A Boolean Extension of KLM-Style Conditional Reasoning

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Abstract. Propositional KLM-style defeasible reasoning involves extending propositional logic with a new logical connective that can express defeasible (or conditional) implications, with semantics given by ordered structures known as ranked interpretations. KLM-style defeasible entailment is referred to as rational whenever the defeasible entailment relation under consideration generates a set of defeasible implications all satisfying a set of rationality postulates known as the KLM postulates. In a recent paper Booth et al. proposed PTL, a logic that is more expressive than the core KLM logic. They proved an impossibility result, showing that defeasible entailment for PTL fails to satisfy a set of rationality postulates similar in spirit to the KLM postulates. Their interpretation of the impossibility result is that defeasible entailment for PTL need not be unique. In this paper we continue the line of research in which the expressivity of the core KLM logic is extended. We present the logic Boolean KLM (BKLM) in which we allow for disjunctions, conjunctions, and negations, but not nesting, of defeasible implications. Our contribution is twofold. Firstly, we show (perhaps surprisingly) that BKLM is more expressive than PTL. Our proof is based on the fact that BKLM can characterise all single ranked interpretations, whereas PTL cannot. Secondly, given that the PTL impossibility result also applies to BKLM, we adapt the different forms of PTL entailment proposed by Booth et al. to apply to BKLM.

Keywords: Non-monotonic reasoning · Defeasible entailment

1 Introduction

Non-monotonic reasoning has been extensively studied in the AI literature, as it provides a mechanism for making bold inferences that go beyond what classical methods can provide, while retaining the possibility of revising these inferences in light of new information. In their seminal paper, Kraus et al. [14] consider a general framework for non-monotonic reasoning, phrased in terms of *defeasible, or conditional implications* of the form $\alpha \mid\sim \beta$, to be read as "*If* α *holds, then typically* β *holds*". Importantly, they provide a set of *rationality conditions*,

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in the form of structural properties, that a reasonable form of entailment for these conditionals should satisfy, and characterise these semantically. Lehmann and Magidor [16] also considered the question of which entailment relations definable in the KLM framework can be considered to be the *correct* ones for non-monotonic reasoning. In general, there is a large class of entailment relations for KLM-style logics [9], and it is widely agreed upon that there is no unique best answer. The options can be narrowed down, however, and Lehmann et al. propose *Rational Closure* (RC) as the minimally acceptable form of rational entailment. Rational closure is based on the principle of *Presumption of Typicality* [15], which states that propositions should be considered typical unless there is reason to believe otherwise. For instance, if we know that birds typically fly, and all we know about a robin is that it is a bird, we should tentatively conclude that it flies, as there is no reason to believe it is atypical. While RC is not always appropriate, there is fairly general consensus that interesting forms of conditional reasoning should extend RC inferentially [9,15].

Since KLM-style logics have limited conditional expressivity (see Sect. 2.1), there has been some work in extending the KLM constructions to more expressive logics. Perhaps the main question is whether entailment relations resembling RC can also be defined for more expressive logics. The first investigation in such a direction was done by Booth and Paris [4], who consider an extension in which both positive ($\alpha \mid\sim \beta$) and negative ($\alpha \not\mid\sim \beta$) conditionals are allowed. Booth et al. [3] later considered a more expressive logic called *Propositional Typicality Logic* (PTL), in which propositional logic is extended with a modal-like typicality operator $\bullet$. This typicality operator can be used anywhere in a formula, in contrast to KLM-style logics, where typicality refers only to the antecedent of conditionals of the form $\alpha \mid\sim \beta$.

The price one pays for this expressiveness is that rational entailment becomes more difficult to pin down. This is shown by Booth et al. [2], who prove that several desirable properties of rational closure are mutually inconsistent for PTL entailment. They interpret this as saying that the correct form of entailment for PTL is contextual, and depends on which properties are considered more important for the task at hand.

In this paper we consider a different extension of KLM-style logics, which we refer to as *Boolean KLM* (BKLM), and in which we allow negative conditionals, as well as arbitrary conjunctions and disjunctions of conditionals. We do not allow the nesting of conditionals, though. We show, perhaps surprisingly, that BKLM is strictly more expressive than PTL by exhibiting an explicit translation of PTL knowledge bases into BKLM. We also prove that BKLM entailment is more restrictive than PTL entailment, in the sense that a stronger class of entailment properties are inconsistent for BKLM. In particular, attempts to extend rational closure to BKLM in the manner of LM-entailment as defined by Booth et al. [2], are shown to be untenable.

The rest of the paper is structured as follows. In Sect. 2 we provide the relevant background on the KLM approach to defeasible reasoning, and discuss various forms of rational entailment. We then define Propositional Typicality

Logic, and give a brief overview of the entailment problem for PTL. In Sect. 3 we define the logic BKLM, an extension of KLM-style logics that allows for arbitrary boolean combinations of conditionals. We investigate the expressiveness of BKLM, and show that it is strictly more expressive PTL by exhibiting an explicit translation of PTL formulas into BKLM. In Sect. 4 we turn to the entailment problem for BKLM, and show that BKLM suffers from stronger versions of the known impossibility results for PTL. Section 5 discusses some related work, while Sect. 6 concludes and points out some future research directions.

2 Background

Let $\mathcal{P}$ be a set of propositional atoms, and let $p, q, \ldots$ be meta-variables for elements of $\mathcal{P}$. We write $\mathcal{L}^{\mathcal{P}}$ for the set of propositional formulas over $\mathcal{P}$, defined by $\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \top \mid \bot$. Other boolean connectives are defined as usual in terms of $\wedge, \neg, \rightarrow$, and $\leftrightarrow$. We write $\mathcal{U}^{\mathcal{P}}$ for the set of valuations of $\mathcal{P}$, which are functions $v : \mathcal{P} \rightarrow \{0, 1\}$. Valuations are extended to $\mathcal{L}^{\mathcal{P}}$ in the usual way, and satisfaction of a formula α will be denoted $v \Vdash \alpha$. For the remainder of this paper we assume that $\mathcal{P}$ is finite, and drop the superscripts where there's no ambiguity.

2.1 The Logic KLM

Kraus et al. [14] study a conditional logic, which we refer to as KLM. It is defined by assertions of the form $\alpha \mid\!\sim \beta$, which are read "if α, then typically β". For example, if $\mathcal{P} = \{\mathsf{b}, \mathsf{f}\}$ refers to the properties of being a bird and flying respectively, then $\mathsf{b} \mid\!\sim \mathsf{f}$ states that birds typically fly. There are various possible semantic structures for this logic, but in this paper we are interested in the case of *rational* conditional assertions. The semantics for rational conditionals is given by *ranked interpretations* [16]. The following is an alternative, but equivalent definition of such a class of interpretations.

Definition 1. *A ranked interpretation $\mathscr{R}$ is a function from $\mathcal{U}$ to $\mathbb{N} \cup \{\infty\}$ satisfying the following convexity condition: if $\mathscr{R}(u) < \infty$, then for every $0 \leq j < \mathscr{R}(u)$, there is some $v \in \mathcal{U}$ for which $\mathscr{R}(v) = j$.*

Given a ranked interpretation $\mathscr{R}$, we call $\mathscr{R}(u)$ the *rank* of u with respect to $\mathscr{R}$. Valuations with a lower rank are viewed as being more typical than those with a higher rank, whereas valuations with infinite rank are viewed as being impossibly atypical. We refer to the set of *possible valuations* as $\mathcal{U}^{\mathscr{R}} = \{u \in \mathcal{U} : \mathscr{R}(u) < \infty\}$, and for any $\alpha \in \mathcal{L}$ we define $[\![\alpha]\!]^{\mathscr{R}} = \{u \in \mathcal{U}^{\mathscr{R}} : u \Vdash \alpha\}$.

Every ranked interpretation $\mathscr{R}$ determines a total preorder on $\mathcal{U}$ in the obvious way, namely $u \leq_{\mathscr{R}} v$ iff $\mathscr{R}(u) \leq \mathscr{R}(v)$. Writing the strict version of this preorder as $\prec_{\mathscr{R}}$, we note that it is *modular*:

Proposition 1. *$\prec_{\mathscr{R}}$ is* modular, *i.e. for all $u, v, w \in \mathcal{U}$, $u \prec_{\mathscr{R}} v$ implies that either $w \prec_{\mathscr{R}} v$ or $u \prec_{\mathscr{R}} w$.*

Lehmann et al. [16] define ranked interpretations in terms of modular orderings on $\mathcal{U}$. The following observation proves that the two definitions are equivalent:

Proposition 2. *Let $\mathscr{R}_1$ and $\mathscr{R}_2$ be ranked interpretations. Then $\mathscr{R}_1 = \mathscr{R}_2$ iff $\prec_{\mathscr{R}_1} = \prec_{\mathscr{R}_2}$.*

We define satisfaction with respect to ranked interpretations as follows. Given any $\alpha \in \mathcal{L}$, we say $\mathscr{R}$ *satisfies* α (written $\mathscr{R} \Vdash \alpha$) iff $[\![\alpha]\!]^{\mathscr{R}} = \mathcal{U}^{\mathscr{R}}$. Similarly, $\mathscr{R}$ satisfies a conditional assertion $\alpha \mid\!\sim \beta$ iff $\min_{\leq_{\mathscr{R}}} [\![\alpha]\!]^{\mathscr{R}} \subseteq [\![\beta]\!]^{\mathscr{R}}$, or in other words iff all of the $\leq_{\mathscr{R}}$-minimal valuations satisfying α also satisfy β.

Example 1. Let $\mathscr{R}$ be the ranked interpretation below. Then $\mathscr{R}$ satisfies $\mathsf{p} \to \mathsf{b}$, $\mathsf{b} \mid\!\sim \mathsf{f}$ and $\mathsf{p} \mid\!\sim \neg\mathsf{f}$. Note that in our diagrams we omit rank ∞ for brevity, and represent a valuation as a string of literals, with $\overline{p}$ indicating the negation of the atom p.

2	pbf
1	$\overline{\mathsf{p}}\mathsf{b}\overline{\mathsf{f}}$, $\mathsf{pb}\overline{\mathsf{f}}$
0	$\overline{\mathsf{p}}\overline{\mathsf{b}}\overline{\mathsf{f}}$, $\overline{\mathsf{p}}\overline{\mathsf{b}}\mathsf{f}$, $\overline{\mathsf{p}}\mathsf{bf}$

A useful simplification is the fact that classical statements (such as $\mathsf{p} \to \mathsf{b}$) can be viewed as special cases of conditional assertions in ranked interpretations:

Proposition 3 *[14, p. 174]. For all $\alpha \in \mathcal{L}$, $\mathscr{R} \Vdash \alpha$ iff $\mathscr{R} \Vdash \neg\alpha \mid\!\sim \bot$.*

In what follows we define a *knowledge base* to be a finite set of conditional assertions. The set of all ranked interpretations over $\mathcal{P}$ is denoted by RI, and we write $\textsc{Mod}(\mathcal{K})$ for the set of ranked models of a knowledge base $\mathcal{K}$. For any $U \subseteq \text{RI}$, we write $U \Vdash \alpha$ to mean $\mathscr{R} \Vdash \alpha$ for all $\mathscr{R} \in U$, and finally the set of formulas satisfied by the ranked interpretation $\mathscr{R}$ is denoted by $\text{sat}(\mathscr{R})$.

2.2 Propositional Typicality Logic

In this paper we are interested in looking at more expressive variations of KLM, as the syntax for conditionals in KLM is somewhat restrained. An early investigations in this direction was done by Booth and Paris [4], who consider an extension of KLM that permits both positive ($\alpha \mid\!\sim \beta$) and negative ($\alpha \not\mid\!\sim \beta$) conditionals.

A more recent variation of KLM is Propositional Typicality Logic (PTL), a logic for defeasible reasoning proposed by Booth et al. [2], in which propositional logic is enriched with a *typicality operator* $\bullet$. The intuition behind a formula $\bullet\alpha$ is that it is true whenever α is *typical* for the world in consideration. In contrast to KLM, however, the typicality operator can be placed anywhere in a formula, as well as nested. Formulas for PTL are defined by the grammar $\alpha ::= \top \mid \bot \mid p \mid \bullet\alpha \mid \neg\alpha \mid \alpha \wedge \alpha$, where p is any propositional atom, and other logical connectives can be defined as usual in terms of $\neg$ and $\wedge$. We denote the set of all PTL formulas by $\mathcal{L}^{\bullet}$.

Satisfaction for PTL formulas is defined with respect to a ranked interpretation $\mathscr{R}$. Given a valuation $u \in \mathcal{U}$ and formula $\alpha \in \mathcal{L}^{\bullet}$, we define $u \Vdash_{\mathscr{R}} \alpha$ inductively in the same manner as propositional logic, with an additional rule for the typicality operator: $u \Vdash_{\mathscr{R}} \bullet\alpha$ if and only if $u \Vdash_{\mathscr{R}} \alpha$ and there is no $v \prec_{\mathscr{R}} u$ such that $v \Vdash_{\mathscr{R}} \alpha$. We say that $\mathscr{R}$ *satisfies* the formula α, written $\mathscr{R} \Vdash \alpha$, iff $u \Vdash_{\mathscr{R}} \alpha$ for all $u \in \mathcal{U}^{\mathscr{R}}$. The following proposition explains why we are viewing PTL as an extension of KLM, rather than as a separate logic in its own right:

Proposition 4 *[3, Proposition 11]. A ranked interpretation $\mathscr{R}$ satisfies $\alpha \mid\!\sim \beta$ iff it satisfies $\bullet\alpha \rightarrow \beta$.*

Proposition 4 can be rephrased as saying that every KLM knowledge base has an equivalent PTL knowledge base, in the sense that they share the same set of ranked models. Note, however, that the converse doesn't hold, which intuitively shows that PTL is strictly more expressive than KLM:

Proposition 5 *[3, Proposition 13]. For any $p \in \mathcal{P}$, the knowledge base consisting of $\bullet p$ has no equivalent* KLM *knowledge base.*

Later, we will show that there is a sense in which PTL is *not* maximally expressive for semantics given by ranked interpretations, a fact that may seem surprising in light of its unrestricted syntax.

2.3 The Entailment Problem

We now turn to a central question in non-monotonic reasoning, namely determining what forms of entailment are appropriate in a defeasible setting. In other words, we wish to understand what it means for a formula α to *follow* from a knowledge base $\mathcal{K}$. We will denote such a relation by $\mathcal{K} \mid\!\approx \alpha$, to be read "$\mathcal{K}$ *defeasibly entails* α".

First steps toward the entailment problem for KLM-style logics were made by Kraus et al. [14], who argue that a defeasible entailment relation should satisfy all of the *rationality properties* listed below. Such relations are said to be *rational*, and one reason for their importance is that they can be characterised precisely by ranked interpretations:

(Refl) $\mathcal{K} \mid\!\approx \alpha \mid\!\sim \alpha$ for all $\alpha \in \mathcal{L}$

(Lle) $\models \alpha \leftrightarrow \beta$ and $\mathcal{K} \mid\!\approx \alpha \mid\!\sim \gamma$ implies $\mathcal{K} \mid\!\approx \beta \mid\!\sim \gamma$

(Rw) $\models \beta \rightarrow \gamma$ and $\mathcal{K} \mid\!\approx \alpha \mid\!\sim \beta$ implies $\mathcal{K} \mid\!\approx \alpha \mid\!\sim \gamma$

(And) $\mathcal{K} \mid\!\approx \alpha \mid\!\sim \beta$ and $\mathcal{K} \mid\!\approx \alpha \mid\!\sim \gamma$ implies $\mathcal{K} \mid\!\approx \alpha \mid\!\sim \beta \wedge \gamma$

(Or) $\mathcal{K} \mid\!\approx \alpha \mid\!\sim \gamma$ and $\mathcal{K} \mid\!\approx \beta \mid\!\sim \gamma$ implies $\mathcal{K} \mid\!\approx \alpha \vee \beta \mid\!\sim \gamma$

(Cm) $\mathcal{K} \mid\!\approx \alpha \mid\!\sim \beta$ and $\mathcal{K} \mid\!\approx \alpha \mid\!\sim \gamma$ implies $\mathcal{K} \mid\!\approx \alpha \wedge \beta \mid\!\sim \gamma$

(Rm) $\mathcal{K} \mid\!\approx \alpha \mid\!\sim \gamma$ implies $\mathcal{K} \mid\!\approx \alpha \mid\!\sim \neg\beta$ or $\mathcal{K} \mid\!\approx \alpha \wedge \beta \mid\!\sim \gamma$

Proposition 6 *[16, Theorem 5]. A defeasible entailment relation $\approx$ is rational iff for each knowledge base $\mathcal{K}$, there is a ranked interpretation $\mathscr{R}_{\mathcal{K}}$ such that $\mathcal{K} \approx \alpha \mid\sim \beta$ iff $\mathscr{R}_{\mathcal{K}} \Vdash \alpha \mid\sim \beta$.*

Note that by Proposition 4, these rationality properties can be considered for PTL entailment relations as well, by replacing each instance of $\alpha \mid\sim \beta$ with the equivalent $\bullet\alpha \to \beta$. An interesting consequence of an entailment relation being rational is *non-monotonicity*, which means that the following Tarskian definition of entailment fails to be rational [16]:

Definition 2. *A formula α is* rank entailed *by a knowledge base $\mathcal{K}$ (written $\mathcal{K} \approx_R \alpha$) iff $\mathscr{R} \Vdash \alpha$ for every ranked model $\mathscr{R}$ of $\mathcal{K}$.*

Despite this, it is generally agreed that defeasible entailment relations should extend rank entailment, a property known as *Ampliativity*. In the context of PTL entailment, Booth et al. [2] consider this, as well as a number of other desirable properties of defeasible entailment:

(Inclusion)	$\mathcal{K} \approx \alpha$ for all $\alpha \in \mathcal{K}$
(Cumulativity)	$\mathcal{K} \approx \alpha$ whenever $\mathcal{K} \approx \beta$ for all $\beta \in \mathcal{K}_2$ and $\mathcal{K}_2 \approx \alpha$
(Ampliativity)	$\mathcal{K} \approx \alpha$ whenever $\mathcal{K} \approx_R \alpha$
(Strict Entailment)	for classical $\alpha \in \mathcal{L}$, $\mathcal{K} \approx \alpha$ iff $\mathcal{K} \approx_R \alpha$
(Typical Entailment)	for classical $\alpha \in \mathcal{L}$, $\mathcal{K} \approx \top \mid\sim \alpha$ iff $\mathcal{K} \approx_R \top \mid\sim \alpha$
(Single Model)	for all $\mathcal{K}$ there's some $\mathscr{R} \in \text{Mod}(\mathcal{K})$ such that $\mathcal{K} \approx \alpha$ iff $\mathscr{R} \Vdash \alpha$

Proposition 6 states that the Single Model property is equivalent to being rational for KLM entailment relations, but note that the properties diverge for more expressive logics. Surprisingly, it turns out that a number of these properties are mutually inconsistent in the case of PTL entailment relations:

Proposition 7 *[2, Theorem 1]. There is no* PTL *entailment relation $\approx$ satisfying the Inclusion, Strict Entailment, Typical Entailment and Single Model properties.*

As a final remark on general entailment relations, we note that this list of properties is by no means exhaustive. Booth et al. [2] consider many variations of the above properties in the context of PTL entailment, whereas Casini et al. [9] study properties of extensions of *Rational Closure*, a well-known entailment relation for KLM.

2.4 Rational Closure

Given the failure of rank entailment to be rational, an interesting question is which rational entailment relation should be considered the right one for defeasible reasoning. In their seminal paper, Lehmann et at. [16] define *Rational Closure*, an entailment relation for KLM that is widely considered to be a minimal acceptable answer to this question [9]. In this section we give a semantic description of Rational Closure in terms of an ordering on ranked interpretations [12]:

Definition 3 *[12, Definition 7]. Given two ranked interpretations $\mathscr{R}_1$ and $\mathscr{R}_2$, we say $\mathscr{R}_1$ is* preferred *to $\mathscr{R}_2$ (written $\mathscr{R}_1 <_G \mathscr{R}_2$) iff for every $u \in \mathcal{U}$ we have $\mathscr{R}_1(u) \leq \mathscr{R}_2(u)$ and there is some $v \in \mathcal{U}$ s.t. $\mathscr{R}_1(v) < \mathscr{R}_2(v)$.*

Intuitively, the lower down a ranked interpretation $\mathscr{R}$ is with respect to the ordering $\leq_G$, the fewer exceptional valuations it has modulo the constraints of $\mathcal{K}$. The $\leq_G$-minimal ranked interpretations can therefore be thought of as the semantic counterpart to the *Presumption of Typicality* mentioned in the introduction. For the case of KLM knowledge bases containing positive and/or negative conditionals, it follows from the work of Booth and Paris [4] that there is always a unique minimal model:

Proposition 8. *Let $\mathcal{K} \subseteq \mathcal{L}^{\mid\sim}$ be a knowledge base. Then if $\mathcal{K}$ is consistent,* $\text{Mod}(\mathcal{K})$ *has a unique $\leq_G$-minimal element, denoted by $\mathscr{R}^{RC}_{\mathcal{K}}$.*

The Rational Closure of a knowledge base is defined to be (or rather, can be characterised as) the set of formulas satisfied by this minimal model:

Proposition 9 *[12, Theorem 2]. A conditional $\alpha \mid\sim \beta$ is in the rational closure of a knowledge base $\mathcal{K} \subseteq \mathcal{L}^{\mid\sim}$ (written $\mathcal{K} \mid\approx_{RC} \alpha \mid\sim \beta$) iff $\mathscr{R}^{RC}_{\mathcal{K}} \Vdash \alpha \mid\sim \beta$.*

Rational Closure satisfies all of the properties given in Sect. 2.3, and has attractive properties in other respects [16]. Nevertheless, it has some well-known shortcomings, such as not providing for the inheritance of generic properties to exceptional individual - a property that is known as the *drowning effect*. To deal with some of these issues, various refinements of Rational Closure have been proposed, such as Lexicographic Closure [15], Relevant Closure [8] and Inheritance-Based Closure [7]. There is a general consensus that interesting forms of defeasible entailment should extend Rational Closure inferentially [9].

3 Boolean KLM

In this section we describe *Boolean* KLM (BKLM), an extension of KLM that permits arbitrary boolean combinations of defeasible conditionals. Syntactically, this goes beyond the extension of Booth and Paris [4] by allowing disjunctive as well as negative assertions in knowledge bases. BKLM formulas are defined by the grammar $A ::= \alpha \mid\sim \beta \mid \neg A \mid A \wedge A$, with other boolean connectives defined as usual in terms of $\neg$ and $\wedge$. For convenience, we use $\alpha \not\mid\sim \beta$ as a synonym for $\neg(\alpha \mid\sim \beta)$, and write $\mathcal{L}^b$ for the set of all BKLM formulas. Hence, for example, $(\alpha \mid\sim \beta) \wedge (\gamma \not\mid\sim \delta)$ and $\neg((\alpha \not\mid\sim \beta) \vee (\gamma \mid\sim \delta))$ are valid BKLM formulas, but the nested conditional $\alpha \mid\sim (\beta \mid\sim \gamma)$ is not.

Satisfaction for BKLM is defined in terms of ranked interpretations, by extending KLM satisfaction in the obvious fashion, namely $\mathscr{R} \Vdash \neg A$ iff $\mathscr{R} \nVdash A$ and $\mathscr{R} \Vdash A \wedge B$ iff $\mathscr{R} \Vdash A$ and $\mathscr{R} \Vdash B$. This leads to some subtle differences between BKLM and the other logics described in this paper. For instance, care must be taken to apply Proposition 3 correctly when translating between propositional formulas and BKLM formulas. The propositional formula $\mathsf{p} \vee \mathsf{q}$ translates to the BKLM formula $\neg(\mathsf{p} \vee \mathsf{q}) \mid\sim \bot$, and *not* to the BKLM formula $(\neg\mathsf{p} \mid\sim \bot) \vee (\neg\mathsf{q} \mid\sim \bot)$, as the following example illustrates:

Example 2. Consider the propositional formula $A = \mathsf{p} \vee \mathsf{q}$ and the BKLM formula $B = (\neg\mathsf{p} \mid\!\sim \bot) \vee (\neg\mathsf{q} \mid\!\sim \bot)$. If $\mathscr{R}$ is the ranked interpretation below, then $\mathscr{R}$ satisfies A but not B, as neither clause of the disjunction is satisfied.

1	$\mathsf{p\overline{q}}$
0	$\mathsf{\overline{p}q}$

To prevent possible confusion, we will avoid mixing classical and defeasible assertions in a BKLM knowledge base. For similar reasons, it's also worth noting the difference between boolean connectives in PTL and the corresponding connectives in BKLM. By Proposition 4, one might expect a BKLM formula such as $\neg(\mathsf{p} \mid\!\sim \mathsf{q})$ to be equivalent to the PTL formula $\neg(\bullet\mathsf{p} \rightarrow \mathsf{q})$. This is not the case in general, however:

Example 3. Consider the formulas $A = \neg(\bullet\mathsf{p} \rightarrow \mathsf{q})$ and $B = \neg(\mathsf{p} \mid\!\sim \mathsf{q})$, and let $\mathscr{R}$ be the ranked interpretation in the example above. Note that A is equivalent to $\bullet\mathsf{p} \wedge \neg\mathsf{q}$, which is not satisfied by $\mathscr{R}$. On the other hand, $\mathscr{R}$ satisfies B.

A natural question is how BKLM compares to PTL in terms of expressiveness. In the next two sections we show that BKLM is strictly more expressive than PTL, and detail an algorithm that converts PTL knowledge bases into equivalent BKLM knowledge bases.

3.1 Expressiveness of BKLM

Satisfaction for KLM, PTL and BKLM formulas is defined in terms of ranked interpretations. This allows us to compare their expressiveness directly, in terms of the sets of models that they can characterise. With the results mentioned earlier, we can already do this for KLM and PTL:

Example 4. Let $\mathcal{K} \subseteq \mathcal{L}^{\mid\sim}$ be a KLM knowledge base. Then the PTL knowledge base $\mathcal{K}' = \{\bullet\alpha \rightarrow \beta : \alpha \mid\!\sim \beta \in \mathcal{K}\}$ has exactly the same ranked models as $\mathcal{K}$ by Proposition 4, and hence PTL is at least as expressive as KLM. Proposition 5 proves that PTL is strictly *more* expressive than KLM.

Our main result in this section is that BKLM is maximally expressive, in the sense that it can characterise *any* set of ranked interpretations. First, we recall that for every valuation $u \in \mathcal{U}$ there is a corresponding characteristic formula $\hat{u} \in \mathcal{L}$, which has the property that $v \Vdash \hat{u}$ iff $v = u$.

Lemma 1. *For any ranked interpretation $\mathscr{R}$ and valuations $u, v \in \mathcal{U}$, the following equivalences hold:*

1. *$\mathscr{R} \Vdash \top \not\mid\!\sim \neg\hat{u}$ iff $\mathscr{R}(u) = 0$.*
2. *$\mathscr{R} \Vdash \hat{u} \mid\!\sim \bot$ iff $\mathscr{R}(u) = \infty$.*
3. *$\mathscr{R} \Vdash \hat{u} \vee \hat{v} \mid\!\sim \neg\hat{v}$ iff $u \prec_{\mathscr{R}} v$ or $\mathscr{R}(u) = \mathscr{R}(v) = \infty$.*

Note that this lemma holds even in the trivial case where $\mathscr{R}(u) = \infty$ for all $u \in \mathcal{U}$. For convenience, in later parts of the paper we will write $\alpha < \beta$ as a standard shorthand for the defeasible conditional $\alpha \vee \beta \mid\!\sim \neg\beta$.

Lemma 2. *Let $\mathscr{R}$ be any ranked interpretation. Then there exists a formula $ch(\mathscr{R}) \in \mathcal{L}^b$ with $\mathscr{R}$ as its unique model.*

We refer to ch($\mathscr{R}$) as the *characteristic formula* of $\mathscr{R}$. Taking a disjunction of characteristic formulas suffices to prove the following more general corollary:

Corollary 1. *Let $U \subseteq \mathrm{RI}$ be a set of ranked interpretations. Then there exists a formula $ch(U) \in \mathcal{L}^b$ with U as its set of models.*

In principle, this corollary shows that for any PTL knowledge base there exists some BKLM formula with the same set of models, and hence BKLM is at least as expressive as PTL. In the next section we make this relationship more concrete, by providing an explicit algorithm for translating PTL knowledge bases into BKLM.

3.2 Translating PTL Into BKLM

In Sect. 2.2, satisfaction for PTL formulas was defined in terms of the possible valuations of a ranked interpretation $\mathscr{R}$. In order to define a translation operator between PTL and BKLM, our main idea is to encode satisfaction with respect to a *particular* valuation $u \in \mathcal{U}$, by defining an operator $\mathrm{tr}_u : \mathcal{L}^\bullet \to \mathcal{L}^b$ such that for each $u \in \mathcal{U}^{\mathscr{R}}$, $\mathscr{R} \Vdash \mathrm{tr}_u(\alpha)$ iff $u \Vdash_{\mathscr{R}} \alpha$.

Definition 4. *We define tr_u by structural induction as follows, where $\alpha, \beta \in \mathcal{L}^\bullet$, $p \in \mathcal{P}$ and $u \in \mathcal{U}$:*

1. $tr_u(p) \stackrel{\text{def}}{=} \hat{u} \mid\!\sim p$
2. $tr_u(\top) \stackrel{\text{def}}{=} \hat{u} \mid\!\sim \top$
3. $tr_u(\bot) \stackrel{\text{def}}{=} \hat{u} \mid\!\sim \bot$
4. $tr_u(\neg\alpha) \stackrel{\text{def}}{=} \neg tr_u(\alpha)$
5. $tr_u(\alpha \wedge \beta) \stackrel{\text{def}}{=} tr_u(\alpha) \wedge tr_u(\beta)$
6. $tr_u(\bullet\alpha) \stackrel{\text{def}}{=} tr_u(\alpha) \wedge \bigwedge_{v \in \mathcal{U}} \left[(\hat{v} < \hat{u}) \to \neg tr_v(\alpha)\right]$

Note that this is well-defined, as each case is defined in terms of strict subformulas. These translations can be viewed as analogues of the definition of PTL satisfaction - case 6 intuitively states that $\bullet\alpha$ is satisfied by a possible valuation u iff u is a minimal valuation satisfying α, for instance. The following lemma confirms that this intuition is correct:

Lemma 3. *Let $\mathscr{R}$ be a ranked interpretation, and $u \in \mathcal{U}^{\mathscr{R}}$ a valuation with $\mathscr{R}(u) < \infty$. Then for all $\alpha \in \mathcal{L}^\bullet$ we have $\mathscr{R} \Vdash tr_u(\alpha)$ if and only if $u \Vdash_{\mathscr{R}} \alpha$.*

A PTL formula $\alpha \in \mathcal{L}^\bullet$ is satisfied by a ranked interpretation $\mathscr{R}$ iff it is satisfied by every possible valuation of $\mathscr{R}$. By combining the translation operators in Definition 4 for each possible world, we can encode this statement as a BKLM formula as follows:

Definition 5. $tr(\alpha) \stackrel{\text{def}}{=} \bigwedge_{u \in \mathcal{U}} \Big((\hat{u} \not\mid\!\sim \bot) \rightarrow tr_u(\alpha) \Big)$

Finally, we can prove that this translation does indeed result in an equivalent BKLM formula:

Lemma 4. *For all $\alpha \in \mathcal{L}^\bullet$ and any ranked interpretation $\mathscr{R}$, $\mathscr{R}$ satisfies α iff $\mathscr{R}$ satisfies $tr(\alpha)$.*

4 The Entailment Problem for BKLM

We now turn to the question of defeasible entailment for BKLM knowledge bases, and in particular whether interesting entailment relations resembling Rational Closure can be defined. As a first observation, Proposition 7 show that there can be no *exact* analogue of Rational Closure for PTL, and thus in light of our translation result there cannot be an exact analogue for BKLM either. In the case of PTL, however, we can get fairly close:

Proposition 10 *[5, Proposition 5.2]. Let $\mathcal{K} \subseteq \mathcal{L}^\bullet$ be a consistent* PTL *knowledge base. Then* Mod($\mathcal{K}$) *has a unique $\leq_G$-minimal element, denoted $\mathscr{R}^{LM}_{\mathcal{K}}$.*

Booth et al. [5] define *LM-entailment* as follows: $\mathcal{K} \approx_{LM} \alpha$ iff $\mathscr{R}^{LM}_{\mathcal{K}} \Vdash \alpha$. While this satisfies many of the desirable properties of Rational Closure, such as the Single Model, Typical Entailment and Ampliativity properties, it fails to satisfy Strict Entailment. Unfortunately, it turns out that the situation is even worse for BKLM:

Lemma 5. *There is no* BKLM *entailment relation $\approx_?$ satisfying Ampliativity, Typical Entailment and the Single Model property.*

This is a concrete sense in which BKLM entailment is more constrained than PTL entailment, and raises the additional question of which of these properties we should commit to giving up. Our main result here, which we will prove in the next two sections, is that satisfying the Single Model property for BKLM entailment incurs heavy costs, and hence it is a reasonable candidate for removal.

4.1 Order Entailment

One way of looking at Rational Closure is as a form of *minimal model entailment*; indeed, this is just Definition 3. In other words, given a knowledge base $\mathcal{K}$, we can construct the Rational Closure of $\mathcal{K}$ by placing an appropriate ordering on its set of ranked models (in this case $\leq_G$), and picking out the consequences common to all the minimal models. In this section we provide a formal definition of this kind of entailment, with a view towards understanding the Single Model property for BKLM.

Definition 6. *Let $<$ be a strict partial order on* RI*. Then for all knowledge bases $\mathcal{K} \subseteq \mathcal{L}^b$ and formulas $\alpha \in \mathcal{L}^b$, we say $\mathcal{K}$ $<$-*entails α *(denoted $\mathcal{K} \approx_< \alpha$) iff $\mathscr{R} \Vdash \alpha$ for all $<$-minimal models $\mathscr{R} \in$* Mod($\mathcal{K}$).

The relation $\approx_<$ will be referred to as an *order entailment relation*. Note that while we have explicitly referred to BKLM knowledge bases here, the construction works identically for KLM and PTL. It is also worth mentioning that the set of models of a consistent knowledge base is always finite, as we have assumed finiteness of $\mathcal{P}$, and hence always has $<$-minimal elements.

Example 5. By Definition 9, the rational closure of any KLM knowledge base $\mathcal{K}$ is the set of formulas satisfied by the (unique) $<_G$-minimal element of Mod($\mathcal{K}$). Thus rational closure is the order entailment relation corresponding to $<_G$ for KLM knowledge bases.

In general, order entailment relations satisfy all of the rationality properties except property Rm (commonly called *rational monotonicity*). Rational monotonicity holds if Mod($\mathcal{K}$) has a unique $<$-minimal model for every knowledge base $\mathcal{K}$, a fact that is closely related to the Single Model property:

Proposition 11. *An order entailment relation $\approx_<$ satisfies the Single Model property iff* Mod($\mathcal{K}$) *has a unique $<$-minimal model for any knowledge base $\mathcal{K}$.*

This is always the case if $<$ is total, for instance, but it is also the case for Rational Closure and LM-entailment. In the next section we will show that, perhaps surprisingly, total order entailment relations are nevertheless (modulo some minor conditions) the *only* entailment relations for BKLM satisfying the Single Model property.

4.2 The Single Model Property

This section is devoted to a proof of the following theorem, mentioned in the preceding discussion:

Theorem 1. *Suppose $\approx_?$ is a* BKLM *entailment relation satisfying Cumulativity, Ampliativity and the Single Model property. Then $\approx_? = \approx_<$, where $<$ is a total ordering of* RI*.*

For the remainder of the proof, we consider a fixed BKLM entailment relation $\approx_?$ satisfying the Cumulativity, Ampliativity and Single Model properties. Corresponding to $\approx_?$ is an associated *consequence operator* $\text{Cn}_?$, defined as follows:

Definition 7. *For any knowledge base $\mathcal{K} \subseteq \mathcal{L}^b$, we define $Cn_?(\mathcal{K}) = \{\alpha \in \mathcal{L}^b : \mathcal{K} \approx_? \alpha\}$.*

In what follows, we will move between the entailment relation and consequence operator notations freely as convenient. To begin with, the following lemma follows easily from our assumptions:

Lemma 6. *For any knowledge base* $\mathcal{K} \subseteq \mathcal{L}^b$, $Cn_?(\mathcal{K}) = Cn_R(Cn_?(\mathcal{K}))$ *and* $Cn_?(\mathcal{K}) = Cn_?(Cn_R(\mathcal{K}))$.

Our approach to proving Theorem 1 is to assign a unique index $\text{ind}(\mathscr{R}) \in \mathbb{N}$ to each ranked interpretation $\mathscr{R} \in \text{RI}$, and then show that $\text{Cn}_?(\mathcal{K})$ corresponds to minimisation of index in $\text{Mod}(\mathcal{K})$. To construct this indexing scheme, consider the following algorithm:

1. Set $M_0 := \text{RI}$, $i := 0$.
2. If $M_i = \emptyset$, terminate.
3. By Corollary 1, there is some $\mathcal{K}_i \subseteq \mathcal{L}^b$ s.t. $\text{Mod}(\mathcal{K}_i) = M_i$.
4. By the Single Model property, there is some $\mathscr{R}_i \in M_i$ s.t. $\text{Cn}_?(\mathcal{K}_i) = \text{sat}(\mathscr{R}_i)$.
5. Set $M_{i+1} := M_i \setminus \{\mathscr{R}_i\}$, $i := i + 1$.
6. Go to step 2, and iterate until termination.

This algorithm is guaranteed to terminate, since M_0 is finite and $0 \leq |M_{i+1}| < |M_i|$. Note that once the algorithm terminates, for each $\mathscr{R} \in \text{RI}$ there will have been a unique $i \in \mathbb{N}$ such that $\mathscr{R} = \mathscr{R}_i$. We will call this i the *index* of $\mathscr{R}$, and denote it by $\text{ind}(\mathscr{R})$. Given a knowledge base $\mathcal{K}$, we define $\text{ind}(\mathcal{K})$ to be the minimum of the indices of each of the models of $\mathcal{K}$.

When we write $\mathscr{R}_n$, $\mathcal{K}_n$ and M_n in the following lemmas, we mean the ranked interpretations, knowledge bases and sets of models constructed in steps 3 to 5 of the algorithm when $i = n$:

Lemma 7. *Given any knowledge base* $\mathcal{K} \subseteq \mathcal{L}^b$, $\text{Mod}(\mathcal{K}) \subseteq M_n$, *where* $n = ind(\mathcal{K})$.

The following lemma proves that entailment under $\approx_?$ corresponds to minimisation of index:

Lemma 8. *Given any knowledge base* $\mathcal{K} \subseteq \mathcal{L}^b$, $Cn_?(\mathcal{K}) = sat(\mathscr{R}_n)$, *where* $n = ind(\mathcal{K})$.

Consider the strict partial order on RI defined by $\mathscr{R}_1 < \mathscr{R}_2$ iff $\text{ind}(\mathscr{R}_1) < \text{ind}(\mathscr{R}_2)$. By construction, the index of a ranked interpretation is unique, and hence $<$ is total. It follows from Lemma 8 that $\approx_? = \approx_<$, and hence $\approx_?$ is equivalent to a total order entailment relation. This completes the proof of Theorem 1.

5 Related Work

The most relevant work w.r.t. the present paper is that of Booth and Paris [4] in which they define rational closure for the extended version of KLM for which negated conditionals are allowed, and the work on PTL [2,5]. The relation this work has with BKLM was investigated in detail throughout the paper.

Delgrande [10] proposes a logic that is as expressive as BKLM. The entailment relation he proposes is different from the minimal entailment relations we consider here and, given the strong links between our constructions and the KLM approach, the remarks in the comparison made by Lehmann and Magidor [16, Sect. 3.7] are also applicable here.

Boutilier [6] defines a family of conditional logics using preferential and ranked interpretations. His logic is closer to ours and even more expressive, since nesting of conditionals is allowed, but he too does not consider minimal constructions. That is, both Delgrande and Boutilier's approaches adopt a Tarskian-style notion of consequence, in line with rank entailment. The move towards a non-monotonic notion of defeasible entailment was precisely our motivation in the present work.

Giordano et al. [13] propose the system P_{min} which is based on a language that is as expressive as PTL. However, they end up using a constrained form of such a language that goes only slightly beyond the expressivity of the language of KLM-style conditionals (their *well-behaved knowledge bases*). Also, the system P_{min} relies on preferential models and a notion of minimality that is closer to circumscription [17].

In the context of description logics, Giordano et al. [11,12] propose to extend the conditional language with an explicit typicality operator $T(\cdot)$, with a meaning that is closely related to the PTL operator $\bullet$. It is worth pointing out, though, that most of the analysis in the work of Giordano et al. is dedicated to a constrained use of the typicality operator $T(\cdot)$ that does not go beyond the expressivity of a KLM-style conditional language, but revised, of course, for the expressivity of description logics.

In the context of adaptive logics, Straßer [18] defines the logic R^+ as an extension of KLM in which arbitrary boolean combinations of defeasible implications are allowed, and the set of propositional atoms has been extended to include the symbols $\{l_i : i \in \mathbb{N}\}$. Semantically, these symbols encode rank in the object language, in the sense that $u \Vdash l_i$ in a ranked interpretation $\mathscr{R}$ iff $\mathscr{R}(u) \geq i$. Straßer's interest in R^+ is to define an adaptive logic ALC^S that provides a dynamic proof theory for rational closure, whereas our interest in BKLM is to generalise rational closure to more expressive extensions of KLM. Nevertheless, the Minimal Abnormality Strategy (see the work of Batens [1], for instance) for ALC^S is closely related to LM-entailment as defined in this paper.

6 Conclusion

The main focus of this paper is exploring the connection between expressiveness and entailment for extensions of the core logic KLM. Accordingly, we introduce the logic BKLM, an extension of KLM that allows for arbitrary boolean combinations of defeasible implications. We take an abstract approach to the analysis of BKLM, and show that it is strictly more expressive than existing extensions of KLM such as PTL [3] and KLM with negation [4]. Our primary conclusion is that a logic as expressive as BKLM has to give up several desirable properties for defeasible entailment, most notably the Single Model property, and thus appealing forms of entailment for PTL such as LM-entailment [2] cannot be lifted to the BKLM case.

For future work, an obvious question is what forms of defeasible entailment *are* appropriate for BKLM. For instance, is it possible to skirt the impossibility results proven in this paper while still retaining the KLM rationality properties? Other forms of entailment for PTL, such as PT-entailment, have also yet to be analysed in the context of BKLM and may be better suited to such an expressive logic. Another line of research to be explored is whether there is a more natural translation of PTL formulas into BKLM than that defined in this paper. Our translation is based on a direct encoding of PTL semantics, and consequently results in an exponential blow-up in the size of the formulas being translated. It is clear that there are much more efficient ways to translate *specific* PTL formulas, but we leave it as an open problem whether this can be done in general. In a similar vein, it is interesting to ask how PTL could be extended in order to make it equiexpressive with BKLM. Finally, it may be interesting to compare BKLM with an extension of KLM that allows for nested defeasible implications, i.e. formulas such as $\alpha \mid\sim (\beta \mid\sim \gamma)$. While such an extension cannot be more expressive than BKLM, at least for a semantics given by ranked interpretations, it may provide more natural encodings of various kinds of typicality, and thus be easier to work with from a pragmatic point of view.

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A Appendix

A.1 Proofs of Lemmas in Sect. 3

Lemma 2. *Let $\mathscr{R}$ be any ranked interpretation. Then there exists a formula $ch(\mathscr{R}) \in \mathcal{L}^b$ with $\mathscr{R}$ as its unique model.*

Proof. Consider the following knowledge bases.

1. $\mathcal{K}_\prec = \{\hat{u} < \hat{v} : u \prec_\mathscr{R} v\} \cup \{\hat{u} \not< \hat{v} : u \not\prec_\mathscr{R} v\}$
2. $\mathcal{K}_\infty = \{\hat{u} \mid\sim \bot : \mathscr{R}(u) = \infty\} \cup \{\hat{u} \not\mid\sim \bot : \mathscr{R}(u) < \infty\}$

By Lemma 1, $\mathscr{R}$ satisfies $\mathcal{K} = \mathcal{K}_\prec \cup \mathcal{K}_\infty$. To show that it is the unique model of $\mathcal{K}$, consider any $\mathscr{R}^* \in \text{Mod}(\mathcal{K})$. Since $\mathscr{R}^*$ satisfies $\mathcal{K}_\infty$, $\mathscr{R}^*(u) = \infty$ iff $\mathscr{R}(u) = \infty$ for any $u \in \mathcal{U}$. Now consider any $u, v \in \mathcal{U}$, and suppose that $\mathscr{R}(u) < \infty$. Then $u \prec_\mathscr{R} v$ iff $\mathcal{K}_\prec$ contains $\hat{u} < \hat{v}$. But $\mathscr{R}^*$ satisfies $\mathcal{K}_\prec$, so this is true iff $u \prec_{\mathscr{R}^*} v$ as $\mathscr{R}^*(u) < \infty$. On the other hand, if $\mathscr{R}(u) = \infty$, then $u \not\prec_\mathscr{R} v$ and $u \not\prec_{\mathscr{R}^*} v$. Hence $\prec_\mathscr{R} = \prec_{\mathscr{R}^*}$, which implies that $\mathscr{R} = \mathscr{R}^*$ by Proposition 2. We conclude the proof by letting $\text{ch}(\mathscr{R}) = \bigwedge_{\alpha \in \mathcal{K}} \alpha$. □

Lemma 3. *Let $\mathscr{R}$ be a ranked interpretation, and $u \in \mathcal{U}^\mathscr{R}$ a valuation with $\mathscr{R}(u) < \infty$. Then for all $\alpha \in \mathcal{L}^\bullet$ we have $\mathscr{R} \Vdash tr_u(\alpha)$ if and only if $u \Vdash_\mathscr{R} \alpha$.*

Proof. We will prove the result by structural induction on the cases in Definition 4:

1. Suppose that $\mathscr{R} \Vdash \mathrm{tr}_u(p)$, i.e. $\mathscr{R} \Vdash \hat{u} \mathrel{|\!\sim} p$. This is true iff $u \models p$, which is equivalent by definition to $u \Vdash_{\mathscr{R}} p$. Cases 2 and 3 are similar.
4. Suppose that $\mathscr{R} \Vdash \mathrm{tr}_u(\neg\alpha)$, i.e. $\mathscr{R} \Vdash \neg\mathrm{tr}_u(\alpha)$. This is true iff $\mathscr{R} \nVdash \mathrm{tr}_u(\alpha)$, which by the induction hypothesis is equivalent to $u \nVdash_{\mathscr{R}} \alpha$. But this is equivalent to $u \Vdash_{\mathscr{R}} \neg\alpha$ by definition. Case 5 is similar.
6. Suppose there exists an $\alpha \in \mathcal{L}^\bullet$ such that $\mathscr{R} \Vdash \mathrm{tr}_u(\bullet\alpha)$ but $u \nVdash_{\mathscr{R}} \bullet\alpha$. Then either $u \nVdash_{\mathscr{R}} \alpha$, which by the induction hypothesis is a contradiction since $\mathscr{R} \Vdash \mathrm{tr}_u(\alpha)$, or there is some $v \in \mathcal{U}$ with $v \prec_{\mathscr{R}} u$ such that $v \Vdash_{\mathscr{R}} \alpha$. But by Lemma 1, $v \prec_{\mathscr{R}} u$ is true only if $\mathscr{R} \Vdash \hat{v} < \hat{u}$. We also have, by the induction hypothesis, that $\mathscr{R} \Vdash \mathrm{tr}_v(\alpha)$ since $v \Vdash_{\mathscr{R}} \alpha$. Hence $\mathscr{R} \Vdash (\hat{v} < \hat{u}) \wedge \mathrm{tr}_v(\alpha)$, which implies that one of the clauses in $\mathrm{tr}_u(\bullet\alpha)$ is false. This is a contradiction, so we conclude that $\mathscr{R} \Vdash \mathrm{tr}_u(\bullet\alpha)$ implies $u \Vdash_{\mathscr{R}} \bullet\alpha$.
   Conversely, suppose that $u \Vdash_{\mathscr{R}} \bullet\alpha$. Then $u \Vdash_{\mathscr{R}} \alpha$, and hence $\mathscr{R} \Vdash \mathrm{tr}_u(\alpha)$ by the induction hypothesis. We also have that if $v \prec_{\mathscr{R}} u$ then $v \nVdash_{\mathscr{R}} \alpha$, which is equivalent to $\mathscr{R} \Vdash \neg\mathrm{tr}_v(\alpha)$ by the induction hypothesis. But by Lemma 1, $v \prec_{\mathscr{R}} u$ iff $\mathscr{R} \Vdash \hat{v} < \hat{u}$. We conclude that $\mathscr{R} \Vdash (\hat{v} < \hat{u}) \to \neg\mathrm{tr}_v(\alpha)$ for all $v \in \mathcal{U}$, and hence $\mathscr{R} \Vdash \mathrm{tr}_u(\bullet\alpha)$. □

Lemma 4. *For all $\alpha \in \mathcal{L}^\bullet$ and any ranked interpretation $\mathscr{R}$, $\mathscr{R}$ satisfies α iff $\mathscr{R}$ satisfies $tr(\alpha)$.*

Proof. Suppose $\mathscr{R} \Vdash \alpha$. Then for all $u \in \mathcal{U}$, either $\mathscr{R}(u) = \infty$ or $u \Vdash_{\mathscr{R}} \alpha$. The former implies $\mathscr{R} \Vdash \hat{u} \mathrel{|\!\sim} \bot$ by Lemma 1, and the latter implies $\mathscr{R} \Vdash \mathrm{tr}_u(\alpha)$ by Lemma 3. Thus $\mathscr{R} \Vdash (\hat{u} \not\mathrel{|\!\sim} \bot) \to \mathrm{tr}_u(\alpha)$ for all $u \in \mathcal{U}$, which proves $\mathscr{R} \Vdash \mathrm{tr}(\alpha)$ as required. Conversely, suppose $\mathscr{R} \Vdash \mathrm{tr}(\alpha)$. Then for any $u \in \mathcal{U}$, either $\mathscr{R} \Vdash \hat{u} \mathrel{|\!\sim} \bot$ and hence $\mathscr{R}(u) = \infty$ by Lemma 1, or $\mathscr{R} \Vdash \hat{u} \not\mathrel{|\!\sim} \bot$ and hence $\mathscr{R} \Vdash \mathrm{tr}_u(\alpha)$ by hypothesis. But then $\mathscr{R} \Vdash \alpha$ by Lemma 3. □

A.2 Proofs of Lemmas in Sect. 4

Lemma 5. *There is no* BKLM *entailment relation $\mathrel{|\!\approx}_?$ satisfying Ampliativity, Typical Entailment and the Single Model property.*

Proof. Suppose that $\mathrel{|\!\approx}_?$ is such an entailment relation, and consider the knowledge base $\mathcal{K} = \{(\top \mathrel{|\!\sim} \mathsf{p}) \vee (\top \mathrel{|\!\sim} \neg\mathsf{p})\}$. Both interpretations in Fig. 1, $\mathscr{R}_1$ and $\mathscr{R}_2$, are models of $\mathcal{K}$. $\mathscr{R}_1$ satisfies $\top \mathrel{|\!\sim} \mathsf{p}$ and not $\top \mathrel{|\!\sim} \neg\mathsf{p}$, whereas $\mathscr{R}_2$ satisfies $\top \mathrel{|\!\sim} \neg\mathsf{p}$ and not $\top \mathrel{|\!\sim} \mathsf{p}$. Thus, by the Typical Entailment property, $\mathcal{K} \not\mathrel{|\!\approx}_? \top \mathrel{|\!\sim} \mathsf{p}$ and $\mathcal{K} \not\mathrel{|\!\approx}_? \top \mathrel{|\!\sim} \neg\mathsf{p}$. On the other hand, by Ampliativity we get

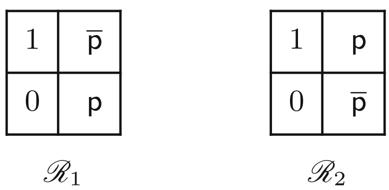

Fig. 1. Ranked models of $\mathcal{K} = \{(\top \mathrel{|\!\sim} \mathsf{p}) \vee (\top \mathrel{|\!\sim} \neg\mathsf{p})\}$.

$\mathcal{K} \approx_? (\top \mid\sim \mathsf{p}) \vee (\top \mid\sim \neg\mathsf{p})$. A single ranked interpretation cannot satisfy all three of these assertions, however, and hence no such entailment relation can exist. □

Lemma 7. *Given any knowledge base $\mathcal{K} \subseteq \mathcal{L}^b$, $\textsc{Mod}(\mathcal{K}) \subseteq M_n$, where $n = ind(\mathcal{K})$.*

Proof. An easy induction on step 5 of the algorithm proves that $M_n = \{\mathscr{R} \in \text{RI} : \text{ind}(\mathscr{R}) \geq n\}$. By hypothesis, $\text{ind}(\mathscr{R}) \geq n$ for all $\mathscr{R} \in \textsc{Mod}(\mathcal{K})$, and hence $\textsc{Mod}(\mathcal{K}) \subseteq M_n$. □

Lemma 8. *Given any knowledge base $\mathcal{K} \subseteq \mathcal{L}^b$, $Cn_?(\mathcal{K}) = sat(\mathscr{R}_n)$, where $n = ind(\mathcal{K})$.*

Proof. For all A, $\mathcal{K}_n \approx_R A$ iff $\mathscr{R} \Vdash A$ for all $\mathscr{R} \in \textsc{Mod}(\mathcal{K}_n) = M_n$. But by Lemma 7, $\textsc{Mod}(\mathcal{K}) \subseteq M_n$ and hence $\text{Cn}_R(\mathcal{K}_n) \subseteq \text{Cn}_R(\mathcal{K})$. On the other hand, $\mathscr{R}_n \in \textsc{Mod}(\mathcal{K})$ by hypothesis and hence $\mathscr{R}_n \Vdash A$ for all $A \in \mathcal{K}$. By the definition of step 4 of the algorithm we have $\text{sat}(\mathscr{R}_n) = \text{Cn}_?(\mathcal{K}_n)$, and thus $\mathcal{K} \subseteq \text{Cn}_?(\mathcal{K}_n)$. Applying Cn_R to each side of this inclusion (using the monotonicity of rank entailment), we get $\text{Cn}_R(\mathcal{K}) \subseteq \text{Cn}_R(\text{Cn}_?(\mathcal{K}_n)) = \text{Cn}_?(\mathcal{K}_n)$, with the last equality following from Lemma 6. Putting it all together, we have $\text{Cn}_R(\mathcal{K}_n) \subseteq \text{Cn}_R(\mathcal{K}) \subseteq \text{Cn}_?(\mathcal{K}_n)$, and hence by Cumulativity we conclude $\text{Cn}_?(\mathcal{K}) = \text{Cn}_?(\mathcal{K}_n) = \text{sat}(\mathscr{R}_n)$. □

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