ does not hold, then there exists a $Q \in \mathcal{S}$ which separates $G_z^{\text{small}} \setminus Q$ from a vertex of $G \setminus B_4(C)$ with an L_ϕ-list of size less than five, contradicting the fact that z is pentagonal. $\square$

Given the result above, it is natural to introduce the following definition.

Definition 10.5.3. Given an end-repelling pentagonal vertex $z \in D_2(C^1)$ and an edge e of $C^1 \cap \tilde{G}_z^{\text{large}}$ which is incident to $\text{Span}(z)$, the notation J_e^0, J_e^1 always refers to the two subgraphs of $\tilde{G}_z^{\text{large}}$ specified in Proposition 10.5.2 and the notation R_e always refers to the 2-chord of $G[V(C_z^1 \cup \text{Span}(z))]$ specified in Proposition 10.5.2. Note that R_e is either a 2-chord of a 3-chord of C^1.

We now prove two propositions about obstruction vertices.

Proposition 10.5.4. *Let $z \in D_2(C^1)$ be a pentagonal vertex, where $|E(C^1 \cap \tilde{G}_z^{\text{large}})| > 1$, and let e be an edge of $C^1 \cap \tilde{G}_z^{\text{large}}$ incident to* $\text{Span}(z)$*, where $x \in V(\text{Span}(z))$. Let wu be the lone edge of $R_e \setminus \text{Span}(z)$, where $u \in V(C^1)$ and w is the unique maximal e-obstruction. Let B be a nonempty subset of $D_2(C) \cap N(w)$, where each $w^* \in B$ has a neighbor in $C_z^1 \setminus J_e^1$. Then there exists a proper 3-chord R of C^1, where wu is one of the terminal edges of R, such that. letting w^*w be the middle edge of R, the following hold.*

1) $\tilde{G}_R^{\text{small}} \cap J_e^1 = wu$*; AND*
2) $w^* \in B$*,* $N(w^*) \cap V(C^1) \subseteq V(\tilde{G}_R^{\text{small}})$*, and* $B \subseteq V(\tilde{G}_R^{\text{small}})$*.*

Proof. Since $B \neq \varnothing$, $\tilde{G}_z^{\text{large}}$ contains a proper 3-chord of C^1 in which wu is a terminal edge and the non-w endpoint of the middle edge lies in B. Let $\mathcal{S}$ be the set of proper 3-chords of C^1 satisfying these properties. To show that Proposition 10.5.4 holds, it suffices to show that, for any $R, R' \in \mathcal{S}$, we have either $\tilde{G}_R^{\text{small}} \subseteq \tilde{G}_{R'}^{\text{small}}$ or $\tilde{G}_{R'}^{\text{small}} \subseteq \tilde{G}_R^{\text{small}}$, and then the element of $\mathcal{S}$ which maximizes $|V(\tilde{G}_Q^{\text{small}})|$ satisfies 1) and 2). If this total ordering of the elements of $\mathcal{S}$ does not hold, then there exists a $R \in \mathcal{S}$ such that R separates z from a vertex of $G \setminus B_4(C)$ with an L_ϕ-list of size less than five, and since R is a 3-chord of C^1, this contradicts the fact that z is pentagonal. $\square$

In general, when we construct (C, z)-openers, we want to avoid deleting the obstruction vertices but sometimes we have to delete them, and Proposition 10.5.4 specifies a natural way to define maximal "second generation" obstruction vertices.

Definition 10.5.5. Let $z \in D_2(C^1)$ be a pentagonal vertex, where $|E(C^1 \cap \tilde{G}_z^{\text{large}})| > 1$, and let e be an edge of $C^1 \cap \tilde{G}_z^{\text{large}}$ incident to $\text{Span}(z)$, where $x \in V(\text{Span}(z))$. Let wu be the lone edge of $R_e \setminus \text{Span}(z)$, where $u \in V(C^1)$ and w is the unique maximal e-obstruction. Let B be a nonempty subset of $D_2(C) \cap N(w)$, where each $w^* \in B$ has a neighbor in $C_z^1 \setminus J_e^1$. We call the 3-chord R of C^1 defined in Proposition 10.5.4 the *e-enclosure of B* and we call the lone edge of $R \setminus \{w, u\}$ the *e-wall of B*.

We now have the following.

Definition 10.5.6. Let $z \in D_2(C^1)$ be an end-repelling pentagonal vertex, and let e be one of the two edges of $C^1 \cap \tilde{G}_z^{\text{large}}$ which is incident to $\text{Span}(z) \cap C^1$. Let u be the unique non-$\text{Span}(z)$ endpoint of R_e. We say that e is *problematic* if $S_\star$ is a nonempty subset (i.e a lone vertex) of $(J_e^1 \cap C^1) \setminus \{u\}$.

Proposition 10.5.7. *Let $z \in D_2(C^1)$ be a pentagonal end-repelling vertex and let $e = xv$ be one of the two edge of $C^1 \cap \tilde{G}_z^{\text{large}}$ which is incident to* $\text{Span}(z)$*, where $x \in V(\text{Span}(z) \cap C^1)$. Let y be the lone neighbor of x in* $\text{Span}(z) \cap D_2(C)$*. Suppose that e is unproblematic and let uw be the lone edge of $R_e \setminus \{x, y\}$, where $u \in V(C^1)$. Then the following hold.*

A) If at most one of x, y is adjacent to w, and there exists an L_ϕ-coloring of $\{u, x, y\}$ which does not extend to an L_ϕ-coloring of $V(J_e^1)$, then $R_e = uwyx$ and, in particular, J_e^1 is a wheel where there is a lone vertex of $J_e^1 \setminus R_e$ adjacent to all the vertices of the cycle $(C^1 \cap J_e^1) + R_e$; AND

B) If x is not adjacent to w, then, for any two distinct colors $c_0, c_1 \in L_\phi(x)$, any L_ϕ-coloring of $\{u, w, z\}$ extends to an L_ϕ-coloring of $V(J_e^1) \cup \{z\}$ using one of c_0, c_1 on x.

Proof. We first prove A). Let σ be an L_ϕ-coloring of $\{u, x, y\}$ which does not extend to an L_ϕ-coloring of $V(J_e^1)$. Suppose first that $y \notin N(w)$. Thus, J_e^1 is a broken wheel with principal path uwx. Since y is not adjacent to w, we have $|L_{\phi \cup \sigma}(w)| \geq 3$, and it follows from 1) of Proposition 1.5.1 that σ extends to L_ϕ-color $V(J_e^1)$, contradicting our assumption. Thus, we have $y \in N(w)$, and, by assumption, $x \notin N(w)$.

Let D be the cycle $(C^1 \cap J_e^1) + R_e$. Note that D is a cyclic facial subgraph of J_e^1. Since σ does not extend to L_ϕ-color $V(J_e^1)$, it follows that $\sigma(u)$ is not highly (xy, R_e)-matchable. By Pm1) of Lemma 10.3.2, u, y have a common neighbor p in $J_e^1 \setminus D$. Since $uy \notin E(G)$, it follows from our triangulation conditions that p is adjacent to w as well. In particular, w has no neighbors on D other than y, u, and since C^1 is an induced cycle and there is no chord of D with y as an endpoint. Thus, D is an induced cycle. We claim now p is adjacent to x.

Suppose that $x \notin N(p)$. In that case, $|L_{\phi \cup \sigma}(p)| \geq 3$ and no vertex of J_e^1 has more than two neighbors among $\{u, x, y\}$. Since D is an induced subgraph of G and $L_\phi(w)| \geq 5$, it follows that $\{v \in V(J_e^1) \setminus \{u, x, y\} : |L_{\phi \cup \sigma}(v)| \leq 2\}$ either consists of a lone vertex of D with an L-list of size at least one or two vertices of D with lists of size at least two. Applying Theorem 0.2.3 in the first case or Theorem 1.3.4 in the second, we get that that σ extends to L_ϕ-color $V(J_e^1)$, contradicting our assumption. Thus, p is adjacent to x.

To finish, we just need to check that $\tilde{G}_{upx}^{\text{small}}$ is a broken wheel with principal path upx. Suppose not. We have $|L_{\phi \cup \sigma}(w)| \geq 3$ and $|L_{\phi \cup \sigma}(p)| \geq 2$. Since D is an induced subgraph of G, it follows from Theorem 1.5.3 that σ extends to L_ϕ-color $V(J_e^1)$, contradicting our assumption. This proves A).

Now we prove B). Since $x \notin N(w)$, we have $R_e = uwy$. Let σ be an L_ϕ-coloring of $\{u, w, z\}$. Let $\hat{v}$ be the lone neighbor of w closest to x on (u, x)-path in $J_e^1 \cap C^1$. Since x is the only neighbor of y on this path and $x \notin N(w)$, it follows from our triangulation conditions that v is not adjacent to x. Furthermore, σ extends to an L_ϕ-coloring σ^* of $V(H) \cup \{z\}$, and $J_e^1 \setminus H$ has a facial subgraph F containing all the vertices of $J_e^1 \setminus H$ with $L_{\phi\cup\sigma^*}$-lists of size less than five. Since C^1 is an induced subgraph of G, and $|L_{\phi\cup\sigma^*}(y)| \geq 3$, every vertex of F has an $L_{\phi\cup\sigma^*}$-list of size at least three, except for at most one vertex of $C^1 \setminus \{\hat{v}, x\}$, which has n $L_{\phi\cup\sigma^*}$-list of size at least two. Thus, by Theorem 1.3.4, σ^* extends to an L_ϕ-coloring of $V(J_e^1) \cup \{z\}$ using one of c_0, c_1 on x. $\square$

10.6 The Trickiest Case

The trickiest case to deal with in the proof of Theorem 10.0.7 is the case where $\text{Span}(z)$ is a proper 4-chord of C^1 such that $\tilde{G}_z^{\text{large}} \cap C^1$ is a subpath of C^1 which differs from C^1 only by an edge (i.e $\tilde{G}_z^{\text{small}}$ is just an edge). In this case, letting $Q := \tilde{G}_z^{\text{large}} \cap C^1$, an element of $\text{Link}(Q)$ is not necessarily be a proper L_ϕ-coloring of its domain in $G \setminus C$, because this partial coloring of Q possibly uses the same color on the endpoints of Q. This is the most difficult and technical aspect of the proof of Theorem 10.0.7.

The purpose of Sections 10.6 and 10.7 is to deal with this obstacle. That is, we show in Sections 10.6-10.7 that, for any pentagonal $z \in D_2(C^1)$, if $\text{Span}(z)$ is a proper 4-chord of C^1 and $\tilde{G}_z^{\text{small}} \cap C^1$ is a path of length one, then there exists a (C, z)-opener. We begin with the following observation, which we use repeatedly.

Observation 10.6.1. *Let $z \in D_2(C^1)$ be a pentagonal vertex, where $\text{Span}(z)$ is a proper 4-chord of C^1 and $\tilde{G}_z^{\text{small}} \cap C^1$ is a path of length one. Let $\text{Span}(z) = xyzy'x'$ for some edge xx' of C^1. If $\text{Span}(z)$ has a chord other than xx', then $\tilde{G}_z^{\text{small}}$ either consists of $\text{Span}(z)$ and the edges $\{xx', xy', yy'\}$, or $\text{Span}(z)$ and the edges $\{xx', x'y, yy'\}$.*

Proof. The 5-cycle $xyzy'x'x'$ is a facial subgraph of $\tilde{G}_z^{\text{small}}$. Since $z \in D_3(C)$, we have $x, x' \notin N(z)$. Furthermore, by Proposition 10.2.8, there is no chord of C_z^1 in $\tilde{G}_z^{\text{large}}$. Since G is short-separation-free and $x, x' \notin N(z)$, the observation immediately follows from our triangulation conditions. $\square$

We now have the following.

Proposition 10.6.2. *Let $z \in D_2(C^1)$ be a pentagonal vertex where $\text{Span}(z)$ is a proper 4-chord of C^1 and $\tilde{G}_z^{\text{small}} \cap C^1$ is a path of length one. Let $\text{Span}(z) := xyzy'x'$ for some $x, x' \in V(C^1)$ and $y, y' \in D_2(C)$. Then, for any $y^* \in \{y, y'\}$, the following hold.*

a) For any L_ϕ-coloring σ of $\{x, x'\}$ there is an extension of σ to a proper L_ϕ-coloring ψ of $V(\text{Span}(z) - y^*)$ such that $|L_{\phi\cup\psi}(y^*)| \geq 3$ and $V(G_z^{\text{small}}) \setminus V(\text{Span}(z))$ is $(L, \phi \cup \psi)$-inert; AND

b) If there is no chord of C^1 with y^* as an endpoint other than yy', then, for any L_ϕ-coloring τ of $V(\text{Span}(z)) \setminus \{y^*, z\}$ there is an extension of τ to a proper L_ϕ-coloring ψ of $V(\text{Span}(z) - y^*)$ such that $|L_{\phi\cup\psi}(y^*)| \geq 3$ and $V(G_z^{\text{small}} - y^*)$ is $(L, \phi \cup \psi)$-inert.

Proof. Suppose without loss of generality that $y^* = y'$. We break this into two cases.

Case 1: There is a chord of C_z^1 with y' as an endpoint

In this case we just need to prove a). Since C_z^1 is an induced cycle of $\tilde{G}_z^{\text{large}}$, it follows from Observation 10.6.1 that $V(\tilde{G}_z^{\text{small}}) = \{x, y, z, y', x'\}$. If $|L_{\phi\cup\sigma}(y)| = 3$, then G_z^{small} contains the edge yx', so $|L_{\phi\cup\sigma}(y')| \geq 4$. Thus, choosing a color $f \in L_{\phi\cup\sigma}(z)$ such that $|L_{\phi^*}(y') \setminus \{f\}| \geq 4$, and coloring y with any remaining color, we have an extension of $\phi \cup \sigma$ to the edge yz which leaves behind at least three colors in the list of y'.

Thus, if $|L_{\phi\cup\sigma}(y)| = 3$, then we are done, so now suppose that $|L_{\phi\cup\sigma}(y)| > 3$. Thus, there is a color $f \in L_{\phi\cup\sigma}(y)$ such that $|L_{\phi\cup\sigma}(y') \setminus \{f\}| \geq 3$, and since $|L_{\phi\cup\sigma}(z) \setminus \{f\}| \geq 4$, there is an extension of $\phi \cup \sigma$ to the edge yz which leaves behind at least three colors in the list of y', so we are done.

Case 2: There is no chord of C^1_z with y as an endpoint

In this case, since C^1_z is an induced subgraph of $\tilde{G}^{\text{large}}_z$, it follows that, for any extension ψ of ϕ^* to an L-coloring of $V(C) \cup \{x, y, z, x'\}$, we have $|L_\psi(y')| \geq 3$, since the only neighbors of y' among the colored vertices are x', z. Now let τ be an L_ϕ-coloring of $\{x, x', y\}$.

If there does not exist a lone vertex of $\tilde{G}^{\text{small}}_z$ adjacent to all five vertices of Span(z), then, by Theorem 1.3.5, any extension of $\phi\cup\tau$ to z satisfies the desired conditions. Now suppose that such a vertex $v^\star$ exists. Since C^1_z is an induced subgraph of $\tilde{G}^{\text{large}}_z$, and $\tilde{G}^{\text{small}}_z$ is a wheel with central vertex $v^\star$, xx' is the only chord of C^1_z in G. Furthermore, we have $|L_{\phi\cup\tau}(v^\star)| \geq 2$, and since $|L_{\phi\cup\tau}(z)| \geq 4$, it immediately follows from Corollary 1.3.6 that there is an extension of τ to an L_ϕ-coloring of $\{x, y, z, x'\}$ satisfying the desired properties. $\square$

Now we prove the first of two lemmas which make up the remainder of Section 10.6.

Lemma 10.6.3. *Let $z \in D_2(C^1)$ be a pentagonal vertex, where* Span(z) *is a proper 4-chord of C^1 such that $\tilde{G}^{\text{small}}_z \cap C^1$ is a path of length one. Suppose that there does not exist a (C, z)-opener. If there is an element of* Link$(C^1 - xx')$ *such that $\sigma(x) \neq \sigma(x')$, then $V(\tilde{G}^z_{\text{small}}) = V(\text{Span}(z))$. In particular, if $S_\star \neq \varnothing$, then $V(\tilde{G}^z_{\text{small}}) = V(\text{Span}(z))$.*

Proof. By Lemma 10.4.2, z is end-repelling, since $C^1 \cap \tilde{G}^{\text{small}}_z$ is an edge. Let e, e' be the two terminal edges of $C^1 \cap \tilde{G}^{\text{large}}_z$, where e is incident to x and e' is incident to x'.

Let $Q := C^1 - xx'$ and let $\sigma \in \text{Link}(Q)$, where $\sigma(x) \neq \sigma(x')$. Suppose toward a contradiction that $V(\tilde{G}^z_{\text{small}}) \neq V(\text{Span}(z))$. By Observation 10.6.1, there is no chord of the path $xyzy'x'$ in G except for xx'.

Applying Proposition 10.5.2, let w be the unique maximal e-obstruction and let w' be the unique maximal e'-obstruction. Let uw be the lone edge of $R_e \setminus \{x, y\}$ and let $u'w'$ be the lone edge of $R_{e'} \setminus \{x', y'\}$. Since G has no chord of Span(z) except for xx', we have $|L_{\phi\cup\sigma}(y)| \geq 4$ and $|L_{\phi\cup\sigma}(y')| \geq 4$. Since each of w, w' has an L_ϕ-list of size at least three, we extend σ to an L_ϕ-coloring of dom$(\sigma) \cup \{y, y'\}$ in the following way. We let $f \in L_{\phi\cup\sigma}(y)$ and $f' \in L_{\phi\cup\sigma}(y')$, where $|L_{\phi\cup\sigma}(w) \setminus \{f\}| \geq 3$ and $|L_{\phi\cup\sigma}(w') \setminus \{f'\}| \geq 3$. Possibly $f = f'$, which is permissible as $yy' \notin E(G)$.

Since $|L_\phi(z) \setminus \{f, f'\}| \geq 3$, it immediately follows from Corollary 1.3.6 that there is an L_ϕ-coloring τ of dom$(\sigma) \cup V(\tilde{G}^{\text{small}}_z)$ using f, f' on the respective vertices y, y'. Let H be the subgraph of G induced by dom$(\phi \cup \tau) \cup V(J^1_e \cup J^1_{e'}) \cup \text{Sh}_2(Q)$.

We claim now that $[H \setminus \{w, w'\}, \phi\cup\tau]$ is a (C, z)-opener. It suffices to check that each of $V(J^1_e - w)$ and $V(J^1_{e'} - w')$ is $(L, \phi \cup \tau)$-inert in G. Without loss of generality, we just show that this holds for $J^1_e - w$. If $R_e = uwx$, then $V(J^1_e \setminus R_e) \subseteq \text{Sh}_2(Q)$ and so it immediately follows from the definition of Link(Q) that $V(J^1_e - w)$ is $(L, \phi\cup\tau)$-inert in G. On the other hand, if $R_e = u'w'x'$, then this immediately follows from Proposition 10.2.9 so we are done. Thus, $[H \setminus \{w, w'\}, \phi \cup \tau]$ is a (C, z)-opener, contradicting our assumption.

We conclude that, if there is an element of Link(Q) such that $\sigma(x) \neq \sigma(x')$, then $V(\tilde{G}^z_{\text{small}}) = V(\text{Span}(z))$. In particular, it immediately follows from 3) of Corollary 10.2.5 that either $S_\star = \varnothing$ or $V(\tilde{G}^z_{\text{small}}) = V(\text{Span}(z))$. $\square$

Now we prove the main result of Section 10.6.

Lemma 10.6.4. *Let $z \in D_2(C^1)$ be a pentagonal vertex, where* $\text{Span}(z)$ *is a proper 4-chord of C^1 and $\tilde{G}_z^{\text{small}} \cap C$ is a path of length one. Suppose that there does not exist a (C, z)-opener. Suppose further that there is a $u_\star \in V(C^1)$ with $S_\star = \{u_\star\}$. Then there exists a terminal edge $e = xv$ of $C^1 \cap \tilde{G}_z^{\text{large}}$, where x is an endpoint of* $\text{Span}(z)$*, such that the following hold.*

1) xv is problematic; AND

2) There is a vertex of $T^{<2}$ which, on the path $C^1 \cap \tilde{G}_z^{\text{large}}$, separates $u_\star$ from the non-x endpoint of $\text{Span}(z)$.

Proof. Since $S_\star \neq \varnothing$, it follows from Co4d) of Definition 10.0.1 that every vertex of $D_2(C)$ has a neighborhood on C^1 consisting precisely of a subpath of C^1. By Lemma 10.4.2, z is end-repelling, since $C^1 \cap \tilde{G}_z^{\text{small}}$ is an edge. Let e, e' be the two terminal edges of $C^1 \cap \tilde{G}_z^{\text{large}}$, where e is incident to x and e' is incident to x'.

Let wu be the unique edge of $R_e \setminus \{x, y\}$ and let $w'u'$ be the unique edge of $R_{e'} \setminus \{x', y'\}$, where w is an e-obstruction, w' is an e'-obstruction, $u \in V(C^1 - x)$ and $u' \in V(C^1 - x')$. By Lemma 10.4.2, we have $u, u' \notin \{x, x'\}$ and $w \neq w'$. Let $Q := C^1 \cap \tilde{G}_z^{\text{large}} = C^1 - xx'$.

Suppose toward a contradiction that the lemma is not satisfied. By Co4c) of Definition 10.0.1, there is a vertex $v^\dagger \in T^{<2}$. Note that $v^\dagger \neq u_\star$. Let q, q' be the endpoints of $S_\star^{\text{path}}$ (possibly $q = q' = u_\star$), where the vertices of $\{q, q', x, x'\}$ have the cyclic order x', q', q, x (possibly one of q, q' lies in $\{x, x'\}$). Since G is $K_{2,3}$-free, $v^\dagger$ is not an internal vertex of $S_\star^{\text{path}}$. Thus, suppose without loss of generality that $v^\dagger \in V(qQx)$. Recall that $w'u'$ is the lone edge of $R_{e'} \setminus \{x', y'\}$. Since the lemma does not hold, and $v^\dagger \in V(qQx)$, we have by assumption that $u_\star \notin V(J_{e'}^1 - u')$, or else e' is problematic and, $v^\dagger$ separates $u_\star$ from x on the path $C^1 - xx'$. Now we have the following:

Claim 10.6.5. *w' is adjacent to each of x', y'*

<u>Proof:</u> We first show that $w'y'$ is an edge of $R_{e'}$. Suppose not. Then $R_{e'} = u'w'x'$ and w' is the unique e'-obstruction. Applying Proposition 10.2.4, there is a $c' \in L_\phi(x')$ and an element $\psi \in \text{Link}(Q)$ such that $\psi(x) \neq c'$, so $\phi \cup \psi$ is a proper L-coloring of its domain in G. By 2) of Proposition 10.6.2, there is an extension σ of $\phi \cup \psi$ to an L-coloring of $\text{dom}(\phi \cup \psi) \cup \{y', z\}$ such that $|L_\sigma(y)| \geq 3$ and $V(\tilde{G}_z^{\text{small}}) \setminus V(\text{Span}(z))$ is (L, σ)-inert

Let $H := \text{dom}(\sigma) \cup V(\tilde{G}_z^{\text{small}} - y) \cup \text{Sh}_{2,}(Q)$. By assumption, $[H, \sigma]$ is not a (C, z)-opener, so there exists a $p \in D_1(H)$ with $|L_\sigma(p)| < 3$. By Proposition 10.2.8, p is not adjacent to z, and since w' is the unique e'-obstruction, we have $p = w'$. Since w' is not adjacent to x', we have $|L_\sigma(p)| \geq 3$, a contradiction. Thus, we indeed have $R_{e'} = u'w'y'$.

By Proposition 10.2.4, there is a $d \in L_\phi(u')$ and a pair of elements ψ_1, ψ_2 in $\text{Link}(u'Qx)$ which use different colors on x and color u' with d. Note that $x'w' \notin E(G)$, or else, since $w' \notin N(x')$ and G is short-separation-free, it follows from our triangulation conditions that $w' \in N(x')$, contradicting our assumption. Now we apply the work of Section 1.6. Since $u_\star \notin V(J_{e'}^1 - u')$ and $N(y') \cap V(C_z^1) = \{z, x'\}$, it follows from Theorem 1.6.1 that there is a color $f \in L_\phi(x')$ such that any L_ϕ-coloring of $u'w'y'x'$ using d, f on the respective vertices u', x' extends to an L_ϕ-coloring of $J_{e'}^1$. Since $\psi_1(x), \psi_2(x)$ are distinct, suppose without loss of generality that $\psi_1(x) \neq f$.

Applying Proposition 10.6.2, let ϕ^* be an extension of ϕ to $V(C) \cup \{x', y', z, x\}$, where ϕ^* uses the colors $\psi_1(x), f$ on the respective vertices x, x', such that $|L_{\phi^*}(y)| \geq 3$ and $V(G_z^{\text{small}}) \setminus V(\text{Span}(z))$ is (L, ϕ^*)-inert. Since C_z^1 is an induced subgraph of $\tilde{G}_z^{\text{large}}$, the union $\phi^* \cup \psi_1$ is a proper L_ϕ-coloring of its domain.

Let H be the subgraph of G induced by $\text{dom}(\phi^* \cup \psi_1) \cup V(J_{e'}^1 - w') \cup \text{Sh}_2(u'Qx) \cup V(G_z^{\text{small}} - y)$. By our construction of $\phi^* \cup \psi_1$, $V(H) \setminus \text{dom}(\phi^* \cup \psi_1)$ is $L_{\phi^* \cup \psi_1}$-inert. By assumption, $[H, \phi^* \cup \psi_1]$ is not a (C, z)-opener so there is a

$p \in D_1(H)$ with $|L_{\phi^* \cup \psi_1}(p)| < 3$. By Proposition 10.2.8, p is not adjacent to z, so $p = w'$. Since $w'x' \notin E(G)$, we have $N(w') \cap \text{dom}(\phi^* \cup \psi_1) = \{u', y'\}$, so $|L_{\phi^* \cup \psi_1}(p)| \geq 3$, a contradiction. ■

Now we have the following:

Claim 10.6.6. *Let $\psi \in \text{Link}(Q)$ with $\psi(x) \neq \psi(x')$. Let σ be an extension of $\phi \cup \psi$ to an L-coloring of* $\text{dom}(\phi \cup \psi) \cup \{y', z\}$. *Suppose that σ is a proper L-coloring of its domain in G, $|L_\sigma(y)| \geq 3$, and $V(\tilde{G}_z^{\text{small}}) \setminus V(\text{Span}(z))$ is (L, σ)-inert. Then $|L_\sigma(w')| < 3$.*

Proof: Let H be the subgraph of G induced by $\text{dom}(\sigma) \cup V(\tilde{G}_z^{\text{small}} - y) \cup \text{Sh}_2(Q)$. By our construction of σ, $V(H) \setminus \text{dom}(\sigma)$ is L_σ-inert. By assumption, $[H, \sigma]$ is not a (C, z)-opener, so there exists a $p \in D_1(H)$ with $|L_\sigma(p)| < 3$. By Proposition 10.2.8, $z \notin N(p)$, so $y' \in N(p)$. Since w' is the unique e'-obstruction, $p = w'$. ■

By Lemma 10.6.3, we have $V(\tilde{G}_z^{\text{small}}) = V(\text{Span}(z))$, so we get $yy' \in E(\tilde{G}_z^{\text{small}})$ by Observation 10.6.1. Applying Claim 10.6.5, $J_{e'}^1 - y'w'$ consists of a broken wheel K with principal path $u'w'x$. Since $u_\star \notin V(J_{e'}^1 - u')$, each vertex on the path $K - \{u', w'\}$ has an L_ϕ-list of size at at least three.

Claim 10.6.7. $y'x \notin E(G)$.

Proof: Suppose toward a contradiction that $y'x \in E(G)$. Thus, $\tilde{G}_z^{\text{small}}$ consists of the path $\text{Span}(z)$ and the edges $xx', y'x, yy'$. The key here is to leave x' uncolored. Since $u_\star \notin V(J_{e'}^1 - u')$, it follows from Proposition 10.2.4 that there is a pair of colorings $\psi_0, \psi_1 \in \text{Link}(u'Qx)$ which use the same color on u' and use different colors on x. Since x' is uncolored, each of ψ_0, ψ_1 is a proper L-coloring of its domain in G. Let c be the colored used by ψ_0, ψ_1 on u', and, for each $i = 0, 1$, let $d_i := \psi_i(x)$.

Now, for each $i = 0, 1$ and $f \in L(y') \setminus \{d_i\}$, we define a partial L_ϕ-coloring σ_i^f of $\tilde{G}$ as follows. We extend ψ_i to an L_ϕ-coloring of $\text{dom}(\psi_i) \cup \{y', z\}$ by coloring y' with f and choose a color $f' \in L(z)$ such that $|L_{\psi_i}(y) \setminus \{f, f'\}| \geq 3$. Such an f' exists since $|L_{\psi_i}(z) \setminus \{f\}| \geq 4$ and $|L_{\psi_i}(y) \setminus \{f\}| \geq 3$. Note that all of colorings of the form σ_i^f have the same domain in $\tilde{G}$ and each is a proper L_ϕ-coloring of its domain. Let $A \subseteq V(\tilde{G})$ be the common domain of all of these colorings and let H be the subgraph of G induced by $V(C) \cup A \cup \text{Sh}_2(Q)$.

By assumption, for each $i \in \{0, 1\}$ and $f \in L(y') \setminus \{d_i\}$, the pair $[H, \sigma_i^f \cup \phi]$ is not a (C, z)-opener, and the only condition which is violated is the inertness of $V(K) \setminus \{u', w'\}$ in G. Since $yx' \notin E(G)$, it follows that, for each $i \in \{0, 1\}$ and $f \in L(y') \setminus \{d_i\}$, there is an extension of $\sigma_i^f \cup \phi$ to an L-coloring τ_i^f of $G \setminus (K \setminus \{u', w'\})$ which does not extend to L-color the rest of K, so the following holds:

$$\mathcal{Z}_K(c, \tau_i^f(w'), \bullet) \subseteq \{d_i, f\} \tag{†}$$

Note that all of these extensions use the color c on u', since u' is already colored.

Subclaim 10.6.8. $\{d_0, d_1\} \subseteq (L(w') \cap L_\phi(x')) \setminus \{c\}$.

Proof: Since $|L(y')| \geq 5$, there exist two distinct colors $f_0, f_1 \in L(y')$ such that $|L_\phi(x') \setminus \{f_0, f_1\}| \geq 3$. Since f_0, f_1 are distinct, suppose without loss of generality that $f_0 \neq d_0$ and $f_1 \neq d_1$. Now consider the two L-colorings $\tau_0^{f_0}$ and $\tau_1^{f_1}$ of $G \setminus (K \setminus \{u', w'\})$.

By Theorem 0.2.3, for each $i = 0, 1$, $\mathcal{Z}_K(c, \tau_i^{f_i}(w'), \bullet)$ contains a color of $L_\phi(x') \setminus \{f_0, f_1\}$, since $|L_\phi(x') \setminus \{f_0, f_1\}| \geq 3$. Let L' be a list-assignment for $V(K)$ where $L'(x') = L_\phi(x') \setminus \{f_0, f_1\}$ and otherwise $L' = L_\phi$. By (†), it follows that, for each $i = 0, 1$, we have $\mathcal{Z}_{K,L'}(c, \tau_i^{f_i}(w'), \bullet) = \{d_i\}$, so $d_0, d_1 \in L_\phi(x')$. Since

$d_0 \neq d_1$ we have $\tau_0^{f_0}(w'), \neq \tau_1^{f_1}(w')$. By 1) of Proposition 1.4.7 applied to K with the list-assignment L', we get that, for each $i = 0, 1$, $\tau_i^{f_i}(w') = \{d_{1-i}\}$. Thus, $d_0, d_1 \neq c$, and $\{d_0, d_1\} \subseteq (L(w') \cap L_\phi(x')) \setminus \{c\}$. ■

Now we return to the main proof of Claim 10.6.7. Applying Theorem 1.5.5, there is a color $c' \in L_\phi(x')$, where $c \neq c'$ if K is a triangle and any L_ϕ-coloring of $u'w'x'$ using c, c' on u', x' respectively extends to an L_ϕ-coloring of K.

Subclaim 10.6.9. $c' \in \{d_0, d_1\}$. *Furthermore,* $L(w') \setminus \{c, c'\} = L(y') \setminus \{d_0, d_1\}$.

Proof: Suppose that at least one of these conditions does not hold. Thus, there exists an $i \in \{0, 1\}$ and an extension ψ_i^* of ψ_i to a proper L-coloring of $\text{dom}(\psi_i) \cup \{y', x'\}$ such that $|L_{\psi_i^*}(w')| \geq 3$. and $\psi_i^*(x') = c'$. Since $|L_{\psi_i^*}(z)| \geq 4$, there is an extension of ψ_i^* to an L-coloring $\psi_i^\dagger$ of $\text{dom}(\psi_i) \cup \{y', x', z\}$ such that $|L_{\psi_i^\dagger}(y)| \geq 3$. Let H be the subgraph of G induced by $\text{dom}(\phi \cup \psi_i^\dagger) \cup \text{Sh}_{2,}(Q)$. Since $xw \notin E(G)$, each of w', y has an $L_{\psi_i^\dagger}$-list of size at least three, and $[H, \phi \cup \psi_i^\dagger]$ is a (C, z)-opener, contradicting our assumption. ■

Now we have enough to finish the proof of Claim 10.6.7. By Subclaim 10.6.9, we have $c' \in \{d_0, d_1\}$, so suppose without loss of generality that $c' = d_0$. By Subclaim10.6.8, we have $d_1 \neq c$ and $d_1 \in L(w)$. Since $d_1 \neq d_0$ we have $d_1 \in L(w') \setminus \{c, c'\}$. Yet, by Subclaim 10.6.9, we have $L(w') \setminus \{c, c'\} = L(y') \setminus \{d_0, d_1\}$, so we have a contradiction. This completes the proof of Claim 10.6.7. ■

We now return to the main proof of Lemma 10.6.4. Since $y'x \notin E(G)$ and $V(G_z^{\text{small}}) = V(\text{Span}(z))$, it follows from Observation 10.6.1 that G_z^{small} consists of the path $\text{Span}(z)$ and the edges $\{xx', yx', yy'\}$.

Claim 10.6.10. $wy \in E(G)$.

Proof: Suppose toward a contradiction that $wy \notin E(G)$. Thus, J_e^1 is a broken wheel with principal path uwx. Applying Proposition 10.2.4, there is an L_ϕ-coloring $\psi \in \text{Link}(Q)$ such that $\psi(x) \neq \psi(x')$. Since $|L_{\psi \cup \phi}(y)| \geq 4$ and $|L_{\psi \cup \phi}(z)| \geq 5$, there is an extension of $\phi \cup \psi$ to an L-coloring σ of $\text{dom}(\phi \cup \psi) \cup \{y, z\}$ such that $|L_{\sigma \cup \phi}(y')| \geq 3$. Since $y \notin N(w)$, we have $|L_{\sigma \cup \phi}(w)| \geq 3$. Let H be the subgraph of G induced by $V(C) \cup \text{Sh}_2(Q) \cup \{y, z\}$. By our construction of σ, $V(H) \setminus \text{dom}(\sigma \cup \phi)$ is $L_{\sigma \cup \phi}$-inert. Since each of y', w has an $L_{\sigma \cup \phi}$-list of size at least three, $[H, \sigma \cup \phi]$ is a (C, z)-opener, contradicting our assumption. ■

Now we have the following:

Claim 10.6.11. $v^\dagger \notin V(J_e^1 - u)$*, and* $u_\star \notin V(J_e^1 - u)$.

Proof: Suppose toward a contradiction that $v^\dagger \in V(J_e^1 - u)$. Let $\hat{v}$ be the unique neighbor of $v^\dagger$ on the path $x'Qv^\dagger$. Since $v^\dagger \in V(J_e^1 - u)$, we have $\hat{v} \in V(J_e^1)$, and furthermore, since $v^\dagger$ lies in the unique subpath of Q which has one endpoint in x and intersects with $S_\star^{\text{path}}$ on precisely an endpoint common to the two paths, it follows that $S_\star^{\text{path}} \subseteq x'Qv^\dagger$. Applying Proposition 10.2.4, we fix an element $\sigma \in \text{Link}(x'Q\hat{v})$.

Subclaim 10.6.12. $wx \notin E(G)$

Proof: Suppose toward a contradiction that $wx \in E(G)$. Thus, $J_e^1 - wy$ consists of a broken wheel with principal path uwx. Since $v^\dagger \in V(J_e^1 - u)$ and $v^\dagger \notin T^{\text{int}}$, we have $v^\dagger = x$ in this case. Possibly, $u_\star$ is an internal vertex of $J_e^1 - w$, i.e $S_\star^{\text{path}} = uQx$. Since $v^\dagger = x$, we have $|L_{\phi \cup \sigma}(x)| \geq 2$.

Since $N(y) \cap V(Q) = \{x, x'\}$ and $N(z) \cap V(Q) = \varnothing$, we have $|L_{\phi \cup \sigma}(y)| \geq 4$ and $|L_{\phi \cup \sigma}(z)| \geq 5$. Furthermore, we have $|L_{\phi \cup \sigma}(y')| \geq 4$. Since $|L_{\phi \cup \sigma}(x)| \geq 2$, we choose a color $d \in L_{\phi \cup \sigma}(y)$ such that $|L_{\phi \cup \sigma}(x) \setminus \{d\}| \geq 2$.

Since $|L_{\phi\cup\sigma}(z)\setminus\{d\}|\geq 4$ and $xy'\notin E(G)$, there is an extension of $\phi\cup\sigma$ to an L-coloring τ of $\text{dom}(\phi\cup\sigma)\cup\{y,z\}$ such that $\tau(y)=d$ and $|L_\tau(y')|\geq 3$.

Let H be the subgraph of G induced by $\text{dom}(\tau)\cup V(J_e^1)\cup\text{Sh}_2(x'Q\hat{v})$. Note that these three vertex sets are not necessarily pairwise-disjoint. By assumption, the pair $[H,\tau]$ is not a (C,z)-opener, so the inertness condition is violated. That is, there is an extension of τ to an L-coloring τ^* of $G\setminus(H\setminus\text{dom}(\tau))$ such that τ^* does not extend to L-color $H\setminus\text{dom}(\tau)$.

By our construction of τ from σ, it follows that τ^* extends to an L-coloring τ^{**} of $\text{dom}(\tau^*)\cup\text{Sh}_2(x'Q\hat{v})$. Note that, since J_e^1 is a broken wheel with principal path xwy, $\text{Sh}_2(x'Q\hat{v})$ contains all the vertices of $J_e^1-\{w,y\}$, except for x. Thus, τ^{**} is an L-coloring of $G-x$. Now, since $y'x\notin E(G)$, $N(v^\dagger)\cap V(G\setminus C)=\{\hat{v},w,y,x'\}$. Thus, by our choice of color $\tau(y)$, it follows that $|L_{\tau^{**}}(x)|\geq 1$, so there is a color left over for x, and τ^{**} extends to an L-coloring of G, contradicting our assumption. ■

Since $wx\notin E(G)$, we have $R_e=uwxy$.

Subclaim 10.6.13. $|N(w)\cap V(C^1)|>1$.

Proof: Suppose not. Thus, we have $N(w)\cap V(C^1)=\{u\}$. By 3) of Corollary 10.2.5, there is a $\zeta\in\text{Link}(Q)$ with $\zeta(x)\neq\zeta(x')$. By Claim 10.6.7, $y'x\notin E(G)$. Since $|L_{\phi\cup\zeta}(y)|\geq 4$ and $|L_{\phi\cup\zeta}(z)|\geq 5$, there is an extension of ζ to an L_ϕ-coloring ζ^* of $\text{dom}(\zeta)\cup\{y,z\}$ such that $|L_{\phi\cup\zeta^*}(y')|\geq 3$. Since $N(w)\cap V(C^1)=\{u\}$, we have $|L_{\phi\cup\zeta^*}(w)|\geq 3$. Letting H^* be the subgraph of G induced by $\text{dom}(\phi\cup\zeta^*)\cup\text{Sh}_2(x'Q\hat{v})\cup V(J_e^1-w)\cup\{z\}$, the pair $[H^*,\phi\cup\zeta^*]$ is a (C,z)-opener, contradicting our assumption. ■

Since $|L_{\phi\cup\sigma}(y)|\geq 4$ and $|L_{\phi\cup\sigma}(z)|\geq 5$, there exists an extension of σ to an L_ϕ-coloring ψ of $\text{dom}(\sigma)\cup\{x,y,z\}$ with $|L_{\phi\cup\psi}(y')|\geq 3$.

Let H be the subgraph of G induced by $\text{dom}(\phi\cup\psi)\cup V(J_e^1-w)\cup\text{Sh}_2(x'Q\hat{v})$. By assumption, the pair $[H,\phi\cup\psi]$ is not a (C,z)-opener, so the inertness condition is violated. Thus, there is an extension of ψ to an L-coloring ψ^* of $\text{dom}(\psi)\cup\{w\}$ which does not extend to L_ϕ-color J_e^1.

Now, J_e^1 has a facial cycle $D:=uQxyw$ which contains all the vertices of $\text{dom}(\phi\cup\psi^*)\cap V(J_e^1)$. Let $X:=V(J_e^1)\cap\text{Sh}_2(uQ\hat{v})$. Note that any extension of ψ^* to $\text{dom}(\psi)\cup V(J_e^1\setminus X)$ also extends to X, since X is $(L,\phi\cup\sigma)$-inert and $X\cap\text{dom}(\phi\cup\psi^*)=\varnothing$. Thus, to prove the subclaim, it just suffices to show that ψ^* extends to an L_ϕ-coloring of $J_e^1\setminus X$.

Consider the list-assignment $L_{\phi\cup\psi^*}$ for $J_e^1\setminus X$. There is a facial subgraph D' of J_e^1 which contains all the vertices of $J_e^1\setminus X$ with $L_{\phi\cup\psi^*}$-lists of size less than five. Furthermore, $\text{dom}(\phi\cup\psi^*)\cap\{u,w,y,x\}=\{u,w,y\}$. Note that x remains uncolored. In particular, since $|L_\phi(v^\dagger)|\geq 4$ and C^1 is an induced cycle of G, any element of $V(D')$ with an $L_{\phi\cup\psi^*}$-list of size less than three is either x or adjacent to all three of u,w,y. Note that there is no vertex of J_e^1 adjacent to all three of u,w,y. To see this, suppose that such a vertex exists. In that case, since $N(w)\cap V(C^1)\subseteq V(J_e^1)$, we have $N(w)\cap V(C^1)=\{u\}$, contradicting Subclaim 10.6.13.

Since no vertex of J_e^1 is adjacent to all three of u,w,y, every vertex of D' has an $L_{\phi\cup\psi^*}$-list of size at least three, except for x. If $x=v^\dagger$ then $|L_\phi(x)|\geq 4$ and $(N(x)\cap\text{dom}(\psi^*))\setminus V(C)=\{x',\hat{v},y\}$, so $|L_{\psi^*}(x)|\geq 1$. If $x\neq v^\dagger$ then $|L_\phi(x)|\geq 3$ and $(N(x)\cap\text{dom}(\psi^*))\setminus V(C)=\{x',y\}$, so, again, $L_{\phi\cup\psi^*}(x)|\geq 1$. In any case, it follows from Theorem 0.2.3 that ψ^* extends to L_ϕ-color $J_e^1\setminus X$ and thus extends to J_e^1, contradicting our choice of ψ^*.

Since $v^\dagger$ lies in the unique subpath of Q which has one endpoint in x and intersects with $S_\star^{\text{path}}$ on precisely an endpoint common to the two paths, it follows that $S_\star^{\text{path}}\subseteq x'Qu$, and, in particular, $u_\star\notin V(J_e^1-u)$. ■

We now have the following:

Claim 10.6.14. $wx \in E(G)$.

Proof: Suppose toward a contradiction that $wx \notin E(G)$. Since w is an e-obstruction, we have $y \in N(w)$. By Claim 10.6.11 $v^\dagger \notin V(J_e^1 - u)$. Thus, applying Proposition 10.2.4, we fix two elements ψ_0, ψ_1 of Link$(x'Qu)$ which use different colors on u, where $\psi_0(x') = \psi_1(x') = c$ for some color c. Since we have not colored x, each of ψ_0, ψ_1 is a proper L_ϕ-coloring of its domain in $\tilde{G}$.

Since $|L_\phi(x)| \geq 3$ and $|L_\phi(y)| \geq 5$, it follows from 1) Theorem 9.0.1 that there is a $\sigma \in$ Corner(R_e, w), where $\sigma(u) \in \{\psi_0(u), \psi_1(u)\}$, and $\sigma(x) \neq c$, and $\sigma(y) \neq c$.

Since $\sigma(u) \in \{\psi_0(u), \psi_1(u)\}$, suppose without loss of generality that $\sigma(u) = \psi_0(u)$. Since $\sigma(y) \neq c$ and $\sigma(x) \neq c$, the union $\psi_0 \cup \sigma$ is a proper L_ϕ-coloring of its domain in G. Now, since $xy' \notin E(G)$, we have $|L_{\sigma \cup \phi}(y')| \geq 3$, and since $|L_{\sigma\cup\phi}(z)| \geq 4$, there is an extension of $\sigma \cup \phi$ to an L-coloring σ^* of dom$(\sigma \cup \phi) \cup \{z\}$, where $|L_{\sigma^*}(y')| \geq 3$. Since $z, y \notin N(w)$, we have $N(w) \cap \text{dom}(\sigma^*) = \{x, u\}$, so $|L_{\sigma^*}(w)| \geq 3$ as well.

Let $H^\star$ be the subgraph of G induced by dom$(\sigma^*) \cup V(J_e^1 - w) \cup \text{Sh}_2(x'Qu)$. By our construction of σ^*, $V(H) \setminus \text{dom}(\sigma^*)$ is L_{σ^*}-inert. Since each of w, y' has an L_{σ^*}-list of size at least three, the pair $[H^\star, \sigma^*]$ is a (C, z)-opener, so we contradict our assumption. ■

Recall that, by Co4d) of Definition 10.0.1, $J_e^1 - wy$ consists of a broken wheel K^* with principal path uwx. By assumption, $u_\star \notin V(J_{e'}^1 - u')$. Thus, by Claim 10.6.11, we have $u_\star \in V(u'Qu)$ and $v^\dagger \in V(u_\star Qu)$. In particular, each vertex of $K^* - \{u, mw\}$ has an L_ϕ-list of size at least three. Applying Proposition 10.2.4, we fix two elements ψ_0, ψ_1 of Link$(x'Qu)$ such that $\psi_0(u') = \psi_1(u')$, $\psi_0(x') = \psi_0(x')$, and ψ_0, ψ_1 use different colors on u'. Let $\psi_0(x') = \psi_0(x') = c$ and $\psi_0(u') = \psi_1(u') = d$ for some colors c, d. For each $i = 0, 1$, let $s_i := \psi_i(u)$.

For each $i \in \{0, 1\}$ and $f \in L_{\phi\cup\psi_i}(y)$, we define an extension of $\phi \cup \psi_i$ to an L-coloring σ_i^f of dom$(\phi \cup \psi_i) \cup \{y, z\}$ in the following way. Since $|L_{\phi\cup\psi_i}(z) \setminus \{f\}| \geq 4$ and $xy' \notin E(G)$, there is an extension of $\phi \cup \psi_i$ to an L-coloring σ_i^f of dom$(\phi \cup \psi_i) \cup \{y, z\}$ such that $\sigma_i^f(y) = f$ and $|L_{\sigma_f^i}(y')| \geq 3$. Note that, for each $i = 0, 1$ and $f \in L_{\phi\cup\psi_i}(y)$, we have $|L_{\sigma_i^f}(w)| \geq 3$, since dom$(\sigma_i^f) \cap N(w) = \{u, y\}$.

We also note that the colorings of the form σ_i^f all have the same domain for any $i = 0, 1$ and $f \in L_{\phi\cup\psi_i}(y)$, so let A be this common domain. Let H be the subgraph of G induced by $A \cup V(K^* - w)$. By assumption, for each $i \in \{0, 1\}$ and $f \in L_{\phi\cup\psi_i}(y)$, the pair $[H, \sigma_i^f]$ is not a (C, z)-opener. The only condition that is violated is the inertness condition, and since Sh$_2(x'Qu)$ is $(L, \phi \cup \psi_i)$-inert for each $i = 0, 1$, it follows that, for each $i \in \{0, 1\}$ and $f \in L_{\phi\cup\psi_i}(y)$, there is an extension of σ_i^f to an L-coloring τ_i^f of $G \setminus (K^* \setminus \{u, w\})$ such that τ_i^f does not extend to L-color the broken wheel K^*. Since $y', z, \notin N(x)$, it follows that $N(x) \cap V(G \setminus C)$ consists of y and the two neighbors of x in K^*. Thus, for each $i \in \{0, 1\}$ and $f \in L_{\phi\cup\psi_i}(y)$, the following is satisfied.

$$\mathcal{Z}_{K^*}(s_i, \tau_i^f(w), \bullet) \subseteq \{c, f\} \tag{†}$$

We now note the following:

Claim 10.6.15. $\{s_0, s_1\} \subseteq L(w) \cap (L_\phi(x) \setminus \{c\})$. *Furthermore, for each* $v \in V(K^* \setminus \{u, w, x\})$, $\{s_0, s_1\} \subseteq L_\phi(v)$.

Proof: Since $|L(y) \setminus \{c\}| \geq 4$, we fix a $g \in L(y) \setminus \{c\}$ with $|L_\phi(x) \setminus \{g\}| \geq 3$. Since $L(y) \setminus \{c\} = L_{\phi\cup\psi_i}(y)$ for each $i = 0, 1$, we have $g \in L_{\phi\cup\psi_i}(y)$. Let L' be a list-assignment for $V(K^*)$ where $L'(x) = L_\phi(x) \setminus \{g\}$ and otherwise

$L' = L_\phi$. For each $i = 0, 1$, we get $\mathcal{Z}_{K^*,L'}(s_i, \tau_i^g(w), \bullet) \neq \varnothing$ by applying Theorem 0.2.3. By (†), since $g \notin L'(x)$, we have $\mathcal{Z}_{K^*,L'}(s_i, \tau_i^g(w), \bullet) = \{c\}$ for each $i = 0, 1$. Applying 2) of Proposition 1.4.7, we have $\tau_i^g(w) = s_{1-i}$ for each $i = 0, 1$. In particular, we have $\{s_0, s_1\} \subseteq L(w)$. Since $\mathcal{Z}_{K^*,L'}(s_i, \tau_i^g(w), \bullet) = \{c\}$ for each $i = 0, 1$, and $s_0 \neq s_1$, it follows from Proposition 1.4.4 that $\{s_0, s_1\} \subseteq L_\phi(x) \setminus \{c\}$.

Now we show that $\{s_0, s_1\} \subseteq L_\phi(v)$ for each $v \in V(K^* \setminus \{u, w, x\})$. If K^* is a triangle, then we are done in that case. Now suppose that K^* is not a triangle. Since $\mathcal{Z}_{K^*,L'}(s_i, \tau_i^g(w), \bullet) = \{c\}$ for each $i = 0, 1$, it follows from Proposition1.4.4 that $s_0, s_1 \in L_\phi(v)$ for each $v \in V(K^* \setminus \{u, w, x\})$, as $\tau_i^g(w) = s_{1-i}$ for each $i = 0, 1$. ■

The last fact we need is the following:

Claim 10.6.16. $\{s_0, s_1\} \cap (L(y) \setminus \{c\}) = \varnothing$.

Proof: Suppose toward a contradiction that $\{s_0, s_1\} \cap (L(y) \setminus \{c\}) \neq \varnothing$, and, without loss of generality, let $s_0 \in L(y) \setminus \{c\}$.

Subclaim 10.6.17. *K^* is not a triangle*

Proof: Suppose toward a contradiction that K^* is a triangle. By Claim 10.6.15, $\{s_0, s_1\} \subseteq L_\phi(x) \setminus \{c\}$. Since $s_0 \in L(y) \setminus \{c\}$, we extend $\phi \cup \psi_0$ to an L-coloring ψ^* of dom$(\phi \cup \psi^*) \cup \{x, y\}$ by coloring x, y with the respectove colors s_1, s_0. We then have $|L_{\phi^*}(w)| \geq 3$, since ϕ^* uses the same color on u, y. Thus, the pair $[H^\star, \phi^*]$ is a (C, z)-opener, contradicting our assumption. ■

Since K^* is not a triangle, let $K^* - w = uv_1 \cdots v_t x$ for some $t \geq 1$. Now, since $s_0 \in L(y) \setminus \{c\}$, and $L(y) \setminus \{c\} = L_{\phi \cup \psi_i}(y)$ for each $i = 0, 1$, consider the two L-colorings $\tau_0^{s_0}, \tau_1^{s_0}$ of $G \setminus (K^* \setminus \{u, w\})$. Applying (†), we have the following:

$$\mathcal{Z}_{K^*}(s_0, \tau_0^{s_0}(w), \bullet) \subseteq \{c, s_0\}$$

$$\mathcal{Z}_{K^*}(s_1, \tau_1^{s_0}(w), \bullet) \subseteq \{c, s_0\}$$

Let $h := \tau_1^{s_0}(w)$. Since $\tau_1^{s_0}$ is a proper L-coloring of $G \setminus (K^* \setminus \{w, u\})$, we have $h \notin \{s_0, s_1\}$, as $\tau_1^{s_0}(y) = s_0$ and $\tau_1^{s_0}(u) = s_1$. Furthermore, since K^* is not a triangle, we have $h \in \bigcap_{k=1}^t L_\phi(v_k)$, or else, by Proposition 1.4.4, we have $s_1 \in \mathcal{Z}_{K^*}(s_1, h, \bullet)$, contradicting the containment above. Applying Claim 10.6.15, we have $\{s_0, s_1, h\} \subseteq L(v_k)$ for each $k = 1, \cdots, t$.

Subclaim 10.6.18. *t is odd.*

Proof: Suppose toward a contradiction that t is even. Since $s_1 \in L_\phi(x) \setminus \{c\}$, we now extend $\phi \cup \psi_0$ to an L-coloring σ^* of dom$(\phi \cup \psi_0) \cup \{x, y\}$ by coloring x, y with the respective colors s_0, s_1. Since $s_0, s_1 \neq c$, σ^* is a proper L-coloring of its domain. By assumption, $[H, \sigma^*]$ is not a (C, z)-opener. Since u, y are colored with the same color, we have $|L_{\sigma^*}(y)| \geq 3$, so the only condition which is violated is the inertness condition. That is, there is an extension of σ^* to an L-coloring τ of $G \setminus (H \setminus \text{dom}(\sigma^*))$ such that τ does not extend to L-color $H \setminus \text{dom}(\sigma^*)$. By our construction of σ^*, the set $\text{Sh}_2(x'Qu)$ is (L, σ^*) inert, so τ extends to an L-coloring τ^* of $G \setminus (K \setminus \{u, w, x\})$, where the principal path uwx of K^* is colored with $(s_0, \tau^*(w), s_1)$. Thus, we have $\tau^*(w) \notin \{s_0, s_1\}$. We now extend this L_ϕ-coloring of uwx to an L_ϕ-coloring of K^* by coloring each of $v_1, v_3, \cdots, v_{t-1}$ with s_1. This leaves a color for each of $v_2, v_4, \cdots, v_t$, since each of $v_2, v_4, \cdots, v_t$ is adjacent to two vertices colored with s_1. But this shows that τ^* extends to an L-coloring of G, which is false. ■

Since t is odd, we now extend the L_ϕ-coloring (s_1, h) of the edge uw to an L_ϕ-coloring of K^* in the following way. We color each of $u_1, u_3, \cdots, u_t$ with s_0 and color x with s_1, which leaves a color for each of $u_2, \cdots, u_{t-1}$, as each of these vertices has two neighbors of the same color. But then we have $s_1 \in \mathcal{Z}_{K^*}(s_1, h, \bullet)$, contradicting the fact that $\mathcal{Z}_{K^*}(s_1, \tau_1^{s_0}(w), \bullet) \subseteq \{c, s_0\}$. This completes the proof of Claim 10.6.16. ■

Now we have enough to finish the proof of Lemma 10.6.4. It follows from Theorem 1.5.10 that one of ψ_0, ψ_1 extends to an element of Link(Q) which uses a color other than c on x, thus, there is a $\sigma \in$ Link(Q) using one of s_0, s_1 on u, where $\sigma(x) \neq c$ and $\sigma(x') = c$.

Suppose without loss of generality that σ is an extension of ψ_0. We note that $|L_{\sigma\cup\phi}(w)| = |L_{\sigma\cup\phi}(y)| = 3$ and $L_{\sigma\cup\phi}(w) = L_{\sigma\cup\phi}(y)$. If one of these conditions does not hold, then there exists an extension of $\sigma \cup \phi$ to an L-coloring σ^* of dom$(\sigma \cup \phi) \cup \{y\}$ such that $L_{\sigma^*}(w)| \geq 3$, and thus $[H, \sigma^*]$ is a (C, z)-opener, contradicting our assumption. Thus, we indeed have $|L_{\sigma\cup\phi}(w)| = |L_{\sigma\cup\phi}(y)| = 3$ and $L_{\sigma\cup\phi}(w) = L_{\sigma\cup\phi}(y)$. In particular, we have $\sigma(x) \neq s_0$, since $\sigma(u) = s_0$. If $\sigma(x) = s_1$, then we have $L(w) \setminus \{s_0, s_1\} = L(y) \setminus \{c, s_1\}$, and thus $s_1 \in L(y) \setminus \{c\}$, contradicting Claim 10.6.16. Thus, $\sigma(x) \neq s_1$. But then $s_1 \in L_{\sigma\cup\phi}(w)$, and since $L_{\sigma\cup\phi}(w) = L_{\sigma\cup\phi}(y)$, we have $s_1 \in L_{\sigma\cup\phi}(y)$, and thus $s_1 \in L(y) \setminus \{c\}$, again contradicting Claim 10.6.16. This completes the proof of Lemma 10.6.4. □

10.7 The Trickiest Case: Part II

This section consists of the following lone result.

Lemma 10.7.1. *Let $z \in D_2(C^1)$ be a pentagonal vertex and suppose that* Span(z) *is a proper 4-chord of C^1 and that $\tilde{G}_z^{\text{small}} \cap C^1$ is a path of length one. Then there exists a (C, z)-opener.*

Proof. Let Span$(z) = xyzy'x'$ for some $x, x' \in V(C^1)$ and $y, y' \in D_2(C)$. Let e, e' be the two terminal edges of $C^1 \cap \tilde{G}_z^{\text{large}}$, where e is incident to x and e' is incident to x'.

Let wu be the unique edge of $R_e \setminus \{x, y\}$ and let $w'u'$ be the unique edge of $R_{e'} \setminus \{x', y'\}$, where w is an e-obstruction, w' is an e'-obstruction, $u \in V(C^1 - x)$ and $u' \in V(C^1 - x')$. By Lemma 10.4.2, we have $u, u' \notin \{x, x'\}$ and $w \neq w'$. Let $Q := C^1 \cap \tilde{G}_z^{\text{large}} = C^1 - xx'$. Finally, let $Q := C^1 \cap \tilde{G}_z^{\text{large}}$. Now we apply the previous lemma. By Lemma 10.6.4, since there does not exist a (C, z)-opener, one of the following holds.

1) $S_\star = \varnothing$; *OR*

2) There is a lone vertex $u_\star$ of $S_\star$ and a vertex $v^\dagger \in T^{<2}$, where $u_\star$ either lies in $J_e^1 - u$ or $u_\star$ lies in $J_{e'}^1 - u'$. In the former case, $v^\dagger$ separates $u_\star$ from x' on Q, and in the latter case, $v^\dagger$ separates $u_\star$ from x on Q.

Thus, we suppose without loss of generality that either $S_\star = \varnothing$, or, letting $S_\star = \{u_\star\}$, we have $u_\star \in V(J_{e'}^1 - u')$, and there is a $v^\dagger \in T^{<2}$ which, on Q, separates $u_\star$ from x.

In the previous lemma, we didn't color any vertices of $V(\tilde{G}_z^{\text{large}}) \cap D_2(C))$ except for y, y'. The trick to this lemma is that we also color w. Possibly, there is a 3-chord of C^1 with wy as a terminal edge, where the other endpoint of this 3-chord does not lie in J_e^1, but crucially, this 3-chord, if it exists, does not separate z from any vertex of $S_\star$, which was not necessarily the case in the situation of Lemma 10.6.4.

Claim 10.7.2. $ww' \notin E(G)$.

Proof: Suppose toward a contradiction that $ww' \in E(G)$. Thus, $\tilde{G}_z^{\text{large}}$ contains the 3-chord $R := u'w'wu$ of C^1 (possibly $u = u'$ and R is a triangle, i.e not a proper 3-chord of C^1). We now define a subgraph H of G as follows. If $u = u'$, we set H to be the triangle $u'w'w$, and otherwise R is a proper 3-chord of C^1, and we set $H := \tilde{G}_R^{\text{small}}$. Note that H intersects with $J^1_{e'}$ precisely on the edge $w'u'$ and intersects with J^1_e precisely on the edge wu. In particular, by our assumption on $S_\star$, every vertex of H has an L_ϕ-list of size at least three. We also have the following, which is an immediate consequence of Proposition10.2.9.

Subclaim 10.7.3. *For any $\sigma \in \text{Link}(x'Qu)$ and $\tau \in \Phi(\sigma, \{w, w', y'\})$, τ extends to an L_ϕ-coloring of $V(J^1_{e'} \cup H)$.*

We now define a cycle D in G in the following way. Let p, p' be the respective endpoints of the edges $R_e - u$ and $R_{e'} - u'$ and let D be the cycle consisting of the unique (p, p')-path in $\text{Span}(z)$ and the path $pww'p'$. Let $G = K^{\text{small}} \cup K^{\text{large}}$ be the natural D-partition of G, where $C \cup C^1 \subseteq K^{\text{small}}$. For any partial L_ϕ-coloring ψ of K^{small}, let U_ψ be the set of vertices of $K^{\text{large}} \setminus D$ with $L_{\phi\cup\psi}$-lists of size less than three.

Subclaim 10.7.4. $\text{Span}(z)$ *has no chord in G other than xx'.*

Proof: Suppose toward a contradiction that this does not hold. By Observation 10.6.1, there is a chord of the cycle $xyzy'x'x$ in $\tilde{G}^{\text{small}}_z$, and yy' is an edge of $\tilde{G}^{\text{small}}_z$. Consider the following cases.

Case 1: $wy \in E(G)$

In this case, since $yy' \in E(G)$, we have $R_{e'} = u'w'x'$, i.e $y' \notin N(w')$. Thus, $yx' \notin E(G)$, so $y'x \in E(G)$. Applying Corollary 10.2.5, let $\sigma \in \text{Link}(x'Qu)$. Since $u \neq x$, σ is a proper L_ϕ-coloring of its domain. Since $|L_{\phi\cup\sigma}(w')| \geq 3$ and $|L_{\phi\cup\sigma}(w)| \geq 4$, let $f \in L_{\phi\cup\sigma}(w)$ with $|L_{\phi\cup\sigma}(w') \setminus \{f\}| \geq 3$. Since $w'x' \in E(G)$, we have $wx \notin E(G)$, or else $xx'w'w$ is a separating cycle of length 4. Thus, by B) of Proposition 10.5.7, σ extends to an L_ϕ-coloring σ^* of $\text{dom}(\sigma) \cup V(J^1_e)$ using f on w.

Now, for any $\psi \in \Phi(\sigma^*, \{y', z\})$, it follows from Subclaim 10.7.3 that $V(K^{\text{small}})$ is $(L, \phi \cup \psi)$-inert in G, and furthermore, since $|L_{\phi\cup\psi}(w')| \geq 3$, we have $U_\psi \neq \varnothing$, otherwise $[K^{\text{small}} \setminus \{w'\}, \phi \cup \psi]$ is a (C, z)-opener, contradicting our assumption that no such pair exists. For any such ψ and any $w^\dagger \in U_\psi$, $w^\dagger$ has at least three neighbors in $\{w, y, z, y'\}$. Since w', y have no common neighbor outside of $\text{Span}(z)$, we have $N(w^\dagger) \cap \text{dom}(\phi \cup \psi) = \{w, y, z\}$, or else we contradict 1) of Proposition 10.6.2. In particular, there exists a $w^\dagger$ such that $\{w^\dagger\} = U_\psi$ for each $\psi \in \Phi(\sigma^*, \{y', z\})$. On the other hand, since $|L_{\phi\cup\sigma^*}(z)| \geq 4$, there is a $\psi \in \Phi(\sigma^*, \{y', z\})$ such that $|L_{\phi\cup\psi}(w^\dagger)| \geq 3$, so we have a contradiction.

Case 2: $wy \notin E(G)$

In this case, we have $R_e = uwx$. Furthermore $x' \notin N(w')$, or else $xx'w'w$ is a separating 4-cycle in G. Thus, $R_{e'} = u'w'y'$. Furthermore, we have $S_\star \neq \varnothing$. To see this, note that if $S_\star = \varnothing$, then J^1_e and $J^1_{e'}$ are symmetric, and, applying the argument of the previous case with the roles of e, e' interchanged, we have $w'y' \notin E(G)$, which is false. Furthermore, $xy' \notin E(\tilde{G}^{\text{small}}_z)$, or else $ww'y'x$ is a separating 4-cycle in G. Thus, by Observation 10.6.1, $\tilde{G}^{\text{small}}_z$ consists of $\text{Span}(z)$ and the edges $\{yy', x'y, xx'\}$. Since $S_\star \neq \varnothing$, we have by the assumption of Lemma 10.7.1 that there is a vertex of $T^{<2}$, which, on Q, separates x from the lone vertex of $S_\star$. By 2) of Proposition 10.2.4, there is a $\sigma \in \text{Link}(Q)$ with $\sigma(x) \neq \sigma(x')$, so σ is a proper L_ϕ-coloring of its domain in Q.

Since $|L_{\phi\cup\sigma}(y)| \geq 4$, and $|L_{\phi\cup\sigma}(z)| \geq 5$, and $xy' \notin E(G)$, there is a $\tau \in \Phi(\sigma, \{y, z\})$ such that $|L_{\phi\cup\tau}(y')| \geq 3$. By Observation 10.7.3, $V(K^{\text{small}}) \setminus \{y', w, w'\}$ is $(L, \phi \cup \tau)$-inert in G. By Proposition 10.2.8, no vertex of K^{large} has more than two neighbors among $\{x, y, z\}$. Since $wy \notin E(G)$, each of w, w' has an $L_{\phi\cup\tau}$-list of size at least three, so $[K^{\text{small}} \setminus \{w, w', y'\}, \phi \cup \tau]$ is a (Cz)-opener, contradicting our assumption. ■

Since $\text{Span}(z)$ has no chord in G other than xx', it follows from Lemma 10.6.3 that $S_\star = \varnothing$. At least one of the edges $wx, w'x'$ does not lie in $E(G)$, or else $ww'x'x$ is a separating 4-cycle. Since $S_\star = \varnothing$, the two sides of Q are symmetric, so suppose without loss of generality that $w'x' \notin E(G)$.

Since $w'x' \notin E(G)$, we have $R_{e'} = u'w'y'$. Since Span(z) has no chord in G except for xx', we have $N(y') \cap V(C^1) = \{x'\}$. We now fix a color $c \in L_\phi(u')$. By Theorem 1.6.1, since $N(y') \cap V(C^1) = \{x'\}$ and $S_\star = \varnothing$, there is a color $f \in L_\phi(w')$, where $f \neq c$ if $u'x' \in E(\tilde{G}^{\text{small}}_{x'y'w'u'})$, and any L_ϕ-coloring of $x'y'w'u'$ using c, f on the repsective vertices u', x' extends to an L_ϕ-coloring of $\tilde{G}^{\text{small}}_{x'y'w'u'})$. Likewise, since $N(w) \cap V(C^1 \cap \tilde{G}^{\text{small}}_R) = \{u\}$ by definition of R, it again follows from Theorem 1.6.1 that there is a $d \in L_\phi(u)$, where $d \neq c$ if $u'u \in E(\tilde{G}^{\text{small}}_R)$, such that any L_ϕ-coloring of R using c, d on the respective vertices u', u extends to an L_ϕ-coloring of $\tilde{G}^{\text{small}}_R$. Thus, let σ be the L_ϕ-coloring of $\{x', u', u\}$ coloring these vertices with the respective colors f, c, d.

Subclaim 10.7.5. $wy \in E(G)$.

Proof: Suppose not. In this case, $R_e = uwx$. Since Span(z) has no chord in G except for xx', we have $|L_{\phi\cup\sigma^*}(y)| \geq 4$. Since $|L_{\phi\cup\sigma}(x)| \geq 2$, there is a color $d^* \in L_{\phi\cup\sigma}(w)$ such that $|L_{\phi\cup\sigma}(x) \setminus \{d^*\}| \geq 2$. Thus, by Observation 1.4.2, σ extends to an L_ϕ-coloring σ^* of $\{x', u'\} \cup V(J^1_e)$.

Since Span(z) has no chord in G except for xx', we have $|L_{\phi\cup\sigma^*}(y)| \geq 4$. Thus, there an extension of σ^* to an L_ϕ-coloring τ of dom$(\sigma^*) \cup \{y\}$ such that either no vertex of $K^{\text{large}} \setminus D$ is adjacent to all three of w, x, y, or, if such a vertex exists, then it has an $L_{\phi\cup\tau}$-list of size at least three. In particular, since any such vertex is unique, we have $U_\tau = \varnothing$. For any extension of τ to an L_ϕ-coloring τ^* of dom$(\tau) \cup \{z\}$, we have $|L_{\phi\cup\tau^*}(y')| \geq 3$, since there is no chord of Span(z) other than xx'. Likewise, since $x' \notin N(w')$, we have $N(w') \cap \text{dom}(\phi \cup \tau^*) = \{u', w\}$, and thus $|L_{\phi\cup\tau^*}(w')| \geq 3$.

Since $|L_{\phi\cup\tau}(z)| \geq 4$, it follows from Corollary 1.3.6 that there exists a $\tau^* \in \Phi(\tau, z)$ such that $V(\tilde{G}^{\text{small}}_z) \setminus V(\text{Span}(z))$ is $L_{\phi\cup\tau^*}$-inert in G, and thus, by our construction of σ, we get that $V(K^{\text{small}}) \setminus \{w', y'\}$ is $(L, \phi\cup\tau^*)$-inert in G and thus $[K^{\text{small}} \setminus \{w', y'\}, \phi \cup \tau^*]$ is a (C, z)-opener, contradicting our assumption. ■

Since $wy \in E(G)$, we have $R_e = uwy$.

Subclaim 10.7.6. *For any $\tau \in \Phi(\sigma, V(J^1_e) \cup \{z\})$, either $U_\tau \neq \varnothing$ or $V(\tilde{G} - y')$ is not $(L, \phi \cup \tau)$-inert in G.*

Proof: Suppose toward a contradiction that $U_\tau = \varnothing$ and $V(\tilde{G} - y')$ is $(L, \phi \cup \tau)$-inert in G. By Subclaim 10.7.4, there is no chord of Span(z) other than xx'. Thus, we have $|L_{\phi\cup\tau}(y')| \geq 3$. Since y' is uncolored, we have $|L_{\phi\cup\tau}(w')| \geq 3$, and since $U_\tau = \varnothing$, the pair $[K^{\text{small}} \setminus \{w', y'\}, \phi \cup \tau]$ is a (C, z)-opener, contradicting our assumption. ■

Now we return to the proof of Claim 10.7.2. Consider the following cases:

Case 1: $\tilde{G}^{\text{small}}_z$ is not a wheel

Since $|L_\phi(x) \setminus \{\sigma(x')\}| \geq 2$, it follows that σ extends to an L_ϕ-coloring σ^* of $V(J^1_e) \cup \{x', u'\}$. Since $|L_{\phi\cup\sigma^*}(z)| \geq 4$, there is a $\tau \in \Phi(\sigma^*, z)$ such that, if there is a vertex $w^\dagger$ of $K^{\text{large}} \setminus D$ adjacent to all three of w, y, z, then $|L_{\phi\cup\tau}(z)| \geq 3$. Thus, $U_\tau = \varnothing$. By Theorem 1.3.5, since $\tilde{G}^{\text{small}}_z$ is not a wheel, $V(\tilde{G}^{\text{small}}_z - y')$ is $(L, \phi \cup \tau)$-inert in G, contradicting Subclaim 10.7.6.

Case 2: $\tilde{G}^{\text{small}}_z$ is a wheel

In this case, there is a lone vertex $x^\dagger$ adjacent to all five vertives of Span(z), where $V(\tilde{G}^{\text{small}}_z) = \{x^\dagger\} \cup V(\text{Span}(z))$. We break this into two subcases.

Subcase 2.1 $L_{\phi\cup\sigma}(x) \subseteq L(x^\dagger)$

In this case, since $|L_{\phi\cup\sigma}(x)| \geq 2$ and $|L(x^{\dagger}) \setminus \{\sigma(x')\}| \geq 4$, there is a color $g \in L(y)$ such that $|L_{\phi\cup\sigma}(x) \setminus \{g\}| \geq 2$ and $|L_{\phi\cup\sigma}(x^{\dagger}) \setminus \{g\}| \geq 4$. Since $|L_{\phi\cup\sigma}(x) \setminus \{g\}| \geq 2$, there is an extension of σ to an L_{ϕ}-coloring σ^* of $\text{dom}(\sigma) \cup V(J_e^1)$ such that $\sigma^*(y) = g$. By our choice of g, we get that $\{x^{\dagger}\}$ is $(L, \phi \cup \sigma^*)$-inert in G.

As above, since $|L_{\phi\cup\sigma^*}(z)| \geq 4$, there is a $\tau \in \Phi(\sigma^*, z)$ such that, if there is a vertex $w^{\dagger}$ of $K^{\text{large}} \setminus D$ adjacent to all three of w, y, z, then $|L_{\phi\cup\tau}(w^{\dagger})| \geq 3$. Thus, $U_{\tau} = \varnothing$. Since $V(\tilde{G}_z^{\text{small}} - y')$ is $(L, \phi \cup \tau)$-inert in G, we contradict Subclaim 10.7.6.

Subcase 2.2 $L_{\phi\cup\sigma}(x) \not\subseteq L(x^{\dagger})$

In this case, there is a $\sigma' \in \Phi(\sigma, x)$ with $|L_{\phi\cup\sigma'}(x^{\dagger})| \geq 4$. As Span$(z)$ has no chord other than xx', we have $x'y \notin E(G)$, so it follows from 2) of Proposition 1.5.1 that σ' extends to an L_{ϕ}-coloring σ^* of $\text{dom}(\sigma') \cup V(J_e^1)$. Again, since Span$(z)$ has no chord other than xx', there is a $\tau \in \Phi(\sigma^*, z)$ such that, if there is a vertex $w^{\dagger}$ of $K^{\text{large}} \setminus D$ adjacent to all three of w, y, z, then $|L_{\phi\cup\tau}(w^{\dagger})| \geq 3$. Thus, $U_{\tau} = \varnothing$. By our choice of $\sigma'(x)$, we get that $V(\tilde{G}_z^{\text{small}} - y')$ is $(L, \phi \cup \tau)$-inert in G, contradicting Subclaim 10.7.6. This completes the proof of Claim 10.7.2. ■

We now make the following definition.

Definition 10.7.7. A partial L_{ϕ}-coloring τ of $V(J_e^1) \cup \{x', y, z\}$ is called an *anchor* if the following hold.

1) $V(\tilde{G}_z^{\text{small}} \cup J_e^1) \setminus \{y'\}$ is $(L, \phi \cup \tau)$-inert*; AND*
2) $|L_{\phi\cup\tau}(y')| \geq 3$ and every vertex of $N(w) \setminus \text{dom}(\phi \cup \tau)$ has an $L_{\phi\cup\tau}$-list of size at least three.

We now have the following facts.

Claim 10.7.8. *Let σ be a partial L_{ϕ}-coloring of $V(x'Qu)$, and let τ be an anchor such that, for each $v \in \{x', u\} \cap \text{dom}(\sigma)$, we have $\sigma(v) = \tau(v)$. Then the following hold.*

1) *$\sigma \cup \tau$ is a proper L_{ϕ}-coloring of its domain in G; AND*
2) *Letting $\psi^* := \phi \cup \sigma \cup \tau$ and $v^{\star} \in V(\tilde{G}_z^{\text{large}}) \setminus \text{dom}(\psi^*)$, if $v^{\star}$ has a neighbor in $\{y, z\}$, then $|L_{\psi^*}(v^{\star})| \geq 3$.*

<u>Proof:</u> Let $\hat{v}$ be the unique element of Q adjacent to x'. Since $\text{dom}(\tau) \subseteq V(J_e^1) \cup \{x', z, y\}$ and C^1 is a chordless cycle, it follows that any edge of G with one endpoint in $V(x'Qu) \setminus \text{dom}(\tau)$ and the other endpoint in $\text{dom}(\tau)$ lies in $\{x'\hat{v}, wu\}$. Thus, $\sigma \cup \tau$ is indeed a proper L_{ϕ}-coloring of its domain in G. This proves 1).

Now we prove 2). Let $v^{\star} \in V(\tilde{G}_z^{\text{large}}) \setminus \text{dom}(\psi^*)$, where $v^{\star}$ has a neighbor in $\{y, z\}$. Suppose that $|L_{\psi^*}(v^{\star})| < 3$. Since $N(y') \cap \text{dom}(\psi^*) \subseteq \text{dom}(\tau)$ and τ is an anchor, we have $|L_{\psi^*}(y')| \geq 3$, so $v^{\star} \neq y'$. Since τ is an anchor, it follows that $v^{\star}$ has a neighbor $v^{\star\star} \in \text{dom}(\phi \cup \sigma) \setminus \{x', u\}$. Suppose first that $v^{\star} \in N(z)$. Then $v^{\star} \notin B_1(C)$, so $v^{\star\star} \in \text{dom}(\sigma) \setminus \{x', u'\}$, contradicting Proposition 10.2.8. Thus, we have $z \notin N(v^{\star})$, and $y \in N(v^{\star})$. Furthermore, $v^{\star}$ has at least two neighbors in $\text{dom}(\sigma)$, so $v^{\star} \in D_2(C)$. But then $v^{\star}$ is an e-obstruction, and every e-obstruction lies in $V(J_e^1)$. Since $V(J_e^1) \cap \text{dom}(\psi^*) \subseteq \text{dom}(\tau)$ and $v^{\star} \notin \text{dom}(\tau)$, we have a contradiction. ■

We now have the following.

Claim 10.7.9. *There exists an L_{ϕ}-coloring σ of $\{x', u\}$ which does not extend to an anchor.*

<u>Proof:</u> We first set $B := \{v \in D_2(C) \setminus V(J_e^1) : |N(v) \cap V(x'Qu)| \geq 2 \text{ and } w \in N(v)\}$. Suppose toward a contradiction that every L_{ϕ}-coloring of $\{x', u\}$ extends to an anchor.

Subclaim 10.7.10. $B \neq \varnothing$.

Proof: Suppose that $B = \varnothing$. By 1) of Corollary 10.2.5, there is a $\sigma \in \text{Link}(x'Qu)$. By assumption, there is an anchor τ using the colors $\sigma(x'), \sigma(u)$ on the respective vertices x', u. By 1) of Claim 10.7.8, $\sigma \cup \tau$ is a proper L_ϕ-coloring of its domain.Thus, let $\psi^* := \sigma \cup \tau \cup \phi$.

Let H be the subgraph of G induced by $(V(\tilde{G}_z^{\text{small}} \cup J_e^1) \setminus \{y'\}) \cup \text{Sh}_2(x'Qu) \cup \text{dom}(\phi \cup \sigma)$. By our construction of ψ^*, $V(H) \setminus \text{dom}(\psi^*)$ is L_{ψ^*}-inert. By assumption, $[H, \psi^*]$ is not a (C, z)-opener, so there exists a $v \in D_1(H)$ with $|L_{\psi^*}(v)| < 3$. By 2) of Claim 10.7.8, $v \notin N(y) \cup N(z)$. In particular, $v \neq y'$, and v has a neighbor in w.

By our construction of J_e^1, we have $N(w) \cap V(C^1) \subseteq V(J_e^1)$, so $v \notin B_1(C)$. Since $|L_{\psi^*}(v)| < 3$ and $y \notin N(v)$, it follows that v has a neighbor in $\text{dom}(\sigma^*)$ and $v \in D_2(C)$. Furthermore, $z \notin N(v)$, or else we contradict Proposition 10.2.8. Since $|L_{\psi^*}(v)| < 3$, it follows that v has at least two neighbors in $V(x'Qu)$, so $v \in B$, contradicting our assumption that $B = \varnothing$. ■

Applying Proposition 10.5.4, since $B \neq \varnothing$, let $R := xww^*v$, where w^*v is the e-wall of B. By Claim 10.7.2, we have $w^* \neq w$. Note that $\tilde{G}_R^{\text{small}}$ contains a cyclic facial subgraph $F := vQxww^*v$, where F contains all the vertices of $\tilde{G}_R^{\text{small}}$ with L_ϕ-lists of size less than five. Since uw is a terminal edge of R and $N(w) \cap V(Q) \subseteq V(J_e^1)$, and thus $N(w) \cap V(F) = \{w^*, u\}$.

Let H be the subgraph of G induced by $(V(\tilde{G}_z^{\text{small}} \cup J_e^1) - y') \cup \text{Sh}_2(x'Qv) \cup V(\tilde{G}_R^{\text{small}} - w^*)\text{dom}(\phi \cup \sigma)$. By 1) of Corollary 10.2.5, there is a $\sigma \in \text{Link}(x'Qv)$. Since $u \neq x$, we have $|L_{\phi \cup \sigma}(u)| \geq 3$. Furthermore, each vertex of $\tilde{G}_R^{\text{small}} \setminus \{v\}$ has an L_ϕ-list of size at least three, since $S_\star \subseteq V(J_{e'}^1 - u')$ by the assumption of Lemma 10.7.1. Now we apply the work of Section 1.6. Since $N(w) \cap V(F) = \{w^*, u\}$, it follows from Theorem 1.6.1 there is a $d \in L_{\phi \cup \sigma}(u)$, where $d \neq \sigma(v)$ if $vu \in E(Q)$, such that any L_ϕ-coloring of R using $\sigma(v), d$ on the respective vertices v, u extends to an L_ϕ-coloring of $\tilde{G}_R^{\text{small}}$.

By assumption, there is an anchor τ using $d, \sigma(x')$ on the respective vertices u, x'. By Claim 10.7.8, the union $\psi^* := \phi \cup \sigma \cup \tau$ is a proper L-coloring of its domain. By our construction of ψ^*, $V(H) \setminus \text{dom}(\psi^*)$ is L_{ψ^*}-inert. By assumption, $[H, \psi^*]$ is not a (C, z)-opener, so there exists a $v^\star \in D_1(H)$ with $|L_{\psi^*}(v^\star)| < 3$. Since $N(w^*) \cap \text{dom}(\psi^*) = \{v, w\}$, we have $|L_{\psi^*}(w^*)| \geq 3$ Thus, we have $v^\star \neq w^*$. Since $B \setminus \{w^*\} \subseteq V(\tilde{G}_R^{\text{small}} - w^*)$, it follows that $v^\star \notin B$. Since $v^\star$ has at least three neighbors in $\text{dom}(\psi^*)$, we have $v^\star \in N(y) \cup N(z)$, contradicting 2) of Claim 10.7.8. ■

Claim 10.7.11. *If at most one of x, y is adjacent to w, then every L_ϕ-coloring of $\{u, x, y\}$ extends to an L_ϕ-coloring of $V(J_e^1)$.*

Proof: Suppose not. By A) of Proposition 10.5.7, we have $R_e = uwyx$, and there is a vertex p of $J_e^1 \setminus R_e$ adjacent to all the vertices of the cycle $D := R_e + (C^1 \cap J_e^1)$. That is, J_e^1 is a wheel with central vertex p. In particular, $N(w) \cap V(C^1) = \{u\}$.

Subclaim 10.7.12. *For any $\sigma \in \text{Link}(Q)$, we have $\sigma(x) = \sigma(x')$. In particular, $S_\star = \varnothing$, and furthermore $L_\phi(x) = L_\phi(x')$ and $|L_\phi(x)| = |L_\phi(x')| = 3$.*

Proof: Suppose toward a contradiction that there is a $\sigma \in \text{Link}(Q)$ with $\sigma(x) \neq \sigma(x')$. Thus, σ is a proper L_ϕ-coloring of its domain in $\tilde{G}$. By a) of Proposition 10.6.2 10.6.2, there is an extension of σ to an L_ϕ-coloring τ of $\text{dom}(\sigma) \cup V(\text{Span}(z) - y')$ such that $|L_{\phi \cup \tau}(y')| \geq 3$ and $V(\tilde{G}_z^{\text{small}}) \setminus V(\text{Span}(z))$ is $(L, \phi \cup \tau)$-inert in G. Let H be the subgraph of G induced by $\text{dom}(\phi \cup \tau) \cup \text{Sh}_2(Q) \cup V(J_e^1 - w) \cup V(\tilde{G}_z^{\text{small}} - y')$. By Proposition 10.2.9,

$V(J_e^1-w)$ is $(L,\phi\cup\tau)$-inert in G, since $y\in\text{dom}(\phi\cup\tau)$. Since $N(w)\cap V(C^1)=\{u\}$, we have $|L_{\phi\cup\tau}(y)|\geq 3$. Since $|L_{\phi\cup\tau}(y')\geq 3$ as well, the pair $[H,\phi\cup\tau]$ is a (C,z)-opener, contradicting our assumption.

We conclude that there is no element σ of Link(C^1-xx') with $\sigma(x)\neq\sigma(x')$,and since $S_\star=\varnothing$, we have $L_\phi(x)=L_\phi(x')$. and $|L_\phi(x)|=|L_\phi(x')|=3$. ■

Now let H^* be the subgraph of G induced by $V(C\cup C^1)\cup\text{Sh}_2(uQx')\cup V(\tilde{G}_z^{\text{small}}-y')\cup V(J_e^1-w)$. By Subclaim 10.7.12, we have $L_\phi(x)=L_\phi(x')$. and $|L_\phi(x)|=|L_\phi(x')|=3$. Thus, there is a set c_0,c_1,c_2 of three colors such that $\{c_0,c_1,c_2\}=L_\phi(x)\cap L_\phi(x')\cap L_\phi(y)$. Since $|L_\phi(y)|\geq 5$, there is a set $\{a_0,a_1\}$ of two colors with $\{a_0,a_1\}\subseteq L_\phi(y)\setminus\{c_0,c_1,c_2\}$.

Subclaim 10.7.13. $xy'\in E(\tilde{G}_z^{\text{small}})$

Proof: Suppose not. Since every chord of Span(z) lies in $\tilde{G}_z^{\text{small}}$, we have $xy'\notin E(G)$. Let $\hat{v}$ be the unique neighbor of u on the path uQx. Since G is short-separation-free, we have $\hat{v}\neq x$. Consider the following cases:

Case 1: Either $L_\phi(x')\not\subseteq L_\phi(y)$ or $V(\tilde{G}_z^{\text{small}})\neq V(\text{Span}(z))$

In this case, there is a $c\in L_\phi(x')$ such that either $c\notin L_\phi(y)$ or $V(\tilde{G}_z^{\text{small}})\neq V(\text{Span}(z))$. Since $S_\star=\varnothing$. Since $S_\star=\varnothing$, it follows from Theorem 1.7.5 that there is a $\sigma\in\text{Link}(uQx')$ with $\sigma(x')=c$.

Since $|L_{\phi\cup\sigma}(p)|\geq 4$, there is a $d\in L_{\phi\cup\sigma}(p)$ such that $|L_\phi(\hat{v})\setminus\{d\}|\geq 3$. Since $|L_{\phi\cup\sigma}(x)|\geq 2$, it follows that there is a $\sigma'\in\Phi(\sigma,V(J_e^1)\setminus\{w\})$. Since we have either $\sigma(x')\notin L_\phi(y)$ or $x'y\notin E(G)$, we have $|L_{\phi\cup\sigma'}(y)|\geq 4$, so there is a $\sigma^*\in\Phi(y)$ such that $|L_{\phi\cup\sigma^*}(w)|\geq 3$.

By b) of Proposition 10.6.2, there is a $\tau\in\Phi(\sigma^*,z)$ such that $|L_{\phi\cup\tau}(y')|\geq 3$ and $V(\tilde{G}_z^{\text{small}}-y')$ is $(L,\phi\cup\tau)$-inert. Since $z\notin N(w)$, we have $|L_{\phi\cup\tau}(w)|\geq 3$. Since J_e^1-w is already colored, $V(H^*)\setminus\text{dom}(\phi\cup\tau)$ is $(L,\phi\cup\tau)$-inert in G and thus $[H^*,\phi\cup\tau]$ is a (C,z)-opener, contradicting our assumption.

Case 2: $L_\phi(x')\subseteq L_\phi(y)$ and $V(\tilde{G}_z^{\text{small}})=V(\text{Span}(z))$

In this case, since $xy'\notin E(G)$, we have $E(\tilde{G}_z^{\text{small}})=E(\text{Span}(z))\cup\{xx',yy',x'y\}$. Since $S_\star=\varnothing$ and no element of Link(uQx') uses a color of $\{a_0,a_1\}$ on x', it follows that, for each $d\in L_\phi(u)$, there is a partial L_ϕ-coloring σ_i^d of $V(uQx')\cup\{y\}$, where $\sigma_i^d(y)=a_i$ and the restriction of σ to $V(uQx')$ is an element of Link(uQx') which uses d on u.

For each $d\in L_\phi(u)$ and $i=0,1$, since $xy'\notin E(G)$, we have $|L_{\phi\cup\sigma_i^d}(y')|\geq 3$ and $|L_{\phi\cup\sigma_i^d}(z)|\geq 4$, so there exists a $\tau_i^d\in\Phi(\sigma_i^d,z)$ such that $|L_{\phi\cup\tau_i^d}(z)|\geq 3$. We have $|L_{\phi\cup\tau_i^d}(w)|\geq 3$ as well. By assumption, for each $d\in L_\phi(u)$ and $i=0,1$, the pair $[H^*,\phi\cup\tau_i^d]$ is not a (C,z)-opener, and thus the inertness condition is violated. since $V(\tilde{G}_z^{\text{small}})=V(\text{Span}(z))$, it follows that, for each $d\in L_\phi(u)$ and $i=0,1$, there is a $\zeta_i^d\in\Phi(\tau_i^d,w)$ which does not extend to L_ϕ-color $V(J_e^1)$.

Since $a_0,a_1\notin L_\phi(x)$, it follows that, for each $i=0,1$ and $d\in L_\phi(u)$, we have $|L_{\phi\cup\zeta_i^d}(p)|=2$ and $L_{\phi\cup\zeta_i^d}(p)\subseteq\{c_0,c_1,c_2\}$, or else we contradict Observation 1.4.2. Furthermore, we have $d\in L_\phi(\hat{v})$ and $L_{\phi\cup\zeta_i^d}(p)\subseteq L_\phi(\hat{v})$ and $|L_\phi(\hat{v})|=3$. Thus, we have $|L_\phi(p)|=5$ and $\{a_0,a_1\}\cup L_\phi(u)=L_\phi(p)$ as a disjoint union. Furthermore, since $d\in L_\phi(u)$, suppose without loss of generality that, for some $i\in\{0,1\}$ and $d\in L_\phi(u)$, we have $\{c_0,c_1\}\subseteq L_\phi(p)\cap L_\phi(\hat{v})$.

By Theorem 1.7.5, since $S_\star=\varnothing$, there is a $\psi\in\text{Link}(uQx')$ with $\psi(x')=c_2$. Thus, there is a $\psi'\in\Phi(\psi,\{x,y\})$ which colors the edge xy with the colors c_0,c_1. Since $|L_{\phi\cup\psi'}(y')|\geq 3$ and $|L_{\phi\cup\psi'}(z)|\geq 4$, there is a $\psi^*\in\Phi(\psi',z)$ with $|L_{\phi\cup\psi^*}(y')|\geq 3$.

Since $[H^*, \psi^*]$ is not a (C, z)-opener and $V(\tilde{G}_z^{\text{small}}) = V(\text{Span}(z))$, there exists a $\psi^\dagger \in \Phi(\psi^*, w)$ which does not extend to L_ϕ-color $V(J_e^1)$. Thus, we have $\psi^*(u) \in L_\phi(\hat{v})$. Since $|L_\phi(\hat{v})| = 3$ and $\{c_0, c_1\} \subseteq L_\phi(\hat{v})$, there is a color in $L_{\phi \cup \psi_*}(p) \setminus L_\phi(\hat{v})$. Note that this is true even if $\psi^*(u) \in \{c_0, c_1\}$, since, in that case, we have $|L_{\phi\cup\psi^*}(p)| \geq 2$. In any case, ψ^* extneds to L_ϕ-color $V(J_e^1)$, a contradiction. ■

It follows from Subclaim 10.7.13 that $V(\tilde{G}_z^{\text{small}}) = V(\text{Span}(z))$ and $E(\tilde{G}_z^{\text{small}}) = E(\text{Span}(z)) \cup \{xx', yy', xy'\}$. Let $\hat{x}$ be t he unique neighbor of x on the path $C^1 \cap J_e^1$. Since p is the central vertex of a wheel and G is short-separation-free, we have $\hat{x} \neq u$.

Subclaim 10.7.14. $L_\phi(\hat{x}) = \{c_0, c_1, c_2\}$

Proof: Suppose not and suppose without loss of generality that $|L_\phi(\hat{x}) \setminus \{c_0\}| \geq 3$. Since $S_\star = \varnothing$, there is a $\sigma \in \text{Link}(uQx')$ with $\sigma(x') \in \{c_1, c_2\}$, so there is a $\sigma^* \in \Phi(\sigma, x)$ with $\sigma^*(x) = c_0$. By a) of Proposition 10.6.2, there is a $\tau \in \Phi(\sigma^*, \{y, z\})$ such that $|L_{\phi\cup\tau}(y')| \geq 3$. Note that $|L_{\phi\cup\tau}(w)| \geq 3$ as well. By our choice of color for x, we get that $V(J_e^1 - w)$ is $(L, \phi \cup \tau)$-inert in G, so $[H^*, \phi \cup \tau]$ is a (C, z)-opener, contradicting our assumption. ■

Now we fix a $\sigma \in \text{Link}(uQx')$. Without loss of generality, let $\sigma(x') = c_2$. Thus, we have $L_{\phi\cup\sigma}(x') = \{c_0, c_1\}$. Since $|L_{\phi\cup\sigma}(p)| \geq 4$, let $d_0, d_1 \in \setminus\{c_0, c_1\}$. By Subclaim 10.7.14, we have $d_0, d_1 \notin L_\phi(\hat{x})$.

Now, since $|L_{\phi\cup\sigma}(y)| \geq 5$, there is a $\sigma^* \in \Phi(\sigma, y)$ such that $\sigma^*(y) \notin \{c_0, c_1, d_0, d_1\}$. The idea here is to leave $J_e^1 \setminus \{u, y\}$ uncolored. Since $|L_{\phi\cup\sigma^*}(z)| \geq 4$, there is a $\tau \in \Phi(\sigma^*, z)$ such that $|L_{\phi\cup\tau}(y')| \geq 3$. By assumption, the pair $[H^*, \phi \cup \tau]$ is not a (C, z)-opener. Since each of w, y' has an $L_{\phi\cup\tau}$-list of size at least three, the inertness condition is violated. Thus, there is a $\tau^* \in \Phi(\tau, \{y', w\})$ which does not extend to L_ϕ-color $V(J_e^1)$. But by our choice of color for y we have $\{d_0, d_1\} \cap L_{\phi\cup\tau^*}(p) \neq \varnothing$, so τ^* does indeed extend to L_ϕ-color $V(J_e^1)$, a contradiction. This completes the proof of Claim 10.7.11. ■

Let U_y be the set of vertices of $V(\tilde{G}_z^{\text{large}}) \setminus V(\text{Span}(z) \cup J_e^1))$ with at least three neighbors among $V(J_e^1 \cup \text{Span}(z)) \setminus \{y'\}$.

Claim 10.7.15. $|U_y| = 1$, *and furthermore, letting* $w_\dagger$ *be the lone vertex of* U_y, *the following hold.*

1) If $wy \in E(G)$ *then* $(N(w_\dagger) \setminus \{y'\}) \cap V(J_e^1 \cup \text{Span}(z)) = \{w, y, z\}$*; AND*

2) If $wy \notin E(G)$ *and* $w_\dagger \in U_y$*, then* $(N(w_\dagger) \setminus \{y'\}) \cap V(J_e^1 \cup \text{Span}(z)) = \{w, x, y\}$.

Proof: Applying Claim 10.7.9, we fix an L_ϕ-coloring σ of $\{x', u\}$ which does not extend to an anchor. We break this into two cases.

Case 1: $wy \notin E(G)$.

Since $wx' \notin E(G)$, we have $|L_{\sigma\cup\phi}(w)| \geq 4$, so there is a a $d \in L_{\sigma\cup\phi}(w) \setminus \{c_0 c_1\}$. Thus, by Observation 1.4.2, there is an extension of σ to an L_ϕ-coloring σ' of $V(J_e^1) \cup \{x'\}$. By Proposition 10.6.2, σ' extends to an L_ϕ-coloring τ of $V(J_e^1) \cup \{x', y, z\}$ such that $|L_{\tau\cup\phi}(y')| \geq 3$ and $V(\tilde{G}_z^{\text{small}}) \setminus V(\text{Span}(z))$ is $(L, \phi \cup \tau)$-inert. By assumption, τ is not an anchor, so there exists a vertex $w_\dagger$ of $N(w) \setminus \text{dom}(\phi \cup \tau)$ such that $|L_{\phi\cup\tau}(w_\dagger)| < 3$. Thus, $w_\dagger$ has at least two neighbors among $\{u, x, y, z, x'\}$. If $w_\dagger$ is adjacent to x', then G contains a 4-cycle $xww_\dagger x'$ which separates z from C, contradicting the fact that $\mathcal{T}$ is a tessellation. Thus, $x' \notin N(w_\dagger)$ and $w_\dagger$ has at least two neighbors among $\{u, x, y, z\}$.

Suppose now that $u \in N(w_\dagger)$. In that case, by definition of Span(z), we have $z \notin N(w_\dagger)$. Furthermore, since w is a maximal e-obstruction, we have $x \notin N(w_\dagger)$, so $w_\dagger$ is adjacent to each of u, w, y, and G contains the 4-cycle $ww_\dagger yx$. Since $wy \notin E(G)$ by assumption, we have $x \in N(w_\dagger)$, which is false, so $u \notin N(w_\dagger)$, and $w_\dagger$ has at least two neighbors among $\{x,y,z\}$. By Proposition 10.2.8, $w_\dagger$ is adjacent to at most one of x,z. Thus, if $z\in N(w_\dagger)$, then $w_\dagger$ is adjacent to w,y,z, and G contains the 4-cycle $ww_\dagger yx$. As above, it follows from our triangulation conditions that $x\in N(w_\dagger)$, contradicting the fact that $w_\dagger\notin N(x)\cap N(z)$. We conclude that there is a vertex $w_\dagger\in U_y$, and furthermore, $w_\dagger$ is the unique vertex of $U_y$since G is $K_{2,3}$-free and any vertex of U_y is adjacent to w,y,z.

Case 2: $wy\in E(G)$.

This case is trickier. Let $c_0,c_1\in L_\phi(x)\setminus\{\sigma(x')\}$. We begin with the following.

Subclaim 10.7.16. *If $U_y=\varnothing$ then the following hold.*

1) $x\in N(w)$ and J_e^1-y is not a triangle; AND

2) $xy'\in E(G)$ and $c_0,c_1\in L_{\phi\cup\sigma}(y')$.

Proof: Suppose that $U_y=\varnothing$ and suppose toward a contradiction that either $N(w)\cap\{x,y\}=\{y\}$ or $J_e^1-y=uxw$. By 2a) of Proposition 10.6.2, there is an extension of σ to an L_ϕ-coloring σ^* of $\{u,x,x',y,z\}$, where $|L_{\sigma^*}(y')|\geq 3$ and $V(\tilde{G}_z^{\text{small}})\setminus V(\text{Span}(z))$ is $(L,\phi\cup\sigma^*)$-inert.

Note that σ^* extends to an L_ϕ-coloring τ of $\text{dom}(\sigma^*)\cup V(J_e^1)$. If $N(w)\cap\{x,y\}=\{y\}$, then this follows from Claim 10.7.11, and if $J_e^1-wy=uxw$, then this just follows from the fact that there is a color left over for w. We have $|L_{\phi\cup\tau}(y')|\geq 3$ and $V(\tilde{G}_z^{\text{small}})\setminus V(\text{Span}(z))$ is $(L,\phi\cup\tau)$-inert. Since $U=\varnothing$, τ is an extension of σ to an anchor, contradicting our choice of σ. Thus, $x\in N(w)$.

Now suppose toward a contradiction that there is a $c\in\{c_0,c_1\}$ such that either $c\notin L_{\phi\cup\sigma}(y')$ or $xy'\notin E(G)$. Since J_e^1-wy is not a triangle and C^1 is an induced subgraph of G, it follows from 2) of Proposition 1.5.1 that there is an extension of σ to an L_ϕ-coloring σ^* of $\{x'\}\cup V(J_e^1)$ using c on x. By 2) of Proposition 10.6.2, since either $c\notin L_{\phi\cup\sigma}(y')$ or $xy'\notin E(G)$, σ^* extends to an anchor, contradicting our choice of σ. ■

We now show that $U_y\neq\varnothing$. Suppose that $U_y=\varnothing$. By Subclaim 10.7.16, we have $x,y\in N(w)$ and $c_0,c_1\in L_{\phi\cup\sigma}(y')$. Furthermore, $\tilde{G}$ consists of Span(z) and the edges xx',yy',xy'. In particular, we have $|L_{\phi\cup\sigma}(y)|\geq 5$, so there is an $f\in L_{\phi\cup\sigma}(y)\setminus\{c_0,c_1\}$ such that $|L_{\phi\cup\sigma}(y')\setminus\{f\}|\geq 4$. Since $|L_{\phi\cup\sigma}(w)|\geq 4$, there is a color $f^*\in L_{\phi\cup\sigma}(w)\setminus\{c_0,c_1,f\}$, and, by Observation 1.4.2, the L_ϕ-coloring $(\sigma(w),f^*)$ of the edge uw extends to an L_ϕ-coloring of J_e^1 using f on y and one of c_0,c_1 on x. But by our choice of f, since $U_y=\varnothing$¡ it follows that σ extends to an anchor, contradicting our choice of σ. Thus, $U_y\neq\varnothing$.

Let $w_\dagger\in U_y$. By definition, $w_\dagger$ has at least three neighbors in $\{w,y,z,x'\}$. If $x'\in N(w_\dagger)$, then, by 3) of Proposition 10.2.8, $w_\dagger$ is not adjacent to z, and since $x'y\notin E(G)$, we have $x'\notin N(w_\dagger)$ and $w_\dagger$ is the unique vertex adjacent to all three of w,y,z. Let τ be an extension of σ to an L_ϕ-coloring of $V(J_e^1)\cup\{x',y,z\}$. Since $U_y=\{w_\dagger\}$, it follows that, if τ satisfies neither 1) nor 2), then τ is an extension of σ to an anchor, contradicting our choice of σ. ■

Applying Claim 10.7.15 and Claim 10.7.9 and , we fix an L_ϕ-coloring σ olf $\{x',u\}$ which does not extend to an anchor, and we fix a vertex $w_\dagger$ such that $U_y=\{w_\dagger\}$. Since $x\notin S_\star$, we fix two colors $c_0,c_1\in L_\phi(x)\setminus\{\sigma(x')\}$. (Possibly J_e^1 is a triangle and $\sigma(u)\in\{c_0,c_1\}$).

Claim 10.7.17. *$E(G)$ contains at most one of wx,wy.*

Proof: Suppose toward a contradiction that $wx,wy\in E(G)$. Applying Claim 10.7.15, $w_\dagger$ is adjacent to each of w,y,z. Let $K:=\tilde{G}_{uwx}^{\text{small}}$. Note that $K=J_e^1-y$.

Subclaim 10.7.18. *G has a chord of* $\text{Span}(z)$ *other than* xx'.

Proof: Suppose not. The trick now is to leave y uncolored. Choosing a color of $L_\phi(w) \setminus \{c_0, c_1, \sigma(u)\}$, it follows from Observation 1.4.2 that there is a $\sigma^* \in \Phi(\sigma, J_e^1 - y)$ using one of c_0, c_1 on x. Since $|L_{\phi\cup\sigma^*}(z)| \geq 5$ and $|L_{\phi\cup\sigma^*}(y)| \geq 3$, there is an extension of σ^* to a $\tau \in \Phi(\sigma^*, z)$ such that $|L_{\phi\cup\tau}(y)| \geq 3$. We claim now that $V(J_e^1 \cup \tilde{G}_z^{\text{small}}) \setminus \{y'\}$ is $(L, \phi\cup\tau)$-inert in G. Applying Corollary 1.3.6, this is immediate if $\tilde{G}_z^{\text{small}}$ is not a wheel with a central vertex adjacent to all five vertices of $xyzy'x'$, so suppose that $\tilde{G}_z^{\text{small}}$ is a wheel with central vertex v^*.

By our choice of color for z, any extension of $\phi\cup\tau$ to an L-coloring of $G \setminus \{v^*, y\}$ also extends to the edge v^*y. Thus, in any case, $V(J_e^1 \cup \tilde{G}_z^{\text{small}}) \setminus \{y'\}$ is $(L, \phi\cup\tau)$-inert in G. Furthermore, $|L_{\phi\cup\tau}(y')| \geq 3$, since $x, y \notin N(y')$. Thus, τ in an anchor, contradicting the fact that σ does not extend to an anchor. ■

We now have the following.

Subclaim 10.7.19. $xy' \in E(G)$

Proof: Suppose not. By Subclaim 10.7.18 G has a chord of $\text{Span}(z)$ other than xx'. Since $xy' \notin E(G)$, it follows from Observation 10.6.1 that $\tilde{G}_z^{\text{small}}$ consists of $\text{Span}(z)$ and the edges $\{xx', yy', x'y\}$. Since $wx \in E(G)$, G contains a 7-wheel in which each vertex of the cycle $wxx'y'zw_\dagger$ is adjacent to y. Consider the following cases.

Case 1: K is a triangle

In this case, we leave w uncolored. Since $x \notin S_\star$, $|L_{\phi\cup\sigma}(x)| \geq 1$. Since $|L_{\phi\cup\sigma}(y)| \geq 4$ and $|L_{\phi\cup\sigma}(z)| \geq 5$, there is an extension of σ to an L_ϕ-coloring τ of $\{x', x, u, y, z\}$ such that $|L_{\phi\cup\tau}(y')| \geq 3$. Since w is uncolored, $|L_{\phi\cup\tau}(w_\dagger)| \geq 3$, so τ is an anchor, contradicting our choice of σ.

Case 2: K is not a triangle.

In this case, we first claim that there exists an $f \in L_{\phi\cup\sigma}(w)$ such that $\{c_0, c_1\} \subseteq \mathcal{Z}_K(\sigma(u), f, \bullet)$. If K is not a broken wheel, then, since C^1 is an induced subgraph of G,it just follows from Theorem 1.5.3 that such an f exists. If K is a broken wheel, then, since K is not a triangle, it follows from Proposition 1.4.4 that such an f exists, or else there is a vertex of $K \setminus \{u, w, x\}$ with an L_ϕ-list of size three which contains both $\{c_0, c_1\}$ and two colors of $L_{\phi\cup\sigma}(w) \setminus \{c_0, c_1\}$, a contradiction.

Let τ be an extension of σ to an L_ϕ-coloring of $\{x', u, w, z\}$ such that $\tau(w) = f$, where f is as above. Since y is uncolored, each of $w_\dagger, y'$ has an $L_{\phi\cup\tau}$-list of size at least three, and, by our choice of τ, $V(\tilde{G}_z^{\text{small}} \cup J_e^1) \setminus \{y'\}$ is $(L, \phi\cup\tau)$-inert in G, so τ is an anchor, contradicting our choice of σ. ■

Since $xy' \in E(G)$, it follows from Observation 10.6.1 that $\tilde{G}_z^{\text{small}}$ consists of $\text{Span}(z)$ and the edges $\{xx', yy', xy'\}$. Thus, G contains a 6-wheel with central vertex y adjacent to all five vertices of the cycle $xww_\dagger zy'$.

Case 1: K is a triangle

In this case, the trick is to leave y uncolored. Since $|L_{\phi\cup\sigma}(x)| \geq 1$, we let σ' be an extension of σ to an L_ϕ-coloring of $\{x', x, u\}$. Since $|L_{\phi\cup\sigma'}(y')| \geq 3$ and $|L_{\phi\cup\sigma}(z)| \geq 5$, let $f_0, f_1 \in L_{\phi\cup\sigma}(z)$, where, for each $i = 0, 1$, $|L_{\phi\cup\sigma'}(y') \setminus \{f_i\}| \geq 3$. Now, since $|L_{\phi\cup\sigma'}(w)| \geq 3$ and $|L_{\phi\cup\sigma'}(y)| \geq 4$, there is an extension of σ' to an L_ϕ-coloring τ of $\{u, x, x', w, z\}$ such that τ uses one of f_0, f_1 on z and such that $|L_{\phi\cup\tau}(y)| \geq 3$. By our choice of τ, we have $|L_{\phi\cup\tau}(y')| \geq 3$ and $\{y\}$ is $(L, \phi\cup\tau)$-inert. Thus, τ is an anchor, contradicting our choice of σ.

Case 2: K is not a triangle

As above, we choose an $f \in L_{\phi\cup\sigma}(w)$ with $\{c_0, c_1\} \subseteq \mathcal{Z}_K(\sigma(u), f, \bullet)$. The trick is to leave $K - \{u, w, y\}$ uncolored. Since $|L_{\phi\cup\sigma}(y) \setminus \{f\}| \geq 4$, there is an $f' \in L_{\phi\cup\sigma}(y) \setminus \{f, c_0, c_1\}$. Let σ^* be an extension of σ to $\{x', u, w, y\}$ obtained by coloring w, y with the respective colors f, f'. Since $|L_{\phi\cup\sigma^*}(z)| \geq 4$, there is an extension of σ^* to an L_ϕ-coloring τ of $\{x', x, u, w, y, z\}$ such that $|L_{\phi\cup\tau}(w_\dagger)| \geq 3$. Since x is uncolored, we have $|L_{\phi\cup\tau}(y')| \geq 3$ as well. By our choice of τ, $V(K) \setminus \{w, y\}$ is $(L, \phi\cup\tau)$-inert in G, so $V(J_e^1 \cup \tilde{G}_z^{\text{small}}) \setminus \{y'\}$ is $(L, \phi\cup\tau)$-inert in G. Thus, τ is an anchor, contradicting our choice of σ. This completes the proof of Claim 10.7.17. ■

Now we have the following.

Claim 10.7.20. *If $wy \notin E(G)$, then $xy' \in E(G)$.*

Proof: Suppose that $wy \notin E(G)$. Thus, w is the lone vertex of $\text{Ob}_e(z)$, and $J_e^1 = \tilde{G}_{uwx}^{\text{small}}$. Furthermore, it follows from Lemma 10.4.2 that w has no neighbor in $\text{Span}(z)$ other than x. Applying Claim 10.7.15, $w_\dagger$ is adjacent to each of w, x, y.

Subclaim 10.7.21. *G has a chord of* Span(z) *other than xx'.*

Proof: Suppose not. Applying Observation 1.4.2, we extend σ to an L_ϕ-coloring σ' of $V(J_e^1) \cup \{x'\}$ by choosing a $d \in L_{\sigma\cup\phi}(w) \setminus \{c_0, c_1\}$. Then $|L_{\phi\cup\sigma'}(w_\dagger)| \geq 3$ and, since $x'y \notin E(G)$, $|L_{\phi\cup\sigma'}(y)| \geq 4$, so there is an extension of σ' to an L_ϕ-coloring σ^* of $\text{dom}(\sigma') \cup \{y\}$ such that $|L_{\sigma^*}(w_\dagger)| \geq 3$.

For any extension τ of σ^* to an L_ϕ-coloring of $\text{dom}(\sigma^*) \cup \{z\}$, we have $|L_{\phi\cup\tau}(y')| \geq 3$, since $\tilde{G}_z^{\text{small}}$ has no chord of $\text{Span}(z)$ other than xx'. By 2b) of Proposition 10.6.2, there is an extension τ of σ^* to an L_ϕ-coloring of $\text{dom}(\sigma^*) \cup \{z\}$ such that $V(\tilde{G}_z^{\text{small}}) \setminus V(\text{Span}(z))$ is $(L, \phi\cup\tau)$-inert, so τ is an anchor, contradicting our choice of σ. ■

Since $\tilde{G}_z^{\text{small}}$ has a chord of $\text{Span}(z)$ other than xx', it follows from Observation 10.6.1 that $\tilde{G}_z^{\text{small}}$ consists of $\text{Span}(z)$ and the edges xx', yy', yx'. We note now the following.

$$\text{For any } \sigma' \in \Phi(\sigma, V(J_e^1)), \text{ we have } |L_{\sigma'\cup\phi}(w_\dagger)| = 2 \tag{‡}$$

To see this, note that if $|L_{\sigma'\cup\phi}(w_\dagger)| \geq 3$, then, since $|L_{\sigma'\cup\phi}(z)| \geq 4$ and $|L_{\sigma'\cup\phi}(y')| \geq 3$, there is an extension of σ' to an $\tau \in \Phi(\sigma, V(J_e^1) \cup \{z\})$ such that each of $y', w_\dagger$ has an $L_{\phi\cup\tau}$-list of size at least three, so τ is an anchor, contradicting our choice of σ.

Recall that we have fixed two colors $c_0, c_1 \in L_\phi(x) \setminus \{\sigma(x')\}$. Since $L_{\phi\cup\sigma}(w)| \geq 4$, let $f_0, f_1 \in L_{\phi\cup\sigma}(w) \setminus \{c_0c_1\}$. It follows from Observation 1.4.2 that, for each $i = 0, 1$, there is a color of $\{c_0, c_1\}$ in $\mathcal{Z}_{J_e^1}(\sigma(u), f_j, \bullet)$. Thus, by (‡), we have $\{f_0, f_1\} \subseteq L(w_\dagger)$ and $|L(w_\dagger)| = 5$. Furthermore, $f_0, f_1 \notin L_{\phi\cup\sigma}(y)$, or else we color w, y with the same color $f \in \{f_0, f_1\} \cap L_{\phi\cup\sigma}(y)$, which leaves a color of $\{c_0, c_1\}$ in $\mathcal{Z}_{J_e^1}(\sigma(u), f, \bullet)$, contradicting (‡). Since $\{f_0, f_1\} \subseteq L(w)$, there is a $c \in L_{\phi\cup\sigma}(y)$ with $c \notin L(w_\dagger)$.

Since $x'y \in E(G)$, we have $c \neq \sigma(x')$, and since $\mathcal{Z}_{J_e^1}(\sigma(u), f_j, \bullet) \setminus \{\sigma(x')\} \neq \varnothing$ for each $j = 0, 1$, we have $\mathcal{Z}_{J_e^1}(\sigma(u), f_j, \bullet) \setminus \{\sigma(x')\} = \{c\}$ for each $j = 0, 1$, or else we contradict (‡). Furthermore, $L(w_\dagger)$ is the disjoint union $\{f_0, f_1\}$ and $L(y) \setminus \{\sigma(x'), c\}$, or else, again, we contradict (‡).

Since $|L_{\phi\cup\sigma}(w)| \geq 4$, let g_0, g_1 be two colors in $L_{\phi\cup\sigma}(w) \setminus \{f_0, f_1\}$. If there is a $g \in \{g_0, g_1\}$ such that $\mathcal{Z}_{J_e^1}(\sigma(u), g, \bullet) \neq \{\sigma(x)\}$, then, since $\mathcal{Z}_{J_e^1}(\sigma(u), g, \bullet) \neq \varnothing$ and $L(w_\dagger)$ is the disjoint union $\{f_0, f_1\}$ and $L(y) \setminus \{\sigma(x'), c\}$, there is an element σ' of $\Phi(\sigma, V(J_e^1))$, using g on w, such that $|L_{\phi\cup\sigma'}(w_\dagger)| \geq 3$, contradicting (‡). We thus have $\mathcal{Z}_{J_e^1}(\sigma(u), g_0, \bullet) = \mathcal{Z}_{J_e^1}(\sigma(u), g_1, \bullet) = \{\sigma(x)\}$.

We note now that there is an $f \in \{f_0, f_1\}$ such that $L_\phi(x) \setminus \{\sigma(x)\} \subseteq \mathcal{Z}_{J_e^1}(\sigma(u), f, \bullet)$. If K is a broken wheel, then this just follows from 2) of Proposition 1.4.5, and if K is not a broken wheel, then this follows from Theorem 1.5.3. In any case, we contradict the fact that $\mathcal{Z}_{J_e^1}(\sigma(u), f, \bullet) \setminus \{\sigma(x')\} = \{c\}$. ■

Now we have the following:

Claim 10.7.22. $x'y \notin E(G)$.

Proof: Suppose that $x'y \in E(G)$. By 1) of Proposition 10.2.8, $x'y$ is a chord of Span(z) in $\tilde{G}_z^{\text{small}}$, and, by Observation 10.6.1, $\tilde{G}_z^{\text{small}}$ consists of Span(z) and the edges $xx', yy', x'y$. In particular, we have $xy' \notin E(G)$, and thus, by Claim 10.7.20, $wy \in E(G)$. Applying Claim 10.7.15, $w_\dagger$ is adjacent to each of w, y, z.

By Claim 10.7.17, $wx \notin E(G)$. Since $xy' \notin E(G)$, we have $|L_{\phi\cup\sigma}(y')| \geq 4$. Since C_z^1 is an induced subgraph of G and $wx \notin E(G)$, it follows from our triangulation conditions that $ux \notin E(G)$. Since $x \notin S_\star$ and $ux \notin E(G)$, we have $|L_{\phi\cup\sigma}(x)| \geq 2$. Since $|L_{\phi\cup\sigma}(y)| \geq 5$, there is an extension of σ to an L_ϕ-coloring σ' of $\{x', u, x, y\}$ such that $|L_{\phi\cup\sigma'}(y')| \geq 3$.

By Claim 10.7.11, σ' extends to an L_ϕ-coloring σ'' of $V(J_e^1) \cup \{x'\}$, as $wx \notin E(G)$. Since $|L_{\phi\cup\sigma''}(z)| \geq 4$, there is an extension of σ'' to an L_ϕ-coloring τ of $V(J_e^1) \cup \{x', z\}$ such that $|L_{\phi\cup\tau}(w_\dagger)| \geq 3$. By our choice of σ'', we have $|L_{\phi\cup\tau}(y')| \geq 3$ as well, so τ is an anchor, contradicting our choice of σ. ■

Now we have the following.

Claim 10.7.23. $xy' \in E(G)$

Proof: Suppose toward a contradiction that $xy' \notin E(G)$. By Claim 10.7.20, we have $wy \in E(G)$. Thus, we have $R_e = ywu$. By Claim 10.7.22, we have $x'y \notin E(G)$. Since neither xy' nor $x'y$ lies in $E(G)$, it follows from Observation 10.6.1 that $xyzy'x$ is an induced cycle of G. Applying Claim 10.7.15, $w_\dagger$ adjacent to each of w, y, z.

We claim now that $\tilde{G}_z^{\text{small}}$ is a wheel with a central vertex adjacent to all five vertices of Span(z). Suppose not. Applying 2) of Proposition 1.5.1, we extend σ to an L_ϕ-coloring σ^* of $\{x'\} \cup V(J_e^1)$. Since $|L_{\phi\cup\sigma^*}(z)| \geq 4$, there is a $\tau \in \Phi(\sigma^*, z)$ such that $|L_{\phi\cup\tau}(w_\dagger)| \geq 3$. Since G has no chord of Span(z) except for xx', we have $|L_{\phi\cup\tau}(y')| \geq 3$, and, by Corollary 1.3.6, $V(\tilde{G}_z^{\text{small}}) \setminus V(\text{Span}(z))$ is $L_{\phi\cup\tau}$-inert. Thus, τ is an anchor, contradicting our choice of σ, so let $v^\star$ be the lone vertex of $\tilde{G}_z^{\text{small}} \setminus \text{Span}(z)$.

Since $yw \in E(G)$, it follows from Claim 10.7.17 that $xw \notin E(G)$. Since $xw \notin E(G)$ and C_z^1 is an induced subgraph of G, it follows from our triangulation conditions that $ux \notin E(G)$. Since $|L_{\phi\cup\sigma}(v^\star)| \geq 4$ and $|L_{\phi\cup\sigma}(y)| \geq 5$, there is a $d \in L_{\phi\cup\sigma}(y)$ such that $L_{\phi\cup\sigma}(v^\star) \setminus \{d\}| \geq 4$.

At least one of c_0, c_1 is distinct from d, and since $xw \notin E(G)$, it follows from Claim 10.7.11 that there is a $\sigma^* \in \Phi(\sigma, J_e^1)$ such that $\sigma^*(y) = d$. Since $|L_{\phi\cup\sigma}(w_\dagger)| \geq 3$ and $|L_{\phi\cup\sigma^*}(z)| \geq 4$, there is an extension of σ^* to an L_ϕ-coloring τ of dom$(\sigma^*) \cup \{z\}$ such that $|L_{\phi\cup\tau}(w_\dagger)| \geq 3$. By our choice of $\sigma^*(y)$, $v^\star$ is $L_{\phi\cup\tau}$-inert, and thus τ is an anchor, contradicting our choice of σ. ■

Since $xy' \in E(G)$, it follows from Observation 10.6.1 that $\tilde{G}_z^{\text{small}}$ consists of Span(z) and the edges yy', xy', xx'. Furthermore, we immediately get $S_\star \neq \varnothing$, or else $J_e^1, J_{e'}^1$ are symmetric, and, interchanging the roles of the two sides in the above, we obtain $x'y \in E(\tilde{G}_z^{\text{small}})$, which is false. Since $S_\star \neq \varnothing$, it follows from 3) of Corollary 10.2.5 that there is a $\psi \in \text{Link}(Q)$ with $\psi(x) \neq \psi(x')$

Claim 10.7.24. $wy \in E(G)$.

Proof: Suppose that $wy \notin E(G)$. Thus, $R_e = uwx$, and $w_\dagger$ is adjacent to each of w, x, y. The trick now is to keep w uncolored. By 2a) of Proposition 10.6.2, there is an extension of ψ to an L_ϕ-coloring τ of $\text{dom}(\psi) \cup \{y, z\}$ such that $|L_{\phi\cup\tau}(y')| \geq 3$. Let H be the subgraph of G induced by $\text{dom}(\phi \cup \tau) \cup \text{Sh}_2(Q)$. Since $wy \notin E(G)$ and w is uncolored, each of $w, w_\dagger$ has an $L_{\phi\cup\tau}$-list of size at least three. Thus, $[H, \phi \cup \tau]$ is a (C, z)-opener, contradicting our assumption. ■

Since $wy \in E(G)$, we have $R_e = uwy$. By Claim 10.7.17, we have $wx \notin E(G)$. Furthermore, $w_\dagger$ is adjacent to each of w, y, z. Furthermore, $N(w) \subseteq V(J_e^1) \cup \{z, y'\}$. The trick now is to keep y uncolored. Since $|L_{\phi\cup\psi}(z)| \geq 5$ and $|L_{\phi\cup\psi}(y)| \geq 4$, there is an extension of ψ to an L_ϕ-coloring τ of $\text{dom}(\psi) \cup \{z\}$ such that $|L_{\phi\cup\tau}(y)| \geq 4$. Let H be the subgraph of G induced by $\text{dom}(\phi\cup\tau) \cup \text{Sh}_2(x'Qu) \cup V(J_e^1)$. Since y is uncolored, each of $y', w_\dagger$ has an $L_{\phi\cup\tau}$-list of size at least three. By assumption $[H, \phi\cup\tau]$ is not a (C, z)-opener. By construction of $\phi\cup\tau$, any extension of $\phi\cup\tau$ to an L-coloring of $\text{dom}(\phi\cup\tau) \cup V(G \setminus H)$ extends to an L-coloring τ^* of $\text{dom}(\phi\cup\tau) \cup \text{Sh}_{2,}(x'Qu)$. By our choice of color for z, τ^* also extends to $\{y\}$, and, by Proposition 10.2.9, the resulting L-coloring of $G \setminus (J_e^1 \setminus R_e)$ extends to $J_e^1 \setminus R_e$ as well. We conclude that $V(H)$ is $(L, \phi\cup\tau)$-inert in G, so $[H, \phi\cup\tau]$ is a (C, z)-opener, contradicting our assumption. This completes the proof of Lemma 10.7.1. □

10.8 Dealing With $\text{Span}(z)$ as a 4-chord: Part III

In this section, we deal with the last difficult case in the proof of 1) of Theorem 10.0.7. The remaining cases are easier and are obtained by similar arguments.

Lemma 10.8.1. *Let $z \in D_2(C^1)$ be a pentagonal vertex, where* $\text{Span}(z) := xyzy'x'$ *is a proper 4-chord of C^1. Suppose further that $E(G)$ contains one of $x'y, xy'$. Then there exists a (C, z)-opener.*

Proof. By 1) of Proposition 10.2.8, we have $x'y, xy' \notin E(\tilde{G}_z^{\text{large}})$, so G contains at most one of $x'y, xy'$. Without loss of generality, let $x'y \in E(\tilde{G}_z^{\text{small}})$. Let $Q^{\text{small}} := C^1 \cap \tilde{G}_z^{\text{small}}$ and let $Q^{\text{large}} := C^1 \cap \tilde{G}_z^{\text{large}}$. By Lemma 10.4.2, since there is no (C, z)-opener z is end-repelling. By Lemma 10.7.1, since there is no (C, z)-opener, we have $|E(Q^{\text{small}})| > 1$. Crucially, since $|E(Q^{\text{small}})| > 1$, every element of $\text{Link}(Q^{\text{large}})$ is a proper L_ϕ-coloring of its domain in G. Since $|E(Q^{\text{large}}| > 1$, let e be the unique terminal edge of Q^{large} incident to x and let e' be the unique terminal edge of Q^{large} incident to x'.

Let $K := \tilde{G}_{x'yx}^{\text{small}}$. Since G is short-separation-free, we have $V(\tilde{G}_z^{\text{small}}) = V(K) \cup \{y', z\}$. Since $z \in D_2(C^1)$, we have $E(\tilde{G}_z^{\text{small}}) = E(K) \cup \{yy', x'y\}$ by our triangulation conditions. In particular, $N(y') \cap V(C^1) = \{x'\}$, and we immediately have the following.

Claim 10.8.2. *For any partial L_ϕ-coloring ψ of $V(C^1) \cup \{y\}$, ψ extends to an L_ϕ-coloring ψ^* of* $\text{dom}(\psi) \cup \{z\}$ *such that $|L_{\phi\cup\psi^*}(y')| \geq 3$.*

Proof: Since $N(y') \cap V(C^1) = \{x'\}$ and $z \in D_2(C^1)$, we have $|L_{\phi\cup\psi}(y')| \geq 3$ and $|L_{\phi\cup\psi}(z)| \geq 4$, so the claim is immediate. ■

Recall that, by Co4d) of Definition 10.0.1, if $S_\star \neq \varnothing$, then every vertex of $D_2(C)$ has a neighborhood on C^1 consisting precisely of a subpath of C^1. In particular, if $S_\star \neq \varnothing$, then K is a broken wheel with principal path $x'yx$. Since G is $K_{2,3}$-free, it immediately follows from Co4 c) and d) of Definition 10.0.1 that the following hold.

Claim 10.8.3. *If $S_\star \neq \varnothing$, then $T^{<2} \cap V(Q^{\text{large}}) \neq \varnothing$.*

Now we have the following.

Claim 10.8.4. *R_e is a 3-chord of C^1, i.e the middle edge of R_e is incident to y.*

Proof: Suppose not. Thus, there is no e-obstruction adjacent to y. Let $H^\dagger$ be the subgraph of G induced by $\text{Sh}_2(Q^{\text{large}}) \cup \text{dom}(\phi \cup \tau)$, and consider the following cases:

Case 1: $S_\star \cap V(\mathring{Q}^{\text{small}}) = \varnothing$

In this case, applying 1) of Corollary 10.2.5, we fix a $\sigma \in \text{Link}(Q^{\text{large}})$. By 2) of Proposition 1.5.1, since $S_\star \cap V(\mathring{Q}^{\text{small}}) = \varnothing$, there is an extension of σ to an L_ϕ-coloring σ^* of $\text{dom}(\sigma) \cup V(K)$. By Claim 10.8.2, σ^* extends to an L_ϕ-coloring τ of $\text{dom}(\sigma^*) \cup \{z\}$ such that $|L_{\phi\cup\tau}(y')| \geq 3$. Since there is no e-obstruction adjacent to y, and $|L_{\phi\cup\tau}(y')| \geq 3$, the pair $[H^\dagger, \phi \cup \tau]$ is a (C, z)-opener, contradicting our assumption.

Case 2: $S_\star \cap V(Q^{\text{small}}) \neq \varnothing$.

In this case, by Claim 10.8.3 there is a $\hat{v} \in T^{<2} \cap V(Q^{\text{large}})$. By Co4d) of Definition 10.0.1, every vertex of $D_2(C)$ has a neighborhood on C^1 consisting precisely of a subpath of C^1. In particular K is a broken wheel.

By 2) of Proposition 10.2.4, there exists a pair of elements σ_0, σ_1 of $\text{Link}(Q^{\text{large}})$ using the same color on x' and different colors on x. By Proposition 1.4.10, there is an $i = 0, 1$ such that σ_i extends to an L_ϕ-coloring σ^* of $\text{dom}(\sigma_i) \cup V(K)$. By Claim 10.8.2 σ^* extends to an L_ϕ-coloring τ of $\text{dom}(\sigma^*) \cup \{z\}$ such that $|L_{\phi\cup\tau}(y')| \geq 3$. Since there is no e-obstruction adjacent to y, and $|L_{\phi\cup\tau}(y')| \geq 3$, the pair $[H^\dagger, \phi \cup \tau]$ is a (C, z)-opener, contradicting our assumption. ■

Applying Claim 10.8.4, let $R_e := xywu$ for some $w \in D_1(C^1)$ and $u \in V(Q^{\text{large}})$. Since z is end-repelling, we have $u \in V(\mathring{Q}^{\text{large}})$. For the remainder of the proof of Lemma 10.8.1, we fix the following notation.

Definition 10.8.5.

1) We set $p_{x'}$ to be the unique vertex of the path Q^{small} which is adjacent to x' and set $q_{x'}$ to be the unique vertex of Q^{large} which is adjacent to x'. Since K is not a triangle, $p_{x'} \neq x$.
2) We set H to be the subgraph of G induced by $V(x'Q^{\text{large}}u) \cup \text{Sh}_2(x'Q^{\text{large}}u) \cup V(J_e^1 - w) \cup V(K \cup C) \cup \{z\}$.

Claim 10.8.6. *Suppose that $wx \in E(G)$ and let $\hat{J}$ be the broken wheel $J_e^1 - y$ with principal path xwu. Let τ be a partial L_ϕ-coloring of $V(Q) \cup \{y, z\}$ with $u, x', y \in \text{dom}(\tau)$ and $|L_{\phi\cup\tau}(y')| \geq 3$. Then there exists an extension of τ to an L_ϕ-coloring τ^* of $\text{dom}(\tau) \cup \{w\}$ such that $\mathcal{Z}_K(\tau(x'), \tau(y), \bullet) \cap \mathcal{Z}_{\hat{J}}(\bullet, \tau^*(w), \tau(u)) = \varnothing$.*

Proof: Suppose there is a τ for which this does not hold. Since $\text{dom}(\phi\cup\tau) \cap N(w) = \{y, u\}$, we have $|L_{\phi\cup\tau}(w)| \geq 3$, and our assumption on τ implies that any extension of τ to an L_ϕ-coloring of $\text{dom}(\tau) \cup \{w\}$ also extends to L_ϕ-color $V(K \cup \hat{J})$, so the inertness condition is satisfied. Since $|L_{\phi\cup\tau}(y')| \geq 3$ as well, it follows that $[H, \phi \cup \tau]$ is a (C, z)-opener, contradicting our assumption. ■

The claim below is the most difficult part of Lemma 10.8.1.

Claim 10.8.7. $S_\star \cap V(\mathring{Q}^{\text{small}}) = \varnothing$.

Proof: Suppose not, and let $S_\star = \{u_\star\}$ for some $u_\star \in V(\mathring{Q}^{\text{small}})$. By Co4d) of Definition 10.0.1, every vertex of $D_2(C)$ has a neighborhood on C^1 consisting precisely of a subpath of C^1. In particular K¡ is a broken wheel. By Claim 10.8.3, there is a $\hat{v} \in T^{<2} \cap V(Q^{\text{large}})$. Now, $\tilde{G}$ contains the 3-chord $M_\star := uwyu_\star$ of C^1. Let $K_\star$ be the broken wheel with principal path $u_\star yx'$, where $K_\star - y = u_\star Q^{\text{small}} x'$.

Subclaim 10.8.8. *For any* $\mathfrak{f} \in \text{Corner}(M_\star, w)$ *and* $\sigma \in \text{Link}(x'Q^{\text{large}}u)$, *we have* $\sigma(x') \notin \mathcal{Z}_{K_\star}(\mathfrak{f}(u_\star), \mathfrak{f}(y), \bullet)$.

Proof: Suppose there is an $\mathfrak{f} \in \text{Corner}(M_\star, w)$ and a $\sigma \in \text{Link}(x'Q^{\text{large}}u)$ such that this does not hold. Thus, $\sigma \cup \mathfrak{f}$ extends to an L_ϕ-coloring $\sigma^\dagger$ of $\text{dom}(\sigma \cup \mathfrak{f}) \cup V(K_\star)$. By Claim 10.8.2, $\sigma^\dagger$ extends to an L_ϕ-coloring τ of $\text{dom}(\sigma^\star) \cup \{z\}$ such that $|L_{\phi\cup\tau}(y')| \geq 3$. Since $N(w) \cap \text{dom}(\phi \cup \tau) = \{y, u\}$, we have $|L_{\phi\cup\tau}(w)| \geq 3$ as well. Thus, the pair $[H, \phi \cup \tau]$ is a (C, z)-opener, contradicting our assumption. ■

We also have the following.

Subclaim 10.8.9. $\hat{v} \notin V(uQx) \setminus \{u\}$

Proof: Suppose toward a contradiction that $\hat{v} \in V(uQx) \setminus \{u\}$. Thus, $\hat{v}$ is an internal vertex of the path $C^1 \cap \tilde{G}^{\text{small}}_{M_\star}$. Let $P := u_\star Q^{\text{small}} x' Q^{\text{large}} u$. By Corollary 10.2.5, there is a $\sigma \in \text{Link}(P)$, and σ is a proper L_ϕ-coloring of its domain, since $uu_\star$ is not an edge of C^1. Since x' is a P-hinge, we have $x' \in \text{dom}(\sigma)$. Since $|L_\phi(y)| \geq 5$, it follows from 2) of Theorem 9.0.1 that there is an $\mathfrak{f} \in \text{Corner}(M_\star, w)$ using $\sigma(u_\star), \sigma(u)$ on the respective vertices $u_\star, u$, where $\mathfrak{f}(y) \neq \sigma(x')$. Since x' is a P-hinge, σ restricts to an element of $\text{Link}(u_\star Q^{\text{small}} x')$, so we have $\sigma(x') \in \mathcal{Z}_{K_\star}(\sigma(u_\star), \mathfrak{f}(y), \bullet)$. Since σ also restricts to an element of $\text{Link}(x'Q^{\text{large}}u)$, we contradict Subclaim 10.8.8. ■

We now have the following.

Subclaim 10.8.10. $L_\phi(x') \subseteq L_\phi(p_{x'})$ *and* $|L_\phi(p_{x'})| = 3$.

Proof: Suppose not. Thus, there is a $d \in L_\phi(x')$ such that either $|L_\phi(p_{x'}) \setminus \{d\}| \geq 3$, or $p_{x'} = u_\star$ and $d \notin L_\phi(p_{x'})$. By i) of Theorem 1.7.5, there is a $\sigma \in \text{Link}(xQ^{\text{large}}u)$ with $\sigma(x') = d$. Consider the following cases.

Case 1: $wx \notin E(G)$

In this case, by Theorem 1.6, there is an extension σ to an L_ϕ-coloring $\sigma^\dagger$ of $\text{dom}(\sigma) \cup \{x\}$ such that $V(J^1_e) \setminus \{w, y\}$ is $(L, \phi \cup \sigma^\dagger)$-inert in G. Now, $L_{\phi\cup\sigma^\dagger}(y)| \geq 3$, so there is a color left in $L_{\phi\cup\sigma^\dagger}(y) \setminus L_\phi(u_\star)$.Thus, by Claim 10.8.2, $\sigma^\dagger$ extends to an L_ϕ-coloring τ of $\text{dom}(\sigma^\dagger) \cup \{y, z\}$ such that $\tau(y) \notin L_\phi(u_\star)$ and $|L_{\phi\cup\tau}(y')| \geq 3$. Note that, since $wx \notin E(G)$, we have $\text{dom}(\phi \cup \tau) \cap N(w) = \{u, y\}$, and $|L_{\phi\cup\tau}(y)| \geq 3$. By our choice of $d, \tau(y)$, we get that τ extends to an L_ϕ-coloring τ^* of $\text{dom}(\tau) \cup V(K)$. Thus, the pair $[H, \phi \cup \tau^*]$ is a (C, z)-opener, contradicting our assumption.

Case 2: $wx \in E(G)$

This case is harder. In this case, $J^1_e - y$ is a broken wheel $\hat{J}$ with principal path xwu. The trick in this case is just to leave x uncolored. By Claim 10.8.2, we get that, for each $c \in L_\phi(y) \setminus \{d\}$, there is an extension of σ to an L_ϕ-coloring τ^c of $\text{dom}(\sigma) \cup \{y, z\}$ such that $\tau^c(y) = c$ and $|L_{\phi\cup\tau^c}(y')| \geq 3$. It follows from Claim 10.8.6 that for each $c \in L_{\phi\cup\sigma}(y)$, there exists an extension of τ^c to an L_ϕ-coloring τ^c_* of $\text{dom}(\tau^c) \cup \{w\}$ such that $\mathcal{Z}_K(d, c, \bullet) \cap \mathcal{Z}_{\hat{J}}(\bullet, \tau^c_*(w), \sigma(u)) = \varnothing$.

By our choice of color for x', we get that, for each $c \in L_\phi(y) \setminus L_\phi(u_\star)$, we have $\mathcal{Z}_K(d, c, \bullet) = L_\phi(x) \setminus \{c\}$. Now, there exist two colors $c_0, c_1 \in L_\phi(y) \setminus \{\sigma(x')\}$ with $c_0, c_1 \notin L_\phi(u_\star)$. Thus, for each $i = 0, 1$, we have

$\mathcal{Z}_{\hat{J}}(\bullet, \tau_*^{c_i}(w), \sigma(u)) = \{c_i\}$. By 1) of Proposition 1.4.7, we have $\tau_*^{c_i}(w) = c_{1-i}$ for each $i = 0, 1$, and, by Proposition 1.4.4, $\{c_0, c_1\}$ lies in the L_ϕ-list of each vertex of $\hat{J} \setminus \{u, y\}$.

Note that, for each $h \in L_{\phi\cup\sigma}(y) \setminus \{c_0, c_1\}$, we have $h \in L_\phi(u_\star)$, or else, if there is an h for which this does not hold, then $\mathcal{Z}_K(\sigma(x'), h, \bullet) = L(x) \setminus \{h\}$ and $\mathcal{Z}_{\hat{J}}(\bullet, \tau_*^h(w), \sigma(u)) = \{h\}$, so $\tau_*^h(w) \notin \{c_0, c_1\}$ and we contradict 1) of Proposition 1.4.5. Let $L_\phi(u_\star) = \{r, s\}$. Thus, we have $L_{\phi\cup\sigma}(y) = \{r, s, c_0, c_1\}$. Recalling that $\sigma(x') = d$, this implies that $L_\phi(y) = \{r, s, c_0, c_1\}$ and, in particular, $d \notin \{r, s\}$. The trick now is simply to switch the colors on x', y. By i) of Theorem 1.7.5, there exist two elements $\zeta_0, \zeta_1 \in \text{Link}(x'Q^{\text{large}}u)$ where $\zeta_0(x'), \zeta_1(x')$ are two distinct colors of $L_\phi(x') \setminus \{d\}$.

By Claim 10.8.2, for each $j = 0, 1$, ζ_j extends to an L_ϕ-coloring $\zeta_j^\dagger$ of $\text{dom}(\zeta_j) \cup \{y, z\}$ such that $|L_{\phi\cup\zeta_j^\dagger}(y')| \geq 3$. It follows from Claim 10.8.6 that, for each $j = 0, 1$, there is an extension of $\phi \cup \zeta_j^\dagger$ to an L_ϕ-coloring ζ_j^* of $\text{dom}(\phi \cup \zeta_j^\dagger) \cup \{w\}$ such that ζ_j^* does not extend to L_ϕ-color the pair of broken wheels $\hat{J} \cup K$.

Since $d \notin L_\phi(p_{x'}) \cup \{r, s\}$, we have $\{c_0, c_1\} \subseteq \mathcal{Z}_K(\zeta_j(x'), d, \bullet)$ for each $j = 0, 1$. Note that this is true even if $u_\star$ is adjacent to x, since $\{c_0, c_1\} \cap \{d, r, s\} = \varnothing$. Let $h \in L_\phi(x') \setminus \{c_0, c_1\}$. For each $j = 0, 1$, we have $\mathcal{Z}_{\hat{J}}(\bullet, d, \zeta_j(u)) = \{h\}$ and thus $\mathcal{Z}_K(\bullet, d, h) \cap \{\zeta_0(x'), \zeta_1(x')\} = \varnothing$. Since $d \notin L_\phi(u_\star)$ and $\{\zeta_0(x'), \zeta_1(x')\}| = 2$, it follows from Observation 1.4.2 that $\mathcal{Z}_K(\bullet, d, h) \cap \{\zeta_0(x'), \zeta_1(x')\} \neq \varnothing$, a contradiction. ■

The subclaim below is an immediate consequence of Subclaim 10.8.10.

Subclaim 10.8.11. *$K_\star$ is not a triangle and $\hat{v} \in V(x'Q^{\text{large}}u) \setminus \{x'\}$.*

Proof: By Subclaim 10.8.9, $\hat{v} \in V(x'Q^{\text{large}}u)$, and, by Subclaim 10.8.10, $|L(x')| < 4$, so $x' \notin T^{<2}$. Thus, $\hat{v} \in V(x'Q^{\text{large}}u) \setminus \{x'\}$. Since $|L_\phi(x')| \geq 3$, it also follows from Subclaim 10.8.10 that $K_\star$ is not a triangle. ■

Applying Subclaim 10.8.10, we fix a set $A \subseteq L_\phi(y)$ with $A \cap (L_\phi(x') \cup L_\phi(p_{x'})) = \varnothing$ and $|A| \geq 2$.

Subclaim 10.8.12. *No element of* Corner$(M_\star, w)$ *uses a color of A on y.*

Proof: Suppose toward a contradiction that there is an $\mathfrak{f} \in \text{Corner}(M_\star, w)$ with $f(y) \in A$. Since $\mathfrak{f}(y) \notin L_\phi(x')$ and $\mathfrak{f}(y) \notin L_\phi(p_{x'})$ we have $\mathcal{Z}_{K_\star}(\mathfrak{f}(u_\star, \mathfrak{f}(y), \bullet) = L_\phi(u)$, contradicting Subclaim 10.8.8. ■

Now we have the following.

Subclaim 10.8.13. $A = L_\phi(u_\star)$

Proof: Suppose not. Since $|A| \geq 2$, let $a \in A \setminus L_\phi(u_\star)$. Let c_0, c_1, c_2 be three distinct colors in $L_\phi(u)$.

By i) of Theorem 1.7.5, for each $i \in \{0, 1, 2\}$, there is a $\sigma^i \in \text{Link}(x'Q^{\text{large}}u)$ with $\sigma^i(u) = c_i$. Since $a \notin L_\phi(x')$, the color a is left over in $L_{\phi\cup\sigma^i}(y)$ and thus, by Claim 10.8.2, there is an extension of σ^i to an L_ϕ-coloring τ^i of $\text{dom}(\sigma^i) \cup \{y, z\}$ such that $\tau^i(y) = a$ and $|L_{\phi\cup\tau^i}(y')| \geq 3$. Since we have not colored any vertex of $J_e^1 \setminus \{u, y\}$, we have $|L_{\phi\cup\tau^i}(w)| \geq 3$ as well. Now consider the following cases.

Case 1: $wx \in E(G)$

In this case, $J_e^1 - w$ is a broken wheel $\hat{J}$ with principal path xwu. Applying Claim 10.8.6, we get that, for each $i = 0, 1, 2$, there is an extension of τ^i to an L_ϕ-coloring τ_*^i of $\text{dom}(\tau^i) \cup \{w\}$ such that $\mathcal{Z}_K(\tau^i(x'), a, \bullet) \cap \mathcal{Z}_{\hat{J}}(\bullet, \tau_*^i(w), c_i) = \varnothing$.

For each $i = 0, 1, 2$, since $a \notin L_\phi(u_\star)$ and $a \notin L_\phi(p_{x'})$, we have $\mathcal{Z}_K(\tau^i(x'), a, \bullet) = L_\phi(x) \setminus \{a\}$ and thus $\mathcal{Z}_{\hat{J}}(\bullet, \tau_*^i(w), c_i) = \{a\}$. But since c_0, c_1, c_2 are three distinct colors, this contradicts 2) of Proposition 1.4.7 1.4.7.

Case 2: $wx \notin E(G)$

Since $N(y)\cap V(C^1)=V(Q^{\text{small}})$, it follows from Theorem 1.6.1 that, for each $i=0,1,2$, there is a $d_i\in L_\phi(x)$, where $d_i\neq c_i$ if ux is an edge of J_e^1, such that any L_ϕ-coloring of $uwyw$ using c_i,d_i on the respective vertices u,w extends to an L_ϕ-coloring of J_e^1. Consider the following subcases.

Subcase 2.1 There exists an $i\in\{0,1,2\}$ such that $d_i\neq a$

In this case, there is an extension of τ_i to an L_ϕ-coloring ζ of dom$(\tau_i)\cup\{x\}$ such that $\zeta(x)=d_i$. Since $wx\notin E(G)$, we have $|L_{\phi\cup\zeta}(w)|\geq 3$. Since $a\notin L_\phi(u_\star)$ and $a\notin L_\phi(p_{x'})$, the L_ϕ-coloring $(\zeta(x'),a,d_i)$ of $x'yx$ extends to L_ϕ-color K, and so ζ extends to an L_ϕ-coloring ζ' of dom$(\zeta)\cup V(K)$. Since each of w,y' has an $L_{\phi\cup\zeta'}$-list of size at least three, $[H,\phi\cup\zeta']$ is a (C,z)-opener, contradicting our assumption.

Subcase 2.2 $d_i=a$ for all $i=0,1,2$

In this case, we choose a color $d^*\in L_\phi(y)\setminus(L_\phi(u_\star)\cup\{a\})$. Let $K_{\star\star}$ be the broken wheel with principal path $u_\star yx'$, where $K_\star\cup K_{\star\star}=K$.

By Observation 1.4.2, since $d^*\notin L_\phi(u_\star)$, the L_ϕ-coloring (d,d^*) of xy extends to an L_ϕ-coloring of $K_{\star\star}$, and thus extends to L_ϕ-color K, so let ζ be an L_ϕ-coloring of K with $\zeta(x)=d$ and $\zeta(y)=d^*$. By i) of Theorem 1.7.5, there is a $\psi\in$ Link$(x'Q^{\text{large}}u)$ with $\psi(x')=\zeta(x')$ and $\psi(u)\in\{c_0,c_1,c_2\}$, and $\zeta^*:=\psi\cup\zeta$ is a proper L_ϕ-coloring of its domain. Since $x\notin N(w)$, we have $|L_{\phi\cup\zeta^*}(w)|\geq 3$. Since each of w,y' has an L_ϕ-list of size at least three and $V(J_e^1-w)$ is $(L,\phi\cup\zeta_*)$-inert in G, the pair $[H,\phi\cup\zeta^*]$ is a (C,z)-opener, contradicting our assumption. ■

Let $p_\star$ be the unique neighbor of $u_\star$ on $K_\star-y$. By Subclaim 10.8.11, $K_\star$ is not a triangle, so $p_\star\neq x'$.

Subclaim 10.8.14. *For any two* $\mathfrak{f}^0,\mathfrak{f}^1\in$ Corner$(M_\star,w)$*, we have* $\mathfrak{f}^0(u_\star)=\mathfrak{f}^1(u_\star)$.

<u>Proof:</u> Suppose not. Thus, by Subclaim 10.8.13, there exist $\mathfrak{f}^0,\mathfrak{f}^1\in$ Corner$(M_\star,w)$ such that $L_\phi(u_\star)=\{\mathfrak{f}^0(u_\star),\mathfrak{f}^1(u_\star)\}=A$. Applying i) of Theorem 1.7.5, for each $i=0,1$, let $\sigma^i\in$ Link$(x'Q^{\text{large}}u)$, where $\sigma^i(u)=\mathfrak{f}^i(u)$. By Subclaim 10.8.8, for each $i=0,1$, we have $\sigma^i(x')\notin\mathcal{Z}_{K_\star}(\mathfrak{f}^i(u_\star),f^i(y),\bullet)$. Since $K_\star$ is not a triangle, it follows from Proposition 1.4.4 that $|L_\phi(p_\star)|=3$, $\{\mathfrak{f}^0(y),\mathfrak{f}^1(y)\}\subseteq L_\phi(p_\star)$ and $\{\mathfrak{f}^0(u_\star),\mathfrak{f}^1(u_\star)\}\subseteq L_\phi(p_\star)$. Since $A=L_\phi(u_\star)$, we have $\{\mathfrak{f}^0(y),\mathfrak{f}^1(y)\}\cap\{\mathfrak{f}^0(u_\star),\mathfrak{f}^1(u_\star)\}=\varnothing$ by Subclaim 10.8.12. Since $\mathfrak{f}^0(u_\star)\neq\mathfrak{f}^1(u_\star)$, we have $\mathfrak{f}^0(y)=\mathfrak{f}^1(y)=c$ for some color c and $L_\phi(p_\star)=L_\phi(u_\star)\cup\{c\}$.

Since $L_\phi(y)|\geq 5$, it thus follows from 1) of Theorem 9.0.1 that there exists a $\mathfrak{g}\in$ Corner$(M_\star,w)$ with $\mathfrak{g}(y)\neq c$. By i) of Theorem 1.7.5, there is a $\tau\in$ Link$(x'Q^{\text{large}}u)$ with $\tau(u)=\mathfrak{g}(u)$. Again by Subclaim 10.8.8, we have $\tau(x')\notin\mathcal{Z}_K(\mathfrak{g}(u_\star),\mathfrak{g}(y),\bullet)$, so, since $K_\star$ is not a triangle, we have $\mathfrak{g}(y)\in L_\phi(p_\star)$. Since $A=L_\phi*u_\star)$, we get $\mathfrak{g}(y)\notin L_\phi(u_\star)$ by Subclaim 10.8.12. Since $L_\phi(p_\star)=L_\phi(u_\star)\cup\{c\}$, we have a contradiction. ■

Now we have enough to finish the proof of Claim 10.8.7. Applying Subclaim10.8.14, let $r\in L_\phi(u_\star)$, where $\mathfrak{f}(u_\star)=r$ for all $\mathfrak{f}\in$ Corner$(M_\star,w)$. Let $P:=u_\star Q^{\text{small}}x'Q^{\text{large}}u_\star$. By Subclaim 10.8.11, $\hat{v}\in V(x'Q^{\text{large}}u)\setminus\{x'\}$. Thus, by Proposition 10.2.4, there exist two elements $\sigma_0,\sigma_1\in$ Link(P), each of which use the color r on $L_\phi(u_\star)$, where σ_0,σ_1 use different colors on u and restrict to the same L_ϕ-coloring of $\{u_\star,x'\}$. Since $x\in V(C^1\setminus P)$, each of σ_0,σ_1 is a proper L_ϕ-coloring of its domain. Let $c\in L_\phi(x')$, where $c=\sigma_0(x')=\sigma_1(x')$. By Theorem 9.0.1, since $|L_\phi(y)\geq 5$, there is a $\mathfrak{g}\in$ Corner$(M_\star,w)$ with $\mathfrak{g}(u)\in\{\sigma_0(u),\sigma_1(u)\}$ and $\mathfrak{g}(y)\neq c$. Recall that $\mathfrak{g}(u_\star)=r$. Since each of σ_0,σ_1 restricts to an element of Link$(x'Qu_\star)$, and $\mathfrak{g}(y)\neq c,r$, the L_ϕ-coloring $(r,\mathfrak{g}(y),c)$ of $u_\star yx'$ extends to L_ϕ-color $K_\star$, contradicting Subclaim 10.8.8. This completes the proof of Claim 10.8.7. ■

Now we have the following.

Claim 10.8.15. *e is problematic.*

Proof: Suppose not. Thus, every vertex of $V(uQ^{\text{large}}x) \setminus \{u\}$ has an L_ϕ-list of size at least three. We now have the following.

Subclaim 10.8.16. *$wx \in E(G)$.*

Proof: Suppose that $wx \notin E(G)$. Applying 1) of Corollary 10.2.5, we first fix a $\sigma \in \text{Link}(x'Q^{\text{large}}u)$. By Theorem 1.6.1 since $u \neq x'$ and $N(y) \cap V(uQ^{\text{large}}x) = \{x\}$, there is an extension of σ to an L_ϕ-coloring $\sigma^\dagger$ of $\text{dom}(\sigma) \cup \{x\}$, such that any extension of $\sigma^\dagger$ to an L_ϕ-coloring of $\text{dom}(\sigma^\dagger) \cup \{w, y\}$ also extend to L_ϕ-color all of J_e^1. By Claim 10.8.7, we have $S_\star \cap V(\mathring{Q}^{\text{small}}) = \varnothing$, so it follows from 2) of Proposition 1.5.1 that $\sigma^\dagger$ extends to L_ϕ-color $V(K)$ as well, and thus, by Claim 10.8.2, $\sigma^\dagger$ extends to an L_ϕ-coloring τ of $\text{dom}(\phi \cup \sigma^\dagger) \cup V(K) \cup \{z\}$ such that $|L_{\phi \cup \tau}(y')| \geq 3$. Since $wx \notin E(G)$, we have $|L_{\phi \cup \tau}(w)| \geq 3$ as well, and the pair $[H, \phi \cup \tau]$ is a (C, z)-opener, contradicting our assumption. ■

Since $wx \in E(G)$, let $\hat{J} := \tilde{G}^{\text{small}}_{xwu}$. Since G is short-separation-free, we have $\hat{J} = J_e^1 - y$. Applying 1) of Corollary 10.2.5, we fix a $\sigma \in \text{Link}(x'Q^{\text{large}}u)$. We now leave x uncolored. Applying Claim 10.8.2 again, we have that, for each $c \in L_\phi(y) \setminus \{\sigma(x')\}$, there is an L_ϕ-coloring τ^c of $\text{dom}(\sigma) \cup \{y, z\}$ with $|L_{\phi \cup \tau^c}(y')| \geq 3$. Applying Claim 10.8.6, we get that, for each $c \in L_\phi(y) \setminus \{\sigma(x')\}$, there exists an extension of τ^c to an L_ϕ-coloring τ^c_* of $\text{dom}(\tau^c) \cup \{w\}$ such that $\mathcal{Z}_K(\sigma(x'), c, \bullet) \cap \mathcal{Z}_{\hat{J}}(\bullet, \tau^c_*(w), \sigma(u)) = \varnothing$.

Subclaim 10.8.17. *Each of K and $\hat{J}$ is a broken wheel.*

Proof: We first show that K is a broken wheel. Suppose not. Since $|L_\phi(y) \setminus \{\sigma(x')\}| \geq 4$, it follows from Theorem 1.5.3 that there exist three distinct colors $c_0, c_1, c_2 \in L_\phi(y) \setminus \{\sigma(x')\}$ such that, for each $i = 0, 1, 2$, we have $\mathcal{Z}_K(\sigma(x'), c, \bullet) = L_\phi(x) \setminus \{c_i\}$. Thus, for each $i = 0, 1, 2$, we have $\mathcal{Z}_{\hat{J}}(\bullet, \tau^{c_i}_*(w), \sigma(u)) = \{c_i\}$. In particular $\{\tau^{c_i}_*(w) : i = 0, 1, 2\}$ are three distinct colors, and, applying Theorem 1.5.3 again, $\hat{J}$ is a broken wheel. But since $\{\tau^{c_i}_*(w) : i = 0, 1, 2\}$ are three distinct colors, we contradict 1) of Proposition 1.4.5 applied to $\hat{J}$. Thus, K is a broken wheel with principal path $x'yx$.

Now we show that $\hat{J}$ is a broken wheel. Suppose not. Since $|L_\phi(y) \setminus \{\sigma(x')\}| \geq 4$, we choose a $c \in L_\phi(y) \setminus \{\sigma(x')\}$ with $|L_\phi(x) \setminus \{c\}| \geq 3$. By Observation 1.4.2, we have $|\mathcal{Z}_K(\sigma(x'), c, \bullet)| \geq 2$. Thus, we have $|\mathcal{Z}_K(\sigma(x'), c, \bullet)| = 2$ and $|\mathcal{Z}_{\hat{J}}(\bullet, \tau^c_*(w), \sigma(u))| = 1$. In particular, since $|L_\phi(x) \setminus \{c\}| \geq 3$ and K is a broken wheel but not a triangle, we have $c, \sigma(x') \in L_\phi(p_{x'})$ and $|L_\phi(x)| = 3$, or else $\mathcal{Z}_K(\sigma(x'), c, \bullet) \cap \mathcal{Z}_{\hat{J}}(\bullet, \tau^c_*(w), \sigma(u)) \neq \varnothing$, which is false.

Let $d \in \mathcal{Z}_K(\sigma(x'), c, \bullet) \setminus \{\tau^c_*(w)\}$. Since $\hat{J}$ is not a broken wheel, it follows that $\tau^c_*(w) \in L_\phi(x)$ and, by Theorem 1.5.3, $(d, \tau^c_*(w), \sigma(u))$ is the unique L_ϕ-coloring of xwu which does not extend to L_ϕ-color $\hat{J}$. Thus, for each $s \in L_\phi(y) \setminus \{\sigma(x')\}$, we have $\mathcal{Z}_K(\sigma(x'), s, \bullet) \subseteq \{\tau^s_*(w), d\}$.

Since $|L_\phi(y)| \geq 5$ and $\{\sigma(x'), c\} \subseteq L_\phi(p_{x'})$, there exist two distinct colors $s_0, s_1 \in L_\phi(y) \setminus L_\phi(p_{x'})$, and $s_0, s_1 \notin \{\sigma(x'), c\}$.

For each $i = 0, 1$, since $s_i \notin L_\phi(p_{x'})$ and K is a broken wheel, but not a triangle, we have $\mathcal{Z}_K(\sigma(x'), s_i, \bullet) = L_\phi(x) \setminus \{s_i\} = \{tau^{s_i}_*(w), d\}$. Since $(d, \tau^c_*(w), \sigma(u))$ is the unique L_ϕ-coloring of xwu which does not extend to L_ϕ-color $\hat{J}$, we have $\tau^{s_0}_*(w) = \tau^{s_1}_*(w) = \tau^c_*(w) = r$ for some color r. But then $\{d, s_0, s_1, r\}$ are four distinct colors all lying in $L_\phi(x)$, which is false, since $|L_\phi(x)| = 3$. ■

Let $X_{\hat{J}} := \bigcap(L_\phi(v); v \in V(\hat{J}) \setminus \{x, w, u\})$. Since $|L_{\phi \cup \sigma}(y)| \geq 4$, there exist two colors $d_0, d_1 \in L_{\phi \cup \sigma}(y)$ such that, for each $i = 0, 1$, $L_\phi(p_{x'}) \setminus \{\sigma(x'), d_i\}| \geq 2$. Since K is not a triangle, it follows that, for each $i = 0, 1$ we have

$\mathcal{Z} = L(x) \setminus \{d_i\}$ and $\mathcal{Z}_{\hat{J}}(\bullet, \tau_*^{d_i}(w), \sigma(u)) = \{d_i\}$. Thus, $\{d_0, d_1\} \subseteq X_{\hat{J}}$.

By 1) of Proposition 1.4.7, we have $\tau_*^{d_i}(w) = d_{1-i}$ for each $i = 0, 1$, and, thus d_0, d_1 are the only vertices in $\{d \in L_\phi(y) \setminus \{\sigma(x') : |L_\phi(p_{x'}) \setminus \{\sigma(x'), d\}| \geq 2\}$. In particular, letting $h_0, h_1 \in L_\phi \setminus \{d_0, d_1\}$, we have $L_\phi(p_{x'}) = \{\sigma(x'), h_0, h_1\}$.

By Proposition 1.4.4 applied to $\hat{J}$, we have $\{d_0, d_1\} \subseteq L_\phi(x)$ and $|L_\phi(x)| = 3$. Thus, we suppose without loss that $h_0 \notin L_\phi(x)$. By Observation 1.4.2, $|\mathcal{Z}_K(\sigma(x'), h_0, \bullet)| \geq 2$, so $\mathcal{Z}_K(\sigma(x'), h_0, \bullet)| = 2$ and $|\mathcal{Z}_{\hat{J}}(\bullet, \tau_*^{h_0}(w), \sigma(u))| = 1$. Thus, $\tau_*^{h_0}(w) \in \{d_0, d_1\}$ so suppose without loss of generality that $\tau_*^{h_0}(w) = d_0$. Let p_x be the unique neighbor of x on Q^{small}. Since K is not a triangle, we have $p_x \neq x'$. Since $\mathcal{Z}_{\hat{J}}(\bullet, d_0, \sigma(u)) = \{d_1\}$, we have $\mathcal{Z}_K(\sigma(x'), h_0, \bullet) = L_\phi(x) \setminus \{d_1\}$ and $L_\phi(p_x) = \{h_0, h_1, d_0\}$.

Now let p_u be the lone neighbor of u on the path $\hat{J} - w$. Possibly $\hat{J}$ is a triangle and $p_u = x$. In any case, since $\mathcal{Z}_{\hat{J}}(\bullet, d_i, \sigma(u)) = \{d_{1-i}\}$ for each $i = 0, 1$, we have $L_\phi(p_u) = \{\sigma(u), d_0, d_1\}$. That is, we have the following.

1. $L_\phi(p_x) = \{h_0, h_1, d_0\}$ and $L_\phi(p_{x'}) = \{\sigma(x'), h_0, h_1\}$; AND
2. $L_\phi(p_u) = \{\sigma(u), d_0, d_1\}$

Now we have the following.

Subclaim 10.8.18.

1. *Every element of* $\text{Link}(x'Q^{\text{large}})$ *uses a color of* $L_\phi(p_{x'})$ *on* x'*; AND*
2. *Every element of* $\text{Link}(x'Q^{\text{large}}u)$ *uses* $\sigma(u)$ *on* u*.*

Proof: We first prove 1). Suppose toward a contradiction that there is a $\zeta \in \text{Link}(x'Q^{\text{large}}u)$ such that $\zeta(x') \notin L_\phi(p_{x'})$. Thus, $\zeta(x') \notin L_\phi(p_{x'})$. Since $|L_{\phi\cup\zeta}(y)| \geq 4$ and $|L_{\phi\cup\zeta}(x)| = 3$, it follows from Claim 10.8.2 that there is an extension of ζ to an L_ϕ-coloring ζ' of $\text{dom}(\zeta) \cup \{y, z\}$ such that $\zeta'(y) \notin L_\phi(x)$ and $|L_{\phi\cup\zeta'}(y')| \geq 3$.

By Claim 10.8.6, there is an extension of ζ' to an L_ϕ-coloring ζ'' of $\text{dom}(\zeta') \cup \{w\}$ such that $\mathcal{Z}_K(\zeta''(x'), \zeta''(y), \bullet) \cap \mathcal{Z}_{\hat{J}}(\bullet, \zeta''(w), \zeta''(u)) = \varnothing$. But since $\zeta''(y) \notin L_\phi(x)$ and $\zeta''(x') \notin L_\phi(p_{x'})$, we have $\mathcal{Z}_K(\zeta''(x'), \zeta''(y), \bullet) = L_\phi(x)$. Since $\mathcal{Z}_{\hat{J}}(\bullet, \zeta''(w), \zeta''(u)) \neq \varnothing$, we have a contradiction. This proves 1).

Now we prove 2). Suppose toward a contradiction that there is a $\zeta \in \text{Link}(x'Q^{\text{large}}u)$with $\zeta(u) \neq \sigma(u)$. By 1), since $N(y) \cap \text{dom}(\phi \cup \zeta) = \{x'\}$ and $d_0, d_1 \notin L_\phi(x')$, we have $\{d_0, d_1\} \subseteq L_{\phi\cup\zeta}(y)$, so, by Claim 10.8.2, ζ extends to an L_ϕ-coloring ζ' of $\text{dom}(\zeta) \cup \{y, z\}$ with $\zeta'(y) \in \{d_0, d_1\}$ and $|L_{\phi\cup\zeta'}(y')| \geq 3$.

By Claim 10.8.6, there is an extension of ζ' to an L_ϕ-coloring ζ'' of $\text{dom}(\zeta') \cup \{w\}$ such that $\mathcal{Z}_K(\zeta''(x'), \zeta''(y), \bullet) \cap \mathcal{Z}_{\hat{J}}(\bullet, \zeta''(w), \zeta''(u)) = \varnothing$. Since $\zeta''(y) \in \{d_0, d_1\}$, we have $\mathcal{Z}_K(\zeta''(x'), \zeta''(y), \bullet) = L_\phi(x) \setminus \{\zeta''(y)\}$ and $\mathcal{Z}_{\hat{J}}(\bullet, \zeta''(w), \zeta''(u)) = \{\zeta'(y)$. Since $L(p_u) = \{\sigma(u), d_0, d_1\}$ and $\zeta''(u) \neq \sigma(u)$, it follows that $\zeta''(u) \in \{d_0, d_1\}$, i.e $\zeta(u) \in \{d_0, d_1\}$.

Now we construct a (C, z)-opener in the following way. We extend ζ to an L_ϕ-coloring ζ^* of $\text{dom}(\zeta) \cup V(uQ^{\text{large}}x) \cup \{y\}$ by 2-coloring the path $uQ^{\text{large}}xy$ with $\{d_0, d_1\}$. By 1), we have $\zeta(x') \notin \{d_0, d_1\}$, so this 2-coloring is indeed permissible, since $uQ^{\text{large}}xy$ is an induced path in G.

Since $\zeta^*(y) \in \{d_0, d_1\}$, ζ^* extends to L_ϕ-color K as well, and, by Claim 10.8.2, ζ^* extends to an L_ϕ-coloring ζ^{**} of $\text{dom}(\zeta^*) \cup V(K) \cup \{z\}$ such that $|L_{\phi\cup\zeta^{**}}(y')| \geq 3$. Since only d_0, d_1 are used among the neighbors of w in $\text{dom}(\phi \cup \zeta^{**})$, we have $|L_{\phi\cup\zeta^{**}}(w)| \geq 3$, so the pair $[H, \zeta^{**}]$ is a (C, z)-opener, contradicting our assumption. This proves 2). ■

Now we have the following.

Subclaim 10.8.19. *$S_\star \cap V(x'Q^{\text{small}}u) \neq \varnothing$, and, letting $S_\star = \{u_\star\}$, we have $T^{<2} \cap V(x'Qu_\star) \neq \varnothing$.*

Proof: If $S_\star = \varnothing$, then it immediately follows from Theorem 1.7.5 that there is an element of Link$(x'Q^{\text{small}}u)$ using a color other than $\sigma(u)$ on u, contradicting 2) of Subclaim 10.8.18. Thus, $S_\star \neq \varnothing$. By Claim 10.8.7, we have $u_\star \in V(Q^{\text{large}})$. By assumption, e is unproblematic so $u_\star \in V(x'Q^{\text{large}}u)$. By Claim 10.8.3, there is a $\hat{v} \in T^{<2} \cap V(Q^{\text{large}})$.

Since $|L_\phi(x)| = 3$, we have $x \notin T^{<2}$, and since no internal vertex of $\hat{J}-y$ lies in $T^{<2}$, we have $\hat{v} \in V(x'Q^{\text{large}}u)$. If $T^{<2} \cap V(u_\star Q^{\text{large}}u) \neq \varnothing$, then, by Proposition 10.2.4, there is an element of Link$(x'Q^{\text{large}}u)$ using a color other than $\sigma(u)$ on u, contradicting 2) of Subclaim 10.8.18. Thus, we have $\hat{v} \in V(x'Q^{\text{large}}u_\star)$. ■

Let $u_\star$ be the lone vertex of $S_\star$ and let $\hat{v} \in T^{<2} \cap V(x'Q^{\text{large}}u_\star)$. By Proposition 10.2.4, there are two elements ψ_0, ψ_1 of Link$(x'Q^{\text{large}}u)$ which use the same color on u and different colors on x'. By 2) of Subclaim 10.8.18, there is a $c \in L_\phi(u)$ such that $c = \sigma(u) = \psi_0(u) = \psi_1(u)$. At least one of $\psi_0(x'), \psi_1(x')$ is distinct from $\sigma(x')$ so let $\psi_0(x') \neq \sigma(x')$. Recall that $L_\phi(p_{x'}) = \{\sigma(x'), h_0, h_1\}$. By 1) of Subclaim 10.8.18, $\psi_0(x') \in \{h_0, h_1\}$, since $\psi_0(x') \neq \sigma(x')$. Thus, there is a color $f \in L_\phi(y) \setminus \{\psi(x'), d_0, d_1, h_0, h_1\}$. By Claim 10.8.2, ψ_0 extends to an L_ϕ-coloring ψ_* of dom$(\psi_0) \cup \{y, z\}$ with $\psi_*(y) = f$ and $|L_{\phi\cup\psi_*}(y')| \geq 3$. By Claim 10.8.6, there is an extension of ψ_* to an L_ϕ-coloring ψ_{**} of dom$(\psi_*) \cup \{w\}$ such that $\mathcal{Z}_K(\psi_0(x'), f, \bullet) \cap \mathcal{Z}_{\hat{J}}(\bullet, \psi^{**}(w), c) = \varnothing$.

Recall that $L_\phi(p_x) = \{h_0, h_1, d_1\}$. Thus, $f \notin L_\phi(p_x)$. Now, K is not a triangle, and thus, as $f \notin \{d_0, d_1\}$ and $\{d_0, d_1\} \subseteq L_\phi(x)$, it follows that $\{d_0, d_1\} \subseteq \mathcal{Z}_K(\psi_0(x'), f, \bullet)$. Recall now that $\mathcal{Z}_{\hat{J}}(\bullet, d_i, c) = \{d_{1-i}\}$ for each $i = 0, 1$, since $c = \sigma(u)$. It follows that $\psi_{**}(w) \notin \{d_0, d_1\}$ or else $\mathcal{Z}_K(\psi_0(x'), f, \bullet) \cap \mathcal{Z}_{\hat{J}}(\bullet, \psi^{**}(w), c) \neq \varnothing$, which is false. Thus, we get $\psi_{**}(w) \notin \{d_0, d_1\}$. But then, since $\{d_0, d_1\} \subseteq \mathcal{Z}_K(\psi_0(x'), f, \bullet)$ and the path $\hat{J} \setminus \{y, u\}$ admits a 2-coloring using $\{d_0, d_1\}$, we again have $\mathcal{Z}_K(\psi_0(x'), f, \bullet) \cap \mathcal{Z}_{\hat{J}}(\bullet, \psi^{**}(w), c) \neq \varnothing$, a contradiction. This completes the proof of Claim 10.8.15. ■

Applying Claim 10.8.15, we now fix $u_\star \in V(uQ^{\text{large}}x) \setminus \{u\}$ with $S_\star = \{u_\star\}$. Since $S_\star \neq \varnothing$, it follows from Claim 10.8.3 that $T^{<2} \cap V(Q^{\text{large}}) \neq \varnothing$. By Co4d) of Definition 10.0.1, every vertex of $D_2(C)$ has a neighborhood on C^1 consisting precisely of a subpath of C^1. In particular K is a broken wheel. An analogous argument to that of Claim 10.8.15 shows the following.

Claim 10.8.20. *No vertex of $uQ^{\text{large}}x$ lies in $T^{<2}$. In particular, $T^{<2} \cap V(x'Q^{\text{large}}u) \setminus \{u\} \neq \varnothing$.*

We now have the following.

Claim 10.8.21. *For any $\psi \in$ Link(Q^{large}) and any extension of ψ to an L_ϕ-coloring ψ_* of dom$(\psi) \cup \{y\}$, either $|L_{\phi\cup\psi_*}(w)| = 2$ or ψ_* does not extend to L_ϕ-color $V(K)$.*

Proof: Suppose there is a ψ for which this does not hold. Thus, there is an extension ψ_* of ψ to an L_ϕ-coloring of dom$(\psi) \cup \{y\}$ such that $|L_{\phi\cup\psi_*}(w)| \geq 3$, where ψ_* also extends to L_ϕ-color $V(K)$. By Claim 10.8.2, there is a $\psi' \in \Phi_{L_\phi}(\psi_*, V(K) \cup \{z\})$ such that $|L_{\phi\cup\psi'}(y')| \geq 3$. Note that $|L_{\phi\cup\psi'}(w)| \geq 3$ as well, since dom$(\psi') \cap N(w) \subseteq$ dom(ψ_*). Since u is a Q^{large}-hinge, ψ restricts to an element of Link$(uQ^{\text{large}}x)$. By Proposition 10.2.9, $V(J^1_e - w)$ is $(L, \phi\cup\psi')$-inert in G. Since each of w, y' has an $L_{\phi\cup\psi'}$-list of size at least three, the pair $[H, \phi\cup\psi']$ is a (C, z)-opener, contradicting our assumption. ■

The above has the following simple consequence.

Claim 10.8.22. *For each $\sigma \in \text{Link}(Q^{\text{large}})$, we have $|L_{\phi\cup\sigma}(w)| = 3$. In particular, $|N(w) \cap V(C^1)| > 1$.*

Proof: Suppose not and let $\sigma \in \text{Link}(Q^{\text{large}})$ with $|L_{\phi\cup\sigma}(w)| \geq 4$. By 2) of Proposition 1.5.1, σ extends to an L_ϕ-coloring σ^* of $\text{dom}(\sigma) \cup V(K)$, as $S_\star \cap V(\mathring{Q}^{\text{small}}) = \varnothing$, and, since $|L_{\phi\cup\sigma}(w)| > 3$, we have $|L_{\phi\cup\sigma^*}(w)| \geq 3$, contradicting Claim 10.8.21. ■

Applying Claim 10.8.22, we introduce the following.

Definition 10.8.23. We define a vertex $p^\dagger$ of $(N(w) \cap V(C^1))$ as follows. If $u_\star \in N(w)$, then we set $p^\dagger = u_\star$. Otherwise, we set $p^\dagger$ to be the non-u endpoint of the path $G[N(w) \cap V(C^1)]$. Since e is problematic and $|N(w) \cap V(C^1)| > 1$, we have $p^\dagger \in V(uQx) \setminus \{u\}$.

We now introduce the following terminology.

Definition 10.8.24. Given a subpath P of Q with $uQx \subseteq P$ and a family $\mathcal{F}$ of elements of $\text{Link}(P)$, a $d \in L_\phi(y)$ is called $\mathcal{F}$-universal if, for each $\psi \in \mathcal{F}$, there is a $\psi^* \in \Phi_{L_\phi}(\psi, y)$ with $\psi^*(y) = d$ and $|L_{\phi\cup\psi^*}(w)| \geq 3$.

Claim 10.8.25. *For any subpath P of $Q^{\text{large}} - x'$ and any $\psi \in \text{Link}_{L_\phi}(P)$, there is a $\psi_* \in \Phi_{L_\phi}(\psi, y)$ such that $|L_{\phi\cup\psi_*}(w)| \geq 3$. In particular, if $uQ^{\text{large}}x \subseteq P$ and $\mathcal{F}$ is a family of elements of $\text{Link}(P)$ which all restrict to the same element of $\text{Link}(uQ^{\text{large}}x)$, then there is an $\mathcal{F}$-universal color in $L_\phi(y)$.*

Proof: For any subpath P of Q^{large} and any $\psi \in \text{Link}(P)$, we have $|L_{\phi\cup\psi}(w)| \geq 3$. Since $N(y) \cap V(Q^{\text{large}} - x') = \{x\}$, we have $|L_{\phi\cup\psi}(y)| \geq 4$, so both parts of the claim trivially follow. ■

We now set $u'w'$ to be the unique edge of $R_{e'} \setminus \{x', y\}$, where $u' \in V(Q^{\text{large}} - x')$ and $w' \in D_2(C)$. Since z is end-repelling, we have $u' \in V(x'Q^{\text{large}}u) \setminus \{x'\}$, and $J^1_e \cap J^1_{e'}$ is either empty or $u = u'$ and the intersection consists of this lone vertex. Let $B := \{w_* \in N(w) \cap : N(w_*) \cap V(Q \setminus J^1_e) \neq \varnothing\}$.

Claim 10.8.26. $ww' \notin E(G)$.

Proof: Suppose toward a contradiction that $ww' \in E(G)$. Note that $w' \notin N(y')$, or else, since $yy' \in E(\tilde{G}^{\text{small}}_z)$, the 4-cycle $ww'y'y$ separates C from a vertex of $G \setminus B_4(C)$ with an L_ϕ-list of size less than five. Thus, $R_{e'} = u'w'x'$. Since $x'y \in E(G^{\text{small}}_z)$, the 4-cycle $ww'x'y$ separates C from a vertex of $G \setminus B_4(C)$ with an L_ϕ-list of size less than five. In any case, we contradict short-separation-freeness. ■

We now have the following.

Claim 10.8.27. $x' \in N(w')$.

Proof: Suppose not. Thus, $R_{e'}$ is a 3-chord of C^1, and $R_{e'} = u'w'y'x'$. Let H^2 be the subgraph of G induced by $V(x'Q^{\text{large}}u') \cup \text{Sh}_2(x'Q^{\text{large}}u') \cup V(J^1_e \cup K) \cup V(J^1_{e'} - w') \cup \{y', z\}$.

Note that $x'u' \notin E(G)$, or else since $N(y) \cap V(C^1) \subseteq V(Q^{\text{small}})$, we have $w' \in N(x')$ by our triangulation conditions, contradicting our assumption. Let $Y_K := \bigcap(L_\phi(v) : v \in V(K) \setminus \{x', y, x\})$ and let $\text{Base}(J^1_{e'})$ be the set of L_ϕ-colorings $\mathfrak{f}$ of $\{u', x'\}$ with the property that any extension of $\mathfrak{f}$ to all of $R^1_{e'}$ extends to L_ϕ-color all of $J^1_{e'}$

Subclaim 10.8.28. $B \neq \varnothing$.

Proof: Suppose toward a contradiction that $B = \varnothing$. Applying 1) of Corollary 10.2.5, we fix a $\sigma \in \text{Link}(u'Q^{\text{large}}x)$. By Theorem 1.6.1, there is an element $\mathfrak{f}$ of $\text{Base}(J^1_{e'})$ such that $\mathfrak{f}(u') = \sigma(u')$. By 2) of Proposition 1.5.1, since $S_\star \cap V(\mathring{Q}^{\text{small}}) = \varnothing$, σ extends to an L_ϕ-coloring σ' of $\text{dom}(\sigma) \cup V(K)$ with $\sigma'(x') = \mathfrak{f}(x')$. Now, since $V(J^1_e) \setminus \{w, y\}$ is $(L, \phi \cup \sigma)$-inert in G, we get that σ' extends to an L_ϕ-coloring σ'' of $\text{dom}(\sigma') \cup V(J^1_e)$. Crucially, for any $\tau \in \Phi_{L_\phi}(\sigma'', \{y', z\})$, we have $|L_{\phi\cup\tau}(w')| \geq 3$, since $x', z \notin N(w')$.

Case 1: w, z have no common neighbor other than y.

In this case, for any $\tau \in \Phi_{L_\phi}(\sigma'', \{y', z\})$, the pair $[H^2, \phi\cup\tau]$ is an (C, z)-opener, since $B = \varnothing$. This contradicts our assumption.

Case 2: w, z have a common neighbor $w^\dagger$ other than y

In this case, $w^\dagger$ is unique, and, since $|L_{\phi\cup\sigma''}(w^\dagger)| \geq 3$ and $|L_{\phi\cup\sigma''}(z)| \geq 4$, there is a $\tau \in \Phi_{L_\phi}(\sigma'', \{y', z\})$ such that $|L_{\phi\cup\tau}(w^\dagger)| \geq 3$, and, as indicated above, $|L_{\phi\cup\tau}(w')| \geq 3$ as well. Since $B = \varnothing$, $[H^2, \phi \cup \tau]$ is a (C, z)-opener, contradicting our assumption. ■

Applying Propsotion 10.5.4, let $w_* v_*$ be the e-wall of B, where $v_* \in V(C^1)$, and let $P := v_* w_* wu$. By Claim 10.8.26, $w_* \neq w'$, so $v_* \in V(u'Q^{\text{large}}u) \setminus \{u\}$. Now we have a nice trick to finish the proof of Claim 10.8.27. Recalling Definition 10.8.23, let $P^\dagger := v_* w_* wp^\dagger$. Note that, by our choice of $p^\dagger$, each vertex of $p^\dagger Q^{\text{large}}x \setminus \{x\}$ has an L_ϕ-list of size at least three. Furthermore, each vertex of $x'Q^{\text{large}}p^\dagger) \setminus \{p^\dagger\}$ has an L_ϕ-list of size at least three.

Subclaim 10.8.29. *There exists a* $\mathfrak{f} \in \text{Corner}(P^\dagger, w_*)$ *such that, for any* $\mathfrak{g} \in \text{Base}(J^1_{e'})$*, the union* $\mathfrak{f} \cup \mathfrak{g}$ *extends to an* L_ϕ*-coloring of* $\text{dom}(\mathfrak{f} \cup \mathfrak{g}) \cup V(J^1_e \cup K)$.

Proof: Since $|L_\phi(p^\dagger)| \geq 2$ and $|L_\phi(u_*)| \geq 3$, it follows from 1) of Theorem 9.0.1 that $\text{Corner}(P^\dagger, w_*) \neq \varnothing$. Consider the following cases.

Case 1: $p^\dagger \neq x$

In this case, we claim that any $\mathfrak{f} \in \text{Corner}(P^\dagger, w_*)$ satisfies the subclaim. Let $\mathfrak{g} \in \text{Base}(J^1_{e'})$ and let $\psi := \mathfrak{f} \cup \mathfrak{g}$; It just suffices to show that the precoloring ψ of $\{p^\dagger, w, x'\}$ extends to L_ϕ-color $\tilde{G}^{\text{small}}_{p^\dagger wyx'}$. Firstly, $D := p^\dagger Q^{\text{large}} x Q^{\text{small}} x' y w p^\dagger$ is a cyclilc facial subgraph of $\tilde{G}^{\text{small}}_{p^\dagger wyx'}$, and every vertex of $D \setminus \{p^\dagger, w, y, x'\}$ has an L_ϕ-list of size at least three. By our choice of $p^\dagger$, we have $N(w) \cap V(D) = \{p^\dagger, y\}$, i.e there is no chord of D with w as an endpoint. Since $p^\dagger \neq x$, we have $N(y) \cap \text{dom}(\phi \cup \psi) = \{x', x\}$ and $|L_{\phi\cup\psi}(y)| \geq 3$, so, by 2) of Proposition 1.5.1, we get that $\phi \cup \psi$ extends to L_ϕ-color $V(\tilde{G}^{\text{small}}_{p^\dagger wyx'})$, as desired.

Case 2: $p^\dagger = x$

In this case, x is the lone vertex of $S_\star$ and $J^1_e - y \subseteq \tilde{G}^{\text{small}}_{P^\dagger}$. Since $|L_\phi(y)| \geq 5$, it follows from 1) of Theorem 9.0.1 there exist two elements $\mathfrak{f}_0, \mathfrak{f}_1$ of $\text{Corner}(P^\dagger, w_*)$ with $\mathfrak{f}_0(w) \neq \mathfrak{f}_1(w)$. We show now that one of $\mathfrak{f}_0, \mathfrak{f}_1$ satisfies the subclaim. Suppose not. Since $J^1_e - y \subseteq \tilde{G}^{\text{small}}_{P^\dagger}$, it follows that, for each $i = 0, 1$, there exists a $\mathfrak{g}_i \in \text{Base}(J^1_{e'})$ such that $\mathfrak{f}_i \cup \mathfrak{g}_i$ does not extend to L_ϕ-color $\{x, w, y'\} \cup V(K)$. In particular, for each $i = 0, 1$, letting $X_i := L_\phi(y) \setminus \{\mathfrak{f}, \mathfrak{f}, \mathfrak{g}\}$, we have $\mathcal{Z}_K(\mathfrak{g}(x'), \bullet, \mathfrak{f}(x)) \cap X_i = \varnothing$. Thus, recalling that $p_x, p_{x'}$ are the respective neighbors of x, x' on $K - y$, and K is not a triangle, we have $L_\phi(p_x) = X_i \cup \{\mathfrak{f}_i(x)\}$ as a disjoint union and $L_\phi(p_{x'}) = X_i \cup \{\mathfrak{g}_i(x')\}$ as a disjoint union, as $|X_i| \geq 2$.

Subcase 2.1: $\mathfrak{f}_0(x) = \mathfrak{f}_1(x)$

In this case, let $f = \mathfrak{f}_0(x) = \mathfrak{f}_1(x)$. We have $X_0 = X_1$ and $\mathfrak{g}_0(x') = \mathfrak{g}_1(x') = g$ for some color g. By 2) of Proposition 1.5.1, the precoloring of $\{x, x'\}$ using f, g on x, x' respectively extends to L_ϕ-color all of K, and since $\mathfrak{f}_0(w) \neq \mathfrak{f}_1(w)$, the color used on y is distinct from at least one of $\mathfrak{f}_0(w), \mathfrak{f}_1(w)$. Thus, for some $i = 0$, $\mathfrak{f}_i \cup \mathfrak{g}_i$ extends to L_ϕ-color K, contradicting our assumption.

Subcase 2.2: $\mathfrak{f}_0(x) \neq \mathfrak{f}_1(x)$

In this case, we have $\mathfrak{f}_0(x) \in X_1$ and $\mathfrak{f}_1(x) \in X_0$, and furthermore, $X_0 \neq X_1$ and $|X_0| = |X_1| = 2$. Thus, we have $\mathfrak{g}_0 \in X_1$ and $\mathfrak{g}_1 \in X_0$, so $\{\mathfrak{f}_0(x), \mathfrak{f}_1(x), \mathfrak{g}_1(x'), \mathfrak{g}_1(x')\} \subseteq X_0 \cup X_1$. Since $|X_0| = |X_1| = 2$, there is a $d \in L_\phi(y)$ with $d \notin X_0 \cup X_1$. Since $\mathfrak{f}_0(w) \neq \mathfrak{f}_1(w)$, there is an $i \in \{0,1\}$ with $d \neq \mathfrak{f}_i(w)$, and since $d \neq \mathfrak{f}_i(x), \mathfrak{g}_i(x')$, we have $d \in X_i$, a contradiction ■

Let $\mathfrak{f} \in \text{Corner}(P^\dagger, w_*)$ be as in Subclaim 10.8.29. Since $S_\star \subseteq V(J_e^1) \setminus \{u\}$, it follows from i) of Theorem 1.7.5 that there exists a $\sigma \in \text{Link}(u'Q^{\text{large}}u_*)$ with $\sigma(u_*) = \mathfrak{f}(u_*)$. By Theorem 1.6.1, there is a $\mathfrak{g} \in \text{Base}(J_{e'}^1)$ such that $\mathfrak{g}(u') = \sigma(u')$. Note that the union $\psi := \mathfrak{f} \cup \sigma \cup \mathfrak{g}$ is a proper L_ϕ-coloring of its domain. Since all the neighbors of $V(K \cup J_e^1) \setminus \text{dom}(\psi)$ in $\text{dom}(\psi)$ lie in $\text{dom}(\mathfrak{f} \cup \mathfrak{g})$, it follows from Subclaim 10.8.29 that ψ extends to an L_ϕ-coloring ψ^* of $\text{dom}(\psi) \cup V(J_e^1)$.

Let H_+^2 be the subgraph of G induced by $\text{dom}(\phi \cup \psi^*) \cup V(\tilde{G}_{P^\dagger}^{\text{small}} - w_*) \cup V(J_{e'}^1 - w') \cup \text{Sh}_2(u'Q^{\text{large}}u_*) \cup \{z\}$. Since $x' \notin N(y')$, it follows that, for any $\tau \in \Phi(\psi^*, \{y', z\})$, we have $|L_{\phi \cup \tau}(w')| \geq 3$, as $N(w') \cap \text{dom}(\phi \cup \tau) = \{u', y'\}$. Furthermore, since $p^\dagger \in V(uQx) \setminus \{u\}$, we have $|L_{\phi \cup \tau}(w_*)| \geq 3$ as well, as $N(w_*) \cap \text{dom}(\phi \cup \tau) = \{u_*, w\}$. Each of $V(\tilde{G}_{P^\dagger}^{\text{small}} - w_*)$ and $V(J_{e'}^1 - w')$ is $(L, \phi \cup \tau)$-inert by our choice of $\mathfrak{f}, \mathfrak{g}$. If w, z have no common neighbor other than y, then, for any $\tau \in \Phi(\psi^*, \{y', z\})$, the pair $[H_+^2, \phi \cup \tau]$ is a (C, z)-opener, contradicting our assumption. Thus, there is a $w^\dagger \in (N(w) \cap N(z)) \setminus \{y\}$, and $w^\dagger$ is unique. We have $|L_{\phi \cup \psi^*}(w^\dagger)| \geq 3$, and $|L_{\phi \cup \psi^*}(y')| \geq 3$ and $|L_{\phi \cup \psi^*}(z)| \geq 4$. Since $y' \notin N(w^\dagger)$, there is a $\tau \in \Phi_{L_\phi}(\psi^*, \{y', z\}$ such that $|L_{\phi \cup \tau}(w^\dagger)| \geq 3$, and $[H_+^2, \phi \cup \tau]$ is a (C, z)-opener, contradicting our assumption. ■

We now have the following.

Claim 10.8.30. *$J_{e'}^1 - y'$ is not a triangle. In particular, the non $\{x', y'\}$-endpoint of $R_{e'}$ is not adjacent to x'.*

<u>Proof:</u> Suppose that either $J_{e'}^1 = \varnothing$ or $J_{e'}^1 - y'$ is a triangle. Applying Claim 10.8.20, we fix a $\hat{v} \in V(x'Q^{\text{large}}u) \setminus \{u\}$. Recall that $q_{x'}$ is the unique neighbor of x' on Q^{large}. Note that, if there is an e'-obstruction, then, letting $u'w'$ be the unique edge of $R_{e'} \setminus \{x', y'\}$, where $w' \in D_2(C)$, we have $u' = q_{x'}$, since $J_{e'}^1 - y = u'y'x'$. In either case, we get that, for any $\sigma \in \text{Link}(Q^{\text{large}} - x')$, any extension of σ ot an L_ϕ-coloring of $\text{dom}(\sigma) \cup \{x'\}$ is an element of $\text{Link}(Q^{\text{large}})$.

Subclaim 10.8.31. *For any $\sigma \in \text{Link}(q_{x'}Q^{\text{large}}x)$ and any $d \in L_{\phi \cup \sigma}(y)$ with $|L_{\phi \cup \sigma}(y)| \geq 3$, we have $\{\sigma(q_{x'})\} = \mathcal{Z}_K(\bullet, d, \sigma(x))$*

<u>Proof:</u> Suppose toward a contradiction that $\{\sigma(q_{x'})\} = \mathcal{Z}_K(\bullet, d, \sigma(x))$. Since $\mathcal{Z}_K(\bullet, d, \sigma(x)) \neq \varnothing$, and any $\sigma' \in \Phi_{L_\phi}(\sigma, x')$ lies in $\text{Link}(Q^{\text{large}})$, we contradict Claim 10.8.21. ■

By 1) of Corollary 10.2.5, $\text{Link}(q_{x'}Q^{\text{large}}x) \neq \varnothing$, and, for any $\sigma \in \text{Link}(q_{x'}Q^{\text{large}}x)$, we have $N(y) \cap \text{dom}(\phi \cup \sigma) = \{x\}$ and thus, $|L_{\phi \cup \sigma}(y)| \geq 4$, so there is a $d \in L_{\phi \cup \sigma}(y)$ such that $|L_{\phi \cup \sigma}(w) \setminus \{d\}| \geq 3$. Thus, it immediately follows from Subclaim 10.8.31 that $\hat{v} \neq x'$, or else, letting $\sigma \in \text{Link}(q_{x'}Q^{\text{large}}x)$, we have $|\mathcal{Z}_K(\bullet, \sigma(y), \sigma(x))| > 1$, as $|L_\phi(\hat{v})| \geq 4$. Since $\hat{v} \neq x'$, we have $\hat{v} \in V(q_{x'}Q^{\text{large}}u) \setminus \{u\}$ by Claim 10.8.20. Thus, by Proposition 10.2.4, there exist two elements ψ_0, ψ_1 of $\text{Link}(q_{x'}Q^{\text{large}}x)$ which use different colors on $q_{x'}$ and restrict to the same element of $\text{Link}(uQ^{\text{large}}x)$. By Claim 10.8.25, there is a $\{\psi_0, \psi_1\}$-universal $d \in L_\phi(y)$. Since $\psi_0(q_{x'}) \neq \psi_1(q_{x'})$, this contradicts

Subclaim 10.8.31. We conclude that $J^1_{e'} \neq \varnothing$ and $J^1_{e'} - y'$ is not a triangle. Since C^1 is an induced subgraph of G, the non $\{x', y'\}$-endpoint of $R_{e'}$ is not adjacent to x'. ■

Analogous to the set B specified above, we now set $B' := \{w_* \in N(w') \cap : N(w_*) \cap V(Q \setminus J^1_{e'}) \neq \varnothing$. By Claim 10.8.27, $x' \in N(w')$. Possibly $y' \in N(w')$ as well, but in any case, $J^1_{e'} - y'$ is a broken wheel $\hat{J}$ with principal path $u'w'x'$, and w' is the unique e'-obstruction

Claim 10.8.32. $B' \neq \varnothing$

Proof: Suppose that $B' = \varnothing$. Applying 1) of Corollary 10.2.5, we fix a $\sigma \in \text{Link}(u'Qx)$. By Claim 10.8.25, there is a $\sigma' \in \Phi_{L_\phi}(\sigma, y)$ with $|L_{\phi\cup\sigma'}(w)| \geq 3$. Since $\mathcal{Z}_K(\bullet, \sigma'(y), \sigma'(x)) \neq \varnothing$, let $f \in \mathcal{Z}_K(\bullet, \sigma'(y), \sigma'(x))$. Let H^* be the subgraph of G induced by $V(H) \cup V(J^1_{e'})$ and let H^{**} be the subgraph of G induced by $V(H) \cup V(J^1_{e'}) \cup \{z\}$. Consider the following cases.

Case 1: $y' \notin N(w')$

In this case, $J^1_{e'}$ is a broken wheel with principal path $u'w'x'$. By Claim 10.8.30, $J^1_{e'}$ is not a triangle, so it follows from 2) of Proposition 1.5.1 that there is an L_ϕ-coloring of $V(J^1_{e'})$ using $\sigma(u'), f$ on the respective vertices $u'x'$, so σ' extends to an L_ϕ-coloring $\sigma^\dagger$ of $\text{dom}(\sigma') \cup V(J^1_{e'} \cup K)$. Since $y', z \notin N(w')$ and $\sigma^\dagger(x') \in \mathcal{Z}_K$ it follows from Claim 10.8.2 that $\sigma^\dagger$ extends to an L_ϕ-coloring τ of $\text{dom}(\sigma^\dagger) \cup \{z\}$ such that $|L_{\phi\cup\tau}(y')| \geq 3$. We have $|L_{\phi\cup\tau}(w)| \geq 3$ as well, since $N(w) \cap \text{dom}(\phi \cup \tau) \subseteq \text{dom}(\sigma')$. Since $B' = \varnothing$, every vertex of $D_1(H^*)$ has an $L_{\phi\cup\tau}$-list of size at least three, and $[H^*, \phi \cup \tau]$ is a (C, z)-opener, contradicting our assumption.

Case 2: $y' \in N(w')$

In this case, if w', y', z have no common neighbor in G, then, for any $\tau \in \Phi_{L_\phi}(\sigma^\dagger, \{y', z\})$, the pair $[H^{\text{aug}}, \phi \cup \tau]$ is a (C, z)-opener, so now suppose that w', y', z have a common neighbor p. Thus, G contains a wheel with central vertex y' adjacent to all the vertices of the 5-cycle $w'pzyx'$. Note that $|L_{\phi\cup\sigma^\dagger}(w')| \geq 2$, and since $|L_{\phi\cup\sigma^\dagger}(z)| \geq 4$, there is a $\tau \in \Phi_{L_\phi}(\sigma^\dagger, z)$ such that $\{y'\}$ is $(L, \phi \cup \tau)$-inert. Since $B = \varnothing$, every vertex of $D_1(H^{**})$ has an $L_{\phi\cup\tau}$-list of size at least three, and the pair $[H^{**}, \phi \cup \tau]$ is a (C, z)-opener, contradicting our assumption. ■

Applying Proposition 10.5.4, let w_*v_* be the e'-wall of B', where $v_* \in V(C^1)$. Note that $v_* \in V(u'Q^{\text{small}}x) \setminus \{u'\}$ and $w_* \in D_2(C) \cap N(w')$. Let $P := v_*w_*w'u'$. By Claim 10.8.26, $w_* \neq w$, so $v_* \in V(u'Q^{\text{large}}u) \setminus \{u\}$. Furthermore, $\tilde{G}$ contains the 3-chord $P^\times := v_*w_*w'x'$ of C^1, with $\tilde{G}^{\text{small}}_P \subseteq \tilde{G}^{\text{small}}_{P^\times}$.

Claim 10.8.33. *Let* $\sigma \in \text{Link}(v_*Q^{\text{large}}x)$ *and let* $\sigma' \in \Phi_{L_\phi}(\sigma, y)$*, where* $|L_{\phi\cup\sigma'}(w)| \geq 3$*. Let* $d' \in \mathcal{Z}_K(\bullet, \sigma'(y), \sigma'(x))$*. Then there is no element of* $\text{Corner}(P^\times, w_*)$ *which uses* $\sigma(v_*), d'$ *on the respective vertices* v_*, x'*.*

Proof: Suppose toward a contradiction that such an element of $\text{Corner}(P^\times, w_*)$ exists. Thus, σ' extends to an L_ϕ-coloring τ of $\text{dom}(\sigma') \cup \{w'\}$, where τ restricts to an element of $\text{Corner}(P^\times, w_*)$. Let $H^\dagger$ be the subgraph of G induced by $\text{dom}(\phi \cup \tau) \cup \text{Sh}_2(v_*Q^{\text{large}}x) \cup V(J^1_e - w) \cup V(J^1_{e'}) \cup V(\tilde{G}^{\text{small}}_P - w_*) \cup V(K) \cup \{z\}$. Now consider the following cases.

Case 1: $w'y' \notin E(G)$

Since $y'w' \notin E(G)$ and $|L_{\phi\cup\tau}(z)| \geq 4$, there is a $\tau' \in \Phi(\tau, z)$ such that $|L_{\phi\cup\tau'}(y')| \geq 3$. We also get that each of w_*, w has an $L_{\phi\cup\tau'}$-list of size at least three, since $N(w) \cap \text{dom}(\phi \cup \tau') \subseteq \text{dom}(\phi \cup \sigma)$, and $N(w_*) \cap \text{dom}(\phi \cup \tau) = \{v_*, w'\}$. Thus, $[H^\dagger, \phi \cup \tau']$ is a (C, z)-opener, contradicting our assumption.

Case 2: $w'y' \in E(G)$

In this case, if w', y', z do not have a common neighbor, then we simply color y' as well, and, for any $\tau' \in \Phi(\tau, \{y', z\})$, the pair $[H^\dagger, \phi \cup \tau']$ is a (C, z)-opener, contradicting our assumption. Now suppose that w', y', z have a common neighbor $w^\dagger$. We have $|L_{\phi\cup\tau}(z)| \geq 4$ and $|L_{\phi\cup\tau}(y')| \geq 2$, and since y' is the universal vertex of a wheel with a 5-cycle,there is a $\tau' \in \Phi(\tau, z)$ such that $\{y'\}$ is $(L, \phi \cup \tau')$-inert. Since y' is uncolored, we have $|L_{\phi\cup\tau'}(w^\dagger)| \geq 3$ as well, and $[H^\dagger, \phi \cup \tau']$ is a (C, z)-opener, contradicting our assumption. ■

Since u is a Q^{large}-hinge, it follows from Propositoon 10.2.4 that there exist a pair of elements ψ_0, ψ_1 of Link$(v_*Q^{\text{large}}x)$ which use different colors on u' and restrict to the same element of Link$(uQ^{\text{large}}x)$. By Claim 10.8.25, there is a $\{\psi_0, \psi_1\}$-universal $d \in L_\phi(w)$. Let $c := \psi_0(x) = \psi_1(x)$ and let $d' \in \mathcal{Z}_K(\bullet, d, c)$. For each $i = 0, 1$, let $\psi_i^\dagger$ be an L_ϕ-coloring of dom$(\psi_i) \cup \{y, x'\}$ using d, d' on the respective vertices y, x'.

Claim 10.8.34. $u' \in N(w_*)$.

<u>Proof:</u> Suppose that $u' \notin N(w_*)$. Since $J^1_{e'}$ is not a triangle, we have $N(u') \cap \text{dom}(\psi_0^\dagger) \subseteq \{v_*\}$, and since $N(w') \cap V(\tilde{G}_P^{\text{small}} \cap C^1)) = \{u'\}$, it follows from Theorem 1.6.1 applied to $\tilde{G}_P^{\text{small}}$ that there is an extension of $\psi_0^\dagger$ to an L_ϕ-coloring τ of dom$(\psi_\dagger^0) \cup \{u'\}$ such that any extension of τ to $V(P)$ extends to L_ϕ-color all of $V(\tilde{G}_P^{\text{small}})$. Possibly $\tau(u') = d'$, but in any case, since $u'w' \notin E(G)$, it follows from 2) of Proposition 1.5.1 that τ extends to an L_ϕ-coloring τ' of dom$(\tau) \cup V(J^1_{e'} - w')$, where $\tau'(x') = d'$.

Since $u' \notin N(w_*)$, it follows that, for any L_ϕ-coloring of $V(P^\times)$ which uses $\tau'(u_\star), \tau'(w'), \tau'(x')$ on the on the respective vertices v_*, w', x', this coloring leaves $\tau(u')$ for u', so τ' restricts to an element of Corner$(P^\times, w_*)$, contradicting Claim 10.8.33. ■

It follows from Claim 10.8.34 that $\tilde{G}_P^{\text{small}} - w'$ is a broken wheel K_P with principal path v_*w_*u'. Let $X_P := \bigcap(L_\phi(v) : v \in V(K_P) \setminus \{v_*, w_*\})$. Recall that $\hat{J}$ is the broken wheel $J^1_{e'} - y'$ with principal path $u'w'x'$, and recall that $q_{x'}$ is the unique neighbor of x' on Q^{large}.

Let $\ell \in \{0, 1\}$. It now follows from Claim 10.8.33 that, for each $r \in L_\phi(w') \setminus \{d'\}$, there is an L_ϕ-coloring τ^r of $V(P_*)$ such that τ^r uses $\psi^\ell(v_*), r, d'$ on the respective vertices v_*, w', x', where $\mathcal{Z}_{K_P}(\psi^\ell(v_*), \tau^r(w_*), \bullet) \cap \mathcal{Z}_{\hat{J}}(\bullet, r, d') = \varnothing$. In particular, there is no $r \in L_\phi(w') \setminus \{d\}$ such that $\mathcal{Z}_{\hat{J}}(\bullet, r, d') = L_\phi(u')$. Thus, each internal vertex of the path $\hat{J} - w'$ has an L_ϕ-list of size precisely three and $d \in L_\phi(q_{x'})$. In particular, since $|L_\phi(w')| \geq 5$, there exist $r_0, r_1 \in L_\phi(w') \setminus \{d'\}$ with $r_0, r_1 \notin L_\phi(q_{x'})$. Since $\hat{J}$ is not a triangle, it follows that, for each $i = 0, 1$, we have $\mathcal{Z}_{\hat{J}}(\bullet, r_i, d') = L_\phi(u') \setminus \{r_i\}$ and $\mathcal{Z}_{K_P}(\psi(v_*)\tau^{r_i}(w_*), \bullet) = \{r_i\}$. By Proposition 1.4.7, we have $\tau^{r_i}(w_*) = r_{1-i}$ for each $i = 0, 1$, and, in particular, letting p_{v_*} be the unique neighbor of v_* on the path $\tilde{G}_P^{\text{small}} - w_*$, we have $L_\phi(p_{v_*}) = \{r_0, r_1, \psi^\ell(v_*)\}$. Now, since $\psi^0(v_*) \neq \psi^1(v_*)$ and $\{r_0, r_1\}$ is independent of the choice of ℓ, we have a contradiction. This completes the proof of Lemma 10.8.1. □

Chapter 11

Constructing a Smaller Counterexample

In this chapter, we complete the proof of Theorem 2.1.7 by starting with a critical mosaic and constructing a smaller counterexample. Chapter 11 consists of four sections. In Sections 11.1 and 11.2, we prove two theorems, which together show that, when we construct a new mosaic from a critical mosaic by deleting a path between two rings (one of which is the outer face) with some additional specified conditions, the resulting graph still satisfies the distance conditions of Definition 2.1.6. In Section 11.1, we deal with the case where the outer face is a closed ring. In Section 11.2, we deal with the case where the outer face is an open ring. The deletion sets we construct in Sections 11.1 and 11.2 possibly require some slight modification away from the outer face in order to construct a desirable coloring of this set. This obstacle is dealt with in Section 11.3 with a technical lemma. The overview of Section 11.4, which is the final section in the proof of Theorem 10.0.7, is as follows: Given a critical mosaic $\mathcal{T} = (G, \mathcal{C}, L, C_*)$, we delete a path in G between C_* and the ring $C \in \mathcal{C} \setminus \{C_*\}$ in such a way as to produce a smaller counterexample. To do this, we apply Theorem 6.0.9 and Theorem 10.0.7.

In Theorems 6.0.9 and Theorem 10.0.7 we showed that, given a $C \in \mathcal{C}$, there is a way to associate a pair consisting of a subgraph of G and a partial coloring of G to vertex of distance two from a specified cycle close to C. We introduce the following terminology to deal with open and closed rings without excessive repetition.

Definition 11.0.1. Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic. Given a $C \in \mathcal{C}$, we define a cycle A called the *collar* of C as follows. If C is a closed ring, then we set A to be the 1-necklace of C, and if C is an open ring, then we set A to be the 2-necklace of C. We define a subgraphs of G called the *large side* and *small side* of A as follows: If C is the outer face of G then we call $\text{Int}(A)$ the large side of A and call $\text{Ext}(A)$ the small side of A, and vice-versa if C is not the outer face of G.

Note that the terms "large side" and "small side" in the definition above are consistent with their uses in Definition 6.0.2.

Observation 11.0.2. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic, let $C \in \mathcal{C}$, and let A be the collar of C. Let G' be the large side of A. Then A is $(4, L)$-short in G'.*

Proof. For any generalized chord Q of A in G' which separates two vertices of $G' \setminus Q$, we get that Q is a proper generalized chord of G', since G' is short-separation-free. Thus, there is a well-defined side of Q in G' which contains all the elements of $\mathcal{C} \setminus \{C\}$, as specified in Definitions 6.0.4 and 8.0.3. $\square$

Given Observation 11.0.2 it is convenient to introduce the following compact notation.

Definition 11.0.3. Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and let $C \in \mathcal{C}$. Let A be the collar of C and let G' be the large side of A. We set $\text{Ann}(C) := V(C) \cup \text{Sh}_{4,L}(A, G') \cup B_1(A, G)$.

The terminology in Definitions 11.0.1 and 11.0.3 allows us to deal with the settings of Theorems 6.0.9 and Theorem 10.0.7 together. In each of these theorems, given a critical mosaic $\mathcal{T} = (G, \mathcal{C}, L, C_*)$, a $C \in \mathcal{C}$ with collar A, and a vertex z on the large side of A which is of distance two from A, we associate to z a pair consisting of a subgraph K of G and a partial L-coloring of K, where $V(K) \subseteq \{z\} \cup \text{Ann}(C)$.

11.1 Dealing With a Closed Outer Face

The lone result of this section is the following.

Theorem 11.1.1. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and suppose that C_* is a closed $\mathcal{T}$-ring. Let C_m be a ring which minimizes the quantity $d_G(w_{\mathcal{T}}(C), w_{\mathcal{T}}(C_*))$ among all the $C \in \mathcal{C} \setminus \{C_*\}$. Let A be the collar of C_m and let A_* be the collar of C_*. Let H be a connected subgraph of G, where $H \setminus (\text{Ann}(C_m) \cup \text{Ann}(C_*))$ is a shortest $(D_2(A), D_2(A_*))$-path in G. Let F be the outer face of $G \setminus H$. Then, for each $C' \in \mathcal{C} \setminus \{C_m, C_*\}$, we have $d(w_{\mathcal{T}}(C'), V(F)) \geq \frac{\beta}{3} + 4N_{\text{mo}}$.*

Proof. Let $H^\dagger$ be the subgraph of G induced by $V(C_* \cup C_m) \cup V(H)$, and let Q be the path $H \setminus (\text{Ann}(C_m) \cup \text{Ann}(C_m))$.

Claim 11.1.2. *For any $C' \in \mathcal{C} \setminus \{C_*, C_m\}$ we have $d(w_{\mathcal{T}}(C'), C_*) + d(w_{\mathcal{T}}(C_m), C_*) \geq \beta + \frac{3N_{\text{mo}}}{2}$. In particular, we have $d(w_{\mathcal{T}}(C'), C_*) \geq \frac{\beta}{2} + \frac{3N_{\text{mo}}}{4}$.*

Proof: By our distance conditions, we have $d(w_{\mathcal{T}}(C'), w_{\mathcal{T}}(C_m)) \geq \beta + 2N_{\text{mo}}$. Since C_* is a closed $\mathcal{T}$-ring, any two vertices of C_* are of distance at most $\frac{N_{\text{mo}}}{2}$-apart, so we have $d(w_{\mathcal{T}}(C'), C_*) + d(w_{\mathcal{T}}(C_m), C_*) \geq \beta + \frac{3N_{\text{mo}}}{2}$. Now suppose toward a contradiction that $d(w_{\mathcal{T}}(C'), C_*) < \frac{\beta}{2} + \frac{3N_{\text{mo}}}{4}$. By the minimality of C_m, we have $d(w_{\mathcal{T}}(C_m), C_*) < \frac{\beta}{2} + \frac{3N_{\text{mo}}}{4}$ as well, so $d(w_{\mathcal{T}}(C'), C_*) + d(w_{\mathcal{T}}(C_m), C_*) < \beta + \frac{3N_{\text{mo}}}{2}$, which is false, as indicated above. ■

We now have the following:

Claim 11.1.3. *For any $C' \in \mathcal{C} \setminus \{C_*, C_m\}$, we have $d(w_{\mathcal{T}}(C'), H^\dagger) > \frac{\beta}{3} + 6N_{\text{mo}}$.*

Proof: Suppose toward a contradiction that there is a $C' \in \mathcal{C} \setminus \{C_*, C_m\}$ violating this inequality. Let P be a shortest $(w_{\mathcal{T}}(C'), H^\dagger)$-path in G, and let $P := v_1 \cdots v_t$, where $v_1 \in V(w_{\mathcal{T}}(C'))$ and $v_t \in V(H^\dagger)$. Then $|E(P)| \leq \frac{\beta}{3} + 6N$. If $v_t \in \text{Ann}(C_m)$, then there is a $(w_{\mathcal{T}}(C'), B_4(C_m))$-path in G of length at most $\frac{\beta}{3} + 6N$. Since each vertex of $B_4(C_m)$ is of distance at most $4 + \frac{N_{\text{mo}}}{3}$ from $w_{\mathcal{T}}(C_m)$, we have $d(w_{\mathcal{T}}(C'), w_{\mathcal{T}}(C_m)) \leq \frac{\beta}{3} + 6N + \frac{N_{\text{mo}}}{3} + 4$, contradicting our distance conditions. Thus, we have $v_t \notin \text{Ann}(C_m)$, so $v_t \in \text{Ann}(C_*) \cup V(Q)$. Consider the following cases:

Case 1: $v_t \in \text{Ann}(C_*)$

In this case, if $v_t \in B_4(C_*)$, then we have $d(w_{\mathcal{T}}(C'), C_*) \leq \frac{\beta}{3} + 6N_{\text{mo}} + 4$, contradicting Claim 11.1.2 Likewise, if $v_t \in \text{Ann}(C_*) \setminus B_4(C_*)$, then there exists a generalized chord R of C^1 in $\text{Int}(C^1)$, where $|E(R)| \leq 4$, such that, in $\text{Int}(C^1)$, R separates v_t from each element of $\mathcal{C} \setminus \{C_*\}$. In that case, since each vertex of R lies in $B_4(C_*)$, it follows that there exists a $(w_{\mathcal{T}}(C'), C_*)$-path of length at most $\frac{\beta}{3} + 6N + 4$, contradicting Claim 11.1.2.

Case 2: $v_t \notin \text{Ann}(C_*)$

In this case, we have $v_t \in V(Q)$. Since Q is a $(D_2(A), D_2(A_*))$-path by assumption, we let $Q := w_1 \cdots w_s$, where $w_1 \in D_2(A)$ and $w_s \in D_2(A_*)$. There exists an index $i \in \{1, \cdots, s\}$ such that $v_t = w_i$. Let $P_1 := v_1 P v_t Q w_s$ and let $P_2 := v_1 P v_t Q w_1$. Since P_2 is a $(w_{\mathcal{T}}(C'), D_2(A))$-path, we have $|E(P_2)| \geq (\beta - 4) - \frac{N_{\text{mo}}}{3}$. Since

$|E(P_1)| + |E(P_2)| = 2|E(P)| + |E(Q)|$, we obtain the inequality $2|E(P)| + |E(Q)| \geq |E(P_1)| + (\beta - 4) - \frac{N_{\text{mo}}}{3}$. Since $|E(P)| \leq \frac{\beta}{3} + 6N$, we then have

$$|E(Q)| - |E(P_1)| \geq (\beta - 4) - \left(\frac{2\beta}{3} + 12N_{\text{mo}}\right) - \frac{N_{\text{mo}}}{3} \geq \left(\frac{\beta}{3} - 13N_{\text{mo}} - 4\right)$$

Since Q is a shortest $(D_2(A), D_2(A_*))$-path in G, we have $d(w_{\mathcal{T}}(C_m), C_*) \geq |E(Q)| + 8$. Likewise, we have $|E(P_1)| \geq d(w_{\mathcal{T}}(C'), C_*) - 4$, so we obtain

$$d(w_{\mathcal{T}}(C_m), C_*) \geq |E(P_1)| + \left(\frac{\beta}{3} + 4 - 13N_{\text{mo}}\right) \geq d(w_{\mathcal{T}}(C'), C_*) + \left(\frac{\beta}{3} - 13N_{\text{mo}}\right)$$

By the minimality of $d(w_{\mathcal{T}}(C_m), C_*)$, we then have $\frac{\beta}{3} - 13N_{\text{mo}} \leq 0$. Recall that $\beta := \frac{17}{15}N_{\text{mo}}^2$ and $N_{\text{mo}} \geq 96$, so the inequality $\beta \leq 39N_{\text{mo}}$ is false, giving us our desired contradiction. This completes the proof of Claim 11.1.3. ■

We now return to the proof of Theorem 11.1.1. Applying Claim 11.1.3, for each vertex $v \in V(H)$, every facial subgraph G containing v, except possibly C_m, C_*, is a triangle, as all the other elements of $\mathcal{C}$ are far from $H^\dagger$. Thus, it immediately follows from Theorem 1.3.2 that $V(F) \setminus V(C_m \cup C_*) \subseteq D_1(H)$, so we have $V(F) \subseteq B_1(H^\dagger)$. Combining this with Claim 11.1.3, it immediately follows that, for any $C' \in \mathcal{C} \setminus \{C_m, C_*\}$, we have $d(F, w_{\mathcal{T}}(C')) \geq \frac{\beta}{3} + 4N_{\text{mo}}$. This completes the proof of Theorem 11.1.1. □

11.2 Dealing With an Open Outer Face

We now prove an analogue to Theorem 11.1.1 for the case where the outer face is an open ring. This is surprisingly technical, and considerable harder than the proof of Theorem 11.1.1. The central obstacle is the fact that, given a critical mosaic $\mathcal{T} := (G, \mathcal{C}, L, C_*)$, it is possible that two elements of $\mathcal{C} \setminus \{C_*\}$ are both close to C_* even though they are far from each other. This is not the case when C_* is closed because the length of C_* is bounded in that case. This obstacle is the reason for the technical conditions in 5) of Definition 6.0.8 which deal with the distance between the precolored path of the outer face and the deletion set we constructed Theorem 6.0.9. We begin this section with the following definition.

Definition 11.2.1. Let Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and suppose that C_* is an open $\mathcal{T}$-ring. Let C_*^2 be the 2-necklace of C, and let $C \in \mathcal{C} \setminus \{C_*\}$. A path P is called *C-monotone* if P is a $(w_{\mathcal{T}}(C), D_3(C_*^2))$-path which satisfies the following.

1) P is a quasi-shortest path*; AND*
2) $|V(P) \cap D_4(C_*^2)| = 1$*; AND*
3) $|E(P)| \leq d(w_{\mathcal{T}}(C), D_3(C_*^2)) + 3$.

We now have the following.

Definition 11.2.2. Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and suppose that C_* is an open $\mathcal{T}$-ring. Let C_*^1 be the 1-necklace of C_* and let C_*^2 be the 2-necklace of C. Given a $C \in \mathcal{C} \setminus \{C\}$, we introduce the following terminology.

1) A subgraph H of G is called a *C-seam* if there is a unique vertex $z \in V(H) \cap D_2(C_*^2)$ such that the following hold:
    a) $H \setminus (\text{Ann}(C_*) \cup \{z\})$ is a C-monotone path and the $D_3(C_*^2)$-endpoint of this path is adjacent to z*; AND*

b) The subgraph of G induced by $V(H) \cap (\text{Ann}(C_*) \cup \{z\})$ is the underlying graph of a (C_*, z)-opener. The head of this (C_*, z)-opener is also called the *head* of H.

2) The vertex z is called the *join* of H. The path $H \setminus (\text{Ann}(C_*) \cup \{z\})$ is called the *tail* of H.

Note that the head of H is indeed well-defined, since, by Definition 6.0.8, $C_*^2 \cap H$ is a path which contains both $\text{Pin}(z)$ and $\mathbf{P}_*^1$. This section is short but somewat technical. The proof of the theorem below, which is an analogue to Theorem 11.1.1 for the case where the outer face is open, makes up the remainder of this section.

Theorem 11.2.3. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic and suppose that C_* is an open $\mathcal{T}$-ring. Let C_*^1 be the 1-necklace of C_* and let C_*^2 be the 2-necklace of C_*. Then there exists a $C \in \mathcal{C} \setminus \{C_*\}$ and a C-monotone path P such that, for any $z \in D_2(C_*^2)$ which is adjacent to the $D_3(C_*^2)$-endpoint of P, there exists a C-seam H such that the following hold.*

1) P is the tail of H and z is the join of H; AND

2) Letting F be the outer face of $G \setminus H$, every $C' \in \mathcal{C} \setminus \{C, C_*\}$ satisfies the inequality $d(F, w_{\mathcal{T}}(C')) \geq \frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C')$.

Proof. Suppose toward a contradiction that the theorem does not hold. Given a $C \in \mathcal{C} \setminus \{C\}$ and a C-seam H, an element $C' \in \mathcal{C} \setminus \{C', C_*\}$ is called an *H-blocker* if, letting F be the outer face of $G \setminus H$, we have $d(F, w_{\mathcal{T}}(C')) < \frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C')$.

Claim 11.2.4. *Let $C \in \mathcal{C} \setminus \{C_*\}$ and let P be a $(w_{\mathcal{T}}(C), D_3(C_*^2))$-path. Then $B_1(P) \cap \text{Ann}(C_*) = \varnothing$.*

<u>Proof:</u> Suppose that $B_1(P) \cap \text{Sh}_4(C_*^2, \text{Int}(C_*^2)) \neq \varnothing$. Since P is a $(w_{\mathcal{T}}(C), D_3(C_*^2))$-path, there is a generalized chord R of C_*^2 in $\text{Int}(C_*^2)$ such that $|E(R)| \leq 4$ and $V(R \cap P) \neq \varnothing$. Each vertex of R has distance at most two from $V(C_*^2)$ so $V(R) \cap B_2(C_*^2) \neq \varnothing$. Since $D_3(C_*^2)$ separates C from $B_2(C_*^2)$, there is an internal vertex of R in $D_3(C_*^2)$, contradicting the fact that R is a $(w_{\mathcal{T}}(C), D_3(C_*^2))$-path. Thus, we have $B_1(P) \cap \text{Sh}_4(C_*^2, \text{Int}(C_*^2)) = \varnothing$

Now suppose that $B_1(P) \cap \text{Ann}(C_*) \neq \varnothing$. S In that case, since $B_1(P) \cap \text{Sh}_4(C_*^2, \text{Int}(C_*^2)) = \varnothing$, there is a vertex p of P of distance at most one from $B_1(C_*^2) \cup V(C_*)$, so $p \in B_2(C_*^2)$. Since the endpoints of P lie in $w_{\mathcal{T}}(C)$ and $D_3(C_*^2)$, p is an internal vertex of P. Since the deletion of $D_3(C_*)$ disconnects $w_{\mathcal{T}}(C)$ from $B_2(C_*^2)$, there is an internal vertex of P lying in $D_3(C_*^2)$, contradicting the fact that P is a $(w_{\mathcal{T}}(C), D_3(C_*^2))$-path. ■

Now we have the following.

Claim 11.2.5. *Let $C \in \mathcal{C} \setminus \{C_*\}$, let P be a C-monotone path, and let z be a vertex of $D_2(C_*^2)$ which is adjacent to the $D_3(C_*^2)$-endpoint of P. Then there exists a C-seam H such that P is the tail of H and z is the join of H.*

<u>Proof:</u> Since P is C-monotone, it follows from Claim 11.2.4 that $z \in D_2(C_*^2) \setminus \text{Sh}_4(C_*^2, \text{Int}(C_*^2))$ by definition. Thus, by Theorem 6.0.9, there exists a (C_*, z)-opener. Let K be the underlying graph of a (C_*, z)-opener, let p be the $D_3(C_*^2)$-endpoint of P, and consider the graph $H := (P \cup K) + zp$. We claim that H is a C-seam in which z is the uniquely specified join. By 5a) of Definition 6.0.8, z is the only vertex of $(V(K) \cap D_2(C_*^2)) \setminus \text{Sh}_4(C_*^2, \text{Int}(C_*^2))$. By Claim 11.2.4, we have $G[V(H) \cap (V(\text{Ann}(C_*) \cup \{z\})] = K$, so H is a C-seam in which z is the uniquely specified join of H and P is the tail of H. ■

Now we introduce the following notation.

Definition 11.2.6. We set $\mathcal{S}$ to be the set of triples (C, P, H) such that the following hold.

1) $C \in \mathcal{C} \setminus \{C_*\}$ and P is a C-monotone path of length at most $\frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C)$*; AND*
2) H is a C-seam with tail P, and there exists an H-blocker.

Now we have the following:

Claim 11.2.7. *Let $(C, P, H) \in \mathcal{S}$ and let F be the outer face of $G \setminus H$. Let $C' \in \mathcal{C} \setminus \{C\}$ be an H-blocker and let R be a shortest $(w_{\mathcal{T}}(C'), F)$-path in $G \setminus H$. Then the following holds:*

1) $d(R, H \setminus \text{Ann}(C_*)) > \frac{\beta}{4} + 2$; AND

2) The F-endpoint of R is of distance precisely one from $V(H) \cap \text{Ann}(C_*)$, and $V(R) \cap V(C_*) = \varnothing$.

Proof: Suppose toward a contradiction that $d(R, H \setminus \text{Ann}(C_*)) \leq \frac{\beta}{4} + 2$. By assumption, R is a path of length at most $\frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C') - 1$, and since $(C, P, H) \in \mathcal{S}$, it follows that $H \setminus \text{Ann}(C_*)$ is a path of length at most $\frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C) + 1$. Thus, we have $d(w_{\mathcal{T}}(C'), w_{\mathcal{T}}(C)) \leq \frac{2\beta}{3} + 4N_{\text{mo}} + \text{Rk}(\mathcal{T}|C) + \text{Rk}(\mathcal{T}|C') + \frac{\beta}{4} + 2$. Since $C \neq C'$ and $C, C' \in \mathcal{C} \setminus \{C_*\}$, it follows from our distance conditions on $\mathcal{T}$ that $\frac{\beta}{4} + 4N_{\text{mo}} + 2 \geq \frac{\beta}{3}$, and thus $48N_{\text{mo}} + 24 \geq \beta$, which is false.

It follows from Theorem 1.3.2 that $V(F) \subseteq D_1(H) \cup V(C_* \setminus H) \cup V(C \setminus H)$. Since $|E(R)| < \frac{\beta}{3} + 2_{\text{mo}} + \text{Rk}(\mathcal{T}|C')$, it follows from our distance conditions on $\mathcal{T}$ that the F-endpoint of R does not lie in $C_* \setminus \mathring{\mathbf{P}}^*$ and does not lie in $w_{\mathcal{T}}(C)$. Since $R \subseteq G \setminus H$, no vertex of R lies in $C_* \cap H$. Since R is a shortest, it follows that $V(R) \cap V(C_*) = \varnothing$. Let z be the join of H. By assumption, the subgraph of G induced by $V(H) \cap (\text{Ann}(C_*) \cup \{z\})$ is the underlying graph of a (C_*, z)-opener, so it follows from 5a) of Definition 6.0.8 that $V(\mathbf{P}_*) \subseteq V(H) \cap \text{Ann}(C_*)$. Since $d(R, H \setminus \text{Ann}(C_*)) > \frac{\beta}{4} + 2$, it follows that the F-endpoint of R is of distance one from $V(H) \cap \text{Ann}(C_*)$. ■

We now have the following.

Claim 11.2.8. *Let $(C, P, H) \in \mathcal{S}$, let z be the join of H, and let F be the outer face of $G \setminus H$. Let $C' \in \mathcal{C} \setminus \{C\}$ be an H-blocker and let $R := p_1 \cdots p_s$ be a shortest $(w_{\mathcal{T}}(C'), F)$-path in $G \setminus H$, where p_s is the F-endpoint of R. Then $H \cap C_*^2$ is a path, and the following hold.*

1) There is a vertex of $p_j \in V(R)$ of distance at most two from $H \cap C_*^2$, where $j \in \{s-2, s-1, s\}$; AND

2) The path $\text{Pin}(z)$ is a terminal subpath of $H \cap C_*^2$, and $\text{Pin}(z) \cap \mathbf{P}_*^1 = \varnothing$; AND

3) The head of H is a path of length at least $\frac{\beta}{4} - N_{\text{mo}}$.

Proof: . Since $G[V(H) \cap \text{Ann}(C_*)]$ is the underlying graph of a (C_*, z)-opener, there is a subpath Q of C_*^2 such that $H \cap C_*^2 = Q$, where each of the paths $\mathbf{P}_1^*$ and $\text{Pin}(z)$ is a subpath of Q.

Subclaim 11.2.9. *There is a vertex of $p_{s-2}p_{s-1}p_s$ of distance at most two from Q.*

Proof: By Claim 11.2.7, p_s is of distance precisely one from $V(H) \cap \text{Ann}(C_*)$, and $V(R) \subseteq V(G \setminus C_*)$. If the F-endpoint of R lies in $C^1 \setminus \mathbf{P}_*^1$ then, by Condition 5e) of Definition 6.0.8, the F-endpoint of R has distance at most two from Q, so we are done in that case. Now suppose that the F-endpoint of R does not lie in $C^1 \setminus \mathbf{P}_1^*$. Since $V(R) \subseteq V(G \setminus C_*)$, the F-endpoint of R lies in $D_1(V(H) \cap \text{Ann}(C_*)) \cap V(\text{Ext}(C_*^2))$. Let p' be a neighbor of p_s of distance one from $V(H) \cap \text{Ann}(C_*)$. If $p' \in B_1(C_*^2) \cap V(H)$, then, again by Condition 5e) of Definition 6.0.8, the F-endpoint of R has distance at most two from Q, so we are done in that case. Since

$p_s \in V(\text{Ext}(C_*^2)$ and $\mathbf{P}_* \subseteq H$, p_s has no neighbors in $C_* \setminus H$, so we just need to deal with the case where $p' \in \text{Sh}_4(C_*^2, \text{Int}(C_*^2))$. By Claim 11.2.7, we have $d(z, p_s) > \frac{\beta}{4}$, so, by Condition 5a) of Definition 6.0.8, we have $p' \in \text{Sh}_4(Q, C_*^2, \text{Int}(C_*^2))$. Thus, there is a subpath of R with one endpoint in $w_{\mathcal{T}}(C')$ and one endpoint q in a generalized chord of C_*^2 of length at most four, where the endpoints of this generalized chord lie in Q. Thus, q has distance at most two from Q. We just need to check that $q \in \{p_{s-2}, p_{s-1}, p_s\}$.

Suppose toward a contradiction that $q \notin \{p_{s-2}, p_{s-1}, p_s\}$. By assumption, we have $|E(R)| < \frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C')$, and thus $E(p_1Rq)| < \frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C') - 3$. Since q has distance at most two from Q, there is a (q, C_*)-path of length at most four. If this path has an endpoint in $C_* \setminus \mathring{\mathbf{P}}$, then we have $(w_{\mathcal{T}}(C'), w_{\mathcal{T}}(C_*))$-path of length at most $\frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C') - 1$, contradicting the distance conditions on $\mathcal{T}$. Thus, no such (q, C_*)-path exists, and thus there is a (q, C_*)-path which has length at most three and contains an internal vertex of $\mathbf{P}_*^1$. But since $\mathbf{P}_*^1 \subseteq Q$, it follows that q has distance at most one from F. Since $q \in V(p_1Rp_{s-3})$, this contradicts the fact that R is a shortest $(w_{\mathcal{T}}(C'), F)$-path. ■

The above proves 1).Now we prove 2). Applying Subclaim 11.2.9, let $\hat{p}$ be a vertex of R of distance at most two from Q. Suppose toward a contradiction that $\text{Pin}(z)$ is not a terminal subpath of Q. In that case, by 5b) of Definition 6.0.8, every vertex of Q is of distance at most 8 from $\mathbf{P}_*$, so the join z of H has distance at most 10 from $\mathbf{P}_*$. Since any two vertices of $\mathbf{P}_*$ are of distance at most $\frac{2N_{\text{mo}}}{3}$ apart, we have $d(\hat{p}, z) \leq 20 + \frac{2N_{\text{mo}}}{3}$. Since $z \in V(H) \setminus \text{Ann}(C_*)$, this contradicts Claim 11.2.7. Thus $\text{Pin}(z)$ is indeed a terminal subpath of $H \cap C_*^2$

Subclaim 11.2.10. *R has distance at most three from $Q \setminus \text{Sh}_4(Q, C_*^2, \text{Int}(C_*^2))$.*

Proof: By Subclaim 11.2.9, there is a vertex $v \in V(Q)$ of distance at most two from R. If $v \in V(Q) \setminus \text{Sh}_4(Q, C_*^2, \text{Int}(C_*^2))$, then we are done, so suppose that $v \in V(Q) \cap \text{Sh}_4(Q, C_*^2, \text{Int}(C_*^2))$. In that case, since one endpoint of R lies in $w_{\mathcal{T}}(C')$, there is a subpath R' of R with one endpoint in $w_{\mathcal{T}}(C')$ and the other endpoint of distance at most one from a proper generalized chord M of C_*^2 in $\text{Int}(C_*^2)$, where M has length at most four and the endpoints of M lie in $Q \setminus \text{Sh}_4(Q, C_*^2, \text{Int}(C_*^2))$. Since each vertex of M has distance at most two from the endpoints of M, it follows that R has distance at most three from $Q \setminus \text{Sh}_4(Q, C_*^2, \text{Int}(C_*^2))$. ■

Now suppose toward a contradiction that $\text{Pin}(z) \cap \mathbf{P}_*^1 \neq \varnothing$. In that case, by 5d) of Definition 6.0.8, every vertex of $Q \setminus \text{Sh}_4(Q, C_*^2, \text{Int}(C_*^2))$ has distance at most 14 from $\mathbf{P}_*$. Since $\text{Pin}(z)$ is a terminal subpath of Q, it follows that z has distance at most two from $Q \setminus \text{Sh}_4(Q, C_*^2, \text{Int}(C_*^2))$. By Subclaim 11.2.10, R has distance at most three from $Q \setminus \text{Sh}_4(Q, C_*^2, \text{Int}(C_*^2))$. Since any two vertices of $\mathbf{P}_*$ are of distance at most $\frac{2N_{\text{mo}}}{3}$ apart, it follows that $d(z, R) \leq \frac{2N_{\text{mo}}}{3} + 19$, contradicting Claim 11.2.7. Thus, we have $\text{Pin}(z) = \varnothing$. This proves 2).

Now we prove 3). Let Q_- be the head of H. Since $\text{Pin}(z) \cap \mathbf{P}_*^1 = \varnothing$, it follows from 5c) of Definition 6.0.8 that every vertex of $Q \setminus (\mathbf{P}_*^1 \cup Q_-)$ has distance at most 8 from $\mathbf{P}_*$, so each vertex of $Q \setminus P$ has distance at most 8 from $\mathbf{P}_*$. By Subclaim 11.2.9, there is a vertex $\hat{p}$ of R of distance at most two from Q. Suppose toward a contradiction that $|E(P)| < \frac{\beta}{4} - N_{\text{mo}}$. In that case, since each of $\hat{p}, z$ has distance at most two from Q, and each vertex of $Q \setminus Q_-$ has distance at most 8 from $\mathbf{P}_*$, we have $d(\hat{p}, z) < 12 + |E(\mathbf{P}_*)| + \frac{\beta}{4} - N_{\text{mo}},$. Since $|E(\mathbf{P}_*)| \leq \frac{2N_{\text{mo}}}{3}$ and $\frac{2N}{3} + 12 < N_{\text{mo}}$, we have $d(\hat{p}, z) < \frac{\beta}{4}$, contradicting Claim 11.2.7. Thus, Q_- is indeed a path of length at least $\frac{\beta}{4} - N_{\text{mo}}$. ■

Now we have the following:

Claim 11.2.11. *Let $C \in \mathcal{C} \setminus \{C_*\}$, let P be a C-monotone path, let H be a C-seam with tail P, and let $C^\dagger \in \mathcal{C} \setminus \{C\}$ be an H-blocker. Then there exists a $(w_{\mathcal{T}}(C^\dagger), D_3(C_*^2))$ path $R^\dagger$ such that the following hold.*

A) $R^\dagger$ is a $C^\dagger$-monotone path and is of length at most $\frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}C^\dagger)$; AND

B) $d(P, R^\dagger) \geq \frac{\beta}{4}$*; AND*

C) If $|E(P)| \leq \frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C)$*, then the* $D_3(C_*^2)$*-endpoint of* $R^\dagger$ *is of distance at most 11 from* $H \cap C_*^2$*.*

Proof: Let z be the join of H. By definition, $G[V(H) \cap (\text{Ann}(C_*) \cup \{z\})]$ is the underlying graph of a (C_*, z)-opener, so $H \cap C_*^2$ is a path Q by 5) of Definition 6.0.8. Let F be the outer face of $G \setminus H$ and let $R := p_1 \cdots p_s$, where $p_1 \in w_{\mathcal{T}}(C^\dagger)$ and $p_s \in V(F)$.

Subclaim 11.2.12. $V(R) \cap B_2(C_*^2) \neq \varnothing$.

Proof: By 2) of Claim 11.2.7, p_s has a neighbor in $V(H) \cap \text{Ann}(C_*)$. If $p_s$has a neighbor in $V(C_*) \cup B_1(C_*^2)$, then we immediately have $V(R) \cap B_2(C_*^2) \neq \varnothing$. Now suppose that p_s has a neighbor in $\text{Sh}_4(C_*^2, \text{Int}(C_*^2))$. Then there is a $j \in \{1, \cdots, s\}$ such that p_j lies on a generalized chord of C_*^2 of length at most four, so $p_j \in B_2(C_*^2)$ and we again have $V(R) \cap B_2(C_*^2) \neq \varnothing$. ■

Since $V(R) \cap B_2(C_*^2) \neq \varnothing$ and $D_4(C_*^2)$ disconnects $C^\dagger$ from $B_2(C_*^2)$, there is an index $m \in \{1, \cdots, s\}$ such that $p_m \in D_4(C_*^2)$. Let m be the minimal index with this property. Since $D_4(C_*^2)$ disconnects $C^\dagger$ from $B_3(C_*^2)$, we have $V(p_1 R p_m) \cap B_3(C_2^*) = \varnothing$, and, in particular, $p_1 R p_m$ is a $(w_{\mathcal{T}}(C^\dagger), D_4(C_*^2))$-path and, for any $q \in D_3(C_*^2) \cap N(p_j)$, $p_1 R p_m q$ is a $(w_{\mathcal{T}}(C^\dagger), D_3(C_*^2))$-path. Furthermore, since $V(R) \cap B_2(C_*^2) \neq \varnothing$, we have $m < s - 1$.

Subclaim 11.2.13. *For any* $q \in D_3(C_*^2) \cap N(p_j)$*, the following hold:*

1. $p_1 R p_m q$ *is an induced path in G; AND*
2. *Either* $|E(P)| > \frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C)$ *or* $d(q, C_*^2 \cap H) \leq 8$

Proof: Firstly, since $q \in D_3(C_*^2) \cap N(p_j)$ and $|V(p_1 R p_m q) \cap B_4(C_*^2) = \{p_m, q\}$, it immediately follows that p_m is the only vertex of $p_1 R p_j q$ adjacent to q. Since R is a shortest $(w_{\mathcal{T}}(C^\dagger), F)$-path, it is an induced path, so $p_1 P p_m q$ is also an induced path.

Now we prove 2). Suppose that $|E(P)| \leq \frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C)$. Thus, we have $(C, P, H) \in \mathcal{S}$. We claim that $d(q, Q) \leq 8$. If p_m has distance four from $\mathbf{P}_*^1$, then we are done, since $\mathbf{P}_*^1 \subseteq Q$ by 5a) Definition 6.0.8. Now suppose that p_m does not have distance three from $\mathbf{P}_*^1$. Thus, p_j has distance four from a vertex of $C_*^2 \setminus \mathbf{P}_*^1$. By assumption $|E(R)| < \frac{\beta}{3} + 2N_{\text{mo}} + w_{\mathcal{T}}(C^\dagger)$. Since each vertex of $C^2 \setminus \mathbf{P}_*^1$ has distance at most two from $w_{\mathcal{T}}(C_*)$, we have $s - 5 \leq m \leq s$, or else we contradict the distance conditions on $\mathcal{T}$. Combining this with 1) of Claim 11.2.8 it follows that p_m has distance at most 7 from from Q, so q has distance at most 8 from $H \cap C_*^2$, as desired. This proves 2). ■

We now have the following.

Subclaim 11.2.14. *For any* $q \in D_3(C_*^2)$*, we have* $|E(p_1 R p_m q)| \leq d(w_{\mathcal{T}}(C^\dagger), D_3(C_*^2)) + 3$*.*

Proof: Suppose toward a contradiction that $|E(p_1 R p_m q)| > d(w_{\mathcal{T}}(C^\dagger), D_3(C_*^2)) + 3$. Since $m < s - 1$, we have $|E(p_1 R p_m q)| \leq \frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C^\dagger) - 1$, so it follows that $d(w_{\mathcal{T}}(C^\dagger), D_3(C_*^2)) < \frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C^\dagger) - 4$. Thus, there exists a $(w_{\mathcal{T}}(C^\dagger), D_3(C_*^2))$-path R_{short} of length at most $\frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C^\dagger) - 5$. Since every vertex of $C_*^2 \setminus \mathbf{P}_*^1$ is of distance two from $w_{\mathcal{T}}(C_*)$, it follows that the $D_3(C_*^2)$-endpoint of R_{short} lies in $D_3(\mathbf{P}_*^1)$, or else we contradict the distance conditions on $\mathcal{T}$.

Since $B_1(\mathbf{P}_*^1) \subseteq V(H \cup F)$, the $D_3(C_*^2)$-endpoint of R_{short} is of distance at most two from $V(H \cup F)$. Since $D_1(H) \subseteq V(F)$, there is a $(w_{\mathcal{T}}(C^\dagger, F)$-path of length at most $|E(R_{\text{short}})| + 3$. By assumption, $|E(R_{\text{short}})| + 3 < |E(p_1 R p_m q)| \leq E(R)$, so this contradicts the fact that R is a shortest $(w_{\mathcal{T}}(C^\dagger), F)$-path. ■

Given a path $T := w_1 \cdots w_t$, we now introduce the following. We set $\text{Def}(T)$ to be the set of $v \in D_1(T)$ such that $G[N(v) \cap V(T)]$ is not a subpath of T of length at most two.

Subclaim 11.2.15. *Let $T := w_1 \cdots w_t$ be an induced $(w_{\mathcal{T}}(C^\dagger), D_3(C_*^2))$-path in G.*

1) If T is not a quasi-shortest path, then $\text{Def}(T) \neq \varnothing$*; AND*

2) For any $v \in \text{Def}(T)$, letting $w_i, w_j \in V(T)$ be the respective vertices of minimal and maximal index in $N(v) \cap V(T)$, we have $|i - j| > 2$

Proof: Since G is short-separation-free and no internal vertex of T lies in any element of $\mathcal{C}$, it follows from our triangulation conditions that any vertex of $D_1(T)$ which is adjacent to two vertices of distance two apart on T is also adjacent to the unique neighbor of these two vertices on T. Both 1) and 2) follow immediately. ■

Since $p_m \in D_4(C_*^2)$, we have $N(p_m) \cap D_3(C_*^2) \neq \varnothing$, so let $q \in N(p_m) \cap D_3(C_*^2)$. Let $R^\dagger := p_1 R p_m q$. Now suppose toward a contradiction that does not exist a $(w_{\mathcal{T}}, D_3(C_*^2))$-path which satisfies all of Conditions A), B), and C) of Claim 11.2.11.

Subclaim 11.2.16. $\text{Def}(R^\dagger) \neq \varnothing$ *and* $\text{Def}(R^\dagger) \subseteq D_4(C_*^2) \cap N(q)$

Proof: Suppose that $\text{Def}(R^\dagger) = \varnothing$. By 1) of Claim 11.2.13, $R^\dagger$ is an induced path, so $R^\dagger$ is a quasi-shortest path. Furthermore, by construction of $R^\dagger$, we have $V(R^\dagger) \cap D_4(C_*^2) = \{p_m\}$, and, by Subclaim 11.2.14, $|E(R^\dagger)| \leq d(w_{\mathcal{T}}(C^\dagger), D_3(C_*^2)) + 3$. Thus $R^\dagger$ is a $C^\dagger$-monotone path. Since R has length at most $\frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}C^\dagger)$, $R^\dagger$ also has length at most $\frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}C^\dagger)$, so $R^\dagger$ satisfies Condition A). Since $|V(R^\dagger) \setminus V(R)| \leq 1$, it follows from 1) of Claim 11.2.7 that $d(R^\dagger, P) \geq \frac{\beta}{4}$, so $R^\dagger$ satisfies B). Finally, by 2) of Subclaim 11.2.13, $R^\dagger$ also satisfies C), contradicting our assumption.

Now, since R is a shortest path between its endpoints, $p_1 R p_m$ is also a shortest path between its endpoints, so we have $\text{Def}(R^\dagger) \subseteq N(q)$, and there exists a $j \in \{1, \cdots, m-2\}$ with $p_j \in N(v)$. Since $|V(R^\dagger) \cap D_4(C_*^2)| = 1$, it follows that v has a neighbor in $D_3(C_*^2)$ and and a neighbor in $R^\dagger \setminus B_4(C_*^2)$, so we have $v \in D_4(C_*^2)$. ■

For each $v \in \text{Def}(R^\dagger)$, let m^v be the minimal index among $\{1 \leq i \leq s : p_i \in N(v)\}$. Since $\text{Def}(R^\dagger) \neq \varnothing$, we choose a $v \in \text{Def}(R^\dagger)$ which minimizes the quantity m^v. Possibly there are two vertices of $\text{Def}(R^\dagger)$ adjacent to p_{m^v}, but we just pick one arbitrarily). Let $R_1^\dagger := p_1 R p_{m^v} v q$.

Subclaim 11.2.17. *All of the following hold.*

1) $V(R_1^\dagger) \cap D_4(C_*^2) = \{v\}$ and $R_1^\dagger$ is an induced $(w_{\mathcal{T}}(C^\dagger), D_3(C_*^2))$-path; AND

2) $\text{Def}(R_1^\dagger) \neq \varnothing$*; AND*

3) For each $w \in \text{Def}(R_1^\dagger)$, we have $w \in N(v) \setminus N(q)$.

Proof: Since $|V(R^\dagger) \cap D_4(C_*^2) = \{p_m\}$, it follows that $v \in D_4(C_*^2)$, since v has a neighbor in $D_3(C_*^2)$ and a neighbor in $G \setminus B_4(C_*^2)$. Since $m^v < m$, it also follows that $|V(R_1^\dagger) \cap D_4(C_*^2) = \{v\}$, and $R_1^\dagger$ is a $(w_{\mathcal{T}}(C^\dagger), D_3(C_*^2))$-path. By Subclaim 11.2.13, $R^\dagger$ is an induced path, so, by our choice of index m^v, $R_1^\dagger$ is also an induced path

Suppose that $\text{Def}(R_1^\dagger) = \varnothing$. Thus, $R_1^\dagger$ is a quasi-shortest path. Since $|E(R_1^\dagger)| < |E(R^\dagger)|$, it follows from Subclaim 11.2.14 that $R^\dagger$ is a $C^\dagger$-monotone path of length at most $\frac{\beta}{3}+2N_{\text{mo}}+\text{Rk}(\mathcal{T}C^\dagger)$, so $R_1^\dagger$ satisfies Condition A). Since $|V(R_1^\dagger) \setminus V(R)| \leq 2$, it follows from 1) of Claim 11.2.7 that $d(R_1^\dagger, P) \geq \frac{\beta}{4}$, so $R_1^\dagger$ satisfies B). FInally, by 2) of Subclaim 11.2.13, $R_1^\dagger$ also satisfies C), contradicting our assumption. Thus, we have $\text{Def}(R_1^\dagger) \neq \varnothing$.

Since R is a shortest path between its endpoints, $p_1 R p_{m^v}$ is also a shortest path between its endpoints, so we have $\text{Def}(R_1^\dagger) \subseteq N(q) \cup N(v)$. Let $w \in \text{Def}(R_1^\dagger)$. To finish, we just need to show that $w \notin N(q)$. Suppose that $w \in N(q)$. If $w \in V(R^\dagger)$, then, since $R^\dagger$ is an induced path, we have $w = p_m$. Since w is adjacent to a vertex of $\{p_1, \cdots, p_{m^v-1}\}$ and $m^v < m$, this contradicts the fact that $R^\dagger$ is an induced path. Thus, we have $w \in \text{Def}(R^\dagger)$. Since $w \in D_1(R^\dagger) \cap N(q)$, and w is adjacent to a vertex of $\{p_1, \cdots, p_{m^v-1}\}$, this contradicts the minimality of m^v. We conclude that $w \notin N(q)$, as desired. ■

We now have the following:

Subclaim 11.2.18. $\text{Def}(R_1^\dagger) \cap B_4(C_*^2) = \varnothing$.

Proof: Let $w \in \text{Def}(R^\dagger)$. Since $\{p_1, \cdots, p_{m-1}\} \subseteq V(G) \setminus B_4(C_*^2)$ and w has a neighbor in $\{p_1, \cdots, p_{m^v-1}$, we have $w \notin B_3(C_*^2)$, so we just need to check that $w \notin D_4(C_*^2)$. Suppose toward a contradiction that $w \in D_4(C_*^2)$. Thus, we have $N(w) \cap D_3(C_*^2) \neq \varnothing$, so let $w^* \in N(w) \cap D_3(C_*^2)$.

Let n be the minimal index of $\{1 \leq i \leq m_v : p_i \in N(w)\}$. By 3) of Subclaim 11.2.17, we have $w \in N(v)$, and, by Subclaim 11.2.15, we have $n \leq m^v - 2$. Now, we have $m^v \leq m - 2$ as well, so it follows from Subclaim 11.2.14 that $p_1 R p_n w w^*$ is a shortest $(w_{\mathcal{T}}(C^\dagger), D_3(C_*^2))$-path. Since $p_1 R p_n w w^*$ is a $(w_{\mathcal{T}}(C^\dagger), D_3(C_*^2))$-path, we get by assumption that $R_*^\dagger := p_1 R p_n w w^*$ violates one of A), B), or C).

Since $R_*^\dagger$ is a shortest $(w_{\mathcal{T}}(C^\dagger), D_3(C_*^2))$-path and $|E(R_*^\dagger)| \leq |E(R)|$, it follows that $R_*^\dagger$ is a $C^\dagger$-monotone path and satisfies Condition A). Since $|V(R_*^\dagger) \setminus V(R)| \leq 2$, it follows from 1) of Claim 11.2.7 that $R_*^\dagger$ satisfies Condition B). Since G contains the path $w^* w v q$, it follows from 2) of Subclaim 11.2.13 that $R_*^\dagger$ also satisfies Condition C), contradicting our assumption. ■

It follows from 3) of Subclaim 11.2.17 that each $w \in \text{Def}(R_1^\dagger)$ is adjacent to v and to a vertex of $\{p_1, \cdots, p_{m^v-2}\}$, so there exists a minimal index n^w among $\{1 \leq i \leq m^v - 2 : p_i \in N(w)\}$. As above, we choose a $w \in \text{Def}(R_1^\dagger)$ which minimizes the quantity n^w and let $R_2^\dagger := q v w p_{n^w} R p_1$. We claim now that $R_2^\dagger$ is a $(w_{\mathcal{T}}(C^\dagger), D_3(C_*^2))$-path which satisfies all of A), B), and C).

By Subclaim 11.2.18, have $w \notin B_4(C_*^2)$ and since $n_w < m$, we have $\{p_1, \cdots, p_{n^w}, w\} \subseteq G \setminus B_4(C_*^2)$. Thus, we have $V(R_\dagger^2) \cap D_4(C_*^2)| = \{v\}$, and $V(R_\dagger^2) \cap D_3(C_*^2) = \{q\}$. In particular, $R_2^\dagger$ is a $(w_{\mathcal{T}}(C^\dagger), D_3(C_\dagger^2))$-path. Since $R_1^\dagger$ is induced, it follows from our choice of index n^w that $R_2^\dagger$ is also an induced path.

Subclaim 11.2.19. $\text{Def}(R_2^\dagger) = \varnothing$.

Proof: Suppose toward a contradiction that there is a $w^* \in \text{Def}(R_2^\dagger)$. Since $p_1 R p_{n^w}$ is a shortest path between its endpoints, we have $w^* \in N(w) \cup N(v) \cup N(q)$. We claim now that $w^* \notin N(v) \cup N(q)$. Suppose that $q \in N(w^*)$. By Subclaim 11.2.15, q is adjacent to a vertex of $\{p_1, \cdots, p_{n^w}\}$, and since $R^\dagger$ is an induced path, we have $q \neq p^m$ and $q \in \text{Def}(R^\dagger)$. But since $n^w < m^v$, this contradicts the minimality of m^v. Thus, we have $w^* \notin N(q)$.

Now suppose that $w^* \in N(v)$. Thus, again by Subclaim 11.2.15, w^* is adjacent to a vertex of $\{p_1, \cdots, p_{n^w-1}\}$. Since $n^w - 1 < m^v - 1$ and $R^\dagger$ is an induced path, we have $w^* \notin V(R_1^\dagger)$, so $w^* \in \text{Def}(R_1^\dagger)$. But since w^* is adjacent to each v and a vertex of $\{p_1, \cdots, p_{n^w-1}\}$, this contradicts the minimality of m^w.

Thus, we conclude that $w^* \notin N(v) \cup N(q)$, so w^* is adjacent to w and also to a vertex of $p \in \{p_1, \cdots, p_{n^w-2}\}$. It follows that G contains the path $p w^* w v q p_m$. Since $n^w \leq m^v - 2$ and $m^v \leq m - 2$, we have $p \in \{p_1, \cdots, p_{m-6}\}$. Since $p_1 R p_m$ is a shortest path between its endpoints, we have a contradiction. ■

Since $\mathrm{Def}(R_2^\dagger) = \varnothing$ and $R_2^\dagger$ is induced, we get that $R_2^\dagger$ is a quasi-shortest path. Since $|ER_2^\dagger)| < |E(R_1^\dagger)| < |E(R^\dagger)| \leq |E(R)|$, it follows that $R_2^\dagger$ is a $C^\dagger$-monotone path and satisfies Condition A). Since $|V(R_2^\dagger \setminus R)| \leq 3$, it follows from 1) of Claim 11.2.7 that $R_2^\dagger$ also satisfies Condition B). Since G contains the path wvq, it follows from 2) of Subclaim 11.2.13 that $R_2^\dagger$ also satisfies Condition C), contradicting our assumption that no such path exists. This completes the proof of Claim 11.2.11. ■

We now define a subset $\mathcal{S}^\circ$ of $\mathcal{S}$ and a binary relation $\otimes$ on $\mathcal{S}$ in the following way:

Definition 11.2.20.

1) We set $\mathcal{S}^\circ$ be the set of triples $(C, P, H) \in \mathcal{S}$ such that, letting q be the $D_3(C_*^2)$-endpoint of P we have $d_G(q, \mathbf{P}_*) \geq 22$.
2) We define a binary relation $\otimes$ on $\mathcal{S}$, where $(C, P, H) \otimes (C', P', H')$ if C' is an H-blocker, and, letting z' be the join of H', we have $d(z', H \cap C_*^2) \leq 12$ and $d(P, P') \geq \frac{\beta}{4}$.

Now we have the following facts.

Claim 11.2.21.

1) $\mathcal{S} \neq \varnothing$; AND

2) For each $(C, P, H) \in \mathcal{S}$, there exists a $(C', P', H') \in \mathcal{S}$ with $(C, P, H) \otimes (C', P', H')$; AND

3) If $(C, P, H) \in \mathcal{S} \setminus \mathcal{S}^\circ$ and $(C', P', H') \in \mathcal{S}$ with $(C, P, H) \otimes (C', P', H')$, then $(C', P', H') \in \mathcal{S}^\circ$.

Proof: Since we have assumed that Theorem 11.2.3 does not hold, it follows from Claim 11.2.5 that, for each $C \in \mathcal{C} \setminus \{C_*\}$ and each C-monotone path P, there is a C-seam H such that there exist an H-blocker and such that P is the tail of H.

Since $\mathcal{C} \setminus \{C_*\} \neq \varnothing$, let $C \in \mathcal{C} \setminus \{C_*\}$ and let P be a C-monotone path. Such a path exists, since a shortest $(w_{\mathcal{T}}(C), D_3(C_*^2))$-path is a candidate. Thus, there is a C-seam H with tail P such that there exists an H-blocker C', and it follows from Claim 11.2.11 that there exists a C'-monotone path R' of length at most $\frac{\beta}{3} + 2N_{\text{mo}} + \mathrm{Rk}(\mathcal{T}|C')$ such that $d(P, R') \geq \frac{\beta}{4}$. Since Theorem 11.2.3 does not hold, it follows from Claim 11.2.5 that there exists a C'-seam H' with tail R' such that there exists an H'-blocker. Thus, we have $(C', R', H') \in \mathcal{S}$. This proves 1).

Now we prove 2). Let $(C, P, H) \in \mathcal{S}$. By definition of $\mathcal{S}$, there exists an H-blocker C'. By definition, we have $C \neq C'$ by definition, since C' is an H-blocker. Since $|E(P)| \leq \frac{\beta}{3} + 2N_{\text{mo}} + \mathrm{Rk}(\mathcal{T}|C)$, it follows from Claim 11.2.11 that there is a C'-monotone path R' of length at most $\frac{\beta}{3} + 2N_{\text{mo}} + \mathrm{Rk}(\mathcal{T}|C')$ such that $d(P, R') \geq \frac{\beta}{4}$ and such that the $D_3(C_*^2)$-endpoint of R' is of distance at most 11 from $H \cap C_*^2$.

Since Theorem 11.2.3 does not hold, it follows from Claim 11.2.5 that there exists a C'-seam H' with tail R' such that there exists an H'-blocker. Since $|E(R')| \leq \frac{\beta}{3} + 2N_{\text{mo}} + \mathrm{Rk}(\mathcal{T}|C')$ and since there exists an H' blocker, we thus have $(C', P', H') \in \mathcal{S}$. As indicated above, we have $d(P, R') \geq \frac{\beta}{4}$ and, letting q be the $D_3(C_*^2)$-endpoint of R', we have $d(q, H \cap C_*^2) \leq 11$, so the join of H' has distance at most 12 from $H \cap C_*^2$. Thus, we have $(C, P, H) \otimes (C', P', H')$. This proves 2).

Now we prove 3). Suppose that $(C', P', H') \notin \mathcal{S}^\circ$. In that case, each of P, P' has distance at most 20 from $V(\mathbf{P}_*)$, so $d(P, P') \leq 42 + \frac{2N_{\text{mo}}}{3}$, contradicting the fact that $d(P, P') \geq \frac{\beta}{4}$. ■

We now fix p_{left} and p_{right} as the endpoints of $\mathbf{P}_*^1$. Since $|E(\mathbf{P}_*^1)| \geq |E(\mathbf{P}_*)| - 2$, these are distinct vertices. It follows from 2) of Claim 11.2.8 that, for each $(C, P, H) \in \mathcal{S}$, the head of H is a path of length at least $\frac{\beta}{4} - N_{\text{mo}}$ whose unique $\mathbf{P}_*^1$-endpoint is one of $p_{\text{left}}, p_{\text{right}}$. Thus, we have the following natural way to partition $\mathcal{S}$. We let $\mathcal{S} = \mathcal{S}_{\text{left}} \cup \mathcal{S}_{\text{right}}$, where these two sets are defines as follows. Given a $(C, P, H) \in \mathcal{S}$, where z is the join of H, we have $(C, P, H) \in \mathcal{S}_{\text{left}}$ if p_{left} is the unique $\mathbf{P}_*^1$-endpoint of the $(\text{Pin}(z), \mathbf{P}_*^1)$-subpath of $C_*^2 \cap H$. Likewise, $(C, P, H) \in \mathcal{S}_{\text{right}}$ if p_{right} is the unique $\mathbf{P}_*^1$-endpoint of the $(\text{Pin}(z), \mathbf{P}_*^1)$-subpath of $C_*^2 \cap H$.

We now have the following.

Claim 11.2.22. *Let $(C, P, H) \in \mathcal{S}$ and $(C', P', H') \in \mathcal{S}^\circ$, where $(C, P, H) \otimes (C', P', H')$. Let Q be the head of H and let Q' be the head of H'. Then Q, Q' have the same unique $\mathbf{P}_*^1$-endpoint and $Q' \subseteq Q$. Furthermore, $|E(Q) \setminus E(Q')| \geq \frac{\beta}{4}$.*

Proof: Since (C, P, H) and (C', P, H') both lie in $\mathcal{S}$, it follows from 3) of Claim 11.2.8 that each of the paths Q, Q' has length at least $\frac{\beta}{4} - N_{\text{mo}}$. Thus, one of $p_{\text{left}}, p_{\text{right}}$ is the unique $\mathbf{P}_*^1$-endpoint of Q, and one of $p_{\text{left}}, p_{\text{right}}$ is the unique $\mathbf{P}_*^1$-endpoint of Q'. Suppose without loss of generality that p_{right} is the unique $\mathbf{P}_*^1$-endpoint of Q. Let z be the join of H and let z' be the join of H'. By 2) of Claim 11.2.8, $\text{Pin}(z)$ is a terminal subpath of $C_*^2 \cap H$. Likewise, $\text{Pin}(z')$ is a terminal subpath of $H' \cap C_*^2$. Since $d(P, P') \geq \frac{\beta}{4}$, we have $d(z, z') \geq \frac{\beta}{4} - 2$.

Again by 2) of Claim 11.2.8, we have $\text{Pin}(z) \cap \mathbf{P}_*^1 = \varnothing$. By 5c) of Definition 6.0.8, $(C_*^2 \cap H) \setminus (Q \cup \mathbf{P}_*^1)$ consists of a path Q_{close} such that every vertex of Q_{close} has distance at most 8 from $V(\mathbf{P}_*)$. Likewise, $C^2 \cap H \setminus (Q' \cup \mathbf{P}_*^1)$ consists of a path Q'_{close} such that every vertex of Q'_{close} has distance at most 8 from $V(\mathbf{P}_*)$.

Now, by definition of $\otimes$, we have $d(z', C_*^2 \cap H) \leq 12$. Since each vertex of $Q_{\text{close}} \cup \mathbf{P}_*^1$ has distance at most 8 from $V(\mathbf{P}_*)$, and $(C', P', H') \in \mathcal{S}^\circ$, any $(z', C_*^2 \cap H)$-path of length at most 12 has its $C_*^2 \cap H$-endpoint in $Q \setminus \{p_{\text{right}}\}$, or else we have a $(z', \mathbf{P}_*)$-path of length at most 20. Thus, let M be a $(z', Q \setminus \{p_{\text{right}}\})$-path of length at most 9 with its non-z'-endpoint lying in $Q \setminus \{p_{\text{right}}\}$. Since each vertex of $Q \setminus \{p_{\text{right}}\}$ has distance two from $C \setminus \mathbf{P}_*$, it follows that M extends to a graph M' such that the following hold:

1. M' is either a cycle of length at most 15 which contains at least one vertex of $V(C_* \cup C_*^1) \setminus V(\mathbf{P} \cup \mathbf{P}_*^1)$, or M' is a path of length at most 15 with both endpoints in $C_* \setminus \mathbf{P}_*$; AND
2. M' contains a path of length four with z' as an endpoint and the other endpoint in $C_* \cap D_2(\text{Span}(z') \cap C_*^2)$

Note that M' possibly intersects with C_* on many vertices, so if it is not a cycle, then it is not necessarily a generalized chord of C_*. In any case, since z' has distance at least $\frac{\beta}{4} - 2$ from $V(P) \cup \{z\}$, we have $d(z, M') \geq \frac{\beta}{4} - 17$ and $d(P, M') \geq \frac{\beta}{4} - 17$. Furthermore, since $d(z', \mathbf{P}_*) \geq 21$, we have $M' \cap (\mathbf{P}_* \cup \mathbf{P}_*^1) = \varnothing$.

Given a subpath M'' of M', we say that M'' is a *touring subpath* of M' if M'' is a generalized chord of C_*, where $M'' \cap Q \neq \varnothing$ and M'' also has nonempty intersection with each of $V(C_*) \cap D_2(\mathring{Q})$ and $V(C_*) \cap D_2(C_*^2 \setminus Q)$.

Subclaim 11.2.23. *There is no touring subpath of M'.*

Proof: . Suppose that there exists a touring subpath M'' of M'. Since $M'' \cap Q \neq \varnothing$, and M'' has nonempty intsersection with each of $V(C_*) \cap D_2(\mathring{Q})$ and $V(C_*) \cap D_2(C_*^2 \setminus Q)$, it follows that $G_{M''}^{\text{small}}$ contains a terminal vertex of Q. Since $\mathcal{C} \setminus \{C_*\} \neq \varnothing$ and $\mathbf{P}_* \cap M'' = \varnothing$, we have $\mathbf{P}_* \cap G_{M''}^{\text{small}} = \varnothing$. Since $d(z, M'') \geq \frac{\beta}{4} - 17$ and $\text{Pin}(z)$ is a terminal subpath of Q, it follows that $z \in V(G_{M''}^{\text{small}}) \setminus V(M'')$. Since $d(P, M'') \geq \frac{\beta}{4} - 17$, we have $C' \subseteq G_{M''}^{\text{small}}$, which is false. ■

By our construction of M', there is a subpath M_{trunc} of length two, where M_{trunc} has one endpoint in $\text{Pin}(z')$ and the other endpoint in $V(C_*) \cap D_2(\text{Pin}(z'))$. Let $K := \text{Ext}(C_*^2) \setminus (\mathbf{P}_* \cup \mathbf{P}_*^1)$. Since $K \cap (\mathbf{P} \cup \mathbf{P}) = \varnothing$, we have $M_{\text{trunc}} \subseteq K$. Let q be the unique non-$\mathbf{P}_*^1$-endpoint of Q and let q' be the non-$\mathbf{P}_*^1$-endpoint of Q'.

Now suppose toward a contradiction that either p_{left} is the unique $\mathbf{P}_*^1$-endpoint of Q', or, if p_{right} is the unique $\mathbf{P}_*^1$-endpoint of Q', then $Q' \not\subseteq Q$. In the latter case, since $q \in \text{Span}(z)$ and $q' \in \text{Span}(z')$, it follows from the definition of $\otimes$ that $|E(Q') \setminus E(Q)| \geq \frac{\beta}{4} - 4$. Note that, in K, $B_2(q)$ disconnects $B_2(\mathring{Q})$ from $V(C_*^2 \cap K) \setminus V(\mathring{Q})$. Thus, in either case, it follows that, in K, the set the set $B_2(q)$ separates $B_2(\text{Span}(z'))$ from $B_2(\mathring{Q})$. Since no touring subpath of M' exists, it follows that M' contains a path M'' with $M'' \subseteq K$, where one endpoint of M'' lies in $V(C_*^2 \cap K) \setminus V(\mathring{Q})$, the other endpoint of M'' lies in $B_2(\mathring{Q})$. Thus, we have $d(M', q) \leq 2$. Since $d(q, z) \leq 2$, we have $d(M', z) \leq 4$, which is false.

Thus, our assumption that $Q \subseteq Q'$ is false. Since Q, Q' share an endpoint $\mathbf{P}_*^1$, and since $\text{Span}(z)$ contains the non-$\mathbf{P}_*^1$ endpoint of Q and $\text{Span}(z')$ contains the non-$\mathbf{P}_*^1$ endpoint of Q', it follows that $|E(Q) \setminus E(Q')|$ is a path of length at least $\frac{\beta}{4} - 4$, since $d(P, P') \geq \frac{\beta}{4}$. This completes the proof of Claim 11.2.22. ■

The claim above has the following immediate consequence;

Claim 11.2.24. *Each of $\mathcal{S}^\circ$ and $\mathcal{S} \setminus \mathcal{S}^\circ$ is nonempty.*

Proof: We first show that $\mathcal{S}^\circ$ is nonempty. By 1) of Claim 11.2.21, $\mathcal{S} \neq \varnothing$ so let $(C, P, H) \in \mathcal{S}$. If $(C, P, H) \in \mathcal{S}^\circ$, then we are done, so suppose that $(C, P, H) \notin \mathcal{S}^\circ$. By 2) of Claim 11.2.21, there exists a $(C', P', H') \in \mathcal{S}$ with $(C, P, H) \otimes (C', P', H')$. By 3) of Claim 11.2.21, we have $(C', P', H') \in \mathcal{S}^\circ$, so we are done. Thus, we indeed have $\mathcal{S}^\circ \neq \varnothing$. Now we show that $\mathcal{S} \setminus \mathcal{S}^\circ \neq \varnothing$. Since $\mathcal{S}^\circ \neq \varnothing$, we choose an element $(C, P, H) \in \mathcal{S}^\circ$ which minimizes the length of the head of H. Let Q be the head of H. By 2) of Claim 11.2.21, there is a $(C', P', H') \in \mathcal{S}$ with $(C, P, H) \otimes (C', P', H')$. We claim now that $(C', P', H') \in \mathcal{S} \setminus \mathcal{S}^\circ$. Let Q' be the head of H'. Suppose toward a contradiction that $(C', P', H') \in \mathcal{S}^\circ$. In that case, by Claim 11.2.22, we have $|E(Q')| < |E(Q)|$, contradicting the minimality of Q. Thus, we indeed have $(C', P', H') \in \mathcal{S} \setminus \mathcal{S}^\circ$, so $\mathcal{S} \setminus \mathcal{S}^\circ \neq \varnothing$. ■

We now have the following.

Claim 11.2.25.

1) *Let $(C, P, H) \in \mathcal{S}_{\text{left}} \cap (\mathcal{S} \setminus \mathcal{S}^\circ)$ and let z be the join of H. Then there is no $(\text{Pin}(z), p_{\text{left}})$-path of length at most 16 on the small side of C_*^2; AND*
2) *Let $(C, P, H) \in \mathcal{S}_{\text{right}} \cap (\mathcal{S} \setminus \mathcal{S}^\circ)$ and let z be the join of H. Then there is no $(\text{Pin}(z), p_{\text{right}})$-path of length at most 16 on the small side of C_*^2.*

Proof: These two statements are symmetric so we just prove 1). Let Q be the head of $C^2 \cap H$. By 3) of Claim 11.2.8, Q is a path of length at least $\frac{\beta}{4} - N_{\text{mo}}$. Let $v, v^\star$ be the vertices of $\text{Span}(z) \cap C_*^2$ (possibly $v = v^\star$). By 2) of Claim 11.2.8, one of $v, v^\star$ is also the non-p_{left} endpoint of Q.

Suppose toward a contradiction that there is a $(\text{Pin}(z), p_{\text{left}})$ path in $\text{Ext}(C_*^2)$ of length at most 16. Let $K := \text{Ext}(C_*^2) \setminus V(\mathring{\mathbf{P}}_* \cup \mathring{\mathbf{P}}_*^1)$. Every vertex of $\mathbf{P}_*^1$ is adjacent to a subpath of $\mathbf{P}_*$ of length at most one, so it follows from 2) of Corollary 2.3.14 that each of $\mathbf{P}_*$ and $\mathbf{P}$ have length at least $\frac{2N_{\text{mo}}}{3} - 2$. Furthermore, in K, the set $B_2(\{v, v^\star\})$ disconnects $B_2(Q)$ from $B_2(K \cap (C_*^2 \setminus Q))$ and disconnects $B_2(C_*^2 \cap \text{Span}(z))$ from $V(K) \setminus B_2(\{v, v^\star\})$. Thus, there exists a $(B_2(\{v, v^\star\}), p_{\text{left}})$-path M in K, where M has length at most 16.

By 2) of Claim 11.2.21, there is a $(C', P', H') \in \mathcal{S}$ with $(C, P, H) \otimes (C', P', H')$. By 3) of Claim 11.2.21, since $(C, P, H) \in \mathcal{S} \setminus \mathcal{S}^\circ$, we have $(C', P', H') \in \mathcal{S}^\circ$. Let z' be the join of H' and let Q' be the head of H'. Since $(C', P', H') \in \mathcal{S}^\circ$, it follows from Claim 11.2.22 that $(C', P', H') \in \mathcal{S}_{\text{left}}$ as well, and Q' is a proper subpath of Q. Let q' be the unique non-$\mathbf{P}^1_*$ endpoint of Q'. Note that, in K, the set $B_2(q', K)$ separates $\text{Pin}(z)$ from $\mathbf{P}_*$. Thus, we have $d(\{v, v^\star\}, q') \leq |E(M)| + 4$. On the other hand, by definition of $\text{Span}(z)$, we have $d(v, v^\star) \leq 4$, and thus since $d(P, P') \geq \frac{\beta}{4}$, we have $d(q, q') \geq \frac{\beta}{4} - 10$, a contradiction. ■

With the above in hand, we prove the following:

Claim 11.2.26. *Let $C \in \mathcal{C} \setminus \{C\}$ and let P be a C-monotone path of length at most $\frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C)$. Let q be the and suppose that $d(q, \mathbf{P}_*) < 22$. Then there exist a pair of elements of $\mathcal{S} \setminus \mathcal{S}^\circ$ of the form (C, P, H) and (C, P, H') for some C-seams H and H', where $(C, P, H) \in \mathcal{S}_{\text{left}} \cap (\mathcal{S} \setminus \mathcal{S}^\circ)$ and $(C, P, H') \in \mathcal{S}_{\text{right}} \cap (\mathcal{S} \setminus \mathcal{S}^\circ)$, and H, H' have the same join.*

<u>Proof:</u> By assumption, Theorem 11.2.3 does not hold, so it follows that there exists a $z \in D_2(C^2_*) \cap N(q)$ such that, for any C-seam H with tail P and join z, there exists an H-blocker. In particular, for any C-seam H with tail P and join z, we have $(C, P, H) \in \mathcal{S}^\circ \setminus \mathcal{S}$. By Claim 11.2.5, there is at least one C-seam H with tail P and join z, so suppose without loss of generality that $(C, P, H) \in (\mathcal{S} \setminus \mathcal{S}^\circ) \cap \mathcal{S}_{\text{left}}$. Combining Claim 11.2.25 with 2) of Theorem 6.0.9, it follows that there exists an element of $(\mathcal{S} \setminus \mathcal{S}^\circ) \cap \mathcal{S}_{\text{right}}$ of the form (C, P, H') for some C-seam H', where H' also has join z, so we are done. ■

By Claim 11.2.24, $\mathcal{S} \setminus \mathcal{S}^\circ \neq \varnothing$, so there exists a $C \in \mathcal{C} \setminus \{C_*\}$ and a C-monotone path P such that, letting q be the $D_3(C^2_*)$-endpoint of P, we have $|E(P)| \leq \frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C)$ and $d(q, \mathbf{P}_*) < 22$. By Claim 11.2.26, there exists a pair of elements of $\mathcal{S} \setminus \mathcal{S}^\circ$ of the form $(C, P, \overleftarrow{H})$ and $(C, P, \overrightarrow{H})$, where $(C, P, \overleftarrow{H}) \in (\mathcal{S} \setminus \mathcal{S}^\circ) \cap \mathcal{S}_{\text{left}}$ and $(C, P, \overrightarrow{H}) \in (\mathcal{S} \setminus \mathcal{S}^\circ) \cap \mathcal{S}_{\text{right}}$, and $\overleftarrow{H}, \overrightarrow{H}$ have the same join.

Let z be the common join of $\overleftarrow{H}$ and $\overrightarrow{H}$. Let $\overleftarrow{Q}$ be the head of $\overleftarrow{H}$ and let $\overrightarrow{Q}$ be the head of $\overrightarrow{H}$. Note that $\overleftarrow{Q} \cup \overrightarrow{Q} = \mathbf{P}^1_* \setminus \{p_{\text{left}}, p_{\text{right}}\}$, and $\overleftarrow{Q} \cap \overrightarrow{Q} = \text{Span}(z) \cap C^2_*$.

Now, by 2) of Claim 11.2.21, there exist a $(C^\ell, P^\ell, H^\ell) \in \mathcal{S}$ and a $(C^r, P^r, H^r) \in \mathcal{S}$ such that $(C, P, \overleftarrow{H}) \otimes (C^\ell, P^\ell, H^\ell)$ and $(C, P, \overrightarrow{H}) \otimes (C^r, P^r, H^r)$. By 3) of Claim 11.2.21, each of (C^ℓ, P^ℓ, H^ℓ) and (C^r, P^r, H^r) lies in $\mathcal{S}^\circ$. Let Q^ℓ be the head of H^ℓ, let Q^r be the head of H^r, and let z^ℓ, z^r be the respective joins of H^ℓ, H^r.

By Claim 11.2.22, Q^ℓ is a proper subpath of $\overleftarrow{Q}$ with p_{left} as an endpoint, and, by 2) of Claim 11.2.8, the other endpoint is a vertex of $\text{Span}(z^\ell) \cap C^2_*)$. Likewise, Q^ℓ is a proper subpath of $\overleftarrow{Q}$ with p_{left} as an endpoint and the other endpoint is a vertex of $\text{Span}(z^\ell) \cap C^2_*$. Let v_ℓ be the vertex of $\text{Span}(z^\ell) \cap C^2_*$ which is closest to $\text{Span}(z) \cap C^2_*$ and let v_r be the vertex of $\text{Span}(z^r) \cap C^2_*$ which is closest to $\text{Span}(z) \cap C^2_*$ on the path $C^2_* \setminus \mathbf{P}^1_*$.

Let M be the unique subpath of $C^2_* \setminus \mathring{\mathbf{P}}^1_*$ with endpoints v_ℓ, v_r. As $\text{Pin}(z)$ is a subpath of M, let x_ℓ, x_r be the endpoints of this subpath, where the sequence $p_{\text{right}}, v_r, x_r, x_\ell, v_\ell, p_{\text{left}}$ indicates the ordering of these vertices on the path $C^2_* \setminus \mathring{\mathbf{P}}^1_*$. Possibly $x_r = x_\ell$, but, in any case, by definition of $\otimes$, each of x_r, x_ℓ has distance at least $\frac{\beta}{4} - 12$ from $\{v_r, v_\ell\}$, as each of x_r, x_ℓ has distance two from z.

Claim 11.2.27. $C^\ell \neq C^r$

<u>Proof:</u> Suppose toward a contradiction that there is a $C^\dagger \in \mathcal{C} \setminus \{C_*, C\}$ such that $C^r = C^\ell = C^\dagger$. Thus, each of P^r, P^ℓ is a $(C^\dagger, D_3(C^2_*))$-path. Now, there exists a $(z^\ell, C_* \setminus \mathbf{P}_*)$-path T^ℓ of length four, where $v_\ell \in V(T^\ell)$. Likewise, there exists a $(z^r, C_* \setminus \mathbf{P}_*)$-path T^r of length four, where $v_r \in V(T_r)$. Let K be the subgraph of G induced

by $V(C^\dagger \cup P^\ell \cup P^r) \cup V(T^\ell \cup T^r)$. Now, K is a connected subgraph of G, since z^ℓ has a neighbor in P^ℓ and z^r has a neighbor in P^r. Since each of $(C^\dagger, P^\ell, H^\ell)$ and $(C^\dagger, P^r, H^r)$ lies in $\mathcal{S}^\circ$, we have $K \cap \mathbf{P}_* = \varnothing$, and K separates $\text{Pin}(z)$ from $\mathbf{P}_*$. Since $(C, P, H) \in \mathcal{S} \setminus \mathcal{S}^\circ$, there is a $(z, \mathbf{P}_*)$-path of length at most 22, so we have $d(z, K) \leq 20$. By Observation 2.1.8, we have $d(z, C^\dagger) \geq \frac{\beta}{3} - 3$, so z has distance at most 26 from $V(P^\ell \cup P^r) \cup V(T^\ell \cup T^r)$. By definition of $\otimes$, z has distance at least $\frac{\beta}{4} - 1$ from $P^\ell \cup P^r$ and thus has distance at least $\frac{\beta}{4} - 5$ from $P^\ell \cup P^r \cup T^\ell \cup T^r$, so we have a contradiction. ■

We now have the following.

Claim 11.2.28. *There is no proper generalized chord of C_*^2 in $\text{Int}(C_*^2)$ which has length at most $\frac{N_{\text{mo}}}{3} - 3$ and which satisfies both of the following conditions.*

1) $d(q, R) \leq \frac{\beta}{5}$; AND

2) R has one endpoint in $\mathring{M}$ and one endpoint in $C_*^2 \setminus M$

Proof: Suppose toward a contradiction that such a proper generalized chord R of C_*^2 exists. Firstly, since q has distance at least $\frac{\beta}{4}$ from each of P^ℓ and P^r and $q \in D_3 * C_*^2)$, it follows from Observation 2.1.8 that q has distance at least $\frac{\beta}{4}$ from each of $P^\ell \cup C^\ell$ and $P^r \cup C^r$. Consider the following cases:

Case 1: Each endpoint of R lies in $C_*^2 \setminus \mathring{\mathbf{P}}_*^1$

In this case, there exists a graph R' with $R \subseteq R'$, where R' is either a cycle of length at most $\frac{N_{\text{mo}}}{3} + 1$ or a proper generalized chord of C_* of length at most $\frac{N_{\text{mo}}}{3} + 1$ which has both endpoints in $C_* \setminus \mathring{\mathbf{P}}_*$. If R' is a proper generalized chord of C_*, then, since q has distance at least $\frac{\beta}{4}$ from each of $P^\ell \cup C^\ell$ and $P^r \cup C^r$, it follows that R' separates at least one of C^r, C^ℓ from $\mathbf{P}_*$, contradicting 3) of Theorem 2.2.4. If R' is a cycle, then at least one of C^r, C^ℓ lies in $\text{Int}(R')$, which is again a consequence of the fact that q has distance at least $\frac{\beta}{4}$ from each of $P^\ell \cup C^\ell$ and $P^r \cup C^r$. Since $d(R', w_{\mathcal{T}}(C_*)) \leq 2$, this contradicts Corollary 2.1.30.

Case 2: The $C_*^2 \setminus M$-endpoint of R lies in $\mathring{\mathbf{P}}_*^1$

In this case, there exists a proper generalized chord R' of C_* with $R \subseteq R'$, where R' has length at most $\frac{N_{\text{mo}}}{3}$, one endpoint of R lies in $D_2(\mathring{M}) \cap (C_* \setminus \mathbf{P}_*)$, and the other endpoint of R' lies in $\mathbf{P}_*$. Since q has distance at least $\frac{\beta}{4}$ from each of $P^\ell \cup C^\ell$ and $P^r \cup C^r$, it follows that R' separates C^ℓ from C^r. Since $|E(\mathbf{P}_*)| \leq \frac{2N_{\text{mo}}}{3}$ and $E(R')| \leq \frac{N_{\text{mo}}}{3}$, this contradicts 4) of Theorem 2.2.4. ■

Now we enough to finish the proof of Theorem 11.2.3. Recall that $d(q, \mathbf{P}_*) < 22$. Thus, there exists a $(\{x_\ell, x_r\}, \mathbf{P}_*)$-path R in G of length at most 24. Since $\frac{N_{\text{mo}}}{3} - 3 \geq 24$ and each vertex of R has distance at most 24 from q, it follows from Claim 11.2.28 that no subpath of R is a proper generalized chord of C_*^2 in $\text{Int}(C_*^2)$ which has one endpoint in $\mathring{M}$ and one endpoint in $C_*^2 \setminus M$. Since $\{v_\ell, v_r\}$ has distance at least $\frac{\beta}{4} - 12$ from $\{x_\ell, x_r\}$, it follows that there is a subpath R' of R such that $R' \subseteq \text{Ext}(C_*^2)$, where R' has an one endpoint in $\mathring{M}$ and one endpoint in $B_2(C_*^2 \setminus M)$. Now, in $\text{Ext}(C_*^2)$, the set $B_2(\{v_\ell, v_r\})$ separates $\mathring{M}$ from $C_*^2 \setminus M$, so $d(R, \{v_r, v_\ell\}) \leq 2$, contradicting the fact that $\{v_\ell, v_r\}$ has distance at least $\frac{\beta}{4} - 12$ from $\{x_\ell, x_r\}$. This completes the proof of Theorem 11.2.3. □

11.3 A Path-Rerouting Result

In the previous two sections, we showed how to construct a deletion set in a critical mosaic by deleting a path between the outer face and an internal ring in such a way that the resulting outer face is sufficiently far away from the remaining internal rings. The path we construct possibly requires some slight modification in a region away from the outer face in order for this path to admit a coloring with desirable properties. The lone result of this short section shows that this is always possible.

Definition 11.3.1. Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical mosaic. Let $C \in \mathcal{C}$ and let A, A_* be the respective collars of C, C_*. A path P is called an *ideal* C-route if it is a $(D_2(A), D_2(A_*))$-path such that the following hold.

1) $|V(P) \cap D_3(A)| = 1$ and $|V(P_*) \cap D_3(A)| = 1$*; AND*
2) P is a quasi-shortest path*; AND*
3) $|E(P)| \leq d(D_2(A), D_2(A_*)) + \frac{2N_{\text{mo}}}{3}$.

We now have the following, which is the lone result of this section.

Lemma 11.3.2. *Let $\mathcal{T}$ be a critical mosaic, let $C \in \mathcal{C} \setminus \{C\}$ and let A, A_* be the respective collars of C, C_*. Let P be an ideal C-route. Let xy, x_*y_* be the two terminal edges of P and let σ be an L-coloring of $\{x, y, x_*, y_*\}$. Then there exists a $(D_2(A), D_2(A_*))$-path P' which also has terminal edges xy, x_*y_*, a $v \in D_1(P')$, and a $\tau \in \Phi_{G,L}(\sigma, V(P'))$ such that the following hold.*

1) $|N(v)| \cap V(\mathring{P}')| \geq 2$; AND
2) P' is also an ideal C-route, and $V(P') \setminus V(P) \subseteq B_{20N_{\text{mo}}}(A)$; AND
3) $|L_\tau(v)| \geq 2$ and, for every $w \in D_1(P') \setminus \{v\}$, we have $|L_\tau(w)| \geq 3$.

Proof. Let $\mathcal{P}_{\text{ideal}}$ be the set of ideal C-routes with terminal edges xy, x_*y_*, where $x \in D_2(A)$ and $y \in D_3(A)$, and likewise, $x_* \in D_2(A_*)$ and $y_* \in D_2(A_*)$. Given a $P' \in \mathcal{P}_{\text{ideal}}$, a $v \in D_1(P')$ with $|N(v) \cap V(\mathring{P}')| \geq 2$, and a $\tau \in \Phi_L(\sigma, V(P'))$, we say that $\langle v, \tau \rangle$ is a P'*-target* if $|L_\tau(v)| \geq 2$ and every vertex of $D_1(P') \setminus \{v\}$ has an L_τ-list of size at least three. By our distance conditions and by Condition 3) of Definition 11.3.1, we immediately have the following.

Claim 11.3.3. *For every $P' \in \mathcal{P}_{\text{ideal}}$, every vertex of $B_2(P')$ has an L-list of size at least five.*

We show that either there exists a P-target, or there exists a $P' \in \mathcal{P}_{\text{ideal}}$ such that there exists a P'-target, where P' differs from P by precisely one vertex, and this deviant vertex lies in $B_{20N_{\text{mo}}}(A)$. Recalling Definition 1.2.2, we first have the following.

Claim 11.3.4. *For any $P' \in \mathcal{P}_{\text{ideal}}$, if there exist two P'-gap vertices of distance either precisely two or precisely four apart in P', then there exists a P'-target.*

Proof: We apply the work of Section 1.2. Let $P' := p_1 \cdots p_s$, where $p_1 \in D_2(A)$ and $p_s \in D_2(A_*)$. Note that $\{p_1, p_2, p_{s-1}, p_s\} = \{x, y, x_*, y_*\}$. Let $j \in \{2, 4\}$ and suppose now that there is an $i \in \{2, \cdots, s-j\}$ such that each of p_i and p_{i+j} is a P'-gap. Since p_i, p_{i+j} are internal vertices of P', we have $xy \in E(p_1P'p_i)$ and $x_*y_* \in E(p_{i+j}P'p_s)$.

Since no vertex of $B_1(P')$ lies in a ring of $\mathcal{C}$, and P' is a quasi-shortest path, it follows from our triangulation conditions that, for every $w \in D_1(P)$, the graph $G[N(w) \cap V(P')]$ is a subpath of P' of length at most two.

By Claim 11.3.3, every vertex of $B_1(p_1P'p_i)$ has an L-list of size at least five. By Proposition 1.2.3, there exists an extension of $\sigma|_{xy}$ to an L-coloring $\psi \in \text{Avoid}(p_1P'p_i)$. Likewise, there is an extension of $\sigma|_{x_*y_*}$ to an L-coloring $\psi_* \in \text{Avoid}(p_{i+j}P'p_s)$ of $V(p_{i+j}P'p_s)$.

Since P' is an induced path, the union $\psi \cup \psi_*$ is a proper L-coloring of $V(p_1P'p_i) \cup V(p_{i+j}P'p_s)$. If $j = 2$, then, since $|L_{\psi\cup\psi_*}(p_{i+1})| \geq 3$ and each of p_i, p_{i+2} is a P'-gap, there is an extension of $\psi \cup \psi^*$ to an element of $\text{Avoid}^\dagger(P')$. Likewise, if $j = 4$, then, applying Proposition 1.2.4, since P' is a quasi-shortest path and each of p_i, p_{i+4} is a P'-gap, it follows that there is an extension of $\psi \cup \psi^*$ to an element of $\text{Avoid}^\dagger(P')$. In either case, there exists an extension of σ to a $\tau \in \text{Avoid}^\dagger(P')$ and a $v \in D_1(P')$ such that $N(v) \subseteq V(p_iP'p_{i+j})$ and such that $\langle v, \tau \rangle$ is a P'-target, so we are done. ■

We now introduce one more piece of terminology. The idea here is that because of Condition 3) of Definition 11.3.1, there is a bound on how much an ideal C-route differs from a shortest $(D_2(A), D_2(A_*))$-path, so there are regions near C in which an ideal C-route behaves like a shortest $(D_2(A), D_2(A_*))$-path.

Definition 11.3.5. Let $P' \in \mathcal{P}_{\text{ideal}}$. Given an integer $1 \leq t \leq 20$ and a subpath $Q \subseteq P'$, we say that Q is a *vertical P'-strip* of length t if Q is a path of length t and there exists an integer $3 \leq k \leq 7N_{\text{mo}}$ such that, for each $i = 0, \cdots, t$, we have $|V(P') \cap D_{k+i}(A)| = 1$, and the lone vertex of $V(P') \cap D_{k+i}(A)$ lies in $V(Q)$.

We now have the following by a simple counting argument.

Claim 11.3.6. *There exists a vertical P-strip Q of length 20.*

<u>Proof:</u> For each integer $3 \leq r \leq 7N_{\text{mo}}$, let $J_r := \{v \in V(P) : r \leq d(v, A) \leq r + 20\}$. The family $\{J_{3+21t} : t \in \{0, 1, \cdots, \lceil \frac{2N_{\text{mo}}}{3} \rceil + 1\}\}$ is a collection of pairwise-disjoint sets. It immediately follows from our distance conditions on $\mathcal{T}$ that $D_k(A) \cap V(P) \neq \varnothing$ for each $3 \leq k \leq 20N_{\text{mo}}$. In particular, for each $t = 0, 1, \cdots, \lceil \frac{2N_{\text{mo}}}{3} \rceil + 1$, there exists a subpath P_t of P where P_t is a $(D_{3+21t}(A), D_{23+21t}(A))$-path, so $V(P_t) \subseteq J_{3+21t}$. Since there does not exist a vertical P-strip of length 20, each of pairwise-disjoint paths in $\{P_t : t \in \{0, 1, \cdots, \lceil \frac{2N_{\text{mo}}}{3} \rceil + 1\}\}$ has a nonzero contribution to $|E(P)| - d(D_2(A), D_2(A_*))$, so we have $|E(P)| \geq d(D_2(A), D_2(A_*)) + \frac{2N_{\text{mo}}}{3} + 1$, contradicting Condition 3) of Definition 11.3.1. ■

Now suppose toward a contradiction that Lemma 11.3.2 does not hold. In particular, there does not exist a P-target. Applying Claim 11.3.6, let Q be a vertical P-strip of length 20. Let $Q := q_0 \cdots q_{20}$.

Claim 11.3.7. *For any five consecutive vertices of Q, at least one of them is a P-gap.*

<u>Proof:</u> Suppose not. Thus, there exists a $0 \leq i \leq 16$ such that no vertex of v_iQv_{i+4} is a P-gap. Let w be the unique vertex of $D_1(P)$ such that $G[N(w) \cap V(P)] = v_{i+1}v_{i+2}v_{i+3}$ and let P' be the path obtained from P by replacing v_{i+2} with w. Since Q is a shortest path between its endpoints and a vertical P-strip, and since $v_{i+1}, v_{i+2}, v_{i+3} \in V(\mathring{Q})$, we have $P' \in \mathcal{P}_{\text{ideal}}$, and it follows from Proposition 1.2.7 that each of v_{i+1}, v_{i+3} is a P'-gap. By Claim 11.3.4, there exists a P'-target, contradicting our assumption that Lemma 11.3.2 does not hold. ■

Likewise, we have the following.

Claim 11.3.8. *For any $0 \leq i \leq 15$, if v_i is a P-gap, then at least one v_{i+1}, v_{i+3} is not a P-gap.*

Proof: Suppose there is an index $0 \leq i \leq 15$ such that v_i is a P-gap and neither of v_{i+1}, v_{i+3} is a P-gap.

Since Lemma 11.3.2 does not hold, it follows from Claim 11.3.4 that v_{i+2} is not a P-gap, and thus none of $v_{i+1}, v_{i+2}, v_{i+3}$ is a P-gap. In particular, there is a $w \in D_1(P)$ such that $G[N(w) \cap V(P)] = v_{i+2}v_{i+3}v_{i+4}$. Let P' be the path obtained from P by replacing v_{i+3} with w. Since each of v_{i+1}, v_{i+3} is an internal vertex of Q and Q is a vertical P-strip, P' is a quasi-shortest path, and $P' \in \mathcal{P}_{\text{ideal}}$. Since Q is a vertical P-strip, it follows from Proposition 1.2.6 that v_{i+2} is a P'-gap. Since v_i is also a P'-gap, it follows from Claim 11.3.4 there exists a P'-target, contradicting our assumption that Lemma 11.3.2 does not hold. ■

By Claim 11.3.4, for any two vertices of Q which are of distance precisely two or four apart on Q, at least one of these two vertices is not a P-gap. Combining this with Claims 11.3.7 and 11.3.8, we have the following.

Claim 11.3.9. *There exists a subpath Q' of Q of length six such that the midpoint of Q and each endpoint of Q' is P-gap.*

Let Q' be as in Claim 11.3.9 and let $0 \leq i \leq 14$, where v_i, v_{i+6} are the endpoints of Q'. Let $P_{xy}, P_{x_*y_*}$ be the two components of $P \setminus \mathring{Q}'$, where $xy \in E(P_{xy})$ and $x_*y_* \in E(P_{x_*y_*})$. Suppose without loss of generality that $v_i \in V(P_{xy})$ and $v_{i+6} \in V(P_{x_*y_*})$

By Proposition 1.2.3, there is an extension of $\sigma|_{xy}$ to a $\psi \in \text{Avoid}(P_{xy})$ and an extension of $\sigma|_{x_*y_*}$ to a $\psi_* \in \text{Avoid}(P_{x_*y_*})$. Since P is a quasi-shortest path, $\psi \cup \psi_*$ is a proper L-coloring of its domain. By Proposition 1.2.5, there is a $\psi' \in \text{Avoid}^\dagger(Q')$ which colors v_i, v_{i+6} with the respective colors $\psi(v_i), \psi_*(v_{i+6})$. Since P is a quasi-shortest path, the union $\psi \cup \psi_* \cup \psi'$ is a proper L-coloring of its domain and lies in $\text{Avoid}^\dagger(P)$. Thus, there exists a P-target, contradicting our assumption. This completes the proof of Lemma 11.3.2. □

11.4 Completing the Proof of Theorem 2.1.7

In this short section, we bring together all the work of the previous chapters to complete the proof of Theorem 2.1.7, which we restate below.

Theorem 2.1.7. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a mosaic. Then G is L-colorable.*

Proof. Suppose not, and let $\mathcal{T} := (G, \mathcal{C}, L, C_*)$ be a critical mosaic. In Sections 11.1 and 11.2, we showed how to construct deletion sets which connect the outer face to an internal ring. In Section 11.3, we showed how to modify these deletion sets slightly away from the outer face in order to produce a desirable coloring. The modifications made in the previous section have no effect on the desired distance conditions, as the following simple result shows.

Claim 11.4.1. *Let K be a connected subgraph of G with $V(K \cap C_*) \neq \varnothing$ and let $C \in \mathcal{C} \setminus \{C_*\}$, where every ring of $\mathcal{C} \setminus \{C, C_*\}$ is disjoint to $V(K)$. Let F be the outer face of $G \setminus K$ and suppose that $d(w_{\mathcal{T}}(C'), F) \geq \frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C')$ for all $C' \in \mathcal{C} \setminus \{C, C_*\}$. Let $K^\dagger$ be a connected subgraph of G such that $V(K^\dagger) \setminus V(K) \subseteq \text{Ann}(C) \cup B_{20N_{\text{mo}}}(C)$. Letting $F^\dagger$ be the outer face of $G \setminus K^\dagger$, we have $d(w_{\mathcal{T}}(C'), F^\dagger) \geq \frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C')$ for all $C' \in \mathcal{C} \setminus \{C, C_*\}$.*

Proof: Suppose toward a contradiction that there is a $C^\dagger \in \mathcal{C} \setminus \{C, C_*\}$ such that $d(F^\dagger, w_{\mathcal{T}}(C^\dagger)) < \frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C')$. It follows from Theorem 11.2.3 that there is a shortest $(F^\dagger, w_{\mathcal{T}}(C^\dagger))$-path Q with an endpoint in $V(F^\dagger) \setminus V(F)$. Let q be the $F^\dagger$-endpoint of Q and let q^* be the $w_{\mathcal{T}}(C^\dagger)$-endpoint of Q. Since $q \notin V(F)$, it follows from Theorem 1.3.2 that $q \notin V(C_* \setminus K) \cup D_1(K)$. We have $q \notin V(C^\dagger)$, or else, since $C, C^\dagger \in \mathcal{C} \setminus \{C_*\}$, we contradict 1) of Observation 2.1.8. Possibly, $K^\dagger$ has nonempty intersection with $V(C)$. In any case, applying Theorem 1.3.2 again, we have $q \in D_1(K^\dagger \setminus K) \cup V(C \setminus K^\dagger)$.

Now, we have $V(K^\dagger \setminus K) \cup V(C \setminus K^\dagger) \subseteq B_{20N_{\text{mo}}}(A)$. Furthermore, any generalized chord of A of length at most four has all of its vertices in $B_4(C)$, and $B_4(C) \subseteq B_{20N_{\text{mo}}}(A)$. By 1) of Observation 2.1.8, $V(C^\dagger) \cap (\text{Ann}(C) \cup B_{8N_{\text{mo}}}(A)) = \varnothing$. Since $q^* \in w_{\mathcal{T}}(C^\dagger)$, there is a subpath of Q with q^* as an endpoint and the other endpoint in $B_{20N_{\text{mo}}}(A)$. Since $|E(P)| < \frac{\beta}{3} + 4N_{\text{mo}}$, we again contradict 1) of Observation 2.1.8. ■

Note now that, applying the terminology introduced at the start of Chapter 11, we have the following combined form of 2) of Theorem 10.0.7 and 1) of Theorem 6.0.9.

Claim 11.4.2. *Let $C \in \mathcal{C}$, let A be the collar of C, let G' be the large side of A, and let z be a vertex of $V(G') \setminus \text{Sh}_{4,L}(A, G')$ which is of distance precisely two from A. Then there exists a (C, z)-opener. In particular, letting $[H, \psi]$ be a (C, z)-opener, H is a subgraph of G and ψ is a partial L-coloring of $V(H)$ such that the following hold.*

1) *H is connected, $\mathbf{P}_{\mathcal{T}}(C) \subseteq H$, and $V(K) \subseteq \text{Ann}(A) \cup \{z\}$; AND*
2) *$z \in \text{dom}(\psi)$ and, for all $v \in D_1(H)$, $|L_\psi(v)| \geq 3$; AND*
3) *$V(H) \setminus \text{dom}(\psi)$ is L_ψ-inert in $G \setminus \text{dom}(\psi)$; AND*
4) *There is at most one vertex of $(\text{dom}(\psi) \cap D_1(A, G')) \setminus \text{Sh}_4(A, G')$ which does not lie in $\text{Span}(z)$; AND*
5) *For any $v \in V(H) \cap \text{Sh}_4(A, G')$, either $v \in \text{Sh}_3(A, G')$, or $\text{Span}(z)$ is a proper 4-chord of A which, in G', separates v from each element of $\mathcal{C} \setminus \{C\}$.*

We also have the following simple facts.

Claim 11.4.3. *Let $C \in \mathcal{C} \setminus \{C_*\}$ and let A, A_* be the respective collars of C, C_*. Let $Q := zp_1 \cdots p_s z_*$ be a $(D_2(A), D_2(A_*))$-path. Then the following hold.*

1) *Q is disjoint to $\text{Ann}(C) \cup \text{Ann}(C_*)$ and no vertex of $p_1 \cdots p_s$ has a neighbor in $\text{Ann}(C) \cup \text{Ann}(C_*)$; AND*
2) *Given a (C, z)-opener $[H, \psi]$ and a (C_*, z_*)-opener $[H_*, \psi_*]$, the subgraph of G induced by $V(H \cup H_* \cup Q)$ is connected.*
3) *If $p_1 \cdots p_s$ is a $(D_3(A), D_3(A_*))$-path, then there is no $v \in V(G)$ with a neighbor in p_2Qp_{s-1} and a neighbor in $\text{Ann}(C) \cup \text{Ann}(C_*)$*

<u>Proof:</u> Note that $\text{Ext}(A)$ is the large side of A and $\text{Int}(A_*)$ is the large side of A_*. For any generalized chord R of A in $\text{Ext}(A)$ with $|E(R)| \leq 4$, each vertex of R lies in $B_2(A) \cap V(\text{Ext}(A))$. Likewise, for any generalized chord R_* of A_* in $\text{Int}(A_*)$ with $|E(R_*)| \leq 4$, each vertex of R_* lies in $B_2(A_*) \cap V(\text{Int}(A_*))$. Since Q intersects with $D_2(A) \cup D_2(A_*)$ precisely on its endpoints, it immediately follows that Q is disjoint $\text{Sh}_4(A, \text{Ext}(A)) \cup \text{Sh}_4(A_*, \text{Int}(A_*))$, and that no internal vertex of Q has a neighbor in $\text{Sh}_4(A, \text{Ext}(A)) \cup \text{Sh}_4(A_*, \text{Int}(A_*))$. Likewise, Q is disjoint to $B_1(A) \cup B_1(A_*)$, and no internal vertex of Q is adjacent to any vertex of $B_1(A) \cup B_1(A_*)$ It immediately follows that Q is disjoint to $\text{Ann}(C) \cup \text{Ann}(C_*)$, and no internal vertex of Q is adjacent to a vertex of $\text{Ann}(C) \cup \text{Ann}(C_*)$. This proves 1).

By Claim 11.4.2, each of H, H_* is connected, and $z \in V(H)$ and $z_* \in V(H_*)$, so it immediately follows that the subgraph of G induced by $V(H \cup H_* \cup Q)$ is connected. Now we prove 3). Let $v \in V(G)$ with a neighbor $p \in \{p_2, \cdots, p_s\}$. Since p_1Qp_s is a $(D_3(A), D_3(A_*))$-path, we have $d(p, A \cup A_*) \geq 4$, so $d(v, A \cup A_*) \geq 3$. Suppose toward a contradiction that v has a neighbor $v' \in \text{Ann}(C) \cup \text{Ann}(C_*)$. Suppose without loss of generality that $v' \in \mathrm{Ann}(C)$. Since $d(v, A) \geq 3$, there is a generalized chord R of A in $\mathrm{Ext}(A)$ of length at most four which separates v' from $p_1 \cdots p_s$. Since $V(R) \subseteq B_2(A)$ and $vv' \in E(G)$, we have a contradiction. ■

Recalling Definition 1.2.8, we have the following.

Claim 11.4.4. *Let $C \in \mathcal{C} \setminus \{C_*\}$ and let A, A_* be the respective collars of C, C_*. Let $z \in D_2(A) \setminus \mathrm{Sh}_4(A, \mathrm{Ext}(A))$ and let $z_* \in D_2(A_*) \setminus \mathrm{Sh}_4(A_*, \mathrm{Int}(A_*))$. Let $[H, \psi]$ be a (C, z)-opener and let $[H_*, \psi_*]$ be a (C_*, z_*)-opener. Let $p \in (N(z) \cap D_3(A)) \setminus \mathrm{Sh}_4(A, \mathrm{Ext}(A))$ and let $p_* \in (N(z_*) \cap D_3(A_*)) \setminus \mathrm{Sh}_4(A_*, \mathrm{Int}(A_*))$, where $|\mathrm{Bar}_A(pz)| \leq 1$ and $|\mathrm{Bar}_{A_*}(p_* z_*)| \leq 1$. Then there exists a $\varphi \in \Phi_{G,L}(\psi \cup \psi_*, \{p, p_*\})$ such that each vertex of $\mathrm{Bar}_A(pz) \cap D_1(H)$ and each vertex of $\mathrm{Bar}_{A_*}(p_* z_*) \cap D_1(H_*)$ has an L_φ-list of size at least three.*

Proof: Firstly, for any $1 \leq k \leq 4$ and any k-chord R of A in $\mathrm{Ext}(A)$, we have $p \notin V(R)$, since $d(p, A) = 3$. Since $p \notin \mathrm{Sh}_4(A, \mathrm{Ext}(A))$, it follows from Claim 11.4.3 that $N(p) \cap \mathrm{dom}(\psi) = \{z\}$. Likewise $N(p_*) \cap \mathrm{dom}(\psi_*) = \{z_*\}$, so it immediately follows from opur distance conditions that each of $N(p)$ and $N(p_*)$ intersect with $\mathrm{dom}(\psi \cup \psi_*)$ on a lone vertex, as $d(A, A_*) \geq \frac{\beta}{3} - 4$, and thus each of p, p_* has an $L_{\psi \cup \psi_*}$-list of size at least four. By 2) of Claim 11.4.2, each vertex of $\mathrm{Bar}_A(pz) \cap D_1(H)$ and each vertex of $\mathrm{Bar}_{A_*}(p_* z_*) \cap D_1(H_*)$ has an $L_{\psi \cup \psi_*}$-list of size at least three. Since each of these vertex sets has size at most one, it immediately follows that there exists a $\varphi \in \Phi_{G,L}(\psi \cup \psi_*, \{p, p_*\})$ which satisfies the claim. ■

Claim 11.4.5. *Let $C \in \mathcal{C} \setminus \{C_*\}$ and let A, A_* be the respective collars of C, C_*. Let $P := p_1 \cdots p_s$ be a $(D_3(A), D_3(A_*))$-path, where $p_1 \in D_3(A)$ and $p_s \in D_3(A_*)$. Suppose further that $B_1(P)$ contains no vertices of any ring of $\mathcal{C}$. Let $z \in D_2(A) \cap N(p_1)$ and let $z_* \in D_2(A_*) \cap N(p_s)$, where $|\mathrm{Bar}_A(p_1 z)| \leq 1$ and $|\mathrm{Bar}_{A_*}(p_s z_*)| \leq 1$. Let $[H, \psi]$ be a (C, z)-opener and let $[H_*, \psi_*]$ be a (C_*, z_*)-opener. Let φ be a extension of $\psi \cup \psi_*$ to an L-coloring of $\mathrm{dom}(\psi \cup \psi_*) \cup \{p_1, p_s\}$ which satisfies Claim 11.4.4, and let F be the outer face of $G \setminus V(H \cup H_* \cup P)$. Then, for any $\varphi^\dagger \in \Phi_{G,L}(\psi \cup \psi_*, V(P))$, if there is a $w \in V(F)$ with $|L_{\varphi^\dagger}(w)| < 3$, then $N(w) \cap \mathrm{dom}(\varphi^\dagger) \subseteq V(z p_1 P p_s z_*)$.*

Proof: Let $w \in V(F)$ with $|L_{\varphi^\dagger}(w)| < 3$. By 1) of Claim 11.4.2, we have $\mathbf{P}_{\mathcal{T}}(C) \cup \mathbf{P}_{\mathcal{T}}(C_*) \subseteq H \cup H_*$. Since $B_1(P)$ contains no vertices of any ring of $\mathcal{C}$, it follows from Theorem 1.3.2 that w has at least three neighbors in $\mathrm{dom}(\varphi^\dagger)$. Furthermore, $N(w) \cap \mathrm{dom}(\varphi^\dagger) \not\subseteq \mathrm{dom}(\psi)$ and $N(w) \cap \mathrm{dom}(\varphi^\dagger) \not\subseteq \mathrm{dom}(\psi_*)$. It immediately follows from our distance conditions that $N(w) \cap \mathrm{dom}(\varphi^\dagger) \not\subseteq \mathrm{dom}(\psi \cup \psi_*)$, so w has a neighbor in P.

Suppose toward a contradiction that $N(w) \cap \mathrm{dom}(\varphi^\dagger) \not\subseteq V(z p_1 P p_s z_*)$. Thus, by 1) of Claim 11.4.2, there is a neighbor of w in $\mathrm{dom}(\varphi^\dagger)) \cap (\mathrm{Ann}(C) \cup \mathrm{Ann}(C_*))$. Since $p_1 \cdots p_s$ is a $(D_3(A), D_3(A_*))$-path, it follows from 3) of Claim 11.4.3 that w has no neighbor in $V(\mathring{P})$. Thus, it immediately follows from our distance conditions that $N(w) \cap \mathrm{dom}(\varphi^\dagger)$ is contained in one of $\mathrm{dom}(\psi) \cup \{p_1\}$ or $\mathrm{dom}(\psi_*) \cup \{p_s\}$, so suppose without loss of generality that $N(w) \cap \mathrm{dom}(\varphi^\dagger) \subseteq \mathrm{dom}(\psi) \cup \{p_1\}$. Thus, p_1 is the unique neighbor of w on P. Note that, since $p_1 \in N(w)$ and $d(p_1, A) = 3$, we have $N(w) \cap V(A) = \varnothing$.

Subclaim 11.4.6. *w has no neighbor in* $\mathrm{Span}(z)$.

Proof: We first show that $z \notin N(w)$. Suppose toward a contradiction that $z \in N(w)$. If w has a neighbor in $D_1(A)$, then $w \in \mathrm{Bar}_A(p_s z)$, and thus $|L_{\varphi^\dagger}(w)| \geq 3$ by our choice of φ, contradicting our assumption. Thus, w has no neighbor in $D_1(A)$, so $w \notin D_2(A)$. Since $w \in N(p_1) \cap N(z)$, we thus have $w \in D_3(A)$, as $p_1 \in D_3(A)$ and $z \in D_2(A)$. Since $|L_{\varphi^\dagger}(w)| < 3$, it follows that w has a neighbor in $\mathrm{dom}(\psi) \cap D_2(A)) \setminus \{z\}$, so, by 1) of Claim 11.4.2, w has a neighbor v in $D_2(A) \cap \mathrm{Sh}_4(A, \mathrm{Ext}(A))$. But since $w \in D_3(A)$, there is no $1 \leq k \leq 4$ such that w lies on a k-chord of A, and since $V(P) \cap \mathrm{Ann}(C) = \varnothing$, this contradicts the fact that $wv \in E(G)$.

Now suppose toward a contradiction that w has a neighbor w' in Span(z). Since $N(w) \cap V(A) = \varnothing$ and $z \notin N(w)$, we have $w' \in D_1(A) \cap V(\text{Span}(z))$, so $zw' \in E(G)$ and G contains the 4-cycle $p_1zw'w$. Since $w' \in D_1(A)$, we have $p_1w' \notin E(G)$, and since G is short-separation-free, it follows from our triangulation conditions that $wz \in E(G)$, which has been ruled out above. Thus, w has no neighbor in Span(z). ■

We now note the following:

Subclaim 11.4.7. $N(w) \cap \text{dom}(\psi) \subseteq D_1(A) \setminus \text{Sh}_4(A, \text{Ext}(A))$.

Proof: Since $z \notin N(w)$ and $N(w) \cap A = \varnothing$, we have $N(w) \cap \text{dom}(\psi) \subseteq D_1(A) \cup \text{Sh}_4(A, \text{Ext}(A))$ by 1) of Claim 11.4.2. We just need to show that no vertex of $N(w) \cap \text{dom}(\psi))$ lies in $\text{Sh}_4(A, \text{Ext}(A))$. Suppose toward a contradiction that there is a $v \in N(w) \cap \text{dom}(\psi))$ with $v \in \text{Sh}_4(A, \text{Ext}(A))$. Since w has no neighbors in Span(z), it follows from 5) of Claim 11.4.2 that, for some $1 \leq k \leq 3$, there is a k-chord R of A such that, in Ext(A), R separates v from each element of $\mathcal{C} \setminus \{C\}$. But since $w \in N(p_1)$, we have $w \notin V(R)$, so R separates v from w and $vw \notin E(G)$. ■

Now, since $|L_{\varphi^\dagger}(w)| < 3$ and $N(w) \cap \text{dom}(\varphi^\dagger) \subseteq \{p_1\} \cup \text{dom}(\psi)$, it follows from Subclaim 11.4.7 that w has two neighbors $v, v' \in (\text{dom}(\psi) \cap D_1(A)) \setminus \text{Sh}_4(A, \text{Ext}(A))$, and, by Subclaim 11.4.6, $v, v' \notin V(\text{Span}(z))$, contradicting 4) of Claim 11.4.2. This completes the proof of Claim 11.4.5. ■

By 2) of Corollary 2.2.29, we have $|\mathcal{C} \setminus \{C_*\}| > 1$. We now have the following.

Claim 11.4.8. *C_* is an open $\mathcal{T}$-ring.*

Proof: Suppose toward a contradiction that C_* is a closed $\mathcal{T}$-ring. Now we apply the work of Section 11.1. We now choose a ring which minimizes the quantity $d_G(w_{\mathcal{T}}(C), w_{\mathcal{T}}(C_*))$ over all $C \in \mathcal{C} \setminus \{C_*\}$. Let C_m be this element of $\mathcal{C} \setminus \{C_*\}$. Let A_m be the collar of C_m and let A_* be the collar of C_*. Since C_* is a closed $\mathcal{T}$-ring, A_* is the 1-necklace of C_*.

Let $P := p_1 \cdots p_s$ be a shortest $(D_3(A_m), D_3(A_*))$-path in G, where $p_1 \in D_3(A_m)$ and $p_s \in D_3(A_*)$. By Observation 1.2.9, there exist $z \in D_2(A_m)$ and $z_* \in D_2(A_*)$ with $|\text{Bar}_{A_m}(p_1z)| \leq 1$ and $|\text{Bar}_{A_*}(p_sz_*)| \leq 1$. Furthermore $P' := zp_1 \cdots p_sz_*$ is a shortest $(D_2(A_m), D_2(A_*))$-path in G, as $d(D_2(A_m), D_2(A_*)) = d(D_3(A_m), D_3(A_*)) + 2$.

By Claim 11.4.3, we have $z, p_1 \notin \text{Sh}_4(A, \text{Ext}(A_m))$ and $z_*, p_s \notin \text{Sh}_4(A_*, \text{Int}(A_*))$. Thus, by Claim 11.4.2, there exists a (C_m, z)-opener $[H, \psi]$, and there exists a (C_*, z_*)-opener $[H_*, \psi_*]$. Furthermore, by Claim 11.4.4, there is an extension of $\psi \cup \psi_*$ to an L-coloring φ of $\text{dom}(\psi \cup \psi_*) \cup \{p_1, p_s\}$ such that each vertex of $\text{Bar}_A(p_1z) \cap D_1(H)$ and each vertex of $\text{Bar}_{A_*}(p_sz_*) \cap D_1(H_*)$ has an L_φ-list of size at least three. Let f be the restriction of φ to $\{z, p_1, p_s, z_*\}$

Now, it follows from Theorem 11.1.1 that no vertex $B_2(P')$ lies in an element of $\mathcal{C}$, as each endpoint of P' is of distance at least three from $C_m \cup C_*$. Since P' is a shortest $(D_2(A_m), D_2(A_*))$-path, it is an ideal C_m-route. By Lemma 11.3.2, there exists a $(D_2(A_m), D_2(A_*))$-path $P^\dagger$ with terminal edges zp_1, z_*p_s, a vertex $v \in D_1(P^\dagger)$, and a $\tau \in \Phi_{G,L}(f, V(P^\dagger))$ such that the following hold.

1) $P^\dagger$ is also an ideal C_m-route; *AND*
2) $V(P^\dagger \setminus P') \subseteq B_{20N_{\text{mo}}}(C_m)$; *AND*
3) $|L_\tau(v)| \geq 2$ and $|N(v) \cap \{p_1, \cdots, p_s\}| \geq 2$.

Let $K^\dagger$ be the subgraph of G induced by $V(H \cup H_* \cup P^\dagger)$ and let $F^\dagger$ be the outer face of $G \setminus K^\dagger$.

Subclaim 11.4.9. *For each $C \in \mathcal{C} \setminus \{C_*, C_m\}$, we have $d(w_{\mathcal{T}}(C), F^\dagger) \geq \frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C)$.*

Proof: Let $C \in \mathcal{C} \setminus \{C_m, C_*\}$ and let K be the subgraph of G induced by $V(H \cup H_* \cup P)$. By 2) of Claim 11.4.3, K is connected. Let F be the outer face of $G \setminus K$. By Theorem 11.1.1, we have $d(C, F) \geq \frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C)$, and, by Claim 11.4.1, we have $d(C, F^\dagger) \geq \frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C)$. ■

It follows from 1) of Claim 11.4.3 that $\varphi^\dagger := \tau \cup \psi \cup \psi_*$ is a proper L-coloring its domain. It follows from Subclaim 11.4.9 that no vertex of $B_1(P^\dagger)$ lies in a ring of $\mathcal{C}$, as $P^\dagger$ has distance at least three from each of C_*, C_m. Since $P^\dagger$ is an ideal C_m-route it follows from Claim 11.4.5 that every vertex of $V(F^\dagger - v$ has an $L_{\varphi^\dagger}$-list of size at least three, and furthermore, since $|N(v) \cap \{p_1, \cdots, p_s\}| \geq 2$, it follows from our distance conditions that v has a neighbor in $\{p_2, \cdots, p_{s-1}\}$, and, by 3) of Claim 11.4.3, we have $N(v) \cap \text{dom}(\varphi^\dagger)) = N(v^\dagger) \cap \text{dom}(\tau)$, so $|L_{\varphi^\dagger}(v)| \geq 2$.

Let $\mathcal{C}_{\text{red}} := \mathcal{C} \setminus \{C_m, C_*\})$ and let $\mathcal{T}_{\text{red}} := (G \setminus K^\dagger, \mathcal{C}_{\text{red}} \cup \{F^\dagger\}, L_{\varphi^\dagger}, F^\dagger)$. Removing some colors from the $L_{\varphi^\dagger}$-list of v so that $|L_{\varphi^\dagger}(v)| = 1$, we get that $\mathcal{T}_{\text{red}}$ is a tessellation in which the outer face is an open ring, and $\mathbf{P}_{\mathcal{T}_{\text{red}}}(F^\dagger)$ is the path v. We claim now that $\mathcal{T}_{\text{red}}$ is a mosaic. It immediately follows from Subclaim 11.4.9 that $\mathcal{T}_{\text{red}}$ satisfies the distance conditions of Definition 2.1.6, since every ring of $\mathcal{C} \setminus \{C_m, C_*\}$ has the same rank in $\mathcal{T}$ and $\mathcal{T}_{\text{red}}$. Since $F^\dagger$ is an open $\mathcal{T}_{\text{red}}$-ring, and $\mathbf{P}_{\mathcal{T}_{\text{red}}}(F)$ is a lone vertex, $\mathcal{T}_{\text{red}}$ trivially satisfies M0)-M2). Thus, $\mathcal{T}_{\text{red}}$ is indeed a tessellation. Since $|V(G \setminus K^\dagger)| < |V(G)|$, it follows from the minimality of $\mathcal{T}$ that $G \setminus K^\dagger$ is $L_{\varphi^\dagger}$-colorable. Thus, there extension of $\varphi^\dagger$ to an L-coloring σ of $V(G \setminus K^\dagger) \cup \text{dom}(\varphi^\dagger)$. By 3) of Claim 11.4.2, σ extends to L-color $V(H) \setminus \text{dom}(\psi)) \cup (V(H_*) \setminus \text{dom}(\psi_*))$ as well, so σ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. ■

Since C_* is an open $\mathcal{T}$-ring, there is a $C \in \mathcal{C} \setminus \{C\}$ and a C-monotone path P which satisfy Theorem 11.2.3. Let C_*^2 be the 2-necklace of C_*. By definition, P is a $(w_{\mathcal{T}}(C), D_3(C_*^2))$-path. By Observation 1.2.9, there is a $z_* \in D_2(C_*^2) \cap N(p_s)$ such that $|\text{Bar}_{C_*^2}(p_s z_*)| \leq 1$. By our choice of P, there exists a C-seam K with tail P and join z_* such that the distance conditions in 2) of Theorem 11.2.3 are satisfied. Thus, there is a (C_*, z_*)-opener $[H_*, \psi_*]$ such that H_* is the subgraph of G induced by $V(K) \cap (\text{Ann}(C_*) \cup \{z_*\})$. Let A be the collar of C.

Claim 11.4.10. *There exist an integer $3 \leq k \leq 9$ and an index $j \in \{1, \cdots, s-2\}$ such that $D_k(A) \cap V(P) = \{p_j\}$ and $D_{k+1}(A) \cap V(P) = \{p_{j+1}\}$.*

Proof: Firstly, since P is a $(w_{\mathcal{T}}(C), D_3(C_*^2))$-path, it is immediate from our distance conditions that, for each $3 \leq k \leq 9$, $\{p_1, \cdots, p_{s-2}\}$ has nonempty intersection with each of $D_k(A)$ and $D_{k+1}(A)$. Thus, if the claim does not hold, then $|E(P)| \geq d(w_{\mathcal{T}}(C), D_3(C_*^2)) + 4$, contradicting the fact that P is a C-monotone path. ■

Let $3 \leq k \leq 9$ and $j \in \{1, \cdots, s-2\}$ be integers satisfying Claim 11.4.10. Let R be a shortest $(D_3(A), p_j)$-path and let q be the $D_3(A)$-endpoint of R. By Observation 1.2.9, there is a $z \in D_2(A) \cap N(q)$ such that $|\text{Bar}_A(qz)| \leq 1$. Let $R' := zqRp_jPp_sz_*$.

Claim 11.4.11. *R' is an ideal C-route.*

Proof: Note that $p_jPp_sz_*$ intersects with $B_k(A)$ precisely on p_j, so R' is a path. Since R is a shortest $(D_3(A), p_j)$-path and P is a C-monotone path, R' intersects with $D_2(A)$ precisely on z and intersects with $D_2(C_*^2)$ precisely on z_*. Thus, R' is a $D_2(A), D_2(C_*^2))$-path. Since P is a C-monotone path, we have $V(R') \cap D_3(C_*^2) = \{p_s\}$ and $V(R") \cap D_4(C_*^2) = \{p_{s-1}\}$. Thus, there is no chord of R' with z_* as an endpoint.

Since R is a shortest path between its endpoints and $V(R) \cap D_3(A) = \{p_j\}$, $zqRp_j$ is also a shortest path between its endpoints. Since $zqRp_j$ is a shortest path and $V(P) \cap D_{k+1} = \{p_{j+1}\}$, there is no chord of R' incident to a vertex of $\{z\} \cup V(R)$. Since P is an induced path, it follows that R' is also an induced path. Suppose toward a contradiction that R' is not a quasi-shortest path. Thus, there is a $v \in D_1(R')$ such that v has two neighbors which are of distance greater than two apart on R'. Since P is a quasi-shortest path, v has a neighbor in $V(R \setminus \{p_j\}) \cup \{z, z_*\}$. Since $V(R') \cap = \{p_j\}$ and $V(R'') \cap D_4 = \{p_{s-1}\}$, we have $z \not\in N(v)$. Since $zqRp_j$ is a shortest path between its endpoints, v has a neighbor $w \in V(R \setminus \{p_j\})$ and a neighbor $w' \in V(P) \setminus \{p_j\}$, where w, w' are of distance greater than two apart on R'', contradicting the fact that $D_k(A) \cap V(P) = \{p_j\}$ and $D_{k+1}(A) \cap V(P) = \{p_{j+1}\}$.

Thus, R' is a quasi-shortest path and a $(D_2(A), D_2(C_*^2))$-path. Since $R' \setminus \{z, z_*\}$ is a $(D_3(A), D_3(C_*^2))$-path, we just need to check the distance bound in Definition 11.3.1. Recall that $|E(P)| \leq d(w_{\mathcal{T}}(C), D_3(C_*^2)) + 3$, as P is a C-monotone path. Every vertex of $C \setminus w_{\mathcal{T}}(C)$ has distance at most $\frac{N_{\text{mo}}}{3}$ from $w_{\mathcal{T}}(C)$, so $|E(P)| \leq d(V(C), D_3(C_*^2)) + \frac{N_{\text{mo}}}{3} + 3$, so $|E(R'')| \leq d(D_2(A), D_2(C_*^2)) + \frac{N_{\text{mo}}}{3} + 3$. We conclude that R' is indeed an ideal C-route. ■

It follows from our choice of P that $B_2(P)$ contains no vertices of any ring in $\mathcal{C} \setminus \{C, C_*\}$, so it follows from our distance conditions that $B_2(R')$ contains no vertices of any ring of $\mathcal{C}$. Since R' is a $(D_2(A), D_2(C_*^2))$-path, $B_2(R')$ contains no vertices of any ring of $\mathcal{C}$. By 1) of Claim 11.4.3, $z, q \not\in \text{Ann}(C)$ and $z_*, p_s \not\in \text{Ann}(C_*)$. By Claim 11.4.2, there exists a (C, z)-opener $[H, \psi]$, and, by Claim 11.4.4, there exists a $\varphi \in \Phi_{G,L}(\psi \cup \psi_*, \{q, p_s\})$ such that every vertex of $\text{Bar}_A(zq) \cap D_1(H)$ and every vertex of $\text{Bar}_{C_*^2}(z_*p_s) \cap D_1(H_*)$ has an $L_{\varphi^\dagger}$-list of size at least three. Let f be the restriction of φ to $\{z, q, p_s, z_*\}$. By Lemma 11.3.2, there exists an ideal C-route R'' with terminal edges zq, z_*p_s, a vertex $v \in D_1(R'')$, and a $\tau \in \Phi_{G,L}(f, V(R''))$ such that the following hold.

1) $V(R'' \setminus R') \subseteq B_{20N_{\text{mo}}}(C)$; *AND*
2) $|L_\tau(v)| \geq 2$ and $|N(v) \cap \{q, p_j, \cdots, p_s\}| \geq 2$.

Let K'' be the subgraph of G induced by $V(H \cup H_* \cup R'')$ and let F'' be the outer face of $G \setminus K''$.

Claim 11.4.12. *For each $C^\dagger \in \mathcal{C} \setminus \{C, C_*\}$, we have $d(C, F'') \geq \frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C^\dagger)$.*

Proof: Let $C^\dagger \in \mathcal{C} \setminus \{C, C_*\}$. Let F be the outer face of $G \setminus K$ and let F' be the outer face of $G \setminus (H_* \cup R')$. By our choice of P, K, it follows from Theorem 11.2.3 that $d(C^\dagger, F) \geq \frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C^\dagger)$. By Claim 11.4.1, we have $d(C^\dagger, F') \geq \frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C^\dagger)$, and, by a second application of Claim 11.4.1, we have $d(C^\dagger, F'') \geq \frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}|C^\dagger)$. ■

It follows from 1) of Claim 11.4.3 that $\varphi^\dagger := \tau \cup \psi \cup \psi_*$ is a proper L-coloring its domain. It follows from Claim 11.4.12 that no vertex of $B_1(R'')$ lies in a ring of $\mathcal{C}$, as R'' has distance at least three from each of C_*, C. Since R'' is an ideal C-route it follows from Claim 11.4.5 that every vertex of $V(F'' - v)$ has an $L_{\varphi^\dagger}$-list of size at least three, and furthermore, since $|N(v) \cap \{q, p_j, \cdots, p_s\}| \geq 2$, it follows from our distance conditions that v has a neighbor in $\{p_j, \cdots, p_{s-1}\}$, and, by 3) of Claim 11.4.3, we have $N(v) \cap \text{dom}(\varphi^\dagger) = N(v^\dagger) \cap \text{dom}(\tau)$, so $|L_{\varphi^\dagger}(v)| \geq 2$.

Let $\mathcal{C}_{\text{red}} := \mathcal{C} \setminus \{C, C_*\}$ and let $\mathcal{T}_{\text{red}} := (G \setminus K'', \mathcal{C}_{\text{red}} \cup \{F''\}, L_{\varphi^\dagger}, F'')$. Removing some colors from the $L_{\varphi^\dagger}$-list of v so that $|L_{\varphi^\dagger}(v)| = 1$, we get that $\mathcal{T}_{\text{red}}$ is a tessellation in which the outer face is an open ring, and $\mathbf{P}_{\mathcal{T}_{\text{red}}}(F'')$ is the path v.

We claim now that $\mathcal{T}_{\text{red}}$ is a mosaic. It immediately follows from Claim 11.4.12 that $\mathcal{T}_{\text{red}}$ satisfies the distance conditions of Definition 2.1.6, as each element of $\mathcal{C} \setminus \{C, C_*\}$ has the same rank in $\mathcal{T}$ and $\mathcal{T}_{\text{red}}$. Since F'' is an open $\mathcal{T}_{\text{red}}$-ring, and $\mathbf{P}_{\mathcal{T}_{\text{red}}}(F'')$ is a lone vertex, $\mathcal{T}_{\text{red}}$ trivially satisfies M0)-M2). Thus, $\mathcal{T}_{\text{red}}$ is indeed a tessellation. Since $|V(G \setminus K'')| < |V(G)|$, it follows from the minimality of $\mathcal{T}$ that $G \setminus K''$ is $L_{\varphi^\dagger}$-colorable. Thus, there extension of $\varphi^\dagger$ to an L-coloring σ of $V(G \setminus K'') \cup \text{dom}(\varphi^\dagger)$. By 3) of Claim 11.4.2, σ extends to L-color $(V(H) \setminus \text{dom}(\psi)) \cup (V(H_*) \setminus \text{dom}(\psi_*))$ as well, so σ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is critical. This completes the proof of Theorem 2.1.7. $\square$

Chapter 12

Lenses and Roulette Wheels

12.1 Introduction

The goal of Chapters 12 and 13 is to complete the proof of Theorem 1.1.3 by reducing from charts to mosaics, i.e we show that Theorem 2.1.7 implies Theorem 1.1.3.

We now provide a brief overview of how this works. Let α be a sufficiently large constant (whose precise value is determined later), and suppose toward a contradiction that there is an $(\alpha, 1)$-chart $(G, \mathcal{C}, L)$ which is not colorable, where this chart has chosen to be vertex-minimal with respect to this property. We show that G contains a family of short separating cycles $B_1, \cdots, B_t$ such that the graph $H := \bigcap_{i=1}^t \text{Ext}(B_i)$ is short-separation-free, and the graph $K := \bigcup_{i=1^t} \text{Int}^+(B_i)$ is admits an L-coloring ϕ such that H is the underlying graph of a mosaic with respect to the list-assignment L_ϕ^K. It then follows that H is L_ϕ^K-colorable, and thus ϕ extends to an L-coloring of G, producing the desired contradiction.

The trickiest part of the argument above is dealing with a short-separation-free subgraph G^* of G obtained from G in the following way: Let D be a separating cycle in G of length at most four, and suppose that $\{D_1, \cdots, D_s\}$ is a collection of separating cycles in G of length at most four, with $\text{Int}(D_i) \subsetneq \text{Int}(D)$ for each $i = 1, \cdots, s$, and the graphs of $\{D_1, \cdots, D_s\}$ are pairwise far apart. Let $G^* := \text{Int}(D) \cap (\bigcap_{i=1}^s \text{Ext}(D_i))$. Since the elements of $\{D_1, \cdots, D_s\}$ are pairwise far apart, there is at most element D^* of $\{D_1, \cdots, D_s\}$ which is close to D (for a definitition of "close" that is made precise later). The main difficulty which arises at the end of Chapter 13 is coloring and deleting a connected subgraph of $B_k(V(D^* \cup D), G^*)$, for some sufficiently small value of k, such that we obtain a graph containing a lone Thomassen facial subgraph which is sufficiently far away from the cycles of $\{D_1, \cdots, D_s\} \setminus \{D^*\}$.

In order to perform the steps above, we prove a sequence of general results about short-separation-free graphs which we need in Chapter 13. The purpose of Chapter 12 is to prove these general results. That is, the work of Chapter 12 is outside of the context of charts with pairwise far-apart rings. We only return to the context of charts with pairwise far-apart rings in Chapter 13

In Section 12.2, we show how, given a short-separation-free graph with a precolored outer cycle, we can color and delete some vertices to produce a Thomassen facial subgraph within a bounded distance of the outer cycle under specified conditions. In Sections 12.3, 12.4, and 12.5, we finally turn our attention to a short-separation-free annulus with two precolored cycles F_0, F_1, each of length at most four, and show that an analogous coloring and deletion can be performed. Finally, in Chapter 13, we apply the results of Chapter 12 to complete the reduction from charts to mosaics described above.

12.2 Precolored Cycles Which Create Many Lists of Size Two

We begin this section by introducing the following natural definition.

Definition 12.2.1. Given a 2-connected planar graph H with outer cycle C, and a facial subgraph D of H, we say that D is *inward-facing* if one the following holds:

1) $H = D = C$*; OR*

2) $H \neq C$ and $D \neq C$.

We now provide a brief overview of this section. In Section 13.4, where we complete the proof of Theorem 1.1.3, there is a step where we need to deal with the following situation: Suppose we have a short-separation-free 2-connected planar graph G with a specified list-assignment L. Suppose further that G contains a 2-connected subgraph K such that $C \subseteq K$, where K is precolored by L, and each vertex of $H \setminus K$ sufficiently close to K has an L-list of size at least five. Suppose further that, for each inward-facing facial subgraph D of K, and each vertex v lying in the open disc bounded by D, the subgraph of K induced by $N(v) \cap V(D)$ is a subpath of D of length at most two. In this section, we show that, in this situation, under some specified additional conditions, we can perform some coloring and deletion of the vertices lying Int(D) which are of distance at most one from D, such that, within the closed disc bounded by D, we obtain a graph whose outer face is a Thomassen facial subgraph of the resulting graph, with respect to the resulting list-assignment. Intuitively, the graph K is a skeleton which partitions G into a collection of closed regions, where we can perform the described coloring and deletion within each of the given regions. We now define the main object of study for Section 12.2.

Definition 12.2.2. Let $k \geq 0$ be an integer. A 4-tuple $\mathcal{L} = (G, C, L, \psi)$ is called a *k-lens* if G is a connected, short-separation-free graph with cyclic outer face C, L is a list-assignment for $V(G)$, and the following conditions are satisfied.

1) ψ is an L-coloring of $V(C)$*; AND*

2) $B_k(C, G)$ is L-colorable, and, in particular, ψ extends to an L-coloring of $B_k(C, G)$*; AND*

3) $|L(v)| \geq 5$ for all $v \in B_{k+1}(C, G) \setminus V(C)$*; AND*

4) For every $v \in B_k(C, G)$, every facial subgraph of G containing v, except possibly C, is a triangle.

We call $\mathcal{L}$ a *lens* if there exists a $k \geq 0$ such that $\mathcal{L}$ is a k-lens.

Note that the definition of a lens does not require the vertices of G outside of the ball of distance k from C to have lists of size at least 5. We begin by analyzing those vertices in the interior of C which have at least three neighbors on C. Thus, we introduce the following useful definition:

Definition 12.2.3. Given a short-separation-free graph G and a cycle C in G, we let $U^{\geq 3}(C) := \{u \in V(\text{Int}(C)) \setminus V(C) : |N(v) \cap V(C)| \geq 3\}$ and we let $U^{2p}(C)$ be the set of $u \in U^{\geq 3}(C)$ such that $C[N(v) \cap V(C)]$ is a subpath of C of length two. Given a vertex $w \in U^{2p}(C)$, we set P_C^w to be the graph $G[N(w) \cap V(C)]$. Note that P_C^w is a path, unless $|V(C)| = 3$.

We build up some more machinery for studying lenses, and then we state the main theorem for Section 12.2. We now introduce the following useful notation.

Definition 12.2.4. Let G be a graph with outer face C, and let $S \subseteq D_1(C, G)$. We let C^S denote the subgraph of G obtained from C by adding to C the vertices of S and all edges of G with one endpoint in S and the other in $V(C)$. If

$S = \{u\}$ is a single vertex, then we denote this graph as C^u.

We also use the following simple observation repeatedly:

Observation 12.2.5. *Let G be a planar graph with outer cycle C and let $u \in U^{\geq 3}(C)$. Let $D_1, \cdots, D_r$ be the inward-facing facial subgraphs of C^u, where $|E(D_1)| \leq |E(D_2)| \leq \cdots \leq |E(D_r)|$. Then the following hold:*

1) *If $|E(D_r)| = |E(D)|$, then $u \in U^{2p}(C)$; AND*
2) *If $|E(D_r)| = |E(C)| - 1$, then one of the following holds.*
   i) *$n = 3$, $|E(D_1)| = 3$, and $|E(D_2)| = 4$; OR*
   ii) *$n = 4$ and $C[N(u)]$ is a subpath of C of length 3.*

Proof. This is an immediate consequence of the equality $\sum_{i=1}^{r} |E(D_i)| = |E(C)| + 2r$, which holds since the sum on the left counts each edge of $E(G) \setminus E(C)$ precisely twice. $\square$

Given a lens $\mathcal{L} = (G, C, L, \psi)$, there is a natural way to associate to $\mathcal{L}$ an ascending sequence of subgraphs of G. We have the following by a simple induction argument:

Observation 12.2.6. *Let $\mathcal{L} = (G, C, L, \psi)$ be a lens. Then there is a sequence of cycles $(C^i : i = 0, 1, 2 \cdots)$ in G, and a sequence of subgraphs $(H^i : i = 0, 1, 2, \cdots)$ of G, such that $H^0 = C$, $C^0 = C$, $H^0 \subseteq H^1 \subseteq \cdots$, and, for each $i = 0, 1, 2, \cdots$, the following hold.*

1) *H^i is 2-connected and $H^i = \mathrm{Ext}(C^i)$; AND*
2) *$|E(C^i)| = |E(C)|$ and every facial subgraph of H^i, except possibly C, C^i, is a triangle; AND*
3) *$H^{i+1} := (C^i)^T$, where $T := U^{2p}(C^i) \cap B_1(C)$; AND*
4) *$V(C^i \setminus C) \subseteq V(C^{i+1} \setminus C)$.*

Given a lens $\mathcal{L} := (G, C, L, \psi)$, since G is a finite graph, there exists an index j such that $U^{2p}(C^j) \cap B_1(C) = \varnothing$, and, in particular, $H^r = H^j$ for all $r \geq j$. We denote the minimal index with this property by $R(\mathcal{L})$ and we call this the *breadth* of $\mathcal{L}$. In some cases, we denote the elements of respective sequences as $C^0_{\mathcal{L}}, C^1_{\mathcal{L}}, \cdots$ and $H^0_{\mathcal{L}}, H^1_{\mathcal{L}}, \cdots$ respectively, where we write the subscript if we need to make clear what the underlying lens is. We are primarily interested in the case where the precolored cycle in a lens admits a subgraph which can be deleted to produce a Thomassen facial subgraph:

Definition 12.2.7. Let $\mathcal{L} = (G, C, L, \psi)$ be a lens. We then have the following definitions.

1) We say that $\mathcal{L}$ is 0-*reducible* if one of the following two statements holds:
   i) ψ extends to an L-coloring of G*; OR*
   ii) There exists a subpath $P \subseteq C$ of length at most one such that $G \setminus (C \setminus P)$ contains a Thomassen facial subgraph F with respect to the list-assignment L^P_{ψ}, where $V(F) = D_1(P, G)$.
2) If P is a subpath of C satisfying condition ii) above, then we call $C \setminus P$ a *reducing path* for $\mathcal{L}$.

Now we define the following.

Definition 12.2.8. Let $\mathcal{L} = (G, C, L, \psi)$ be a lens and let $k \geq 1$. We say that $\mathcal{L}$ is *k-reducible* if there exists a 2-connected subgraph $H \subseteq G[B_k(F_0 \cup F_1, G)]$, with $C \subseteq H$, and a $\psi' \in \Phi(\psi, H)$ such that, for every inward-facing facial subgraph D of H, the tuple $(\text{Int}(D), D, L, \psi'|_{V(D)})$ is a 0-reducible lens. The pair (H, ψ') is called a *k-reducing pair* for $\mathcal{L}$.

In general, given a k-lens $\mathcal{L} = (G, C, L, \psi)$, a 2-connected subgraph $H \subseteq G[B_k(F_0 \cup F_1)]$, with $C \subseteq H$, and a $\psi' \in \Phi(\psi, H)$, the primary obstacle preventing (H, ψ') from being a k-reducing pair for $\mathcal{L}$ is the existence of an inward-facing facial subgraph D of H such that Int(D) contains many vertices of $U^{2p}(D)$. One way to deal with this obstacle is to partially $L_{\psi'}$-color $B_1(\text{Int}(D))$ in such a way that we obtain from Int(D) a graph whose outer face is a Thomassen facial subgraph with respect to the resulting list-assignment. We thus introduce the following useful notion analogous to k-reducibility.

Definition 12.2.9. Let $\mathcal{L} = (G, C, L, \psi)$ be a lens and let $k \geq 1$. We say that $\mathcal{L}$ is *k-partionable* if there exists a 2-connected subgraph $K \subseteq G[B_{k-1}(C)]$, with $C \subseteq K$, and a $\psi' \in \Phi(\psi, K)$, such that, for every inward-facing facial subgraph D of K, conditions 1) and 2) below are satisfied. We call the pair (K, ψ') a *k-partitioning pair* for $\mathcal{L}$.

1) $\mathcal{L}_D := (\text{Int}(D), D, L, \psi'|_{V(D)})$ is a lens*; AND*
2) There exist a $j \in \{0, \cdots, R(\mathcal{L}_D)\}$, a subset $Z \subseteq V(C^j_{\mathcal{L}_D}) \setminus V(D)$, a subpath A of $C^j_{\mathcal{L}_D} \setminus Z$ of length at most one, and a partial $L_{\psi'}$-coloring ϕ of $V(C^j_{\mathcal{L}_D}) \setminus (V(D) \cup Z)$, such that conditions i)-iii) below hold. We call the tuple (j, Z, A, ϕ) a *(K, ψ')-boundary cutter* for D.
   i) $V(A) \subseteq V(C^j_{\mathcal{L}_D}) \setminus (Z \cup \text{dom}(\phi))$*; AND*
   ii) Z is $(L, \psi' \cup \phi)$-inert in G*; AND*
   iii) The outer face of Int$(D) \setminus ((\text{dom}(\psi' \cup \phi) \cup Z) \setminus V(A))$ is a Thomassen facial subgraph with respect to the list-assignment $L^A_{\psi' \cup \phi}$.

Note that if $k \geq 1$ and $\mathcal{L}$ is k-reducible, then $\mathcal{L}$ is k-partitionable. To see this, suppose that $\mathcal{L}$ is k-reducible, and let (H, ψ') be a k-reducing pair for $\mathcal{L}$. Let D be an inward-facing facial subgraph of H. Then $(\text{Int}(D), D, L, \psi'|_{V(D)})$ is a 0-reducible lens. Thus, $(0, \varnothing, \varnothing, \psi')$ is a (H, ψ')-boundary cutter for D. Thus, (H, ψ') is also a 1-partitioning pair for $\mathcal{L}$. With the above machinery in hand, we are finally ready to state our main result for Section 12.2.

Theorem 12.2.10. *Let $\mathcal{L} = (G, C, L, \psi)$ be an 11-lens with $|V(C)| \leq 11$. Then $\mathcal{L}$ is 11-partitionable.*

The majority of the proof of Theorem 12.2.10 consists of an intermediate result which we state below. To state this result, we first introduce the following two definitions:

Definition 12.2.11. Let $\mathcal{L} := (G, C, L, \psi)$ be a lens of breadth r.

1) We say that $\mathcal{L}$ is *non-split* if it satisfies i) and ii) below.
   i) C^r is a chordless cycle and $U^{\geq 3}(C^r) = U^{2\text{p}}(C^r)$; AND
   ii) For any $\phi \in \Phi(\psi, V(C^r))$, the tuple $(\text{Int}(C^r), C^r, L, \phi|_{V(C^r)})$ is a lens.
2) We say that $\mathcal{L}$ is *split* if it not non-split.

One particular property of non-split lens that we use is the following simple observation.

Observation 12.2.12. *Let $\mathcal{L} = (G, C, L, \psi)$ be a non-split lens of breadth r. Then any two vertices of $U^{2p}(C)$ are nonadjacent in G.*

Proof. By definition of the sequence $C^0, C^1, \cdots, C^r$, for each $i \in \{0, \cdots, r\}$, there is no edge ww' in $E(C_i)$ such that each of w, w' lies in $C^i \setminus C$ and is adjacent to a subpath of C of length two. Since C^r is an induced subgraph of

G, any two vertices of $U^{2p}(C)$ are nonadjacent in G. $\square$

We now introduce the following notation:

Definition 12.2.13. Let $\mathcal{L} = (G, C, L, \psi)$ be a lens. Given a subgraph H of G, we let $\text{Part}(H)$ denote the graph $\bigcup(P_C^w : w \in V(H) \cap U^{2p}(C))$.

We now state our intermediate result, the proof of which takes up the majority of Section 12.2.

Proposition 12.2.14. *Let $\mathcal{L} = (G, C, L, \psi)$ be a non-split lens of breadth r such that either:*

1) $\text{Part}(C^r)$ *has at most two connected components; OR*

2) $\text{Part}(C^r)$ *has precisely three connected components, at least one of which is a subpath of C of length two.*

Then $\mathcal{L}$ is 1-partitionable, and, in particular, (C, ψ) is a 1-partitioning pair for $\mathcal{L}$.

The proof of Proposition 12.2.14 consists of a sequence of four lemmas, which we state and prove below. We begin by introducing the following notation.

Definition 12.2.15. Given a non-split lens $\mathcal{L} = (G, C, L, \psi)$ of breath r and a subgraph Q of $C^r \setminus C$, we have the following notation.

1) Let $V^{\geq 1p}(Q) := \{v \in V(Q) : |N(v) \cap V(C)| \geq 2\}$.

2) Let $\text{Mid}(Q)$ be the set of vertices $v \in V(Q)$ such that there exists a $w \in U^{2\text{p}}(C^r)$ with $P_{C^r}^w \subseteq Q$ and v is the middle vertex of $P_{C^r}^w$.

3) If Q is a path, let $\mathcal{E}(Q)$ denote the set of pairs (Z, ϕ) satisfying the following conditions.

   i) ϕ is a partial L_ψ-coloring of Q, and $Z \subseteq V(Q) \setminus \text{dom}(\phi)$*; AND*

   ii) Each endpoint of Q lies in $\text{dom}(\phi)$*; AND*

   iii) For each $w \in U^{2p}(C^r)$ with $P_{C^r}^w \subseteq Q$, we have $|L_{\psi \cup \phi}(w)| \geq 3$. Furthermore, for each $v \in V(Q) \setminus (\text{dom}(\phi) \cup Z)$, we have $|L_{\psi \cup \phi}(v)| \geq 3$*; AND*

   iv) $Z \subseteq \text{Mid}(Q)$, and furthermore, for any $y \in Z$, if w is the unique vertex of $U^{2p}(C^r)$ such that y is the midpoint of $P_{C^r}^w$, then the endpoints of $P_{C^r}^w$ lie in $\text{dom}(\phi)$, and $|L_{\psi \cup \phi}(w)| \geq 2$.

4) If Q is a path, then let $\mathcal{E}_{\text{col}}(Q)$ be the set of partial L_ψ-colorings ϕ of Q such that there exists a $Z \subseteq V(Q) \setminus \text{dom}(\phi)$ with $(Z, \phi) \in \mathcal{E}(Q)$.

Given a pair (Z, ϕ), if Z is a singleton $\{a\}$, then we generally write (a, ϕ) to mean $(\{a\}, \phi)$. Note that, for any $(Z, \phi) \in \mathcal{E}(Q)$, Z is L_ψ-inert.

Given a non-split lens $\mathcal{L} = (G, C, L, \psi)$ of breath r, the majority of the work needed to prove Proposition 12.2.14 consists of finding partial L_ψ-colorings of subpaths of $C^r \setminus C$. Possibly, the entire set $D_1(C)$ consists of vertices of C^r, so a path Q in C^r possibly differs from all of C^r by precisely an edge and induces all of C^r. Much of the analysis below deals with the case of subpaths Q of $C^r \setminus C$ for which this does not happen and thus, in particular, it is permissible to construct partial colorings of Q in which the endpoints share a color. This is the motivation for the definition below:

Definition 12.2.16. Let $\mathcal{L} = (G, C, L, \psi)$ be a non-split lens of breadth r. A subpath Q of $C^r \setminus C$ is called *end-separated* if $V(Q) \neq V(C^r)$ and furthermore, either $|V(Q)| \leq 3$, or the endpoints of Q do not have a common neighbor in C.

We also note the following:

Observation 12.2.17. *Let $\mathcal{L} = (G, C, L, \psi)$ be a non-split lens of breadth r. For any end-separated subpath Q of $C^r \setminus C$, we have $\text{Mid}(Q) \subseteq V^{\geq 1p}(Q)$.*

Proof. If this does not hold, then there is a vertex w with three consecutive neighbors $x_1x_2x_3$ on Q, such that x_2 has only one neighbor on C. Thus, by our triangulation conditions, x_1, x_2, x_3 have a common neighbor in C, and G contains a copy of $K_{2,3}$, contradicting the fact that G is short-separation-free. $\square$

We now prove the first of four lemmas that we need for Proposition 12.2.14:

Lemma 12.2.18. *Let $\mathcal{L} = (G, C, L, \psi)$ be a non-split lens of breadth r and let Q be an end-separated subpath of $C^r \setminus C$. Then the following facts hold.*

1) *Let w be an endpoint of Q, and, for each $v \in V(Q) \setminus \{w\}$, let $B_v \subseteq L_\psi(v)$ be a set of colors with $|B_v| \geq 3$. Then $\text{Col}(w, \mathcal{E}_{\text{col}}(Q)) = L_\psi(w)$ and, in particular, for each $c \in L_\psi(w)$, there exists a pair $(Z, \phi) \in \mathcal{E}(Q)$ such that $\psi(w) = c$, $Z \cup \text{dom}(\phi) = V(Q)$, and $\phi(v) \in B_v$ for each $v \in \text{dom}(\phi) \setminus \{w\}$; AND*
2) *Let $x_1x_2x_3$ be a subpath of Q and suppose that at least one of x_2, x_3 lies in $V(Q) \setminus V^{\geq 1p}(Q)$. Let $A \subseteq L_\psi(x_1)$ with $|A| \geq 2$. Then there is a $(Z, \phi) \in \mathcal{E}(x_1x_2x_3)$ with $\phi(x_1) \in A$.*

Proof. We first prove the following intermediate result.

Claim 12.2.19. *Let $x_1x_2x_3$ be a subpath of $C^r \setminus C$ and suppose that $x_2 \in \text{Mid}(Q)$. Suppose further that both x_2, x_3 have L_ψ-lists of size at least three, and let $B \subseteq L_\psi(x_3)$ with $|B| \geq 3$. Then $\text{Col}(x_1, \mathcal{E}_{\text{col}}(P^w_{C^r})) = L_\psi(x_1)$ and, in particular, for each $c \in L_\psi(x_1)$, there is a pair $(Z, \phi) \in \mathcal{E}(P^w_{C^r})$ with $\phi(x_1) = c$, $Z = \{x_2\}$, and $\phi(x_3) \in B$.*

<u>Proof:</u> Let $c \in L_\psi(x_1)$. We show there is a pair $(Z, \phi) \in \mathcal{E}(x_1x_2x_3)$ with $\phi(x_1) = c$, $Z = \{x_2\}$, and $\phi(x_3) \in B$. If either $c \notin L_\psi(x_2)$ or $|L_\psi(x_2)| \geq 4$, then, for each $d \in B$, there is a pair $(x_2, \phi) \in \mathcal{E}(x_1x_2x_3)$ such that $\phi(x_1) = c$ and $\phi(x_3) = d$, so we are done in that case. Now suppose that $c \in L_\psi(x_2)$ and $|L_\psi(x_2)| = 3$. If there is a color $d \in B \setminus L_\psi(x_2)$, then, again, there is a pair $(x_2, \phi) \in \mathcal{E}(x_1x_2x_3)$ with $\phi(x_1) = c$ and $\phi(x_3) = d$, so we are done in that case. So now suppose that $B \subseteq L_\psi(x_2)$. Since $|B| \geq 3$ and $|L_\psi(x_2)| = 3$, we have $L_\psi(x_2) = B$. Thus, there is a pair $(x_2, \phi) \in \mathcal{E}(x_1x_2x_3)$ with $\phi(x_1) = \phi(x_3) = c$, and $c \in B$. $\blacksquare$

Let $Q := v_1 \cdots v_k$ and let $c \in L(v_1)$. We show by induction on the length of k that there is a pair $(Z, \phi) \in \mathcal{E}(Q)$ such that $\phi(v_1) = c$, $\text{dom}(\phi) \cup Z = V(Q)$, and $\phi(v) \in B_v$ for all $v \in \text{dom}(\phi) \setminus \{v_1\}$. If $k = 1$, then the claim is trivial. Now let $1 \leq i < k$, and let $(Z, \phi) \in \mathcal{E}(Qv_i)$ such that $\phi(v_1) = c$, $Z \cup \text{dom}(\phi) = V(Qv_i)$, and $\phi(v) \in B_v$ for each $v \in \text{dom}(\phi) \setminus \{v\}$.

It suffices to show that there exists a pair $(Z^\dagger, \phi^\dagger) \in \mathcal{E}(Qv_{i+1})$, such that $\phi^\dagger(v_{i+1}) = c$, $Z^\dagger \cup \text{dom}(\phi^\dagger) = V(Qv_{i+1})$, and $\phi^\dagger(v) \in B_v$ for all $v \in \text{dom}(\phi) \setminus \{v\}$. If $v_i \notin \text{Mid}(Q)$, then any extension of ϕ to v_1Qv_{i+1} lies in $\mathcal{E}_{\text{col}}(Q'v_{i+1})$, so we are done in that case.

Now suppose that $v_i \in \text{Mid}(Q)$. We then have $v_{i-1} \notin Z$, since $v_{i-1} \notin \text{Mid}(Q)$. Thus, $v_{i-1} \in \text{dom}(\phi)$. Let ϕ' be the restriction of ϕ to Qv_{i-1}. By Claim 12.2.19, there is a pair $(v_i, \phi^*) \in \mathcal{E}(v_{i-1}v_iv_{i+1})$ with $\phi^*(v_{i-1}) = \phi'(v_{i-1})$, and $\phi^*(v_{i+1}) \in B_{v_{i+1}}$.

Since $V(Q) \neq V(C^r)$ and C^r is a chordless cycle, $\phi' \cup \phi^*$ is a proper L_ψ-coloring of its domain, and the pair $(Z \cup Z^*, \phi \cup \phi^*)$ lies in $\mathcal{E}_{\text{col}}(Qv_{i+1})$. Furthermore, $(Z \cup Z^*) \cup \text{dom}(\phi' \cup \phi^*) = V(Qv_{i+1})$, and $(\phi' \cup \phi^*)(v) \in B_v$ for each $v \in \text{dom}(\phi' \cup \phi^*) \setminus \{v_1\}$. This completes the proof of Fact 1.

Now we prove Fact 2. Suppose toward a contradiction that no element of $\mathcal{E}_{\text{col}}(x_1x_2x_3)$ colors x_1 with a color from A. If $x_2 \notin \text{Mid}(Q)$, then any L_ψ-coloring of $x_1x_2x_3$ lies in $\mathcal{E}_{\text{col}}(x_1x_2x_3)$, contradicting our assumption. Thus, we have $x_2 \in \text{Mid}(Q)$. By Observation 12.2.17, we then have $x_2 \in V^{\geq 1p}(Q)$ and thus $x_3 \in V(Q) \setminus V^{\geq 1p}(Q)$.

Since $x_2 \in \text{Mid}(Q)$, let $w \in U^{2p}(C^r)$ with $P^w_{C^r} = x_1x_2x_3$. If $|L(w)| > 5$, then any L_ψ-coloring of $x_1x_2x_3$ lies in $\mathcal{E}_{\text{col}}(Q)$, contradicting our assumption. Thus, we get $|L(w)| = 5$. Since $|A| \geq 2$, let $c_1, c_2 \in A$. If there is a $c \in L_\psi(x_2)$ with $c \notin L_\psi(w)$, then, taking $i \in \{1,2\}$ with $c_i \neq c$, and letting $d \in L_\psi(x_3) \setminus \{c\}$, the coloring (c_i, c, d) of Q lies in $\mathcal{E}_{\text{col}}(x_1x_2x_3)$, contradicting our assumption. Since c_1, c_2 are distinct and $|L_\psi(x_3)| \geq 4$, there is an $i \in \{1,2\}$ and a $q \in L_\psi(x_3)$ with $|L(w) \setminus \{c_i, q\}| \geq 4$. Since $L_\psi(x_2) \subseteq L(w)$ and $|L(w)| = 5$, we have $L_\psi(x_2) \neq \{c_i, q\}$. Thus, there is an L_ψ-coloring ϕ of $x_1x_2x_3$ with $\phi(x_1) = c_i$ and $\phi(x_3) = q$, contradicting our assumption. This completes the proof of Fact 2. $\square$

The second lemma we need is the following.

Lemma 12.2.20. *Let $\mathcal{L} = (G, C, L, \psi)$ be a non-split lens of breadth r and let Q be an end-separated subpath of $C^r \setminus C$. Let $Q := v_1 \cdots v_k$ and let $A \subseteq L_\psi(v_1)$ with $|A| \geq 2$. Suppose that $V^{\geq 1p}(\mathring{Q}) \subseteq U^{2p}(C)$, and suppose further that there is an internal vertex v' of Q such that $V(\mathring{Q}) \setminus V^{\geq 1p}(Q) = \{v'\}$. Let $B' \subseteq L_\psi(v')$ and $B'' \subseteq L_\psi(v_k)$, where $|B'| \geq 3$ and $|B''| \geq 3$. Then at least one of the following two statements holds.*

1) *There exists a pair $(Z, \phi) \in \mathcal{E}(v_1Qv')$ such that $\phi(v_1) \in A$ and $\phi(v') \in B'$; OR*
2) *There exists a pair $(Z, \phi) \in \mathcal{E}(v_1Qv_k)$ such that $\phi(v_1) \in A$ and $\phi(v_k) \in B''$.*

Proof. By Observation 12.2.12, no two vertices of $U^{2p}(C)$ are adjacent in G. Since $V^{\geq 1p}(Q - v_1) \subseteq U^{2p}(C)$, and $V(Q - v_1) \setminus V^{\geq 1p}(Q) = \{v', v_k\}$, it follows that $|V(Q)| \leq 5$, or else there are two vertices of $V(Q) \cap U^{2p}(C)$ which are consecutive in Q. Now suppose toward a contradiction that the lemma does not hold.

Claim 12.2.21. $k = 5$*,* $v' = v_3$*, and* $v_2, v_4 \in \text{Mid}(Q)$.

<u>Proof:</u> We first show that $|V(Q)| = 5$. We have $|V(Q)| \geq 3$, since v' is an internal vertex of Q. If $|V(Q)| = 3$, then we have $v' = v_2$. But then, since $|L_\psi(v')| \geq 4$, there exists a pair $(Z, \phi) \in \mathcal{E}(v_1Qv_3)$ with $\phi(v_1) \in A$ and $\phi(v_3) \in B''$ by Fact 1 of Lemma 12.2.18, so 2) is satisfied, contradicting our assumption. Suppose now that $|V(Q)| = 4$. Thus, at least one of v_2, v_3 lies in $V(Q) \setminus V^{\geq 1p}(Q)$. On the other hand, at least one of v_2, v_3 lies in $\text{Mid}(Q)$, or else every L_ψ-coloring of Q lies in $\mathcal{E}_{\text{col}}(Q)$, contradicting our assumption.

Suppose that $v_2 \in V(Q) \setminus V^{\geq 1p}(Q)$. Thus $v_3 \in \text{Mid}(Q)$ by Observation 12.2.17. Since $v_2 \notin V^{\geq 1p}(Q)$, we apply Fact 2 of Lemma 12.2.18 to obtain a pair $(Z, \phi) \in \mathcal{E}(v_2v_3v_4)$ with $\phi(v_4) \in B''$. Since $|A| \geq 2$, let $c \in A \setminus \{\phi(v_2)\}$. Recalling Definition 1.1.9, we have the following. Since $v_2 \notin \text{Mid}(Q)$, the pair $(Z, \phi\langle v_1 : c\rangle)$ lies in $\mathcal{E}(Q)$, and colors v_1 with a color of A, and v_4 with a color of B'', contradicting our assumption.

Now suppose that $v_2 \in \mathrm{Mid}(Q)$ and $v_3 \in V(Q) \setminus V^{\geq 1p}(Q)$. By Fact 2 of Lemma 12.2.18, there is a pair $\phi \in \mathcal{E}_{\mathrm{col}}(v_1 Q v_3)$ with $\phi(v_1) \in A$. For any $b \in B'' \setminus \{\phi(v_3)\}$, we then have $\phi\langle v_k : b\rangle \in \mathcal{E}_{\mathrm{col}}(Q)$, which again contradicts our assumption. We conclude that $|V(Q)| > 4$. Since $|V(Q)| \leq 5$, we have $|V(Q)| = 5$.

We now rule out the possibility that $v_3 \in \mathrm{Mid}(Q)$. Suppose toward a contradiction that $v_3 \in \mathrm{Mid}(Q)$. Since $|V(Q)| = 5$, we then have $\{v_3\} = \mathrm{Mid}(Q)$. Since $V^{\geq 1p}(Q)\setminus\{v_1, v_5\} \subseteq U^{2p}(C)$, we have $v_3 \in U^{2p}(C)$ by Observation 12.2.17, and thus $v_2, v_4 \in V(Q) \setminus V^{\geq 1p}(Q)$, contradicting our assumption that $|V(Q \setminus \{v_1, v_5\}) \setminus V^{\geq 1p}(Q)| = 1$. Thus, we have $v_3 \notin \mathrm{Mid}(Q)$, so $\mathrm{Mid}(Q) \subseteq \{v_2, v_4\}$. We claim now that $\mathrm{Mid}(Q) = \{v_2, v_4\}$.

Suppose toward a contradiction that $v_2 \notin \mathrm{Mid}(Q)$. Thus, every L_ψ-coloring of $v_1v_2v_3$ lies in $\mathcal{E}_{\mathrm{col}}(v_1v_2v_3)$, contradicting our assumption, so we have $v_2 \in \mathrm{Mid}(Q)$. Now suppose toward a contradiction that $v_4 \notin \mathrm{Mid}(Q)$.

> **Subclaim 12.2.22.** $v_3 \notin V^{\geq 1p}(Q)$.
>
> Proof: If $v_3 \in V^{\geq 1p}(Q)$, then, by assumption, we have $v_3 \in U^{2p}(C)$, since $v_3 \in V(\mathring{Q})$ and $V^{\geq 1p}(\mathring{Q}) \subseteq U^{2p}(C)$. Since $v_2 \in \mathrm{Mid}(Q)$, we have $v_2 \in V^{\geq 1p}(Q)$ by Observation 12.2.17, and thus, since $v_2 \in V(\mathring{Q})$ as well, we have $v_2 \in U^{2p}(C)$, contradicting Observation 12.2.12. ■

Since $v_3 \notin V^{\geq 1p}(Q)$, there is a pair $(Z, \phi) \in \mathcal{E}(v_1v_2v_3)$ with $\phi(v_1) \in A$, by Fact 2 of Lemma 12.2.18. Since $v_4 \notin \mathrm{Mid}(Q)$, every extension of ϕ to $\mathrm{dom}(\phi) \cup \{v_4, v_5\}$ lies in $\mathcal{E}_{\mathrm{col}}(Q)$, so there is a $\phi' \in \mathcal{E}_{\mathrm{col}}(Q)$ with $\phi'(v_1) \in A$ and $\phi'(v_5) \in B''$, contradicting our assumption. Thus, we have $v_2, v_4 \in \mathrm{Mid}(Q)$. By Observation 12.2.17, we thus have $v_2, v_4 \notin V(Q) \setminus V^{\geq 1p}(Q)$, so we have $v' = v_3$. This completes the proof of Claim 12.2.21. ■

Now we have the following:

Claim 12.2.23.

1) $|L_\psi(v_2)| = |L_\psi(v_4)| = 2$*; AND*
2) $L_\psi(v_3)$ *is the disjoint union of* $L_\psi(v_2)$ *and* $L_\psi(v_4)$*; AND*
3) $A \cap L_\psi(v_2) = \varnothing$.

Proof: Suppose toward a contradiction that there exists a pair $(q, q') \in L_\psi(v_2) \times L_\psi(v_4)$ such that $|L_\psi(v_3)\setminus\{q, q'\}| \geq 3$. Since $|A| \geq 2$, let $a \in A \setminus \{q\}$, and likewise, since $|B''| \geq 3$, let $b \in B'' \setminus \{q'\}$. Now let ϕ be an L_ψ-coloring of $\{v_1, v_2, v_4, v_5\}$ with $\phi(v_1) = a$, $\phi(v_2) = q$, $\phi(v_4) = q'$, and $\phi(v_5) = b$. Then $(\varnothing, \phi) \in \mathcal{E}(Q)$, with $\phi(v_1) \in A$ and $\phi(v_5) \in B''$, contradicting our assumption. Thus, there does not exists such a pair of colors $(q, q') \in L_\psi(v_2)\times L_\psi(v_4)$.

Since $v_3 \notin V^{\geq 1p}(Q)$, we have $|L_\psi(v_3)| \geq 4$, and thus, since no pair of colors (q, q') satisfying the conditions above exists, $L_\psi(v_3)$ is the disjoint union of $L_\psi(v_2)$ and $L_\psi(v_4)$, and furthermore, $|L_\psi(v_3)| = 4$ and $|L_\psi(v_2)| = |L_\psi(v_4)| = 2$. This proves 1) and 2).

Now we prove 3). Suppose toward a contradiction that there is a color $c \in A \cap L_\psi(v_2)$. By 2), we have $c \in L_\psi(v_3)$ and $c \notin L_\psi(v_4)$. Since $|B''| \geq 3$ and $|L_\psi(v_4)| = 2$, there is a $b \in B'' \setminus L_\psi(v_4)$. Let $d \in L_\psi(v_2) \setminus \{c\}$, and let ϕ be an L_ψ-coloring of $\{v_1, v_2, v_3, v_5\}$ obtained by setting $\phi(v_1) = \phi(v_3) = c$, $\phi(v_2) = d$, and $\phi(v_5) = b$. Then the pair (v_4, ϕ) lies in $\mathcal{E}(Q)$. Yet $\phi(v_1) \in A$ and $\phi(v_5) \in B''$, so this contradicts our assumption. This completes the proof of Claim 12.2.23. ■

Now we return to the main proof of Lemma 12.2.20. By 2) of Claim 12.2.23, $|L_\psi(v_2)| = 2$. Since $|B'| \geq 3$, there is a $b \in B'$ such that $b \notin L_\psi(v_2)$. Let $a \in A$ and let ϕ be an L_ψ-coloring of $\{v_1, v_3\}$ obtained by setting $\phi(v_1) = a$

and $\phi(v_3) = b$. By 3) of Claim 12.2.23, we have $a \notin L_\psi(v_2)$, and thus the pair (v_2, ϕ) lies in $\mathcal{E}(v_1v_2v_3)$. Yet since $\phi(v_1) \in A$ and $\phi(v_3) \in B'$, this contradicts our assumption. This completes the proof of Lemma 12.2.20. $\square$

We now combine the two lemmas above in the third lemma in our sequence of four lemmas in the proof of Proposition 12.2.14. We begin by introducing the following definition.

Definition 12.2.24. Let $\mathcal{L} = (G, C, L, \psi)$ be a non-split lens of breadth r. For any subpath Q of $C^r \setminus C$, let $\mathcal{E}^{\text{end}}(Q)$ denote the set of pairs (Z, ϕ) obtained by dropping from $\mathcal{E}(Q)$ the condition that the endpoints of Q lie in $\text{dom}(\phi)$. That is, $\mathcal{E}^{\text{end}}(Q)$ is the superset of $\mathcal{E}(Q)$ consisting of pairs (Z, ϕ) satisfying the following conditions:

1) ϕ is a partial L_ψ-coloring of Q, and $Z \subseteq V(Q) \setminus \text{dom}(\phi)$*; AND*
2) For each $w \in U^{2p}(C^r)$ with $P^w_{C^r} \subseteq Q$, we have $|L_{\psi\cup\phi}(w)| \geq 3$. Furthermore, for each $v \in V(Q) \setminus (\text{dom}(\phi) \cup Z)$, we have $|L_{\psi\cup\phi}(v)| \geq 3$; *AND*
3) $Z \subseteq \text{Mid}(Q)$, and furthermore, for any $y \in Z$ and $w \in U^{2p}(C^r)$ such that y is the midpoint of $P^w_{C^r}$, the endpoints of $P^w_{C^r}$ lie in $\text{dom}(\phi)$ and $|L_{\psi\cup\phi}(y)| \geq 2$.

Let $\mathcal{E}^{\text{end}}_{\text{col}}(Q)$ be the set of partial L_ψ-colorings ϕ of Q such that there exists a $Z \subseteq V(Q) \setminus \text{dom}(\phi)$ with $(Z, \phi) \in \mathcal{E}^{\text{end}}(Q)$.

The third lemma we need for Proposition 12.2.14 is the following. This lemma is the lengthiest of the four lemmas.

Lemma 12.2.25. *Let $\mathcal{L} = (G, C, L, \psi)$ be a non-split lens of breadth r. Let Q be an end-separated subpath of $C^r \setminus C$, with $Q = v_1 \cdots v_k$. Let $f : V(C^r \setminus C) \to \mathbb{N}$ be a function defined as follows:*

$$f(v) := \begin{cases} |L_\psi(v)| & \textit{if } v \in U^{2p}(C) \\ |L_\psi(v)| - 1 & \textit{otherwise} \end{cases}$$

Then the following facts hold.

1) *If $V^{\geq 1p}(Q) \subseteq U^{2p}(C)$ and w is an endpoint of Q, then $|\text{Col}(w, \mathcal{E}_{\text{col}}(Q))| \geq f(w)$; AND*
2) *If* $\text{Part}(Q)$ *has at most one connected component, then, for each $w \in V(Q)$, we have $|\text{Col}(w, \mathcal{E}^{\text{end}}_{\text{col}}(Qw))| \geq f(w)$ and $|\text{Col}(w, \mathcal{E}^{\text{end}}_{\text{col}}(wQ))| \geq f(w)$.*

Proof. Note that, for each $v \in V(C^r \setminus C)$, we have $f(v) \geq 2$. We now have the following simple fact.

Claim 12.2.26. *Let $Q = v_1 \cdots v_k$ be an end-separated subpath of $C^r \setminus C$, and let $j \in \{1, \cdots, k\}$ with $v_{j+1} \notin \text{Mid}(Q)$. Then the following hold.*

1) *If $|\text{Col}(v_i, \mathcal{E}_{\text{col}}(v_iQ))| \geq 2$ for all $i \in \{j+1, \cdots, k\}$, then $\text{Col}(v_j, \mathcal{E}_{\text{col}}(v_jQ)) = L_\psi(v_j)$; AND*
2) *If $|\text{Col}(v_i, \mathcal{E}^{\text{end}}_{\text{col}}(v_iQ))| \geq 2$ for all $i \in \{j+1, \cdots, k\}$, then $\text{Col}(v_j, \mathcal{E}^{\text{end}}_{\text{col}}(v_jQ)) = L_\psi(v_j)$.*

<u>Proof:</u> Let $c \in L_\psi(v_j)$. Since $|\text{Col}(v_j, \mathcal{E}_{\text{col}}(v_{j+1}Q))| \geq 2$, let $(Z, \phi) \in \mathcal{E}(v_{j+1}Q)$ with $\phi(v_{j+1}) \neq c$. Since $v_{j+1} \notin \text{Mid}(Q)$, the pair $(Z, \phi\langle v_j : c\rangle)$ lies in $\mathcal{E}_{\text{col}}(v_jQ)$, and thus $\text{Col}(v_j, \mathcal{E}_{\text{col}}(v_jQ)) = L_\psi(v_j)$. An identical argument shows the analogous statement with $\mathcal{E}_{\text{col}}$ replaced by $\mathcal{E}^{\text{end}}_{\text{col}}$. $\blacksquare$

Now we prove Fact 1. We first have the following:

Claim 12.2.27. *Let Q be an end-separated subpath of $C^r \setminus C$ with $V^{\geq 1p}(Q) \subseteq U^{2p}(C)$ and $|V(Q)| \leq 3$. Then $|\mathrm{Col}(w, \mathcal{E}_{\mathrm{col}}(Q))| \geq f(w)$ for each $w \in V(Q)$.*

Proof: If $|V(Q)| \leq 2$, for each $w \in V(Q)$ and each $c \in L_\psi(w)$, there is an element of $\mathcal{E}_{\mathrm{col}}(wQ)$ using c on w, as any remaining vertex w' of Q has at least one color left in $|L_\psi(w') \setminus \{c\}$. Thus, in that case, we have $|\mathrm{Col}(w, \mathcal{E}_{\mathrm{col}}(Q))| \geq f(w)$.

Now suppose that $k = 3$, so that $Q = v_1v_2v_3$. If the claim does not hold, then there is an endpoint w of Q such that $|\mathrm{Col}(w, \mathcal{E}_{\mathrm{col}}(Q))| < f(w)$. Suppose toward a contradiction that such a w exists, and suppose without loss of generality that $w = v_1$. Note then that $v_2 \in \mathrm{Mid}(Q)$, or else every L_ψ-coloring of Q lies in $\mathcal{E}_{\mathrm{col}}(Q)$, contradicting our assumption. Thus, by Observation 12.2.17, we have $v_2 \in U^{2p}(C)$, and thus $v_1, v_3 \notin V^{\geq 1p}(Q)$. Thus, since $|\mathrm{Col}(v_1, \mathcal{E}_{\mathrm{col}}(Q)| < f(v_1) \leq 3$, there is a set $A \subseteq L_\psi(v_1)$ with $A \cap \mathrm{Col}(v_1, \mathcal{E}_{\mathrm{col}}(Q)) = \varnothing$. Yet, by Fact 2 of Lemma 12.2.18, there is a $(Z, \phi) \in \mathcal{E}(Q)$ with $\phi(v_1) \in A$, so we have a contradiction. ■

Now let $Q := v_1 \cdots v_k$ be an end-separated subpath of C^r with $V^{\geq 1p}(Q) \subseteq U^{2p}(C)$. By Claim 12.2.27, we have $|\mathrm{Col}(v_j, \mathcal{E}_{\mathrm{col}}(v_jQ))| \geq f(v_j)$ for all $j \geq k-2$. If $|V(Q)| \leq 3$, then we are done, so suppose that $|V(Q)| > 3$. Let $j \in \{1, \cdots, k-3\}$, and suppose that, for each index $i \in \{j, \cdots, k\}$, we have $|\mathrm{Col}(v_i, \mathcal{E}_{\mathrm{col}}(v_iQ))| \geq f(v_i)$. It suffices to show now that $|\mathrm{Col}(v_j, \mathcal{E}_{\mathrm{col}}(v_jQ))| \geq f(v_j)$.

If $v_{j+1} \notin \mathrm{Mid}(Q)$, then we are immediately done by Claim 12.2.26, so suppose now that $v_{j+1} \in \mathrm{Mid}(Q)$. By Observation 12.2.17, we have $v_{j+1} \in U^{2p}(C)$, and thus $v_j \in V(Q) \setminus V^{\geq 1p}(Q)$ and $v_{j+2} \in V(Q) \setminus V^{\geq 1p}(Q)$. Suppose that $j = k-4$. Applying Claim 12.2.27, we have $|\mathrm{Col}(v_j, \mathcal{E}_{\mathrm{col}}(v_jQv_{k-1}))| \geq f(v_j)$. Let $(Z, \phi) \in \mathcal{E}_{\mathrm{col}}(v_jQv_{k-1})$. Since $|L_\psi(v_k)| \geq 2$, let $a \in L_\psi(v_k) \setminus \{\phi(v_{k-1})\}$. Then $(Z, \phi^a_{v_k}) \in \mathcal{E}(v_jQ)$, since $v_{k-1} \notin \mathrm{Mid}(Q)$. Thus, in that case, we have $\mathrm{Col}(v_j, \mathcal{E}_{\mathrm{col}}(v_jQv_{k-1})) \subseteq \mathrm{Col}(v_j, \mathcal{E}_{\mathrm{col}}(v_jQ))$, and thus $|\mathrm{Col}(v_j, \mathcal{E}_{\mathrm{col}}(v_jQ))| \geq f(v_j)$, so we are done.

Now suppose that $|V(Q)| > 4$ and that $j \leq k-4$. By Observation 12.2.12, no two vertices of $U^{2p}(C)$ are adjacent in G. Thus, since $j \leq k-4$, there is a minimal index $t \in \{j+3, \cdots, k\}$ such that $v_t \notin U^{2p}(C)$. Suppose toward a contradiction that $\mathrm{Col}(v_j, \mathcal{E}_{\mathrm{col}}(v_jQ))| < f(v_j) = |L_\psi(v_j)| - 1$. Now we apply Lemma 12.2.20 to the path v_jQv_t. In the statement of Lemma 12.2.20, we set $v' := v_{j+2}$, and we set

$$\begin{aligned}
A &:= L_\psi(v_j) \setminus \mathrm{Col}(v_j, \mathcal{E}_{\mathrm{col}}(v_jQ)) \\
B' &:= \mathrm{Col}(v_{j+2}, \mathcal{E}_{\mathrm{col}}(v_{j+2}Q)) \\
B'' &:= \mathrm{Col}(v_t, \mathcal{E}_{\mathrm{col}}(v_tQ))
\end{aligned}$$

Note that v_{j+2} is an internal vertex of v_jQv_t, and $v_{j+2} \notin U^{2p}(C)$. By definition of t, we have $V(v_jQv_t \setminus \{v_j, v_t\}) \setminus V^{\geq 1p}(Q) = \{v_{j+2}\}$, and by assumption, we have $V^{\geq 1p}(Q - \{v_j, v_t\}) \subseteq U^{2p}(C)$. Furthermore, since $v_{j+2}, v_t \notin V^{\geq 1p}(Q)$, we have $f(v_{j+2}) \geq 3$ and $f(v_t) \geq 3$, and thus $|B'| \geq 3$ and $|B''| \geq 3$. By assumption, we have $|\mathrm{Col}(v_j, \mathcal{E}_{\mathrm{col}}(v_jQ))| < f(v_j)$, so $|A| \geq 2$. Thus, either Statement 1 or Statement 2 of Lemma 12.2.20 applies to the given sets A, B', B'' above.

Suppose first that Statement 1 holds. Thus, there is a pair $(Z, \phi) \in \mathcal{E}_{\mathrm{col}}(v_jQv_{j+2})$ with $\phi(v_j) \in A$ and $\phi(v_{j+2}) \in B'$. In that case, there is a pair $(Z*, \phi^*) \in \mathcal{E}_{\mathrm{col}}(v_{j+2}Q)$ with $\phi^*(v_{j+2}) = \phi(v_{j+2})$. But then, since $v_{j+2} \notin \mathrm{Mid}(Q)$, we have $(Z \cup Z^*, \phi \cup \phi^*) \in \mathcal{E}_{\mathrm{col}}(v_jQ)$, contradicting the fact that $\phi(v_j) \in A$. Thus, since no such pair (Z, ϕ) exists, Statement 2 of Lemma 12.2.20 holds, and there exists a pair $(Z', \phi') \in \mathcal{E}_{\mathrm{col}}(v_jQv_t)$ with $\phi'(v_j) \in A$ and $\phi'(v_t) \in B''$. Thus, there is a pair $(Z^*, \phi^*) \in \mathcal{E}(v_tQ)$ with $\phi^*(v_t) = \phi'(v_t)$. But then, since $v_t \notin \mathrm{Mid}(Q)$, the pair $(Z' \cup Z^*, \phi' \cup \phi^*)$ lies in $\mathcal{E}(v_jQ)$, contradicting the fact that $\phi(v_j) \in A$. This completes the proof of Fact 1 of Lemma 12.2.25.

In order to prove Fact 2, we first prove the following intermediate result:

Claim 12.2.28. *If there is an index $m \in \{1, \cdots, k\}$ such that $U^{2p}(C) \subseteq V(Qv_m)$ and $V^{\geq 1p}(Q) \setminus U^{2p}(C) \subseteq V(v_{m+1}Q)$, then, for each $w \in V(Qv_m)$, we have $|\text{Col}(w, \mathcal{E}_{\text{col}}^{\text{end}}(wQ))| \geq f(w)$, and furthermore, for each $w \in V(v_{m+1}Q)$, we have $|\text{Col}(w, \mathcal{E}_{\text{col}}(Qw))| \geq f(w)$.*

Proof: We first deal with the possibility that v_m is an endpoint of Q. If $m = k$, then the claim immediately follows from Fact 1, so we are done in that case. If $m = 1$, then we have $|L_\psi(v)| \geq 3$ for all $v \in V(Q) \setminus \{v_1\}$, and thus the claim follows from Fact 1 of Lemma 12.2.18, so we suppose for the remainder of the proof of Claim 12.2.28 that $1 < m < k$.

Subclaim 12.2.29. *For each $j \in \{m-1, m\}$, we have $|\text{Col}(v_j, \mathcal{E}_{\text{col}}^{\text{end}}(v_jQ))| \geq f(v_j)$.*

Proof: Firstly, for each $j \in \{m, \cdots, k\}$, we have $|\text{Col}(v_j, \mathcal{E}_{\text{col}}^{\text{end}}(v_jQ))| \geq f(v_j)$ by Fact 1 of Claim 12.2.18. Suppose toward a contradiction that $|\text{Col}(v_{m-1}, \mathcal{E}_{\text{col}}^{\text{end}}(v_{m-1}Q))| < f(v_{m-1})$. In that case, by Claim 12.2.26, we have $v_m \in \text{Mid}(Q)$. Furthermore, we have $L_\psi(v_m) \subseteq L_\psi(v_{m+1})$. To see this, suppose there is a $c \in L_\psi(v_m)$ with $c \notin L_\psi(v_{m+1})$. Then, for each color $d \in L_\psi(v_{m-1}) \setminus \{c\}$, the coloring (c, d) of $v_{m-1}v_m$ lies in $\mathcal{E}_{\text{col}}^{\text{end}}(v_{m-1}Q)$, and thus $|\text{Col}(v_{m-1}, \mathcal{E}_{\text{col}}^{\text{end}}(v_{m-1}Q))| \geq |L_\psi(v_{m-1})| - 1 = f(v_{m-1})$, contradicting our assumption, so we indeed have $L_\psi(v_m) \subseteq L_\psi(v_{m+1})$.

Furthermore, if $c \in L_\psi(v_{m-1}) \setminus L_\psi(v_m)$, then $c \in \text{Col}(v_{m-1}, \mathcal{E}_{\text{col}}^{\text{end}}(v_{m-1}Q))$. To see this, note that, since $|L_\psi(v_{m+1})| \geq 3$, there is a $d \in L_\psi(v_{m+1})$ such that $|L_\psi(v_m) \setminus \{d\}| \geq 2$, and, by Fact 1 of Lemma 12.2.18, there is a pair $(Z, \phi) \in \mathcal{E}(v_{m+1}Q)$ with $\phi(v_{m+1}) = d$. Then the pair $(Z \cup \{v_m\}, \phi^c_{v_{m-1}})$ lies in $\mathcal{E}(v_{m-1}Q)$, so we indeed have $L_\psi(v_{m-1}) \setminus L_\psi(v_m) \subseteq \text{Col}(v_{m-1}, \mathcal{E}_{\text{col}}^{\text{end}}(v_{m-1}Q))$.

We also have $L_\psi(v_{m-1}) \cap L_\psi(v_m) \subseteq \text{Col}(v_{m-1}, \mathcal{E}_{\text{col}}^{\text{end}}(v_{m-1}Q))$. To see this, let $c \in L_\psi(v_{m-1}) \cap L_\psi(v_m)$. Since $L_\psi(v_m) \subseteq L_\psi(v_{m+1})$, we have $c \in L_\psi(v_{m+1})$. Let $d \in L_\psi(v_m) \setminus \{c\}$ and let ϕ be the coloring (c, d, c) of $v_{m-1}v_mv_{m+1}$. Then $(\varnothing, \phi) \in \mathcal{E}(v_{m-1}v_mv_{m+1})$. Furthermore, by Fact 1 of Lemma 12.2.18, there is a pair $(Z', \phi') \in \mathcal{E}(v_{m+1}Q)$ with $\phi'(v_{m+1}) = c$. Since $v_{m+1} \notin \text{Mid}(Q)$, the pair $(Z \cup Z', \phi \cup \phi')$ lies in $\mathcal{E}(v_{m-1}Q)$, so we indeed have $L_\psi(v_{m-1}) \cap L_\psi(v_m) \subseteq \text{Col}(v_{m-1}, \mathcal{E}_{\text{col}}^{\text{end}}(v_{m-1}Q))$.

Combining the facts above, we have $\text{Col}(v_{m-1}, \mathcal{E}_{\text{col}}^{\text{end}}(v_{m-1}Q)) = L_\psi(v_{m-1})$, contradicting our assumption that $|\text{Col}(v_{m-1}, \mathcal{E}_{\text{col}}^{\text{end}}(v_{m-1}Q))| < |f(v_{m-1})$. This completes the proof of Subclaim 12.2.29. ■

Now we return to the proof of Claim 12.2.28. We first show that $|\text{Col}(v_i, \mathcal{E}_{\text{col}}^{\text{end}}(v_iQ))| \geq f(v_i)$ for all $i \in \{1, \cdots, m\}$. If $m \leq 2$, then we are done by applying Subclaim 12.2.29, so suppose now that $m \geq 3$. Let $i \in \{1, \cdots, m\}$. If $i \in \{m-1, m\}$, then we are again done by Subclaim 12.2.29, so suppose that $i \leq m-2$ and that, for each index $j \in \{i+1, \cdots, m\}$, we have $|\text{Col}(v_j, \mathcal{E}_{\text{col}}^{\text{end}}(v_jQ))| \geq f(v_j)$. It suffices to show that $|\text{Col}(v_i, \mathcal{E}_{\text{col}}^{\text{end}}(v_iQ))| \geq f(v_i)$.

Suppose toward a contradiction that $|\text{Col}(v_i, \mathcal{E}_{\text{col}}^{\text{end}}(v_iQ))| < |f(v_i)|$. By Claim 12.2.26, we have $v_{i+1} \in \text{Mid}(Q)$. Since $i+1 \leq m$, and $v_{i+1} \in V^{\geq 1p}(Q)$ by Observation 12.2.17, we have $v_i \in U^{2p}(C)$, and thus, since $i+2 \leq m$, we have $v_{i+2} \notin V^{\geq 1p}(Q)$.

Let p be the minimal index among $\{i+3, \cdots, k\}$ such that $v_p \notin U^{2p}(C)$. By Observation 12.2.12, no two vertices of $U^{2p}(C)$ are adjacent in G, so we have $|V(v_iQv_p)| \leq 5$, and $p \in \{i+3, i+4\}$. Now consider the following cases:

Case 1: $v_p \in \text{Mid}(Q)$

In this case, we have $v_p \in V^{\geq 1p}(Q) \setminus U^{2p}(C)$ by Observation 12.2.17, and thus $m = p-1$. Note that $|\text{Col}(v_i, \mathcal{E}_{\text{col}}(v_iQv_{p-1})| \geq f(v_i)$ by Fact 1, so there exists a $(Z, \phi) \in \mathcal{E}_{\text{col}}(v_iQv_{p-1})$ such that $\phi(v_i) \notin \text{Col}(v_i, \mathcal{E}_{\text{col}}^{\text{end}}(v_iQ))$. Since $|L_\psi(v_p)| \geq 3$

and $|L_\psi(v_{p+1})| \geq 3$, there is a $c \in L_\psi(v_{p+1})$ such that $|L_\psi(v_p) \setminus \{c, \phi(v_{p-1})\}| \geq 2$. By Fact 1 of Lemma 12.2.18, there is a pair $(Z', \phi') \in \mathcal{E}_{\text{col}}(v_{p+1}Q)$ with $\phi'(v_{p+1}) = c$, and thus the pair $(Z \cup \{v_p\} \cup Z', \phi \cup \phi')$ lies in $\mathcal{E}(v_iQ)$, contradicting the fact that $\phi(v_i) \notin \text{Col}(v_i, \mathcal{E}^{\text{end}}_{\text{col}}(v_iQ))$. This completes Case 1.

Case 2: $v_p \notin \text{Mid}(Q)$

In this case, we apply Lemma 12.2.20. We have $V(v_iQv_p \setminus \{v_i, v_p\}) \setminus V^{\geq 1p}(Q) = \{v_{i+2}\}$ by the choice of p, and we have $V^{\geq 1p}(v_iQv_p - \{v_i, v_p\}) \subseteq U^{2p}(C)$, again, by the definition of p. In the statement of Lemma 12.2.20, we set $v' := v_{i+2}$ and we set

$$\begin{aligned}
A &:= L_\psi(v_i) \setminus \text{Col}(v_i, \mathcal{E}^{\text{end}}_{\text{col}}(v_iQ)) \\
B' &:= \text{Col}(v_{i+2}, \mathcal{E}^{\text{end}}_{\text{col}}(v_{i+2}Q)) \\
B'' &:= \text{Col}(v_p, \mathcal{E}^{\text{end}}_{\text{col}}(v_pQ))
\end{aligned}$$

Since $v_i \notin U^{2p}(C)$ and $|\text{Col}(v_i, \mathcal{E}^{\text{end}}_{\text{col}}(v_iQ))| < f(v_i)$, we have $|A| \geq 2$. By induction, we have $|B'| \geq f(v_{i+2})$. Since $v_{i+2} \notin V^{\geq 1p}(Q)$, we have $|B'| \geq 3$. Thus, we just need to check that $|B''| \geq 3$. If $v_p \notin V^{\geq 1p}(Q)$, then this immediately follows by induction, since $|B''| \geq f(v_p)$ and, if $v_p \notin V^{\geq 1p}(Q)$ then $f(v_p) \geq 3$. Now suppose that $v_p \in V^{\geq 1p}(Q)$. In that case, we have $m = p - 1$, and $|L_\psi(v_j)| \geq 3$ for all $j \in \{m, \cdots, k\}$. Thus, by Fact 1 of Lemma 12.2.18, we have $\text{Col}(v_p, \mathcal{E}_{\text{col}}(v_pQ)) = L_\psi(v_p)$, and thus $\text{Col}(v_p, \mathcal{E}^{\text{end}}_{\text{col}}(v_pQ)) = L_\psi(v_p)$, so again, we have $|B''| \geq 3$.

Thus Lemma 12.2.20 applies to the sets A, B', B'', so there is either a pair $(Z', \phi') \in \mathcal{E}_{\text{col}}(v_iQv_{i+2})$ with $\phi'(v_i) \in A$ and $\phi'(v_{i+2}) \in B'$, or there is a pair $(Z', \phi') \in \mathcal{E}_{\text{col}}(v_iQv_p)$ with $\phi'(v_i) \in A$ and $\phi'(v_p) \in B''$. In either case, since $v_{i+2}, v_p \notin \text{Mid}(Q)$, the pair (Z', ϕ') can be combined with a pair $(Z^*, \phi^*) \in \mathcal{E}^{\text{end}}_{\text{col}}(v_{i+2}Q) \cup \mathcal{E}^{\text{end}}_{\text{col}}(v_pQ)$, where the right endpoint of $\text{dom}(\phi')$ coincides with the left endpoint of $\text{dom}(\phi^*)$, and the two colorings agree on this vertex. In both cases, we produce an element $\phi' \cup \phi^*$ of $\mathcal{E}^{\text{end}}_{\text{col}}(v_iQ)$ which colors v_i with an element of A, a contradiction. This completes Case 2, and thus completes the proof of the first statement of Claim 12.2.28.

Now we prove the second part of Claim 12.2.28 using an indiction argument similar to the one above, but proceeding in the other direction along the path. The base case in this induction argument deals with the vertex v_{m+1}.

Subclaim 12.2.30. $|\text{Col}(v_{m+1}, \mathcal{E}_{\text{col}}(Qv_{m+1}))| \geq f(v_{m+1})$.

Proof: Let $A := L_\psi(v_{m+1}) \setminus \text{Col}(v_{m+1}, \mathcal{E}_{\text{col}}(Qv_{m+1}))$. Suppose toward a contradiction that $|A| \geq 2$. By Fact 1, we have $|\text{Col}(v_j, \mathcal{E}_{\text{col}}(Qv_j))| \geq f(v_j)$ for each $j \in \{1, \cdots, m\}$. Thus, by Claim 12.2.26, we have $v_m \in \text{Mid}(Q)$, and furthermore, by definition of m, we have $v_m \in U^{2p}(C)$, or else we contradict Observation 12.2.17. Since $v_m \in \text{Mid}(Q)$, we have $m \geq 2$. Now consider the following cases:

Case 1: $m = 2$

In this case, we have $v_1 \notin V^{\geq 1p}(Q)$. Thus, we simply apply Fact 2 of Lemma 12.2.18 to obtain a pair $(Z, \phi) \in \mathcal{E}_{\text{col}}(v_1v_2v_3)$ with $\phi(v_3) \in A$, contradicting the definition of A.

Case 2: $m = 3$

In this case, we have $v_2 \notin V^{\geq 1p}(Q)$. Thus, we simply apply Fact 2 of Lemma 12.2.18 to obtain a pair $(Z, \phi) \in \mathcal{E}_{\text{col}}(v_2v_3v_4)$ with $\phi(v_4) \in A$. Since $|L_\psi(v_1)| \geq 2$, let $c \in L_\psi(v_1) \setminus \{\phi(v_2)\}$. Since $v_3 \in \text{Mid}(Q)$, we have $v_2 \notin \text{Mid}(Q)$, and thus the pair $(Z, \phi^c_{v_1})$ lies in $\mathcal{E}_{\text{col}}(Qv_4)$, contradicting the fact that $\phi^c_{v_1}(v_4) \in A$.

Case 3: $m > 3$

In this case, let p be the maximal index among $\{1, \cdots, m-2\}$ such that $v_p \notin V^{\geq 1p}(Q)$. Such a p exists, since

$m-2 \geq 2$ and no two vertices of $U^{2p}(C)$ are adjacent in G. Now we set $B' := \text{Col}(v_{m-1}, \mathcal{E}_{\text{col}}(Qv_{m-1}))$ and $B'' := \text{Col}(v_p, \mathcal{E}_{\text{col}}(Qv_p))$. Since $v_{m-1}, v_p \notin V^{\geq 1p}(Q)$, we have $|B'| \geq 3$ and $|B''| \geq 3$ by Fact 1, so at least one of Statement 1 or Statement 2 of Lemma 12.2.20 applies to the sets A, B', B''.

Thus, there is either a pair $(Z', \phi') \in \mathcal{E}_{\text{col}}(v_{m-1}Qv_{m+1})$ with $\phi'(v_{m-1}) \in A$ and $\phi'(v_{m-1}) \in B'$, or there is a pair $(Z', \phi') \in \mathcal{E}_{\text{col}}(v_pQv_{m+1})$ with $\phi'(v_{m+1}) \in A$ and $\phi'(v_p) \in B''$. In either case, since $v_{m-1}, v_p \notin \text{Mid}(Q)$, the pair (Z', ϕ') can be combined with a pair $(Z^*, \phi^*) \in \mathcal{E}_{\text{col}}(Qv_{m-1}) \cup \mathcal{E}_{\text{col}}(Qv_p)$, where the right endpoint of $\text{dom}(\phi')$ coincides with the left endpoint of $\text{dom}(\phi^*)$, and the two colorings agree on this vertex. In both cases, we produce an element $\phi' \cup \phi^*$ of $\mathcal{E}_{\text{col}}(Qv_{m+1})$ which colors v_{m+1} with an element of A, contradicting the definition of A.

Thus, our assumption that $|A| \geq 2$ is false. Since $L_\psi(v_{m+1}) \notin U^{2p}(C)$, we have $f(v_{m+1}) = |L_\psi(v_{m+1})| - 1$, and thus $|\text{Col}(v_{m+1}, \mathcal{E}_{\text{col}}(Qv_{m+1}))| \geq f(v_{m+1})$. This completes the proof of Subclaim 12.2.30. ■

Combining Subclaim 12.2.30 with Fact 1, we have $|\text{Col}(v_i, \mathcal{E}_{\text{col}}(Qv_i))| \geq f(v_i)$ for each $i \in \{1, \cdots, m+1\}$. We now finish the proof of Claim 12.2.28 by induction. If $k = m+1$, then we are done, so suppose now that $k > m+1$. Let $j \in \{m+2, \cdots, k-1\}$ and suppose that, for each index $i \in \{1, \cdots, j-1\}$, we have $|\text{Col}(v_i, \mathcal{E}_{\text{col}}(Qv_i))| \geq f(v_i)$. It suffices to show that $|\text{Col}(v_j, \mathcal{E}_{\text{col}}(Qv_j))| \geq f(v_j)$.

Suppose toward a contradiction that $|\text{Col}(v_j, \mathcal{E}_{\text{col}}(Qv_j))| < f(v_j)$. Thus, by Claim 12.2.26, we have $v_{j-1} \in \text{Mid}(Q)$. Let $A := L_\psi(v_j) \setminus \text{Col}(v_j, \mathcal{E}_{\text{col}}(Qv_j))$. Since $|L_\psi(v_j)| \geq 3$, we have $|A| \geq 2$. Yet, by induction, we have $|\text{Col}(v_{j-2}, \mathcal{E}_{\text{col}}(Qv_{j-2}))| \geq f(v_{j-2}) \geq 2$. Since $j-1 \geq m+1$, we have $|L_\psi(v_{j-1})| \geq 3$. Thus, there is a pair of colors (c, d) with $c \in \text{Col}(Qv_{j-2}, \mathcal{E}_{\text{col}}(v_{j-2}))$ and $d \in A$ such that $|L_\psi(v_j) \setminus \{c, d\}| \geq 2$. Let $(Z, \phi) \in \mathcal{E}(Qv_{j-2})$ with $\phi(v_{j-2}) = c$. Then the pair $(Z \cup \{v_{j-1}\}, \phi\langle v_j : d\rangle)$ lies in $\mathcal{E}_{\text{col}}(Qv_j)$, contradicting the fact that $d \in A$. This completes the proof of Claim 12.2.28. ■

With the intermediate result above in hand, we prove Fact 2 of Lemma 12.2.25.

Claim 12.2.31. *Let Q be an end-separated subpath of $C^r \setminus C$ such that* $\text{Part}(Q)$ *has one connected component. If $w \in V^{\geq 1p}(Q) \setminus U^{2p}(C)$, then there exists a connected component Q' of $Q \setminus \{w\}$ such that $V(Q) \cap U^{2p}(C) \subseteq V(Q')$.*

Proof: If w is an endpoint of Q, then there is nothing to prove, so suppose that $|V(Q)| \geq 3$ and w is an internal vertex of Q. Since $w \in V(C^r) \setminus U^{2p}(C)$, there is an edge $xy \in E(C)$ such that $N(w) \cap V(C) = \{x, y\}$. Let P_1, P_2 be the connected components of $Q \setminus \{w\}$, and suppose towards a contradiction that $V(P_i) \cap U^{2p}(C) \neq \varnothing$ for each $i = 1, 2$. Since the endpoints of Q do not share a neighbor in C, $\text{Part}(Q)$ is a subpath of C. It follows that the edge xy lies in $E(\text{Part}(Q))$, or else the deletion of xy separates $\text{Part}(P_1)$ from $\text{Part}(P_2)$ in C. Thus, there is a $w' \in U^{\geq 3}(C) \cap V(Q)$ such that $xy \in E(P_C^{w'})$. Since $w \neq w'$, we contradict short-separation-freeness. ■

Now we return to the proof of Fact 2. Let $Q = v_1 \cdots v_k$. It suffices to show that $\text{Col}(v_i, \mathcal{E}^{\text{end}}_{\text{col}}(v_iQ))| \geq f(v_i)$ for each $i \in \{1, \cdots, k\}$. If $U^{2p}(C) = \varnothing$, then we have $|L_\psi(v)| \geq 3$ for each $v \in V(Q)$. In that case, for each $i \in \{1, \cdots, k\}$, we have $\text{Col}(v_i, \mathcal{E}_{\text{col}}(v_iQ)) = L_\psi(v_i)$ by Fact 1 of Lemma 12.2.18, so we are done in that case. So now suppose that $U^{2p}(C) \cap V(Q) \neq \varnothing$.

We call a subpath Q' of Q an *alternating subpath* of Q if $V^{\geq 1p}(Q')$ is nonempty and $V^{\geq 1p}(Q') \subseteq U^{2p}(C)$. Let Q' be a vertex-maximal alternating subpath of Q. Such a Q' exists since $U^{2p}(C) \cap V(Q) \neq \varnothing$. Let $s, t \in \{1, \cdots, k\}$ be indices such that $Q' = v_sQv_t$. Note that if either $s = 1$ or $t = k$, then Fact 2 immediately follows from Claim 12.2.28, so we suppose now that $1 < s \leq t < k$.

Claim 12.2.32. *For each $i \in \{1, \cdots, s-1\}$, we have $v_i \notin U^{2p}(C)$. Likewise, for each $j \in \{t+1, \cdots, k\}$, we have $v_j \notin U^{2p}(C)$.*

Proof: If $s = 1$, then the first statement is vacuously true, so suppose that $1 < s$. By the maximality of Q', we have $v_{s-1} \in V^{\geq 1p}(Q) \setminus U^{2p}(C)$. Thus, by Claim 12.2.31, we have $\{v_1, \cdots, v_{s-1}\} \cap U^{2p}(C) = \varnothing$. An identical argument shows the analogous statement for $v_{t+1}Qv_k$. ■

Combining Claim 12.2.32 with Claim 12.2.28, we have $\text{Col}(v_j, \mathcal{E}_{\text{col}}(v_jQ))| \geq f(v_j)$ for each $j \in \{s, \cdots, k\}$. If $s = 1$, then we are done, so suppose now that $s > 1$. Now we have the following:

Claim 12.2.33. $|\text{Col}(v_{s-1}, \mathcal{E}^{\text{end}}_{\text{col}}(v_{s-1}Q))| \geq f(v_{s-1})$.

Proof: Suppose toward a contradiction that $|\text{Col}(v_{s-1}, \mathcal{E}^{\text{end}}_{\text{col}}(v_{s-1}Q))| < f(v_{s-1})$. In that case, we have $v_s \in \text{Mid}(Q)$ by Claim 12.2.26. Since $v_s \in V(Q')$, we have $v_s \notin V^{\geq 1p}(Q) \setminus U^{2p}(C)$, and thus $v_s \in U^{2p}(C)$ by Observation 12.2.17.

To see that $L_\psi(v_s) \subseteq L_\psi(v_{s+1})$, suppose there is a color $d \in L_\psi(v_s) \setminus L_\psi(v_{s+1})$. Then, for each $c \in L_\psi(v_{s-1}) \setminus \{d\}$, the pair $(\varnothing, (c, d))$ lies in $\mathcal{E}^{\text{end}}(v_{s-1}Q)$, where (c, d) is a coloring of $v_{s-1}v_s$, and thus thus $|\text{Col}(v_{s-1}, \mathcal{E}^{\text{end}}_{\text{col}}(v_{s-1}Q))| \geq |L_\psi(v_{s-1})| - 1 = f(v_{s-1})$, contradicting our assumption. Now consider the following cases:

Case 1: $v_{s+1} \in V^{\geq 1p}(Q)$

In this case, since $v_{s+1} \notin U^{2p}(C)$, we have $s = t$ and $s + 1 = k$ by Claim 12.2.32. Now, each color $c \in L_\psi(v_{s-1}) \cap L_\psi(v_s)$ lies in $\text{Col}(v_{s-1}, \mathcal{E}^{\text{end}}_{\text{col}}(v_{s-1}Q))$. To see this, let $c \in L_\psi(v_{s-1}) \cap L_\psi(v_s)$. Then, as shown above, $c \in L_\psi(v_{s+1})$. Let ϕ be an L_ψ-coloring of $v_{s-1}v_sv_{s+1}$ with $\phi(v_{s-1}) = \phi(v_{s+1}) = c$. Then $\phi \in \mathcal{E}_{\text{col}}(v_{s-1}Q)$, and thus $c \in \text{Col}(v_{s-1}, \mathcal{E}^{\text{end}}_{\text{col}}(v_{s-1}Q))$.

Now let $c \in L_\psi(v_{s-1}) \setminus L_\psi(v_s)$ and let $d \in L_\psi(v_{s+1})$. Let ϕ be an L_ψ-coloring of $\{v_{s-1}, v_{s+1}\}$ with $\phi(v_{s-1}) = c$ and $\phi(v_{s+1}) = d$. Then $|L_\psi(v_s) \setminus \{c, d\}| \geq 2$, so $(\{v_s\}, \phi)$ lies in $\mathcal{E}(v_{s-1}Q)$, and thus $c \in \text{Col}(v_{s-1}, \mathcal{E}^{\text{end}}_{\text{col}}(v_{s-1}Q))$. We conclude that $\text{Col}(v_{s-1}, \mathcal{E}^{\text{end}}_{\text{col}}(v_{s-1}Q)) = L_\psi(v_{s-1})$, contradicting our assumption. This completes Case 1.

Case 2: $v_{s+1} \notin V^{\geq 1p}(Q)$

In this case, the edge v_sv_{s+1} lies in $E(Q')$, and thus $s + 1 \leq t < k$ (recall that $1 < s \leq t < k$ by assumption). Set $A := L_\psi(v_{s-1}) \setminus \text{Col}(v_{s-1}, \mathcal{E}^{\text{end}}_{\text{col}}(v_{s-1}Q))$. Since $|\text{Col}(v_{s-1}, \mathcal{E}^{\text{end}}_{\text{col}}(v_{s-1}Q))| < f(v_{s-1})$ by assumption, we have $|A| \geq 2$. Now consider the following subcases.

Subcase 2.1: $v_{s+2} \notin \text{Mid}(Q)$

Since $v_{s+1} \notin V^{\geq 1p}(Q)$, we apply Fact 2 of Lemma 12.2.18: There is a pair $(Z, \phi) \in \mathcal{E}(v_{s-1}Qv_{s+1})$ with $\phi(v_{s-1}) \in A$. By our induction hypothesis, we have $\text{Col}(v_{s+2}, \mathcal{E}^{\text{end}}_{\text{col}}(v_{s+2}Q)| \geq f(v_{s+2}) \geq 2$, and thus there is a pair $(Z', \phi') \in \mathcal{E}^{\text{end}}(v_{s+2}Q)$ with $v_{s+2} \in \text{dom}(\phi')$ and $\phi'(v_{s+2}) \neq \phi(v_{s+1})$. By Observation 12.2.17, $v_{s+1} \notin \text{Mid}(Q)$, and since $v_{s+2} \notin \text{Mid}(Q)$ by assumption, the pair $(Z \cup Z', \phi \cup \phi')$ lies in $\mathcal{E}^{\text{end}}(v_{s-1}Q)$, contradicting the fact that $\phi(v_{s-1}) \in A$. This completes Subcase 2.1.

Subcase 2.2: $v_{s+2} \in \text{Mid}(Q) \setminus U^{2p}(C)$

In this case, we have $v_{s+2} \in V^{\geq 1p}(Q) \setminus U^{2p}(C)$ by Observation 12.2.17. Thus, Q' consists of the edge v_sv_{s+1}. Again applying Fact 2 of Lemma 12.2.18, there is a pair $(Z, \phi) \in \mathcal{E}(v_{s-1}Qv_{s+1})$ with $\phi(v_{s-1}) \in A$. Since $Q' = v_sv_{s+1}$, we have $|L_\psi(v_i)| \geq 3$ for all $i \in \{s+2, \cdots, k\}$ by Claim 12.2.32. Thus, by Fact 1 of Lemma 12.2.18, there is a pair

$(Z', \phi') \in \mathcal{E}(v_{s+1}Q)$ with $\phi'(v_{s+1}) = \phi(v_{s+1})$. Since $v_{s+1} \notin \text{Mid}(Q)$, the pair $(Z \cup Z', \phi \cup \phi')$ lies in $\mathcal{E}(v_{s-1}Q)$, contradicting the fact that $\phi(v_{s-1}) \in A$. This completes Subcase 2.2.

Subcase 2.3: $v_{s+2} \in \text{Mid}(Q) \cap U^{2p}(C)$

In this case, we have $k \geq s+3$, and furthermore, $v_{s+1}, v_{s+3} \notin U^{2p}(C) \cup \text{Mid}(Q)$. We apply Lemma 12.2.20 directly to the path $v_{s-1}Qv_{s+3}$. Note that $V(v_sQv_{s+3}) \setminus V^{\geq 1p}(Q) \subseteq \{v_{s+1}, v_{s+3}\}$, and that $V^{\geq 1p}(v_sQv_{s+3}) \setminus \{v_s, v_{s+3}\} \subseteq \{v_s, v_{s+2}\} \subseteq U^{2p}(C)$.

In the statement of Lemma 12.2.20, we set $v' := v_{s+1}$, $B' := \text{Col}(v_{s+1}, \mathcal{E}^{\text{end}}_{\text{col}}(v_{s+1}Q))$, and $B'' := \text{Col}(v_{s+3}, \mathcal{E}^{\text{end}}_{\text{col}}(v_{s+3}Q))$. We just need to check that $|B'| \geq 3$ and $|B''| \geq 3$. By our induction hypothesis, we have $\text{Col}(v_{s+1}, \mathcal{E}^{\text{end}}_{\text{col}}(v_{s+1}Q))| \geq f(v_{s+1})$. Since $v_{s+1} \notin V^{\geq 1p}(Q)$, we have $f(v_{s+1}) \geq 3$, so we indeed have $|B'| \geq 3$. If $v_{s+3} \notin V^{\geq 1p}(Q)$, then we have $f(v_{s+3}) \geq 3$, and we have $|B''| \geq f(v_{s+3})$ by our induction hpothesis, so we are done in that case. Now suppose that $v_{s+3} \in V^{\geq 1p}(Q)$. In that case, we have $v_{s+3} \in V^{\geq 1p}(Q) \setminus U^{2p}(C)$, since $v_{s+2} \in U^{2p}(C)$, so we have $Q' = v_sv_{s+1}v_{s+2}$. By Claim 12.2.32, we have $|L_\psi(v_i)| \geq 3$ for all $i \geq s+3$, and thus, by Fact 1 of Lemma 12.2.18, we have $\text{Col}(v_{s+3}, \mathcal{E}_{\text{col}}(v_{s+3}Q)) = L_\psi(v_{s+3})$, and thus $\text{Col}(v_{s+3}, \mathcal{E}^{\text{end}}_{\text{col}}(v_{s+3}Q)) = L_\psi(v_{s+3})$, so we again have $|B''| \geq 3$.

Thus, in any case, $|B'| \geq 3$ and $|B''| \geq 3$, so Lemma 12.2.20 applies to the sets A, B', B''. Thus, there is either a pair $(Z', \phi') \in \mathcal{E}_{\text{col}}(v_{s-1}Qv_{s+1})$ with $\phi'(v_{s-1}) \in A$ and $\phi'(v_{s+1}) \in B'$, or there is a pair $(Z', \phi') \in \mathcal{E}_{\text{col}}(v_{s-1}Qv_{s+3})$ with $\phi'(v_{s-1}) \in A$ and $\phi'(v_{s+3}) \in B''$. In either case, since $v_{s+1}, v_{s+3} \notin \text{Mid}(Q)$, the pair (Z', ϕ') can be combined with a pair $(Z^*, \phi^*) \in \mathcal{E}^{\text{end}}_{\text{col}}(v_{s+1}Q) \cup \mathcal{E}^{\text{end}}_{\text{col}}(v_{s+3}Q)$, where the right endpoint of $\text{dom}(\phi')$ coincides with the left endpoint of $\text{dom}(\phi^*)$, and the two colorings agree on this vertex. In both cases, we produce an element $\phi' \cup \phi^*$ of $\mathcal{E}^{\text{end}}_{\text{col}}(v_{s-1}Q)$ which colors v_{s-1} with an element of A, a contradiction. This completes the proof of Claim 12.2.33. ■

Now we return to the proof of Fact 2 of Lemma 12.2.25. If $s = 2$, we are done, so suppose that $s \geq 3$. It suffices to show that $|\text{Col}(v_j, \mathcal{E}^{\text{end}}_{\text{col}}(v_jQ))| \geq f(v_j)$ for all $j \in \{1, \cdots, s-2\}$.

Let $j \in \{1, \cdots, s-2\}$, and suppose that $|\text{Col}(v_i, \mathcal{E}^{\text{end}}_{\text{col}}(v_iQ))| \geq f(v_i)$ for all $i \in \{j+1, \cdots, k\}$. It suffices to show that $|\text{Col}(v_j, \mathcal{E}^{\text{end}}(v_jQ))| \geq f(v_j)$. If $v_{j+1} \notin \text{Mid}(Q)$, then we are done by Claim 12.2.26, so now suppose that $v_{j+1} \in \text{Mid}(Q)$. Now, since $j+1 \leq s-1$, we have $|L_\psi(v_{j+1})| \geq 3$ by Claim 12.2.32. Suppose toward a contradiction that $|\text{Col}(v_j, \mathcal{E}^{\text{end}}_{\text{col}}(v_jQ))| < f(v_j)$. In that case, since $f(v_j) = |L_\psi(v_j)| - 1$, there are two colors $c_1, c_2 \in L_\psi(v_j)$ with $c_1, c_2 \notin \text{Col}(v_j, \mathcal{E}^{\text{end}}_{\text{col}}(v_jQ)))$. Since $v_{j+1} \in \text{Mid}(Q)$, we have $j+1 < k$, and $|\text{Col}(v_{j+2}, \mathcal{E}^{\text{end}}_{\text{col}}(v_{j+2}Q))| \geq f(v_{j+2}) \geq 2$ by induction.

Since $|L_\psi(v_{j+1})| \geq 3$, there is a $q \in \text{Col}(v_{j+2}, \mathcal{E}^{\text{end}}_{\text{col}}(v_{j+2}Q))$ and an $n \in \{1, 2\}$ such that $L_\psi(v_{j+1}) \setminus \{q, c_n\}| \geq 2$. Let $(Z, \phi) \in \mathcal{E}^{\text{end}}_{\text{col}}(v_{j+2}Q)$ with $\phi(v_{j+2}) = q$. But then, the pair $(Z \cup \{v_{j+1}\}, \phi\langle v_j : c_n\rangle)$ lies in $\mathcal{E}^{\text{end}}_{\text{col}}(v_jQ)$, contradicting the fact that $c_n \notin \text{Col}(v_j, \mathcal{E}^{\text{end}}_{\text{col}}(v_jQ)))$. This completes the proof of Fact 2 and thus completes the proof of Lemma 12.2.25. □

The fourth and final lemma we need for the proof of Proposition 12.2.14 is the following.

Lemma 12.2.34. *Let $\mathcal{L} = (G, C, L, \psi)$ be a non-split lens of breadth r and let Q be an end-separated subpath of $C^r \setminus C$. If* $\text{Part}(Q)$ *has at most two connected components, then $\mathcal{E}^{\text{end}}(Q) \neq \varnothing$.*

Proof. If Q has at most one connected component, then this immediately follows from Fact 2 of Lemma 12.2.25. Now suppose that $\text{Part}(Q)$ has precisely two connected components. Let P_1, P_2 be the connected components of

Part(Q). In that case, there exists an internal vertex v of Q with $v \in V(Q) \setminus U^{2p}(C)$ such that Part(Qv) $= P_1$ and Part(vQ) $= P_2$. Let $Q = v_1 \cdots v_k$ for some $k \geq 3$ and let $1 < i < k$ with $v = v_i$. Now consider the following cases:

Case 1: $v_i \notin \text{Mid}(Q)$

In this case, we apply Fact 2 of Lemma 12.2.25 to obtain $|\text{Col}(v_i, \mathcal{E}^{\text{end}}(Qv_i))| \geq |L_\psi(v_i)|-1$, and $|\text{Col}(v_i, \mathcal{E}^{\text{end}}(v_iQ))| \geq |L_\psi(v_i)| - 1$. Since $|L_\psi(v_i)| \geq 3$, there exists a pair $(Z, \phi) \in \mathcal{E}^{\text{end}}(Qv_i)$ and a pair $(Z', \phi') \in \mathcal{E}^{\text{end}}(v_iQ)$ such that $\phi(v_i) = \phi'(v_i)$. Since $v_i \notin \text{Mid}(Q)$, the pair $(Z \cup Z', \phi \cup \phi')$ lies in $\mathcal{E}^{\text{end}}(Q)$. This completes Case 1.

Case 2: $v \in \text{Mid}(Q)$.

In this case, we again apply Fact 2 of Lemma 12.2.25 to obtain $|\text{Col}(v_{i-1}, \mathcal{E}^{\text{end}}(Qv_{i-1}))| \geq 2$ and $|\text{Col}(v_{i+1}, \mathcal{E}^{\text{end}}(v_{i+1}Q))| \geq 2$. Since $|L_\psi(v_i)| \geq 3$, there exists a pair of colors (q, q') with $q \in \text{Col}(v_{i-1}, \mathcal{E}^{\text{end}}(Qv_{i-1}))$ and $q' \in \text{Col}(v_{i+1}, \mathcal{E}^{\text{end}}(v_{i+1}Q))$ such that $|L_\psi(v_i) \setminus \{q, q'\}| \geq 2$. Let $(Z, \phi) \in \mathcal{E}^{\text{end}}(Qv_{i-1})$ and $(Z', \phi') \in \mathcal{E}^{\text{end}}(v_{i+1}Q)$ with $\phi(v_{i-1}) = q$ and $\phi'(v_{i+1}) = q'$. Then $(Z \cup \{v_i\} \cup Z', \phi \cup \phi')$ lies in $\mathcal{E}^{\text{end}}(Q)$. This completes the proof of Lemma 12.2.34. $\square$

With all of the above in hand, we are finally ready to prove Proposition 12.2.14, which we restate below.

Proposition 12.2.14. *Let $\mathcal{L} = (G, C, L, \psi)$ be a non-split lens of breadth r such that either:*

1) Part(C^r) *has at most two connected components; OR*

2) Part(C^r) *has precisely three connected components, at least one of which is a subpath of C of length two.*

Then $\mathcal{L}$ is 1-partitionable, and, in particular, (C, ψ) is a 1-partitioning pair for $\mathcal{L}$.

Proof. We break the proof of this proposition into three intermediate results, the first of which is as follows:

Claim 12.2.35. *If $C^r \cap C \neq \varnothing$ and, for each connected component Q of $C^r \setminus C$,* Part(Q) *has at most two connected components, then $\mathcal{L}$ is 1-partitionable, and (C, ψ) is a 1-partitioning pair for $\mathcal{L}$.*

Proof: If $C^r = C$, then, by the definition of C^r, we have $U^{\geq 3}(C^r) = \varnothing$, and thus any subpath of C of length $|V(C)| - 3$ is a reducing path for $\mathcal{L}$ (recall Definition 12.2.7). Thus, (C, ψ) is a 1-reducing path and thus a 1-partitioning pair for $\mathcal{L}$. So now suppose that $C^r \neq C$. Note that, since $C^r \neq C$, we have $U^{2p}(C) \neq \varnothing$. We also note that, for any end-separated subpath Q' of $C^r \setminus C$, Part(Q') has at most two connected components.

Recalling Definition 12.2.9, it suffices to construct a (C, ψ)-boundary cutter for C. Since $C^r \neq C$ and $C^r \cap C \neq \varnothing$, each connected component of $C^r \setminus C$ is an induced path in G, as $\mathcal{L}$ is nonsplit. Let $P_1, \cdots, P_\ell$ be the connected components of $C^r \setminus C$. Now, Part(P_i) has at most two connected components for each $i = 1, \cdots, \ell$. Thus, applying Lemma 12.2.34, we have the following: For each $i = 1, \cdots, \ell$, let $(Z_i, \phi_i) \in \mathcal{E}^{\text{end}}(P_i)$. Let $Z = \bigcup_{i=1}^{\ell} Z_i$ and let $\phi = \bigcup_{i=1}^{\ell} \phi_i$.

Since $\mathcal{L}$ is nonsplit, C^r is a chordless cycle, so, for any $i, j \in \{1, \cdots, \ell\}$ with $i \neq j$, there is no edge of G with one endpoint in dom(ϕ_i) and the other in dom(ϕ_j). Thus, ϕ is a proper L_ψ-coloring of its domain. Furthermore, since Z_i is $L_{\psi\cup\phi_i}$-inert for each $i = 1, \cdots, \ell$, Z is $L_{\psi\cup\phi}$-inert.

Subclaim 12.2.36. $|L_{\psi\cup\phi}(w)| \geq 3$ *for all* $w \in B_1(C^r) \setminus (V(C) \cup Z \cup \text{dom}(\phi))$.

Proof: Suppose toward a contradiction that there is a $w \in B_1(C^r) \setminus (V(C) \cup Z \cup \text{dom}(\phi))$ such that $|L_{\psi\cup\phi}(w)| < 3$. Suppose first that $w \in B_1(C, G)$. In that case, by the construction of $H^r_{\mathcal{L}}$, we have $w \in V(C^r) \cup V(C)$, and thus there is an $i \in \{1, \cdots, \ell\}$ with $w \in V(P_i)$. Since $C^r \setminus C$ is a chordless path, we have $N(w) \cap V(C^r \setminus C) \subseteq$

P_i, and thus $L_{\psi\cup\phi}(w) = L_{\psi\cup\phi_i}(w)$. But then, since $(Z_i, \phi_i) \in \mathcal{E}^{\text{end}}(P_i)$, and $w \notin V(C) \cup Z \cup \text{dom}(\phi))$ we have $|L_{\psi\cup\phi_i}(w)| \geq 3$, contradicting our assumption.

Thus, we have $w \notin B_1(C)$, so we get that $N(w)\cap\text{dom}(\psi\cup\phi)) \subseteq V(C^r)\setminus V(C)$. Since $|L_{\psi\cup\phi}(w)| < 3$, we have $w \in U^{\geq 3}(C^r)$, and thus, since $\mathcal{L}$ is a non-split lens, we have $w \in U^{2p}(C^r)$. But then, since $N(w)\cap V(C^r\setminus C) = \varnothing$, there exists an $i \in \{1, \cdots, \ell\}$ such that $P^w_{C^r}$ is a subpath of P^i, and thus $L_{\psi\cup\phi}(w) = L_{\psi\cup\phi_i}(w)$. Since $(Z_i, \phi_i) \in \mathcal{E}^{\text{end}}(P_i)$, we have $|L_{\psi\cup\phi_i}(w)| \geq 3$, contradicting our assumption. ■

Since $|L_{\psi\cup\phi}(w)| \geq 3$ for all $w \in B_1(C^r) \setminus (V(C) \cup Z \cup \text{dom}(\phi))$, it follows that $(r, Z, \varnothing, \phi)$ is a (C, ψ)-boundary cutter for C. Thus, (C, ψ) is indeed a 1-partitioning pair for $\mathcal{L}$. This completes the proof of Claim 12.2.35. ■

The second intermediate result we need for Proposition 12.2.14 is the following:

Claim 12.2.37. *If* Part(C^r) *has at most two connected components, then $\mathcal{L}$ is 1-partitionable, and (C, ψ) is a 1-partitioning pair for $\mathcal{L}$.*

Proof: If $C^r \cap C \neq \varnothing$, then each connected component of $C^r \setminus C$ is a path Q such that Part(Q) has at most two connected components, so in that case, we are done by Claim 12.2.35. Now suppose that $C^r \cap C = \varnothing$.

Subclaim 12.2.38. *For any $v \in V^{\geq 1p}(C^r)$, $C^r - v$ is an end-separated subpath of $C^r \setminus C$.*

Proof: Let v', v'' be the endpoints of $C^r - v$. Note that $|V(C^r - v)| \geq 3$, or else G contains a cycle of length 4 which separates C from a vertex of $U^{2p}(C^r)$. Since $v \in V^{\geq 1p}(C^r)$, there is a path $P \subseteq C$ of length at least one such that $P = C[N(v)]$. Each of $C[N(v')]$ and $C[N(v'')]$ is a subpath of C, neither of which contains an edge of P. Thus, if v', v'' share a neighbor u of C, then the deletion of the vertices u, v and the edges of $E(P)$ separates v' from v'', contradicting the fact that $C^r \cap C = \varnothing$. Thus, the vertices v', v'' do not have a common neighbor in C, so $C^r - v$ is indeed an end-separated subpath of $C^r \setminus C$. ■

Now we return to the proof of Claim 12.2.37. We break this into two cases:

Case 1 $V^{\geq 1p}(C^r) \setminus U^{2p}(C) = \varnothing$

In this case, let $v \in U^{2p}(C)$. Note that $U^{2p}(C) \neq \varnothing$, or else $C^r = C$. Applying Subclaim 12.2.38, $C^r - v$ is an end-separated subpath of C^r.

Let v', v'' be the endpoints of $C^r - v$. Since $V^{\geq 1p}(C^r) \setminus U^{2p}(C) = \varnothing$, we apply Fact 1 of Lemma 12.2.25. Since $v' \notin V^{\geq 1p}(C^r)$, we have $|\text{Col}(v', \mathcal{E}_{\text{col}}(C^r - v))| \geq f(v') \geq 3$. Thus, there is a pair $(Z, \phi) \in \mathcal{E}(C^r - v)$ such that $|L_\psi(v) \setminus \{\phi(v')\}| \geq 2$. Thus, we have $|L_{\psi\cup\phi}(v)| \geq 1$. We claim now that (r, Z, v, ϕ) is a (C, ψ)-boundary cutter for C. If we show this, then it follows that (C, ψ) is a 1-partitioning pair for $\mathcal{L}$, and then we are done. By definition, for any $w \in B_1(C^r, \text{Int}(C^r)) \setminus (\text{dom}(\psi \cup \phi) \cup Z \cup \{v\})$, we have $|L_{\psi\cup\phi}(w)| \geq 3$, since $(Z, \phi) \in \mathcal{E}(C^r)$. Furthermore, Z is $L_{\psi\cup\phi}$-inert, so (r, Z, v, ϕ) is a (C, ψ)-boundary cutter for C. This completes Case 1.

Case 2: $V^{\geq 1p}(C^r) \setminus U^{2p}(C) \neq \varnothing$

In this case, let $v \in V^{\geq 1p}(C^r) \setminus U^{2p}(C)$. Note then that Part$(C^r - v)$ = Part(C^r). Furthermore, again applying Subclaim 12.2.38, $C^r - v$ is an end-separated subpath of C^r. Applying Lemma 12.2.34, there is a pair $(Z, \phi) \in \mathcal{E}^{\text{end}}(C^r - v)$. Since $(Z, \phi) \in \mathcal{E}^{\text{end}}(C^r - v)$, we have $|L_{\psi\cup\phi}(w)| \geq 3$ for all $w \in B_1(C^r, \text{Int}(C^r)) \setminus (\text{dom}(\phi) \cup Z \cup \{v\})$. Furthermore, since C^r is a chordless cycle, we have $|L_{\psi\cup\phi}(v)| \geq |L_\psi(v)| - 2 \geq 1$, so $\{v\}$ is indeed $L_{\psi\cup\phi}$-colorable. Thus, (r, Z, v, ϕ) is a (C, ψ)-boundary cutter for C, so (C, ϕ) is indeed a 1-partitioning pair for $\mathcal{L}$. This completes the proof of Claim 12.2.37. ■

To complete the proof of Proposition 12.2.14, it suffice to prove the following:

Claim 12.2.39. *If* Part(C^r) *has three connected components, at least one of which is a path of length two, then $\mathcal{L}$ is 1-partitionable, and (C, ψ) is a 1-partitioning pair for $\mathcal{L}$.*

Proof: Let P be a connected component of Part(C^r) of length two. Since P has length two, there is a lone vertex $w \in U^{2p}(C)$ such that $P = P_C^w$. Now consider the following cases. In each case below, we show that (C, ψ) is a 1-partitioning pair for $\mathcal{L}$.

Case 1: $C^r \cap C \neq \varnothing$

In this case, let $Q_1, \cdots, Q_\ell$ be the connected components of $C^r \setminus C$. Since $C^r \cap C \neq \varnothing$, each of $Q_1 \cdots Q_\ell$ is a proper subpath of C. If, for each $i = 1, \cdots, \ell$, Part(Q_i) has at most two connected components, then we are done by Claim 12.2.35, so suppose without loss of generality that Part(Q_1) has three connected components. Thus, we have $w \in V(Q_1)$. Note that, for each $i > 1$, Part(Q_i) has at most two connected components, since $w \notin V(Q_i)$. Thus, by Lemma 12.2.34 we have $\mathcal{E}^{\text{end}}(Q_i) \neq \varnothing$ for each $i = 2, \cdots, \ell$. For each $i = 2, \cdots, \ell$, let $(Z_i, \phi_i) \in \mathcal{E}^{\text{end}}(Q_i)$. Now we have the following subcases.

Subcase 1.1 w is an endpoint of Q_1

In this case, since the path Part$(Q_1 - w)$ has two connected components, we have $\mathcal{E}^{\text{end}}(Q_1 - w) \neq \varnothing$ by Lemma 12.2.34. Let $(Z_1, \phi_1) \in \mathcal{E}^{\text{end}}(Q_1 - w)$. Let $Z := \bigcup_{i=1}^{\ell} Z_i$ and let $\phi := \bigcup_{i=1}^{\ell} \phi_i$. Note that ϕ is indeed a proper L_ψ-coloring of its domain. Furthermore, at most one element of dom(ϕ) is adjacent to w, so $\{w\}$ is $L_{\psi\cup\phi}$-colorable, and, if $x \in B_1(C^r) \setminus (\text{dom}(\psi \cup \phi) \cup Z \cup \{w\})$, then $|L_{\psi\cup\phi}(x)| \geq 3$. Thus, (r, Z, w, ϕ) is a (C, ψ)-boundary cutter for C, so (C, ψ) is indeed a 1-partitioning pair for $\mathcal{L}$.

Subcase 1.2 w is an internal vertex of Q_1

In this case, since Part(Q_1) has three connected components, there is a neighbor v of w in Q_1 which is also an internal vertex of Q_1. Let Q', Q'' be the connected components of $Q \setminus \{w, v\}$. Each of Part(Q') and Part(Q'') has at most two connected components, so, by Lemma 12.2.34, let $(Z', \phi') \in \mathcal{E}^{\text{end}}(Q')$ and $(Z'', \phi'') \in \mathcal{E}^{\text{end}}(Q'')$. Now set $Z := (Z' \cup Z'')$ and set $\phi := (\phi' \cup \phi'')$. Each of w, v has at most one neighbor in dom(ϕ), and $|L_\psi(v)| \geq 3$, so wv is $L_{\psi\cup\phi}$-colorable. As above, $|L_{\psi\cup\phi}(x)| \geq 3$ for all $x \in B_1(C^r) \setminus (\text{dom}(\psi \cup \phi) \cup Z \cup \{w, v\})$, so (r, Z, wv, ϕ) is a (C, ψ)-boundary cutter for C. Thus, (C, ψ) is indeed a 1-partitioning pair for $\mathcal{L}$. This completes Case 1.

Case 2: $C^r \cap C = \varnothing$

In this case, let v be one of the two neighbors of w in C^r. Note that $C^r \setminus \{w, v\}$ is an end-separated path, and Part$(C^r \setminus \{w, v\})$ has two connected components so we have $\mathcal{E}^{\text{end}}(C^r \setminus \{w, v\}) \neq \varnothing$. Let $(Z, \phi) \in \mathcal{E}^{\text{end}}(C^r \setminus \{w, v\})$. Note that wv is $L_{\psi\cup\phi}$-colorable, and $|L_{\psi\cup\phi}(x)| \geq 3$ for all $x \in B_1(C^r) \setminus (\text{dom}(\psi \cup \phi) \cup Z \cup \{w, v\})$. Thus, (r, Z, wv, ϕ) is a (C, ψ)-boundary cutter for C, so (C, ψ) is indeed a 1-partitioning pair for $\mathcal{L}$. This completes the proof of Claim 12.2.39. ■

Combining Claim 12.2.37 and Claim 12.2.39, we complete the proof of Proposition 12.2.14. □

We are now almost ready to prove Theorem 12.2.10. We first gather several additional useful facts.

Lemma 12.2.40. *Let $\mathcal{L} = (G, C, L, \psi)$ be a non-split lens with $|V(C)| \leq 11$. Then $\mathcal{L}$ is 1-partitionable, and, in particular, (C, ψ) is a 1-partitioning pair for $\mathcal{L}$.*

Proof. Let r be the breadth of $\mathcal{L}$. If Part(C^r) has at least 4 connected components, then $|V(C)| \geq (4)(3) = 12$, contradicting our assumption, so Part(C^r) has at most 3 connected components. If Part(C^r) has at most two connected components, then we are done by Proposition 12.2.14. If Part(C^r) has three connected components, and Q is a connected component of Part(C^r) with $|V(Q)| > 2$, then $|V(Q)| \geq 5$, since there exist at least two vertices $w, w' \in U^{2p}(C)$ such that $P_C^w \cup P_C^{w'} \subseteq Q$. Thus, if at least two connected components of Part(C^r) are paths of length greater than 2, then $|V(C)| \geq 5 + 5 + 2 = 12$, contradicting our assumption. Thus, by Proposition 12.2.14, we are done. $\square$

We also have the following very useful fact, which is an immediate consequence of Theorem 1.3.5.

Lemma 12.2.41.

1) *Let $\mathcal{L} = (G, C, L, \psi)$ be a 0-lens with $|V(C)| \leq 4$. Then $\mathcal{L}$ is 0-reducible; AND*

2) *Let $\mathcal{L} = (G, C, L, \psi)$ be a 1-lens with $5 \leq |V(C)| \leq 6$. Then $\mathcal{L}$ is 0-reducible.*

With the above in hand, we now prove Theorem 12.2.10, which we restate below:

Theorem 12.2.10. *Let $\mathcal{L} = (G, C, L, \psi)$ be an 11-lens with $|V(C)| \leq 11$. Then $\mathcal{L}$ is 11-partitionable.*

Proof. We first show the following:

Claim 12.2.42. *Let $t \geq 3$ be an integer, and let $\mathcal{L} = (G, C, L, \psi)$ be an t-lens with $5 < |V(C)| \leq 11$. Suppose further that, for every $(t-2)$-lens $\mathcal{L}' = (G', C', L', \psi')$ with $|V(C')| < |V(C)|$, $\mathcal{L}'$ is $(t-2)$-partitionable. Then $\mathcal{L}$ is t-partitionable.*

Proof: If $\mathcal{L}$ is non-split, then, by Proposition 12.2.14, $\mathcal{L}$ is 1-partitionable, and thus t-partitionable, so we are done in that case. Now suppose that $\mathcal{L}$ is split, and let r be the breadth of $\mathcal{L}$. Note that, since $t \geq 3$, $\Phi(\psi, V(C^r)) \neq \varnothing$, and furthermore, for every $\phi \in \Phi(\psi, V(C^r))$, the tuple $(\text{Int}(C^r), C^r, L, \phi|_{V(C^r)})$ is a 1-lens. Thus, since $\mathcal{L}$ is a split lens, there is either a chord of C^r in G, or $U^{\geq 3}(C^r) \setminus U^{2p}(C^r) \neq \varnothing$. In either case, there is a 2-connected subgraph H of G, where $H^r \subseteq H$, and H is obtained from H^r by adding to H^r either a chord of C^r or a vertex $w \in U^{\geq 3}(C^r) \setminus U^{2p}(C^r)$, together with all edges of $E(w, V(C^r))$.

Let $D_1, \cdots, D_\ell$ be the facial subgraphs of H, other than C. Since $\mathcal{L}$ is a split lens, we have $|V(D_i)| < |V(C^r)| = |V(C)|$ for each $i = 1, \cdots, \ell$. Since $\mathcal{L}$ is a t-lens and $V(H) \subseteq B_2(C)$, there exists a $\phi \in \Phi(\psi, H)$. Furthermore, note that, for each $i = 1, \cdots, \ell$, the tuple $\mathcal{L}_i := (\text{Int}(D_i), D_i, L, \phi|_{V(D_i)})$ is a $(t-2)$-lens, since each vertex of $V(D_i)$ lies in $B_2(C, G)$. By hypothesis, we get that, for each $i = 1, \cdots, \ell$, there exists a $(t-2)$-partitioning pair (K_i, ϕ_i) for $\mathcal{L}_i$. For each $i = 1, \cdots, \ell$, let r_i be the breadth of $\mathcal{L}_i$.

Now set $K^* := \bigcup_{i=1}^{\ell} K_i$ and set $\phi^* := \bigcup_{i=1}^{\ell} \phi_i$. Note that K^* is a 2-connected subgraph of G with $C \subseteq K^*$, and ϕ^* is a proper L-coloring of its domain. For each $i = 1, \cdots, \ell$, we have $V(K_i) \subseteq B_{t-3}(D_i, \text{Int}(D_i))$ by defnition, and thus $V(K_i) \subseteq B_{t-1}(C, G)$, so $(VK^*) \subseteq B_{t-1}(C)$. Thus, (K^*, ϕ^*) is a t-partitioning pair for $\mathcal{L}$. ■

We are now ready to finish the proof of Theorem 12.2.10. We show something slightly stronger. For any lens $\mathcal{L}$, set $k(\mathcal{L}) := \max\{1, 2|V(C)| - 11\}$. We show that, if $\mathcal{L}$ is a $k(\mathcal{L})$-lens bounded by an outer cycle of length at most 11, then $\mathcal{L}$ is $k(\mathcal{L})$-partitionable. We show this by induction on $|V(C)|$. If $|V(C)| \leq 6$, then $k(\mathcal{L}) = 1$, so $\mathcal{L}$ is a 1-lens. By Lemma 12.2.41, $\mathcal{L}$ is 0-reducible, and thus 1-partitionable, so we are done in that case.

Now suppose that $7 \leq |V(C)| \leq 11$ and suppose that, for all lenses $\mathcal{L}' = (G', C', L', \psi')$, with $|V(C')| < |V(C)|$, if $\mathcal{L}'$ is a $k(\mathcal{L}')$-lens, then $\mathcal{L}'$ is $k(\mathcal{L}')$-partitionable. Note that, since $7 \leq |V(C)|$, we have $k(\mathcal{L}) \geq 3$, and for any lens $\mathcal{L}' = (G', C', L', \psi')$, with $|V(C')| < |V(C)|$, we have $k(\mathcal{L}') \leq k(\mathcal{L}) - 2$. Thus, for any $(k(\mathcal{L}) - 2)$-lens $\mathcal{L}' = (G', C', L', \psi')$, with $|V(C')| < |V(C)|$, $\mathcal{L}'$ is $(k(\mathcal{L}) - 2)$-partitionable. By Claim 12.2.42, $\mathcal{L}$ is $k(\mathcal{L})$-partitionable, so we are done. This completes the proof of Theorem 12.2.10. $\square$

12.3 Roulette Wheels and Cycle Connectors: Preliminaries

In this section and the next section, we analyze short-separation-free planar graphs having two designated precolored facial cycles of length at most four. Our goal is to show that, under certain conditions, we can color and delete a path between the two precolored cycles to obtain a single Thomassen facial subgraph. We begin with the following definition.

Definition 12.3.1. Let $\beta := \frac{17}{15}N_{\text{mo}}^2$ and $\beta' := \beta + 4_{\text{mo}}$. A *roulette wheel* is a tuple $\mathcal{A} := (G, F_0, F_1, L, \psi)$ such that the following hold:

Ro1: G is a connected, short-separation-free graph with distinct cyclic facial subgraphs F_0, F_1, each having length at most four; *AND*

Ro2: $V(G) \neq V(F_0 \cup F_1)$ *and* $d(F_0, F_1) \leq \beta' + 1$*; AND*

Ro3: L is a list-assignment for $V(G)$, and ψ is an L-coloring of $V(F_0 \cup F_1)$, such that we have the following for all $v \in B_{\frac{\beta'+3}{2}}(F_0 \cup F_1)$:

- i) If $v \notin V(F_0 \cup F_1)$, then $|L(v)| \geq 5$*; AND*
- ii) Every facial subgraph of G containing v, except possibly F_0, F_1, is a triangle.

Ro4: If $d(F_0, F_1) \geq 2$, then, for each $i = 0, 1$ the following hold.

- i) There is no generalized chord of F_i of length at most six which separates F_{1-i} from a vertex of $G \setminus F_i$ with an L-list of size less than five*; AND*
- ii) $G[D_1(F_i)]$ is an induced cycle.

The cycles F_0, F_1 are the *boundary cycles* of $\mathcal{A}$.

The following definition makes precise the idea of deleting a path connecting the two boundary cycles in a roulette wheel to produce a Thomassen facial subgraph.

Definition 12.3.2. Let $\mathcal{A} := (G, F_0, F_1, L, \psi)$ be a roulette wheel, let $\beta := \frac{17}{15}N_{\text{mo}}^2$, and $\beta' := \beta + 4N_{\text{mo}}$. A *cycle connector* for $\mathcal{A}$ is a tuple $[K; Q; \phi; Z]$ such that K is a subgraph of $G[B_{\frac{\beta'+1}{2}}(F_0 \cup F_1))]$, Q is either a subpath of $F_0 \cup F_1$ of length at most one, or a lone vertex of $D_1(K)$, Z is a vertex set with $Z \subseteq V(K \setminus Q) \setminus V(F_0 \cup F_1)$, and the following hold.

i) $G[V(F_0 \cup F_1 \cup K)] \setminus Q$ is connected*; AND*

ii) $\phi \in \Phi(\psi, K \setminus Z)$ and Z is (L, ϕ)-inert*; AND*

iii) For all $w \in D_1(\text{dom}(\phi)) \setminus V(Q)$, $|L_\phi^Q(w)| \geq 3$. Furthermore, if Q is a lone vertex of $D_1(K)$, then Q is L_ϕ-colorable.

If K just consists of a single vertex z, then we write $[z; Q; \phi; Z]$ in place of $[\{z\}; Q; \phi; Z]$. In some cases in the analysis below, to avoid clutter, it is easier to specify the first coordinate of a cycle connector as a vertex-set rather than a graph. If $S \subseteq V(G)$ is a vertex set, then the notation $[S; Q; \phi, Z]$ is always understood to mean $[G[S]; Q; \phi, Z]$.

The goal of sections 12.3, 12.4, and 12.5 is to prove the following theorem.

Theorem 12.3.3. *Let $\mathcal{A} := (G, F_0, F_1, L, \psi)$ be a roulette wheel, let $\beta := \frac{17}{15}N_{\text{mo}}^2$ and let $\beta' := \beta + 4N_{\text{mo}}$. Then one of the following two statements holds.*

- S1: *There exists a 2-connected subgraph H of G with $F_0 \cup F_1 \subseteq H$ and $V(H) \subseteq B_{\frac{\beta'}{3}}(F_0 \cup F_1)$ such that, for every facial subgraph C of H, C is a cycle of length at most 11; OR*
- S2: *There exists a cycle connector for $\mathcal{A}$.*

In the remainder of Section 12.3, we gather the preliminary facts we need for the proof of Theorem 12.3.3. We begin with the following.

Lemma 12.3.4. *Let $\mathcal{A} := (G, F_0, F_1, L, \psi)$ be a roulette wheel. Then the following hold.*

1) *For each $i = 0, 1$ F_i is an induced cycle in G and furthermore, if $x, y \in V(F_i)$ have a common neighbor in $D_1(F_i)$, then $xy \in E(F_i)$; AND*
2) *For any $i \in \{0, 1\}$ and $w \in D_1(F_i)$, $G[N(w) \cap V(F_i)]$ is a subpath of F_i of length at most one. In particular, no vertex of G has more than two neighbors in F_i.*

Proof. 1) is an immediate consequence of our triangulation conditions, together with the fact that G is short-separation-free. Now let $i \in \{0, 1\}$ and $w \in D_1(F_i)$. Suppose that w has at least three neighbors in F_i. If $|V(F_i)| = 3$, then G contains a copy of K_4, and thus, by short-separation-freeness, we have $G = K^4$. Since $|V(F_0) \cap V(F_1)| \leq 2$, we have $V(G) = V(F_0 \cup F_1)$, contradicting the definition of $\mathcal{A}$. If $|V(F_i)| = 4$, then G contains a 2-chord of F_i of the form xwx', where x, x' are not adjacent in F_i, contradicting 1). This proves 2). $\square$

In view of Ro4 of Definition 12.3.1, we introduce the following terminology.

Definition 12.3.5. Let $\mathcal{A} := (G, F_0, F_1, L, \psi)$ be a roulette wheel with $d(F_0, F_1) \geq 2$. For each $i \in \{0, 1\}$, the *1-band* of C^i is the unique cycle of G such that $V(C^i) = D_1(F_i)$.

We now have the following

Proposition 12.3.6. *Let $\mathcal{A} := (G, F_0, F_1, L, \psi)$ be a roulette wheel and suppose that $d(F_0, F_1) \geq 2$. For each $i \in \{0, 1\}$, letting C^i be the 1-band of F_i, we have the following.*

1) *$|V(C)| \geq 5$; AND*
2) *For any $v, w \in V(C^i)$, if v, w have common neighbors in $V(F_i)$ and $D_2(F_i)$, then $vw \in E(C^i)$.*

Proof. 1) is trivial, since, if $|V(C^i)| \leq 4$, then there is a 4-cycle separating F_0 from F_1. Now let $v, w \in V(C^i)$ and suppose that x, x' have a common neighbor $x \in V(F_0)$ and a common neighbor $z \in D_2(F_0)$. Then G contains the 4-cycle $xvzw$, and since $z \in D_2(F_0)$, we have $zx \notin E(G)$, and thus $vw \in E(G)$ by our triangulation conditions. By Ro4 of Definition 12.3.1, C^i has no chords, and since v, w share a neighbor in F_0, we have $vw \in E(C^i)$, as desired. $\square$

We now introduce the following notation, which we use both in this section and the next one:

Definition 12.3.7. For a roulette wheel $\mathcal{A} := (G, F_0, F_1, L, \psi)$, we have the following notation.

1) For each $i \in \{0, 1\}$, let $A^i := \{v \in D_1(F_i) \setminus V(F_{1-i}) : |V(F_i) \cap N(v)| = 2\}$ and, for each $w \in D_1(F_i)$, let $A^i_w := A^i \cap N(w)$.
2) For each $i \in \{0, 1\}$ and $w \in D_1(F_i)$, let $R^i_w := G[N(w) \cap V(F_i)]$.
3) For any subgraphs K, H of G, we set $T(K; H)$ to be the set of $v \in V(G \setminus (F_0 \cup F_1 \cup K))$ such that v has at least three neighbors in $V((F_0 \cup F_1 \cup K) \setminus H)$.
4) For any $\phi \in \Phi(\psi, K)$, we let $T'(K; H; \phi) := \{z \in T(K; H) : |L^H_\phi(z)| < 3\}$.

In particular, note that $T(\varnothing; \varnothing)$ is the set of vertices in $V(G) \setminus V(F_0 \cup F_1)$ with at least three neighbors on $V(F_0 \cup F_1)$. If either of the graphs K, H in the notation above consists of a lone vertex z, then we write this coordinate of $T(K : H)$ or $T(K; H; \psi)$ as just z.

By Lemma 12.3.4, R^i_w is a subpath of F_i of length at most one for each $i \in \{0, 1\}$. The motivation for the notation above is as follows. Given a roulette wheel $\mathcal{A} := (G, F_0, F_1, L, \psi)$ and a candidate $[K; Q; \phi; Z]$ for a cycle connector for $\mathcal{A}$, we look for vertices of $G \setminus (V(F_0 \cup F_1 \cup K) \setminus Z)$ with at least three neighbors in $\text{dom}(\phi) \setminus V(Q)$. If there is such a vertex w and $|L^Q_\phi(w)| < 3$, then $[K; Q; \phi; Z]$ is not a cycle connector, and we extend our coloring ϕ to include z and try again. We prove two more propositions, and then we proceed with the proof of Theorem 12.3.3. We use the following facts repeatedly in this section and in the next one. We state these without proof as all of them are immediate consequences of Ro4 of Definition 12.3.1

Lemma 12.3.8. *Let $\mathcal{A} := (G, F_0, F_1, L, \psi)$ be a roulette wheel with $d(F_0, F_1) \geq 2$. For each $i \in \{0, 1\}$ and $w \in D_1(F_i)$, the following hold.*

1) *$|A^i_w| \leq 2$, and, for all $v \in A^i_w$, we have $R^i_v \cap R^i_w \neq \varnothing$; AND*
2) *If R^i_w is an edge and $v \in A^i_w$, then we have $|V(F_i)| = 4$, and there is an endpoint of the edge $F_i \setminus R^i_w$ which is adjacent to v, and furthermore, $D_1(R^i_w \cap R^i_v) \subseteq V(F_i) \cup \{w, v\}$; AND*
3) *If R^i_w is a vertex x and $|A^i_w| = 2$, then we have $N(x) \subseteq V(F_i) \cup \{w\} \cup A^i_w$.*
4) *Each connected component of $G[A^i]$ is an induced subpath of C^i of length at most $|V(F_i)| - 1$*

Applying 4) of Lemma 12.3.8, we introduce the following notation which we retain for the remainder of this section and the next.

Definition 12.3.9. Let $\mathcal{A} := (G, F_0, F_1, L, \psi)$ be a roulette wheel with $d(F_0, F_1) \geq 2$. Let C^i be the 1-band of F_i for each $i = 0, 1$.

1) For each $v \in A^i$, H^i_v is the connected component of $G[A^i]$ containing v*; AND*
2) For each $i \in \{0, 1\}$ and each connected subgraph K of C^i, $\text{Mid}^i(K)$ is a subset of of $V(K)$ where, for any $w \in V(K)$, $w \in \text{Mid}^i(K)$ if and only if there is a vertex $z \in D_2(F_i)$ such that $G[N(z) \cap V(C^i)]$ is a subpath of K and w is an internal vertex of this path.

Note that the notation Mid^i is analogous to that of Definition 12.2.15.

12.4 Roulette Wheels with Close Boundary Cycles

In this section, we prove Theorem 12.3.3 holds in the special case where the boundary cycles of the roulette wheels are close. This is trickier than the case where they boundary cycles are not close, which we deal with in Section 12.5, because in that case we apply 1) of Theorem 10.0.7. The lone result of this section is the following.

Theorem 12.4.1. *Let $\mathcal{A} := (G, F_0, F_1, L, \psi)$ be a roulette wheel and suppose that $d(F_0, F_1) \leq 2$. Then one of the following two statements holds.*

- S1: *There exists a 2-connected subgraph H of G with $F_0 \cup F_1 \subseteq H$ and $V(H) \subseteq B_2(F_0 \cup F_1)$ such that, for every facial subgraph C of H, C is a cycle of length at most 11; OR*
- S2: *There exists a cycle connector for $\mathcal{A}$.*

Given a roulette wheel $\mathcal{A}$, the result above states that $\mathcal{A}$ either admits a cycle connector, or we can partition a small ball around $V(F_0 \cup F_1)$ into regions bounded by cycles of of length at most 11. The usefulness of the latter possibility lies in the fact that we can apply the work of Section 12.2 to color and delete some vertices in each of these regions to obtain a Thomassen facial subgraph near $V(F_0 \cup F_1)$. In the last section of Chapter 13, when we prove complete the proof of Theorem 1.1.3, we use this work to produce a smaller counterexample from a critical chart.

Note that, if S1 holds, there is no guarantee that ψ extends to an L-coloring of G. Indeed, it is easy to construct an example of a roulette wheel where $d(F_0, F_1) \leq 2$ and ψ does not extend to an L-coloring of G, but when we apply Theorem 12.3.3 to a critical chart in the last section of Chapter 13, we begin with an L-coloring of $V(F_0 \cup F_1)$ which already extends to a small ball around in $F_0 \cup F_1$ in G.

We break the proof of Theorem 12.4.1 into several propositions, which we then combine at the end of this section to prove Theorem 12.4.1. We now introduce the following definition:

Definition 12.4.2. A roulette wheel is *defective* if its boundary cycles are of distance at most two apart but it does not satisfy either S1 or S2 of Theorem 12.3.3.

The trickiest case is the case where the boundary cycles are of distance precisely two apart. We now gather some sufficient conditions for constructing a subgraph of G satisfying S1.

Proposition 12.4.3. *Let $\mathcal{A} := (G, F_0, F_1, L, \psi)$ be a roulette wheel with $d(F_0, F_1) \leq 2$. If there exist two disjoint (F_0, F_1)-paths P_1, P_2 satisfying either of the following conditions, then there exists a subgraph H of G satisfying S1.*

1) $|V(P_1)| + |V(P_2)| \leq 7$*; OR*
2) $|V(P_1)| + |V(P_2)| \leq 8$ *and, for some $i \in \{0, 1\}$, either $|V(F_i)| = 3$ or P_1, P_2 have non-adjacent endpoints on F_i.*

Proof. Let P_1, P_2 be a pair of disjoint (F_0, F_1)-paths and let H be the subgraph of G induced by $V(F_0 \cup F_1) \cup V(P_1 \cup P_2)$. Since P_1, P_2 are vertex-disjoint, H is 2-connected . Let C be a facial subgraph of H. Since H is 2-connected, C is a cycle. Since $d(F_0, F_1) \geq 1$, we have $|V(P_j)| \geq 2$ for each $j = 1, 2$. Furthermore, $|V(H)| = |V(F_0)| + |V(F_1)| + (|V(P_1)| - 2) + (|V(P_2)| - 2)$. Thus, if $|V(P_1)| + |V(P_2)| \leq 7$ then $|V(H)| \leq 11$ and thus $|V(C)| \leq 11$, so we are done in that case. Now let $|V(P_1)| + |V(P_2)| = 8$. If $|V(F_i)| = 3$ for some $i \in \{0, 1\}$ then $|V(H)| \leq 7 + (8 - 4) = 11$, so again, we are done in that case. Now suppose that $|V(F_0)| = |V(F_1)| = 4$ and that there exists an $i \in \{0, 1\}$ such that P_1, P_2 have non-adjacent-endpoints in F_i. Then $|V(H)| = 12$, and H contains a generalized chord of F_i whose endpoints are non-adjacent in F_i, so there does not exist a facial subgraph C of H such that $V(C) = V(H)$. Thus, for all facial subgraphs C of H, we have $|V(C)| \leq 11$, so again, we are done. $\square$

With the above in hand, we deal with the case where F_0, F_1 are of distance at most one apart:

Proposition 12.4.4. *Let $\mathcal{A} := (G, F_0, F_1, L, \psi)$ be a roulette wheel and suppose that $d(F_0, F_1) \leq 1$. Then $\mathcal{A}$ is not defective.*

Proof. Suppose toward a contradiction that $\mathcal{A}$ is defective. We first deal with the case where F_0, F_1 share a vertex. Suppose there is a vertex $x \in V(F_0) \cap V(F_1)$. Firstly, $|V(F_0) \cap V(F_1)| \leq 2$, or else at least one of F_0, F_1 has a chord, contradicting 1) of Lemma 12.3.4. Now suppose $|V(F_0) \cap V(F_1)| = 2$. Then $F_0 \cap F_1$ is a path of length one, or else, for some $i \in \{0, 1\}$, we get that $|V(F_i)| = 4$ and $V(F_0) \cap V(F_1)$ consists of two vertices x, y of F_i which are not adjacent in F_i. In that case, $xy \notin E(F_{1-i}))$, since F_i is induced, so $|V(F_{1-i})| = 4$ and x, y have a common neighbor in $V(F_{1-i}) \setminus V(F_i)$, contradicting 1) of Lemma 12.3.4. If $V(F_0 \setminus F_1)$ and $V(F_1 \setminus F_0)$ have a common neighbor, then there is an (F_0, F_1)-path of length at most two, disjoint to $\{x\}$, so $\mathcal{A}$ satisfies S1 by Proposition 12.4.3, contradicting our assumption that $\mathcal{A}$ is defective.

Since $V(F_0 \setminus F_1)$ and $V(F_1 \setminus F_0)$ have no common neighbor, we set Q to be an edge of $F_0 \setminus F_1$. Then the tuple $[\varnothing; Q; \psi; \varnothing]$ is a cycle connector for $\mathcal{A}$, contradicting our assumption, so we have $d(F_0, F_1) = 1$. Thus, let $x \in V(F_0)$ and $y \in V(F_1)$, where $xy \in E(G)$. Then $G[V(F_0 \cup F_1)]$ is connected. By assumption, the tuple $[\varnothing; \varnothing; \varnothing, \psi]$ is not a cycle connector for $\mathcal{A}$, so there is a vertex $z \in V(G) \setminus V(F_0 \cup F_1)$ with at least three neighbors on $V(F_0 \cup F_1)$. Thus, $N(z)$ has nonempty intersection with each of $V(F_0), V(F_1)$, since $N(z)$ intersects each of F_0, F_1 on at most an edge. We then have either $N(z) \cap V(F_0 = \{x\}$ or $N(z) \cap V(F_1) = \{y\}$, or else the conditions of Proposition 12.4.3 are satisfied, contradicting our assumption that $\mathcal{A}$ is defective.

Thus, suppose without loss of generality that $N(z) \cap V(F_0) = \{x\}$. In that case, R^1_z is an edge of F_1, and thus, for each $w \in D_1(F_0) \cap D_1(F_1)$, we have $N(w) \cap V(F_0) = \{x_1\}$, or else, if there is a $w \in D_1(F_0) \cap D_1(F_1)$ with $N(w) \cap V(F_0) \neq \{x\}$, then $w \neq z$ and the conditions of Proposition 12.4.3 are satisfied, contradicting our assumption that $\mathcal{A}$ is defective.

We claim now that, for each $z \in T(\varnothing; \varnothing)$, the edge R^1_z has y as an endpoint. Suppose not. Then there is a $z \in T(\varnothing; \varnothing)$ such that y is not an endpoint of R^1_z. Let $R^1_z = y'y''$. At least one endpoint of R^1_z is adjacent to y in F_1, so suppose for the sake of definiteness that $yy' \in E(F_1)$. Thus, G contains the 4-cycle $xzy'y$. Since $y \notin N(z)$ we have $xy' \in E(G)$ by our triangulation conditions. Now let $Q := F_1 \setminus \{y, y'\}$. By assumption, the tuple $[\varnothing; Q; \psi; \varnothing]$ is not a cycle connector for $\mathcal{A}$. Since $G[V(F_0 \cup F_1)] \setminus Q$ is connected, there exists a vertex $w \in T(\varnothing; Q)$.

Note that $w \neq z$ since $N(z) \cap V((F_0 \cup F_1) \setminus Q) = \{x, y'\}$. Since $w \in D_1(F_0) \cap D_1(F_1)$ we have $N(w) \cap V(F_0) = \{x\}$ as shown above. Since $|N(w) \cap V((F_0 \cup F_1) \setminus Q)| \geq 3$, we thus have $y, y' \in N(w)$, and thus G contains a $K_{2,3}$ with bipartition $\{x, y'\}, \{z, w, y\}$, contradicting short-separation-freeness. Thus our assumption that there is a $z \in T(\varnothing; \varnothing)$ such that $y \notin V(R^1_z)$ is false.

Now, if $T(\varnothing; \varnothing)| = 1$, then we let z be the lone vertex of $T(\varnothing; \varnothing)$ and let $\phi \in \Phi(\psi, z)$. Then the tuple $[z; Q; \phi; \varnothing]$ is a cycle connector for $\mathcal{A}$, contradicting our assumption that $\mathcal{A}$ is defective. Thus, we have $|T(\varnothing; \varnothing)| \geq 2$. Let y', y'' be the two neighbors of y in F_1. Since $y \in V(R^1_z)$ for each $z \in T(\varnothing; \varnothing)$, we thus have $|T(\varnothing; \varnothing)| = 2$, and, as shown above, there exist $z_1, z_2 \in V(G)$ such that $\{z_1, z_2\} = T(\varnothing; \varnothing)$, where $N(z_1) \cap V(F_0 \cup F_1) = \{x, y, y'\}$ and $N(z_2) \cap V(F_0 \cup F_1) = \{x, y, y''\}$.

Note that $z_1z_2 \notin E(G)$ or else the four vertices $\{x, z_1, z_2, y\}$ induce a K_4 in G, contradicting short-separation-freeness. Now let $Q^* := F_1 \setminus \{y, y''\}$ and let $\psi^* \in \Phi(\psi, z_2)$. Consider the tuple $[z_2; Q^*; \psi^*; \varnothing)]$. Since $\mathcal{A}$ is defective, $[z_2; Q^*; \psi^*; \varnothing)]$ is not a cycle connector for $\mathcal{A}$, so there exists a $w \in T(z_2; Q^*)$. Note that $w \neq z_1$, since z_1 has only two neighbors among $V((F_0 \cup F_1) \setminus Q^*) \cup \{z_1\}$.

Since $w \neq z_1$, we have $w \notin T(\varnothing; Q^*)$, and thus $wz_2 \in E(G)$ and $|N(w) \cap V(F_0 \cup F_1) \setminus V(Q)| = 2$. If w has a neighbor in $F_0 \setminus \{x\}$ then G contains an (F_0, F_1)-path of length four which is disjoint to xy, and thus the conditions of Proposition 12.4.3 are satisfied, contradicting our assumption that $\mathcal{A}$ is defective.

Thus, we have $N(w) \cap V(F_0) \subseteq \{x\}$. Yet we also have $N(w) \cap V(F_1 \setminus Q^*) \subseteq \{y''\}$, since the 4-chord $y'z_1xz_2y''$ of F_1 separates y_1 from $G \setminus (V(F_1) \cup \{z_1, z_2, x\})$. Thus, we have $\{x, z_2, y''\} \subseteq N(w)$, so G contains a $K_{2,3}$ with bipartition $\{y, z_2, w\}$, $\{x, y''\}$, contradicting short-separation-freeness. This completes the proof of Proposition 12.4.4. $\square$

Thus, for the remainder of Section 12.4, we deal with roulette wheels of the form $\mathcal{A} := (G, F_0, F_1, L, \psi)$, where $d(F_0, F_1) = 2$.

Given a roulette wheel $\mathcal{A} := (G, F_0, F_1, L, \psi)$ and an $i \in \{0, 1\}$, a vertex $x \in V(F_i)$ is called an *anchor vertex* if x has a neighbor in $D_1(F_0) \cap D_1(F_1)$. Since $D_1(F_0) \cap D_1(F_1) \neq \varnothing$, each of F_0, F_1 contains at least one anchor vertex. We now prove the following.

Proposition 12.4.5. *Let $\mathcal{A} := (G, F_0, F_1, L, \psi)$ be a roulette wheel with $d(F_0, F_1) = 2$. Suppose further that $|D_1(F_0) \cap D_1(F_1)| \geq 2$ and that there exists an $i \in \{0, 1\}$ such that F_i has at least two anchor vertices. Then $\mathcal{A}$ is not defective.*

Proof. Suppose toward a contradiction that $\mathcal{A}$ is defective. Suppose for the sake of definiteness that there are two anchor vertices in F_1. Then there exists a pair of distinct vertices $w, w' \in D_1(F_0) \cap D_1(F_1)$ and a pair of distinct vertices $y, y' \in V(F_1)$ with $yw, y'w' \in E(G)$. If G contains two disjoint (F_0, F_1)-paths of length two, then, by Proposition 12.4.3, $\mathcal{A}$ is not defective, contradicting our assumption. Thus, no such pair of paths exists, so F_0 has precisely one anchor vertex x. Let $\psi' \in \Phi(\psi, w)$. We now note the following:

Claim 12.4.6.

1) $|V(F_1)| = 4$ and, for any subpath Q of of $F_1 \setminus \{u\}$ of length at most one, and any $z \in T(w; Q)$, we have $N(z) \cap V(F_0) \subseteq \{x\}$; AND

2) There exists a subpath Q of $F_1 \setminus \{y\}$ of length one such that $|T(w; Q)| = 1$.

<u>Proof:</u> Let Q be a subpath of $F_1 \setminus \{y\}$ and let $z \in T(w; Q)$. Suppose toward a contradiction that there is an $x' \in V(F_0 - x)$ with $x' \in N(z)$. Then we have $N(z) \cap V(F_1) \subseteq \{y\}$, or else G contains an (F_0, F_1)-path of length two disjoint to xwy, so S1 is satisfied by Proposition 12.4.3, contradicting our assumption that $\mathcal{A}$ is defective.

Since $N(z) \cap V(F_1) \subseteq \{y\}$, we have $z \neq w'$. As z has at least three neighbors in $(V(F_0 \cup F_1) \cup \{w\}) \setminus Q$, we have $y \in (V(F_0 \cup F_1) \cup \{w\}) \setminus Q$ and z is adjacent to each of $\{x, w, u\}$. But then, the path $x'zwy$ is disjoint to $xw'y'$, contradicting our assumption that $\mathcal{A}$ is defective.

Now suppose toward a contradiction that $|V(F_1)| = 3$. Then $Q = F_1 \setminus \{y\}$. Since $\mathcal{A}$ is defective, the tuple $[w; Q; \psi'; \varnothing]$ is not a cycle connector for $\mathcal{A}$, so there exists a $z \in T(w; Q)$. Since $N(z) \cap V(F_0) \subseteq \{x\}$, we have $(V(F_0 \cup F_1) \cup \{w\}) \cap (N(z) \setminus V(Q)) = \{x, w, y\}$.

Let $\psi'' \in \Phi(\psi, \{w, z\})$. Again, since $\mathcal{A}$ is defective, the tuple $[\{w, z\}; Q; \psi''; \varnothing]$ is not a cycle connector for $\mathcal{A}$, so there exists a $z' \in T(\{w, z\}; Q)$. Then z' is not adjacent to both of $\{x, y\}$, or else G contains a $K_{2,3}$ with bipartition $\{x, y\}$, $\{z, z', w\}$, contradicting short-separation-freeness. Thus, the set $(N(z') \setminus V(Q)) \cap (V(F_0 \cup F_1) \cup \{w, z\})$ is either $\{x, w, z\}$ or $\{y, w, z\}$. In either case, G contains a $K_{2,3}$ with bipartition $\{w, z\}$, $\{x, y, z'\}$, contradicting short-separation-freeness. Thus, we have $|V(F_1)| = 4$, as desired. This completes the proof of 1).

Now we prove 2) of Claim 12.4.6. Since $|V(F_1)| = 4$, let Q_1, Q_2 be the two paths of length one in $F_1 \setminus \{y\}$. For each $j = 1, 2$, $G[V(F_0 \cup F_1) \cup \{w\})] \setminus Q_j$ is connected, since it contains the path xwy. Since $\mathcal{A}$ is defective, it follows that, for each $j = 1, 2$, the tuple $[w; Q_j; \psi'; \varnothing]$ is not a cycle connector for $\mathcal{A}$. Thus, we have $T(w; Q_j) \neq \varnothing$ for each $j = 1, 2$. To finish, it suffices to show that $|T(w; Q_j)| = 1$ for some $j \in \{1, 2\}$. Suppose toward a contradiction that $T(w; Q_j)| > 1$ for each $j = 1, 2$. Let $F_1 := y_1y_2y_3y_4$, where $y = y_1$.

Subclaim 12.4.7.

1) There does not exist a $z \in V(G)$ such that $\{x, w, y_1, y_4\} \subseteq N(z)$. Likewise, there does not exist a $z \in V(G)$ such that $\{x, w, y_1, y_2\} \subseteq N(z)$; AND

2) There does not exist a pair of vertices $z, z' \in D_1(F_0 \cup F_1)$ such that $\{w, y_1, y_4\} \subseteq N(z)$ and $\{w, y_1, y_2\} \subseteq N(z')$; AND

3) There does not exist a $z \in D_1(F_0 \cup F_1)$ with $\{x, y_1, y_2\} \subseteq N(z)$. Likewise, there does not exist a $z \in D_1(F_0 \cup F_1)$ with $\{x, y_1, y_4\} \subseteq N(z)$.

Proof: Suppose towards a contradiction that 1) does not hold. Then there exists a $z \in V(G)$ adjacent to each of x, w, y_1 and one of y_2, y_4. Thus, there exists a $j \in \{1, 2\}$ such that $z \in T(w; Q_j)$ and, since $|T(w; Q_j)| > 1$, there is a $z' \in T(w; Q_j) \setminus \{z\}$. But then z, z' have at least three common neighbors, so G contains a copy of $K_{2,3}$, contradicting short-separation-freeness. This proves 1).

Now we prove 2). Suppose toward a contradiction that such a pair of vertices z, z' exists. Then $z \in T(w; Q_1)$ and $z' \in T(w; Q_2)$. Furthermore, we have $z \notin T(w; Q_2)$, and $z' \notin T(w; Q_1)$ since, by 1), no vertex of $D_1(F_1)$ is adjacent to each of y_2, y_4. By 1) and 3) of Lemma 12.3.8, we have $\{z, z'\} = A^1_w$, and $N(y_1) \subseteq V(F_1) \cup \{z, z', w\}$. Since $z \notin T(w; Q_2)$, and $|T(w; Q_2)| > 1$, there is a $z'' \in T(w; Q_2) \setminus \{z, z'\}$. But then $z'' \notin N(y_1)$, and thus $\{w, x, y_2\} \subseteq N(z'')$. Thus, G contains a $K_{2,3}$ with bipartition $\{z'', z', y_1\}$, $\{w, y_2\}$, contradicting short-separation-freeness. This proves 2).

To prove 3), suppose there exists a $z \in D_1(F_0 \cup F_1)$ with $\{x, y_1\} \subseteq N(z)$ and a vertex $y \in \{y_2, y_4\} \cap N(z)$. Then G contains the 4-cycle $xzyw$. Thus, by our triangulation conditions, G either contains the edge xy_1 or the edge zw. Since $d(F_0, F_1) = 2$, we have $zw \in E(G)$, so $N(z)$ contains $\{x, w, y_1, y\}$, contradicting 1), so we have $x_1y_1 \in E(G)$, contradicting the fact that $d(F_0, F_1) = 2$. This completes the proof of Subclaim 12.4.7. ■

Now we combine the facts from this subclaim. Since $|T(w; Q_1)| > 1$, it follows from 3) of Subclaim 12.4.7 that $T(w, Q_1)$ consists of two vertices v, v' with $(N(v) \setminus V(Q_1)) \cap (V(F_0 \cup F_1) \cup \{w\}) = \{x, y_1, w\}$ and $N(v') \setminus V(Q_1)) \cap (V(F_0 \cup F_1) \cup \{w\}) = \{x, y_1, w\}$. Thus, G contains the 4-cycle $xv'y_4y_1$. By 1) of Subclaim 12.4.7, v' is adjacent to at most three vertices in $\{x, w, y_1, y_4\}$ and thus, by our triangulation conditions, G contains the edge wy_4. Note that G contains the 5-cycle xvy_1y_4v', and since $wy_4 \in E(G)$, w is adjacent to each vertex in the cycle xvy_1y_4v'. Thus, since G is short-separation-free, we have $N(w) = \{x, v, y_1, y_4, v'\}$.

We claim now that $v' \notin T(w; Q_2)$. Suppose toward a contradiction that $v' \in T(w, Q_2)$. Since no vertex of $D_1(F_1)$ is adjacent to each of y_2, y_4, we then we have $\{x, w, y_1\} \subseteq N(v')$, and thus G contains a $K_{2,3}$ with bipartition

$\{v, v'\}$, $\{x_1, w, y_1\}$, contradicting short-separation-freeness. Thus, since $|T(w, Q_2) \setminus \{v\}| \geq 1$ by assumption, let $v'' \in T(w; Q_2) \setminus \{v\}$. Then $v'' \notin N(w)$ and $(N(v'') \setminus V(Q_2)) \cap (V(F_0 \cup F_1) \cup \{w\}) = \{x, y_1, y_2\}$. But then G contains a $K_{2,3}$ with bipartition $\{w, v, v''\}$, $\{x, y_1\}$, contradicting short-separation-freeness. This completes the proof of Claim 12.4.6. ■

Applying 2) of Claim 12.4.6, let Q be a subpath of $F_1) \setminus \{y_1\}$ of length one such that $|T(w; Q)| = 1$. Let $T(w; Q) = \{q_0\}$. By 1) of Claim 12.4.6, we have $N(q_0) \cap V(F_0) \subseteq \{x_1\}$, so q_0 has at least three neighbors among $\{x, w\} \cup V(F_1 \setminus Q)$. Suppose without loss of generality that $Q = y_3y_4$. Let $\psi' \in \Phi(\psi, \{w, q_0\})$.

Since $\mathcal{A}$ is defective, the tuple $[\{w, q_0\}; Q; \psi'; \varnothing]$ is not a cycle connector for $\mathcal{A}$, so $T(\{w, q_0\}; Q) \neq \varnothing$. Let $q_1 \in T(\{w, q_0\}; Q)$ and let $\psi^* \in \Phi(\psi, \{w, q_0, q_1\})$. As above, the tuple $[\{w, q_0, q_1\}; Q; \psi^*; \varnothing]$ is not a cycle connector for $\mathcal{A}$. Since $V(F_0 \cup F_1) \cup \{w, q_0\}) \setminus V(Q)$ has precisely two vertices in $D_1(F_0 \cup F_1)$, namely $\{w, q_0\}$, and q_1 has at least three neighbors in $(V(F_0 \cup F_1) \setminus V(Q)) \cup \{w, q_0\}$. Thus, since the tuple $[\{w, q_0, q_1\}; Q; \psi^*; \varnothing]$ is not a cycle connector for $\mathcal{A}$, there exists a $q_2 \in T(\{w, q_0, q_1\}; Q)$.

Note that $q_0q_1 \in E(G)$, or else $q_1 \in T(w; Q)$, contradicting the fact that $T(w; Q) = \{q_0\}$. Furthermore, q_2 is adjacent to at least one of $\{q_0, q_1\}$, or else, again, we have $q_2 \in T(w; Q)$, contradicting the fact that $T(w; Q) = \{q_0\}$. Thus, G contains either the path $q_0q_1q_2$, or the path $q_2q_0q_1$.

Claim 12.4.8. *For each $j = 0, 1, 2$, we have $N(q_j) \cap V(F_0) \subseteq \{x\}$.*

Proof: The case where $j = 0$ is done above. Suppose toward a contradiction that there is an $x' \in V(F_0 \setminus \{x\}) \cap N(q_1)$. Since $q_0q_1 \in E(G)$ and $w \neq q_0, q_1$, neither q_0, q_1 have a neighbor in $\{y_2, y_3, y_4\}$, or else G contains an (F_0, F_1)-path of length at most three which is disjoint to xwy_1. Thus, $w' \notin \{q_0, q_1\}$, and $\{x, w, y_1\} \subseteq N(q_0)$. But then the two disjoint (F_0, F_1)-paths $xw'u'$ and $x'q_1q_0y_1$ satisfy Proposition 12.4.3, contradicting our assumption that $\mathcal{A}$ is defective. We conclude that $N(q_1) \cap V(F_0) \subseteq \{x\}$, as desired.

Now suppose toward a contradiction that there is an $x' \in V(F_0 \setminus \{x\}) \cap N(q_2)$. Since q_2 is adjacent to at least one of $\{q_0, q_1\}$, let $q' \in \{q_0, q_1\}$ with $q'q_2 \in E(G)$. Then neither q' nor q_2 has a neighbor in $\{y_2, y_3, y_4\}$ or else G contains an (F_0, F_1)-path of length at most three which is disjoint to xwy_1. Applying Proposition 12.4.3, this contradicts our assumption that $\mathcal{A}$ is defective. Thus, we have $w' \notin \{q', q_2\}$, and $q' \in \{q_0, q_1\}$.

If $q_0q_2 \in E(G)$, then G contains the two disjoint (F_0, F_1)-paths $xw'u'$ and $x'q_2q_0y_1$, again contradicting our assumption that $\mathcal{A}$ is defective. Thus $q' = q_1$, and furthermore, $y_1 \notin N(q_1)$, or else G contains an (F_0, F_1)-path of length three disjoint to either xwu or $xw'u'$. Applying Proposition 12.4.3, this contradicts our assumption that $\mathcal{A}$ is defective. Thus $V(F_1) \cap N(q_1) = \varnothing$. Since $q_1 \in T(\{w, q_0\}; Q)$ and q_1 has at most two neighbors on F_0, q_1 is adjacent to w and an edge of F_0. But then G contains an (F_0, F_1)-path of length three disjoint to $x_1w'u'$. Again applying Proposition 12.4.3, this contradicts our assumption that $\mathcal{A}$ is defective. ■

Set $P := xwy_1y_2$. Then q_0 is adjacent to at least one endpoint of P, since $|N(q_0) \cap V(P)| \geq 3$. To make the following claim easier to read, we label P as $P := p_1p_2p_3p_4$, where $p_1 \in N(q_0)$. Now we have the following:

Claim 12.4.9. *Let H be the subgraph of G induced by the vertices $V(P) \cup \{q_0, q_1, q_2\}$. Then the following hold.*

1) $N(q_0) \cap V(P) = \{p_1, p_2, p_3\}$; AND

2) $\{q_0, q_1, q_2\}$ induce a triangle in H; AND

3) $N(q_2) \cap V(P) = \{p_1\}$; AND

4) $N(q_1) \cap V(P) = \{p_3, p_4\}$.

Proof: As shown above, H either contains a path R that is either $q_0q_1q_2$ or the path $q_2q_0q_1$.

Subclaim 12.4.10. *For any vertices $q, q' \in V(R)$ and any distinct vertices $p, p' \in N(q) \cap N(q')$, then $pp' \notin E(H)$ and p, p' are the endpoints of P.*

Proof: Suppose that $pp' \in E(H)$. In that case, $qq' \notin E(H)$, or else H contains a copy of K_4, contradicting short-separation-freeness, so q, q' are the end vertices of R. Let q_m be the middle vertex of R. Thus, H contains the non-induced 4-cycles $qq_mq'p$ and $qq_mq'p'$. Since $qq' \notin E(G)$, we get that q_m is adjacent to p, p', so p, p' are both adjacent to each vertex of R. Thus, H contains a copy of $K_{2,3}$, contradicting short-separation-freeness. Now suppose toward a contradiction that at least one of p, p' is not an endpoint of P. Then, since p, p' are not adjacent, they have a common neighbor $p'' \in V(P)$. In that case, H contains the non-induced 4-cycles $pp''p'q$ and $pp''p'q'$. Since $pp' \notin E(H)$, p'' is adjacent to each of q, q', so H contains a $K_{2,3}$ with bipartition $\{q, q'\}$, $\{p, p', p''\}$, contradicting short-separation-freeness. ■

With the facts above in hand, we prove the following:

Subclaim 12.4.11. *For any two vertices $q, q' \in V(R)$, we have $|N(q) \cap N(q') \cap V(P)| \leq 1$.*

Proof: Suppose toward a contradiction that this does not hold. Then there exist $q, q' \in V(R)$ with $|N(q) \cap N(q') \cap V(P)| \geq 2$. By Subclaim 12.4.10, we have $p_1, p_4 \in N(q) \cap N(q')$. Since $p_1p_4 \notin E(H)$ and H contains the non-induced 4-cycle p_1qp_4q', we have $qq' \in E(H)$.

We claim now that $q_0 \in \{q, q'\}$. Suppose that $q_0 \notin \{q, q'\}$. Then at least one of q, q' is adjacent to q_0, so, without loss of generality, let $qq_0 \in E(G)$. By Subclaim 12.4.10, each of q, q' is adjacent to p_1, p_4, so $\{p_1, p_4\} not \subseteq N(q_0)$, or else H contains a copy of $K_{2,3}$, contradicting short-separation-freeness. Since $|N(q_0) \cap V(P)| \geq 3$, we then have $N(q_0) \cap V(P) = \{p_1, p_2, p_3\}$. Now, H contains the non-induced 4-cycle $q_0p_3p_4q$, and thus, as $q_0p_4 \notin E(H)$, we have $p_3q \in E(H)$. But then H contains a $K_{2,3}$ with bipartition $\{p_1, q_0, p_3\}$, $\{p_2, q\}$, contradicting short-separation-freeness. Thus, $q_0 \in \{q, q'\}$, say $q_0 = q$, and let q'' be the lone vertex of $R \setminus \{q_0, q'\}$.

Since $|N(q_0) \cap V(P)| \geq 3$ and $p_1, p_4 \in N(q_0)$, suppose without loss of generality that $p_2 \in N(q_0)$. Since the 4-cycle $q_0p_2p_3p_4$ is not induced, we have either $p_2p_4 \in E(H)$ or $q_0p_3 \in E(H)$.

If $p_2p_4 \in E(H)$, then $q_0q' \notin E(G)$, or else H contains a $K_{2,3}$ with bipartition $\{q', p_2\}, \{q_0, p_1, p_4\}$, contradicting short-separation-freeness. Thus, q_0 is not the midpoint of R, so H contains the path $q_0q_1q_2$, and $q' = q_2$. But then G contains a $K_{2,3}$ with bipartition $\{q_0, q_2\}$, $\{p_1, p_4, q_1\}$, contradicting short-separation-freeness. It follows that $q_0p_3 \in E(H)$, so q_0 is adjacent to each vertex of P. Now let q'' be the lone vertex of $R \setminus \{q_0, q'\}$.

We claim now that q'' has precisely one neighbor in P. Suppose that $|N(q'') \cap V(P)| \geq 2$. In that case, since q_0 is adjacent to every vertex of P, we have $|N(q_0) \cap N(q'') \cap V(P)| \geq 2$, so, by Subclaim 12.4.10, we have $N(q'') \cap V(P) = \{p_1, p_4\}$. Yet by our assumption on q, q', we have $p_1, p_4 \in N(q')$ as well, so H contains a $K_{2,3}$ with bipartition $\{p_1, p_4\}, \{q_0, q_1, q_2\}$, contradicting short-separation-freeness. Thus, q'' has precisely one neighbor on P, so we have $q'' = q_2$, $q' = q_1$, and $\{q_0, q_1, q_2\}$ induces a triangle in H. But then H contains a $K_{2,3}$ with bipartition $\{q_0, q_1\}$, $\{q_2, p_1, p_4\}$, contradicting short-separation-freeness. We conclude that our assumption that $|N(q) \cap N(q') \cap V(P)| \geq 2$ is false. This completes the proof of Subclaim 12.4.11. ■

With the above in hand, we prove the following:

Subclaim 12.4.12. *$N(q_0)$ contains a subpath of P of length two in H.*

Proof: Suppose not. In that case, since $|N(q_0) \cap V(P)| \geq 3$ and $p_1 \in N(q_0)$, so, without loss of generality, we suppose that $N(q_0) \cap V(P) = \{p_1, p_2, p_4\}$ without loss of generality. Since H contains the non-induced 4-cycle $p_2p_3p_4q_0$ and $p_3q_0 \notin E(H)$, we have $p_2p_4 \in E(H)$. Furthermore, since $|N(q_1) \cap N(q_0) \cap V(P)| \leq 1$ by Subclaim 12.4.11, we have $p_3 \in N(q_1)$.

Thus, we have $q_0q_1 \notin E(H)$, or else H contains a $K_{2,3}$ with bipartition $\{q_0, p_3\}$, $\{p_2, p_4, q_1\}$, so H contains the path $q_2q_0q_1$. In that case, q_1 is adjacent to at most one of $\{p_1, p_2, p_4\}$, or else q_0, q_1 are each adjacent to q_2 and two vertices among $\{p_1, p_2, p_4\}$, contradicting short-separation-freeness.

Since $|N(q_1) \cap V(P)| \geq 2$, we have $q_1p_3 \in E(H)$, so H contains the 4-cycle $p_3p_4q_0q_1$. Since $q_0p_3 \notin E(H)$, we have $q_1p_4 \in E(H)$, and thus H contains a $K_{2,3}$ with bipartition $\{p_2, q_1\}$, $\{p_3, p_4, q_0\}$, contradicting short-separation-freeness. We conclude that our assumption that $N(q_0)$ does not contain a subpath of P of length two is false. ■

Since $p_1 \in N(q_0)$, it follows from Subclaim 12.4.12 that $N(q_0) \cap V(P)$ contains $\{p_1, p_2, p_3\}$. By Subclaim 12.4.11, we have $|N(q_1) \cap N(q_0) \cap V(P)| \leq 1$. Since $|N(q_1) \cap V(P)| \geq 2$, we have $N(q_0) \cap V(P) = \{p_1, p_2, p_3\}$ and $N(q_1) \cap V(P) = \{p_3, p_4\}$.

Subclaim 12.4.13. *q_2 has precisely one neighbor in P.*

Proof: Suppose that $|N(q_2) \cap V(P)| \geq 2$. By Subclaim 12.4.11, we have $|N(q_2) \cap N(q_0) \cap V(P)| \leq 1$ and $|N(q_2) \cap N(q_1) \cap V(P)| \leq 1|$, so we have either $N(q_2) \cap V(P) = \{p_1, p_4\}$ or $N(q_2) \cap V(P) = \{p_2, p_4\}$. If $N(q_2) \cap V(P) = \{p_2, p_4\}$, then H contains the non-induced 4-cycle $q_2p_2p_3p_4$. Since $p_3 \notin N(q_2)$, we then have $p_2p_4 \in E(H)$, and thus H contains a $K_{2,3}$ with bipartition $\{p_2, q_1\}$, $\{p_3, p_4, q_0\}$, contradicting short-separation-freeness.

Thus, we have $N(q_2) \cap V(P) = \{p_1, p_4\}$. Furthermore, we have $q_0q_2 \notin E(H)$, or else H contains a $K_{2,3}$ with bipartition $\{q_0, p_4\}$, $\{p_3, q_1, q_2\}$, contradicting short-separation-freeness. It follows that H contains the path $q_0q_1q_2$. But then H contains the non-induced 4-cycle $q_0q_1q_2p_1$, and $q_0q_2 \notin E(H)$, we have $p_1q_1 \in E(H)$, contradicting the fact that $N(q_1) \cap V(P) = \{p_3, p_4\}$. We conclude that q_2 does indeed have precisely one neighbor on P. ■

Since q_2 has precisely one neighbor on P, we get that $\{q_0, q_1, q_2\}$ induces a triangle in H, since $|N(q_2) \cap V(H)| \geq 3$ and q_2 is an endpoint of R. To finish the proof of Claim 12.4.9, it suffices to show that $N(q_2) \cap V(P) = \{p_1\}$. We rule out the possibility that any of the other three vertices of P lie in $N(q_2)$:

- If $q_2p_2 \in E(H)$, then H contains a $K_{2,3}$ with bipartition $\{q_1, p_2\}$, $\{q_0, q_2, p_3\}$.
- If $q_2p_3 \in E(H)$,then H contains a $K_{2,3}$ with bipartition $\{q_0, q_2, p_4\}$, $\{q_1, p_3\}$.
- If $q_2p_4 \in E(H)$, then H contains a $K_{2,3}$ with bipartition $\{q_0, p_4\}$, $\{p_3, q_1, q_2\}$.

In any of the cases above, we contradict the fact that H is short-separation-free. This completes the proof of Claim 12.4.9. ■

Now we return to the proof of Proposition 12.4.5. By Claim 12.4.9, G contains the 5-cycle $p_1p_2p_3q_1q_2$ and q_0 is adjacent to each vertex of $\{p_1, p_2, p_3, q_1, q_2\}$. Thus, since G is short-separation-free, we get that $N(q_0) = \{p_1, p_2, p_3, q_1, q_2\}$. By Corollary 1.3.6, there is a $\phi \in \Phi(\psi, \{q_1, w\})$ such that $\{q_0\}$ is L_ϕ-inert. Since $\mathcal{A}$ is defective, the tuple $[\{q_1, w\}; Q; \phi; q_0]$ is not a cycle connector for $\mathcal{A}$, so there is a $v \in V(G \setminus \{q_0, q_1, w\})$ with at least three neighbors in $(V(F_0 \cup F_1) \cup \{q_1, w\}) \setminus V(Q)$. Since $T(w, Q) = \{q_0\}$, v has at most two neighbors in $\{x_1, w, y_1, y_2\}$. Thus, $vq_1 \in E(G)$ and v

has precisely two neighbors in $\{x, w, y_1, y_2\}$. By Claim 12.4.9, q_2 has precisely one neighbor in $\{x, w, y_1, y_2\}$, so we have $v \neq q_2$.

Now, if $p_4 \notin N(v)$, then v has two neighbors z, z' in $\{p_1, p_2, p_3\}$. But then G contains a $K_{2,3}$ with bipartition $\{z, z', q_1\}$, $\{q_0, v\}$, contradicting short-separation-freeness. Thus, we have $p_4 \in N(v)$. Since $|N(v) \cap V(P)| = 2$, there is a $j \in \{1, 2, 3\}$ with $p_j v \in E(G)$. We complete Proposition 12.4.5 by producing a contradiction for each possible value of j:

- If $j = 1$, then G contains a $K_{2,3}$ with bipartition $\{q_0, q_2, v\}$, $\{p_1, q_1\}$.
- If $j = 2$, then G contains a $K_{2,3}$ with bipartition $\{p_2, p_4, q_1\}$, $\{p_3, v\}$.
- If $j = 3$, then the four vertices $\{p_3, p_4, v, q_1\}$ induce a K_4 in G.

In any case, we contradict short-separation-freeness. This completes the proof of Proposition 12.4.5. $\square$

We now prove the following:

Lemma 12.4.14. $\mathcal{A} := (G, F_0, F_1, L, \psi)$ *be a defective roulette wheel with* $d(F_0, F_1) = 2$. *Suppose that there exists an* $i \in \{0, 1\}$ *and a* $w \in D_1(F_0) \cap D_1(F_1)$ *such that* $A^i_w = \varnothing$. *Then the following hold.*

1) $A^{1-i}_w \neq \varnothing$*; AND*

2) For any $w^* \in (D_1(F_0) \cap D_1(F_1)) \setminus \{w\}$, $A^i_{w^*} = \varnothing$.

Proof. For the sake of definiteness, let $i = 0$, and suppose toward a contradiction that $A^1_w = \varnothing$. Let $\psi_1 \in \Phi(\psi, w)$, let $y \in V(F_1) \cap N(w)$ and let e be an edge of $F_1 \setminus \{y\}$. If $D_1(F_0) \cap D_1(F_1) = \{w\}$, then, since $A^0_w \cup A^1_w = \varnothing$, the tuple $[w; e; \psi_1; \varnothing]$ is a cycle connector for $\mathcal{A}$, contradicting our assumption that $\mathcal{A}$ is defective. Thus, $|D_1(F_0) \cap D_1(F_1)| = 2$, so there exists a $w^* \in V(G)$ with $D_1(F_0) \cap D_1(F_1) = \{w, w^*\}$. In that case, we retain the vertex w^* and delete the vertex w. Since $|L_{\psi_1}(w^*)| \geq 2$ and $G[V(F_0 \cup F_1 \cup \{w\}]$ is connected, the tuple $[\{w, w^*\}; w^*; \psi_1 \varnothing]$ is a cycle connector for $\mathcal{A}$, contradicting our assumption that $\mathcal{A}$ is defective. This proves 1).

Now we prove 2). Let $w^* \in (D_1(F_0) \cap D_1(F_1)) \setminus \{w\}$. Since $\mathcal{A}$ is defective, it follows from Proposition 12.4.5 that there exist vertices x, y which are the unique anchor vertices of F_0, F_1 respectively. Since G is $K_{2,3}$-free, we have $N(w) \cap N(w^*) = \{x, y\}$. By Lemma 12.3.8, each vertex of $A^0_w \cup A^0_{w^*}$ is adjacent to an edge of F_0 with x as an endpoint, and each vertex of $A^1_w \cup A^1_{w^*}$ is adjacent to an edge of F_0 with y as an endpoint. In particular, each of the four sets $A^0_w, A^0_{w^*}, A^1_w, A^1_{w^*}$ has size at most one, or else, for some $i \in \{0, 1\}$, there is an edge of F_i whose endpoints have two common neighbors in $G \setminus F_i$, which is false, as G is short-separation-free. Thus, by 1), we have $|A^1_w| = 1$ so let v be the unique vertex of A^1_w. Then R^1_v is an edge of F_1 with y as an endpoint. Let y_v be the other endpoint of R^1_v. Now set Q_1 to be the unique edge of $F_1 - y$ with y_v as an endpoint. . Note that $T(\{w, w^*\}; Q_1) \neq \varnothing$, or else, for any $\phi \in \Phi(\psi, \{w, w^*\})$, the tuple $[\{w, w^*\}; Q; \phi; \varnothing]$ is a cycle connector for $\mathcal{A}$, contradicting the fact that $\mathcal{A}$ is defective.

Claim 12.4.15. *For any* $z \in T(\{w, w^*\}; Q_1)$, $w^* \in N(z)$ *and* $w \notin N(z)$, *and exactly one of the following holds.*

1. $N(z) \cap V(F_0 \cup F_1)$ *consists of the unique edge of* $F_1 \setminus \{y_v\}$ *with* y *as an endpoint; OR*
2. $N(z) \cap V(F_0 \cup F_1)$ *consists of an edge of* F_0 *with* x *as an endpoint.*

<u>Proof:</u> Let $z \in T(\{w, w^*\}; Q_1)$. Since $z \notin \{w, w^*\}$, we have $z \notin D_1(F_0) \cap D_1(F_1)$. Since G is $K_{2,3}$-free, z is adjacent to at most one of w, w^*. Thus, z has at least two neighbors among $V(F_0 \cup F_1) \setminus V(Q_1)$.

Suppose toward a contradiction that $w \in N(z)$. Then $w^* \notin N(z)$ and $z \in A_w^0 \cup A_w^1$. Since $A_w^0 = \varnothing$, we have $z = v$. Yet by our choice of Q_1, we have $v \notin T(\{w, w^*\}; Q_1)$, a contradiction. Thus, $w \notin N(z)$. Since $z \notin D_1(F_0) \cap D_1(F_1)$, it follows from Lemma 12.3.4 that $|N(z) \cap V(F_0 \cup F_1)| = 2$ and $z \in N(w^*)$. Thus, $z \in A_{w^*}^0 \cup A_{w^*}^1$. If $z \in A_{w^*}^0$, then, by Lemma 12.3.8, $N(z) \cap V(F_0 \cup F_1)$ consists of an edge of F_0 with x as an endpoint. If $z \in A_{w^*}^1$ then, again by Lemma 12.3.8, $N(z) \cap V(F_0 \cup F_1)$ consists of an edge of F_1 with y as an endpoint, and this edge is not yy_v, or else we contradict the fact that G is short-separation-free. This completes the proof of Claim 12.4.15. ■

Now we return to the proof of 2) of Lemma 12.4.14. Note that by Claim 12.4.15, we have $|T(\{w, w^*\}; Q_1)| \leq 2$, since each of $A_{w^*}^0$ and $A_{w^*}^1$ has size at most one. Suppose toward a contradiction that there exists a $z \in A_{w^*}^0$. Since $N(z) \cap V(F_0)$ consists of an edge of F_0 with x as an endpoint, let $G[N(z) \cap V(F_0)] = xx_z$ for some x_z.

Recall that H_v^1 is the connected component of $G[A^1]$ containing v and is a subpath of the 1-band C^1 of F_1. Let $P := H_v^1 + vw$, and note that each vertex of P has an L_ψ-list of size at least three, since $N(w) \cap V(F_0 \cup F_1) = \{x, y\}$. Since y is the unique anchor vertex of F_0, we have $w^* \notin H_v^1$. By Theorem 1.7.5, there is a $\phi \in \text{Link}(P, C^1)$.

The graph $G[V(P) \cup V(F_0 \cup F_1)] \setminus \{w^*\}$ is connected, since P has a neighbor in each of F_0, F_1. Since $\mathcal{A}$ is not defective, the tuple $[P; w^*; \phi;\ \text{Mid}^1(P)]$ is not a cycle connector for $\mathcal{A}$. Thus, there exists a $u \in V(G) \setminus (\text{dom}(\phi) \cup \text{Mid}^1(P) \cup \{w^*\})$ such that $|L_\phi(u)| < 3$. Since $\phi \in \text{Link}(P, C^1)$, we have either $u \in V(C^1)$, or u has a neighbor in F_0.

Now we claim that u has a neighbor in F_0. Suppose not. Then we have $u \in V(C^1)$, and $N(u) \cap \text{dom}(\phi)$ consists of $N(u) \cap V(F_1)$ and at most the endpoints of P. Thus, u is adjacent to at least one endpoint of P, and, by definition of H_v^1, we have $u \notin A^1$, so u has a lone neighbor in F_1 and is adjacent to both endpoints of P. As one endpoint of P lies in A^1, u is adjacent to the A^1-endpoint of P on the cycle C^1, since C^1 is an induced subgraph of G.

If u is adjacent to w on C^1, then, since $u \neq v$, we have $u = w^*$, which is false. The only remaining possibility is that u is the endpoint of a chord of C^1 whose other endpoint is w. In this case, we have $|V(F_1)| = 4$ and $u \in N(y')$, where y' is the unique vertex of F_1 not adjacent to y. Thus, G contains the two disjoint (F_0, F_1)-paths yw^*zx_z and $y'uwx$. Applying 2) of Proposition 12.4.3, this contradicts our assumption that $\mathcal{A}$ is defective. Thus, u has a neighbor in F_0, as desired.

Since u has a neighbor in F_0, and $u \notin \{w, w^*\}$, we have $N(u) \cap V(F_1) = \varnothing$. Since u has at most two neighbors in F_0, u also has at least one neighbor in $P \setminus \text{Mid}^1(P)$. If w is the lone neighbor of u in $P \setminus \text{Mid}^1(P)$, then, since u has at least three neighbors among $\text{dom}(\phi)$, u has at least two neighbors in F_0, so $u \in A_w^0$, contradicting our assumption that $A_w^0 = \varnothing$. Thus, there is at least one vertex of $H_v^1 \setminus \text{Mid}^1(P)$ in $N(u)$. Note now that $N(u) \cap V(F_0) = \{x\}$, or else, since $N(u) \cap V(F_0) \neq \varnothing$ and u is adjacent to a vertex of C^1 with two neighbors in F_1, there exists an (F_0, F_1)-path of length at most three which is disjoint to xwy. Applying Proposition 12.4.3, this contradicts our assumption that $\mathcal{A}$ is defective.

Claim 12.4.16. $N(u) \cap A^1 = \{v\}$.

Proof: Suppose not. Since there is at least one vertex of $H_v^1 \setminus \text{Mid}^1(P)$ in $N(u)$, there exists a $u' \in N(u) \cap A^1$ with $u' \neq v$, so u' has a neighbor $y' \in V(F_1 \setminus \{y\})$. Since $\mathcal{A}$ is defective, the two disjoint (F_0, F_1)-paths yw^*zx_z and $y'uwx$ do not satisfy 2) of Proposition 12.4.3, so $|V(F_1)| = 4$ and $R_{u'}^1 = yy'$ is an edge of F_1 with y as an endpoint.

Since $u' \neq v$, $R_{u'}^1 \neq yy_v$, so y' is the unique vertex of F_1 opposite to y_v. Set K to be the subgraph of G induced by $V(F_0 \cup F_1) \cup \{u', u, z, w, w^*\}$. This graph is 2-connected, since it contains the (F_0, F_1)-paths yw^*zx_z and $y'u'ux$. Since $\mathcal{A}$ is not defective, there is a facial cycle $D \subseteq K$ of length at least 12.

Now, K contains the 5-chord $R := yw^*xuu'y'$ of F_1. Let $K = K^1 \cup K^2$ be the natural $yw^*xuu'y'$ -partition of K. Note that D lies on one side of the partition, since D is a facial subgraph of K, so let $D \subseteq K^1$. Thus, we have $F_0 \subseteq K^1$ as well, or else $|V(F_0 \cap K^1)| \leq 1$, since $V(F_0 \cap R)| = 1$. If that holds, then $|V(D)| \leq 1 + 4 + 4$, contradicting our assumption. Thus, we indeed have $F_0 \subseteq K^1$, and since z is adjacent to each of x_1, x_z, we have $z \in K^1 \setminus R$, as $x_z \notin V(R)$. Since $z \in K^1 \setminus R$, we have $w \in K^2 \setminus R$, or else K^1 has a facial subgraph containing an edge w^*x adjacent to two vertices of K^1, which is false, as G is short-separation-free. Thus, we get $w \in V(K^2)$. Since w is adjacent to v, we have $v \in V(K^2)$ as well. But then, since $w, v \notin V(R)$ and $D \subseteq K^1$, we have $|V(D)| \leq |V(K)| - 2$, so $|V(D)| \leq 11$, a contradiction. ■

Now we return to the proof of Lemma 12.4.14. Since $N(u) \cap V(F_0) = \{x\}$, we have $N(u) \cap \text{dom}(\phi) = \{v, w, x\}$. Since G is $K_{2,3}$-free, u is the unique vertex of $T(P; w^*) \setminus \text{Mid}^1(P)$, and G contains the 5-cycle $yvuxw^*$, each vertex of which is adjacent to w. Thus, since G is short-separation-free, we have $N(w) = \{y, v, u, x, w^*\}$.

Claim 12.4.17. $|L_\psi(w)| = 3$ and $L_\psi(v) = L_\psi(w)$.

Proof: Suppose that at least one of these conditions does not hold. Then we choose a color $d \in L_\psi(v)$ such that $|L_\psi(w) \setminus \{d\}| \geq 3$. By Theorem 1.7.5, there exists a $\sigma \in \text{Link}(H^1_v, C^1)$ with $\sigma(v) = d$. Since $\mathcal{A}$ is defective, the tuple $[H^1_v; w^* : \sigma; \text{Mid}^1(H^1_v) \cup \{w\}]$ is not a cycle connector for $\mathcal{A}$, so $T(H^1_v; w^*) \setminus (\text{Mid}^1(H^1_v) \cup \{w\}) \neq \varnothing$. Yet we have $T(H^1_v; w^*) \setminus (\text{Mid}^1(H^1_v) \cup \{w\}) \subseteq T(P; w^*) \setminus \text{Mid}^1(P)$, so u is the lone vertex of $T(H^1_v; w^*) \setminus (\text{Mid}^1(H^1_v) \cup \{w\})$. Since $w \notin \text{dom}(\sigma)$, u only has two neighbors in $\text{dom}(\sigma)$, so we have a contradiction. ■

Since$|L_\psi(u)| \geq 4$, it follows from Claim 12.4.17 there is a color $d \in L_\psi(u)$ such that $d \notin L_\psi(w) \cup L_\psi(v)$. Let $\psi^\dagger \in \Phi(\psi, u)$ with $\psi^\dagger(u) = d$. Now set Q_0 to be an edge of $F_0 - x$ which contains all the vertices of $F_0 \setminus \{x, x_z\}$. Since $\mathcal{A}$ is defective, the tuple $[\{u, w\}; Q_0; \psi^\dagger; w]$. By our choice of $\psi^\dagger$, w is $L_{\psi^\dagger}$-inert. Since $G[V(F_0 \cup F_1) \cup \{u, w\}] \setminus Q_0$ is connected, there exists a $u^* \in T(u; Q_0) \setminus \{w\}$ with $|L^{Q_0}_{\psi^\dagger}(u^*)| < 3$. Since $uw^* \notin E(G)$, w^* only has two neighbors among $\text{dom}(\psi^\dagger) \setminus V(Q_0)$. Thus, $u^* \notin \{w, w^*\}$, so $(N(u^*) \setminus V(Q^\dagger)) \cap \text{dom}(\psi^\dagger)$ consists of u and either two vertices of $F_0 \setminus Q_0$ or two vertices of F_1. By our choice of $\psi^\dagger(u)$, we have $|L_{\psi^\dagger}(v)| \geq 3$, so $u^* \neq v$.

Suppose that u^* has a neighbor in $V(F_0)$. Then $|V(F_0)| = 4$ and u^* is adjacent to each vertex of $F_0 \setminus Q_0$. Yet then, by our choice of Q_0, u^* is adjacent to both endpoints of xx_z. Since z is also adjacent to both of these vertices, we contradict short-separation-freeness. Thus, u^* has a neighbor in $V(F_1)$, and $R^1_{u^*}$ is an edge of F_1. Since $u^* \neq v$ and $u^* \in N(u)$, this contradicts Claim 12.4.16. This completes the proof of Lemma 12.4.14. □

With the above in hand, we prove the following:

Lemma 12.4.18. *Let $\mathcal{A} := (G, F_0, F_1, L, \psi)$ be a roulette wheel with $d(F_0, F_1) = 2$ and suppose that $\mathcal{A}$ is defective. Let $w \in D_1(F_0) \cap D_1(F_1)$. Then $A^i_w \neq \varnothing$ for each $i \in \{0, 1\}$.*

Proof. Suppose toward a contradiction that there is an $i \in \{0, 1\}$ with $A^i_w = \varnothing$, say $i = 0$ without loss of generality.

Claim 12.4.19. $|N(w) \cap V(F_1)| = 1$.

Proof: Suppose not. Then, by 2) of Lemma 12.3.4, $G[N(w) \cap V(F_1)]$ is an edge of F_1. Since R^1_w is an edge, we have $\{w\} = D_1(F_0) \cap D_1(F_1)$, or else, if $|D_1(F_0) \cap D_1(F_1)| \geq 2$, then each vertex of R^1_w is an anchor vertex of F_1 and thus, , applying Proposition 12.4.5, $\mathcal{A}$ is not defective. This contradicts our assumption.

Let Q be an edge of F_1, where Q is disjoint to R^1_w if $|V(F_1)| = 4$, and otherwise Q contains at most one endpoint of R^1_w. Then, by our choice of Q, the graph $G[V(F_0 \cup F_1) \cup \{w\} \setminus Q$ is connected. Let $\psi_1 \in \Phi(\psi, w)$. Since $\mathcal{A}$ is defective, the tuple $[w; Q; \psi_1; \varnothing]$ is not a cycle connector for $\mathcal{A}$, so there exists a $z \in T(w; Q)$. Since $D_1(F_0) \cap D_1(F_1) = \{w\}$, there exists a $j \in \{0, 1\}$ such that $N(z) \cap V(F_0 \cup F_1) \subseteq V(F_j)$. Thus, $(N(z) \setminus V(Q)) \cap (V(F_0 \cup F_1)$ consists of w and a subpath of F_j of length one. Since $A^0_w = \varnothing$, z is adjacent to w and, by 1) of Lemma 12.3.8, R^1_z intersects withg R^0_w on precisely one common endpoint of the edges. By 2) of Lemma 12.3.8, we have $|V(F_1)| = 4$, and, by our choice of Q, R^1_z has at least one endpoint in Q, so $\text{dom}(\phi) \cap (N(z) \setminus V(Q))| = 2$, contradicting the fact that $z \in T(w; Q)$ Thus, our assumption that $|N(w) \cap V(F_1)| \neq 1$ is false. ■

Since $|N(w) \cap V(F_1)| = 1$, we fix a $y \in V(F_1)$ and an $x \in V(F_0)$ such that $N(w) \cap V(F_1) = \{y\}$ and $x \in N(w) \cap V(F_0)$. Applying 1) of Lemma 12.4.14, there exists a $v \in A^1_w$. By 1) of Lemma 12.3.8, R^1_v is an edge of F_1 with y as an endpoint. Let y_v be the other endpoint of this edge.

Claim 12.4.20. $D_1(F_0) \cap D_1(F_1) = \{w\}$.

Proof: Suppose toward a contradiction that there is a $|D_1(F_0) \cap D_1(F_1)| \geq 2$, and let $w^* \in D_1(F_0) \cap D_1(F_1)$ with $w^* \neq w$. We then have $N(w^*) \cap V(F_0 \cup F_1) = N(w) \cap V(F_0 \cup F_1) = \{x, y\}$, or else there is an $i \in \{0, 1\}$ such that F_i has more than one anchor vertex. Applying Proposition 12.4.5, this contradicts our assumption that $\mathcal{A}$ is defective. Since G is $K_{2,3}$-free, w^* is the lone vertex of $(D(F_0) \cap D(F_1)) \setminus \{w\}$.

By 2) of Lemma 12.4.14, we have $A^0_{w^*} = \varnothing$. Furthermore, each of $A^1_w, A^1_{w^*}$ is of size at most one, or else, since $N(w) \cap V(F_1) = N(w^*) \cap V(F_1) = \{y\}$, it follows from of Lemma 12.3.8 that there is an edge of F_1 which has y as an endpoint and which has at least two neighbors in $G \setminus V(F_1)$, contradicting short-separation-freeness. In particular, $A^1_w = \{v\}$.

Now set Q to be the unique edge of $F \setminus \{y\}$ with y_v as an endpoint and let $\psi' \in \Phi(\psi, \{w, w^*\})$. The graph $G[V(F_0 \cup F_1) \cup \{w\}] \setminus Q$ is connected, as it contains the path xwy. Since $\mathcal{A}$ is defective, the tuple $[w; Q; \psi'; \varnothing]$ is not a cycle connector for $\mathcal{A}$, so $T(\{w, w^*; Q) \neq \varnothing$. By 2) of Lemma 12.4.14, we have $A^0_{w^*} = \varnothing$. Since w, w^* have no common neighbors other than x, y, it follows that any vertex of $T(\{w, w^*\}; Q)$ lies in $A^1_w \cup A^1_{w^*}$. By our choice of Q, v only has two neighbors in $(V(F \cup F_1) \cup \{w, w^*\}) \setminus V(Q)$, so $A^1_{w^*} \neq \varnothing$ and $T(\{w, w^*\}; Q)$ consists of the lone vertex of $A^1_{w^*}$. Let $T\{w, w^*\}; Q) = A^1_{w^*} = \{z\}$ for some vertex z. Since z has at least three neighbors in $(V(F_0 \cup F_1) \cup \{w, w^*\}) \setminus V(Q)$, we have $|V(F_1)| = 4$, and R^1_z consists of the two vertices of $F_1 \setminus Q$. By our choice of Q, R^1_z is an edge with y as an endpoint. Let y_z be the other endpoint of this edge.

Subclaim 12.4.21. *There exists a unique vertex z^* such that $\{z^*\} = T(\{w, w^*, z\}; Q)$. Furthermore, $N(z^*) \cap (V(F_0 \cup F_1) \cup \{z, w^*, w\} = \{x, w^*, z\}$.*

Proof: Let $\psi'' \in \Phi(\psi, \{w, w^*, z\})$. Since $\mathcal{A}$ is defective, the tuple $[\{w, w^*, z\}; Q; \psi''; \varnothing]$ is not a cycle connector for $\mathcal{A}$, so there exists a $z^* \in T(\{w, w^*, z\}; Q)$. Since z is the unique vertex of $T(\{w, w^*\}; Q)$, and $z \neq z^*$, we have $z \in N(z^*)$, and z^* has precisely two neighbors in $V((F_0 \cup F_1) \setminus Q) \cup \{w, w^*\}$.

If z^* has a neighbor $x' \in V(F_0 \setminus \{x\})$, then G contains the two disjoint (F_0, F_1)-paths $x'z^*zy_z$ and x_1wy_1. Applying Proposition 12.4.3, this contradicts our assumption that $\mathcal{A}$ is defective. Thus, we have $N(z^*) \cap V(F_0 \setminus \{x_1\}) = \varnothing$, and z^* has precisely two neighbors among $\{y, y_z, x, w^*, w\}$. Now, $z^* \neq v$, since $vz \notin E(G)$, and G contains the 5-chord $y_z z w^* w v y_v$ of F_1, each vertex of which is adjacent to y. Thus, since G is short-separation-free, we have $N(y) = \{y_z, z, w^*, w, v, y_v\}$, and $z^* \notin N(y)$. We conclude that z^* has precisely two neighbors among $\{x, y_z, w^*, w\}$.

Now, if $w \in N(z^*)$, then G contains a $K_{2,3}$ with bipartition $\{w, z\}, \{w^*, y, z^*\}$, contradicting short-separation-freeness, so $w \notin N(z^*)$ and z^* has precisely two neighbors among $\{x, y_z, w^*\}$. If $y_z, w^* \in N(z^*)$, then G contains a $K_{2,3}$ with bipartition $\{w^*, y_z\}$, $\{z^*, z, y\}$, contradicting short-separation-freeness. If $x, y_z \in N(z^*)$, we contradict the fact that $D_1(F_0) \cap D_1(F_1) = \{w\}$. The only remaining possibility is that $N(z) \cap (V(F_0 \cup F_1) \cup \{w, w^*\}) = \{w^*, y, y_z\}$. Since G is $K_{2,3}$-free, z is the unique vertex of $T(\{w, w^*\}; Q)$. ■

Now we are ready to finish the proof of Claim 12.4.20. Applying Subclaim 12.4.21, let z^* be the unique vertex of $\{z^*\} = T(\{w, w^*, z\}; Q)$. Then G contains the 5-cycle xz^*zyw, and w^* is adjacent to each vertex of this cycle. Since G is short-separation-free, we have $N(w^*) = \{x_1, z^*, z, y_1, w\}$. Applying Corollary 1.3.6, there is a $\psi^\dagger \in \Phi(\psi, \{z, w\})$ such that w^* is $L_{\psi^\dagger}$-inert. Since $\mathcal{A}$ is defective, the tuple $[\{w^*, w, z\}; Q; \psi^\dagger; w^*]$ is not a cycle connector for $\mathcal{A}$, so there exists a vertex $u \in T(\{w, z\}; Q)$ with $|L^Q_{\psi^\dagger}(u)| < 3$. But then $u = z^*$, since $\{z^*\} = T(\{w, w^*, z\}; Q)$. Yet z^* only has two neighbors in $\text{dom}(\psi^\dagger) \setminus V(Q)$, so we have a contradiction. This completes the proof of Claim 12.4.20. ■

Now we return to the proof of Lemma 12.4.18. We set $Q' := F_1 \setminus \{y, y_v\}$ and let $\psi' \in \Phi(\psi, \{w, v\})$. Since $\mathcal{A}$ is defective, the tuple $[\{w, v\}; Q'; \psi'; \varnothing]$ is not a cycle connector for $\mathcal{A}$. The graph $G[V(F_0 \cup F_1) \cup \{w, v\}] \setminus Q'$ is connected, so $T(\{w, v\} : Q') \neq \varnothing$.

Claim 12.4.22. *There exists a unique vertex z such that $T(\{w, v\}; Q') = \{z\}$. Furthermore, the following hold.*

1) $N(z) \cap (V(F_0 \cup F_1) \cup \{w, v\}) = \{x, w, v\}$; AND

2) $L_\psi(v) = L_\psi(w)$ and $|L_\psi(v)| = 3$; AND

3) $|L_\psi(z)| = 4$, and $L_\psi(v) \cup L_\psi(w) \subseteq L_\psi(z)$.

Proof: Since $T(\{w, v\}; Q') \neq \varnothing$, there exists a $z \in T(\{w, v\}; Q')$. Thus, z has at least three neighbors among $V(F_0) \cup \{w, v, y, y_v\}$. Suppose toward a contradiction that there is an $x' \in V(F_0 \setminus \{x\})$ with $x' \in N(z)$. Since $D_1(F_0) \cap D_1(F_1) = \{w\}$, we have $N(z) \cap V(F_1) = \varnothing$, so z has at least three neighbors among $V(F_0) \cup \{w, v\}$. If $v \in N(z)$, then G contains the two disjoint (F_0, F_1)-paths xwy and $x'zvy_v$. Applying Proposition 12.4.3, this contradicts our assumption that $\mathcal{A}$ is defective. Thus, $v \notin N(z)$, and z contains at least three neighbors among $V(F_0) \cup \{w\}$. Since z has at most two neighbors in F_0, we have $z \in A^0_w$, contradicting our assumption that $A^0_w = \varnothing$.

We conclude that $N(z) \cap V(F_0) \subseteq \{x\}$, so z has at least three neighbors among $\{x, w, v, y, y_v\}$. Suppose that $y_1 \in N(z)$. Then $v \notin N(z)$, or else G contains a $K_{2,3}$ with bipartition $\{w, y_v, z\}$, $\{y, v\}$. Furthermore, $x \notin N(z)$, since $D_1(F_0) \cap D_1(F_1) = \{w\}$. But then z is adjacent to all three of w, y, y_v, and G contains a $K_{2,3}$ with bipartition $\{w, y, y_v\}$, $\{v, z\}$, contradicting short-separation-freeness.

Thus, $y \notin N(z)$, and z has at least three neighbors among $\{x, w, v, y_v\}$. Since $xwvy_v$ is an induced path in G, and $N(y_v) \cap N(x) = \varnothing$, it follows from our triangulation conditions that $G[N(z) \cap \{x, w, v, y_v\}]$ is a subpath of $xwvy_v$ of length precisely two. This path is not wvz, or else G contains a copy of $K_{2,3}$, as w, v, z are all adjacent to y. Thus, this path is xwv, and $N(z) \cap (V(F_0 \cup F_1) \cup \{w, v\}) = \{x, w, v\}$. Since G is $K_{2,3}$-free, z is unique and $T(\{w, v\}; Q') = \{z\}$. This proves 1).

Suppose now that 2) does not hold. Since each of w, v has an L_ψ-list of size at least three, there is a $d \in L_\psi(w)$ such that $|L_\psi(v) \setminus \{d\}| \geq 3$. Let $\phi \in \Phi(\psi, w)$ with $\psi(w) = d$. Since $T(\{w, v\}; Q') = \{z\}$ and $L_\phi(v)| \geq 3$, the tuple $[w; Q'; \phi; \varnothing]$ is a cycle connector for $\mathcal{A}$, contradicting the fact that $\mathcal{A}$ is defective.

If 3) does not hold, then ψ extends to an L-coloring $\psi^\dagger \in \Phi(\psi, \{w, v\})$ such that $|L_{\psi^\dagger}(z)| \geq 3$. Since z is the unique vertex of $T(\{w, v\}; Q')$, we then get that $[\{w, v\}; Q'; \psi^\dagger; \varnothing]$ is a cycle connector for $\mathcal{A}$, contradicting our assumption that $\mathcal{A}$ is defective. ■

Applying Claim 12.4.22, let z be the unique vertex in $T(\{w, v\}; Q')$.

Claim 12.4.23. *For any vertex $z^* \in T(\{w, v, z\}; Q')$, z^* satisfies precisely one of the following.*

1) $N(z^*) \cap (V(F_0 \cup F_1) \cup \{w, v, z\})$ consists of z and an edge of F_0 with x as an endpoint, and $\{z^*\} = A^0_z$; OR

2) $N(z^*) \cap (V(F_0 \cup F_1) \cup \{w, v, z\}) = \{z, v, y_v\}$.

Proof: Let $\psi'' \in \Phi(\psi, \{w, v, z\})$. Since $\mathcal{A}$ is defective, $[\{w, v, z\}; Q'; \psi''; \varnothing]$ is not a cycle connector for $\mathcal{A}$. Since $G[V(F_0 \cup F_1) \cup \{w, v, z\}] \setminus Q'$ is connected, we have $T(\{w, v, z\}; Q') \neq \varnothing$ so there exists a $z^* \in T(\{w, v, z\}; Q')$. Since $z \neq z^*$, it follows from Claim12.4.22 that z^* has precisely two neighbors in $V((F_0 \cup F_1) \setminus Q') \cup \{w, v\}$, and $z \in N(z^*)$. Note that $w \notin N(z^*)$, or else G contains a $K_{2,3}$ with bipartition $\{x, v, z^*\}$, $\{z, w\}$, contradicting short-separation-freeness. Thus, z^* has precisely two neighbors among $V((F_0 \cup F_1) \setminus Q') \cup \{v\}$.

Suppose now that $x \in N(z^*)$. Since $z^* \notin D_1(F_0) \cap D_1(F_1)$, we have $N(z^*) \cap V(F_1) = \varnothing$, and, by 1) of Lemma 12.3.8, $N(z^*) \cap (V(F_0 \cup F_1) \cup \{w, v, z\})$ consists of z and an edge of F_0 with x as an endpoint, so $z^* \in A^0_z$. Now suppose toward a contradiction that there is a $z^\dagger \in A^0_z$ with $z^\dagger \neq z$. By By Ro4 of Definition 12.3.1, each of zz^*, zw, and $zz^\dagger$ is an edge of the 1-band C^1 of C, which is false since every vertex of $V(C^1)$ has degree two in C^1.

Thus, if $x \in N(z^*)$, then we are done. Suppose now that $x \notin N(z^*)$. We claim now that $z^* \cap V(F_0) = \varnothing$. Suppose not. Then z^* has a neighbor $x' \in V(F_0 \setminus \{x\})$. We have $N(z) \cap V(F_1) = \varnothing$, since $z^* \notin D_1(F_0) \cap D_1(F_1)$. Thus, z^* has precisely two neighbors among $V(F_0 \setminus \{x\}) \cup \{v\}$. Now, if x' is adjacent to x, then G contains the 4-cycle xzz^*x'. Since $x \notin N(z^*)$, we then have $x' \in N(z)$ by our triangulation conditions. This contradicts Claim 12.4.22. Thus, $|V(F_0)| = 4$, x' is opposite to x in F_0, and $\{x'\} = N(z^*) \cap V(F_0)$. But then $v \in N(z^*)$ as well, since z^* has precisely two neighbors among $V(F_0 \setminus \{x\}) \cup \{v\}$. Thus, G contains the (F_0, F_1)-paths xwy and $x'z^*vy_v$. Applying Proposition 12.4.3, this contradicts our assumption that $\mathcal{A}$ is defective.

Thus, we have $N(z) \cap V(F_0) = \varnothing$, and so z^* has precisely two neighbors among $\{y, y_v, z\}$. If $y \in N(z^*)$, then G contains a $K_{2,3}$ with bipartition $\{y, z\}$, $\{w, v, z^*\}$, contradicting short-separation-freeness. We conclude that $y \notin N(z)$, and so $N(z) \cap (V((F_0 \cup F_1) \setminus Q') \cup \{w, v\}) = \{v, y_v\}$. This completes the proof of Claim 12.4.23. ■

We claim now that there is a $z^* \in T(\{w, v, z\}; Q')$ such that $N(z^*) \cap (V(F_0 \cup F_1) \cup \{w, v, z\}) = \{z, v, y_v\}$. Suppose toward a contradiction that no such z exists. Then, by Claim 12.4.22, there exists a $z^\dagger$ such that $\{z^\dagger\} = A^0_z = T(\{w, v\}; Q')$, and $N(z^\dagger) \cap (V(F_0 \cup F_1) \cup \{w, v, z\})$ consists of z and an edge of F_0. Thus, we have $|L_\psi(z^\dagger)| \geq 3$. Since $|L_\psi(z)| \geq 4$ and $w, v \notin N(z^\dagger)$, there is a $\phi \in \Phi(\psi, \{w, v, z\})$ such that $|L_\phi(z^\dagger)| \geq 3$. But then, since $z^\dagger$ is the unique vertex of $T(\{w, v, z\}; Q')$, the tuple $[\{w, v, z\}; Q'; \phi; \varnothing]$ is a cycle connector for $\mathcal{A}$, contradicting the fact that $\mathcal{A}$ is defective.

Thus, there is a $z^* \in T(\{w, v, z\}; Q')$ such that $N(z^*) \cap (V(F_0 \cup F_1) \cup \{w, v, z\}) = \{z, v, y_v\}$, and G contains the 5-cycle $z^*zwy_1y_2$, each vertex of which is adjacent to v. Since G is short-separation-free, we have $N(v) = \{z^*, z, w, y, y_v\}$.

Claim 12.4.24. *There is an L-coloring $\phi \in \Phi(\psi, \{w, z^*\})$ such that v is L_ϕ-inert and $|L_\phi(z)| \geq 3$.*

Proof: If $L_\psi(w) \cap L_\psi(z^*) \neq \varnothing$, then, since $wz^* \notin E(G)$, there is a $\phi \in \Phi(\psi, \{w, z^*\})$ with $\phi(w) = \phi(z^*) = d$. Thus, v is L_ϕ-inert and $|L_\phi(v)| \geq 3$, so we are done in this case. So now suppose that $L_\psi(w) \cap L_\psi(v^*) = \varnothing$. Since $|L_\psi(z^*)| \geq 4$, it follows from 2) and 3) of Claim 12.4.22 that there exists a color $d^* \in L_\psi(z^*)$ such that $d^* \notin L_\psi(v)$ and $d^* \notin L_\psi(z)$. Then, for any $\phi \in \Phi(\psi, \{w, z^*\})$ with $\phi(z^*) = d^*$, ϕ again satisfies the desired properties. ■

Let $\phi \in \Phi(\psi, \{w, z^*\})$ be as in Claim 12.4.24. Since $\mathcal{A}$ is defective, the tuple $[\{z^*, w\}; Q'; \phi; v]$ is not a cycle connector for $\mathcal{A}$, so there is a $u \in V(G) \setminus (\text{dom}(\phi) \cup \{v\})$ with $|L_\phi^{Q'}(u)| < 3$, so $u \in T(\{w, z^*\}; Q')$. By our choice of ϕ, we have $|L_\phi^{Q'}(z)| \geq 3$. Thus, $u \neq z$. Since $u \neq z$, we have $z^* \in N(u)$ and u has precisely two neighbors in $T(\{w, v\}; Q')$, or else we contradict Claim 12.4.22. Thus, u has precisely two neighbors among $V(F_0 \cup F_1) \setminus Q') \cup \{w, v\}$.

If u has a neighbor $x' \in V(F_0 \setminus \{x\})$, then G contains the two disjoint (F_0, F_1)-paths $x'uz^*y_v$ and xwy. Applying Proposition 12.4.3, this contradicts our assumption that $\mathcal{A}$ is defective. Thus, u has precisely two neighbors among $\{x, w, v\} \cup \{y, y_v\}$. We now rule out the following adjacencies by producing a $K_{2,3}$ in each case:

- If $v \in N(u)$, then G contains a $K_{2,3}$ with bipartition $\{v, z^*\}$, $\{z, y_v, u\}$.
- If $y \in N(u)$, then G contains a $K_{2,3}$ with bipartition $\{y, z^*\}$, $\{v, y_v, u\}$.
- If $w \in N(u)$, then G contains a $K_{2,3}$ with bipartition $\{w, z^*\}$, $\{z, v, u\}$.

The only remaining possibility is that u is adjacent to each of x, y_v, contradicting the fact that $u \notin D_1(F_0) \cap D_1(F_1)$. This completes the proof of Lemma 12.4.18. □

With the above in hand, we now prove the following:

Proposition 12.4.25. *Let $\mathcal{A} := (G, F_0, F_1, L, \psi)$ be a roulette wheel with $d(F_0, F_1) = 2$ and suppose that, for each $i = 0, 1$, F_i has two anchor vertices. Then $\mathcal{A}$ is not defective.*

Proof. Suppose toward a contradiction that $\mathcal{A}$ is defective. By Proposition 12.4.5, there exists a $w \in V(G)$ such that $D_1(F_0) \cap D_1(F_1) = \{w\}$. For each $i = 0, 1$, since F_i has two anchor vertices, R_w^i is an edge. By Lemma 12.4.18, we have $|A_w^i| \geq 1$ for each $i = 0, 1$. Thus, we have $|V(F_0)| = |V(F_1)| = 4$ by 2) of Lemma 12.3.8. Now we write $F_0 := x_1x_2x_3x_4$ and $F_: = y_1y_2y_3y_4$, and, without loss of generality, let $R_w^0 = x_1x_2$ and $R_w^1 = y_1y_2$. Applying 2) of Lemma 12.3.8 again, $R_{u_0}^0$ intersects with R_w^0 on a vertex, so, without loss of generality, let $R_{u_0}^0 = x_1x_4$ and $R_{u_1}^1 = y_1y_4$. By 2) of Lemma 12.3.8, we have $N(x_1) \subseteq \{w, u_0\} \cup V(F_0)$ and $N(y_1) \subseteq \{w, u_1\} \cup V(F_1)$. For each $i = 0, 1$, let C^i be the 1-band of F_i. Since w has precisely four neighbors in $V(F_0 \cup F_1)$, we have $|L_\psi(w)| \geq 1$, so we fix a color $c \in L_\psi(w)$.

Claim 12.4.26. *For each $i = 0, 1$, $w \notin \text{Mid}^i(C^i)$, and there exists a $\phi \in \text{Link}(H_{u_i}^i, C^i)$ with $\phi(w) = c$.*

Proof: Suppose that $w \in \text{Mid}^i(C^i)$. then there is a vertex $z \in D_2(F_i)$ such that w is an internal vertex of $G[V(H_{u_i}^i) \cap N(z)]$. But then $G \setminus F_1$ contains a wheel with central vertex w, and thus, since G is short-separation-free, we have $N(w) \cap V(F_1) = \varnothing$, which is false.

If w is an endpoint of $H_{u_i}^i$, then, applying Theorem 1.7.5, it immediately follows that there exists a $\phi \in \text{Link}(H_{u_i}^i, C^i)$ with $\phi(w) = c$, since each vertex of $H_{u_i}^i$ other than w has an L_ψ-list of size at least three. Now suppose that w is not an endpoint of $H_{u_i}^i$, and let P_1, P_2 be the two subpaths of $H_{u_i}^i$ intersecting on w.

For each $j = 12$, there is a $\phi_j \in \text{Link}(P_j, C^i)$ with $\phi_j(w) = c$ by Theorem 1.7.5. Since $w \notin \text{Mid}^i(C^i)$, the union $\phi_1 \cup \phi_2$ also lies in $\text{Link}(H_{u_i}^i, C^i)$, so we are done. ■

By Claim 12.4.26, there is a $\phi \in \text{Link}(H^0_{u_0}, C^0)$ with $\phi(w) = c$. Now let $Q := y_3y_4$, and consider the tuple $[H^0_{u_0}, Q; \phi; \text{Mid}^0(H^0_{u_0})]$. Note that the subgraph of G induced by $V(F_0 \cup F_1) \cup V(H^0_{u_0})$ is connected, as it contains the path x_1wy_1. Since $\mathcal{A}$ is defective, this tuple is not a cycle connector for $\mathcal{A}$, so there exists a $z \in T(H^0_{u_0}; Q)$ with $|L^Q_\phi(z)| < 3$. Since $\phi \in \text{Link}(H^0_{u_0}, C^0)$, we have either $z \in D_1(F_1)$, or $z \in V(C^0 \setminus H^0_{u_0})$.

Suppose that $z \in D_1(F_1)$ and let $y \in N(z) \cap V(F_1)$. Since z has at most two neighbors in F_1, z also has a neighbor in $\text{dom}(\phi) \setminus V(F_1)$. Since $\{w\} = D_1(F_0) \cap D_1(F_1)$, z has a neighbor in $H^0_{u_0}$. If z has a neighbor in $H^0_{u_0} \setminus \{w\}$, then, since each vertex of $H^0_{u_0}$ has at least two neighbors in F_0, it follows that G contains an (F_0, F_1)-path of length three which is disjoint to one of the four (F_0, F_1)-paths of length with midpoint w. Applying Proposition 12.4.3, this contradicts our assumption that $\mathcal{A}$ is defective. Thus, w is the lone neighbor of z in $\text{dom}(\phi) \setminus V(F_1)$, and since $|L^Q_\phi(z)| < 3$, z is adjacent to each of y_1, y_2. Since w is also adjacent to each of y_1, y_2, we contradict short-separation-freeness. Thus, our assumption that $z \in D_1(F_1)$ is false.

Since $z \notin D_1(F_1)$, we have $z \in V(C^0 \setminus H^0_{u_0}))$, and thus, by definition of $H^0_{u_0}$, z has precisely one neighbor in F_0 and two neigbhbors in $H^0_{u_0}$. Since C^0 has no chords, $N(z) \cap V(H^0_{u_0})$ consists of the endpoints of $H^0_{u_0}$, so $V(C^0) = V(H^0_{u_0}) \cup \{z\}$ and $|V(C^0)| = 5$.

Claim 12.4.27. *There exists a $\phi \in \text{Link}(C^0)$ with $\phi(w) = c$.*

Proof: We break the proof of the claim into two cases:

Case 1: w is an endpoint of $H^0_{u_0}$

Since $|V(C^0_| = 5$ and G is short-separation-free, we get that, for each $u \in D_2(F_0)$, $G[N(u) \cap V(C^0)]$ is a subpath of C^0 of length at two. Since $|L_\psi(z)| \geq 4$ and all the other vertices of the path $C^0 - wz$ have L_ϕ-lists of size at least three, it follows from Theorem 1.7.5 that there exists a pair of elements $\phi_1, \phi_2 \in \text{Link}(C^0 - wz, C^0)$ which use distinct colors on z, so say without loss of generality that $\phi_1(z) \neq c$. Then ϕ_1 is a proper L-coloring of $C^0 \setminus \text{Mid}^0(C^0)$, and $\phi_1 \in \text{Link}^0(C^0)$.

Case 2: w is not an endpoint of $H^0_{u_0}$

In this case, let p, p^* be the endpoints of $H^0_{u_0}$, let P_1 be the subpath of $H^0_{u_0}$ with endpoints p, w, and let P_2 be the subpath of $H^0_{u_0}$ with endpoints w, p^*. Then one of P_1, P_2 has length one and the other has length two, so suppose without loss of generality that $|E(P_1)| = 1$ and $|E(P_2)| = 2$. Note that $z \notin \text{Mid}^0(P)$, or else G contains a copy of $K_{2,3}$, since z, p, p^* have a common neighbor in F_0.

Since G is short-separation-free and $|V(C^0)| = 5$, it follows from the above that, for each $u \in D_2(F_0)$, the graph $G[N(u) \cap V(C^0)]$ is either a subpath of $P_1 + pz$ of length two, or a subpath of $P_2 + pz$ of length two. Applying Theorem 1.7.5, there is a pair of colors $\psi_1, \psi_2 \in \text{Link}(P_1 + pz, C^0)$ with $\psi_1(w) = \psi_2(w) = c$ and $\psi_1(z) \neq \psi_2(z)$, as $|L_\psi(z)| \geq 4$.

We first show that he claim holds if $p^* \notin \text{Mid}^0(C^0)$. Suppose that $p^* \notin \text{Mid}^0(P)$. Applying Theorem 1.7.5, there is a $\phi \in \text{Link}(P_2, C^0)$ with $\phi(w) = c$. There exists a $j \in \{1, 2\}$ such that $\psi_j(z) \neq \phi(p^*)$, and since $p^*, z \notin \text{Mid}^0(C^0)$, the union $\phi \cup \psi_i$ lies in $\text{Link}(C^0)$, and uses c on w, so we are done on that case.

Now suppose that $p^* \in \text{Mid}^0(C^0)$. Let z be the unique vertex of $D_2(F_0)$ such that p^* is the midpoint of $G[N(z) \cap V(C^0)]$. Now let u be the midpoint of P_2. Recall that, since $|V(C^0)| = 5$, $G[N(z) \cap V(C^0)]$ is a path of length two, so $G[N(z) \cap V(C^0)] = up^*z$. Since $|L_\psi(u)| \geq 2$, there is a color $d \in L_\psi(u)$ and a $j \in \{1, 2\}$ such that

$L_\psi(p^*) \setminus \{d, \psi_j(z)\}| \geq 2$. Let ψ'_j be the resulting extension of ψ_j to $\text{dom}(\psi_j) \cup \{u\}$. Then p^* is $L_{\psi'_j}$-inert, and since neither u nor w lies in $\text{Mid}^0(C^0)$, we have $\psi'_j \in \text{Link}(C^0)$, and ψ'_j uses the color c on w. ■

Thus, let $\phi \in \text{Link}(C^0)$ with $\phi(w) = c$. Now consider the tuple $[C^0; Q; \phi; \text{Mid}^0(C^0)]$. The graph $G[V(F_0 \cup F_1 \cup C^0)]$ is connected, and, since $\mathcal{A}$ is defective, there exists a $z' \in T(C^0; Q)$ with $|L^Q_{\phi^*}(z')| < 3$. Since $\phi^* \in \text{Link}(C^0)$, z' has a neighbor in F_1, and $N(z') \cap V(F_0) = \varnothing$, as $\{w\} = D_1(F_0) \cap D_1(F_1)$.

If z' has a neighbor in $V(C^0 \setminus \{w\})$, then G contains an (F_0, F_1)-path of length three which is disjoint to one of the four (F_0, F_1)-paths of length two with midpoint w. Applying Proposition 12.4.3, this contradicts our assumption that $\mathcal{A}$ is defective. Thus, $N(z') \cap (\text{dom}(\phi^*) \setminus V(F_1)) = \{w\}$, so z' is adjacent to each vertex of $F_1 \setminus Q$. But then each of z', w is adjacent to y_1, y_2, contradicting short-separation-freeness. This completes the proof of Proposition 12.4.25. □

We now prove the following lemma:

Lemma 12.4.28. *Let $\mathcal{A} := (G, F_0, F_1, L, \psi)$ be a defective roulette wheel with $d(F_0, F_1) = 2$ and suppose that at least one of F_0, F_1 has precisely one anchor vertex. Let $i \in \{0, 1\}$ and let C^i be the 1-band of F_i. Then $V(C^i \setminus A^i)| > 1$.*

Proof. Without loss of generality, let $i = 0$, and suppose toward a contradiction that $V(C^i \setminus A^i)| \leq 1$. By 1) of Proposition 12.3.6, we have $|V(C^0)| \geq 5$, so $|V(F_0)| = 4$, $V(C^0 \setminus A^0)| = 1$, and $|V(C^0)| = 5$. Let q be the lone vertex of $C^0 \setminus A^0$. Let $w \in D_1(F_0) \cap D_1(F_1)$ (possibly $w = q$) and let $x \in N(w) \cap V(F_0)$.

Claim 12.4.29. $\{w\} = D_1(F_0) \cap D_1(F_1)$.

Proof: Suppose not, and let $w^* \in D_1(F_0) \cap D_1(F_1)$ with $w^* \neq w$. If F_0 has only one anchor vertex, then $w^* = q$, since $w^* \notin A^0$. But then $w \in A^0$, contradicting our assumption that F_0 has only one anchor vertex. Thus, F_0 has more than one anchor vertex, and thus, by Proposition 12.4.5, $\mathcal{A}$ is not defective, contradicting our assumption. ■

Let C^1 be the 1-band of F_1. Applying Lemma 12.4.18, we fix a vertex $v \in A^1_w$. Then R^1_v is an edge which intersects with R^1_w on an endpoint, so let $y \in V(F_1) \cap N(w)$ and let y_v be the neighbor of y in F_1 such that $R^1_v = yy_v$.

Claim 12.4.30. *Let $u_1 \in V(C^1 - w)$. Then the following hold.*

1) *$N(u_1) \cap A^0 \subseteq \{w\}$; AND*
2) *If either $|N(w) \cap V(F_1)| > 1$ or $N(u_1) \cap V(F_1) \not\subseteq N(w) \cap V(F_1)$, then*
    i) *For any $u_0 \in A^0 \setminus \{w\}$, $N(u_0) \cap N(u_1) \subseteq \{w\}$; AND*
    ii) *For any $z \in D_1(C^0) \setminus V(F_0)$ with $|N(z) \cap V(C^0)| \geq 3$, we have $N(z) \cap N(u_1) \subseteq \{w\}$.*

Proof: Suppose toward a contradiction that there is a $u' \in N(u_1) \cap A^0$ with $u' \neq w$. Let K be the subgraph of G induced by $V(F_0 \cup F_1) \cup V(C^0) \cup \{u_1, v\}$. Since u' has two neighbors in F_0, K contains an (F_0, F_1)-path which has internal vertices $u'u_1$ and which is disjoint to either xwy or $xwvy_v$. Thus, K is 2-connected. Let D be a facial subgraph of K. Since C^0 separates $V(F_0)$ from $V(F_1) \cup \{u_1, v\}$, we have $V(D) \subseteq V(F_0) \cup V(C^0)$ or $V(D) \subseteq V(F_1) \cup V(C^0) \cup \{z, v\}$. In either case, since $|V(C^0)| \leq 5$, we have $|V(D)| \leq 11$, so $\mathcal{A}$ satisfies S1, contradicting the fact that $\mathcal{A}$ is defective. This proves 1).

Now we prove 2). Suppose that either $|N(w) \cap V(F_1)| > 1$ or $N(u_1) \cap V(F_1) \not\subseteq N(w) \cap V(F_1)$. Let $u_0 \in A^0 \setminus \{w\}$ and suppose toward a contradiction that u_0, u_1 have a common neighbor z with $z \neq w$.

Since $\{w\} = D_1(F_0) \cap D_1(F_1)$ we have $z \notin V(F_0 \cup F_1)$. Let K be the subgraph of G induced by $V(F_0 \cup F_1) \cup V(C^0) \cup \{u_1, z\}$. Since u_0 has two neighbors in F_0 and either $|N(w) \cap V(F_1)| > 1$ or $N(u_1) \cap V(F_1) \not\subseteq |N(w) \cap V(F_1)$, there exists a vertex $y^* \in V(F_1) \cap N(w)$ such that K contains an (F_0, F_1)-path which has internal vertices u_0zu_1 and which is disjoint to xwy^*. Thus, K is 2-connected.

Let D be a facial subgraph of K. Since C^0 separates $V(F_0)$ from $V(F_1) \cup \{u_1\}$, we have $V(D) \subseteq V(F_0) \cup V(C^0) \cup \{z\}$ or $V(D) \subseteq V(F_1) \cup V(C^0) \cup \{z, u_1\}$. In either case, since $|V(C^0)| \leq 5$, we have $|V(D)| \leq 11$, so $\mathcal{A}$ satisfies S1, contradicting the fact that $\mathcal{A}$ is defective. This proves i).

Now we prove ii). Suppose toward a contradiction that there is a $z \in D_1(C^0) \setminus V(F_0)$ such that $|N(z) \cap V(C^0)| \geq 3$ and $N(z) \cap N(u_1) \not\subseteq \{w\}$. Let $z' \in N(z) \cap N(u_1)$ with $z' \neq w$. Since $|V(C^0)| = 5$ and G is short-separation-free, it follows that $G[N(z) \cap V(C^0)]$ is a subpath of C^0, so $G[N(z) \cap V(C^0)]$ is a path of length precisely two in this case. Let v^m be the middle vertex of this subpath. Let K_* be the subgraph of G induced by $V(F_0 \cup F_1) \cup V(C^0 \setminus \{q\}) \cup \{z, z'\}$. Since z' has a neighbor in $A^0 \setminus \{v^m\}$ and either $|N(w) \cap V(F_1)| > 1$ or $N(u_1) \cap V(F_1) \not\subseteq N(w) \cap V(F_1)$, there exists a vertex $y^* \in V(F_1) \cap N(w)$ such that K contains an (F_0, F_1)-path which has internal vertices $zz'u_1$ and which is disjoint to xwy^*. Thus, K_* is 2-connected. Let C^0_* be the cycle of K_* obtained from C^0 by replacing v^m with z.

Let D be a facial subgraph of K_*. Since C^0_* separates $V(F_0)$ from $V(F_1) \cup \{u_1\}$, we have $V(D) \subseteq V(F_0) \cup V(C^0_*) \cup \{z'\}$ or $V(D) \subseteq V(F_1) \cup V(C^0_*) \cup \{z', u_1\}$. In either case, since $|V(C^0_*)| \leq 5$, we have $|V(D)| \leq 11$, so $\mathcal{A}$ satisfies S1, contradicting the fact that $\mathcal{A}$ is defective. This proves ii) and thus completes the proof of 2). ■

Let Q be an edge of F_1, where Q intersects yy_v precisely on y if $|V(F_1)| = 3$, and Q is disjoint to yy_v if $|V(F_1)| = 4$. We now have the following:

Claim 12.4.31. $\text{Link}^0(C^0) = \varnothing$.

Proof: Suppose toward a contradiction that there is a $\psi' \in \text{Link}(C^0)$. Then the subgraph of G induced by $(V(F_0 \cup F_1 \cup C^0) \cup \{v\}) \setminus V(Q)$ is connected, since it contains the path $xwvy_v$. By 2) of Claim 12.4.30, v has no neighbors in $C^0 \setminus \{w\}$, so $|L_{\psi'}(w)| \geq 2$. Thus, let $\psi'' \in \Phi(\psi', v)$. Since $\mathcal{A}$ is defective, the tuple $[V(C^0) \cup \{v\}; \psi''; Q; \text{Mid}^0(C^0)]$ is a cycle connector for $\mathcal{A}$, so $T(V(C^0) \cup \{v\}; Q; \psi'') \setminus \text{Mid}^0(C^0) \neq \varnothing$.

Let $z \in T'(V(C^0) \cup \{v\}; Q; \psi') \setminus \text{Mid}^0(C^0)$. We claim now that $z \in D_1(F_1)$. If z is adjacent to v, then, by 2) of Claim 12.4.30, v has no neighbors in $C_0 \setminus \{w\}$, so $N(z) \cap \text{dom}(\psi'') \subseteq B_1(F_1)$. Since $\text{dom}(\psi'') \cap B_1(F_1) = V(F_1) \cup \{v, w\}$ and z has at least three neighbors in $\text{dom}(\psi'')$, z has at least one neighbor in F_1, so $z \in B_1(F_1)$. Since $z \notin V(F_1)$, we have $z \in D_1(F_1)$. On the other hand, if z is not adjacent to v, then we have $|L_{\psi'}(z)| < 3$, and, by definition of $\text{Link}(C^0)$, we have $z \in D_1(F_1)$.

In any case, we have $z \in D_1(F_1)$. By 1) of Claim 12.4.30, z has no neighbors in $A^0 \setminus \{w\}$. Since $|L^Q_{\psi''}(z)| < 3$, it follows that $(N(z) \setminus V(Q)) \cap \text{dom}(\psi'')$ consists of w and an edge of $F_1 \setminus Q$. Thus, $|V(F_1)| = 4$, and, by our choice of Q, both of w, z are adjacent to both endpoints of yy_v, contradicting the fact that G is short-separation-free. ■

Now let U be the set of vertices of $D_1(C^0) \setminus V(F_0)$ with at least three neighbors in $V(C^0)$. Since G is short-separation-free, it follows from Ro4 that, for each $z \in U$, $G[N(z) \cap V(C^0)]$ is a subpath of C^0 of length two. Since $|V(C^0)| = 5$, we have $|U| \leq 2$.

Claim 12.4.32. $|U| = 2$.

Proof: Suppose toward a contradiction that $|U| < 2$ and there exists a $p \in V(C^0 \setminus \{w\})$ such that, for each $z \in U$, p does not lie in $N(z)$. Let p', p'' be the two neighbors of p on C^0. Let P' be the subpath of $C^0 - p$ with endpoints p', w, and let P'' be the subpath of $C^0 - p$ with endpoints p'', w. Since $|L_\psi(w)| \geq 2$, let $c \in L_\psi(w)$. By Theorem 1.7.5, there exist $\psi' \in \text{Link}(P', C^0)$ and $\psi'' \in \text{Link}(P'', C^0)$ such that $\psi'(w) = \psi''(w) = c$. Since C^0 is a chordless cycle, the union $\psi' \cup \psi''$ is a proper L-coloring of its domain. Since $|L_\psi(d)| \geq 3$, we have $|L_\psi(p) \setminus \{\psi'(p'), \psi''(p'')\}| \geq 1$, so $\psi' \cup \psi''$ extends to a proper L-coloring ψ^* of $\text{dom}(\psi') \cup \text{dom}(\psi'') \cup \{p\}$. Since $U \cap N(p) = \varnothing$, we have $\psi^* \in \text{Link}(C^0)$, contradicting Claim 12.4.31. ■

Applying Claim 12.4.32, let $U = \{z_0, z_1\}$. For each $j = 0, 1$, we have the following. Set $P_j := G[N(z_j) \cap V(C^0)]$. By Theorem 1.7.5, $\text{Link}(P^j, C^0) \neq \varnothing$, so let $\psi_j \in \text{Link}(P_j, C^0)$. Since $|L_\psi(w)| \geq 2$ and each vertex of $C^0 - w$ has an L_ψ-list of size at least three, ψ_j extends to an L-coloring ψ_j' of $V(C^0 \setminus \text{Mid}^0(P_j)) \cup \{z_{1-j}\}$.Since $\mathcal{A}$ is defective, the tuple $[V(C^0) \cup \{z_{1-j}\}; Q; \psi_j'; \text{Mid}^0(P_j)]$ is not a cycle connector for $\mathcal{A}$. Since $\text{Mid}^0(P_j)$ is $L_{\psi_j'}$-inert, there exists a $q_j \in T'(V(C^0) \cup \{z_{1-j}\}; Q; \psi_j') \setminus \text{Mid}^0(P_j)$.

Claim 12.4.33. $\{q_0, q_1\} \cap D_1(F_1) \neq \varnothing$.

Proof: Suppose toward a contradiction that $q_0, q_1 \notin D_1(F_1)$. For each $j = 0, 1$, let $C^0_{z_j}$ be the 5-cycle obtained from C^0 by replacing the middle vertex of $G[N(z_j) \cap V(C^0)]$ with z_j. Then $\text{dom}(\psi_j') \cap N(q_j) \subseteq V(C^0_{z_{1-j}})$, since $C^0_{z_{1-j}}$ separates q_j from F_0.

Since $|L^Q_{\psi_j'}(q_j)| < 3$, q_j has at least three neighbors in $V(C^0_{z_{1-j}})$. Since $C^0_{z_{1-j}}$ is a 5-cycle and G is short-separation-free, the graph $G[N(q_j) \cap V(C^0_{z_{1-j}})]$ is a subpath of $C^0_{z_{1-j}}$ of length precisely two. We claim now that, for each $j = 0, 1$, $q_j \in N(z_{1-j})$. Suppose there is a $j \in \{0, 1\}$ with $q_j \notin N(z_{1-j})$. Then $q_j \in U$ and thus $q_j = z_j$. Since $\psi_j \in \text{Link}(P_j, C^0)$, we have $|L_{\psi_j}(z_j)| \geq 3$, and, since $|V(C^0)| = 5$ and each of z_0, z_1 is adjacent to a subpath of C^0 of length precisely two, we have $z_0 z_1 \notin E(G)$, or else G contains a separating cycle of length at most 4. Since $z_0 z_1 \notin E(G)$ and $|L_{\psi_j}(z_j)| \geq 3$, we have $|L_{\psi_j'}(z_j)| \geq 3$ as well, so $z_j \neq q_j$. Thus, our assumption that $q_j \notin N(z_{1-j})$ is false.

Thus, for each $j = 0, 1$, $G[N(q_j) \cap V(C^0_{z_{1-j}}]$ is a subpath of $C^0_{z_{1-j}}$ containing z_{1-j}. Furthermore, z_{1-j} is also not the midpoint of $G[N(q_j) \cap V(C^0_{z_{1-j}})]$, or else q_j, z_{1-j} and the midpoint of P_{1-j} are all adjacent to the endpoints of P_{1-j}, contradicting the fact that G is $K_{2,3}$-free. Likewise, w is not the midpoint of the path $G[N(q_j) \cap V(C^0_{z_{1-j}})]$, or else the deletion of $V(C^0_{z_{1-j}} \setminus \{w\})$ separates w from F_1, which is false since w has a neighbor in F_1.

Thus, for each $j = 0, 1$, the graph $G[N(q_j) \cap V(C^0_{z_{1-j}})]$ is a subpath of $C^0_{z_{1-j}} - w$ with z_{1-j} as an endpoint, and the other two vertices of $G[N(q_j) \cap V(C^0_{z_{1-j}})]$ are the endpoints of an edge of C^0. In particular, we have $q_0 \neq q_1$. If each of z_0, z_1 is adjacent to w, then, since $w \notin \text{Mid}^0(C^0)$, each of q_0, q_1 is adjacent to both vertices of the lone edge of $C^0_{z_0} \cap C^0_{z_1}) \setminus \{w\}$. Since C^0 is a facial subgraph of $G \setminus F_0$ and each of q_0, q_1 is adjacent to both endpoints of an edge of C^0, which is false, as G is short-separation-free.

Thus, there is at least one $j \in \{0, 1\}$ such that $w \notin V(P_j)$, say $j = 1$ without loss of generality. Since $|V(C^0)| = 5$ and $w \notin \text{Mid}^0(C^0)$, w is an endpoint of P_0. Let p be the non-z_0-endpoint of $G[N(q_1) \cap V(C^0_{z_0})]$. Since $w \notin V(P_1)$, the paths P_0, P_1 intersect on a common endpoint which is not w, so p is also the midpoint of P_1, contradicting the fact that G contains a 2-chord of C^0 with midpoint z_1 which separates q_1 from the midpoint of P_1. This completes the proof of Claim 12.4.33. ■

Applying Claim 12.4.33, let $j \in \{0, 1\}$ with $q_j \in D_1(F_1)$. Since $q_j \in D_1(F_1)$, it follows from 1) of Claim 12.4.30 that q_j has no neighbors in A^0. Since $|L^Q_{\psi_j'}(q_j)| < 3$, it follows that $(N(q_j) \setminus V(Q)) \cap \text{dom}(\psi_j')$ consists of w and an edge of $F_1 \setminus Q$. Thus, $|V(F_1)| = 4$, and, by our choice of Q, both of q_j, v are adjacent to both endpoints of the edge yy_v, which is false, as G is short-separation-free. This completes the proof of Lemma 12.4.28. $\square$

We use the following smple observation several times in the remainder of Section 12.4.

Observation 12.4.34. *Let $\mathcal{A} := (G, F_0, F_1, L, \psi)$ be a defective roulette wheel with $d(F_0, F_1) = 2$ and suppose that, for some $i \in \{0, 1\}$, F_i has precisely one anchor vertex. Let C^i be the 1-band of F_i. If there exists a $w^* \in D_1(F_0) \cap D_1(F_1)$ with $w \neq w^*$, then ww^* is an edge of C^i and there exists an $x \in V(F_0)$ and a $y \in V(F_1)$ with $N(w) \cap V(F_1) = N(w^*) = \{y\}$ and $N(w) \cap V(F_0) \cap N(w^*) \cap V(F_0) = \{x\}$. In particular, $N(w) \cap N(w^*) = \{x, y\}$.*

Proof. Suppose without loss of generality that $i = 0$ and let x be the lone ahcor vertex of F_0. Since $|D_1(F_0) \cap D_1(F_1)| \geq 2$ and $\mathcal{A}$ is defective, it follows from Proposition 12.4.5 that there is a lone anchor vertex in F_1, so let $y \in V(F_1)$ with $N(w) \cap V(F_1) = N(w^*) \cap V(F_1) = \{y\}$. Then G contains the 4-cycle $xwyw^*$. We have $xy \notin E(G)$ since $d(F_0, F_1) = 2$. By our triangulation conditions, we have $ww^* \in E(G)$. By By Ro4 of Definition 12.3.1, ww^* is not a chord of C^0, so $ww^* \in E(C^0)$. Since G is $K_{2,3}$-free, we have $N(w) \cap N(w^*) = \{x, y\}$. $\square$

With the above in hand, we prove the following lemma:

Lemma 12.4.35. *Let $\mathcal{A} := (G, F_0, F_1, L, \psi)$ be a defective roulette wheel with $d(F_0, F_1) = 2$ and suppose that, for some $i \in \{0, 1\}$, F_i has precisely one anchor vertex. Let C^i be the 1-band of F_i and let $S := D_1(F_0) \cap D_1(F_1)$ and let H_* be the subgraph of G induced by the vertex set $S \cup \bigcup_{w \in S, u \in A^i_w} V(H^i_w)$. Then there exists a subpath R of C^0 such that the following holds:*

i) Either $V(R) = V(C^0)$ or $V(R) = V(H_*)$. In the former case, there is a $v \in V(C^0 \setminus H_*)$ such that $V(C^0) = V(H_*) \cup \{v\}$. In the latter case, R is a proper subpath of C^0 whose endpoints do not have a common neighbor in C^0; AND

ii) There exists a $\phi \in \text{Link}(R, C^0)$ such that $|L_\phi(v)| \geq 3$ for all $v \in V(C^0) \setminus (\text{dom}(\phi) \cup \text{Mid}^0(R))$; AND

iii) For any $w \in S$ and $u \in A^0_w$, if there exists a $v \in A^{1-i}_w$ such that $N(v) \cap N(u) \not\subseteq \{w\}$, then, for all $c \in L_\psi(u)$, there exists a $\phi \in \text{Link}(P, C^0)$ such that either $\phi(c) = c$ or $u \in \text{Mid}^0(R)$, and furthermore, $|L_\phi(v)| \geq 3$ for all $v \in V(C^0) \setminus (\text{dom}(\phi) \cup \text{Mid}^0(R))$; AND

iv) If $R \neq C^0$ then R is an induced subpath of C^0, and if $R = C^0$ then C^0 is induced in in G.

Proof. Without loss of generality, let $i = 0$. and we fix a lone anchor vertex $x \in V(F_0)$. We fix an element $w \in D_1(F_0) \cap D_1(F_1)$. Applying Applying Lemma 12.4.18, we also fix vertices $u_0 \in A^0$ and $u_1 \in A^1_w$. Applying Observation 12.4.34, we fix a vertex $y \in V(F_1)$ such that $y \in N(w') \cap V(F_1)$ for each $w' \in D_1(F_0) \cap D_1(F_1)$. We now have the following:

Claim 12.4.36. *$V(H_*) \neq V(C^0)$, and H_* is an induced subpath of C^0 of length at most five.*

<u>Proof:</u> Suppose toward a contradiction that H_* is a cycle. If $D_1(F_0) \cap D_1(F_1) = \{w\}$, then $H^0_{u_0}$ is a path of length $|V(F_i)|$ whose endpoints are both adjacent to w, contradicting Lemma 12.4.28, so there exists a $w^* \neq w$ such that $D_1(F_0) \cap D_1(F_1) = \{w, w^*\}$, and, for each $u \in A^0$, H^i_u is a path of length $|V(F_i)|$ with one endpoint adjacent to w and the other endpoint adjacent to w^*. In particular, we have $|V(C^0)| \leq 6$.

Let K be the subgraph of G induced by $V(F_0 \cup F_1) \cup V(C^0) \cup \{u_1\}$. Since u_0 has two neighbors in F_0 and u_1 has two neighbors in F_1, K contains an (F_0, F_1)-path with internal vertices u_0wu_1 which is disjoint to xw^*y. Thus,

K is 2-connected. Let D be a facial subgraph of K. Since C^0 separates $V(F_0)$ from $V(F_1) \cup \{v\}$, we have either $V(D) \subseteq V(F_0) \cup V(C^0)$ or $V(D) \subseteq V(F_1 \cup C^0)) \cup \{u_1\}$. In either case, since $|V(C^0)| \leq 6$, we have $|V(D)| \leq 11$, so our choice of K satisfies S1, contradicting the fact that $\mathcal{A}$ is defective. We conclude that H_* is not a cycle. By By Ro4 of Definition 12.3.1, H_* is an induced subpath of C^0. ■

Now we return to the proof of Lemma 12.4.35. We have the following:

Claim 12.4.37.

1) $\text{Link}(H_*, C^0) \neq \varnothing$*;AND*

2) For any $w' \in D_1(F_0) \cap D_1(F_1)$ *and* $u \in A^0_{w'}$*, if* Q *is the unique subpath of* $H_* - u$ *with* w' *as an endpoint, then, for each* $c \in L_\psi(w')$*, there is a* $\phi \in \text{Link}(Q, C^0)$ *with* $\phi(w') = d$*;' AND*

3) For any $w' \in D_1(F_0) \cap D_1(F_1)$ *and* $u \in A^0_{w'}$*, if* $u \in \text{Mid}^0(C^0)$ *and there exists a* $u' \in A^1_{w'}$ *such that* u, u' *have a common neighbor* z *with* $z \neq w$*, then* $G[N(z) \cap V(C^0]$ *is a subpath of* C^0 *of length precisely two, with* u *as its midpoint and* w' *as an endpoint; AND*

4) For any $w' \in D_1(F_0) \cap D_1(F_1)$*, if there exists a* $u \in A^0_w$ *and a color* $c \in L_\psi(u)$ *such that no element of* $\text{Link}(H_*, C^0)$ *uses the color* c *on* u*, then* $u \in \text{Mid}^0(H_*)$*.*

Proof: If each vertex of H_* has an L_ψ-list of size at least three, then, by Theorem 1.7.5, we immediately have $\text{Link}(H_*, C^0) \neq \varnothing$ in that case. Now suppose there is a vertex of H_* with an L_ψ-list of size less than three. Then, by Observation 12.4.34, this vertex is the lone vertex of $D_1(F_0) \cap D_1(F_1)$, and $D_1(F_0) \cap D_1(F_1) = \{w\}$.

Let P, P' be the two subpaths of H_* with endpoint w. Since w has at most four neighbors in $V(F_0 \cup F_1)$, let $c \in L_\psi(w)$. By Theorem 1.7.5, there exist $\phi \in \text{Link}(P, C^0)$ and $\phi' \in \text{Link}(P', C^0)$ with $\phi(w) = \phi'(w) = c$. Since $w \notin \text{Mid}^0(C^0)$, the union $\phi' \cup \phi''$ lies in $\text{Link}(H_*, C^0)$. This proves 1).

Now we prove 2). Let $c \in L_\psi(w')$. If w' is the oly vertex of $D_1(F_0) \cap D_1(F_1)$ lying in Q, then it immediate follows from Theorem 1.7.5 that there is a $\phi \in \text{Link}(Q, C^0)$ with $\phi(w') = c$. Now suppose that there is a $w^* \in D_1(F_0) \cap D_1(F_1)$ with $w^* \neq w'$ and $w^* \in V(Q)$. By Observation 12.4.34, we have $\{w', w^*\} = D_1(F_0) \cap D_1(F_1)$, $w'w^*$ is a terminal edge of Q, and each fo w', w^* has an L_ψ-list of size at least three. Let $d \in L_\psi(w^*)$ with $d \neq c$. Again applying Theorem 1.7.5, we get that that there is a $\phi \in \text{Link}(Q - w', C^0)$ with $\phi(w^*) = d$. Let ϕ' be the extension of ϕ to $\text{dom}(\phi) \cup \{w'\}$ obtained by coloring w' with c. Since neither w^* nor w' lies in $\text{Mid}^0(Q)$, we have $\phi' \in \text{Link}(Q, C^0)$, so we are done.

Now we prove 3). Let z be a common neighbor of u, u'. Since $u \in \text{Mid}(C^0)$, we have $\deg_G(u) = 5$, and z is the unique vertex of $D_1(C^0) \setminus V(F_0)$ such that $G[N(z) \cap V(C^0)]$ is a subpath of C^0 with u as an internal vertex. Thus, $u' \notin N(u)$. Furthermore, G contains the 4-cycle $zuu'w'$, and since $uu' \notin E(G)$, we have $zw' \in E(G)$ by our triangulation conditions. Let p be the unique neighbor of u on the path $C^0 - w$, and let p' be the other neighbor of p on the cycle C^0. Then the path $w'zp$ lies in $N(u)$, and since $w \notin \text{Mid}(C^0)$, the path $G[N(z) \cap V(C^0)]$ has w' as an endpoint. We just need to show that $p' \notin N(z)$.

Firstly, since $u' \in A^0$, there is a $y' \in N(u') \cap V(F_0)$ with $y' \neq y$. Suppose that $p' \in N(z)$. It follows that $p \in A^0$, or else we contradict short-separation-freeness. and, since p is adjacent to u, it follows that p has a neighbor $x' \in V(F_0)$ such that either x' is opposite to x in F_0 or $|V(F_0)| = 3$. In either case, since G contains the (F_0, F_1)-paths $x'pzu'y'$ and $xw'y$, it follows from Proposition 12.4.3 that $\mathcal{A}$ is not defective, contradicting our assumption. This proves 2).

Now we prove 4). Let $u \in A^0_w$ and $c \in L_\psi(u)$. Suppose that $u \notin \text{Mid}^0(H_*)$. We now construct an element of $\text{Link}(H_*, C^0)$ using the color c on u. Let P, P' be the two subpaths of H_* with w' as an endpoint, where $u \in V(P)$. Applying Theorem 1.7.5, there is a $\phi \in \text{Link}(P - w', C^0)$ with $\phi(u) = c$. Since $|L_\psi(w')| \geq 2$, let $d \in L_\psi(w) \setminus \{w'\}$. Applying Fact 2), there is a $\phi' \in \text{Link}(P', C^0)$ with $\phi'(w') = d$. Since H_* is an induced proper subpath of C^0, the union $\phi \cup \phi'$ is a proper L-coloring of its domain. Since neither u nor w' lies in $\text{Mid}^0(H_*)$, we have $\phi \cup \phi' \in \text{Link}(H_*, C^0)$, so we are done. ■

We now break Lemma 12.4.35. We deal with the easier case first:

Case 1 of Lemma 12.4.35: The endpoints of H_* do not have a common neighbor in C^0.

In this case, we claim that the choice of path $R := H_*$ satisfies Lemma 12.4.35. By Claim 12.4.36, H_* is an induced subpath of C^0, so condition iv) of Lemma 12.4.35 is satisfied.

Let p, p' be the endpoints of H_*. Let q be the unique neighbor of p on the path $C^0 \setminus H_*$, and let q' be the unique neighbor of p' on the path $C^0 \setminus H_*$. By assumption, we have $q \neq q'$, and, by definition of H_*, each of q, q' has precisely one neighbor in F_0. Since $q, q' \notin D_1(F_0) \cap D_1(F_1)$, each of q, q' has an L_ψ-list of size at least four.

By 1) of Claim 12.4.37, there exists a $\phi \in \text{Link}(H_*, C^0)$. By By Ro4 of Definition 12.3.1, there is no chord of C^0, so each of q, q' has an L_ϕ-list of size at least three, and each vertex of $C^0 \setminus (V(H_*) \cup \{q, q'\})$ also has an L_ϕ-list of size at least three. Thus, R satisfies condition ii) of 12.4.35.

Now we show that iii) holds. Fix a color $c \in L_\psi(u_0)$. Suppose further that there exists a $u' \in A^1_w$ such that $N(u_0) \cap N(u') \not\subseteq \{w\}$. Let $z \in N(u_0) \cap N(u')$ with $z \neq w$. Since the endpoints of H_* do not have a common neighbor, we just need to show that there is a $\phi \in \text{Link}(H_*, C^0)$ such that either $\phi(u_0) = c$ or $u_0 \in \text{Mid}^0(H_*)$.This immediately follows from 4) of Claim 12.4.37, so we are done.

Case 2 of Lemma 12.4.35: The endpoints of H_* have a common neighbor in C^0.

Let p, p' be the endpoints of H_* and let q be their common neighbor in C^0. Note that, by definition of H_*, we have $p, p' \notin A^0$, so three vertices p, q, p' have a common neighbor in F_0. In particular, since G is $K_{2,3}$-free, we have $q \notin \text{Mid}^0(C^0)$.

We now show that our choice $R = C^0$ satisfies the requirements of Lemma 12.4.35. Since C^0 is an induced subgraph of G, Condition iv) of Lemma 12.4.35 is satisfied. We just need to check ii) and iii). Let $U \subseteq D_1(C^0) \setminus V(F_0)$ be the set of vertices with at least three neighbors in C^0.

Claim 12.4.38. $|L_\psi(w)| > 2$.

Proof: Suppose toward a contradiction that $|L_\psi(w)| \leq 2$. Since F_0 has precisely one anchor vertex, w is adjacent to an edge of F_1, and $|L_\psi(w)| = 2$. Thus, $w \in A^1$.

Let $c \in L_\psi(w)$ and let P, P' be the two subpaths of H^1_w with w as an endpoint. By Theorem 1.7.5, there is a $\phi \in \text{Link}(P, C^1)$ and a $\phi' \in \text{Link}(P', C^1)$ with $\phi(w) = \phi'(w) = c$. Since $w \notin \text{Mid}^1(C^1)$, we have $\phi \cup \phi' \in \text{S}^1(H^1_w)$. Let v be the lone vertex of $C^1 \setminus H^1_w$ adjacent to the non-w-endpoint of P, and let v' be the lone vertex of $C^1 \setminus H^1_w$ adjacent to the non-w-endpoint of P'. By definition of H^1_w, each vertex of v, v' has an L_ψ-list of size at least four, since C^1 has no chord with an endpoint in A^1. Furthermore, if $v = v'$, then we contradict Lemma 12.4.28. Thus, $v \neq v'$, and each vertex of $C^1 \setminus H^1_w$ has an $L_{\phi \cup \phi'}$-list of size at least three.

Then there is a $\phi \in \text{Link}(H^1_w, C^1)$ such that $L_\phi(v)| \geq 3$ for all $v \in V(C^1 \setminus R^1)$. By Ro4 of Definition 12.3.1, H^1_w is chordless subpath of C^1 so ϕ is a proper L-coloring of its domain in G. Now let P, P' be the two subpaths of $C^0 - q$ which intersect precisely on w and whose union is $C^0 - q$. By Theorem 1.7.5, there is a $\sigma \in \text{Link}(P, C^0)$ with $\sigma(w) = \phi(w)$ and a $\sigma'(w) \in \text{Link}(P', C^0)$ with $\sigma'(w) = \phi(w)$. Since $w \notin \text{Mid}^0(C^0)$, we have $\sigma \cup \sigma' \in \text{Link}(C^0 - w, C^0)$.

We claim now that $\sigma \cup \sigma' \cup \phi$ is a proper L-coloring of its domain. If this does not hold, then there is an edge of G with one endpoint in $A^0 \setminus \{w\}$ and one endpoint in $A^1 \setminus \{w\}$. In that case, G contains an (F_0, F_1)-path of length three which is disjoint to xwy. Applying Proposition 12.4.3, this contradicts the fact that $\mathcal{A}$ is defective. Thus, $\tau := \sigma \cup \sigma' \cup \phi$ is indeed a proper L-coloring of its domain. Now, if q has a neighbor q' in $\text{dom}(\tau) \setminus V(C^0)$, then, since $w \in A^1$ and $q \notin D_1(F_1)$, G contains two disjoint (F_0, F_1)-paths of length three which respective internal edges qq' and wu_0, where either $|V(F_1)| = 3$ or the F_1-endpoints of these two paths are nonadjacent. In either case, applying 2) of Proposition 12.4.3, we contradict the fact that $\mathcal{A}$ is defective. Thus, we have $|L_\tau(q)| \geq 1$.

Now, the tuple $[V(C^0 - q) \cup V(H^1_w); q; \tau; \text{Mid}^0(C^0) \cup \text{Mid}^1(H^1_w)]$ is not a cycle connector for $\mathcal{A}$, so there is a vertex $z \notin V(C^0) \cup V(H^1_w)$ with at least three neighbors in $\text{dom}(\tau)$ and $|L_\tau(z)| < 3$. Since $\sigma \cup \sigma' \in \text{Link}(C^0 - w, C^0)$ and $\phi \in \text{Link}(H^1_w, C^1)$, and each vertex of $C^1 \setminus H^1_w$ has an L_ϕ-list of size at least three, it follows that z has at least one neighbor in $\text{dom}(\phi) \setminus \{w\}$ and at least one neighbor in $\text{dom}(\sigma \cup \sigma') \setminus \{w\}$, or else $|L_\tau(z)| \geq 3$.

Let K be the subgraph of G induced by $V(F_0 \cup F_1 \cup C^0) \cup \{z\}$. Since z has a neighbor in A^0 and a neighbor in $A^1 \setminus \{w\}$, K contains an (F_0, F_1)-path of length three which is disjoint to xwy, so K is 2-connected. By Observation 12.4.34, since w has two neighbors in F_1, we have $\{w\} = D_1(F_0) \cup D_1(F_1)$, and thus $|V(C^0)| \leq 6$. Since C^0 separates F_0 from $F1$, any facial subgraph of K has length at most 11, contradicting the fact that $\mathcal{A}$ is defective. ■

Applying Claim 12.4.38, each vertex of C^0 has an L_ψ-list of size at least three. Now we show that $\text{Link}(C^0) \neq \varnothing$. Consider the following cases:

Case 1: $U \cap N(q) = \varnothing$

In this case, neither neighbor of q in C^0 lies in $\text{Mid}^0(C^0)$, and, by Theorem 1.7.5, since each vertex of $C^0 - q$ has an L_ψ-list of size at least three, there is a $\phi \in \text{Link}(C^0 - q, C^0)$. Since $|L_\psi(q)| \geq 4$, there is a color left over for q, so we extend ϕ to q and let ϕ' be the resulting coloring. Then $\phi' \in \text{Link}(C^0)$, and we are done.

Case 2: $U \cap N(q) \neq \varnothing$

In this case, let P, P' be the two subpaths of C^0 with endpoints w, q and let $z \in U \cap N(q)$. Since $q \notin \text{Mid}^0(C^0)$, $G[N(z) \cap V(C^0)]$ is a subpath of C^0 with q as an endpoint, so suppose for the sake of definiteness that $G[N(z) \cap V(C^0)]$ is a subpath of P'. Let v^m be the lone vertex of $G[N(z) \cap V(C^0)]$ adjacent to q. Since $|L_\psi(q)| \geq 4$, let $d \in L_\psi(q)$ be such that $L_\psi(v^m) \setminus \{d\}| \geq 3$. Now, by Theorem 1.7.5, since $|L_\psi(w)| \geq 3$, there is a $\phi \in \text{Link}(P', C^0)$ such that $\phi(q) = d$. Let v be the non-q-endpoint of $G[N(z) \cap V(C^0)]$. Again by Theorem 1.7.5, there is a $\phi^* \in \text{Link}(wPv, C^0)$ such that $\phi(w) = \phi^*(w)$. Thus, $\phi \cup \phi^*$ is a proper L-coloring of its domain, and, by our choice of d, the path $G[N(z) \cap V(C^0)] \setminus \{q, v\}$ is $L_{\phi \cup \phi^*}$-inert, so $\phi \cup \phi^* \in \text{Link}(C^0)$.

Thus, our choice of R satisfies ii) of our Lemma. Now suppose that $u_0 \notin \text{Mid}^0(C^))$ and fix a $c \in L_\psi(u_0)$. To finish, it suffices to show that there is a $\phi \in \text{Link}(C^0)$ using c on u_0. Let P be the subpath of $C^0 - u_0w$ with endpoints u_0, q, and let P' be the subpath of $C^0 - u_0w$ with endpoints q, w. By Theorem 1.7.5, since $|L_\psi(q)| \geq 4$, there is a pair of elements $\phi, \phi' \in \text{Link}(P, C^0)$ which both use c on u_0 and which use distinct colors on q. By Claim 12.4.38, we have $L_\psi(w) \setminus \{c\}| \geq 2$. Thus, by Theorem 1.7.5, there is a $\sigma \in \text{Link}(P', C^0)$ using one of $\{\phi(q), \phi'(q)\}$ on q and a color of $L_\psi(w) \setminus \{c\}$ on w, say $\sigma(q) = \phi(q)$ without loss of generality. The union $\sigma \cup \phi$ is a proper L-coloring of its domain. Since $w, u_0, q \notin \text{Mid}^0(C^0)$, we have $\sigma \cup \phi \in \text{Link}(C^0)$. This completes the proof of Lemma 12.4.35. □

We now come to the final proposition we need in order to prove Theorem 12.4.1.

Proposition 12.4.39. *Let $\mathcal{A} := (G, F_0, F_1, L, \psi)$ be a roulette wheel with $d(F_0, F_1) = 2$ and suppose that, for some $i \in \{0, 1\}$, F_i has precisely one anchor vertex. Then $\mathcal{A}$ is not defective.*

Proof. Suppose without loss of generality that F_0 has precisely one anchor vertex x. By Proposition 12.4.5, either $|D_1(F_0) \cap D_1(F_1)| = 1$ or there exists a $y \in V(F_1)$ such that $N(w) \cap V(F_1) = \{y\}$ for all $w \in D_1(F_0) \cap D_1(F_1)$. In the latter case, $|D_1(F_0)) \cap D_1(F_1)| \leq 2$, since G is $K_{2,3}$-free. Thus, in any case, we fix a vertex $y \in V(F_1)$ such that $y \in N(w)$ for all $w \in D_1(F_0) \cap D_1(F_1)$. We also fix a vertex $w \in D_1(F_0) \cap D_1(F_1)$.

Let H_* be the subgraph of G induced by $\bigcup(\{w'\} \cup A^0_{w'} : w' \in D_1(F_0) \cap D_1(F_1))$, and let R be a subgraph of C^0 satisfying Lemma 12.4.35, where either $R = C^0$ or $R = H_*$, and in the latter case, R is a subpath of C^0 consisting of all but one vertex of C^0, and $R = H_*$. Let P be the set of vertices of $C^1 \setminus (D_1(F_0) \cap D_1(F_1))$ with at least two neighbors in R.

Claim 12.4.40. *Let $u_0 \in A^0_w$ and $u_1 \in A^1_w$. Then the following hold.*

1) For any $z \in N(u_1) \cap V(C^0)$, we have $N(z) \cap V(F_0) = \{x\}$. In particular, $N(u_1) \cap A^0 = \varnothing$, and $N(u_1) \cap V(R)| \leq 2$; AND

2) If there is a $p \in P$, then $D_1(F_0) \cap D_1(F_1) = \{w\}$, $N(p) \cap V(F_1) = N(w) \cap V(F_1) = \{y\}$, and $G[N(p) \cap V(R)]$ is an edge with w as an endpoint. In particular, $P = \{p\}$; AND

3) There is at most one vertex lying in $N(u_1) \cap (B_1(C^0) \setminus R)$, and, if z is such a vertex, then $N(z) \cap V(R)$ consists of at most w and one vertex of A^0_w.

Proof: Let z be a neighbor of u_1 in $V(C^0)$ and suppose that there is an $x' \neq x$ with $x' \in N(z)$. Then G contains the (F_0, F_1)-path $x'zu_1$. Ssince u_1 has a neighbor in $F_1 \setminus \{y\}$ and G contains the path xwy, it follows from 1) of Proposition 12.4.3 that $\mathcal{A}$ is not defective, contradicting our assumption. Thus, $N(z) \cap V(F_0) = \{x\}$, so we have $N(u_1) \cap A^0 = \varnothing$. By definition of R from Lemma 12.4.35, $R \setminus A^0$ consists of $D_1(F_0) \cap D_1(F_1)$ and at most one other vertex. By Observation 12.4.34, u_1 has at most one neighbor in $D_1(F_0) \cap D_1(F_1)$, so we indeed have $|N(u_1) \cap V(R)| \leq 2$. This proves 1).

Now we prove 2). Let $p \in V(C^1) \setminus (D_1(F_0) \cap D_1(F_1))$ and suppose that $N(p) \cap V(R)| > 1$.

Subclaim 12.4.41. $N(p) \cap V(F_1) = \{y\}$.

Proof: Suppose that there is a $y' \in N(p) \cap V(F_1)$ with $y \neq y'$. In that case, for any $u \in N(p) \cap V(C^0)$, we have $N(u) \cap V(F_0) = \{x\}$, or else G contains an (F_0, F_1)-path of length at most three which is disjoint to xwy. Applying Proposition 12.4.3, this contradicts the fact that $\mathcal{A}$ is defective. Since each neighbor of p in C^0 is adjacent to x, the graph $G[N(p) \cap V(C^0)]$ is a subpath of C^0 of length at most one, or else G contains a separating cycle of length at most four. Thus, since $N(p) \cap V(R)| > 1$, the graph $G[N(p) \cap V(C^0)]$ is an edge of $R \setminus \text{Mid}^0(R)$, each vertex of which lies outside of A^0. By Observation 12.4.35, z is adjacent to at most one vertex of $D_1(F_0) \cap D_1(F_1)$. Thus, by definition of R, there exists a $w' \in D_1(F_0) \cap D_1(F_1)$ and a neighbor v of w' such that $G[N(p) \cap V(C^0)] = w'v$, and $R = C^0 = H_* + v$.

Now let K be the subgraph of G induced by $V(F_0 \cup F_1 \cup C^0) \cup \{p\}$. Note that K is 2-connected, since K contains the paths $xvpy'$ and u_0wy, where u_0 has two neighbors in F_0 . Since $\mathcal{A}$ is defective, there is a facial subgraph D of K with $|V(D)| > 11$. Since C^0 separates F_0 from $V(F_1) \cup \{p\}$, we have either $V(D) \subseteq V(F_0) \cup V(C^0)$ or

$V(D) \subseteq V(F_1 \cup C^0) \cup \{p\}$. Since $R = H_* + v$, we have $|V(C^0)| \leq 7$, so $V(D) = V(F_1 \cup C^0) \cup \{p\})$ and $|V(C^0)| = 7$. In particular, $|D_1(F_0) \cap D_1(F_1)| = 2$, or else, since $C^0 = H_* + v$, we have $|V(C^0)| \leq 6$. THus, let $D_1(F_0) \cap D_1(F_1) = \{w, w^*\}$.

Since D is a facial subgraph of K and $V(D) = V(F_1 \cup C^0) \cup \{p\})$, y' is adjacent to y, since K does not contain a generalized chord of F_1 whose endpoints are notadjacent in F_1. Thus, G contains the 4-cycle $yy'pw'$. By Observation 12.4.34, we have $y' \notin N(w')$, so $y \in N(p)$ by our triangulation conditions. Furthermore, the two neighbors of w' on the cycle C^0 are v and the lone element of $(D_1(F_0) \cap D_1(F_1)) \setminus \{w'\}$, so K contains a 5-cycle, each vertex of which is adjacent to w'. But then, since $|V(D)| > 11$, we have $w' \notin V(D)$, contradicting the fact that $V(D) = V(F_1 \cup C^0) \cup \{p\})$. ■

Subclaim 12.4.41 implies that $|D_1(F_0) \cap D_1(F_1)| = 1$. If thius does not hold, then, by Observation 12.4.34, there exists a $w^* \neq w$ such that $N(w) \cap N(w^*) = \{x, y\}$. By Lemma 12.4.18, there is a $u_1^* \in A^1_{w^*}$, and since $N(w) \cap N(w^*) = \{x, y\}$, we have $u_1^* \neq u_1$. Let y_1 be the non-y-endpoint of the edge R^1_y and let y_1^* be the non-y-endpoint of the edge $R^1_{u_1^*}$. Then G contains the 4-chord $y_1 u_1 w u_1^* y_1^*$, each vertex of which is adjacent to y. Since G is short-separation-free, we have $N(y) = \{y_1, u_1, w, u_1^*, y_1^*\}$. Since $|N(p) \cap V(F_1)| = 1$ and $p \notin D_1(F_0) \cap D_1(F_1)$, this contradicts the fact that $p \in N(y)$. Thus, we have $D_1(F_0) \cap D_1(F_1)| = 1$. In particular, $D_1(F_0) \cap D_1(F_1) = \{w\}$.

Subclaim 12.4.42. *For each $u \in N(p) \cap V(R)$, either $u \in A^0_w$ or $N(u) \cap V(F_0) = \{x\}$.*

Proof: We first note that for any $u \in N(p)$, each vertex of $N(p) \cap V(F_0)$ is either x or adjacent to x. If this does not hold, then there is a $u \in N(p)$ and an $x' \in V(F_0) \cap N(u)$ nonadjacent to x. In that case, since $N(p) \cap V(F_1) = \{y\}$, G contains the disjoint paths xwu_1 and $x'upy$. Since $u_1 \in A^1$, u_1 has a neighbor in $F_1 \setminus \{y\}$. Applying 2) of Proposition 12.4.3, we contradict the fact that $\mathcal{A}$ is defective.

Thus, if the subclaim does not hold, then there is an edge xx^* of F_0 incident to x and a $v^* \in N(p) \cap V(R)$ with $N(v^*) \cap V(F_0) = \{x^*\}$. Since $v^* \in V(R)$ and $v^* \notin A^0$, it then follows by definition of H_*, R that $R = H_* + v^*$. Let K be the subgraph of G induced by $V(F_0 \cup F_1) \cup V(C^0) \cup \{p, u_1\}$. Since G contains the paths xwu_1 and x^*v^*py, and u_1 has a neighbor in $F_1 \setminus \{y\}$, K is 2-connected. Since $\mathcal{A}$ is defective, let D be a facial subgraph of K with $|V(D)| > 11$. Since C^0 separates $V(F_1)$ from $V(F_1) \cup \{p, u_1\}$, we have either $V(D) \subseteq V(F_0 \cup C^0)$ or $V(D) \subseteq V(F_1 \cup C^0) \cup \{p, u_1\}$. Since $|D_1(F_0) \cap D_1(F_0)| = 1$ and $R = H_* + v_*$, we have $|V(C^0)| \leq 6$, so $V(D) \subseteq V(F_1 \cup C^0) \cup \{p, u_1\}$. Since K contains the 5-chord x^*v^*pywx of F_0, D does not contain all of the vertices of C^0, so $|V(C^0 \cap D)| \leq 5$, and thus $|V(D)| \leq 11$, which is false. ■

Applying Subclaim 12.4.42, each neighbor of p in $V(R)$ is adjacent to x. By 2) of Proposition 12.3.6, $G[N(p) \cap V(R)]$ is an edge of C^0, and, by definition of R, this edge has w as an endpoint. If $|P| \geq 2$, then w, y have two common neighbors in P. Since w, y are both adjacent to u_1 and $A^1 \cap P = \varnothing$, this contradicts the fact that G is $K_{2,3}$-free. Thus, $|P| \leq 1$. To finish the proof of 2), it just suffices to check that, if $P \neq \varnothing$, then $N(w) \cap V(F_1) = \{y\}$. Suppose there is a $y' \in N(w)$ with $y' \neq y$. By 2) of Lemma 12.3.8, we then have $N(y) \subseteq V(F_1) \cup \{w, u_1\}$, contradicting the fact that $p \in N(y)$. This proves 2) of Claim 12.4.40. Now we prove 3). Let $z \in N(u_1) \cap (B_1(C^0) \setminus R)$.

Subclaim 12.4.43. $z \notin V(C^0 \setminus R)$.

Proof: Suppose that $z \in V(C^0 \setminus R)$. By 1), we have $N(z) \cap V(F_0) = \{x\}$. Thus, G contains the 4-cycle xwu_1z. Since $u_1 \notin N(x)$, we have $wz \in E(G)$, and, by Ro4 of Definition 12.3.1, we have $wz \in E(C^0)$. Since u_0 is the other neighbor of w in the cyclic order, it follows from Observation 12.4.34 that $D_(F_0) \cap D_1(F_1) = \{w\}$.

By Lemma 12.4.35, there is no chord of C^0 with an endpoint in R. Furthermore, since $z \notin V(R)$, we have $R = H_*$ by definition of R, so the neighbors of z in R consist of w and the lone endpoint of $H^0_{u_0}$ which is not adjacent to w. Thus $|V(C^0)| = 6$. Let K be the subgraph of G induced by $V(F_0 \cup F_1 \cup C^0) \cup \{u_1\}$. Since z has a neighbor in A^0 and a neighbor in A^1, K contains an (F_0, F_1)-path disjoint to xwy, so K is 2-connected. Since C^0 separates $V(F_0)$ from $V(F_1) \cup \{u_1\}$, every facial subgraph of K has length at most 11, contradicting the fact that $\mathcal{A}$ is defective. ■

Thus we have $z \in D_1(C^0)$. Now we claim that $N(z) \cap A^0 \subseteq A^0_w$. Suppose not. Let $u \in N(z)$ with $u \in A^0 \setminus A^0_w$. Since $u \in A^0 \setminus A^0_w$, we have either $|V(F_0)| = 3$, or u has a neighbor in F_0 which is not adjacent to x. Thus, since G contains the paths xwy and uzu_1, it follows from 2) of Proposition 12.4.3 that $\mathcal{A}$ satisfies S1, contradicting the fact that $\mathcal{A}$ is defective. Thus, we have $N(z) \cap A^0 \subseteq A^0_w$. If $|N(z) \cap A^0_w| > 1$, then, by 2) of Proposition 12.3.6, there is an edge of C^0 with both endpoints in A^0_w, which is false, so $N(z) \cap A^0$ consists of at most one vertex of A^0_w.

Subclaim 12.4.44. $N(z) \cap V(R \setminus A^0) \subseteq \{w\}$.

Proof: Suppose not, and let $v \in N(z) \cap V(R \setminus A^0)$ with $v \neq w$. If $x \in N(v)$, then G contains the 4-cycle $xvzu_0$, and since $v \in D_1(C^0) \setminus F_0$, we have $xv \notin E(G)$, so $vu_0 \in E(G)$. Letting x^* be the other endpoint of $G[N(u_0) \cap V(F_0)]$, each of x^*, v, w is adjacent to each of x, u_0, contradicting the fact that G is $K_{2,3}$-free. Thus, $x \notin N(z)$, so $v \notin D_1(F_0) \cap D_1(F_1)$.

By definition of R, C^0 is a cycle with $R = C^0 = H_* + v$. Let K be the subgraph of G induced by $V(F_0 \cup F_1 \cup C^0) \cup \{z, u_1\}$. Then K is 2-connected, since there is an (F_0, F_1)-path in K with internal vertices vzu_1 which is disjoint to xwy. Since $\mathcal{A}$ is defective, let D be a facial subgraph of K with $|V(D)| > 11$. Since C^0 separates F_0 from $V(F_1) \cup \{z, u_1\}$, we have either $V(D) \subseteq V(F_0 \cup C^0)$ or $V(D) \subseteq VF_1 \cup (C^0) \cup \{z, u_1\}$. Since $R = C^0 = H_* + v$, we have $|V(C^0)| \leq 7$, so $V(D) \subseteq V(F_1 \cup C^0) \cup \{z, u_1\}$. Now, K contains the 3-chord wu_1zv of C^0 with internal vertices , and since $x \notin N(v)$, the vertices w, v are not adjacent in the cyclic order of C^0, so $|V(D \cap C^0)| < |V(C^0)|$. Since $|V(D)| > 11$, we then have $|V(C^0)| = 7$, $|VC^0 \cap D)| = 6$, and $D_1(F_0) \cap D_1(F_1)$ consists of two vertices, so let $w^* \in (D_1(F_0) \cap D_1(F_1)) \setminus \{w\}$.

By Observation 12.4.34, ww^* is an edge of G and $y \in N(w^*)$. Since $V(C^0 \cap D)| = 6$, w, v have a common neighbor in R, and, by iv) or Lemma 12.4.35, $R = C^0$ is an induced cycle, so this lone neighbor is next to w in the cyclic order. Since $w^* \notin N(v)$, this common neighbor is u_0. But then G contains the 5-chord vzu_1yw^* of C^0. Since $|V(C^0)| \geq 7$ and C^0 contains the path u_0ww^*v, we then have $|V(C^0 \cap D)| \leq |V(C^0| - 2$, contradicting the fact that $|V(C^0 \cap D)| = 6$. ■

To finish, we just need to check that z is unique. Suppose not. Then there is a $z' \in N(u_1) \cap D_1(C^0)$ with $z' \neq z$, and each of z, z' has at least one neighbor among $\{w\} \cup A^0_w$. If each of z, z' are adjacent to w, then each of z, z', y is adjacent to each of w, u_1, contradicting the fact that G is $K_{2,3}$-free. Thus, we suppose without loss of generality that $z \notin N(w)$, so z has a neighbor $u \in A^0_w$, and G contains the 4-cycle $wuzu_1$. Since $N(u_1) \cap A^0 = \varnothing$, it then follows from our triangulation conditions that $wz \in E(G)$¡ which is false. This completes the proof of Claim 12.4.40. ■

We claim now that it suffices to prove that Proposition 12.4.39 in the case where $P = \varnothing$. Suppose that $P \neq \varnothing$. By 2) of Claim 12.4.40, there is a lone vertex p such that $P = \{p\}$, and $D_1(F_0) \cap D_1(F_1) = \{w\}$, where $N(w) \cap V(F_1) = \{y\}$. In particular, by the symmtery of F_0, F_1, we define a subgraph R' of C^1 satisfying Lemma 12.4.35, and we let P' be the set of vertices of $C^0 \setminus \{w\}$ with at least two neighbors in R'. We just need to show that $P' = \varnothing$.

Claim 12.4.45. $P' = \varnothing$.

Proof: Suppose that $P' \neq \varnothing$. By 2) of Claim 12.4.40, it follows from the symmetry of F_0, F_1 in this case that there is a unique vertex p' such that $P' = \{p'\}$, where $N(p') \cap V(F_0) = \{x\}$ and $G[N(p') \cap V(R')]$ is an edge of R'

with w as an endpoint. Let $G[N(p')\cap V(R')] = wv$ and $G[N(p')\cap V(R')] = wv'$ for some $v\in V(R-w)$ and $v'\in V(R'-w)$. If $p=v'$, then each of p', v is adjacent to each of x,w,p, contradicting the fact that G is $K_{2,3}$-free, so $p\neq v'$. Likewise, we have $p'\neq v$. Consider the following cases.

Case 1: $v\notin A^0$ and $v'\notin A^1$

In this case, by definition of R, R', since $D_1(F_0)\cap D_1(F_1)=\{w\}$, we have $V(C^0)=V(H^0_{u_0})\cup\{w,v\}$ and $V(C^1)=V(H^1_{u_1})\cup\{w,v'\}$, and each of C^0, C^1 is a cycle of length at most six. Let $K:=G[V(F_0\cup F_1)\cup V(C^0\cup C^1)]$. Then K is 2-connected, since it contains the paths u_0py and xwu_1, where u_0 has a neighbor in F_0-x and u_1 has a neighbor in F_1-y. Since C^0, C^1 each have length six and intersect on w, every facial subgraph of K has length at most 11, contradicting the fact that $\mathcal{A}$ is defective.

Case 2: Either $v\in A^0$ or $v'\in A^1$.

In this case, suppose without loss of generality that $v\in A^0$. If $v'\notin A^1$, then, since p, v' are the two vertices adjacent to w in the cyclic order of C^0 and $A^0_w\neq\varnothing$, we have $p\in A^1_w$. But then pv is an edge of G with one endpoint in A^0 and one endpoint in A^1, so there is an (F_0,F_1)-path in G of length three which is disjoint to xwy. Applying 1) of Proposition 12.4.3, this contradicts the fact that $\mathcal{A}$ is defective. Thus, we have $v'\in A^1$, so let xx_v be the neighborhood of v in F_0 and let $yy_{v'}$ be the neighborhood of v' in F_1.

Note that G contains a 6-cycle $xp'v'ypv$, each vertex of which is adjacent to w, so $N(w)=\{x,p',v',y,p,v\}$, as G is short-separation-free. Furthermore, we have $|V(F_0)|=|V(F_1)|=4$, or else, since G contains the paths $xp'u_1$ and u_0py, with $u_1\in A^1$ and $u_0\in A^1$, it follows from 2) of Proposition 12.4.3 that $\mathcal{A}$ is not defective. Now, let Q be the lone edge of $F_0\setminus N(v)$.

Since $L_\psi(p')|\geq 4$ and $|L_\psi(w)|\geq 3$, we fix a $d'\in L_{\psi'}(p')$ with $L_\psi(w)\setminus\{d'\}|\geq 3$. Since neither p' nor v' is adjacent to v, there is an L_ψ-coloring ϕ of $\{p',v,v'\}$ using d' on p' such that $|L_\psi(w)\setminus\{\phi(p'),\phi(v),\phi(v')\}|\geq 2$. In particular, w is $L_{\psi\cup\phi}$-inert since it has only one uncolored neighbor.

By Theorem 1.7.5, there is a $\phi^*\in\text{Link}(H^1_{v'},C^1)$ using $\phi(v')$ on v'. Since $H^1_{v'}\subseteq R'$, p' is not not adjacent to any vertex of $H^1_{v'}$ except v', and, since G has no (F_0,F_1)-path of length three disjoint to xwy, u_0 has no neighbors in H^1_v. Thus, $\phi\cup\phi^*$ is a proper L-coloring of its domain. Let $\tau=\psi\cup\phi\cup\phi^*$ and consider the tuple $[\{p',v\}\cup V(H^1_{v'}); Q:\tau; w]$. Since $\mathcal{A}$ is defective, this is not a cycle connector for $\mathcal{A}$, so there exists a $z\in V(G)\setminus(\text{dom}(\phi^*)\cup\{w\})$ such that $|L^Q_\tau(z)|<3$. Since $z\notin D_1(F_0)\cap D_1(F_1)$, z has a neighbor in $\{p',v\}\cup V(H^1_{v'})$.

Subclaim 12.4.46. *$N(z)\cap(\{p',v\}\cup V(H^1_{v'})$ consists of either $\{v\}$ or a subset of $\{p',v'\}$.*

Proof: Suppose that z has a neighbor $q\in V(H^1_{v'})-v')$. Since $\phi^*\in\text{Link}(H^1_{v'},C^1)$ and $|L^Q_\tau(z)|<3$, we have $N(z)\cap\text{dom}(\tau)\not\subseteq V(H^1_{v'})$, so z also has a neighbor in $V(F_0\cup F_1)\cup\{p',v\}$. If $z=p$, then, since there is no chord of C^1 with q as an endpoint, C^1 is a 6-cycle, and then $K:=G[V(F_0\cup F_1\cup C^1)\cup\{v\}]$ is a 2-connected graph in which every face has length at most 11, since C^1 separates F_0 from F_1 and K contains the two disjoint (F_0,F_1)-paths $xwv'y_{v'}$ and x_vvpy. This contradicts our assumption that $\mathcal{A}$ is defective, so $z\neq p$. If $z\in V(C^1)$, then z has one neighbor in $H^1_{v'}$, since there is no chord of C^1 with an endpoint in $H^1_{v'}$. But then, since z is also adjacent to one of p', v, since $|L^Q_\tau(z)|<3$. Thus, we have $N(z)\cap V(F_1)=\{y\}$.

Since $N(z)\cap V(F_1)=\{y\}$, we have $v\notin N(z)$, or else the three vertices w,p,z are each adjacent to both of y,v, contradicting the fact that G has no $K_{2,3}$-free. But then z is adjacent to each of y,p'. Since y,p' are each adjacent to w,v, this contradicts the fact that G is $K_{2,3}$-free. We conclude that $z\notin V(C^1)$, so z has a neighbor in $V(F_0\setminus Q)\cup\{v,p'\}$. Since $q\in A^1$, $x_v\notin N(z)$, or else there is an $(F_), F_1)$-path of length four disjoint to xwy, contradicting the fact that $\mathcal{A}$ is defective. Thus, z has a neighbor $q' \in \{x, v, p'\}$.

If $q' = x$, then $y \in N(q)$, or else G contains two disjoint (F_0, F_1)-paths of length four with nonadjacent endpoints in F_1, contradicting the fact that $\mathcal{A}$ is defective. In that cae, let K be the subgraph of G induced by $V(F_0 \cup F_1) \cup \{q, z, p, v, w\}$. Then K is 2-connected since it contains the paths xzq and $x_v vp$, and $q \in A^1$. Since $\mathcal{A}$ is defective, let D be a facial subgraph of G with length $|V(D)| > 11$. Then $w \notin V(D)$. Yet K also contains a generalized chord of F_0 separating p from v', so $|V(D) \cap \{q, z, p, v, w\}| < 5$, and thus $|V(D)| \leq 11$, a contradiction. A similar argument shows that $v, p' \notin N(z)$.

Thus, $N(z) \cap (\{p', v\} \cup V(H^1_{v'}) \subseteq \{v, p', v'\}$. Since p', v are adjacent to each of x, w and G is has no $K_{2,3}$, z is adjacent to at most one of p', v. If z is adjacent to each of v, v', then G contains the 4-cycle $wvz'v'$. But then, since $N(w) = \{x, p', v', y, p, v\}$, it follows from our triangulation conditions that $vv' \in E(G)$, and thus G contains a $K_{2,3}$ with bipartiton $\{x, w, v'\}$, $\{p', v\}$, contradicting the fact that G is $K_{2,3}$-free. ■

Since z is not adjacent to all three of $\{p', v, v'\}$, z has a neighbor in $V(F_0 \setminus Q) \cup V(F_1)$. Suppose first that z has a neighbor in $x^* \in V(F_0 \setminus Q)$. In that case, by our choice of Q, z has precisely one in neighbor in $F_0 \setminus Q$, or else each of v, q is adjacent to both endpoints of $G[N(v) \cap V(F_0)]$, contradicting the fact that G is short-separation-free. Thus, since z has at least three neighbors in $\text{dom}(\tau \setminus V(Q)$, it follows from Subclaim 12.4.46 that $p', v' \in N(z)$. If $x^* = x$, then G contains a $K_{2,3}$ with bipartition $\{w, p', z\}$, $\{x, v'\}$. Thus, $x^* \neq x$. But then, since $v' \in N(z)$, G contains an (F_0, F_1)-path of length three disjoint to xwy, contradicting the fact that $\mathcal{A}$ is defective.

We conclude that z does not have a neighbor in $F_0 \setminus Q$, so z has a neighbor in F_1. If $v \in N(z)$, then, by Subclaim 12.4.46, we have $p', v' \notin N(z)$, so z has a neighbor in $F_1 - y$. But then G contains an (F_0, F_1)-path of length three disjoint to xwy, contradicting the fact that $\mathcal{A}$ is defective. Thus, $v \notin N(z)$, and z has a neighbor among $\{p', v'\}$. If $v' \in N(z)$, then, by definition of $H^1_{v'}$, we have $|N(z) \cap V(F_1)| = 1$, so $p' \in N(z)$ as well, and $N(z) \cap V(F_1) = \{y_{v'}\}$ by 1) of Lemma 12.3.8.

Likewise, if $p' \in N(z)$, then $y \notin N(z)$. Thus, z has a neighbor $y' \in V(F_1 - y)$, and this neighbor y' is unique, or else, if $z \in A^1$, then z is adjacent to an edge of $F_1 - y$, and$G$ contains an (F_0, F_1) of length four which is disjoint to $x_v vwy$ and whose F_1-endpoint is nonadjacent to y. Applying 2) of Proposition 12.4.3, this contradicts. Since $|N(z) \cap V(F_1)| = 1$, we have $v' \in N(z)$ as well, and $y' = y_v$ by 1) of Lemma 12.3.8.

Thus, in any case, we conclude that $H^1_{v'} = v'$, and that $(N(z) \setminus V(Q)) \cap \text{dom}(\phi^*)$ consists of $p', v', v_{v'}$. Thus, z is the unique vertex of $T(\{p', v, v'\}; Q)$, and furthermore, G contains the 7-cycle $p'zy_{v'}ypu_0x$. Note now that there is an L_ψ-coloring σ of $\{p', v\}$ such that the edge wv' is L_σ-inert. To construct σ, we choose $\sigma(p') = d'$ as above, and let c, c' be two colors in $L_\psi(v') \setminus \{d\}$. Then,since $|L_\psi(w) \setminus \{d'\}| \geq 3$, there is a color $r \in L_\psi(w) \setminus \{c, c', d\}$, and we simply choose $\sigma(v)$ to be distinct from r, and then the edge wv' is L_σ-inert. Since $\{z\} = T(\{p', v, v'\}; Q)$ and $(N(z) \setminus V(Q)) \cap \text{dom}(\phi) = \{p', v', y'\}$, we have $T(\{p', v\}; Q) = \varnothing$. Bu then $[\{p', v\}; Q; \sigma; \{w, v'\}]$ is a cycle connector for $\mathcal{A}$, contradicting the fact that $\mathcal{A}$ is defective. ■

Thus, we suppose for the remainder of Proposition 12.4.39 that $P = \varnothing$. Applying Lemma 12.4.18, we fix a $u_0 \in A^0_w$ and $u_1 \in A^1_w$ for the remainder of the proof of Proposition 12.4.39.

et Q be an edge of F_1, where Q is the lone edge of $F_1 \setminus N(u_1)$ if $|V(F_1)| = 4$, and Q intersects with $N(u_1) \cap V(F_1)$ precisely on y if $|(F_1)| = 3$. Applying ii) of Lemma 12.4.35, we let $\phi \in \text{Link}(R, C^0)$ such that $|L_\phi(v)| \geq 3$ for all $v \in V(C^0 \setminus R)$. By 1) of Claim 12.4.40, we have $|L_\phi(u_1)| \geq 1$. Thus, ϕ extends to an L-coloring ϕ' of $\text{dom}(\phi) \cup \{u_1\}$.

Now consider the tuple $[V(R + u_1); \phi'; Q; \text{Mid}^0(R)]$. Since $\mathcal{A}$ is connected, this is not a cycle connector for $\mathcal{A}$. By our choice of Q, the graph $G[V(F_0 \cup F_1 \cup R) \cup \{u_1\}] \setminus V(Q)$ is connected, so there exists a $z \in V(G) \setminus (V(R) cup \{u_1\})$

with $|L^Q_{\phi'}(z)| < 3$.

Claim 12.4.47. *For any $\sigma \in \text{Link}(R, C^0)$ such that each vertex of $C^0 \setminus R$ has an L_σ-list of size at least three, and any extension of σ to an L-coloring of σ' of $\text{dom}(\phi) \cup \{u_1\}$, z is unique the unique vertex of $T'(R + u_1; Q : \sigma')$, and $N(z) \cap \text{dom}(\sigma')$ consists of w, u_1, and one vertex of A^0_w.*

Proof: Firstly,by our choice of Q, z has at most one neighbor in $F_1 \setminus Q$, or else $F_1 \setminus Q$ is an edge of F_1 where each of u_1, z is adjacent to both endpoints of $F_1 \setminus Q$, contradicting the fact that G is short-separation-free. Since $|L^Q_{\sigma'}(z)| < 3$, it follows that z has at least two neighbors among $V(R) \cup \{u_1\}$. If z is not adjacent to u_1, then we have $|L^Q_\sigma(z)| < 3$, contradicting our choice of σ, since $\sigma \in \text{Link}(R, C^0)$ and each vertex of $C^0 \setminus R$ also has an L_σ-list of size at least three. Thus, $u_1 \in N(z)$.

If $z \in V(C^1 \setminus R)$, then, since $P = \varnothing$ by assumption, z has at most one neighbor in R. In that case, since $|L^Q_{\phi'}(z)| \leq 3$, $N(z) \setminus V(Q)) \cap \text{dom}(\phi')$ consists of u_1, one vertex of $F_1 \setminus Q$, and one vertex of R, contradicting 3) of Claim 12.4.40. Thus, $z \notin V(C^1 \setminus R)$, so, $(N(z) \setminus V(Q)) \cap \text{dom}(\sigma')$ consists of u_1 and at least two vertices of R. By 3) of Claim 12.4.40, z is unique, and $N(z) \cap \text{dom}(\sigma')$ consists of w, u_1, and one vertex of A^0_w. ■

Applying Claim 12.4.47, z has precisely one neighbor in $A^0_w \cap (R \setminus \text{Mid}^0(R))$, so suppose without loss of generality that this neighbor is u_0. Applying iii) of Lemma 12.4.35, we have the following: For each $c \in L_\psi(u_0)$, there is an element $\phi^c \in \text{Link}(R, C^0)$ such that $\phi^c(u_0) = c$ and $|L_{\phi^c}(v)| \geq 3$ for all $v \in V(C^0 \setminus R)$. Note that w is in the domain of each coloring in $\text{Link}(R, C^0)$, since $w \notin \text{Mid}^0(R)$.

Claim 12.4.48. *For each $c \in L_\psi(u_0)$ and $d \in L_\psi(u_1) \setminus \{\phi^c(u_1)\}$, we have $L(z) \setminus \{c, d, \phi^c(u_1)\}| = 2$. In particular, $c \notin L_\psi(u_1)$.*

Proof: If any of these conditions do not hold, then, since no vertex of P is adjacent to u_1 or has a common neighbor with u_1 outside of R, there is a $c \in L_\psi(u_0)$ and an extension of ϕ^c to an L-coloring ϕ^c_* of $\text{dom}(\phi^c) \cup \{u_1\} \cup V(P)$ such that $|L_{\phi^c_*}(z)| \geq 3$. Possibly $\phi^*_c(u_1) = c$. This is permissible as $u_0u_1 \notin E(G)$ by 1) of Claim 12.4.40. Yet by Claim 12.4.47, we have $\{z\} = T(R + u_1; Q : \phi^c_*)$, so we have a contradiction. ■

We claim now that $L_\psi(u_1) \subseteq L(z)$. Suppose not, and let $d \in L_\psi(u_1)$ with $d \notin L(z)$. Thus, for all $c \in L_\psi(u_0)$, we have $\phi^c(w) = d$, otherwise we get $|L(z) \setminus \{c, d, \phi^c(u_1)\}| \geq 3$, contradicting Claim 12.4.48. In particular $\{\phi^c(w) : c \in L_\psi(u_0)\}$ is a constant color, and $d \notin L_\psi(u_0)$. Since each of u_0, u_1 has an L_ψ-list of size at least three, there exist a $c \in L_\psi(u_))$ and $c' \in L_\psi(u_1) \setminus \{d\}$ such that $|L(z) \setminus \{c, c'\}| \geq 4$ (possibly $c = c'$). Then $|L(w) \setminus \{c, d, c'\}| \geq 3$, contradicting Claim 12.4.48. Thus, we indeed have $L_\psi(u_1) \subseteq L(z)$. By Claim 12.4.48, we have $L_\psi(u_0) \subseteq L(z)$ as well, and $L(z) = 5$, so there is a color $c \in L_\psi(u_0) \cap L_\psi(u_1)$, contradicting Obsevation 12.4.48. This completes the proof of Proposition 12.4.39. □

Combining Proposition 12.4.5, Proposition 12.4.25, and Proposition 12.4.39, we complete the proof of Theorem 12.4.1.

12.5 Roulette Wheels with Distant Boundary Cycles

In this section, we complete the proof of Theorem 12.3.3, which we restate below.

Theorem 12.3.3. *Let $\mathcal{A} := (G, F_0, F_1, L, \psi)$ be a roulette wheel, let $\beta := \frac{17}{15}N_{\text{mo}}^2$ and let $\beta' := \beta + 4N_{\text{mo}}$. Then one of the following two statements holds.*

S1: *There exists a 2-connected subgraph H of G with $F_0 \cup F_1 \subseteq H$ and $V(H) \subseteq B_{\frac{\beta'}{3}}(F_0 \cup F_1)$ such that, for every facial subgraph C of H, C is a cycle of length at most 11; OR*

S2: *There exists a cycle connector for $\mathcal{A}$.*

Proof. Let $\mathcal{A} := (G, F_0, F_1, L, \psi)$ be a roulette wheel and suppose toward a contradiction that $\mathcal{A}$ satisfies neither S1 nor S2. Without loss of generality, we suppose that F_0 is the outer face of G. By Theorem 12.4.1, we have $d(F_0, F_1) \geq 3$. Let C^i be the 1-band of F_i for each $i = 0, 1$.

Now we apply the work of Chapter 10. Since $d(F_0, F_1) \geq 3$, no vertex of $C_0 \cup C_1$ has an L_ψ-list of size less than three. In particular, recalling Definition 10.0.1, we immediately have the following.

Claim 12.5.1. *For each $i = 0, 1$, F_i is an L-coil of G*

Proof: Let $i \in \{0, 1\}$. Note that we have $L_\psi^{F_{1-i}}(v) = Ł_\psi(v)$ for each $v \in V(C^i)$. By Observation 2.1.2, F_i is a highly predictable, and thus L-predictable, cyclic facial subgraph of G. Since each F_i is an induced subgraph of G, Co1 of Definition 10.0.1 is satisfied. Since each of F_0, F_1 is precolored and $d(F_0, F_1) \geq 3$, Co3 is also satisfied. Since no vertex of $D_1(F_i)$ has more than two neighbors in F_i, the rest just follows from the definition of a roulette wheel. ■

Recalling Definition 1.2.8, we now have the following.

Claim 12.5.2. $3 \leq d(F_0, F_1) \leq 15$.

Proof: Suppose toward a contradiction that $d(F_0, F_1) > 15$ and let $P := v_1 \cdots v_k$ be a shortest $(D_3(C^0), D_3(C^1))$-path, where $v_1 \in D_3(C^0)$ and $v_k \in D_3(C^1)$. By Observation 1.2.9, there exists a $z \in D_2(C^0) \cap N(v_1)$ and a $z^* \in D_2(C^1) \cap N(v_k)$ such that each of $\text{Bar}_{C^0}(v_1 z)$ and $\text{Bar}_{C^1}(v_k z^*)$ has size at most one.. Let $P^\dagger := z v_1 \cdots v_k z^*$. By Claim 12.5.1, each of F_0, F_1 is an L-coil of G. Since $d(F_0, F_1) > 15$, we get that, for each $i \in \{0, 1\}$, every vertex of $D_3(F_i)$ is F_i-pentagonal. Now, applying Theorem 10.0.7, let $[H_0, \sigma_0]$ be an (F_i, z)-opener and let $[H_1, \sigma_1]$ be an (F_1, z^*)-opener. Since $d(F_0, F_1) > 15$, the union $\tau := \sigma_0 \cup \sigma_1$ is a proper L-coloring of its domain, and furthermore, we have $v_1 v_k \notin E(G)$ and each of v_1, v_K has an L_τ-list of size at least four. Since each vertex of $D_1(H_0) \cup D_1(H_1)$ has an L_τ-list of size at least three and $v_1 v_k \notin E(G)$, there is an extension of τ to an L-coloring τ' of $\text{dom}(\tau) \cup \{v_1, v_k\}$ such that each vertex of $\text{Bar}_{C^0}(v_1 z) \cup \text{Bar}_{C^1}(v_k z^*)$ has an $L_{\tau'}$-list of size at least three.

Subclaim 12.5.3. *There exists a shortest $(D_2(C^0), D_2(C^1))$-path P^* such that the following hold.*

1) *P^* has terminal edges $z v_1, z^* v_k$; AND*

2) *There exists a $\varphi \in \text{Avoid}^\dagger(P^*) \neq \varnothing$ such that φ and τ' restrict to the same L-coloring of $\{z, v_1, v_k, z^*\}$ and $\varphi \cup \tau'$ is a proper L-coloring of its domain*

Proof: Given a shortest $(D_2(C^0), D_2(C^1))$-path P^* and a subpath Q of $\mathring{P}^*$, we say that Q is a *sectioned* subpath of P^* if both endpoints of Q are P-gaps and one of the following holds.

1) Q has length either two or four; *OR*

2) Q has length six and the midpoint of Q is a P^*-gap.

Case 1: There is a sectioned subpath of $\mathring{P}$

Let Q be a sectioned subpath of $\mathring{P}$, where $Q := v_i \cdots v_j$ for some $j \in \{i, i+2, i+4\}$. If $|E(Q)| = 2$, then it immediately follows from Proposition 1.2.3 that $zv_1Pv_kz_*$ satisfies 1) and 2), since we extend $\tau'|_{zv_1}$ to an element of Avoid(zv_1Pv_i) and extend $\tau'|_{v_kz_*}$ to an element of Avoid$(v_{i+2}Pv_kz_*)$. The union of these two colorings leaves at least three colors for v_{i+1}. Likewise, if $|E(Q)| = 4$, then it follows from Proposition 1.2.4 that $zv_1Pv_kz_*$ satisfies 1) and 2) above. Finally, if $|E(Q)| = 6$, then, applying Proposition 1.2.5, it follows that $zv_1Pv_kz_*$ satisfies 1) and 2) above, so we are done in this case.

Case 2: There is no sectioned subpath of $\mathring{P}$

In this case, since $k \geq 7$, there either exists a P-gap vertex of $\mathring{P}$ followed by three consecutive vertices of $v_2 \cdots v_{k-1}$ which are not P-gaps, or there exist five consecutive vertices of $v_2 \cdots v_{k-1}$ which are not P-gaps. In either case, applying Proposition 1.2.6 or Proposition 1.2.7 respectively, there is a shortest $(D_2(C^0), D_2(C^1))$-path P^* which differs from P by one vertex, where P^* has terminal edges zv_1, z^*v_k, and $\mathring{P}^*$ contains two P^*-gap vertices of distance two apart, so we are back to Case 1 with the role of $zv_1Pv_kz_*$ replaced by P^*. ■

Let P^*, φ be as in Subclaim 12.5.3 and let $K^\dagger := (H_0 \cup H_1) \setminus \text{dom}(\sigma_0 \cup \sigma_1)$. Since $\varphi \in$ and P^* is a shortest (D_2, D_2)-path, there exists a $v^\dagger \in D_1(P^*)$ such that $G[N(v^\dagger) \cap V(P^*)]$ is a subpath of P^* length at most two and such that, for each $v \in D_1(P^*) \setminus \{v^\dagger\}$, we have $|L_\varphi(v)| \geq 3$.

By our choice of z, z^* and our construction of τ', it then follows that, for each $v \in D_1(K^\dagger) \setminus B_2(C^0 \cup C^1)$ with $v \neq v^\dagger$, we have $|L_{\varphi \cup \tau'}(v)| \geq 3$. Likewise, for each $v \in D_1(K^\dagger) \cap B_2(C^0 \cup C^1)$ with $v \neq v^\dagger$, we have $N(v) \cap \text{dom}(\varphi \cup \tau') \subseteq \text{dom}(\tau')$ and $|L_{\varphi \cup \tau'}(v)| \geq 3$.

Finally, if $v^\dagger \in B_2(C^0 \cup C^1)$, then $|L_{\varphi \cup \tau'}(v^\dagger)| \geq 3$, and if $v^\dagger \notin B_2(C^0 \cup C^1)$, then $N(v^\dagger) \cap \text{dom}(\varphi \cup \tau') \subseteq \text{dom}(\varphi)$ and thus $|L_{\varphi \cup \tau'}(v^\dagger)| \geq 2$. In any case, the tuple $[K; v^\dagger; \varphi \cup \tau', V(K^\dagger)]$ is a cycle connector, contradicting our assumption that S2 of Theorem 12.3.3 is not satisfied. ■

We deal with the remaining distance cases via a similar argument, where we retain a precolored edge of one of the boundary cycles instead of reducing to a graph whose outer face has a lone 2-list. □

Chapter 13

Reduction to Mosaics

In this chapter, we complete the proof of Theorem 1.1.3 by showing the following result.

Theorem 13.0.1. *Let γ be as in Theorem 0.2.6 and let $\beta := \frac{17}{15}N_{\text{mo}}$. Let $\alpha := \frac{9}{2}(\beta + 4N_{\text{mo}}) + 3\gamma + 18$. Then every $(\alpha, 1)$-chart is colorable.*

Chapter 13 consists of four sections. The first ingredient we need is a simple edge-maximality lemma which is proven in Section 13.1. In Section 13.2, we prove some basic properties of minimal counterexamples to Theorem 13.0.1. In Section 13.3, we show that, under certain circumstances, an annulus between two short separating cycles in a minimal counterexample to Theorem 13.0.1 is a roulette wheel (so that we can apply the work of Chapter 12). Finally, in Section 13.4, we put all of these together to complete the proof of Theorem 13.0.1.

13.1 A Simple Edge-Maximality Lemma

In this short section, we prove the following simple lemma that we need for Theorem 13.0.1.

Lemma 13.1.1. *Let $\alpha \geq 1$ be an integer, let G be a connected planar graph, and let $C_1, \cdots, C_m$ be a collection of facial subgraphs of G such that $d_G(C_i, C_j) \geq \alpha$ for each $1 \leq i < j \leq m$. There exist a graph planar G', such that the following hold.*

1) *G' is an embedding obtained from G by adding edges to G; AND*
2) *For each $i = 1, \cdots, m$, C_i is also a a facial subgraph of G'; AND*
3) *with $d_{G'}(C_i, C_j) \geq \alpha$ for each $1 \leq i < j \leq m$; AND*
4) *For every facial subgraph H of G', with $H \notin \{C_1, \cdots, C_m\}$, and every block H' of H, every face of the induced graph $G[V(H')]$, except possibly H', is a triangle.*

Proof. If every facial subgraph of G with $H \notin \{C_1, \cdots, C_m\}$ satisfies property 4) above, then we take $G' = G$ and we are done. Now suppose there exists a facial subgraph H of G, with $H \notin \{C_1, \cdots, C_m\}$, and there exists a block H' of H such that at least one facial subgraph of $G[V(H')]$ is not a triangle. Thus, there is a subset $S \subseteq V(H)$ with $|S| > 3$ such that the induced graph $G[S]$ is a chordless cycle.

Since H is a facial subgraph of G, there is an open connected component U of $\mathbb{R}^2 \setminus G$ with $H = \partial(U)$. Let $G[S] := v_1 \cdots v_k$. To prove Lemma 13.1.1, it suffices to show that there exists an index $j \in \{1, \cdots, k\}$ such that, reading the indices mod k and setting $G^\dagger := G + v_j v_{j+2}$, we have $d_{G^\dagger}(C_s, C_t) \geq \alpha$ for any pair of distinct indices $s, t \in \{1, \cdots, m\}$, where $G + v_i v_j$ denotes an embedding obtained by drawing an arc $v_i v_j$ whose interior lies in U.

If we show that the above holds, then we simply iterate until we obtain a drawing from G which satisfies properties 1)-4) above. At each stage of the construction, each graph in the sequence satisfies properties 1)-3) of Lemma 13.1.1, and the sequence terminates in at most $3|V(G)| - 6$ steps in an embedding which satisfies 1)-4). We first note that, for any distinct indices $s, t \in \{1, \cdots, m\}$ and any $j \in \{1, \cdots, k\}$, since $d_G(C_s, C_t) \geq \alpha$, we have $d_G(v_j, C_s) + d_G(v_{j+2}, C_t) \geq \alpha - 2$.

Suppose toward a contradiction that there does not exist an index $j \in \{1, \cdots, k\}$ satisfying the above. For the remainder of the proof of Lemma 13.1.1, a distance between two vertices of $V(G)$ without a subscript denotes a distance between these two vertices in the initial graph G. For each $j \in \{1, \cdots, k\}$, let B_j be the set of pairs $(s, t) \in \{1, \cdots, m\} \times \{1, \cdots, m\}$ such that $d(C_s, v_j) + d(C_t, v_{j+2}) = \alpha - 2$. If there exists a $j \in \{1, \cdots, k\}$ such that $B_j = \varnothing$, then, setting $G^\dagger := G + v_j v_{j+2}$, we have $d_{G^\dagger}(C_s, C_t) \geq \alpha$ for all $1 \leq s < t \leq m$, contradicting our assumption. Thus, we have $B_j \neq \varnothing$ for each $j \in \{1, \cdots, k\}$. Let $B := \bigcup_{j=1}^{k} B_j$.

Claim 13.1.2. *Let $j \in \{1, \cdots, k\}$ and let $(s, t) \in B_j$. Then the following distance conditions hold:*

1) $d(C_t, v_j) = d(C_t, v_{j+2}) + 2$*; AND*
2) $d(C_s, v_{j+2}) = d(C_s, v_j) + 2$*; AND*
3) $d(C_t, v_{j+1}) = d(C_t, v_{j+2}) + 1$*; AND*
4) $d(C_s, v_{j+1}) = d(C_s, v_j) + 1$.

Proof: We have $d(C_t, v_j) \leq d(C_t, v_{j+2}) + 2$, and if $d_G(C_t, v_j) < d_G(C_t, v_{j+2}) + 2$ then $d(C_t, v_j) + d(C_s, v_j) < d(C_t, v_{j+2}) + d(C_s, v_j) + 2 = \alpha$, and thus $d(C_t, C_s) < \alpha$, contradicting our distance conditions. The same argument shows 2). We have $d(C_t, v_{j+1}) \leq d(C_t, v_{j+2}) + 1$, and if $d(C_t, v_{j+1}) < d(C_t, v_{j+2}) + 1$, then we have $d(C_s, v_{j+1}) + d(C_t, v_{j+1}) < d(C_s, v_{j+1}) + d(C_t, v_{j+2}) + 1$. Since $d(C_s, v_j) \geq d(C_s, v_{j+1}) - 1$, we then have $d(C_s, v_{j+1}) + d(C_t, v_{j+1}) < d(C_s, v_j) + d(C_t, v_{j+2}) = \alpha$, contradicting our distance conditions. The same argument shows 4). ■

It immediately follows from Claim 13.1.2 that $d(C_s, v_r) + d(C_t, v_r) = \alpha$ for each $r \in \{j, j+1, j+2\}$.

Claim 13.1.3. *Let $j \in \{1, \cdots, k\}$, let $(s, t) \in B_j$, and suppose that $d(C_s, v_j) \leq \frac{\alpha}{2} - 1$. Then the following hold.*

1) *For every pair $(p, q) \in B_{j-1}$, either $p = s$ or $q = s$; AND*
2) *For every $(p, q) \in B_{j+1}$, either $p = s$ or $q = s$.*

Proof: Since $B_{j-1} \neq \varnothing$, there is a pair $(p, q) \in B_{j-1}$. Suppose that $p \neq s$. Now, by Claim 13.1.2, we have $d(C_p, v_j) = d(C_p, v_{j-1}) + 1$. Since $d(C_s, v_j) \leq \frac{\alpha}{2} - 1$ and $s \neq q$, we have $d(C_p, v_j) \geq \frac{\alpha}{2} + 1$, or else $d(C_p, C_s) < \alpha$. Thus, we have $d(C_p, v_{j-1}) \geq \frac{\alpha}{2}$. Since $d(C_p, v_{j-1}) + d(C_q, v_{j+1}) = \alpha - 2$, we have $d(C_q, v_{j+1}) \leq \frac{\alpha}{2} - 2$. Thus we have $d(C_q, v_j) \leq \frac{\alpha}{2} - 1$, so $q = s$, or else we have distinct cycles C_s, C_q such that $d(C_s, C_q) \leq \alpha - 2$, violating our distance conditions.

Now let $(p, q) \in B_{j+1}$ and suppose that $p \neq s$. As above, since $d(C_s, v_j) \leq \frac{\alpha}{2} - 1$ and $p \neq s$, we have $d(C_p, v_j) \geq \frac{\alpha}{2} + 1$, or else $d(C_p, C_s) < \alpha$. Thus, we have $d(C_p, v_{j+1}) \geq \frac{\alpha}{2}$. Since $d(C_p, v_{j+1}) + d(C_q, v_{j+3}) = \alpha - 2$, we have $d(C_q, v_{j+2}) \leq \frac{\alpha}{2} - 2$. Now, since $d(C_s, v_j) \leq \frac{\alpha}{2} - 1$, we have $d(C_s, v_{j+2}) \leq \frac{\alpha}{2} + 1$. Thus, we have $q = s$, or else there are distinct cycles C_s, C_q such that $d(C_s, C_q) \leq \alpha - 1$, contradicting our distance conditions. ■

Now we choose an index $j^\star \in \{1, \cdots, k\}$ and a pair $(s^\star, t^\star) \in B$ such that the quantity $\min\{d(C_{s^\star}, v_{j^\star}), d(C_{t^\star}, v_{j^\star+2})\}$ is minimized. Consider the following cases:

Case 1: $\min\{d(C_{s^\star}, v_{j^\star}), d(C_{t^\star}, v_{j^\star+2})\} = d(C_{t^\star}, v_{j^\star+2})$.

In this case, since $d(C_{s^\star}, v_{j^\star}) + d(C_{t^\star}, v_{j^\star+2}) = \alpha - 2$, we have $d(C_{s^\star}, v_{j^\star}) \leq \frac{\alpha}{2} - 1$.

Claim 13.1.4. $d(C_{s^\star}, v_{j^\star-1}) = d(C_{s^\star}, v_{j^\star})$.

Proof: Suppose not. Then we have $d(C_{s^\star}, v_{j^\star-1}) = d(C_{s^\star}, v_j) \pm 1$. If $d(C_{s^\star}, v_{j^\star-1}) = d(C_{s^\star}, v_{j^\star}) - 1$, then applying Claim 13.1.3, $B_{j^\star-1}$ either contains a pair of the form $(s^\star, q)$, or a pair of the form $(p, s^\star)$. In either case, we contradict the minimality of $(s^\star, t^\star)$. Thus, we have $d(C_{s^\star}, v_{j^\star-1}) = d(C_{s^\star}, v_{j^\star}) + 1$.

Applying Claim 13.1.2, we have $d(C_{s^\star}, v_{j^\star-1}) = d(C_{s^\star}, v_{j^\star+1}) = d_G(C_{s^\star}, v_{j^\star}) + 1$. By Claim 13.1.3, $B_{j^\star-1}$ either contains a pair of the form (s, q) or a pair of the form (q, s). If $B_{j^\star-1}$ contains a pair of the form $(s^\star, q)$, then, by Claim 13.1.2, we have $d(C_{s^\star}, v_{j^\star+1}) = d(C_s, v_{j^\star-1})$, contradicting the fact that $d(C_{s^\star}, v_{j^\star-1}) = d(C_{s^\star}, v_{j+1}) = d(C_{s^\star}, v_{j^\star}) + 1$. Thus, $B_{j^\star-1}$ contains a pair of the form $(q, s^\star)$ for some $q \in \{1, \cdots, m\}$. Thus, by Claim 13.1.2, we have $d(C_{s^\star}, v_{j^\star-1}) = d(C_{s^\star}, v_{j^\star+1}) + 2$. But we also have $d(C_{s^\star}, v_{j^\star+1}) = d(C_{s^\star}, v_{j^\star}) + 1$, applying Claim 13.1.2 to the pair $(s^\star, t^\star)$, so $d(C_{s^\star}, v_{j^\star-1}) = d(C_{s^\star}, v_{j^\star}) + 3$, which is false since $v_{j^\star}, v_{j^\star-1}$ are adjacent. ■

Since $B_{j^\star-1} \neq \varnothing$ by assumption, there exists a $(p, q) \in B_{j^\star-1}$. Since $d(C_{s^\star}, v_{j^\star}) \leq \frac{\alpha}{2} - 1$, it follows from Claim 13.1.3 that either $p = s^\star$ or $q = s^\star$. If $p = s^\star$ then we have $d(C_{s^\star}, v_{j^\star}) = d(C_{s^\star}, v_{j^\star-1}) + 1$, contradicting the fact that $d(C_{s^\star}, v_{j^\star}) = d(C_{s^\star}, v_{j^\star-1})$. Thus, we have $q = s^\star$, and it follows from Claim 13.1.2 applied to $(p, s^\star)$ that $d(v_{j^\star}, C_{s^\star}) = d(v_{j^\star+2}, C_{s^\star}) + 2$. Yet, by Claim 13.1.2 applied to $(s^\star, t^\star)$, we also have $d(v_{j^\star+2}, C_{s^\star}) = d(v_{j^\star}, C_{s^\star}) + 2$, a contradiction.

Case 2: $\min\{d(C_{s^\star}, v_{j^\star}), d(C_{t^\star}, v_{j^\star+2})\} = d(C_{t^\star}, v_{j^\star+2})$.

In this case, we simply reverse the orientation and apply the same argument as above. For each $j \in \{1, \cdots, k\}$, we set $\hat{B}_j$ to be the set of pairs $(s, t) \in \{1, \cdots, m\} \times \{1, \cdots, m\}$ such that $d(C_s, v_j) + d(C_s, v_{j-2}) = \alpha - 2$. Then we are back to Case 1 with $B_1, \cdots, B_k$ replaced by $\hat{B}_1, \cdots, \hat{B}_k$. This completes the proof of Lemma 13.1.1. □

13.2 Properties of Critical Charts

We now ready to return to the context of charts and prove our main theorem for Chapter 13, which we restate below.

Theorem 13.0.1. *Let γ be as in Theorem 0.2.6 and let $\beta := \frac{17}{15}N_{\text{mo}}$. Let $\alpha := \frac{9}{2}(\beta + 4N_{\text{mo}}) + 3\gamma + 18$. Then every $(\alpha, 1)$-chart is colorable.*

We now set $\alpha := \frac{9}{2}(\beta + 4N_{\text{mo}}) + 3\gamma + 18$ and $\beta' := \beta + 4N_{\text{mo}}$. To prove Theorem 13.0.1, we begin by introducing the following definition.

Definition 13.2.1. Given an oriented chart $\mathcal{T} = (G, \mathcal{C}, L, C_*)$, we say that $\mathcal{T}$ is a *critical chart* if, letting α be as in the statement of Theorem 13.0.1, the following hold.

1) $\mathcal{T}$ is an $(\alpha, 1)$-chart and G is not L-colorable*; AND*
2) For any $(\alpha, 1)$-chart $(G', \mathcal{C}', L')$, if $|V(G')| < |V(G)|$, then G' is L'-colorable*; AND*
3) For any $(\alpha, 1)$-chart $(G', \mathcal{C}', L')$, if $|V(G')| = |V(G)|$ and $|E(G')| > |E(G)|$, then G' is L'-colorable.

Over the course of Sections 13.2-13.4, we show that no critical charts exist. We prove a sequence of propositions in which we gather some facts about critical charts, and then, at the end of Section 13.4, we combine these results in a one-paragraph proof which shows that no critical charts exist. More precisely, given a critical chart $\mathcal{T} = (G, \mathcal{C}, L, C_*)$, we show that G contains a family of short separating cycles $M_1, \cdots, M_s$ such that the graph $H := \bigcap_{i=1}^{s} \text{Ext}(M_i)$ is short-separation-free, and the graph $K := \bigcup_{i=1}^{s} \text{Int}^+(M_i)$ admits an L-coloring ϕ such that H is the underlying graph of a mosaic with respect to the list-assignment L_ϕ^K. It then follows from Theorem 2.1.7 that H is L_ϕ^K-colorable, and thus ϕ extends to an L-coloring of G, producing the desired contradiction.

For the proof of Theorem 13.0.1, we need some facts about intersections of short cycles. The motivation for this is as follows: When we deal with a critical chart $\mathcal{T} = (G, \mathcal{C}, L, C_*)$, one of the steps requires us to deal with two separating cycles D_0, D_1 in G, each of length at most four, such that $D_i \not\subseteq \text{Int}(D_{1-i})$ for each $i \in \{0, 1\}$, and such that $V(D_0) \cap V(D_1) \neq \varnothing$. The following fact is very simple and is stated without proof.

Proposition 13.2.2. *Let G be graph, let C_0, C_1 be cycles in G with $|V(C_j)| \leq 4$ for each $j \in \{0, 1\}$, and suppose that there are edges $e, f \in E(C_1) \setminus E(C_0)$ such that $e \in E(\text{Int}(C_0))$ and $f \in E(\text{Ext}(C_0))$. Let $A_{ii}, A_{ie}, A_{ei}, A_{ee}$ be the four cycles contained in the graph $C_0 \cup C_1$, such that*

1) $\text{Int}(A_{ii}) = \text{Int}(C_0) \cap \text{Int}(C_1)$ and $\text{Int}(A_{ie}) = \text{Int}(C_0) \cap \text{Ext}(C_1)$*; AND*

2) $\text{Int}(A_{ei}) = \text{Ext}(C_0) \cap \text{Ext}(C_1)$ and $\text{Int}(A_{ee}) = \text{Int}(A_{ie}) \cup \text{Int}(A_{ei}) \cup \text{Int}(A_{ie})$*; AND*

3) $\text{Ext}(A_{ee}) = \text{Ext}(C_0) \cap \text{Ext}(C_1)$.

Then the following hold.

1) If C_0, C_1 are edge-disjoint then $|E(A_{ee})| + |E(A_{ii})|$ and $|E(A_{ei})| + |E(A_{ie})|$ are both equal to $|E(C_0)| + |E(C_1)|$; AND

2) If C_0, C_1 are not edge disjoint then $|E(A_{ee})| + |E(A_{ii})| = E(C_0)| + |E(C_1)|$ and $|E(A_{ei})| + |E(A_{ie})| = |E(C_0)| + |E(C_1)| - 2$; AND

3) If $|E(C_0)| = |E(C_1)| = 4$, then the lengths of the four cycles $A_{ii}, A_{ie}, A_{ei}, A_{ee}$ have the same parity.

We now have the following simple facts.

Lemma 13.2.3. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical chart. Then the following hold.*

1) G is connected; AND

2) Each element of $\mathcal{C}$ is a cyclic facial subgraph of G and has no chords; AND

3) Suppose P is a path in G with $1 \leq |V(P)| \leq 2$, and there is a partition $G = G_0 \cup G_1$ with $G_0 \cap G_1 = P$ and $G_i \setminus P \neq \varnothing$ for each $i \in \{0, 1\}$. Then $\mathcal{C}^{\subseteq G_i} \neq \varnothing$ for each $i = 0, 1$, and $\mathcal{C} = \mathcal{C}^{\subseteq G_0} \cup \mathcal{C}^{\subseteq G_1}$.

Proof. It is an immediate consequence of the minimality of $\mathcal{T}$ that G is connected. If there is a $C \in \mathcal{C}$ such that C is either not a cycle or G contains a chord of C, then G admits a partition $G = G_0 \cup G_1$, where $G_0 \cap G_1$ is a path in G of length at most one, $V(G_0 \cap G_1) \subseteq V(C)$, and $V(G_i) \setminus V(G_0 \cap G_1) \neq \varnothing$ for each $i = 0, 1$. Without loss of generality, let $\mathbf{P}_\mathcal{T}(C) \subseteq G_0$. By the minimality of $\mathcal{T}$, G_0 is L-coloring, so let ϕ be an L-coloring of G_0. Let $P := G_0 \cap G_1$ and let $C^* := (C \cap G_1) + P$. Then $(G_1, \mathcal{C}^{\subseteq G_1} \cup \{C^*\}, L_\phi)$ is an $(\alpha, 1)$-chart, and thus G_1 is L_ϕ-colorable, so G is L-colorable, contradicting our assumption. This proves 2).

Now we prove 3). Let P be a path in G with $1 \leq |V(P)| \leq 2$ and let $G = G_0 \cup G_1$ with $G_0 \cap G_1 = P$ and $G_i \setminus P \neq \varnothing$ for each $i \in \{0, 1\}$. For each $C \in \mathcal{C}$, since C is a chordless cyclic facial subgraph of G, we have either

$C \subseteq G_0$ or $C \subseteq G_1$. Suppose toward a contradiction that 3) does not hold, and suppose without loss of generality that $C \subseteq G_0$ for each $C \in \mathcal{C}$. Then $(G_0, \mathcal{C}, L)$ is an $(\alpha, 1)$-chart with $|V(G_0)| < |V(G)|$, so G_0 is L-colorable. Let ψ be an L-coloring of G_0. Then G_1 is L_ψ-colorable, as every vertex of G_1 has an L_ϕ-list of size 5, except for a properly precolored path of length at most one. Thus, G is L-colorable, contradicting our assumption. □

We now introduce the following two pieces of notation, the second of which generalizes the notion of an annulus in a planar graph.

Definition 13.2.4. Given a planar graph H, we let $Sep(H)$ denote the set of separating cycles of length at most four in H. Given a cycle C in H and a family $\mathcal{D}$ of cycles in H, we set $A_H(C|\mathcal{D}) := \text{Int}_H(C) \cap \left(\bigcap_{D \in \mathcal{D}} \text{Ext}_H(D)\right)$. We call $A_H(C|\mathcal{D})$ the *annulus of* $(C, \mathcal{D})$.

Now we have the following.

Proposition 13.2.5. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical chart. Let F_0 be a cycle in G, and let $\mathcal{F}$ be a collection of cycles in G such that, for each $F \in \mathcal{F}$, $F \subseteq \text{Int}(F_0)$. Let $A := A_G(F_0|\mathcal{F})$. Let $\mathcal{F}^* := (\mathcal{F} \cup \{F_0\}) \setminus \mathcal{C}^{\subseteq A}$ and suppose that the following conditions hold.*

1) *A is short-separation-free and $\mathcal{C}^{\subseteq A} \subseteq \mathcal{F}$; AND*

2) *$3 \leq |V(F)| \leq 4$ for each $F \in \mathcal{F}^*$.*

Then every face of A, except possibly those of $\{F_0\} \cup \mathcal{F}$, is bounded by a triangle. Furthermore, if $d(F, F') \geq \beta'$ for all $F \in \mathcal{F}^*$ and $F' \in \mathcal{F}^* \cup \mathcal{C}^{\subseteq A}$, then any L-coloring of $\bigcup_{F \in \mathcal{F}^*} V(F)$ extends to an L-coloring of A.

Proof. We begin with the first part of the proposition:

Claim 13.2.6. *Every face of A, except possibly those of $\{F_0\} \cup \mathcal{F}$, is bounded by a triangle, .*

Proof: We first note the following:

> **Subclaim 13.2.7.** *Let $K \subseteq A$ be a cycle with $3 \leq |V(K)| \leq 4$ with $K \notin \{F_0\} \cup \mathcal{F}$. Then K is not a separating cycle in G.*
>
> Proof: Since A is short-separation-free, K is a facial subgraph of A. Since $K \notin \{F_0\} \cup \mathcal{F}$, this means that K is a facial subgraph of G as well, so K is not a separating cycle of G. ■

Since $\mathcal{C}^{\subseteq A} \subseteq \mathcal{F}$, we apply Lemma 13.1.1 and the edge-maximality of G to obtain the following: For every facial subgraph K of G, with $K \notin \{F_0\} \cup \mathcal{F}$ and every induced cycle D of G with $V(D) \subseteq V(K)$, D is a triangle. Now, let K be a facial subgraph of A, with $K \notin \{F_0\} \cup \mathcal{F}$. In that case, K is also a facial subgraph of G. We claim that K is a cycle.

Suppose that K is not a cycle. Thus, there is a vertex $v \in V(K)$ which is a cut-vertex of G. Thus, by Lemma 13.2.3, let $G = G_0 \cup G_1$, where $G_0 \cap G_1 = v$, and let $C, C' \in \mathcal{C}$ with $C \subseteq G_0$, and $C' \subseteq G_1$. Now, $K \cap G_0$ is a subgraph of K and a facial subgraph of G_0, and there exists a subgraph $H \subseteq G_0$ with $C \subseteq H$ such that $K \cap G_0$ contains a cycle which is a facial subgraph of H. Since every induced cycle in K is a triangle, K contains a cycle which, in G, separates C from C', contradicting Subclaim 13.2.7.

We conclude that K is a cycle, and thus every induced cycle in $G[V(K)]$ is a triangle. If K is not a triangle, then, applying Lemma 13.2.3, K has a chord U separating C from C' in G, and thus, there is a triangle in G whose vertices lie in $V(K)$ and which separates C from C' in G, contradicting Subclaim 13.2.7. Thus, K is a triangle, so every face of A, except possibly those of $\{F_0\} \cup \mathcal{F}$, is bounded by a triangle. ■

Now we return to the proof of Proposition 13.2.5. Let $S := \bigcup_{F \in \mathcal{F}^*} V(F)$ and ϕ be an L-coloring of S. Consider the tuple $(A, \mathcal{F} \cup \{F_0\}, L^S_\phi, F_0)$. Since every facial subgraph of A other than those of $\mathcal{F}_0 \cup \{F_0\}$ is a triangle, it follows from Observation 2.1.2 that each cycle of $\mathcal{F}^*$ is a highly predictable facial subgraph of A and thus an L^S_ϕ-predictable facial subgraph of A. Since the elements of $\mathcal{C}^{\subseteq A}$ are all Thomassen facial subgraphs of A with respect to the list-assignment L^S_ϕ, and are pairwise of distance at least α apart, it follows that $(A, \mathcal{F} \cup \{F_0\}, L^S_\phi, F_0)$ is a mosaic in which each element of $\mathcal{F}^*$ is a closed ring and each element of $\mathcal{C}^{\subseteq A}$ is an open ring. Thus, by Theorem 2.1.7, ϕ extends to an L-coloring of A. □

We now have the following easy facts.

Lemma 13.2.8. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical chart. Then the following holds.*

1) *$Sep(G) \neq \varnothing$; AND*
2) *For every $D \in Sep(G)$, there exist cycles $C, C' \in \mathcal{C}$ such that $C \subseteq \text{Int}(D)$ and $C' \subseteq \text{Ext}(D)$, and furthermore, both of the graphs $\text{Int}^+(D)$ and $\text{Ext}^+(D)$ are L-colorable.*

Proof. We first prove 1). Applying Proposition 13.2.5 where we set $F_0 := C_*$ and $\mathcal{F} := \mathcal{C}$, we get that each facial subgraph of G, except those among $\mathcal{C}$, is a triangle (the conditions of Proposition 13.2.5 are trivially satisfied since $\mathcal{F} = \mathcal{F} \cup \{F_0\} = \mathcal{C}$). Thus, $\mathcal{T}$ is an $(\alpha, 1)$-tessellation, so $\mathcal{T}$ is a $(\beta', 1)$-tessellation and thus a mosaic. By Theorem 2.1.7, G is L-colorable, contradicting the fact that $\mathcal{T}$ is a critical chart. Thus, $Sep(G) \neq \varnothing$.

Let $D \in Sep(G)$. Since D is a separating cycle in G, we have $|V(\text{Int}^+(D))| < |V(G)|$. Since $(\text{Int}^+(D), \mathcal{C}^{\subseteq \text{Int}^+(D)}, L)$ is also an $(\alpha, 1)$-chart, $\text{Int}^+(D)$ is indeed L-colorable by the minimality of $\mathcal{T}$. The same argument shows that $\text{Ext}^+(D)$ is L-colorable. Now suppose toward a contradiction that $C \subseteq \text{Int}(D)$ for all $C \in \mathcal{C}$. Let ϕ be an L-coloring of $\text{Int}^+(C)$. Then $\text{Ext}(D)$ is L^D_ϕ-colorable, since $\text{Ext}(D)$ has a properly precolored facial cycle of length at most 4, and every other vertex of $\text{Ext}(D)$ has an L^D_ϕ-list of size at least 5. Thus, G is L-colorable, contradicting our assumption. The same argument shows that it is not the case that $C \subseteq \text{Ext}(D)$ for all $C \in \mathcal{C}$. □

We now introduce the following definitions and notations.

Definition 13.2.9. Let $\mathcal{T} := (G, \mathcal{C}, L)$ be a chart.

1) Given a cycle $F \subseteq G$, we define the following.
   a) A cycle $D \in Sep(G)$ is called a *descendant* of F if $D \neq F$ and $D \subseteq \text{Int}(F)$. We denote the set of descendants of F by $\mathcal{I}(F)$.
   b) A cycle $D \in \mathcal{I}(F)$ is called an *immediate descendant* of F if, for any $D' \in \mathcal{I}(F)$ such that $D \subseteq \text{Int}(D')$, we have $D' = D$. We denote the set of immediate descendants of F by $\mathcal{I}^m(F)$.
2) Give a cycle $D \in Sep(G)$, we define the following.
   a) We say that D is *minimal* if $\mathcal{I}(D) = \varnothing$. Likewise, we say that D is *maximal* if there does not exist a $D' \in Sep(G)$ such that $D \in \mathcal{I}(D')$.
   b) We say that D is a *blue cycle* if, for every $C \in \mathcal{C}^{\subseteq \text{Int}(D)}$, there exists a $D' \in \mathcal{I}(D)$ such that $C \subseteq \text{Int}(D')$. Otherwise, we say that D is a *red* cycle.

c) We let $Sep_r(G)$ denote the set of red cycles in $Sep(G)$, and we let $Sep_b(G)$ denote the set of blue cycles of $Sep(G)$.

We now have the following easy facts:

Lemma 13.2.10. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical chart. Then the following hold.*

1) *For any $D \in Sep_b(G)$, there exists a $D' \subseteq \text{Int}(D)$ with $D' \in Sep_r(G)$; AND*

2) *For any minimal $D \in Sep(G)$, there exists a $C \in \mathcal{C}^{\subseteq \text{Int}(D)}$ such that $d(C, D) \leq \beta'$.*

Proof. Let $D \in Sep_b(G)$. Since D is blue, we have $\mathcal{I}(D) \neq \varnothing$ by definition. Since G is finite, let D' be a minimal descendant of D. By the minimality of D', we have $\mathcal{I}(D') = \varnothing$, and thus $D' \in Sep_r(G)$. This proves 1). Now we prove 2). Let $D \in Sep(G)$ be minimal. Since $\mathcal{I}(D) = \varnothing$, we have $D \in Sep_r(G)$. Now, by Lemma 13.2.8, there is an L-coloring ϕ of $\text{Ext}^+_G(D)$. Suppose toward a contradiction that $d(C, D) > \beta'$ for all $C \in \mathcal{C}^{\subseteq \text{Int}(D)}$. Since D is minimal, we have $\mathcal{I}(D) = \varnothing$, and thus $A(D|\mathcal{I}(D)) = \text{Int}(D)$. By Proposition 13.2.5 applied to $A(D, \mathcal{I}(D))$, we get that ϕ extends to an L-coloring of $\text{Int}(D)$, and thus G is L-colorable, which is false. $\square$

Recalling the fact that γ is the constant defined in Theorem 0.2.6, we now introduce the following definitions.

Definition 13.2.11. *Let $\mathcal{T} := (G, \mathcal{C}, L, C_*)$ be a chart.*

1) We say that a cycle $D \in Sep(G)$ is $\mathcal{C}$-*close* if one of the following holds.

   a) $D \in Sep_r(G)$ and there exists a $C \in \mathcal{C}^{\subseteq A(D|\mathcal{I}^m(D))}$ such that $d(D, C) \leq \beta'$; *OR*

   b) $D \in Sep_b(G)$ and there exists a red descendant D' of D such that $d(D, D') \leq \gamma + 1$.

2) We define a binary relation $\sim$ on $Sep(G)$ as follows: For $D_1, D_2 \in Sep(G)$, we say that $D_1 \sim D_2$ if there is an element $C \in \mathcal{C}$ such that $C \subseteq \text{Int}(D_1) \cap \text{Int}(D_2)$ and $d(C, D_i) \leq \beta' + \gamma + 3$ for each $i \in \{1, 2\}$.

Now we have the following observation:

Lemma 13.2.12. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical chart. Then the following hold.*

1) *If $D \in Sep(G)$, H_1, H_2 are two subgraphs of G, then $d(H_1, H_2) \leq d(H_1, D) + d(H_2, D) + 2$; AND*

2) *For any $D \in Sep(G)$ such that every cycle in $\{D\} \cup \mathcal{I}(D)$ is $\mathcal{C}$-close, there is a unique $C \in \mathcal{C}$ with $C \subseteq \text{Int}(D)$ such that $d(C, D) \leq \beta' + \gamma + 3$.*

Proof. 1) is trivial, since D is a cycle of length at most four. Now we prove 2). If $D \in Sep_r(G)$ then, since D is $\mathcal{C}$-close, there is a $C \in \mathcal{C}^{\subseteq A(D, \mathcal{I}^m(D))}$ such that $d(C, D) \leq \beta'$, so we are done in that case. If $D \in Sep_b(G)$, then, since D is $\mathcal{C}$-close, there is a $D' \in Sep_r(G) \cap \mathcal{I}(D)$ such that $d(D, D') \leq \gamma + 1$. Since D' is $\mathcal{C}$-close, there is a $C \in \mathcal{C}^{\subseteq A(D', \mathcal{I}^m(D'))}$ with $d(C, D') \leq \beta'$. By 1), we have $d(C, D) \leq \beta' + \gamma + 3$. Now suppose there is another cycle $C' \in \mathcal{C}^{\subseteq \text{Int}(D)}$ with $d(C', D) \leq \beta' + \gamma + 3$. Applying 1) again, we have $d(C, C') \leq 2(\beta' + \gamma + 3) + 2 < \alpha$, contradicting the fact that $(G, \mathcal{C}, L)$ is an $(\alpha, 1)$-chart. Thus, C is unique. $\square$

It immediately follows from 2) of Lemma 13.2.12 that the relation $\sim$ partitions the set of $\mathcal{C}$-close cycles of $Sep(G)$ into equivalence classes. We now have the following:

Proposition 13.2.13. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical chart and let $\mathcal{M} \subseteq Sep(G)$ be a collection of short separating cycles in G such that the following hold.*

1) For any distinct $D, D' \in \mathcal{M}$, $D \notin \mathcal{I}(D')$; AND

2) For each $D \in \mathcal{M}$, every cycle of $\{D\} \cup \mathcal{I}(D)$ is $\mathcal{C}$-close.

Let $D_1, \cdots, D_k \in \mathcal{M}$ be a set of representatives of distinct equivalence classes of $\mathcal{M}$. For each $i = 1, \cdots, k$, let $[D_i] = \{D \in \mathcal{M} : D \sim D_i\}$. Then the following hold.

1) For each $1 \leq i < j \leq k$, the graphs $\text{Int}(D_i)$ and $\text{Int}(D_j)$ are disjoint; AND

2) There exist k distinct element $C_1, \cdots, C_k \in \mathcal{C}$ such that, for each $j \in \{1, \cdots, k\}$, $C_j \subseteq \bigcap_{D^* \in [D_j]} \text{Int}(D^*)$ and $d(C_j, D^*) \leq \beta' + \gamma + 3$ for each $D^* \in [D_j]$; AND

3) $\bigcup_{i=1}^{k} \text{Int}^+(D_i)$ is L-colorable.

Proof. Applying 2) of Lemma 13.2.12, there exist k distinct element $C_1, \cdots, C_k \in \mathcal{C}$ such that, for each $j \in \{1, \cdots, k\}$, $C_j \subseteq \bigcap_{D^* \in [D_j]} \text{Int}(D^*)$ and $d(C_j, D^*) \leq \beta' + \gamma + 3$ for each $D^* \in [D_j]$. Suppose toward a contradiction that there exists a pair of indices $1 \leq i < j \leq k$ such that $\text{Int}(D_i) \cap \text{Int}(D_j) \neq \varnothing$. Then, since $D_i \notin \mathcal{I}(D_j)$ and $D_j \notin \mathcal{I}(D_i)$, we have $D_i \cap D_j \neq \varnothing$, and thus $d(C_i, C_j) \leq 2(\beta' + \gamma + 3) + 2 < \alpha$, contradicting the fact that $(G, \mathcal{C}, L)$ is an $(\alpha, 1)$-chart.

To finish, it suffices to check that $\bigcup_{i=1}^{k} \text{Int}^+(D_i)$ is L-colorable. For any distinct $i, j \in \{1, \cdots, k\}$, we have $d(D_i, C_i) \leq \beta' + \gamma + 3$ and $d(D_j, C_j) \leq \beta' + \gamma + 3$. Since $\mathcal{T}$ is an $(\alpha, 1)$-chart, we have $d(C_i, C_j) \geq \alpha$. By two successive applications of 1) of Lemma 13.2.12, we have $d(C_i, D_i) + d(D_i, D_j) + d(D_j, C_j) \geq \alpha - 4$, so $d(D_i, D_j) \geq (\alpha - 4) - 2(\beta' + \gamma + 3)$. Since $D_i \notin \mathcal{I}(D_j)$ and $\mathcal{I}(D_j)$, the graph $\bigcup_{i=1}^{k} \text{Int}^+(D_i)$ is a union of k connected components, pairwise of distance at least $(\alpha - 4) - 2(\beta' + \gamma + 3)$ apart. For each $i = 1, \cdots, k$, $\text{Int}^+(D_i)$ is L-colorable by Lemma 13.2.8. Since $(\alpha - 4) - 2(\beta' + \gamma + 3) > 1$, the union $\bigcup_{i=1}^{k} \text{Int}^+(D_i)$ is also L-colorable. ☐

Note that, by Proposition 13.2.13, it follows that, for any cycle $D \subseteq G$, the relation $\sim$ partitions $\mathcal{I}^m(D)$ into equivalence classes, since no cycle of $\mathcal{I}^m(D)$ lies in the interior of another. Now we have the following:

Proposition 13.2.14. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical chart. Let D be a cycle in G and suppose that, for every cycle $D' \in \mathcal{I}(D)$, every element of $\{D'\} \cup \mathcal{I}(D')$ is $\mathcal{C}$-close. We then have the following.*

1) Let $D_1, D_2 \in \mathcal{I}_m(D)$, where D_1, D_2 lie in different equivalence classes of $\mathcal{I}^m(D)$ under $\sim$. Let $\mathcal{R} \in \{0, 1, 2\}$ be the number of red cycles in $\{D_1, D_2\}$. Then $d(D_1, D_2) \geq \frac{5\beta'}{2} + \mathcal{R}(\gamma + 3) + \gamma + 8$.

Furthermore, for any $D' \in \mathcal{I}^m(D)$ and $C \in \mathcal{C}$ with $C \subseteq G[A(D|\mathcal{I}^m(D))]$, we have the following.

2) If $D' \in Sep_r(G)$, then $d(D', C) \geq \frac{7\beta'}{2} + 3\gamma + 16$; AND

3) If $D' \in Sep_b(G)$, then $d(D', C) \geq \frac{7\beta'}{2} + 2\gamma + 13$.

Proof. We first prove 1). We first note that $\text{Int}(D_1) \cap \text{Int}(D_2) = \varnothing$ by Proposition 13.2.13. Combining Definition 13.2.11 with 2) of Lemma 13.2.12, we have the following. For each $j \in \{1, 2\}$, there exists an element $C_j \in \mathcal{C}$ with $C_j \subseteq \text{Int}(D_j)$, where $d(C_j, D_j) \leq \beta' + \gamma + 3$, and, if D_j is red, then we have the stronger condition $d(C_j, D_j) \leq \beta'$. Thus, we obtain $d(C_1, D_1) + d(C_2, D_2) \leq 2\beta' + (2 - \mathcal{R})(\gamma + 3)$. Since $\text{Int}(D_1) \cap \text{Int}(D_2) = \varnothing$ and each of C_1, C_2 is a cycle, we have $C_1 \neq C_2$, and thus $d(C_1, C_2) \geq \alpha$, as $\mathcal{T}$ is an $(\alpha, 1)$-chart. By two successive applications of 1) of Lemma 13.2.12, we have $d(C_1, D_1) + d(D_1, D_2) + d(C_2, D_2) \geq \alpha - 4$. Thus, we obtain $d(D_1, D_2) \geq (\alpha - 4) - 2\beta' - (2 - \mathcal{R})(\gamma + 3)$, so $d(D_1, D_2) \geq \frac{5\beta'}{2} + \mathcal{R}(\gamma + 3) + \gamma + 8$, as desired.

Now we prove 2). Since $D' \in Sep_r(G)$, and D' is $\mathcal{C}$-close by assumption, there exists a $C' \in \mathcal{C}$ such that $C' \subseteq \text{Int}(D')$ and $d(C', D') \leq \beta$. Since $C' \not\subseteq A(D|\mathcal{I}^m(D))$, we have $C \neq C'$. Thus, $d(C, C') \geq \alpha$. By 1) of Lemma 13.2.12, we then have $d(C, D') + d(D', C') \geq \alpha - 2$, and thus $d(C, D') \geq (\alpha - 2) - \beta' = \frac{7\beta'}{2} + 3\gamma + 16$. This proves 2).

Now suppose that $D' \in Sep_b(G)$ and let $C \in \mathcal{C}$ with $C \subseteq G[A(D|\mathcal{I}^m(D))]$. Since $D' \in Sep_b(G)$ and each element of $\{D\} \cup \mathcal{I}(D)$ is $\mathcal{C}$-close, there is a $C' \in \mathcal{C}$ with $C' \subseteq \text{Int}(D')$ and $d(C', D') \leq \beta' + \gamma + 3$ by 2) of Lemma 13.2.12. Since $C' \subseteq \text{Int}(D')$, we have $C \neq C'$, and thus $d(C, C') \geq \alpha$. By 1) of Lemma 13.2.12, we have $d(C, D') + d(D', C') \geq \alpha - 2$, so $d(C, D') \geq (\alpha - 2) - (\beta' + \gamma + 3) = \frac{7\beta'}{2} + 2\gamma + 13$. This proves 3), and completes the proof of Proposition 13.2.14. $\square$

Now we have the following key fact:

Proposition 13.2.15. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical chart. Let D be a cycle in G and suppose that, for each $D' \in \mathcal{I}(D)$, every element of $\{D'\} \cup \mathcal{I}(D')$ is $\mathcal{C}$-close. Then there exists a system $\mathcal{D}$ of distinct representatives of the $\sim$-equivalence classes of $\mathcal{I}^m(D)$ under the relation $\sim$ such that $A(D|\mathcal{D})$ is short-separation-free.*

Proof. Let $\mathcal{D}$ be a system of distinct representatives of distinct equivalence classes of $\mathcal{I}^m(D)$, and, among all choices of systems of distinct representatives of $\sim$-equivalence classes in $\mathcal{I}^m(D)$, we choose $\mathcal{D}$ so as to minimize the quantity $|Sep(A(D|\mathcal{D}))|$. We claim now that $Sep(A(D|\mathcal{D})) = \varnothing$.

Suppose toward a contradiction that $Sep(A(D|\mathcal{D})) \neq \varnothing$, and let $T \in Sep(A(D|\mathcal{D}))$. Note that T is also a separating cycle of length at most 4 in G, and since $T \subseteq \text{Int}(D)$, we have $T \in \mathcal{I}(D)$, so there exists a $D^* \in \mathcal{I}^m(D)$ such that $T \subseteq \text{Int}(D^*)$. Now, for some unique $F \in \mathcal{D}$, we have $D^* \sim F$. Note that $D^* \neq F$, or else T is not a separating cycle of $A(D|\mathcal{D})$, since $\text{Int}(F) \cap A(D|\mathcal{D}) = F$, which is a facial subgraph of $A(D|\mathcal{D})$. By Proposition 13.2.13, there is a unique element $C^* \in \mathcal{C}$ such that $C^* \subseteq \bigcap(\text{Int}(D') : D' \sim F \text{ and } D' \in \mathcal{I}^m(D))$, and each element of $\mathcal{I}^m(D)$ which is equivalent to D^* under $\sim$ is of distance at most $\beta' + \gamma + 3$ from C^*.

Now, $C^* \subseteq \text{Int}(F) \cap \text{Int}(D^*)$, and, since both D^*, F lie in $\mathcal{I}^m(D)$, we have $D^* \notin \mathcal{I}(F)$ and $F \notin \mathcal{I}(D^*)$. Thus, we get $V(D^*) \cap V(F) \neq \varnothing$. so we apply Proposition 13.2.2. There exist four cycles $A_{ii}, A_{ie}, A_{ei}, A_{ee}$ in G, each of which is a subgraph of $D^* \cup F$, such that $\text{Int}(A_{ii}) = \text{Int}(D^*) \cap \text{Int}(F)$ and $\text{Int}(A_{ie}) = \text{Int}(D^*) \cap \text{Ext}(F)$, and, analogously, $\text{Int}(A_{ei}) = \text{Ext}(D^*) \cap \text{Int}(F)$ and $\text{Int}(A_{ee}) = \text{Ext}(D^*) \cap \text{Ext}(F)$. We now have the following.

Claim 13.2.16. $T \subseteq \text{Ext}(F)$ *and* $|V(A_{ie})| \geq 5$. *Furthermore,* $|V(A_{ei})| \leq 4$.

Proof: Since $T \subseteq A(D|\mathcal{D})$, we have $T \subseteq \text{Ext}(F)$ and thus $T \subseteq \text{Int}(A_{ie})$. Suppose towards a contradiction that $|V(A_{ie})| \leq 4$. Since $T \in Sep(G)$, there is a $v \in V(\text{Int}(T) \setminus V(T))$. Thus, $A_{ie} \in Sep(G)$, since A separates v from a point of $\text{Ext}(F) \setminus V(F)$. Since $A_{ie} \in Sep(G)$ we have $A_{ie} \in \mathcal{I}(D^*)$. Thus, since A_{ie} is $\mathcal{C}$-close by assumption, there is a cycle $C' \in \mathcal{C}^{\subseteq \text{Int}(A_{ie})}$ with $d(C', A_{ie}) \leq \beta + \gamma + 3$ by Lemma 13.2.12. Note that $C' \neq C^*$ since $C^* \subseteq \text{Int}(F)$. Since $A_{ie} \subseteq D^* \cup F$, we get $d(C', D^* \cup F) \leq \beta' + \gamma + 3$, and we have $d(C^*, D^*) \leq \beta' + \gamma + 3$ and $d(C^*, F) \leq \beta' + \gamma + 3$, so $d(C', C^*) \leq 2(\beta' + \gamma + 3) + 2 < \alpha$, contradicting the fact that $\mathcal{T}$ is an $(\alpha, 1)$-chart. Thus, $|V(A_{ie})| \geq 5$. By Proposition 13.2.2, at least one of A_{ie}, A_{ei} has length at most 4, so $|V(A_{ei})| \leq 4$. $\blacksquare$

Applying the above, we have the following.

Claim 13.2.17. $V(\text{Int}(A_{ei})) = V(A_{ei})$.

Proof: Suppose toward a contradiction that $V(\text{Int}(A_{ei})) \neq V(A_{ei})$. In that case, since $|V(A_{ei})| \leq 4$, we have $A_{ei} \in \mathit{Sep}(G)$. Since $A_{ei} \in \mathit{Sep}(G)$ we get that A_{ei} is $\mathcal{C}$-close by assumption, and so there is a cycle $C^\dagger \in \mathcal{C}^{\subseteq \text{Int}(A_{ei})}$ with $d(C^\dagger, A_{ei}) \leq \beta' + \gamma + 3$ by Lemma 13.2.12. Note that $C^\dagger \neq C^*$, since $C^\dagger \subseteq \text{Ext}(D^*)$ and $C^* \subseteq \text{Int}(D^*)$ by definition. But since $A_{ei} \subseteq D^* \cup F$, we have $d(C^\dagger, D^* \cup F) \leq \beta' + \gamma + 3$, and we also have $d(C^*, D^*) \leq \beta + \gamma + 3$ and $d(C^*, F) \leq \beta' + \gamma + 3$, so $d(C^\dagger, C^*) \leq 2(\beta' + \gamma + 3) + 2$ by 1) of Lemma 13.2.12, contradicting the fact that $(G, \mathcal{C}, L, C_*)$ is an $(\alpha, 1)$-chart. Thus, our assumption that $V(\text{Int}(A_{ei})) \neq V(A_{ei})$ is false. ■

Now consider the set $\mathcal{D}^\dagger := (\mathcal{D} \cup \{D^*\}) \setminus \{F\}$. This is also a system of distinct representatives of the equivalence classes of $\mathcal{I}^m(D)$. Furthermore, $T \notin \mathit{Sep}(A(D|\mathcal{D}^\dagger))$, since $T \subseteq \text{Int}(D^*)$ by assumption. If $\mathit{Sep}(A(D|\mathcal{D}^\dagger)) \subseteq \mathit{Sep}(A(D|\mathcal{D}))$, then we have $|\mathit{Sep}(A(D|\mathcal{D}^\dagger))| < |\mathit{Sep}(A(D|\mathcal{D}))|$, contradicting the minimality of $|\mathit{Sep}(A(D|\mathcal{D}))|$. Thus, there is a $T^\dagger \in \mathit{Sep}(A(D|\mathcal{D}^\dagger))$ with $T^\dagger \notin \mathit{Sep}(A(D|\mathcal{D}))$.

Claim 13.2.18. $T^\dagger \subseteq \text{Ext}(D^*) \cap \text{Int}(F)$.

Proof: Firstly, since $T^\dagger \subseteq A(D|\mathcal{D}^\dagger)$, we have $T^\dagger \subseteq \text{Ext}(D^*)$. Now suppose toward a contradiction that $T^\dagger \not\subseteq \text{Int}(F)$. Note that $T^\dagger$ is also an element of $\mathit{Sep}(G)$, and $T^\dagger \subseteq \text{Int}(D)$. Thus, there is a $D^{**} \in \mathcal{I}^m(D)$ such that $T^\dagger \subseteq \text{Int}(D^{**})$, and there is a unique $F^{**} \in \mathcal{D}$ with $D^{**} \sim F^{**}$.

> **Subclaim 13.2.19.** *$F^{**} = F$, and furthermore, $D^{**} \neq D^*$, and $T^\dagger \neq D^{**}$.*
>
> Proof: If $F^{**} \neq F$, then $T^\dagger$ is a separating cycle of $A(D|\mathcal{D})$ if and only if $T^\dagger$ is a separating cycle of $A(D|\mathcal{D}^\dagger)$, contradicting our assumption. Thus, we indeed have $F^{**} = F$, and $D^{**} \sim D^* \sim F$. Since $T^\dagger \subseteq \text{Int}(D^{**})$ and $T^\dagger \in \mathit{Sep}(A(D|\mathcal{D}^\dagger))$, it follows that D^{**} is not a facial subgraph of $A(D|\mathcal{D}^\dagger)$, and thus $D^{**} \neq D^*$. Suppose now that $T^\dagger = D^{**}$. Since $D^{**} \sim D^*$ and $D^{**} \neq D^*$, it follows that $E(T^\dagger)$ has nonempty intersection with $E(\text{Int}(D^*)) \setminus E(D^*)$, contradicting the fact that $T^\dagger \subseteq A(D|\mathcal{D}^\dagger)$. Thus, we have $T^\dagger \neq D^{**}$. ■

We claim now that $T^\dagger \subseteq \text{Ext}(F)$. Suppose not. Then, since $T^\dagger \not\subseteq \text{Int}(F)$ by assumption, $T^\dagger$ has an edge in $\text{Int}(F) \setminus E(F)$, and an edge in $\text{Ext}(F) \setminus E(F)$, and thus, $T^\dagger$ is also an immediate descentant of D, so we get $T^\dagger = D^{**}$, contradicting Subclaim 13.2.19. Thus, we have $T^\dagger \subseteq \text{Ext}(F)$. Since $T^\dagger \subseteq \text{Int}(D) \cap \text{Ext}(F)$ and $D^{**} \sim D^*$, we have $\text{Int}(T^\dagger) \subseteq A(D|\mathcal{D})$. Since $T^\dagger \in \mathit{Sep}(A(D|\mathcal{D}^\dagger))$, we have $V(\text{Int}(T^\dagger) \setminus V(T^\dagger) \neq \varnothing$. Since $T^\dagger \subseteq \text{Int}(D^{**})$ and $T^\dagger \neq D^{**}$, it follows that $T^\dagger$ separates a vertex of $\text{Int}(T^\dagger) \setminus T^\dagger$ from D, and thus $T^\dagger$ is a separating cycle of $A(D|\mathcal{D})$, contradicting our assumption that $T^\dagger \notin \mathit{Sep}(A(D|\mathcal{D}))$. ■

Applying Claim 13.2.18, we have $T^\dagger \subseteq \text{Ext}(D^*) \cap \text{Int}(F)$, and so $T^\dagger \subseteq \text{Int}(A_{ei})$. By Claim 13.2.17, we have $V(\text{Int}(A_{ei})) = V(A_{ei})$. Since $T^\dagger$ is a separating cycle in G, we have a contradiction. Thus, our assumption that $\mathit{Sep}(A(D|\mathcal{D})) \neq \varnothing$ is false, and $\mathit{Sep}(A(D|\mathcal{D}))$ is indeed empty, so $A(D|\mathcal{D})$ is short-separation-free, as desired. This completes the proof of Proposition 13.2.15. □

13.3 Boundary Analysis for Critical Charts

In order to complete the proof of Theorem 13.0.1, we need to apply the work of Sections 12.2-12.5, and, in particular, we need a result that states that, under certain conditions, the annulus between two short separating cycles in a critical chart behaves like a roulette wheel. The key is to check that the "short side property" of Definition 12.3.1 is satisfied by this annulus. The lone result of Section 13.3 is the following.

Lemma 13.3.1. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical chart. Let $D_0 \in \mathit{Sep}(G)$, where, for each $F \in \mathcal{I}(F_0)$, every element of $\{F\} \cup \mathcal{I}(F)$ is $\mathcal{C}$-close. Let $\mathcal{M}_0$ be a complete set of representatives of the $\sim$-equivalence classes of*

$\mathcal{I}^m(F_0)$, *where* $A := A(F_0|\mathcal{M}_0)$ *is short-separation-free. Let* $F_1 \in \mathcal{M}_0$, *suppose that* $2 \leq d(F_0, F_1) \leq \beta' + 1$. *For each* $i \in \{0, 1\}$, *the following hold.*

1) *For each* $i = 0, 1$ *and any generalized chord* P *of* D_i *in* A *of length at most six, letting* $A = A^- \cup A^+$ *be the natural* P*-partition of* A, *where* $F_{1-i} \subseteq A^+$, *every element of* $\mathcal{C}^{\subseteq A} \cup (\mathcal{M}_0 \setminus \{F_1\})$ *also lies in* A^+*; AND*

2) *For each* $i = 0, 1$, *the subgraph of* A *induced by* $D_1[V(F_i)]$ *is a chordless cycle.*

Proof. Firstly, by Proposition 13.2.5, every facial subgraph of A, except those of $\{\mathcal{M}_0\} \cup \{F_0\} \cup \mathcal{C}^{\subseteq A}$, is a triangle. Since F_1 is $\mathcal{C}$-close, we fix a $C^\dagger \in \mathcal{C}$ with $C^\dagger \subseteq \text{Int}_G(F_1)$ such that $d(C^\dagger, F_1) \leq \beta' + \gamma + 3$. We now have the following.

Claim 13.3.2. *Let* $i \in \{0, 1\}$ *and* $\mathcal{F} := \mathcal{C}^{\subseteq A} \cup (\mathcal{M}_0 \setminus \{F_1\})$. *Then the following inequalities hold.*

1) *If* $i = 0$, *then, for each* $F \in \mathcal{F}$, *we have* $d(F, F_i) \geq \frac{\beta}{3} + 4N_{\text{mo}}$*; AND*

2) *If* $i = 1$, *then, for each* $F \in \mathcal{F}$, *we have* $d(F, F_i) \geq \beta'$.

Proof: Consider the following cases.

Case 1: $i = 0$

Note that $\frac{\beta}{3} + 4N_{\text{mo}} = \frac{\beta' + 8N_{\text{mo}}}{3}$. Suppose toward a contradiction that there is an $F^\dagger \in \mathcal{F}$ with $d(F^\dagger, F_0) < \frac{\beta' + 8N_{\text{mo}}}{3}$. Since $C^\dagger \subseteq \text{Int}_G(F_1)$, we have $C^\dagger \neq F^\dagger$. Since $D(F_0, F_1) \leq \beta' + 1$, it follows from two successive applications 1) of Lemma 13.2.12 that $d(C^\dagger, F^\dagger) \leq 2\beta' + \frac{\beta'}{3} + \gamma + \frac{8N_{\text{mo}}}{3} + 7$. If $F^\dagger \in \mathcal{C}$, then we contradict Proposition 13.2.14, so $F^\dagger \in \mathcal{M}_0 \setminus \{F_1\}$. But, by assumption, each element of $\mathcal{M}_0$ is $\mathcal{C}$-close, so we again contradict Proposition 13.2.14.

Case 2: $i = 1$

Suppose toward a contradiction that there is an $F^\dagger \in \mathcal{F}$ with $d(F^\dagger, F_1) < \beta'$. As above, since $C^\dagger \subseteq \text{Int}_G(F_1)$, we have $D^\dagger \neq C^\dagger$. By 1) of Lemma 13.2.12, we have $d(F^\dagger, C^\dagger) \leq 2\beta' + \gamma + 2$, which, as above, contradicts Proposition 13.2.14. ■

To prove 1) of Lemma 13.3.1, we prove something stronger in the form of the claim below. This is similar to the arguments of the main results of Section 2.1 but simpler because this argument takes place in a minimal chart, not a minimal mosaic.

Claim 13.3.3. *Let* $M \subseteq A$ *be a cycle which does not separate* F_0 *from* F_1 *and suppose that* $|V(M)| \leq 10$. *Then at least one of the following holds.*

1) *For any* $F^\dagger \in \mathcal{C}^{\subseteq A} \cup (\mathcal{M}_0 \setminus \{F_1\})$, *we have* $F^\dagger \subseteq \text{Ext}(M)$*; OR*

2) *There exists an* $F^\dagger \in \mathcal{C}^{\subseteq A} \cup (\mathcal{M}_0 \setminus \{F_1\})$ *such that* $F^\dagger \subseteq \text{Int}(D)$ *and* $\max\{d(v, F^\dagger) : v \in V(M)\} < \frac{\beta}{3} + \frac{3}{2}|V(M)| + 2N_{\text{mo}}$.

Proof: Given a cycle $M \subseteq A$, we say that M is *broken* if $|V(M)| \leq 10$ and M separates F_0 from F_1, but M satisfies neither 1) nor 2) above. Suppose toward a contradiction that there exists a broken cycle, and, among all broken cycles, we choose M to minimize the quantity $|V(\text{Int}_A(M))|$. Since M is a broken cycle, there exists at least one $F^\dagger \in \mathcal{C}^{\subseteq A} \cup (\mathcal{M}_0 \setminus \{F_1\})$ with $F^\dagger \subseteq \text{Int}_A(M)$, and, for any such $F^\dagger$, we have $\max\{d(v, F^\dagger) : v \in V(M)\} < \frac{\beta}{3} + \frac{3}{2}|V(M)| + 2N_{\text{mo}}$. Let $\mathcal{F} := \{F^\dagger \in \mathcal{C}^{\subseteq A} \cup (\mathcal{M}_0 \setminus \{F_1\}) : F^\dagger \subseteq \text{Int}_A(M)\}$. The minimality of $|V(\text{Int}_A(M))|$ immediately implies the following.

1) M has no chord in $\text{Int}_A(M)$; *AND*

2) For any $v \in D_1(M) \cap V(\text{Int}_A(M))$, the graph $G[N(v) \cap V(M)]$ is a subpath of M of length at most one; *AND*

3) There is at most one vertex of $D_1(M) \cap V(\text{Int}_A(M))$ adjacent to a subpath of M of length precisely one.

In particular, M is a highly predictable cyclic facial subgraph of $\text{Int}(M)$. Let $G' := G \setminus (V(\text{Int}_A(M)) \setminus V(M))$. Since M a broken cycle, we have $|V(G')| < |V(G)|$, and since $(G', \mathcal{C} \setminus \mathcal{F}, L, C_*)$ is an $(\alpha, 1)$-chart, it follows from the minimality of $\mathcal{T}$ that G' admits an L-coloring ϕ. Since M is an induced subgraph of $\text{Int}_A(M)$, ϕ restricts to a proper L-coloring of $V(M)$.

Now consider the tuple $\mathcal{T}^M := (\text{Int}_A(M), \mathcal{F} \cup \{M\} L_\phi^{V(M)}, M)$. We claim that $\mathcal{T}^M$ is a mosaic in which M is a closed ring and each element of $\mathcal{F} \cap \mathcal{M}_0$ is also a closed ring. Each cycle in $\{M\} \cup (\mathcal{F} \cap \mathcal{M}_0)$ is precolored by ϕ. Since A is short-separation-free, no element of $\mathcal{M}_0 \cap \mathcal{F}$ has a chord in A, so ϕ properly precolors each element of $\mathcal{M}_0 \cap \mathcal{F}$.

By Observation 2.1.2, each element of $\mathcal{M}_0 \cap \mathcal{F}$ is a highly predictable cyclic facial subgraph of $\text{Int}_A(M)$ and thus an $L_\phi^{V(D)}$-predictable facial subgraph of $\text{Int}A$. Since M is a highly predictable facial subgraph of $\text{Int}_A(M)$, it is also $L_\phi^{V(M)}$-predictable, and we also have $|V(M)| \leq 10 < N_{\text{mo}}$, so M0) of Definition 2.1.6 is satisfied. In particular, $\mathcal{T}^M$ is a tessellation, by the triangulation conditions satisfied by A. The only nontrivial part of Definition 2.1.6 to check is that the distance conditions are satisfied.

Since M is a broken cycle and any two vertices of M are of distance at most $\frac{|V(M)|}{2}$ apart, every element of $\mathcal{F}$ has distance at least $\frac{\beta}{3} + 2N_{\text{mo}} + \text{Rk}(\mathcal{T}^M | F^\dagger)$ from M. Since $\mathcal{T}$ is an $(\alpha, 1)$-chart, all of the elements of $\mathcal{F} \cap \mathcal{C}$ have distance at least α from each other. Furthermore, since every element of $\mathcal{M}_0$ is $\mathcal{C}$-close, it follows from Proposition 13.2.14 that all of the elements of $\mathcal{F}$ have distance at least β' from each other. Thus, $\mathcal{T}^M$ does indeed satisfy all the conditions of 2.1.6, so $\mathcal{T}^M$ is a mosaic. By Theorem 2.1.7, ϕ extends to an L-coloring of $\text{Int}_A(M)$, so ϕ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is a counterexample. ■

We now have enough to finish the proof of 1) of Lemma 13.3.1.

Claim 13.3.4. *For each $i \in \{0, 1\}$ and proper generalized chord P of F_i in A, if P has length at most six, then P does not separate F_{1-i} from any element of $\mathcal{C}^{\subseteq A} \cup (\mathcal{M}_0 \setminus \{F_1\})$*

Proof: Suppose toward a contradiction that there exists a generalized chord P of F_i of length at most six which separates F_{1-i} from an element $F^\dagger$ of $\mathcal{C}^{\subseteq A} \cup (\mathcal{M}_0 \setminus \{F_1\})$. Since each of F_0, F_1 has length at most four, there is a cycle $M \subseteq A$ of length at most 10, where M does not separate F_0 from F_1 and $V(M) \subseteq V(F_0 \cup F_1 \cup P)$, and $F^\dagger \subseteq \text{Int}(M)$. By Claim 13.3.3, there exists an $F^{\dagger\dagger} \in \mathcal{C}^{\subseteq A} \cup (\mathcal{M}_0 \setminus \{F_1\})$ such that $d(F^{\dagger\dagger}, D) < \frac{\beta}{3} + 15 + 2N_{\text{mo}}$ and since every vertex of P has distance at most 3 from $F_0 \cup F_1$, we contradict Claim 13.3.2. ■

Given the result of Claim 13.3.4, we now introduce the following notation. For each $i \in \{0, 1\}$ and proper generalized chord P of F_i of length at most six, we let $A = A_{i,P}^- \cup A_{i,P}^+$ be the natural P-partition of A, where $F_{1-i} \subseteq A_{i,P}^+$, and, furthermore, each element of $\mathcal{C}^{\subseteq A} \cup (\mathcal{M}_0 \setminus \{F_1\})$ also lies in $A_{i,P}^+$. We now prove 2) of Lemma 13.3.1.

Let $i \in \{0, 1\}$. Since $d(F_0, F_1)| \geq 2$, it follows from our triangulation conditions, together with Observation 2.1.2, that there is a cycle C^i such that $V(C^i) = D_1(F_i, A)$, where, for each $v \in (C^i)$, the graph $A[N(v) \cap V(F_i)]$ is a subpath of F_i of length at most one. We just need to check that C^i is an induced subgraph of A. Suppose toward a contradiction that there is a chord ww' of C^i in A. Let $p \in N(w) \cap V(F_i)$ and $p' \in N(w') \cap V(F_i)$. Since A is short-separation-free, we have $|V(F_i)| = 4$ and p, p' are opposing vertices of F_i. Furthermore, $N(w) \cap V(F_i) = \{p\}$ and $N(w') \cap V(F_i) = \{p'\}$. Let $S := V(A^i_{i,P}) \setminus (V(F_i) \cup \{w, w'\})$. Since ww' is a chord of C^i, we have $S \neq \varnothing$.

Let $P := pww'p'$. Since $|V(F_i)| = 4$ and p, p' are nonadjacent vertices of F_i, let q be the midpoint of the 2-path $F_i \cap A^-_{i,P}$. Now, $D := pww'p'q$ is a cyclic facial subgraph of $A^-_{i,P}$. Since $S \neq \varnothing$, there is no chord of D in $A^-_{i,P}$, or else there is a cycle in A of length at most four which separates S from F_{1-i}, contradicting the fact that A is short-separation-free.

Let $G' := G \setminus S$. By Claim 13.3.4, each element of $\mathcal{C}$ lies in G'. Now, $(G', \mathcal{C}, L, C_*)$ is an $(\alpha, 1)$-chart. Since $S \neq \varnothing$, it follows from the minimality of G that G' admits an L-coloring ϕ. Since D has no chords in $A^-_{i,P}$, ϕ restricts to a proper L-coloring of $V(D)$. Since G is a counterexample, ϕ does not extend to L-color the vertices of S. Since all the vertices of S have L-lists of size at least five, it follows from Theorem 1.3.5 that there is a lone vertex of S adjacent to all five vertices of D, so there is a vertex of S adjacent to all of p, q, p', so A contains a 4-cycle which separates q from F_{1-i}, contradicting the fact that A is short-separation-free. This completes the proof of Lemma 13.3.1. $\square$

13.4 Completing the proof of Theorem 1.1.3

We now prove the following key proposition, which is the last ingredient we need in order to complete the proof of Theorem 13.0.1, and thus the proof of Theorem 1.1.3 as well.

Proposition 13.4.1. *Let $\mathcal{T} = (G, \mathcal{C}, L, C_*)$ be a critical chart. Then every $D \in Sep(G)$ is $\mathcal{C}$-close.*

Proof. Suppose toward a contradiction that there exists a $D^{\text{mc}} \in Sep(G)$ which is not $\mathcal{C}$-close, and furthermore, among all elements of $Sep(G)$ which are not $\mathcal{C}$-close, we choose D^{mc} so as to minimize $|V(\text{Int}(D^{\text{mc}}))|$. Note that $\mathcal{I}^m(D^{\text{mc}}) \neq \varnothing$ or else D^{mc} is a minimal element of $Sep(G)$ and is thus $\mathcal{C}$-close by 2) of Lemma 13.2.10. For each $D \in \{D^{\text{mc}}\} \cup \mathcal{I}(D^{\text{mc}})$, every element of $\mathcal{I}^m(D)$ is $\mathcal{C}$-close by our choice of D^{mc}. By Proposition 13.2.15, there exists a system $\mathcal{M}_D \subseteq \mathcal{I}^m(D)$ of distinct representatives of the $\sim$-equivalence classes of $\mathcal{I}^m(D)$ such that $A(D|\mathcal{M}_D)$ is short-separation-free.

Given a $D \in \{D^{\text{mc}}\} \cup \mathcal{I}(D^{\text{mc}})$, we say that D is an *obstructing cycle* if there exists a $D' \in \mathcal{M}_D$ such that $d(D, D') \leq \beta' + 1$. If there exist two distinct $D', D'' \in \mathcal{M}_D$ such that $d(D, D'') \leq \beta' + 1$ and $d(D, D') \leq \beta' + 1$, then $d(D', D'') \leq 2(\beta' + 1) + 2$ by 1) of Lemma 13.2.12, contradicting Proposition 13.2.14. Thus, if D is an obstructing cycle, the corresponding $D' \in \mathcal{M}_D$ is unique.

Claim 13.4.2. *D^{mc} is an obstructing cycle.*

<u>Proof:</u> Suppose toward a contradiction that D^{mc} is not an obstructing cycle, and set $A^* := A(D^{\text{mc}}|\mathcal{M}_{D^{\text{mc}}})$.

Subclaim 13.4.3. *For each $D \in \{D^{\text{mc}}\} \cup \mathcal{M}_{D^{\text{mc}}}$, the following hold.*

1) *For each $C \in \mathcal{C}^{\subseteq A^*}$, we have $d(C, D) \geq \beta' + 1$; AND*

2) *For each $D' \in \{D^{\text{mc}}\} \cup \mathcal{M}_{D^{\text{mc}}}$ with $D' \neq D$, we have $d(D, D') \geq \beta' + 2$.*

<u>Proof:</u> We break this into two cases.

Case 1: $D = D^{\text{mc}}$.

Since D^{mc} is not an obstructing cycle, we have $d(D^{\text{mc}}, D') \geq \beta' + 2$ for each $D' \in \mathcal{M}_{D^{\text{mc}}}$. This proves 2). Now we check that $d(C, D^{\text{mc}}) \geq \beta' + 1$ for all $C \in \mathcal{C}^{\subseteq A^*}$. If D^{mc} is a blue cycle, then this immediately follows from the fact that D^{mc} is not an obstructing cycle, since each element of $\mathcal{C}^{\subseteq A^*}$ is separated from D^{mc} by an element of $\mathcal{M}_{D^{\text{mc}}}$. On the other hand, if D^{mc} is a red cycle, then this is true by our assumption that D^{mc} is not $\mathcal{C}$-close.

Case 2: $D \neq D^{\text{mc}}$

In this case, for any $C \in \mathcal{C}^{\subseteq A^*}$, we immediately have $d(C, D) \geq \beta' + 1$ by Proposition 13.2.14. This proves 1). To finish, it suffices to prove that 2) holds for each $D' \in \mathcal{M}_{D^{\text{mc}}}$, since, if $D' = D^{\text{mc}}$, then we are back to Case 1 with the roles of D, D' interchanged. Applying Proposition 13.2.14 again, it follows that, for any $D' \in \mathcal{M}_{D^{\text{mc}}}$, we have $d(D, D') \geq \beta' + 2$, so we are done. ■

By Proposition 13.2.13, there is an L-coloring ϕ of $\bigcup(\text{Int}^+(D) : D \in \mathcal{M}_{D^{\text{mc}}})$. By Lemma 13.2.8, $\text{Ext}^+(D^{\text{mc}})$ is L-colorable. By Subclaim 13.4.3, the graphs $\bigcup(\text{Int}^+(D) : D \in \mathcal{M}_{D^{\text{mc}}})$ and $\text{Ext}^+(D^{\text{mc}})$ are of distance at least $\beta' + 2$ apart in G, so $\phi \cup \psi$ is a proper L-coloring of its domain. Applying Subclaim 13.4.3, together with Proposition 13.2.5, $\phi \cup \psi$ extends to an L-coloring of A^*, and thus G is L-colorable, which is false. ■

We now break the remainder of the proof of Proposition 13.4.1 into two cases.

Case 1 of Proposition 13.4.1: There exists an obstructing cycle $D^\dagger \in \{D^{\text{mc}}\} \cup \mathcal{I}(D^{\text{mc}})$ such that either $D^\dagger$ is red or $|\mathcal{M}_{D^\dagger}| > 1$.

In this case, since $D^\dagger$ is an obstructing cycle, let D' be the unique element of $\mathcal{M}_{D^\dagger}$ be such that $d(D', D^\dagger) \leq \beta' + 1$. Let $A^\dagger := A(D^\dagger | \mathcal{M}_{D^\dagger})$, and let $R^\dagger$ be the subgraph of G induced by set $V(\text{Ext}(D^\dagger)) \cup \bigcup(V(\text{Int}(D)) : D \in \mathcal{M}_{D^\dagger})$. Note that $R^\dagger$ consists of the family $\{D^\dagger\} \cup \mathcal{M}_{D^\dagger}$ of boundary cycles of $A^\dagger$ and their chords, together with all of the edges and vertices of G that intersect with $A^\dagger$ precisely on this family of boundary cycles. Let $R^\dagger_{\text{aug}}$ be the subgraph of G induced by the vertex set $V(R^\dagger) \cup B_{11+(\beta'/3)}(D' \cup D^\dagger, A^\dagger)$. That is, we augment $R^\dagger$ by a set of vertices within a small ball within $A^\dagger$ around $D' \cup D^\dagger$. The idea here is that we show that there is an L-coloring of $R^\dagger$ which extends to a small ball in $A^\dagger$ around $D' \cup D^\dagger$, and then apply the results of Sections 12.2-12.5.

Claim 13.4.4. *$|V(R^\dagger_{\text{aug}})| < |V(G)|$ and $R^\dagger_{\text{aug}}$ is L-colorable.*

<u>Proof:</u> We first show that $(R^\dagger_{\text{aug}}, \mathcal{C}^{\subseteq G \setminus A^\dagger}, L)$ is an $(\alpha, 1)$-chart. To prove this, it suffices to show that, for any $C \in \mathcal{C}$, if C has nonempty intersection with $R^\dagger_{\text{aug}}$, then $C \subseteq G \setminus A^*$. If this holds then it immediately follows that $(R^*_{\text{aug}}, \mathcal{C}^{\subseteq G \setminus A^*}, L)$ is an $(\alpha, 1)$-chart., since $(G, \mathcal{C}, L)$ is an $(\alpha, 1)$-chart.

Suppose toward a contradiction that there is a $C \in \mathcal{C}$ with nonempty intersection with $R^\dagger_{\text{aug}}$ such that $C \not\subseteq G \setminus A^\dagger$. In that case, since C is a facial subgraph of G, and each cycle in $\{D^\dagger\} \cup \{\mathcal{M}_{D^\dagger}\}$ is a separating cycle of G, we have $C \subseteq A^\dagger$. Since $V(C) \cap V(R^*_{\text{aug}}) \neq \varnothing$, we have $d(C, D' \cup D^\dagger) \leq 11 + \frac{\beta'}{3}$. Since $D^\dagger$ is not $\mathcal{C}$-close, we have $d(C, D') \leq 11 + \frac{\beta'}{3}$, contradicting Proposition 13.2.14. Thus, we conclude that the tuple $(R^\dagger_{\text{aug}}, \mathcal{C}^{\subseteq G \setminus A^\dagger}, L)$ is indeed an $(\alpha, 1)$-chart. To finish the proof of Claim 13.4.4, it just suffices to check that $|V(R^\dagger_{\text{aug}})| < |V(G)|$, and it then follows from the minimality of $\mathcal{T}$ that $R^\dagger_{\text{aug}}$ is L-colorable.

We break this into two cases. Suppose first that $D^\dagger$ is red. In this case, by definition, there exists a $C^* \in \mathcal{C}$ with $C^* \subseteq A^\dagger$. Furthermore, $d(C^*, D') \geq \frac{7\beta'}{2} + 3\gamma + 16$ by Proposition 13.2.14, and thus, since $d(D', D^\dagger) \leq \beta' + 1$, C^* is disjoint to $R^\dagger_{\text{aug}}$. Thus, we indeed have $|V(R^\dagger_{\text{aug}})| < |V(G)|$ in this case.

Now suppose that $D^\dagger$ is blue. Thus, we have $|\mathcal{M}_{D^\dagger}| > 1$ by the assumption of Case 1, and by Proposition 13.2.14, there exists a $D'' \in \mathcal{M}_{D^\dagger}$ such that D'' is of distance at least $\frac{5\beta'}{2} + 8$ from D'. Furthermore, since D' is the unique cycle of $\mathcal{I}^m(D^\dagger)$ of distance at most $\beta'+1$ from $D^\dagger$, we have $d(D'', D^\dagger) > \beta'+1$. Thus, again, we have $|V(R^\dagger_{\text{aug}})| < |V(G)|$, as desired. ■

Since $R^\dagger_{\text{aug}}$ is L-colorable, let ϕ be an L-coloring of $R^\dagger_{\text{aug}}$ and let $\phi^\dagger$ be the restriction of ϕ to $V(D^\dagger)\cup(\bigcup V(D) : D \in \mathcal{M}_{D^\dagger})$, and let $L^\dagger$ be a list-assignment for $V(A^\dagger)$ in which each vertex of dom$(\phi^\dagger)$ is precolored by $\phi^\dagger$, and otherwise $L^\dagger = L$. Recalling Definition 12.3.1, we now want apply the work of Sections 12.2-12.4. To do this, we need to check that the annulus between $D^\dagger$ and D' is a roulette wheel, i.e that the tuple $(A^\dagger, D^\dagger, D', L^\dagger, \phi^\dagger)$.

Claim 13.4.5. *$A^\dagger$ is not $L^\dagger$-colorable*

Proof: if $A^\dagger$ is $L^\dagger$-colorable, then, since $\phi^\dagger$ extends to an L-coloring of $R^\dagger$, $\phi^\dagger$ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is a critical chart. ■

Now we can apply the work of Sections 12.2-12.4.

Claim 13.4.6. *$(A^\dagger, D^\dagger, D', L^\dagger, \phi^\dagger)$ is a roulette wheel.*

Proof: We first check that Ro1), Ro2), and Ro3) of Definition 12.3.1 hold. By Claim 13.4.4, we have $|V(R^\dagger_{\text{aug}})| < |V(G)|$, and 2) follows immediately. By Claim 13.2.3, G is connected, so $A^\dagger$ is also connected. By our choice of $\mathcal{M}_{D^\dagger}$, $A^\dagger$ is short-separation-free, so we have 1) of Definition 12.3.1 as well. Furthermore, it follows from Proposition 13.2.5 that each facial subgraph of $A^\dagger$, except those of $\mathcal{C}^{\subseteq A^\dagger} \cup \{D^\dagger\} \cup \mathcal{M}_{D^\dagger}$, is a triangle. Since $d(D', D^\dagger) \leq \beta+1$, it follows from Proposition 13.2.14 that, for all $v \in B_{\frac{\beta'+3}{2}}(D^\dagger \cup D', A^\dagger) \setminus V(D^\dagger \cup D')$, we have $|L(v)| \geq 5$ and thus $|L^\dagger(v)| \geq 5$. Since each facial subgraph of $A^\dagger$, except those of $\mathcal{C}^{\subseteq A^\dagger} \cup \{D^\dagger\} \cup \mathcal{M}_{D^\dagger}$, is a triangle, it also follows from Proposition 13.2.14 that, for all $v \in B_{\frac{\beta'+3}{2}}(D^\dagger \cup D', A^\dagger)$, every facial subgraph of $A^\dagger$ containing v, except $D^\dagger, D'$, is a triangle. Thus, $(A^\dagger, D^\dagger, D', L^\dagger, \phi^\dagger)$ satisfies Ro1)-Ro3) of Definition 12.3.1. By Lemma 13.3.1, $(A^\dagger, D^\dagger, D', L^\dagger, \phi^\dagger)$ also satisfies Ro4) of Definition 12.3.1, so we are done. ■

Claim 13.4.7. *For each $D \in \mathcal{M}_{D^\dagger} \setminus \{D'\}$ and $C \in \mathcal{C}^{\subseteq A^\dagger}$, the following conditions hold.*

1) $d(D, D^\dagger \cup D') > \frac{3\beta'}{2} + 4$*; AND*
2) $d(C, D^\dagger \cup D') > \frac{5\beta'}{2}$.

Proof: Suppose toward a contradiction that there is a $D \in \mathcal{M}_{D^\dagger} \setminus \{D'\}$ such that $d(D, D^\dagger \cup D') \leq \frac{3\beta'}{2} + 4$. Since $d(D', D^\dagger) \leq \beta'+1$, it follows from 1) of Lemma 13.2.12 that $d(D', D) \leq \frac{5}{2}\beta' + 7$, contradicting Proposition 13.2.14. Now suppose toward a contradiction that there is a $C \in \mathcal{C}^{\subseteq A^\dagger}$ such that $d(C, D^\dagger \cup D') \leq \frac{5\beta'}{2}$. Again, since $d(D^\dagger, D') \leq \beta'+1$, it follows from 1) of Lemma 13.2.12 that $d(D', C) \leq \frac{7\beta'}{2} + 3$, which again contradicts Proposition 13.2.14. Thus, condition 2) holds. ■

Appplying Claim 13.4.6, since $(A^\dagger, D^\dagger, D', L^\dagger, \phi^\dagger)$ is a roulette wheel, it satisfies either S1 or S2 of Theorem 12.3.3, so we break Case 1 into two subcases.

Subcase 1.1: $(A^\dagger, D^\dagger, D', L^\dagger, \phi^\dagger)$ satisfies S2 of Theorem 12.3.3.

In this case, let $[K; Q; \sigma; Z]$ be a cycle connector for $(A^\dagger, D^\dagger, D', L^\dagger, \phi^\dagger)$. Consider the graph $A^* := A^\dagger \setminus (V(K \setminus Q))$. Since $K \setminus Q$ is connected and has nonempty intersection with each of $D^\dagger, D'$, it follows from Theorem 1.3.2 that graph A^* contains a facial subgraph F^* such that $V(F^*) \subseteq V(D^\dagger \cup D') \cup D_1(K \setminus Q, A^*)$ and $Q \subseteq F^*$. Furthermore, F^* is a Thomassen facial subgraph of A^* with respect to the list-assignment $(L^\dagger)^Q_\sigma$.

For each $D \in \mathcal{M}_{D^\dagger} \setminus \{D'\}$, let P_D be a subpath of D of length $|V(D)| - 3$ and let $P := \bigcup(P_D : D \in \mathcal{M}_{D^\dagger} \setminus \{D'\})$. Since the disjoint paths in P are pairwise far apart, we obtain the following by Theorem 1.3.2: For each $D \in \mathcal{M}_{D^\dagger} \setminus \{D'\}$, there is a facial subgraph F_D of $A^* \setminus P$ such that $V(F_D) = V(D \setminus P) \cup D_1(P_D, A^*)$. Let $\mathcal{F} = \{F^*\} \cup \{F_D : D \in \mathcal{M}_{D^\dagger} \setminus \{D'\}\}$. Let $P^* := Q \cup \bigcup(D \setminus P_D : D \in \mathcal{M}_{D^\dagger} \setminus \{D'\})$. That is, P is the union of all the paths in A^* that we delete in order to produce our Thomassen facial subgraphs and P^* is the union of the precolored edges that we are retaining.

Claim 13.4.8. *$(A^* \setminus P,\ \mathcal{C}^{\subseteq A^*} \cup \mathcal{F}, (L^\dagger)^{P^*}_{\phi^\dagger \cup \sigma})$ is a $(\beta', 1)$-tessellation.*

Proof: To show this, it suffices to check that the following distance conditions hold.

1) $d(F^*, F_D) \geq \beta'$ for all $D \in \mathcal{M}_{D^\dagger} \setminus \{D'\}$; *AND*
2) $d(F^*, C) \geq \beta'$ for all $C \in \mathcal{C}^{\subseteq A^*}$; *AND*
3) $d(F_D, C) \geq \beta'$ for all $C \in \mathcal{C}^{\subseteq A^*}$ and $D \in \mathcal{M}_{D^\dagger} \setminus \{D'\}$.

Let $r := \lfloor \frac{\beta'+1}{2} \rfloor$. Since $d(D', D^\dagger) \leq \beta' + 1$, it follows from the definition of $[K; Q; \sigma; Z]$ that $V(K) \subseteq B_{r+1}(D^\dagger \cup D', A^*)$. Recall that $V(F^*) \subseteq V(D^\dagger \cup D') \cup D_1(K \setminus Q, A^*)$. Thus, we have $V(F^*) \subseteq B_{r+2}(D^\dagger \cup D', A^*)$. Suppose toward a contradiction that there is a $D \in \mathcal{M}_{D^\dagger} \setminus \{D'\}$ such that $d(F^*, F_D) < \beta'$. Thus, we have $d(F^*, D) < \beta' + 1$, so D is of distance at most $\frac{3\beta'}{2} + 4$ from $V(D^\dagger \cup D')$, contradicting 1) of Claim 13.4.7 Thus, condition 1) holds.

Now suppose toward a contradiction that there is a $C \in \mathcal{C}^{\subseteq A^*}$ such that $d(F^*, C) < \beta'$. In that case, we have $d(D^\dagger \cup D', C) \leq \frac{3}{2}\beta' + 2$, contradicting 2) of Claim 13.4.7. Likewise, if there exists a $C \in \mathcal{C}^{\subseteq A^*}$ and a $D \in \mathcal{M}_{D^\dagger} \setminus \{D'\}$ such that $d(F_D, C) < \beta'$, then $d(D, C) \leq \beta'$, contradicting Proposition 13.2.14. ■

Since $(A^* \setminus P,\ \mathcal{C}^{\subseteq A^*} \cup \mathcal{F}, (L^\dagger)^{P^*}_{\phi^\dagger \cup \sigma})$ is a $(\beta', 1)$-tessellation, it is a mosaic, and thus, by Theorem 2.1.7, $A^* \setminus P$ is $(L^\dagger)^{P^*}_{\phi^\dagger \cup \sigma})$-colorable, so A^* is $(L^\dagger)^{Q}_{\sigma})$-colorable. Let ψ be an $(L^\dagger)^{Q}_{\sigma})$-coloring of A^*. Since Z is $(L^\dagger, \sigma)$-inert, and σ is an $L^\dagger$-coloring of $K \setminus Z$, it follows that ψ extends to an $L^\dagger$-coloring of $A^\dagger$, contradicting Claim 13.4.5. This completes Subcase 1.1.

Subcase 1.2: $(A^\dagger, D^\dagger, D', L^\dagger, \phi^\dagger)$ satisfies S1 of Theorem 12.3.3.

In this case, by Theorem 12.3.3, there exists a 2-connected subgraph H of $A^\dagger$ such that $D^\dagger \cup D' \subseteq H$, $V(H) \subseteq B_{\beta'/3}(D^\dagger \cup D', A^\dagger)$ and each facial subgraph of H is a cycle of length at most 11. Now we apply the work of Section 12.2. Since $\phi^\dagger$ extends to an L-coloring of $B_{(\beta'/3)+11}(D^\dagger \cup D', A^\dagger)$, there exists an $L^\dagger$-coloring τ of $V(H)$ such that, for each facial subgraph K of H, the tuple $(\text{Int}_{A^\dagger}(K), K, L^\dagger, \tau)$ is an 11-lens.

Recalling Definition 12.2.1, there exists an inward-facing cyclic facial subgraph K of H such that τ does not extend to an $L^\dagger$-coloring of $\text{Int}_{A^\dagger}(K)$, or else $A^\dagger$ is $L^\dagger$-colorable, contradicting Claim 13.4.4. By Theorem 12.2.10 since $|V(K)| \leq 11$, the tuple $\mathcal{L} := (\text{Int}_{A^\dagger}(K), K, L^\dagger, \tau)$ is 11-partitionable. Let $\mathcal{L} := (\text{Int}_{A^\dagger}(K), K, L^\dagger, \tau)$. By Definition 12.2.9, there exists an 11-partitioning pair (K^*, τ^*) for $\mathcal{L}$. Since τ does not extend to an $L^\dagger$-coloring of $\text{Int}_{A^\dagger}(K)$, there exists an inward-facing facial subgraph F of K^* such that τ^* does not extend to an $L^\dagger$-coloring of $\text{Int}_{A^\dagger}(F)$.

By Definition 12.2.9, there exist a subset $Z \subseteq B_1(F) \cap V(\text{Int}_{A^\dagger}(K^*))$, a subpath Q of $F \setminus Z$ of length at most one, and a partial $L^\dagger_{\tau^*}$-coloring ψ of $B_1(F) \cap V(\text{Int}_{A^\dagger}(K^*))$ such that, letting $A^* := \text{Int}_{A^\dagger}(F) \setminus ((Z \cup \text{dom}(\tau^* \cup \psi)) \setminus V(Q))$, the following holds:

1. The outer face F^* of A^* is a Thomassen facial subgraph of A^* with respect to $(L^\dagger)^{Q}_{\tau^* \cup \psi}$; *AND*
2. Z is $(L^\dagger, \tau^* \cup \psi)$-inert in $\text{Int}_{A^\dagger}(K)$.

Let $\mathcal{M}^* := \{D \in \mathcal{M}_{D^\dagger} : D \subseteq \text{Int}_{A^\dagger}(F)\}$. For each $D \in \mathcal{M}^*$, let P_D be a subpath of D of length $|V(D)| - 3$ and let $P := \bigcup(P_D : D \in \mathcal{M}^*)$. Since the disjoint paths in P are pairwise far apart, we obtain the following by Theorem 1.3.2: For each $D \in \mathcal{M}^*$, there is a facial subgraph F_D of $A^* \setminus P$ such that $V(F_D) = V(D \setminus P) \cup D_1(P_D, A^*)$. Let $\mathcal{F} := \{F^*\} \cup \{F_D : D \in \mathcal{M}^*\}$ and let $\psi^\dagger := \phi^\dagger \cup \tau^* \cup \psi$. Finally, let $P^* := Q \cup \bigcup(D \setminus P_D : D \in \mathcal{M}^*)$. That is, as the previous subcase, P is the union of all the paths in A^* that we delete in order to produce our Thomassen facial subgraphs and P^* is the union of the precolored edges that we are retaining.

Claim 13.4.9. *$(A^* \setminus P, \mathcal{C}^{\subseteq A^*} \cup \mathcal{F}, (L^\dagger)^{P^*}_{\psi^\dagger})$ is a $(\beta', 1)$-tessellation.*

Proof: As in the previous subcase, it suffices to check that the following distance conditions hold.

1) $d(F^*, F_D) \geq \beta'$ for all $D \in \mathcal{M}^*$; *AND*
2) $d(F^*, C) \geq \beta'$ for all $C \in \mathcal{C}^{\subseteq A^*}$; *AND*
3) $d(F_D, C) \geq \beta'$ for all $C \in \mathcal{C}^{\subseteq A^*}$ and $D \in \mathcal{M}^*$.

Recall that $V(H) \subseteq B_{\beta/3}(D^\dagger \cup D', A^\dagger)$, K is a facial subgraph of H, and, since (K^*, τ^*) is an 11-partitioning pair for $\mathcal{L}$, we have $V(F) \subseteq B_{10}(K)$, so $V(F) \subseteq B_{(\beta'/3)+10}(D^\dagger \cup D', A^\dagger)$. By definition of (K^*, τ^*), we have $V(F^*) \subseteq B_2(F, A^*)$, and thus $V(F^*) \subseteq B_{(\beta'/3)+12}(D^\dagger \cup D', A^\dagger)$.

Suppose toward a contradiction that there is a $D \in \mathcal{M}^*$ with $d(F^*, F_D) < \beta'$. Thus, we have $d(F^*, D) < \beta' + 1$, so $d(D^\dagger \cup D', D) < (\beta' + 1) + \frac{\beta'}{3} + 12$, contradicting Claim 13.4.7. This proves 1). Now suppose toward a contradiction that there is a $C \in \mathcal{C}^{\subseteq A^*}$ with $d(F^*, C) < \beta'$. Then we have $d(D \cup D', C) < (\beta' + 1) + \frac{\beta'}{3} + 12$, which again contradicts Claim 13.4.7. Finally ,if there is a $D \in \mathcal{M}^*$. and a $C \in \mathcal{C}^{\subseteq A^*}$ such that $d(F_D, C) < \beta'$, then we have $d(D, C) \leq \beta'$, contradicting Proposition 13.2.14. ■

Since $(A^* \setminus P, \mathcal{C}^{\subseteq A^*} \cup \mathcal{F}, (L^\dagger)^{P^*}_{\psi^\dagger})$ is a $(\beta', 1)$-tesselllation, it is a mosaic, and thus, by Theorem 2.1.7, $A^* \setminus P$ is $(L^\dagger)^{P^*}_{\psi^\dagger}$-colorable, and thus $\psi^\dagger$ extends to an $L^\dagger$-coloring of A^*. Since Z is $(L^\dagger, \tau^* \cup \psi)$-inert in $\text{Int}_{A^\dagger}(K)$, it follows that $\tau^* \cup \psi$ extends to an $L^\dagger$-coloring of A^*, contradicting our choice of F. Thus, we have ruled out Case 1 of Proposition 13.4.1.

Case 2 of Proposition 13.4.1: For all obstructing cycles $D^\dagger \in \{D^{\text{mc}}\} \cup \mathcal{I}(D^{\text{mc}})$, $D^\dagger$ is blue and $|\mathcal{M}_{D^\dagger}| = 1$.

In this case, since D^{mc} is an obstructing cycle by assumption, we have $D^{\text{mc}} \in Sep_b(G)$, and so $Sep_r(G) \cap \mathcal{I}(D^{\text{mc}}) \neq \varnothing$ by 1) of Lemma 13.2.10. Let D_r be a maximal element of $Sep_r(G) \cap \mathcal{I}(D^{\text{mc}})$.

Claim 13.4.10. *For each $D \in \{D^{\text{mc}}\} \cup \mathcal{I}(D^{\text{mc}})$, if D is blue, then D is an obstructing cycle.*

Proof: If $D = D^{\text{mc}}$, then this holds by assumption. If $D \neq D^{\text{mc}}$, then D is $\mathcal{C}$-close by assumption, and thus there exists a $D^* \in \mathcal{I}^m(D)$ such that $d(D, D^*) \leq \gamma + 1$. Since $\gamma \leq \beta'$, D is indeed an obstructing cycle. ■

Now set $A^* := \text{Int}(D^{\text{mc}}) \cap \text{Ext}(D_r)$.

Claim 13.4.11. *For every $v \in V(A^*) \setminus V(D^{\text{mc}} \cup D_r)$, $|L(v)| \geq 5$.*

Proof: Since D_r is a red cycle and D_r is $\mathcal{C}$-close by assumption, there exists a $C^* \in \mathcal{C}$ with $C^* \subseteq \text{Int}(D_r)$ and $d(C^*, D_r) \leq \beta'$. To prove the claim, it suffices to show there does not exist a $C \in \mathcal{C}$ such that $C \subseteq A^*$. Suppose toward a contradiction that such a C exists. Since D^{mc} is a blue cycle, there exists a red cycle $D'_r \in \mathcal{I}(D^{\text{mc}})$ such that $D'_r \neq D_r$, $D'_r \subseteq A^*$, and D'_r separates C from D^{mc}. Since D'_r is $\mathcal{C}$-close by assumption, there exists a cycle $C^\dagger \in \mathcal{C}$

with $C^\dagger \subseteq \text{Int}(D'_r)$ and $d(C^\dagger, D'_r) \leq \beta$. Since $C^\dagger \neq C^*$, we have $d(C^\dagger, C^*) \geq \alpha$ and thus $d(D_r, D'_r) \geq (\alpha - 4) - 2\beta$ by 1) of Lemma 13.2.12. It follows that $D_r \not\sim D'_r$.

Let $U := \{D \in \mathit{Sep}(G) : D_r \cup D'_r \subseteq \text{Int}(D)\}$. Note that $U \neq \varnothing$, since $D^{\text{mc}} \in U$. Among all elements of U, choose $D^u \in U$ so as to minimize the quantity $|E(\text{Int}_G(D^u))|$. Since D_r is a maximal element of $\mathit{Sep}_r(G) \cap \mathcal{I}(D^{\text{mc}})$ and $D'_r \not\subseteq \text{Int}(D_r)$, we have $D^u \in \mathit{Sep}_b(G)$. By Claim 13.4.10, every blue cycle in $\{D^{\text{mc}}\} \cup \mathcal{I}(D^{\text{mc}})$ is an obstructing cycle, so we have $|\mathcal{M}_{D^u}| = 1$. If $D_r \in \mathcal{I}^m(D^u)$, then, since $|\mathcal{M}_{D^u}| = 1$ and $D_r \not\sim D'_r$, it follows that D'_r is a descendant of D_r, which is false. The same argument shows that $D'_r \notin \mathcal{I}^m(D^u)$.

Let D' be the lone element of $\mathcal{M}_{D^u}$. Note that $D_r \cup D'_r \not\subseteq \text{Int}(D')$, or else D' contradicts the minimality of $|E(\text{Int}_G(D^u))|$. Since neither D_r nor D'_r lies in $\mathcal{I}^m(D^u)$, we have $D_r \not\sim D'$ and $D'_r \not\sim D'$. Thus, at least one of D_r, D'_r is separated from D^u by the deletion of D', and $\mathcal{I}^m(D^u)$ contains at least one equivalence class distinct from that of D', contradicting the fact that $|\mathcal{M}_{D^u}| = 1$. ■

Since $D^{\text{mc}} \in \mathit{Sep}_b(G)$ and D^{mc} is not $\mathcal{C}$-close, we have $d(D_r, D^{\text{mc}}) > \gamma + 1$. By Lemma 13.2.8, there is an L-coloring ϕ of $\text{Int}^+(D_r)$ and an L-coloring ψ of $\text{Ext}^+(C)$. Since D_r, D^{mc} are of distance at least $\gamma + 2$ apart, $\phi \cup \psi$ a proper L-coloring of its domain. Since each vertex of $A^* \setminus (D^{\text{mc}} \cup D_r)$ has an L-list of size at least five, it follows from Theorem 0.2.6 that $\phi \cup \psi$ extends to an L-coloring of A^*, so G is L-colorable, contradicting the fact that $\mathcal{T}$ is a counterexample. This completes the proof of Proposition 13.4.1. □

With the results above in hand, we are ready to finish the proof of Theorem 13.0.1.

Theorem 13.0.1. *Let γ be as in Theorem 0.2.6 and let $\beta := \frac{17}{15} N_{\text{mo}}$. Let $\alpha := \frac{9}{2}(\beta + 4N_{\text{mo}}) + 3\gamma + 18$. Then every $(\alpha, 1)$-chart is colorable.*

Proof. Suppose not. Thus, there exists a critical chart $\mathcal{T} = (G, \mathcal{C}, L, C_*)$. By Lemma 13.2.8, $\mathit{Sep}(G) \neq \varnothing$, so let $\mathcal{M}$ be the set of maximal elements of $\mathit{Sep}(G)$. By Proposition 13.4.1, for each $M \in \mathcal{M}$ and each $D \in \{M\} \cup \mathcal{I}(M)$, D is $\mathcal{C}$-close. Thus, $\mathcal{M}$ admits a partition into equivalence classes under the relation $\sim$, and furthermore, by Proposition 13.2.15, there exists a system $\mathcal{M}^* \subseteq \mathcal{M}$ of distinct representatives of the $\sim$-equivalence classes of $\mathcal{M}$ such that $A(C^o|\mathcal{M}^*)$ is short-separation-free. Note that $A(C_*|\mathcal{M}^*) = \bigcap_{M \in \mathcal{M}^*} \text{Ext}(M)$. By Proposition 13.2.13, there is an L-coloring ϕ of $\bigcup_{M \in \mathcal{M}^*} \text{Int}^+(M)$. By Proposition 13.2.14, the following distance conditions are satisfied.

1) For any $D \in \mathcal{M}^*$ and any $C \in \mathcal{C}$ with $C \subseteq A(C_*|\mathcal{M}^*)$, we have $d(C, D) \geq \beta' + 1$; *AND*
2) For any distinct $D, D' \in \mathcal{M}^*$, we have $d(D, D') \geq \beta' + 2$.

Thus, by Proposition 13.2.5, ϕ extends to an L-coloring of G, contradicting the fact that $\mathcal{T}$ is a critical chart. This completes the proof of Theorem 13.0.1. □

With the above in hand, we finally complete the proof of Theorem 1.1.3, which we restate below.

Theorem 1.1.3. *Every $(48749 + 3\gamma, 1)$-chart is colorable, where γ is as in Theorem 0.2.6.*

Proof. As indicated at the start of Chapter 2, the lower bound we need on N_{mo} in order to ensure that all mosaics are colorable is $N_{\text{mo}} = 96$, so we obtain $\lceil \beta + 4N_{\text{mo}} \rceil = 10829$ and thus, by Theorem 13.0.1, all $(\alpha, 1)$-charts are colorable for $\alpha \geq 48749 + 3\gamma$. This completes the proof of Theorem 1.1.3. □

Chapter 14

Drawings with Pairwise Far-Apart Nonplanar Regions

In this section, we apply Theorem 1.1.3 to obtain a result about the 5-choosability of graphs G which are not too far from being planar in the sense that there is a drawing of G in the plane such that the pairs of crossing edges can be partitioned into a collection of pairwise far-apart sets where each set in the partition satisfies some additional constraints. Theorem 0.2.5 from [6] is an example of a result of this form where each set in the partition of pairs of crossing edges has size one. The lone result of this chapter is Theorem 0.3.5, which we restate below.

Theorem 0.3.5. *There exists a constant α' such that the following holds: Let G be a drawing on the sphere of a graph and let $C_1, \cdots, C_m$ be a collection of cycles in G such that $d(C_i, C_j) \geq \alpha'$ for each $1 \leq i < j \leq m$. Suppose that, for each $1 \leq i \leq m$, there is a connected component U_i of $\mathbb{S}^2 \setminus C_i$ such that the following hold.*

1) For each crossing point x of G, there is an $i \in \{1, \cdots, m\}$ such that $x \in U_i$; AND

2) For each $v \in V(C_i)$, $V(G) \cap U_i = \varnothing$ and there is at most one chord of C lying in $\mathrm{Cl}(U_i)$ which is incident to v.

Then G is 5-choosable. In particular, letting γ be as in Theorem 0.2.6, the choice $\alpha' = 48751 + 3\gamma$ suffices.

We begin by introducing the following definition.

Definition 14.0.1. Given a drawing G and a cycle $C \subseteq G$, we say that C is *uncrossed* in G, if no crossing point of G is an internal point of an edge in $E(C)$. C is called *vertex-partitioning* in G if, letting U_0, U_1 be the two connected open components of $\mathbb{R}^2 \setminus C$, there is an $i \in \{0, 1\}$ such that $V(G) \subseteq \mathrm{Cl}(U_i)$. The set U_{1-i} is called a *vertex-free side* of C in G. Note that if $V(G) = V(C)$ then each of U_0, U_1 is a vertex-free side of C in G.

Theorem 0.3.5 is a broad generalization of Theorem 0.2.5. A graph G satisfying the conditions of Theorem 0.3.5 has a collection of vertex-partitioning cycles such that, for each cycle C in the collection, the vertex-free side of of C contains arbitrarily many crossings, each of which is the intersection of two chords of C. In order to prove this, we prove something stronger. We allow our graph G to also contain some facial subgraphs with lists of less less than five and we also allow some of the cycles of G to contain some lists of size at least four and a precolored edge. The distance constant in Theorem 0.3.5 is clearly a function of the distance constant obtained in Theorem 0.3.1, so, for the remainder of Chapter 14, we set α to be the distance constant obtained in Theorem 0.3.1. We introduce the following definition:

Definition 14.0.2. A tuple $(G, \mathcal{F}, \mathcal{T}, L)$ is called a *tennis court* if G be a drawing in the sphere of a graph, $\mathcal{F} = \{F_1, \cdots, F_k\}$ is a collection of subgraphs of G, and $\mathcal{T} = \{C_1, \cdots, C_m\}$ is a collection of vertex-partitioning cycles in G such that the following hold.

1) For each $i = 1, \cdots, k$, there is a facial subgraph F'_i of G such that $F_i \subseteq F'_i$*; AND*

2) $d(C_i, C_j) \geq \alpha + 2$ for each $1 \leq i < j \leq m$ and $d(F_i, F_j) \geq \alpha$ for each $1 \leq i < j \leq k$*; AND*

3) For each $1 \leq i \leq m$ and each $1 \leq j \leq k$, we have $d(C_i, F_j) \geq \alpha + 1$*; AND*

4) For each $v \in V(G)$, if $v \notin \bigcup_{i=1}^{m} V(C_i)$ and $v \notin \bigcup_{j=1}^{l} V(F_i)$, then $|L(v)| \geq 5$*; AND*

5) For each $1 \leq i \leq k$, there is a path $P_i \subseteq F_i$ of length one such that P_i is L-colorable and such that $|L(v)| \geq 3$ for all $v \in V(F_i \setminus P_i)$*; AND*

6) For each $1 \leq i \leq m$, there exists a path $P_i \subseteq C_i$ of length one such that P_i is L-colorable and such that, for each $v \in V(C_i \setminus P_i)$, $|L(v)| \geq 4$ and furthermore, if v is incident to a chord of C_i lying in $E(\text{Int}(C_i))$, then $|L(v)| \geq 5$*; AND*

7) For each $i \in \{1, \cdots, m\}$, there is a vertex-free side U_i of C_i in G such that the following hold.

   i) For each $v \in V(C_i)$, there is at most one chord of C lying in $\text{Cl}(U_i)$ which is incident to v*; AND*

   ii) For each crossing point x of G, there is an $i \in \{1, \cdots, m\}$ such that $x \in U_i$.

Note that, given a tennis court $(G, \mathcal{F}, \mathcal{T}, L)$, where $\mathcal{F} = \{F_1, \cdots, F_k\}$, possibly G contains a facial subgraph F' such that F' contains more than one of the subgraphs $F_1, \cdots, F_k$. That is, we do not require $F_1, \cdots, F_k$ to be a collection of pairwise far-apart facial subgraphs of G. Given $1 \leq i < j \leq k$, F_i and F_j are far apart, but possibly there is a facial subgraph of G containing both of them.

We now introduce the following intuitive definition analogous to Definition 0.1.3.

Definition 14.0.3. Given a drawing G, an uncrossed cycle $C \subseteq G$, and a pair $\{G_0, G_1\}$ of subgraphs of G with $G = G_0 \cup G_1$, we say that $\{G_0, G_1\}$ is the *natural C-partition* of G if $G_0 \cap G_1 = C$ and, for each $i \in \{0, 1\}$, there exists a simply connected region U of $\mathbb{S}^2 \setminus C$ such that G_i is the subgraph of G consisting of all the edges and vertices of G in $\text{Cl}(U)$.

We also have the following.

Definition 14.0.4. Let G be a drawing and let $C \subseteq G$ be an uncrossed cycle.

1) We call C a *cyclic facial subgraph* of G if there exists a $U \subseteq \mathbb{R}^2 \setminus G$ such that $C = \partial(U)$*; AND*

2) If C is a cyclic facial subgraph of G and Q is a proper generalized chord of C, then the *natural (C, Q)-partition* of G is a pair of subgraphs $\{G_0, G_1\}$ of G defined analogously to Definition 0.1.6.

We now have the following simple observation.

Proposition 14.0.5. *If $(G, \mathcal{F}, \mathcal{T}, L)$ is a tennis court and G is not L-colorable, then $\mathcal{T} \neq \varnothing$*

Proof. Suppose that $\mathcal{T} = \varnothing$ and suppose toward a contradiction that G is not L-colorable. Since $\mathcal{T} = \varnothing$, G is a planar embedding, and thus $\mathcal{F} \neq \varnothing$, or else G is a planar embedding in which every vertex has an L-list of size at least five, contradicting our assumption that G is not L-colorable. Let $\mathcal{F} = \{F_1, \cdots, F_k\}$. For each $i = 1, \cdots, k$, let $P_i \subseteq F_i$ be a path of one such that P_i is L-colorable and, for each $v \in V(F_i \setminus P_i)$, $|L(v)| \geq 3$. We partition $\{1, \cdots, k\}$ into two sets as $\{1, \cdots, k\} = I \cup J$, where $I := \{1 \leq i \leq k : V(F_i) \neq V(P_i)\}$.

By adding edges to G if necessary, we obtain a graph G' such that, for each $i \in I$, G' contains a cyclic facial subgraph F_i^* such that $P_i \subseteq F_i^*$ and $V(F_i^*) = V(F_i)$. Let G'' be a planar embedding obtained from G', where, for each $j \in J$, we add a vertex of degree two to G' to produce a cyclic facial subgraph F_j^* of length three with $P_j \subseteq F_j^*$. Then we have $d_{G''}(F_i^*, F_j^*) = d_G(F_i, F_j)$, since, for each edge e of $E(G'' \setminus G)$, there is either an $i \in I$ such that both endpoints of e lie in $V(F_i)$, or there is a $j \in J$ such that e has one endpoint in $V(P_j)$ and one endpoint in one of the degree two vertices of $V(G'' \setminus G)$. Let L'' be a list-assignment for G'', where, for each $x \in V(G)$, $L''(x) = L(x)$, and, for each new vertex y added to G'', $L''(y)$ is an arbitrary 3-list.

Thus, G'' is a planar embedding with a collection of cyclic facial subgraphs $F_1^*, \cdots, F_k^*$, each of which is a Thomassen facial subgraph of G'' with respect to the list-assignment L'', and each vertex of $V(G'') \setminus \left(\bigcup_{i=1}^k V(F_k^*)\right)$ has an L''-list of size at least 5. Since $d_{G''}(F_i^*, F_j^*) \geq \alpha$ for each $1 \leq i < j \leq k$, it follows from [reference tag for main result] that G'' is L-colorable, and thus G is L-colorable, contradicting our assumption. $\square$

To prove Theorem 0.3.5, it suffices to show that, for every tennis court $(G, \mathcal{F}, \mathcal{T}, L)$, G is L-colorable.

Definition 14.0.6. Let $(G, \mathcal{F}, \mathcal{T}, L)$ be a tennis court. For each $C \in \mathcal{T}$, letting $P \subseteq C$ be a subpath of C of length one such that P satisfies 6) of Definition 14.0.2, we call P the *precolored subpath* of C. We say that $(G, \mathcal{F}, \mathcal{T}, L)$ is a *minimal counterexample* if the following hold.

1) G is not L-colorable*; AND*
2) Subject to 1), $|E(G)|$ is minimized*; AND*
3) Subject to 1) and 2), $\sum_{v \in V(G)} |L(v)|$ is minimized.

We now have the following:

Proposition 14.0.7. *Let $(G, \mathcal{F}, \mathcal{T}, L)$ be a minimal counterexample. Then G is connected, and, for each $C \in \mathcal{T}$, $V(C) \neq V(G)$. In particular, there is a unique vertex-free side of $\mathbb{R}^2 \setminus C$, and $G[V(C)]$ is L-colorable.*

Proof. We obtain connectivity immediately from the minimality of $(G, \mathcal{F}, \mathcal{T}, L)$. Suppose toward a contradiction that there is a $C \in \mathcal{T}$ such that $V(C) = V(G)$. By assumption, there is an open connected component U of $\mathbb{R}^2 \setminus C$ such that, for each $v \in V(C)$, there is at most one chord of C lying in $\text{Cl}(U)$ with v as an endpoint. Furthermore, the drawing consisting of C and all chords of C lying in $\mathbb{R}^2 \setminus U$ is a planar embedding. Let H be the embedding consisting of C and all chords of C lying in $\mathbb{R}^2 \setminus U$, and let P be the precolored path of C.

We now note that there exists a $v \in V(C \setminus P)$ with degree at most 3 in G. If there is no chord of C in $\mathbb{R}^2 \setminus U$, then every vertex of $G[V(C)]$ has degree at most three, so we are done in that case. Now suppose there is a chord of C in $\mathbb{R}^2 \setminus U$. Since H is a planar embedding ,it follows that, for each chord Q of C in $\mathbb{R}^2 \setminus U$, H admits a natural Q-partition $H = H_Q^0 \cup H_Q^1$, where $H_Q^0 \cap H_Q^1 = Q$ and $P \subseteq H_Q^0$. Among all such chords, we choose Q so that $|V(H_Q^1)|$ is minimized. Since $|V(C \cap H_Q^1)| \geq 3$, let $v \in V(C \cap H_Q^1) \setminus V(Q)$. By the minimality of Q, there is no chord of C in $\mathbb{R}^2 \setminus U$ which has v as an endpoint, so v has degree at most 3 in G.

Thus, in any case, let $v \in V(C)$ have degree at most three in G. If $|V(C)| = 3$ then $G = C$, and G is trivially L-colorable, contradicting the fact that $(G, \mathcal{F}, \mathcal{T}, L)$ is a minimal counterexample. Thus, we have $|V(C)| \geq 4$. Let G' be the graph obtained from G by deleting any chord of C in $\mathbb{R}^2 \setminus U$ with v as an endpoint, if such a chord exists, and suppressing the resulting degree two vertex. Since $|V(C)| \geq 4$, let $C' \subseteq G'$ be the cycle obtained from C by this suppression. Then $(G', \varnothing, \{C'\}, L)$ is also a tennis court, and $|E(G')| < |E(G)|$. Thus, by the minimality of $(G, \mathcal{F}, \mathcal{T}, L)$, G' admits an L-coloring ϕ. Since $|L(v)| \geq 4$ and $\deg_G(v) \leq 3$, ϕ extends to an L-coloring of G, contradicting the fact that $(G, \mathcal{F}, \mathcal{T}, L)$ is a counterexample. Thus, our assumption that $V(C) = V(G)$ is false. Since G is connected, we have $|E(G[V(C)])| < |E(G)|$. Since $(G[V(C)], \varnothing, \{C\}, L)$ is also a tennis court, we get that

$V(C)$ is L-colorable by the minimality of $(G, \mathcal{F}, \mathcal{T}, L)$. Furthermore, there is a unique connected component U of $\mathbb{R}^2 \setminus C$ such that $V(G) \cap U = \varnothing$. $\square$

We now gather the following facts:

Proposition 14.0.8. *Let $(G, \mathcal{F}, \mathcal{T}, L)$ be a minimal counterexample, let $C \in \mathcal{T}$ and let $P \subseteq C$ be the precolored subpath of C. Let $U \subseteq \mathbb{R}^2 \setminus C$ be the unique vertex-free side of C. Then, letting $C := v_1 \cdots v_n$, the following hold.*

1) *$|L(v)| = 1$ for each $v \in V(P)$, and furthermore, there is no chord of C lying in* Cl(U) *with an endpoint in P; AND*

2) *$|V(C)| \geq 5$; AND*

3) *There is no chord of C lying in $\mathbb{R}^2 \setminus U$; AND*

4) *For any $w \in D_1(C, G)$, if there is an index $a \in \{1, \cdots, n\}$ and an $j \in \{1, 2\}$ such that w is adjacent to each of v_a, v_{a+j}, where the indices are read mod n, then the cycle $v_a v_{a+1} \cdots v_{a+j} w$ is uncrossed and not a separating cycle in G; AND*

5) *Let $w \in D_1(C, G)$ such that, for some $a \in \{1, \cdots, n\}$, we have $w \in N(v_a) \cap N(v_{a+2})$, where the indices are read mod n. Then $v_{a+1} \in V(P)$.*

Proof. By the minimality of $\sum_{v \in V(G)} |L(v)|$, we immediately have $|L(v)| = 1$ for each $v \in V(P)$. Let $v \in V(P)$ and suppose toward a contradiction that there is a chord uv of C lying in Cl(U) with v as an endpoint. Then $u \notin V(P)$ and, by definition, uv is the unique chord of C lying in Cl(U) which has v as an endpoint. Let $|L(v)| = c$ and let L^* be a list-assignment for $V(G)$ where $L^*(u) = L(u) \setminus \{c\}$ and $L^*(x) = L(x)$ for all $x \in V(G) \setminus \{u\}$. Since $|L(u)| \geq 5$, and thus $|L^*(u)| \geq 4$, $(G - uv, \mathcal{F}, \mathcal{T}, L^*)$ is also a tennis court, and thus $G - uv$ admits an L^*-coloring ϕ by the edge-minimality of $(G, \mathcal{F}, \mathcal{T}, L)$. Since $L^*(v) = L(v) = \{c\}$, ϕ is also an L-coloring of G, contradicting the fact that $(G, \mathcal{F}, \mathcal{T}, L)$ is a counterexample.

Now we prove 2). Suppose toward a contradiction that $|V(C)| \leq 4$. In that case, since $C \setminus P$ is a path of length at most one, it follows from 1) that there is no chord of C lying in Cl(U). That is, C is a facial subgraph of G with $C = \partial(U)$. Let ϕ be an L-coloring of $V(C \setminus P)$ which extends to an L-coloring of $G[V(C)]$. Let $G' := G \setminus V(C \setminus P)$. Then G' contains a facial subgraph F such that $P \subseteq F$ and $D_1(C \setminus P, G) \subseteq V(F)$. Let F^* be a subgraph of F containing P and containing each vertex of $D_1(C \setminus P, G)$.

Since $|V(C \setminus P)| \leq 2$, we have $|L_\phi(v)| \geq 3$ for all $v \in V(F^* \setminus P)$, and P is L_ϕ-colorable. Furthermore, since each vertex of F^* is of distance at most one from C, the tuple $(G', \mathcal{F} \cup \{F^*\}, \mathcal{T} \setminus \{C\}, L_\phi)$ satisfies the distance conditions of Definition 14.0.2. Thus, $(G', \mathcal{F} \cup \{F^*\}, \mathcal{T} \setminus \{C\}, L_\phi)$ is also a tennis court. Since $|E(G')| < |E(G)|$, G' admits an L_ϕ-coloring, so ϕ extends to an L-coloring of G, contradicting the fact that $(G, \mathcal{F}, \mathcal{T}, L)$ is a minimal counterexample. This proves 2).

Now we prove 3). Let G^* be the graph obtained from G by deleting from G all chords of C lying in Cl(U). Suppose toward a contradiction that there is a chord Q of C lying in Cl(U). Note that C is a cyclic facial subgraph of G^*. Let $G^* = G_0^* \cup G_1^*$ be the natural (C, Q)-partition of G^*, where $G_0^* \cap G_1^* = Q$ and $P \subseteq G_0^*$.

Let G_0^{**} be the graph obtained from G_0^* by adding to G_0^* all the chords of C in Cl(U) with both endpoints in $V(C \cap G_0^*)$. For each $i = 0, 1$, let C_i^* be the cycle $C_i^* := (C \cap G_i^*) + Q$. For each $i = 0, 1$, let $\mathcal{T}_i^* := \{C' \in \mathcal{T} : C' \subseteq G_i^*\}$ and let $\mathcal{F}_i^* := \{F \in \mathcal{F} : F \subseteq G_i^*\}$. Note that, since each cycle of $\mathcal{T}$ is vertex-separating in G, we have by our distance conditions that $\mathcal{T} \setminus \{C\} = \mathcal{T}_0^* \cup \mathcal{T}_1^*$ as a disjoint union, and furthermore, since each element of $\mathcal{F}$ is contained in a facial subgraph of G, and thus contained in a facial subgraph of G^*, $\mathcal{F}=\mathcal{F}_0^*\cup\mathcal{F}_1^*$ as a disjoint union. In particular, $(G_0^{**}, \mathcal{F}_0^*, \mathcal{T}_0^*\cup\{C_0^*\}, L)$ is also a tennis court, and since $|E(G_0^{**})|<|E(G)|$, G_0^{**} admits an L-coloring ϕ.

Let G_1^{**} be the graph obtained from G_1^* by adding to G_1^* all the chords of C in Cl(U) with both endpoints in $V(C\cap G_1^*)$. Consider the tuple $(G_1^{**}, \mathcal{F}_1^*, \mathcal{T}_1^*\cup\{C_1^*\}, L_\phi^Q)$. Let U' be the unique connected component of $\mathbb{R}^2\setminus C_1^*$ such that $U\subseteq U'$. Then, in G_1^{**}, for each $v\in V(C_1^*)$, there is at most one chord of C_1^* in Cl(U') with v as an endpoint.

Given $v\in V(C_1^*\setminus Q)$, if there does not exist a chord of C lying in Cl(U) with v as an endpoint and the other endpoint in $V(C_0^*\setminus Q)$, then we have $L_\phi^Q(v)=L(v)$. On the other hand, if there is a such a chord, then we have $|L(v)|\geq 5$ and $|L_\phi^Q(v)|\geq 4$, and furthermore, there is no chord of C in Cl$(U')\cap E(G_1^{**})$ with v as an endpoint, so $(G_1^{**}, \mathcal{F}_1^*, \mathcal{T}_1^*\cup\{C_1^*\}, L_\phi^Q)$ is a tennis court. Thus, since $|E(G_1^{**})|<|E(G)|$, G_1^{**} is L_ϕ^Q-colorable, so ϕ extends to an L-coloring of G, contradicting the fact that $(G, \mathcal{F}, \mathcal{T}, L)$ is a minimal counterexample.

Now we prove 4). Let $w\in D_1(C,G)$ and $j\in\{1,2\}$ with $v_a, v_{a+j}\in N(w)$. Since U is the vertex-free side of C, we have $w\in\mathbb{R}^2\setminus U$, and, by definition of $(G, \mathcal{F}, \mathcal{T}, L)$, the cycle $D:=v_av_{a+1}\cdots v_{a+j}w$ is uncrossed. Thus, since $v_av_{a+1}\cdots v_{a+j}$ is a subpath of C, let $G=G_0\cup G_1$ be the natural D-partition of G, where $C\subseteq G_0$. We claim now that $V(G_1)=V(D)$.

Suppose not. Let $G':=G\setminus(V(G_1)\setminus V(D))$. Since G is connected, we have $|E(G')|<|E(G)|$. By our distance conditions on $\mathcal{T}$, each element of $\mathcal{T}$ is either a subgraph of G_0 or a subgraph of G_1, and likewise, each element of $\mathcal{F}$ is either a subgraph of G_0 or a subgraph of G_1, since each element of $\mathcal{F}$ is contained in a facial subgraph of G. For each $i=0,1$, let $\mathcal{T}_i':=\{C^*\in\mathcal{T}: C^*\subseteq G_i\}$, and let $\mathcal{F}_i':=\{F\in\mathcal{F}: F\subseteq G_i\}$.

Now, the tuple $(G', \mathcal{T}_0', \mathcal{F}_0', L)$ is a tennis court, and since $|E(G')|<|E(G)|$, G' admits an L-coloring ϕ. Let $Q:=v_aw$. The graph $G_1\setminus V(D\setminus Q)$ contains a facial subgraph F' such that F' contains the edge v_aw and F' contains each vertex of $D_1(D\setminus Q, G_1)\setminus V(Q)$. Let F'' be a subgraph of F' with $Q\subseteq F''$ and $V(F'')=V(Q)\cup D_1(D\setminus Q, G_1)$.

Consider the tuple $(G_1\setminus V(D\setminus Q), \{F''\}\cup\mathcal{F}_1', \mathcal{T}_1', L_\phi^Q)$. By the distance conditions on $(G, \mathcal{F}, \mathcal{T}, L)$, we have $d(C,C')\geq\alpha+2$ for each $C'\in\mathcal{T}_1'$, and thus $d(F'',C')\geq\alpha+1$ for each $C'\in\mathcal{T}_1$. Likewise, $d(C,F)\geq\alpha+1$ for each $F\in\mathcal{F}_1'$, and thus $d(F'',F)\geq\alpha$ for each $F\in\mathcal{F}_1'$. Thus, $(G_1\setminus V(D\setminus Q), \{F''\}\cup\mathcal{F}_1', \mathcal{T}_1', L_\phi^Q)$ is also a tennis court, and since $|E(G_1\setminus V(D\setminus Q))|<|E(G)|$, $G_1\setminus V(D\setminus Q)$ admits an L_ϕ^Q-coloring, so ϕ extends to an L-coloring of G, contradicting the fact that $(G, \mathcal{F}, \mathcal{T}, L)$ is a minimal counterexample. Thus, $V(G_1)=V(D)$. This proves 4).

Now we prove 5). Let $w\in D_1(C,G)$ such that, for some $a\in\{1,\cdots,n\}$, we have $w\in N(v_a)\cap N(v_{a+2})$, and suppose toward a contradiction that $v_{a+1}\not\in V(P)$.. Let $D:=v_av_{a+1}v_{a+2}w$. Since $V(G)\cap U=\varnothing$, we have $w\in\mathbb{R}^2\setminus U$, and, by definition of $\mathcal{T}$, D is an uncrossed cycle in G.

Let $G=G_0\cup G_1$ be the natural D-partition of G, where $C\subseteq G_0$. By 4), we have $V(G_1)=V(D)$. Let $G^\dagger$ be the graph obtained from G by deleting from G any edge of $E(G)\setminus E(C)$ incident to v_{a+1} and then suppressing the resulting vertex of degree 2. Let $C^\dagger$ be the cycle $v_1\cdots v_{a-1}v_{a+1}\cdots v_n$ obtained from this suppression.

The tuple $(G^\dagger, L)$ is a tennis court with $|E(G^\dagger)|<|E(G)|$, so $G^\dagger$ admits an L-coloring ϕ by the minimality of $(G, \mathcal{F}, \mathcal{T}, L)$. If v_{a+1} is not an endpoint of a chord of C lying in Cl(U), then, since $V(G_1)=V(D)$, we have $N(v_{a+1})\subseteq\{v_a, v_{a+2}, w\}$. Thus, since $|L(v_{a+1})|\geq 4$ in this case, ϕ extends to an L-coloring of G, contradicting the fact that $(G, \mathcal{F}, \mathcal{T}, L)$ is a minimal counterexample. Likewise, if v_{a+1} is the endpoint of a chord uv_{a+1} of C lying in Cl(U), then, since $V(G_1)=V(D)$, we have $N(v_{a+1})\subseteq\{u, v_a, v_{a+2}, w\}$. Since $|L(v_{a+1})|\geq 5$ in this case, ϕ again extends to an L-coloring of G, contradicting the fact that $(G, \mathcal{F}, \mathcal{T}, L)$ is a minimal counterexample. This completes the proof of Proposition 14.0.8. $\square$

In order to continue, we need the following purely combinatorial fact.

Lemma 14.0.9. *Let $m \geq 1$ and let $\mathcal{P}$ be a partition of $\{1, \cdots, 2m\}$ into a collection of m pairwise-disjoint sets of size 2. Then there exists an $S \subseteq \{1, \cdots, 2m\}$ such that $|S| = m$, S contains precisely one element of each pair in $\mathcal{P}$, and for each odd integer $j \in \{1, \cdots, 2m\}$, $|S \cap \{j, j+1\}| \leq 1$.*

Proof. Given an $m \geq 1$ and a partition $\mathcal{P}$ of $\{1, \cdots, 2m\}$ into m pairwise-disjoint sets of size two, we say that $S \subseteq \{1, \cdots, 2m\}$ is a *$\mathcal{P}$-sampling* if S is a set of size m such that S contains precisely one element of each pair in $\mathcal{P}$, and for each odd integer $j \in \{1, \cdots, 2m\}$, $|S \cap \{j, j+1\}| \leq 1$. Thus, we claim that, for each $m \geq 1$ and each partition $\mathcal{P}$ of $\{1, \cdots, 2m\}$ into m pairwise-disjoint sets of size two, there exists a $\mathcal{P}$-sampling.

We show this by induction on m. If $m = 1$ then the claim is trivially true since $|\mathcal{P}| = 1$, so we just choose a single element from the lone set of $\mathcal{P}$. Now let $m > 1$ and suppose that, for any $1 \leq m' < m$ and any partition $\mathcal{P}'$ of $\{1, \cdots, 2m'\}$ into m' pairwise-disjoint sets of size two, there exists a $\mathcal{P}'$-sampling. Let $\mathcal{P}$ be a partition of $\{1, \cdots, 2m\}$ into m pairwise-disjoint sets of size 2. For each $j \in \{1, \cdots, 2m\}$. Now consider the following cases:

Case 1: $\{2m-1, 2m\} \in \mathcal{P}$

In this case, let $\mathcal{P}^* := \mathcal{P} \setminus \{\{2m-1, 2m\}\}$. Applying our induction hypothesis to $\mathcal{P}^*$, there is a $\mathcal{P}^*$-sampling S^*. Then $S^* \cup \{2m-1\}$ consists of precisely one element from each $A \in \mathcal{P}$, and, for each odd $j \in \{1, \cdots, 2m\}$, we have $|S^* \cap \{j-1, j\}| \leq 1$. Thus, $S^* \cup \{2m-1\}$ is a $\mathcal{P}$-sampling, so we are done in this case.

Case 2: $\{2m-1, 2m\} \notin \mathcal{P}$

In this case, there exist distinct $a, b \in \{1, \cdots, 2m-2\}$ such that $\{2m-1, a\} \in \mathcal{P}$ and $\{2m, b\} \in \mathcal{P}$. Let $\mathcal{P}^*$ be a partition of $\{1, \cdots, 2m-2\}$ into $m-1$ sets of size 2, where $\mathcal{P}^*$ is obtained from $\mathcal{P}$ by removing $\{2m-1, a\}$ and $\{2m, b\}$, and replacing them with $\{a, b\}$. Applying our induction hypothesis to $\mathcal{P}^*$, there is a $\mathcal{P}^*$-sampling $S^* \subseteq \{1, \cdots, 2m-2\}$, so S^* contains precisely one of $\{a, b\}$. If $a \in S^*$, then $S^* \cup \{2m\}$ is a $\mathcal{P}$-sampling, and, if $b \in S^*$, then $S^* \cup \{2m-1\}$ is a $\mathcal{P}^*$-sampling, so we are done. This completes the proof of Lemma 14.0.9. $\square$

The main result we need in order to prove Theorem 0.3.5 is the following.

Proposition 14.0.10. *Let $(G, \mathcal{F}, \mathcal{T}, L)$ be a minimal counterexample, let $C \in \mathcal{T}$ and let $P \subseteq C$ be the precolored subpath of C. Then there exists a subset $S \subseteq V(C \setminus P)$ and an L-coloring ϕ of $G[S]$ such that the following hold.*

1) For each $w \in D_1(S, G) \setminus V(P)$, $|L_\phi(w)| \geq 3$; AND

2) $V(P)$ is L_ϕ-colorable; AND

3) For each chord e of C_i with $e \in \mathrm{Cl}(U_i)$, at least one endpoint of e lies in S

Proof. Let $U \subseteq \mathbb{R}^2 \setminus C$ be the unique vertex-free side of C. Given a subset $S \subseteq V(C)$, we say that S is a *covering set* if S consists of precisely one endpoint of each chord of C in $\mathrm{Cl}(U)$. It is clear that such a subset of $V(C)$ exists, since each vertex of C is incident to at most one chord of C in $\mathrm{Cl}(U)$. Furthermore, for any covering set S, we have $S \cap V(P) = \varnothing$ by 1) of Proposition 14.0.8.

Applying Lemma 14.0.9 to the path $C \setminus P$, there exists a covering set $S \subseteq V(C \setminus P)$ such that every connected component of $C[S]$ is a path of length at most one. Now set $T := \{w \in D_1(C) : |N(w) \cap S| \geq 3\}$. Let $C := v_1 \cdots v_n$. Without loss of generality, let $P := v_{n-1}v_n$. For each vertex $v \in V(C)$, we say that v is *matched* if there is a chord of C with v as an endpoint. Otherwise we say v is *unmatched*. By 3) of Proposition 14.0.8, each matched vertex is the endpoint of precisely one chord of C, and this chord lies in $\mathrm{Cl}(U)$. In particular, every edge of G with both endpoints in S lies in $E(C)$, since S consists of precisely one endpoint from each chord of C. Given a matched vertex $v \in V(C)$, if $u \in V(C)$ is the other endpoint of the unique chord of C incident to v, we say that u is *matched to* v.

Claim 14.0.11. *If $T = \varnothing$, then there exists an L-coloring ϕ of S satisfying Proposition 14.0.10.*

Proof: By Proposition 14.0.7, $G[V(C)]$ is L-colorable, so let ϕ be an L-coloring of S which extends to an L-coloring of $G[V(C)]$. Thus, $V(P)$ is L_ϕ-colorable, and, since $T = \varnothing$, we have $|L_\phi(w)| \geq 3$ for all $w \in D_1(S, G) \setminus V(P)$, so we are done. ■

Thus, for the remainder of the proof of Proposition 14.0.10, we suppose that $T \neq \varnothing$. For each element $w \in T$, we let Q_w be the unique subpath of $C \setminus P$ such that each endpoint of Q_w lies in $N(w) \cap S$ and $N(w) \cap S \subseteq V(Q_w)$. Since Q_w contains at least three vertices of S, it follows from Proposition 14.0.8 that $|V(Q_w)| \geq 5$ for each $w \in T$. Furthermore, for each $w \in T$, and any two distinct $v, v' \in V(Q_w)$, we define an open subset $A_{vwv'}$ of $\mathbb{R}^2$ as follows: $wv'W_wv$ is a cycle, and we set A_w to be the unique open connected component of $\mathbb{R}^2 \setminus (wv'P_wv)$ which does is disjoint to $V(P)$. Furthermore, for each $w \in T$, we define an open subset $A_w \subseteq \mathbb{R}^2$ as follows: Let v, v' be the endpoints of Q_w. Then we let $A_w := A_{vwv'}$.

We now define an ordering relation $<$ on T as follows. Given $w, w' \in T$, we say that $w' < w$ if $w' \in A_w$. We say that $w' \leq w$ if either $w' = w$ or $w' < w$. $\leq$ is clearly a well-defined partial order on T. We define a sequence $T_0, T_1, \cdots$ of subsets of T inductively as follows. Let T_0 be the set of $\leq$-maximal elements of T. For each $j \geq 0$, if T_j is well-defined and nonempty, then we set T_{j+1} be the set of $w \in T$ such that, for some $w' \in T_j$ we have $w < w'$, and there does not exist a $w'' \in T$ such that $w < w'' < w'$. Since $\leq$ is a well-defined partial order on T, the sets $T_0, T_1, \cdots$ are pairwise-disjoint. Let ℓ be the minimal index such that $T_{\ell+1} = \varnothing$. Such an ℓ exists as T is finite.

Definition 14.0.12. For each $j \in \{0, \cdots, \ell\}$, we let S_j be the set of S-endpoints of the edges of G connecting $T_0 \cup \cdots T_j$ to $V(C)$. Given a $j \in \{0, \cdots, \ell\}$, a subset $A \subseteq S_j$, and an L-coloring ϕ of A, we say that ϕ is a *match-valid* L-coloring of A if the following hold.

V1) $V(P)$ is L_ϕ-colorable; *AND*

V2) $|L(x)| \geq 3$ for all $v \in D_1(A, G) \setminus V(P)$; *AND*

V3) For any index $a \in \{1, \cdots, n\}$, if $v_a, v_{a+2} \in A$ and v_{a+1} is a matched vertex of C, then we have $|L(v_{a+1}) \setminus \{\phi(v_a), \phi(v_{a+2})\}| \geq 4$.

Note that Property V3) above is stronger than the condition that $|L_\phi(v_{a+1})| \geq 3$, since, if v_{a+1} is matched to some vertex $u \in V(C)$, then we have $|L_\phi(v_{a+1})| \geq 3$ for any choice of color used by ϕ on u. Since no three consecutive vertices of C lie in S, we have $v_{a+1} \notin \text{dom}(\phi)$ if $v_a, v_{a+2} \in \text{dom}(\phi)$. For each $j \in \{0, \cdots, \ell\}$, and two distinct vertices $v, v' \in S_j$, we say that v, v' are *S_j-consecutive* if no internal vertex of the unique subpath of $C \setminus P$ with endpoints v, v' lies in S_j. The following facts are immediate:

Claim 14.0.13. *For each $j \in \{0, \cdots, \ell\}$, the following hold.*

1) *For any two $w, w' \in T_j$, $A_w \cap A_{w'} = \varnothing$; AND*

2) *For any two vertices $v, v' \in S_j$, v, v' have at most one common neighbor in T_j; AND*

3) *For any $0 < j \leq \ell$ and any $w \in T_j$, there is a unique $w' \in T_{j-1}$ such that $w < w'$. In particular, there is a unique pair of S_{j-1}-consecutive vertices $v, v' \in V(Q_{w'})$ such that $w \in A_{vw'v'}$.*

We now claim the following:

Claim 14.0.14. *There exists a match-valid L-coloring of S_0.*

Proof: Recall that $P := v_{n-1}v_n$. Thus, let $m_1, \cdots, m_r$ be a set of indices with $1 \leq m_1 < m_2 < \cdots < m_r \leq n-2$, where $S_0 = \{v_{m_1}, \cdots, v_{m_r}\}$. It is clear that, for any L-coloring ϕ of $V(P) \cup \{v_{m_1}\}$, the coloring $\phi(v_{m-1})$ is a match-valid L-coloring of $\{v_{m_1}\}$. Now let $i \in \{1, \cdots, r\}$ and suppose there is a match-valid L-coloring ϕ of $\{v_{m_1}, \cdots, v_{m_i}\}$. If $i = r$, then we have a match-valid L-coloring of S_0, so we are done. Now suppose that $1 \leq i < r$. We claim there exists a match-valid L-coloring of $\{v_{m_1}, \cdots, v_{m_i}, v_{m_{i+1}}\}$. We first note that, for any $v \in S_0$, either $|N(v) \cap T_0| = 1$, or there exists a pair of vertices $w, w' \in T_0$ such that $N(v) \cap T_0 = \{w, w'\}$, v is the right endpoint of Q_w, and v is the left endpoint of $Q_{w'}$.

Since every edge of G with both endpoints in S is an edge of C, we have $L_\phi(v_{m_{i+1}}) = L(v) \setminus \{\phi(v_{m_i})\}$ if $m_{i+1} = m_i + 1$, and otherwise $L_\phi(v_{m_{i+1}}) = L(v_{m_{i+1}})$. Now consider the following cases:

Case 1 of Claim 14.0.14: $v_{m_{i+1}}, v_{m_i}$ do not have a common neighbor in T_0

In this case, there is a $w \in T_0$ such that $v_{m_{i+1}}$ is the left endpoint of Q_w.

Subclaim 14.0.15. *For any extension ϕ^* of ϕ to an L-coloring of $\{v_{m_1}, \cdots, v_{m_{i+1}}\}$, if ϕ^* is not a match-valid L-coloring of its domain, then $m_i = m_{i+1} - 2$ and one of the following holds.*

1) *v_{m_i+1} is a matched vertex of C, and $|L(v_{m_i+1}) \setminus \{\phi(v_{m_i}), \phi^*(v_{m_{i+1}})\}| = 3$; OR*
2) *v_{m_i+1} is an unmatched vertex of C and $L(v_{m_i+1}) \setminus \{\phi(v_{m_i}), \phi^*(v_{m_{i+1}})\}| = 2$.*

Proof: Let ϕ^* be an extension of ϕ to an L-coloring of $\{v_{m_1}, \cdots, v_{m_{i+1}}\}$ and suppose that ϕ^* is not a match-valid L-coloring of its domain. Since $v_{m_{i+1}}$ is the left endpoint of Q_w, there are at least two vertices in $\{v_{m_{i+1}+1}, \cdots, v_{n-2}\}$ adjacent to w. Thus, since $m_i < m_{i+1}$ and there is no chord of C with an endpoint in $V(P)$, $v_{m_{i+1}}$ is not adjacent to any vertex of $V(P)$, so, since ϕ extends to an L-coloring of $V(P) \cup \text{dom}(\phi)$, ϕ^* extends to an L-coloring of $V(P) \cup \text{dom}(\phi^*)$.

Thus, ϕ^* satisfies V1) of Definition 14.0.12. If ϕ^* does not satisfy V3) of Definition 14.0.12, then $m_{i+1} = m_i + 2$, v_{m_i+1} is the endpoint of a chord e of C with $e \in \text{Cl}(U)$, and $L(v_{m_i+1}) \setminus \{\phi(v_{m_i}), \phi^*(v_{m_i+2})\}| = 3$. Note that $v_{m_i+1} \notin \text{dom}(\phi)$ since S contains no three consecutive vertices of $C \setminus P$. Thus, if ϕ^* does not satisfy V3) of Definition 14.0.12, then we are done.

Suppose now that ϕ^* satisfies V1) and V3) of Definition 14.0.12, but not V2) of Definition 14.0.12. Then there is a vertex $z \in D_1(\text{dom}(\phi), G) \setminus V(P)$ such that $|L_{\phi^*}(z)| < 3$. Since there is no chord of C in $\mathbb{R}^2 \setminus U$ and each vertex of $C \setminus P$ has an L-list of size at least four, we have $z = v_{m_i+1}$ and $m_{i+1} = m_i + 2$. Since ϕ^* satisfies V3) of Definition 14.0.12, v_{m_i+1} is unmatched, or else we have $|L_{\phi^*}(v_{m_i+1})| \geq 3$. Since, we have $|L(v_{m_i+1})| \geq 4$ and $L(v_{m_i+1}) \setminus \{\phi(v_{m_i}), \phi^*(v_{m_{i+1}})\}| = 2$, so we are done. ■

Now we finish Case 1 of of Claim 14.0.14. If $m_{i+1} \neq m_i + 2$, then any extension of ϕ to $\{v_{m_1}, \cdots, v_{m_{i+1}}\}$ is a match-valid L-coloring of $\{v_{m_1}, \cdots, v_{m_{i+1}}\}$, so we are done in that case. Now suppose that $m_{i+1} = m_i + 2$. Since every edge of G with both endpoints in S is an edge of C, we have $|L_\phi(v_{m_{i+1}})| \geq 5$.

If v_{m_i+1} is a matched vertex, then we have $|L(v_{m_i+1}) \setminus \{\phi(v_{m_i})\}| \geq 4$, and since $|L_\phi(v_{m_{i+1}})| \geq 5$, we choose a color $d \in |L_\phi(v_{m_{i+1}})$ such that $|L(v_{m_i+1}) \setminus \{\phi(v_{m_i}), d\}| \geq 4$. By Subclaim 14.0.15, the resulting extension of ϕ to $\text{dom}(\phi) \cup \{v_{m_{i+1}}\}$ is then a match-valid L-coloring of its domain, so we are done in that case. Now suppose that

v_{m_i+1} is unmatched. In that case, since $|L(v_{m_i+1})| \geq 4$, we have $|L_\phi(v_{m_i+1})| \geq 3$, so we simply choose a color $d \in L_\phi(v_{m_{i+1}})$ such that $L_\phi(v_{m_i+1}) \setminus \{d\}| \geq 3$, and, again by Subclaim 14.0.15, the resulting extension of ϕ to $\text{dom}(\phi) \cup \{v_{m_{i+1}}\}$ is a match-valid L-coloring of its domain. This completes Case 1 of Claim 14.0.14.

Case 2 of Claim 14.0.14: $v_{m_{i+1}}, v_{m_i}$ have a common neighbor in T_0

In this case, by Claim 14.0.13, let $w \in T_0$ be the unique common neighbor of $v_{m_i}, v_{m_{i+1}}$. Now consider the following subcases:

Subcase 2.1 Either $m_{i+1} < n-2$ or $|L_\phi(v_{m_{i+1}})| \geq 5$

By 1) of Proposition 14.0.8, there is no chord of C with an endpoint in P. Thus, since $|L(v_{n-1})| = 1$, $|L_\phi(w)| \geq 3$, and $|L_\phi(v_{m_{i+1}})| \geq 4$, there is a color $c \in L_\phi(v_{m_{i+1}})$ such that $|L_\phi(w) \setminus \{c\}| \geq 3$, and such that either $v_{m_{i+1}}$ is not adjacent to a vertex of P or $N(v_{m_{i+1}}) \cap V(P) = \{v_{n-1}\}$ and $\{c\} \neq L(v_{n-1})$. In either case, letting ϕ' be the extension of ϕ obtained by coloring $v_{m_{i+1}}$ with c, ϕ' extends to an L-coloring of $\text{dom}(\phi') \cup V(P)$.

By definition of S, together with 1) of Proposition 14.0.8, there is a unique chord $v_{m_{i+1}}v_\ell$ of C lying in $\text{Cl}(U)$, where $\ell \in \{1, \cdots, n-2\}$. Since $v_{m_{i+1}}$ is the other end of the unique chord of C in $\text{Cl}(U)$ which is incident to v_ℓ, we have $\text{dom}(\phi) \cap N(v_\ell) \subseteq \{v_{\ell-1}, v_{\ell+1}\}$, where the indices are read mod n. Thus, since ϕ is a match-valid L-coloring of $\{v_{m_1}, \cdots, v_{m_i}\}$, we have $|L_\phi(v_\ell)| \geq 4$, and thus $|L_{\phi'}(v_\ell)| \geq 3$. Thus, ϕ' satisfies conditions V1) and V2) of Definition 14.0.12. If V3) is not satisfied, then we have $v_{m_{i+1}-1} \notin \text{dom}(\phi)$ and $v_{m_{i+1}-2} \in \text{dom}(\phi)$, so w is adjacent to each of $v_{m_{i+1}-2}, v_{m_{i+1}}$. Since $v_{m_{i+1}-1} \notin V(P)$, this contradicts 5) of Proposition 14.0.8. Thus, ϕ' is indeed a match-valid L-coloring of $\{v_{m_1}, \cdots, v_{m_{i+1}}\}$.

Subcase 2.2 $m_{i+1} = n-2$ and $|L_\phi(v_{m_{i+1}})| = 4$

In this case, we have $N(v_{m_{i+1}}) \cap S = \{v_{m_i}\}$ and $m_{i+1} = m_i + 1$. Furthermore, we have $S_0 = \{m_1, \cdots, m_{i+1}\}$, and $v_{m_{i+1}}$ is the right endpoint of Q_w. Since w has at least three neighbors in S_0, v_{m_i} is an internal vertex of Q_w. Let $L(v_{n-1}) = \{q\}$ and let ϕ' be the restriction of ϕ to $\{v_{m_1}, \cdots, v_{m_{i-1}}\}$. Then we have the following:

> **Subclaim 14.0.16.** *Let $c \in L(v_{n-2}) \setminus \{q\}$ and let $d \in L_{\phi'}(v_{m_i})$, where $c \neq d$. Let ϕ^* be an extension of ϕ' to* $\text{dom}(\phi') \cup \{v_{m_i}, v_{m_{i+1}}\}$ *obtained by coloring $v_{m_{i+1}}$ with c and coloring v_{m_i} with d. if ϕ^* is not a match-valid L-coloring of its domain, then $|L_{\phi^*}(w)| < 3$.*
>
> Proof: Firstly, since v_{m_i} is an internal vertex of Q_w, and there is no chord of C with an endpoint in $V(C)$, the L-coloring ϕ^* extends to an L-coloring of $\text{dom}(\phi^*) \cup V(P)$, as $q \neq c$. Thus, if ϕ^* is not a match-valid L-coloring of $\{v_{m_1}, \cdots, v_{m_i}\}$, then either V2) or V3) of Definition 14.0.12 is not satisfied. If 3) is not satisfied, then, since $m_{i+1} = m_i + 1$ and ϕ' is a match-valid L-coloring of its domain, we have $m_i = m_{i-1} + 2$ and $L(v_{m_i+1}) \setminus \{\phi'(v_{m_{i-1}}), d\}| = 3$. But then, since v_{m_i} is an internal vertex of Q_w, we contradict 5) of Proposition 14.0.8. Thus, ϕ^* satisfies 3) of Definition 14.0.12. Since ϕ^* satisfies V3) of Definition 14.0.12, we have $|L_{\phi^*}(v)| \geq 3$ for all $v \in V(C) \setminus (V(P) \cup \text{dom}(\phi^*))$. Thus, since ϕ' is a match-valid L-coloring of $\{v_1, \cdots, v_{m_{i-1}}\}$, the only remaining possibility is that $|L_{\phi^*}(w)| < 3$. ■

Since there is no chord of C with both endpoints in S, and no three consecutive vertices of C lie in S, we have $|L_{\phi'}(v_{m_i})| \geq 5$, as $v_{m_i}, v_{m_{i+1}}$ are consecutive in C. Likewise, $|L_{\phi'}(v_{m_{i+1}})| \geq 5$ and thus $|L_{\phi'}(v_{m_{i+1}}) \setminus \{q\}| \geq 4$.

Now we extend ϕ' to an L-coloring of $\text{dom}(\phi') \cup \{v_{m_i}, v_{m_{i+1}}\}$ in the following way: Since ϕ' is a match-valid L-coloring of its domain, we have $|L_{\phi'}(w)| \geq 3$, and since $|L_{\phi'}(v_{m_{i+1}}) \setminus \{q\}| \geq 4$, we choose a color $c \in L_{\phi'}(v_{m_{i+1}}) \setminus \{q\}$ such that $|L_{\phi'}(w) \setminus \{c\}| \geq 3$. Finally, since $|L_{\phi'}(v_{m_i}) \setminus \{c\}| \geq 4$, we choose a color $d \in L_{\phi'}(v_{m_i}) \setminus \{c\}$

such that $|L_{\phi'}(w) \setminus \{c, d\}| \geq 3$. By Subclaim 14.0.16, the resulting extension of ϕ' is a match-valid L-coloring of $\{v_{m_1}, \cdots, v_{m_{i+1}}\}$, so we are done. This completes the proof of Claim 14.0.14. ■

Claim 14.0.14 is the base case of an induction argument on the sequence of sets $S_0, S_1, \cdots, S_\ell$. We complete the argument with the following claim:

Claim 14.0.17. *Let $j \in \{0, \cdots, \ell\}$ and suppose there is a match-valid L-coloring ϕ of S_j. Then ϕ extends to a match-valid L-coloring of S_{j+1}.*

Proof: By Claim 14.0.13, for each $w \in T_{j+1}$, there is a unique $w' \in T_j$ such that $w < w'$, and, given this w', there is a unique pair v, v' of S_j-consecutive vertices such that $w' \in A_{vw'v'}$.

Subclaim 14.0.18.

1) For any extension of ϕ to an L-coloring ϕ^* of S_{j+1}, ϕ^* extends to an L-coloring of $S_{j+1} \cup V(P)$; AND

2) Suppose that, for each $w' \in T_j$ and each pair v, v' of S_j-consecutive vertices in $Q_{w'}$, ϕ extends to a match-valid L-coloring of $S_j \cup (V(vQ_w v') \cap S_{j+1})$. Then ϕ extends to a match-valid L-coloring of S_{j+1}.

Proof: We first prove 1). Since there is no chord of C with an endpoint in P, it follows that, for each $w' \in T_j$, no internal vertex of $Q_{w'}$ is adjacent to a vertex of P. Thus, since ϕ extends to an L-coloring of dom$(\phi) \cup V(P)$, ϕ^* also extends to an L-coloring of dom$(\phi^*) \cup V(P)$.

Now we prove 2). For each $w' \in T_j$ and each pair v, v' of S_j-consecutive vertices in $Q_{w'}$, let $\phi_{vw'v'}$ be an extension of ϕ to a match-valid L-coloring of $S_j \cup (V(vQ_{w'}v') \cap S_{j+1})$, and let ϕ^* be the union of these extensions, taken over each $w' \in T_j$ and each pair v, v' of S_j-consecutive vertices of $Q_{w'}$. Since there is no chord of C with both endpoints in S, ϕ^* is a proper L-coloring of S_{j+1}. We claim now that ϕ^* is a match-valid L-coloring of S_{j+1}.

By 1), ϕ^* extends to an L-coloring of $S_{j+1} \cup V(P)$, so we just need to check V2) and V3) of Definition 14.0.12. Let $a \in \{1, \cdots, n-2\}$ with $v_a, v_{a+2} \in$ dom(ϕ^*), where v_{a+1} is a matched vertex of C. We claim that $|L(v_{a+1}) \setminus \{\phi^*(v_a), \phi^*(v_{a+2})\}| \geq 4$. If $v_a, v_{a+2} \in$ dom(ϕ), then we are done, since ϕ is a match-valid L-coloring of S_j. If not, then at least one of v_a, v_{a+2} lies in S_{j+1}, so suppose without loss of generality that $v_{a+2} \in S_{j+1} \setminus S_j$. In that case, there is a $w \in T_{j+1}$ adjacent to v_{a+2}, and, by Claim 14.0.13, there is a unique $w' \in T_j$ with $w < w'$ and a unique pair v, v' of S_j-consecutive vertices of $Q_{w'}$ such that $w \in A_{vw'v'}$. Since w is adjacent to v_{a+2}, v_{a+2} is an internal vertex of $vQ_{w'}v'$. If v_a also lies in the path $vQ_{w'}v'$, then we immediately have $|L(v_{a+1}) \setminus \{\phi^*(v_a), \phi^*(v_{a+2})\}| \geq 4$, since $\phi_{vw'v'}$ is a match-valid L-coloring of its domain by assumption. If $v_a \notin V(vQ_{w'}v')$, then $v_{a+1} \in \{v, v'\}$, contradicting the fact that no three consecutive vertices of $C \setminus P$ lie in S. Thus, ϕ^* satisfies V3) of Dfinition 14.0.12.

Now we just check that ϕ^* satisfies V2) of Definition 14.0.12. Let $v_a \in V(C \setminus P) \setminus$ dom(ϕ^*), where $a \in \{1, \cdots, n-2\}$. We claim that $|L_{\phi^*}(v_a)| \geq 3$. If v_a is a matched vertex of C, then, since ϕ^* satisfies V3) of Definition 14.0.12, we have $|L_{\phi^*}(v_a)| \geq 3$. Now suppose that v_a is not a matched vertex and suppose toward a contradiction that $|L_{\phi^*}(v_a)| < 3$. Since $|L(v_a)| \geq 4$ and no chord of C is incident to v_a, we have $v_{a-1}, v_{a+1} \in$ dom(ϕ^*). If $v_{a-1}, v_{a+1} \in$ dom(ϕ), then, since ϕ is a match-valid L-coloring of its domain, we have $|L_{\phi^*}(v_a)| \geq 3$, contradicting our assumption.

Thus, suppose without loss of generality that $v_{a+1} \in S_{j+1} \setminus S_j$. As above, there is a $w \in T_{j+1}$ adjacent to v_{a+1}, and, by Claim 14.0.13, there is a unique $w' \in T_j$ with $w < w'$ and a unique pair v, v' of S_j-consecutive vertices of $Q_{w'}$ such that $w \in A_{vw'v'}$, so v_{a+1} is an internal vertex of $vQ_{w'}v'$. If v_{a-1} lies in $vQ_{w'}v'$, then we immediately have $|L_{\phi^*}(v_a)| \geq 3$, since $\phi_{vw'v'}$ is a match-valid L-coloring of its domain. If v_{a-1} does not lie in $vQ_{w'}v'$, then v_a is an endpoint of $vQ_{w'}v'$, which is false, since $v_a \notin \text{dom}(\phi^*)$.

Thus, we have $|L_{\phi^*}(v)| \geq 3$ for each $v \in V(C \setminus P) \setminus \text{dom}(\phi^*)$. For each $w \in T_0 \cup \cdots T_j$, we have $N(w) \cap \text{dom}(\phi^*) \subseteq S_j$, and thus $|L_{\phi^*}(w)| \geq 3$. For each $w \in T_{j+1}$, there exists a $w' \in T_j$ and a pair v, v' of S_j-consecutive vertices of Q_w such that $N(w) \cap \text{dom}(\phi^*) \subseteq \text{dom}(\phi_{vw'v'})$, so we have $|L_{\phi^*}(w)| \geq 3$, since $\phi_{vw'v'}$ is a match-valid L-coloring of its domain. Thus, $|L_{\phi^*}(w)| \geq 3$ for all $w \in D_1(S_{j+1}, G) \setminus V(P)$. Thus, ϕ^* is indeed a match-valid L-coloring of its domain. This proves 2), and completes the proof of Subclaim 14.0.18. ■

Now we fix a $w' \in T_j$ and a pair v_a, v_b of S_j-consecutive vertices of $Q_{w'}$, where $1 \leq a < b \leq n-2$. Let $m_0, m_1, \cdots, m_r$ be a set of indices with $a = m_0, b = m_r$, and $m_0 < m_1 < \cdots < m_r$, where $S_{j+1} \cap V(v_aQ_{w'}v_b) = \{v_{m_0}, \cdots, v_{m_r}\}$. Applying Subclaim 14.0.18, we just need to show that ϕ extends to a match-valid L-coloring of $\text{dom}(\phi) \cup \{v_{m_1}, \cdots, v_{m_{r-1}}\}$.

Let $i \in \{0, \cdots, r\}$ and suppose there is an extension of ϕ to a match-valid L-coloring of $\text{dom}(\phi) \cup \{v_{m_0}, v_{m_1}, \cdots, v_{m_i}\}$. This holds for $i = 0$, since $v_{m_0} \in \text{dom}(\phi)$. We claim now that if $0 \leq i < r-1$ and this holds for i, then it also holds for $i+1$. If we show this, then there exists a match-valid L-coloring of $\text{dom}(\phi) \cup \{v_{m_1}, \cdots, v_{m_{r-1}}\}$, so we are done. Fix an $i \in \{0, \cdots, r-2\}$ and an extension ϕ^* of ϕ to a match-valid L-coloring of $\text{dom}(\phi) \cup \{v_{m_0}, \cdots, v_{m_i}\}$. Now we break the proof into the following cases.

Case 1 of Claim 14.0.17: $v_{m_i}, v_{m_{i+1}}$ do not have a common neighbor in T_{j+1}

In this case, there is a $w \in T_{j+1}$ such that $v_{m_{i+1}}$ is the left endpoint of Q_w.

Subclaim 14.0.19. *For any extension ϕ^{**} of ϕ^* to an L-coloring of* $\text{dom}(\phi^*) \cup \{v_{m_{i+1}}\}$*, if ϕ^{**} is not a match-valid L-coloring of its domain, then $m_i = m_{i+1} - 2$ and one of the following holds.*

1) *v_{m_i+1} is a matched vertex of C, and $|L(v_{m_i+1}) \setminus \{\phi^*(v_{m_i}), \phi^{**}(v_{m_{i+1}})\}| = 3$; OR*
2) *v_{m_i+1} is an unmatched vertex of C and $L(v_{m_i+1}) \setminus \{\phi^*(v_{m_i}), \phi^{**}(v_{m_{i+1}})\}| = 2$.*

Proof: Suppose that ϕ^{**} is not a match-valid L-coloring of its domain. By Subclaim 14.0.18, ϕ^{**} satisfies V1) of Definition 14.0.12. If ϕ^{**} does not satisfy V3) of Definition 14.0.12, then $m_{i+1} = m_i + 2$, v_{m_i+1} is the endpoint of a chord e of C with $e \in \text{Cl}(U)$, and $L(v_{m_i+1}) \setminus \{\phi^*(v_{m_i}), \phi^{**}(v_{m_i+2})\}| = 3$. Note that $v_{m_i+1} \notin \text{dom}(\phi^*)$ since S contains no three consecutive vertices of $C \setminus P$. Thus, if ϕ^{**} does not satisfy V3) of Definition 14.0.12, then we are done.

Suppose now that ϕ^{**} satisfies V1) and V3) of Definition 14.0.12, but not V2). Then there is a vertex $z \in D_1(\text{dom}(\phi), G) \setminus V(P)$ such that $|L_{\phi^{**}}(z)| < 3$. Since there is no chord of C in $\mathbb{R}^2 \setminus U$ and each vertex of $C \setminus P$ has an L-list of size at least four, we have $z = v_{m_i+1}$ and $m_{i+1} = m_i + 2$. Since ϕ^{**} satisfies V3), v_{m_i+1} is unmatched, or else we have $|L_{\phi^*}(v_{m_i+1})| \geq 3$. Since, we have $|L(v_{m_i+1})| \geq 4$ and $L(v_{m_i+1}) \setminus \{\phi(v_{m_i}), \phi^*(v_{m_{i+1}})\}| = 2$, so we are done. ■

Now we finish Case 1 of of Claim 14.0.17. If $m_{i+1} \neq m_i + 2$, then any extension of ϕ^* to $\text{dom}(\phi^*) \cup \{v_{m_{i+1}}\}$ is a match-valid L-coloring of its domain, so we are done in that case. Now suppose that $m_{i+1} = m_i + 2$. Note that, since $v_{m_{i+1}}$ is the left endpoint of Q_w, and $w \in T_{j+1}$, there are at least two vertices of Q_w lying to the right of $v_{m_{i+1}}$ on the path $v_a \cdots v_b$. Thus, since every edge of G with both endpoints in S is an edge of C, we have $|L_{\phi^*}(v_{m_{i+1}})| \geq 5$.

If v_{m_i+1} is a matched vertex, then we have $|L(v_{m_i+1}) \setminus \{\phi^*(v_{m_i})\}| \geq 4$, and since $|L_{\phi^*}(v_{m_{i+1}})| \geq 5$, we choose a color $d \in |L_{\phi^*}(v_{m_{i+1}})$ such that $|L(v_{m_i+1}) \setminus \{\phi(v_{m_i}), d\}| \geq 4$. By Subclaim 14.0.19, the resulting extension of ϕ

to dom$(\phi^*) \cup \{v_{m_{i+1}})$ is then a match-valid L-coloring of its domain, so we are done in that case. Now suppose that v_{m_i+1} is unmatched. In that case, since $|L(v_{m_i+1})| \geq 4$, we have $|L_{\phi^*}(v_{m_i+1})| \geq 3$, so we simply choose a color $d \in L_{\phi^*}(v_{m_{i+1}})$ such that $L_{\phi^*}(v_{m_i+1}) \setminus \{d\}| \geq 3$, and, again by Subclaim 14.0.19, the resulting extension of ϕ^* to dom$(\phi^*) \cup \{v_{m_{i+1}}\}$ is a match-valid L-coloring of its domain. This completes Case 1 of Claim 14.0.17.

Case 2 of Claim 14.0.17: $v_{m_{i+1}}, v_{m_i}$ have a common neighbor in T_{j+1}

In this case, by Claim 14.0.13, let $w \in T_{j+1}$ be the unique common neighbor of $v_{m_i}, v_{m_{i+1}}$. Now consider the following subcases.

Subcase 2.1 Either $m_{i+1} < m_r - 1$ or $|L_{\phi^*}(v_{m_{i+1}})| \geq 5$

Since $|L_{\phi^*}(w)| \geq 3$, and $|L_{\phi^*}(v_{m_{i+1}})| \geq 4$, there is a color $c \in L_{\phi^*}(v_{m_{i+1}})$ such that $|L_{\phi^*}(w) \setminus \{c\}| \geq 3$, and such that either $v_{m_{i+1}}$ is not adjacent to v_{m_r} or, if $v_{m_{i+1}}$ is adjacent to v_{m_r}, then $c \neq \phi(v_{m_r})$. In either case, letting ϕ' be the extension of ϕ^* obtained by coloring $v_{m_{i+1}}$ with c, we claim that ϕ' is a match-valid L-coloring of its domain.

By definition of S, together with 1) of Proposition 14.0.8, there is a unique chord $v_{m_{i+1}}v_\ell$ of C lying in Cl(U), where $\ell \in \{1, \cdots, n-2\}$. Since $v_{m_{i+1}}$ is the other end of the unique chord of C in Cl(U) which is incident to v_ℓ, we have dom$(\phi^*) \cap N(v_\ell) \subseteq \{v_{\ell-1}, v_{\ell+1}\}$, where the indices are read mod n. Thus, since ϕ^* is a match-valid L-coloring of its domain, we have $|L_{\phi^*}(v_\ell)| \geq 4$, and thus $|L_{\phi'}(v_\ell)| \geq 3$. Thus, ϕ' satisfies condition V2) of Definition 14.0.12. If V3) is not satisfied, then we have $v_{m_{i+1}-1} \notin$ dom(ϕ^*) and $v_{m_{i+1}-2} \in$ dom(ϕ^*), so w is adjacent to each of $v_{m_{i+1}-2}, v_{m_{i+1}}$. Since $v_{m_{i+1}-1} \notin V(P)$, this contradicts 5) of Proposition 14.0.8. Thus, ϕ' is indeed a match-valid L-coloring of its domain, so we are done in this case.

Subcase 2.2 $m_{i+1} = m_r - 1$ and $|L_{\phi^*}(v_{m_{i+1}})| = 4$

In this case, we have $N(v_{m_{i+1}}) \cap S = \{v_{m_i}\}$ and $m_{i+1} = m_i + 1$.

> **Subclaim 14.0.20.** $m_i < m_{i+1} - 2$.
>
> Proof: Firstly, if w is adjacent to v_{m_r}, then, since $V(P) \cap \{v_a, v_{a+1}, \cdots, v_b\} = \varnothing$, and $v_{m_i}, v_{m_{i+1}}$ are adjacent to w, we immediately have $m_{i+1} < m_i - 2$ by Proposition 14.0.8. Now suppose that w is not adjacent to v_{m_r}. In that case, since $N(w) \cap S \subseteq \{v_{m_0}, \cdots, v_{m_r}\}$, $v_{m_{i+1}}$ is the right endpoint of Q_w. Suppose toward a contradiction that $m_i \geq m_{i+1} - 2$. In that case, again by Proposition 14.0.8, the only possibility is that $m_i = m_{i+1} - 1$, and thus S contains $\{v_{m_i}, v_{m_{i+1}}, v_{m_r}\}$. As these are consecutive vertices of C, we have a contradiction. ■

Now, since $|L_{\phi^*}(v_{m_{i+1}})| \geq 4$, we choose a color $c \in |L_{\phi^*}(v_{m_{i+1}})$ such that $|L_{\phi^*}(w) \setminus \{c\}| \geq 3$. Let ϕ' be the resulting extension of ϕ^* to dom$(\phi^*) \cup \{v_{m_{i+1}}\}$. We claim that ϕ' is a match-valid L-coloring of its domain. It just suffices to check V2) and V3) of Definition 14.0.12. Since $m_i < m_{i+1} - 2$, it immediately follows that ϕ' satisfies V3). Since $|L_{\phi'}(w)| \geq 3$, to finish, it just suffices to check that $|L_{\phi'}(v)| \geq 3$ for each $v \in V(C \setminus P) \setminus$ dom(ϕ'). Suppose there is a $v \in V(C \setminus P) \setminus$ dom(ϕ') with $|L_{\phi'}(v)| < 3$. Since $|L_{\phi'}(v^*)| \geq 3$, v is adjacent to $v_{m_{i+1}}$.

If v is matched to $v_{m_{i+1}}$, then since ϕ' satisfies V2), we have $|L_{\phi'}(v)| \geq 3$. Since $v_{m_r} \in$ dom(ϕ'), the only remaining possibility is that $v = v_{m_{i+1}-1}$. Since $m_i < m_{i+1} - 2$, if v is unmatched, then we have dom$(\phi^*) \cap N(v) = \varnothing$, and thus $|L_{\phi^*}(v)| = |L(v)| \geq 4$, so $|L_{\phi'}(v)| \geq 3$. On the other hand, if v is matched, then, since ϕ' satisfies V2), we again have $|L_{\phi'}(v)| \geq 3$. Thus ϕ' satisfies V2), as desired. This completes the proof of Claim 14.0.14. ■

Now we finish the proof of Proposition 14.0.10. Combining Claim 14.0.14 and Claim 14.0.17, there is a match-valid L-coloring ϕ of S_ℓ. Since $S_\ell = S$, the set S and the L-coloring ϕ of S satisfy Proposition 14.0.10, so we are done. This completes the proof of Proposition 14.0.10. □

Now we are ready to finish the proof of Theorem 0.3.5. Let $(G, \mathcal{F}, \mathcal{T}, L)$ be a minimal counterexample. Since G is not L-colorable, we have $\mathcal{T} \neq \varnothing$ by Proposition 14.0.5. Thus, let $C \in \mathcal{T}$ and let U be the unique vertex-free side of C and let P be the precolored path of C. Given this element $C \in \mathcal{T}$, let $S \subseteq V(C \setminus P)$ and let ϕ be an L-coloring of $G[S]$ such that Proposition 14.0.10 is satisfied. Let G^* be the graph obtained from G by deleting each chord of C in $\text{Cl}(U)$. Then C is a cyclic facial subgraph of G^*. Applying Theorem 1.3.2, $G^* \setminus S$ contains a facial subgraph F such that $P \subseteq F$, $V(C \setminus S) \subseteq V(F)$, and $D_1(S, G) \subseteq V(F)$.

Since S has an endpoint from each chord of C in $\text{Cl}(U)$, we have $G \setminus S = G^* \setminus S$, so F is also a facial subgraph of $G \setminus S$. Let F' be a subgraph of F with $P \subseteq F'$ and $V(F') = V(C \setminus S) \cup D_1(S, G)$. Now consider the tuple $(G \setminus S, \mathcal{F} \cup \{F'\}, \mathcal{T} \setminus \{C\}, L_\phi)$. For each $F^* \in \mathcal{F}$, we have $d(C, F^*) \geq \alpha + 1$ and thus $d(F', F^*) \geq \alpha$. Likewise, for each $C' \in \mathcal{T} \setminus \{C\}$, we have $d(C, C') \geq \alpha + 2$ and thus $d(F', C') \geq \alpha + 1$. Furthermore, P is L_ϕ-colorable and $|L_\phi(v)| \geq 3$ for all $v \in V(F' \setminus P)$. Thus, $(G \setminus S, \mathcal{F} \cup \{F'\}, \mathcal{T} \setminus \{C\}, L_\phi)$ is also a tennis court, and since $|E(G \setminus S)| < |E(G)|$, it follows from the minimality of $(G, \mathcal{F}, \mathcal{T}, L)$ that $G \setminus S$ is L_ϕ-colorable. Thus, G is L-colorable, contradicting our assumption. This completes the proof of Theorem 0.3.5.

We now briefly discuss some potential future work along the same lines as Theorem 0.3.5. The goal of this future work is to use Theorem 0.3.1 to obtain more results about the 5-choosability of drawings which differ from a planar embeddings by some pairwise far apart region. One such conjecture we have is the following.

Conjecture 14.0.21. *There exists a constant* d *such that the following holds: Let* G *be a drawing on the sphere of a graph and let* $C_1, \cdots, C_m$ *be a collection of cycles in* G *such that* $d(C_i, C_j) \geq d$ *for each* $1 \leq i < j \leq m$*. Suppose that, for each* $1 \leq i \leq m$*, there is a connected component* U_i *of* $\mathbb{S}^2 \setminus C_i$ *such that the following hold.*

1) *For each crossing point* x *of* G*, there is an* $i \in \{1, \cdots, m\}$ *such that* $x \in U_i$ *; AND*
2) *For each* $i = 1, \cdots, m$*, the underlying graph* $G \cap \text{Cl}(U_i)$ *has girth at least five and admits a planar embedding.*

Then G *is 5-choosable.*

In the statement above, G does not necessarily admit a planar embedding, and, for each $1 \leq i \leq m$, the drawing of $G \cap \text{Cl}(U_i)$ is not necessarily planar, or else the result would trivially follow from Theorem 0.2.3. Although $G \cap \text{Cl}(U_i)$ admits a planar embedding, it does not necessarily admit a planar embedding in which C_i is a facial cycle, so the drawing G possibly has arbitrarily many crossings and the underlying abstract graph of G possibly has an arbitrarily large crossing number.

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Index

This index consists of non-symbolic terminology and definitions which are specific to this thesis. The pages referenced indicate where each piece of terminology is first defined.