Scuola Internazionale Superiore di Studi Avanzati – Trieste
Physics area
Ph.D. course in Astroparticle Physics

Doctoral Thesis

Aspects of the Modular Symmetry Approach to Lepton Flavour

Candidate:
Pavel Novichkov

Advisor:
Prof. Serguey Petcov

Academic Year 2020-2021

Contents

Abstract iv

Publications v

List of symbols vi

List of abbreviations vii

I. Introduction 1

1. Flavour puzzle 2
1.1. Symmetry approach to flavour 2
1.2. Modular symmetry 4

2. Framework [P5] 7
2.1. The modular group and transformation of fields 7
2.2. Modular forms and modular-invariant actions 9

II. Theory 11

3. Construction of modular forms [P2, P5] 12
3.1. $\Gamma'_4 \simeq S'_4$ [P5] 12
"Weight 1/2" 12
Weight 1 14
Higher weights 15
3.2. $\Gamma_5 \simeq A_5$ [P2] 17
Weight 2 17
Higher weights 20

4. Combining CP and modular symmetry [P4, P5] 21
4.1. CP transformations 21
Transformation of the modulus 21

Action on the modular group 23
CP_1 24
CP_2 25
Transformation of modular form multiplets 27
4.2. CP-invariant theories 28
Implications for the couplings 28
Implications for the mass matrices 29

5. Residual symmetries [P1, P5] **32**
5.1. Symmetric points in the fundamental domain 32
5.2. CP-conserving points 33
5.3. Classification of the residual symmetry groups 35

6. Mass and mixing patterns [P6] **37**
6.1. Mass hierarchies without fine-tuning 37
Mass matrices close to symmetric points 37
Decomposition under residual symmetries 40
Hierarchical structures 42
6.2. Large mixing angles without fine-tuning 46

III. Phenomenology 49

7. General setup [P1] **50**
7.1. Field content 50
7.2. Potential sources of corrections 51
7.3. Numerical procedure 52

8. Predictive model based on S_4 [P1, P4] **56**
8.1. Model description 56
8.2. Numerical result 57

9. Trimaximal mixing from modular A_4 group [P3] **63**
9.1. Modular forms of level 3 63
9.2. Lepton mass matrices at the symmetric points 64
Charged leptons 65
Neutrinos 66
9.3. Model with two moduli 67
Neutrino mixing 68

Neutrino masses and Majorana phases 70

10. S_4' model free from fine-tuning [P6] **75**

Summary and conclusions **80**

Appendix 82

A. Dedekind eta and Jacobi theta **83**

B. Finite modular group theory **85**
B.1. $\Gamma_3 \simeq A_4$ 85
B.2. $\Gamma_4' \simeq S_4'$ 85
B.3. $\Gamma_5 \simeq A_5$ 87

C. Residual group decompositions **100**

D. Possible hierarchical patterns **104**

Index **113**

Bibliography **114**

Abstract

A new bottom-up approach to the flavour problem based on modular invariance has been recently proposed and has gained considerable attention in the literature. In the present thesis we develop basic aspects of the requisite modular symmetry formalism and explore its application to the lepton flavour problem. After introducing the relevant notions (the modular group, the modulus field and modular forms), we concentrate on the theoretical tools required for model-building such as explicit construction of the modular forms, the interplay of modular and CP transformations and of the related symmetries, classification of residual symmetries and their possible relation to the observed hierarchical flavour patterns. Armed with these tools, we construct and discuss three examples of viable models of lepton flavour: a simple predictive model depending on a small number of parameters, a model with an unbroken residual symmetry which leads to trimaximal neutrino mixing, and a model with a slightly broken residual symmetry which explains the observed pattern of charged-lepton masses without fine-tuning.

Publications

This thesis is based on the following publications listed in chronological order:

[P1] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, "Modular S_4 models of lepton masses and mixing", JHEP **04**, 005 (2019), arXiv:1811.04933 [hep-ph].

[P2] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, "Modular A_5 symmetry for flavour model building", JHEP **04**, 174 (2019), arXiv:1812.02158 [hep-ph].

[P3] P. P. Novichkov, S. T. Petcov, and M. Tanimoto, "Trimaximal Neutrino Mixing from Modular A4 Invariance with Residual Symmetries", Phys. Lett. B **793**, 247 (2019), arXiv:1812.11289 [hep-ph].

[P4] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, "Generalised CP Symmetry in Modular-Invariant Models of Flavour", JHEP **07**, 165 (2019), arXiv:1905.11970 [hep-ph].

[P5] P. P. Novichkov, J. T. Penedo, and S. T. Petcov, "Double cover of modular S_4 for flavour model building", Nucl. Phys. B **963**, 115301 (2021), arXiv:2006.03058 [hep-ph].

[P6] P. P. Novichkov, J. T. Penedo, and S. T. Petcov, "Fermion mass hierarchies, large lepton mixing and residual modular symmetries", JHEP **04**, 206 (2021), arXiv:2102.07488 [hep-ph].

List of symbols

α_{21}, α_{31}	leptonic Majorana CPV phases	57
$\mathcal{D}$	the fundamental domain of Γ	32
δ	leptonic Dirac CPV phase	57
δm^2, Δm^2	neutrino mass-squared differences	53
Γ	modular group	7
$\Gamma(N)$	principal congruence subgroup of Γ	8
$\overline{\Gamma}$	inhomogeneous modular group	8
Γ_N	inhomogeneous finite modular group	8
Γ'_N	homogeneous finite modular group	8
$\mathcal{H}$	upper half-plane	7
$\mathbb{1}$	identity element of a group, also identity matrix	7
m_1, m_2, m_3	neutrino masses	57
$\mathcal{M}_k(\Gamma(N))$	linear space of modular forms of level N and weight k	10
$\lvert\langle m\rangle\rvert$	neutrinoless double beta decay effective Majorana mass	57
ω	3rd root of unity $\omega \equiv \exp(2i\pi/3)$	32
q_N	q-expansion parameter defined as $q_N \equiv \exp(2i\pi\tau/N)$	13, 19, 63
r	ratio of neutrino mass-squared differences	53
S, T, R	generators of the modular group	7
τ	the modulus field	7
θ_{12}, θ_{13}, θ_{23}	neutrino mixing angles	57
$Y^{(N,k)}_{\mathbf{r}(,i)}$	modular form multiplet of level N, weight k and irrep $\mathbf{r}$ (i labels linearly independent multiplets)	10
ζ	5th root of unity $\zeta \equiv \exp(2i\pi/5)$	19
ζ_N	N-th root of unity $\zeta_N \equiv \exp(2i\pi/N)$	38

List of abbreviations

CPV CP violating

IO inverted ordering

MSSM minimal supersymmetric Standard Model

NO normal ordering

RG renormalisation group

SUSY supersymmetry

VEV vacuum expectation value

irrep irreducible representation

Part I.

Introduction

1. Flavour puzzle

1.1. Symmetry approach to flavour

Understanding the origins of flavour in both the quark and lepton sectors, i.e., the origins of the patterns of quark masses and mixing, of the charged-lepton and neutrino masses, of neutrino mixing and of the CP violation in the two sectors is one of the most challenging unresolved fundamental problems in particle physics [1].

Within the reference three-neutrino mixing scheme, the lepton flavour problem consists of three basic elements or sub-problems, namely, understanding:

1. the origin of the hierarchical pattern of charged-lepton masses: $m_e \ll m_\mu \ll m_\tau$, $m_e/m_\mu \simeq 1/200$, $m_\mu/m_\tau \simeq 1/17$;

2. why neutrino masses m_{ν_j} are much smaller than the masses of charged leptons and quarks, $m_{\nu_j} \lll m_{\ell,q}$, $q = u, c, t, d, s, b$ and $\ell = e, \mu, \tau$, with $m_{\nu_j} \lesssim 0.5$ eV, $m_\ell \geq 0.511$ MeV, $m_q \gtrsim 2$ MeV;

3. the origins of the patterns of neutrino mixing of 2 large and 1 small angles, and of the two independent neutrino mass squared differences, $\Delta m^2_{21} \ll |\Delta m^2_{31}|$ with $\Delta m^2_{21}/|\Delta m^2_{31}| \simeq 1/30$, where $\Delta m^2_{ij} \equiv m^2_i - m^2_j$.

Each of these three sub-problems is by itself a formidable problem. As a consequence, individual solutions to each of them have been proposed. The hierarchical pattern of charged-lepton masses can most naturally be understood within the Froggatt-Nielsen mechanism based on the $\mathrm{U}(1)_{\mathrm{FN}}$ flavour symmetry [2] and its extensions. The enormous disparity between the neutrino masses and the masses of the charged leptons and quarks can be understood within the seesaw or radiative models of neutrino mass generation or else employing the Weinberg effective operator idea [3] (for a concise review see, e.g., [4]). All these approaches lead naturally to massive Majorana neutrinos. Arguably the most elegant and natural explanation of the observed pattern of neutrino (or lepton) mixing of two large and one small mixing angles is obtained within the non-Abelian discrete symmetry approach to the problem (see, e.g., [5–9]).

In the case of the quark sector, the flavour problem similarly has two basic sub-problems, namely, understanding:

1. the origins of the mass hierarchies of the charge $2/3$ and of the charge $(-1/3)$ quarks;

2. the origins of the relatively small values of the three quark mixing angles.

The most natural qualitative solution of these two problems is arguably provided by the Froggatt-Nielsen approach [2], although the approach based on non-Abelian discrete symmetries has been applied to the quark flavour problem as well. Solutions to the two flavour problems within the theories with extra dimensions have also been proposed.[1]

The specific solutions to the individual lepton flavour sub-problems listed above become problematic when applied to the sub-problems they were not intended to solve. The seesaw and the radiative neutrino mass models do not lead to understanding of the origin of the neutrino mixing pattern without additional input, consisting typically of imposing specific additional symmetries (of GUT or flavour type) on the relevant constructions. Within the Froggatt-Nielsen approach one most naturally obtains small values of the three neutrino mixing angles and while the charged-lepton and quark mass hierarchies can be qualitatively understood within this approach, the specific predictions suffer from relatively large uncertainties. The symmetry breaking in the lepton and quark flavour models based on non-Abelian discrete symmetries is impressively cumbersome: it requires the introduction of a plethora of "flavon" scalar fields having elaborate potentials, which in turn require the introduction of large shaping symmetries to ensure the requisite breaking of the symmetry leading to correct mass and mixing patterns.

There have been also attempts to make progress, e.g., on the lepton flavour problem by combining the proposed "solutions" of the three related sub-problems. In these combined approaches it is difficult, if not impossible, to avoid the drawbacks of each of the sub-problem "solutions". In some cases this can be achieved at the cost of severe fine-tuning. Thus, a universal, elegant, natural and viable theory of flavour that is free from undesired drawback features is still lacking. Constructing such a theory would be a major breakthrough in particle physics.

[1] A rather comprehensive discussion of the past proposed approaches to the lepton and quark flavour problems can be found in the review article [1].

1.2. Modular symmetry

The unsatisfactory status of the flavour problem and the remarkable progress made in the studies of neutrino oscillations (see, e.g., [10]), which began 22 years ago with the discovery of oscillations of the atmospheric ν_μ and $\bar{\nu}_\mu$ by the Super-Kamiokande experiment [11] and lead, in particular, to the determination of the pattern of neutrino mixing, stimulated renewed attempts to seek alternative viable approaches to the flavour problem. A step in this direction was made in [12] where the idea of using modular invariance as a flavour symmetry was put forward. This new original approach opened up a promising direction in the studies of the flavour problem and correspondingly in flavour model building.

The main feature of the approach proposed in [12] is that the elements of the Yukawa coupling and fermion mass matrices in the Lagrangian of the theory are modular forms of a certain level N which are functions of a single complex scalar field τ — the modulus — and have specific transformation properties under the action of the modular group. In addition, both the couplings and the matter fields are assumed to furnish representations of an inhomogeneous (homogeneous) finite modular group $\Gamma_N^{(\prime)}$.

For $N \leq 5$, the finite modular groups Γ_N are isomorphic to the permutation groups S_3, A_4, S_4 and A_5 (see, e.g., [13]), while the groups Γ'_N are isomorphic to the double covers of the indicated permutation groups, $S'_3 \equiv S_3$, $A'_4 \equiv T'$, S'_4 and A'_5. These discrete groups are widely used in flavour model building.

The theory is assumed to possess the modular symmetry described by the finite modular group $\Gamma_N^{(\prime)}$, which plays the role of a flavour symmetry. In the simplest class of such models, the VEV (vacuum expectation value) of the modulus τ is the only source of flavour symmetry breaking, such that no flavons are needed. Another appealing feature of the proposed framework is that the VEV of τ can also be the only source of breaking of the CP symmetry [P4].

When the flavour symmetry is broken, the elements of the Yukawa coupling and fermion mass matrices get fixed, and a certain flavour structure arises. As a consequence of the modular symmetry, in the lepton sector, for example, the charged-lepton and neutrino masses, neutrino mixing and the leptonic CPV (CP violating) phases are simultaneously determined in terms of a limited number of coupling constant parameters. This together with the fact that they are also functions of a single complex VEV — that of the modulus τ — leads to experimentally testable correlations between, e.g., the neutrino mass and

mixing observables. Models of flavour based on modular invariance have then an increased predictive power.

The modular symmetry approach to the flavour problem has been widely implemented so far primarily in theories with global SUSY (supersymmetry). Within the SUSY framework, modular invariance is assumed to be a feature of the Kähler potential and the superpotential of the theory. Bottom-up modular invariance approaches to the lepton flavour problem have been exploited first using the groups $\Gamma_3 \simeq A_4$ [12, 14], $\Gamma_2 \simeq S_3$ [15], $\Gamma_4 \simeq S_4$ [16].

After the first studies, the interest in the approach grew significantly and models based on the groups $\Gamma_4 \simeq S_4$ [P1, 17–25], $\Gamma_5 \simeq A_5$ [P2, 23, 26], $\Gamma_3 \simeq A_4$ [P3, 20, 27–49], $\Gamma_2 \simeq S_3$ [50, 51] and $\Gamma_7 \simeq \mathrm{PSL}(2, \mathbb{Z}_7)$ [52] have been constructed and extensively studied. Similarly, attempts have been made to construct viable models of quark flavour [53] and of quark-lepton unification [54–65]. The formalism of the interplay of modular and CP symmetries has been developed and first applications made in [P4]. It was explored further in [66–69], as was the possibility of coexistence of multiple moduli [70–74], considered first phenomenologically in [P1, P3]. Moduli-mediated SUSY breaking effects have been studied in [75, 76]. Such bottom-up analyses are expected to eventually connect with top-down results [77–103] based on ultraviolet-complete theories.

While the aforementioned finite quotients Γ_N of the modular group have been widely used in the literature to construct modular-invariant models of flavour from the bottom-up perspective, top-down constructions typically lead to their double covers Γ'_N (see, e.g., [80, 82, 83, 104]). The formalism of such double covers has been developed and viable flavour models constructed in refs. [105], [P5, 106], [107, 108], and [109] for the cases of $\Gamma'_3 \simeq T'$, $\Gamma'_4 \simeq S'_4$, $\Gamma'_5 \simeq A'_5$, and $\Gamma'_6 \simeq S_3 \times T'$ respectively.

In almost all phenomenologically viable flavour models based on modular invariance constructed so far the hierarchy of the charged-lepton and quark masses is obtained by fine-tuning some of the constant parameters present in the models.[2] One way to overcome this drawback is to utilise the hierarchical mass matrix structures in the vicinity of the so-called symmetric points [P6, 110, 111]. Another possibility is to use modular weights as Froggatt-Nielsen charges, with additional scalar fields of non-zero modular weights playing the role of flavons [112–114].

[2] By fine-tuning we refer to either i) high sensitivity of observables to model parameters or ii) unjustified hierarchies between parameters which are introduced in the model on an equal footing.

The aim of this study is to investigate certain aspects of the modular symmetry framework as well as its applications to the flavour problem. The thesis is structured as follows.

We start by reviewing the framework in chapter 2. Then, in part II we introduce several theoretical tools required for model-building. Namely, we show in chapter 3 how basic blocks of such theories — modular forms — are constructed. In chapter 4 we extend modular symmetry with a CP transformation to increase predictivity of the models. Next, in chapter 5 we classify the possible unbroken residual symmetries, which can be used to explain the observed flavour patterns as shown later in chapter 6.

We turn to construction of viable models of lepton flavour in part III. After describing the general model-building setup in chapter 7, we proceed with three examples of such models: a simple predictive model based on $\Gamma_4 \simeq S_4$ modular group (chapter 8), a model with an unbroken residual symmetry which leads to trimaximal neutrino mixing (chapter 9), and a model with a slightly broken residual symmetry which explains the observed pattern of charged-lepton masses without fine-tuning (chapter 10). Finally, we summarise our results and present our conclusions.

2. Framework [P5]

In this chapter we describe the framework of modular symmetry in the context of $\mathcal{N} = 1$ SUSY. The discussion follows the original bottom-up construction of ref. [12] as well as its extension including odd-weight modular forms [105].

2.1. The modular group and transformation of fields

We introduce a chiral supermultiplet τ, called the *modulus*, whose scalar component is restricted to the upper half-plane $\mathcal{H} \equiv \{\tau \in \mathbb{C} : \mathrm{Im}\,\tau > 0\}$. With some abuse of notation, we will use the same symbol τ to denote the modulus itself, its scalar component and its VEV, depending on the context. The modulus τ transforms non-trivially under the *modular group* Γ, which is the special linear group of 2×2 integer matrices with unit determinant, i.e.,

$$\Gamma \equiv \mathrm{SL}(2, \mathbb{Z}) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}. \tag{2.1}$$

The group Γ is generated by three matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.2}$$

subject to the following relations:

$$S^2 = R, \quad (ST)^3 = \mathbb{1}, \quad R^2 = \mathbb{1}, \quad RT = TR, \tag{2.3}$$

where $\mathbb{1}$ denotes the identity element of a group.

The modular group Γ acts on the modulus with fractional linear transformations:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \ \tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}\,. \tag{2.4}$$

Matter chiral superfields transform under Γ as "weighted" multiplets [12, 104, 115]:

$$\psi_i \to (c\tau + d)^{-k}\, \rho_{ij}(\gamma)\, \psi_j\,, \tag{2.5}$$

where $(c\tau + d)^{-k}$ is the *automorphy factor*, $k \in \mathbb{Z}$ is the *modular weight*[1] and ρ is a unitary representation of Γ.

Note that the group action (2.4) has a non-trivial kernel $\mathbb{Z}_2^R = \{\mathbb{1}, R\}$, i.e., the modulus τ does not transform under the action of R. For this reason one typically defines the *inhomogeneous modular group* as the quotient $\overline{\Gamma} \equiv \mathrm{PSL}(2,\mathbb{Z}) \equiv \mathrm{SL}(2,\mathbb{Z}) \,/\, \mathbb{Z}_2^R$, which is the projective version of $\mathrm{SL}(2,\mathbb{Z})$ with matrices γ and $-\gamma$ being identified. However, matter fields of a modular-invariant theory are in general allowed to transform under R, as can be seen from (2.5). Therefore the symmetry group of such theory is Γ rather than $\overline{\Gamma}$ (see, e.g., [83]).

We assume that representations of matter fields are trivial when restricted to the so-called *principal congruence subgroup*,

$$\Gamma(N) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}),\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathrm{mod}\ N) \right\}, \tag{2.6}$$

with a fixed integer $N \geq 2$ called the *level*. In other words, $\rho(\gamma)$ of (2.5) is the identity matrix whenever $\gamma \in \Gamma(N)$, so that ρ is effectively a representation of the quotient group

$$\Gamma'_N \equiv \Gamma \,\big/\, \Gamma(N) \simeq \mathrm{SL}(2,\mathbb{Z}_N), \tag{2.7}$$

called the *homogeneous finite modular group*. Unlike Γ, Γ'_N is finite as the name suggests. For $N \leq 5$, this group admits the presentations

$$\begin{aligned} \Gamma'_N &= \left\langle S, T, R \mid S^2 = R,\ (ST)^3 = \mathbb{1},\ R^2 = \mathbb{1},\ RT = TR,\ T^N = \mathbb{1} \right\rangle \\ &= \left\langle S, T \mid S^4 = \mathbb{1},\ (ST)^3 = \mathbb{1},\ S^2T = TS^2,\ T^N = \mathbb{1} \right\rangle, \end{aligned} \tag{2.8}$$

where with a slight abuse of notation we denote by S, T, R the equivalence classes of the corresponding generators (2.2) of the full modular group. For $N > 5$, additional relations are needed in order to render the group finite [13].

In the special case when ρ does not distinguish between γ and $-\gamma$, i.e., $\rho(R)$ is identity, we see that ρ is a representation of a smaller quotient group

$$\Gamma_N \equiv \Gamma \,\big/\, \left\langle\, \Gamma(N) \cup \mathbb{Z}_2^R \,\right\rangle \simeq \mathrm{SL}(2,\mathbb{Z}_N) \,\big/\, \langle R \rangle\,, \tag{2.9}$$

called the *inhomogeneous finite modular group*. For $N \leq 5$, Γ_N has the following presentation:

$$\Gamma_N = \left\langle S, T \mid S^2 = \mathbb{1},\ (ST)^3 = \mathbb{1},\ T^N = \mathbb{1} \right\rangle. \tag{2.10}$$

Note that $R \in \Gamma(2)$, hence $\Gamma_2 = \Gamma'_2$. In contrast, for $N \geq 3$ one has $R \notin \Gamma(N)$, and Γ'_N is a double cover of Γ_N. For small values of N, the groups Γ_N and Γ'_N are isomorphic to permutation groups and their double covers, see table 2.1.

[1] While we restrict ourselves to integer k, it is also possible for weights to be fractional [83, 116–119].

Table 2.1.: Finite modular groups and dimensionality of the corresponding spaces of modular forms, for $N \leq 5$. Note that for $N = 2$ only even-weighted modular forms exist.

N	2	3	4	5
Γ_N	S_3	A_4	S_4	A_5
Γ'_N	S_3	$A'_4 \equiv T'$	$S'_4 \equiv \mathrm{SL}(2,\mathbb{Z}_4)$	$A'_5 \equiv \mathrm{SL}(2,\mathbb{Z}_5)$
$\dim \mathcal{M}_k(\Gamma(N))$	$k/2+1$	$k+1$	$2k+1$	$5k+1$

As a final remark, let us stress that the level N defining the finite modular group is common to all matter fields ψ_I, which may however carry different modular weights k_I.

2.2. Modular forms and modular-invariant actions

The Lagrangian of an $\mathcal{N} = 1$ global SUSY theory is given by

$$\mathcal{L} = \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, K(\Phi, \bar{\Phi}) + \left[\int \mathrm{d}^2\theta\, W(\Phi) + \mathrm{h.c.} \right], \tag{2.11}$$

where K is the Kähler potential, W is the superpotential, θ and $\bar{\theta}$ are Grassmann variables, and Φ collectively denotes chiral superfields of the theory, including the modulus τ. In theories with global SUSY, modular symmetry requires the superpotential to be modular-invariant [115]. In theories of supergravity, the superpotential is instead coupled to the Kähler potential and has to transform with a certain weight $-h$ under modular transformations (up to a field-independent phase $\alpha(\gamma)$ called the *multiplier system*) [104, 115]:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \; W(\Phi) \to e^{i\alpha(\gamma)}\, (c\tau + d)^{-h}\, W(\Phi). \tag{2.12}$$

The superpotential can be expanded in powers of matter superfields ψ_I as:

$$W(\tau, \psi_I) = \sum \left(Y_{I_1 \ldots I_n}(\tau)\, \psi_{I_1} \ldots \psi_{I_n} \right)_{\mathbf{1}}, \tag{2.13}$$

where the sum is taken over all possible combinations of fields $\{I_1, \ldots, I_n\}$ and all independent singlets of Γ'_N, denoted by $(\ldots)_{\mathbf{1}}$.

In order to satisfy (2.12) given the field transformation rules (2.5), the field couplings $Y_{I_1\ldots I_n}(\tau)$ have to be *modular forms* of level N and weight $k_Y = k_{I_1} + \ldots + k_{I_n} - h$, i.e., transform under Γ as

$$Y_{I_1\ldots I_n}(\tau) \;\rightarrow\; Y_{I_1\ldots I_n}(\gamma\tau) = (c\tau + d)^{k_Y}\rho(\gamma)\, Y_{I_1\ldots I_n}(\tau), \tag{2.14}$$

where ρ is a unitary representation of the homogeneous finite modular group Γ'_N such that $\rho \otimes \rho_{I_1} \otimes \ldots \otimes \rho_{I_n} \supset \mathbf{1}$. Apart from that, due to holomorphicity of the superpotential, modular forms have to be holomorphic functions of τ. Together with the transformation property (2.14), this significantly constrains the space of modular forms.

In fact, non-trivial modular forms of a given level N exist only for positive integer weights $k \in \mathbb{N}$. They span finite-dimensional linear spaces $\mathcal{M}_k(\Gamma(N))$ arranged into multiplets of Γ'_N. In the following, we will denote the modular form multiplet of level N, weight k and irrep (irreducible representation) $\mathbf{r}$ of Γ'_N as $Y^{(N,k)}_{\mathbf{r}(,i)}$, with i labelling linearly independent multiplets of the same irrep type in case more than one exists.

By analysing (2.14), one notes that odd-weighted modular forms necessarily have $\rho(R) = -\mathbb{1}$ in order to compensate the minus sign arising from the automorphy factor, while for even-weighted modular forms one has $\rho(R) = \mathbb{1}$. It follows that in modular-invariant theories based on inhomogeneous modular groups Γ_N only even-weighted modular forms appear.

As can be seen from table 2.1, the spaces $\mathcal{M}_k(\Gamma(N))$ have low dimensionalities for small values of k and N. Therefore it is possible to form only a few independent Yukawa couplings, which yields predictive models of flavour.

We should note however that the most general Kähler potential consistent with modular symmetry contains multiple terms which may jeopardise the predictive power of the framework [120]. This problem is a subject of ongoing research (see, e.g., [121]). In what follows, we will assume the *minimal form* of the Kähler potential which reads

$$K(\tau, \overline{\tau}, \psi, \overline{\psi}) = -\Lambda_0^2 \log(-i\tau + i\overline{\tau}) + \sum_I \frac{|\psi_I|^2}{(-i\tau + i\overline{\tau})^{k_I}}\,, \tag{2.15}$$

with Λ_0 having mass dimension one.

Part II.

Theory

3. Construction of modular forms [P2, P5]

We have seen in the previous chapter that basic building blocks of a modular-invariant theory of flavour are modular forms. These functions have been explicitly constructed and arranged into multiplets of the corresponding finite modular groups for the first few levels N; the relevant references are collected in table 3.1. In this chapter we will consider two specific examples of such constructions — integral-weight modular forms of level 4, corresponding to the finite group $\Gamma'_4 \simeq S'_4$, and even-weight modular forms of level 5, corresponding to $\Gamma_5 \simeq A_5$. The discussion follows refs. [P5] and [P2], respectively.[1]

Table 3.1.: References for explicit modular form constructions

N	2	3	4	5	6	7
Γ_N	[15]	[12]	[16]	[P2]	[109]	[52]
Γ'_N	[15]	[105]	[P5]	[107, 108]	[109]	—

3.1. $\Gamma'_4 \simeq S'_4$ [P5]

"Weight 1/2"

Modular forms of level 4 and weight k form a linear space of dimension $2k+1$ given by [122]:

$$\begin{aligned} \mathcal{M}_k(\Gamma(4)) &= \bigoplus_{\substack{m+n=2k,\\ m,n\geq 0}} \mathbb{C}\, \frac{\eta^{2n-2m}(4\tau)\,\eta^{5m-n}(2\tau)}{\eta^{2m}(\tau)} \\ &= \bigoplus_{\substack{m+n=2k,\\ m,n\geq 0}} \mathbb{C} \left(\frac{\eta^5(2\tau)}{\eta^2(\tau)\eta^2(4\tau)}\right)^m \left(\frac{\eta^2(4\tau)}{\eta(2\tau)}\right)^n , \end{aligned} \tag{3.1}$$

[1] After the appearance of [P5] and [P2], modular form multiplets of level 4, corresponding to Γ'_4, and of level 5, corresponding to Γ_5, were constructed also in refs. [106] and [26], respectively.

where m and n are non-negative integers, and $\eta(\tau)$ is the Dedekind eta function (we collect all the necessary definitions and properties of special functions in appendix A). In other words, $\mathcal{M}_k(\Gamma(4))$ is spanned by polynomials of even degree $2k$ in two functions $\theta(\tau)$ and $\varepsilon(\tau)$ defined as

$$\theta(\tau) \equiv \frac{\eta^5(2\tau)}{\eta^2(\tau)\eta^2(4\tau)} = \Theta_3(2\tau), \quad \varepsilon(\tau) \equiv \frac{2\,\eta^2(4\tau)}{\eta(2\tau)} = \Theta_2(2\tau). \tag{3.2}$$

Here $\Theta_2(\tau)$ and $\Theta_3(\tau)$ are the Jacobi theta constants related to the Dedekind eta by (A.6). In particular, we conclude from (3.1) that the space of weight-1 modular forms of level 4 is spanned by the homogeneous quadratic polynomials in θ and ε, or equivalently, in the theta constants Θ_2 and Θ_3 of double argument.

From (3.2) and (A.3) we find immediately that $\theta(\tau)$ and $\varepsilon(\tau)$ admit the following q-expansions, i.e., power series expansions in $q_4 \equiv \exp(i\pi\tau/2)$:

$$\begin{aligned} \theta(\tau) &= 1 + 2\sum_{k=1}^{\infty} q_4^{(2k)^2} = 1 + 2q_4^4 + 2q_4^{16} + \ldots, \\ \varepsilon(\tau) &= 2\sum_{k=1}^{\infty} q_4^{(2k-1)^2} = 2q_4 + 2q_4^9 + 2q_4^{25} + \ldots, \end{aligned} \tag{3.3}$$

so that $\theta \to 1$, $\varepsilon \to 0$ in the "large volume" limit $\operatorname{Im}\tau \to \infty$. In fact, $\varepsilon \sim 2q_4$ in this limit, so it can be used as an expansion parameter instead of q_4, which justifies the notation. Note that, due to quadratic dependence in the exponents of q_4, the series (3.3) converge rapidly in the fundamental domain of the modular group, where one has $|q_4| \le \exp(-\pi\sqrt{3}/4) \simeq 0.26$. We give below the values of $\theta(\tau)$ and $\varepsilon(\tau)$ at the so-called *symmetric points* (see chapter 5 for details):

$$\begin{aligned} \theta(\tau_C) &= 1 + 2e^{-2\pi} + O(10^{-11}) \simeq 1.00373, \\ \varepsilon(\tau_C) &= 2e^{-\pi/2} + O(10^{-6}) \simeq 0.415761, \\ \theta(\tau_L) &= 1 - 2e^{-\sqrt{3}\pi} + O(10^{-9}) \simeq 0.991333, \\ \varepsilon(\tau_L) &= 2e^{-i\pi/4}\left[e^{-\sqrt{3}\pi/4} + O(10^{-5})\right] \simeq 0.512152\, e^{-i\pi/4}, \\ \theta(\tau_T) &= 1, \\ \varepsilon(\tau_T) &= 0, \end{aligned} \tag{3.4}$$

where $\tau_C \equiv i$, $\tau_L \equiv e^{2i\pi/3}$, and $\tau_T \equiv i\infty$. We further find the exact relations at symmetric points:

$$\frac{\varepsilon(\tau_C)}{\theta(\tau_C)} = \frac{1}{1+\sqrt{2}}, \qquad \frac{\varepsilon(\tau_L)}{\theta(\tau_L)} = \frac{1-i}{1+\sqrt{3}}. \tag{3.5}$$

The action of the T generator on θ and ε follows from the corresponding transformation of the theta constants (A.4):

$$\theta(\tau) \xrightarrow{T} \theta(\tau), \quad \varepsilon(\tau) \xrightarrow{T} i\varepsilon(\tau). \tag{3.6}$$

Similarly, one can obtain the action of the S generator on θ from (A.4) with the help of identity (A.7):

$$\begin{aligned}\theta(\tau) = \Theta_3(2\tau) &= \frac{1}{2}\left[\Theta_3\left(\frac{\tau}{2}\right) + \Theta_4\left(\frac{\tau}{2}\right)\right] \xrightarrow{S} \frac{1}{2}\left[\Theta_3\left(-\frac{1}{2\tau}\right) + \Theta_4\left(-\frac{1}{2\tau}\right)\right] \\ &= \frac{1}{2}\sqrt{-i2\tau}\left[\Theta_3(2\tau) + \Theta_2(2\tau)\right] = \sqrt{-i\tau}\,\frac{\theta(\tau)+\varepsilon(\tau)}{\sqrt{2}}\,.\end{aligned} \tag{3.7}$$

By requiring that the second action of S should transform the result back to $\theta(\tau)$, we find the corresponding action on $\varepsilon(\tau)$, and conclude that

$$\theta(\tau) \xrightarrow{S} \sqrt{-i\tau}\,\frac{\theta(\tau)+\varepsilon(\tau)}{\sqrt{2}}\,, \quad \varepsilon(\tau) \xrightarrow{S} \sqrt{-i\tau}\,\frac{\theta(\tau)-\varepsilon(\tau)}{\sqrt{2}}\,. \tag{3.8}$$

From the transformation properties (3.6) and (3.8), one sees that θ and ε transform as "weight-1/2" modular forms. This statement can be made precise in the context of the *metaplectic group* $\mathrm{Mp}_2(\mathbb{Z})$, which is a double cover of the modular group [119].

Weight 1

We have seen that the linear space of weight-1 modular forms of level 4 is spanned by three quadratic monomials in $\theta(\tau)$ and $\varepsilon(\tau)$, namely:

$$\theta(\tau)^2, \quad \theta(\tau)\varepsilon(\tau), \quad \varepsilon(\tau)^2, \tag{3.9}$$

such that the linear space of weight $k = 1$ has the correct dimension, $2k + 1 = 3$.

These three functions can be arranged into a triplet furnishing a representation of $S_4' \equiv \mathrm{SL}(2, \mathbb{Z}_4)$, which is a double cover[2] of the permutation group S_4 [105]. We summarise the group theory of S_4' in appendix B.2.

[2] Strictly speaking, the term "double cover of symmetric group" is used for a special kind of a double cover called the Schur cover. There are two double covers of S_4 of this kind: the binary octahedral group (group ID `(48,28)` in GAP [123, 124]) and $\mathrm{GL}(2,3)$ (group ID `(48,29)`). Our double cover $\mathrm{SL}(2, \mathbb{Z}_4)$ is not a Schur cover of S_4. It has group ID `(48,30)`, hence it is a double cover of S_4 in a broader sense.

In the group representation basis of table B.4, the relevant triplet has the form

$$Y^{(4,1)}_{\hat{\mathbf{3}}}(\tau) = \begin{pmatrix} \sqrt{2}\varepsilon\theta \\ \varepsilon^2 \\ -\theta^2 \end{pmatrix} \tag{3.10}$$

and furnishes an irrep $\hat{\mathbf{3}}$. Indeed, using the transformation rules (3.6), (3.8) it is easy to check that the triplet (3.10) transforms under the generators of the modular group as expected:

$$\begin{aligned} Y^{(4,1)}_{\hat{\mathbf{3}}}(\tau) &\xrightarrow{T} Y^{(4,1)}_{\hat{\mathbf{3}}}(\tau+1) &&= \rho_{\hat{\mathbf{3}}}(T)\, Y^{(4,1)}_{\hat{\mathbf{3}}}(\tau), \\ Y^{(4,1)}_{\hat{\mathbf{3}}}(\tau) &\xrightarrow{S} Y^{(4,1)}_{\hat{\mathbf{3}}}(-1/\tau) &&= (-\tau)\, \rho_{\hat{\mathbf{3}}}(S)\, Y^{(4,1)}_{\hat{\mathbf{3}}}(\tau), \\ Y^{(4,1)}_{\hat{\mathbf{3}}}(\tau) &\xrightarrow{R} Y^{(4,1)}_{\hat{\mathbf{3}}}(\tau) &&= (-1)\, \rho_{\hat{\mathbf{3}}}(R)\, Y^{(4,1)}_{\hat{\mathbf{3}}}(\tau). \end{aligned} \tag{3.11}$$

The $\hat{\mathbf{3}}$ modular triplet of (3.10) is the base result of our construction. It can be used to generate all modular forms entering and determining the fermion Yukawa couplings and mass matrices, as we will see in what follows.

Higher weights

Modular multiplets of higher weights $Y^{(4,k>1)}_{\mathbf{r},\,i}$ may be obtained from those of lower weight via tensor products. The lowest weight multiplet in (3.10) works then as a "seed" multiplet, since all higher weight modular multiplets can be recovered from a sufficient number of tensor products of $Y^{(4,1)}_{\hat{\mathbf{3}}}(\tau)$ with itself. Note that the latter has been written in terms of a minimal set of functions of τ from the start, namely $\theta(\tau)$ and $\varepsilon(\tau)$. By doing so, tensor products directly provide spaces of modular forms with the correct dimensions, bypassing the typical need to look for constraints relating redundant higher weight multiplets (cf. section 3.2). In other words, these constraints are manifestly verified given the explicit forms of the multiplet components.

First of all, we recover the previously known [16] modular S_4 lowest-weight multiplets, a doublet and a triplet($'$), which are now expressed in terms of $\theta(\tau)$ and $\varepsilon(\tau)$ and read

$$Y^{(4,2)}_{\mathbf{2}}(\tau) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\theta^4+\varepsilon^4\right) \\ -\sqrt{6}\varepsilon^2\theta^2 \end{pmatrix}, \quad Y^{(4,2)}_{\mathbf{3}'}(\tau) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\theta^4-\varepsilon^4\right) \\ -2\varepsilon\theta^3 \\ -2\varepsilon^3\theta \end{pmatrix}. \tag{3.12}$$

Our construction reduces to that of modular $\Gamma_4 \simeq S_4$ for even weights (see also appendix B.2). In order to compare the results in (3.12) with those of ref. [16], one needs to work in compatible group representation bases, i.e., bases in which the representation matrices $\rho_{\mathbf{r}}(S)$ and $\rho_{\mathbf{r}}(T)$ coincide, for irreducible representations $\mathbf{r}$ common to S_4 and S_4' (those without hats). The basis for S_4 compatible with the one for S_4' we here consider, together with the expressions for modular multiplets in that basis, can be found in ref. [P4] (see appendices B and C therein). Then, by looking at the q-expansions,

$$\begin{aligned} Y^{(4,2)}_{\mathbf{2}}(\tau) &= \begin{pmatrix} \frac{1}{\sqrt{2}}\left(1 + 24\,q_4^4 + 24\,q_4^8 + 96\,q_4^{12} + 24\,q_4^{16} + \ldots\right) \\ -4\sqrt{6}\left(q_4^2 + 4\,q_4^6 + 6\,q_4^{10} + 8\,q_4^{14} + 13\,q_4^{18} + \ldots\right) \end{pmatrix}, \\ Y^{(4,2)}_{\mathbf{3'}}(\tau) &= \begin{pmatrix} \frac{1}{\sqrt{2}}\left(1 - 8\,q_4^4 + 24\,q_4^8 - 32\,q_4^{12} + 24\,q_4^{16} + \ldots\right) \\ -4\left(q_4 + 6\,q_4^5 + 13\,q_4^9 + 14\,q_4^{13} + 18\,q_4^{17} + \ldots\right) \\ -16\left(q_4^3 + 2\,q_4^7 + 3\,q_4^{11} + 6\,q_4^{15} + 5\,q_4^{19} + \ldots\right) \end{pmatrix}, \end{aligned} \tag{3.13}$$

one can see that the modular multiplets in question indeed match, up to normalisation.

Further tensor products with $Y^{(4,1)}_{\hat{\mathbf{3}}}$ produce modular multiplets of higher weights. At weight 3, a non-trivial singlet and two triplets exclusive to S_4' arise:

$$\begin{aligned} Y^{(4,3)}_{\hat{\mathbf{1}}'}(\tau) &= \sqrt{3}\left(\varepsilon\theta^5 - \varepsilon^5\theta\right), \\ Y^{(4,3)}_{\hat{\mathbf{3}}}(\tau) &= \begin{pmatrix} \varepsilon^5\theta + \varepsilon\theta^5 \\ \frac{1}{2\sqrt{2}}\left(5\varepsilon^2\theta^4 - \varepsilon^6\right) \\ \frac{1}{2\sqrt{2}}\left(\theta^6 - 5\varepsilon^4\theta^2\right) \end{pmatrix}, \quad Y^{(4,3)}_{\hat{\mathbf{3}}'}(\tau) = \frac{1}{2}\begin{pmatrix} -4\sqrt{2}\,\varepsilon^3\theta^3 \\ \theta^6 + 3\varepsilon^4\theta^2 \\ -3\varepsilon^2\theta^4 - \varepsilon^6 \end{pmatrix}. \end{aligned} \tag{3.14}$$

Finally, at weight 4 one again recovers the S_4 result. We obtain:

$$\begin{aligned} Y^{(4,4)}_{\mathbf{1}}(\tau) &= \frac{1}{2\sqrt{3}}\left(\theta^8 + 14\varepsilon^4\theta^4 + \varepsilon^8\right), \quad Y^{(4,4)}_{\mathbf{2}}(\tau) = \begin{pmatrix} \frac{1}{4}\left(\theta^8 - 10\varepsilon^4\theta^4 + \varepsilon^8\right) \\ \sqrt{3}\left(\varepsilon^2\theta^6 + \varepsilon^6\theta^2\right) \end{pmatrix}, \\ Y^{(4,4)}_{\mathbf{3}}(\tau) &= \frac{3}{2\sqrt{2}}\begin{pmatrix} \sqrt{2}\left(\varepsilon^2\theta^6 - \varepsilon^6\theta^2\right) \\ \varepsilon^3\theta^5 - \varepsilon^7\theta \\ -\varepsilon\theta^7 + \varepsilon^5\theta^3 \end{pmatrix}, \quad Y^{(4,4)}_{\mathbf{3'}}(\tau) = \begin{pmatrix} \frac{1}{4}\left(\theta^8 - \varepsilon^8\right) \\ \frac{1}{2\sqrt{2}}\left(\varepsilon\theta^7 + 7\varepsilon^5\theta^3\right) \\ \frac{1}{2\sqrt{2}}\left(7\varepsilon^3\theta^5 + \varepsilon^7\theta\right) \end{pmatrix}, \end{aligned} \tag{3.15}$$

which can be seen to match known multiplets (up to normalisation) by comparing q-expansions. For the explicit expressions of S_4' modular multiplets with

higher weights, up to $k = 10$ and written in terms of $\theta(\tau)$ and $\varepsilon(\tau)$, we refer the reader to appendix D of ref. [P5]. Note that odd(even)-weighted modular forms always furnish (un)hatted representations, since in our notation hatted representations are exactly the ones for which $\rho(R) = -\mathbb{1}$.

3.2. $\Gamma_5 \simeq A_5$ [P2]

Weight 2

Weight-2 modular forms of level 5 can be constructed in a way analogous to the cases of $\Gamma_2 \simeq S_3$ [15], $\Gamma_3 \simeq A_4$ [12], and $\Gamma_4 \simeq S_4$ [16]. To do that, one has to find a set of "seed" functions $\alpha_{i,j}(\tau)$ such that the sought-after modular forms are linear combinations of their logarithmic derivatives. At level N, $\alpha_{i,j}(\tau)$ should form a set which is in a certain sense closed under the action of Γ_N. As can be inferred from the results in ref. [125], a convenient choice for $\alpha_{i,j}(\tau)$ at level 5 is given by the Jacobi theta functions $\Theta_3(z,\tau)$ (see appendix A), and they explicitly read:

$$
\begin{aligned}
\alpha_{1,-1}(\tau) &\equiv \Theta_3\left(\frac{\tau+1}{2}, 5\tau\right), & \alpha_{2,-1}(\tau) &\equiv e^{2\pi i\tau/5}\Theta_3\left(\frac{3\tau+1}{2}, 5\tau\right),\\
\alpha_{1,0}(\tau) &\equiv \Theta_3\left(\frac{\tau+9}{10}, \frac{\tau}{5}\right), & \alpha_{2,0}(\tau) &\equiv \Theta_3\left(\frac{\tau+7}{10}, \frac{\tau}{5}\right),\\
\alpha_{1,1}(\tau) &\equiv \Theta_3\left(\frac{\tau}{10}, \frac{\tau+1}{5}\right), & \alpha_{2,1}(\tau) &\equiv \Theta_3\left(\frac{\tau+8}{10}, \frac{\tau+1}{5}\right),\\
\alpha_{1,2}(\tau) &\equiv \Theta_3\left(\frac{\tau+1}{10}, \frac{\tau+2}{5}\right), & \alpha_{2,2}(\tau) &\equiv \Theta_3\left(\frac{\tau+9}{10}, \frac{\tau+2}{5}\right),\\
\alpha_{1,3}(\tau) &\equiv \Theta_3\left(\frac{\tau+2}{10}, \frac{\tau+3}{5}\right), & \alpha_{2,3}(\tau) &\equiv \Theta_3\left(\frac{\tau}{10}, \frac{\tau+3}{5}\right),\\
\alpha_{1,4}(\tau) &\equiv \Theta_3\left(\frac{\tau+3}{10}, \frac{\tau+4}{5}\right), & \alpha_{2,4}(\tau) &\equiv \Theta_3\left(\frac{\tau+1}{10}, \frac{\tau+4}{5}\right).
\end{aligned}
\tag{3.16}
$$

Under the action of the generators S and T of Γ_5 (see appendix B.3), each of these functions is mapped to another, up to (possibly τ-dependent) multiplicative factors. A diagram of said map is given in fig. 3.1, and one can check that the actions of S^2, $(ST)^3$ and T^5 applied to each element correspond to the identity. Taking logarithmic derivatives, one obtains:

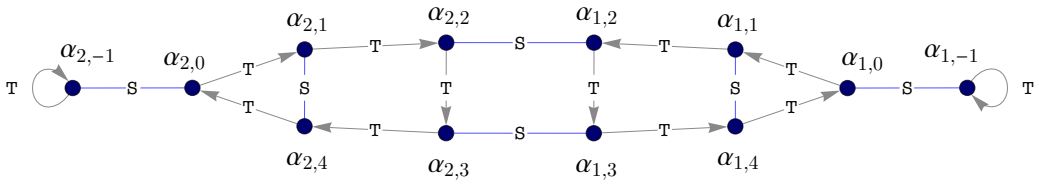

Figure 3.1.: Graph illustrating the automorphisms of the set of seed functions $\alpha_{i,j}(\tau)$, defined in (3.16), under the actions of $\Gamma_5 \simeq A_5$ generators S and T.

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\log\alpha_{i,j}(-1/\tau) = \frac{i\pi}{20}\left(1-\frac{1}{\tau^2}\right)+\frac{1}{2\tau}+\frac{\mathrm{d}}{\mathrm{d}\tau}\log\alpha^S_{i,j}(\tau), \tag{3.17}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\log\alpha_{i,j}(\tau+1) = \frac{\mathrm{d}}{\mathrm{d}\tau}\log\alpha^T_{i,j}(\tau), \tag{3.18}$$

where $\alpha^S_{i,j}$ and $\alpha^T_{i,j}$ are the images of $\alpha_{i,j}$ under the S and T maps of fig. 3.1, respectively.

It then follows that the functions

$$Y(c_{1,-1},\ldots,c_{1,4};c_{2,-1},\ldots,c_{2,4}|\tau) \equiv \sum_{i,j} c_{i,j}\frac{\mathrm{d}}{\mathrm{d}\tau}\log\alpha_{i,j}(\tau), \text{ with } \sum_{i,j} c_{i,j} = 0\,, \tag{3.19}$$

span the sought-after 11-dimensional space of weight-2 modular forms of level $N = 5$. Under S and T, one has the following transformations:

$$\begin{aligned} S : {} & Y(c_{1,-1},\ldots,c_{1,4};c_{2,-1},\ldots,c_{2,4}|\tau) \to \\ & Y(c_{1,0},c_{1,-1},c_{1,4},c_{2,2},c_{2,3},c_{1,1};c_{2,0},c_{2,-1},c_{2,4},c_{1,2},c_{1,3},c_{2,1}|\tau), \end{aligned} \tag{3.20}$$

$$\begin{aligned} T : {} & Y(c_{1,-1},\ldots,c_{1,4};c_{2,-1},\ldots,c_{2,4}|\tau) \to \\ & Y(c_{1,-1},c_{1,4},c_{1,0},c_{1,1},c_{1,2},c_{1,3};c_{2,-1},c_{2,4},c_{2,0},c_{2,1},c_{2,2},c_{2,3}|\tau). \end{aligned} \tag{3.21}$$

The space in question is divided into the following multiplets of A_5:

$$Y_{\mathbf{5}}^{(5,2)}(\tau) = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{pmatrix} \equiv \begin{pmatrix} -\frac{1}{\sqrt{6}}\, Y(-5,1,1,1,1,1;-5,1,1,1,1,1|\tau) \\ Y(0,1,\zeta^4,\zeta^3,\zeta^2,\zeta\,;\,0,1,\zeta^4,\zeta^3,\zeta^2,\zeta\,|\,\tau) \\ Y(0,1,\zeta^3,\zeta,\zeta^4,\zeta^2\,;\,0,1,\zeta^3,\zeta,\zeta^4,\zeta^2\,|\,\tau) \\ Y(0,1,\zeta^2,\zeta^4,\zeta,\zeta^3\,;\,0,1,\zeta^2,\zeta^4,\zeta,\zeta^3\,|\,\tau) \\ Y(0,1,\zeta,\zeta^2,\zeta^3,\zeta^4\,;\,0,1,\zeta,\zeta^2,\zeta^3,\zeta^4\,|\,\tau) \end{pmatrix}, \tag{3.22}$$

$$Y_{\mathbf{3}}^{(5,2)}(\tau) = \begin{pmatrix} Y_6 \\ Y_7 \\ Y_8 \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} Y\left(-\sqrt{5},-1,-1,-1,-1,-1;\sqrt{5},1,1,1,1,1\middle|\tau\right) \\ Y(0,1,\zeta^4,\zeta^3,\zeta^2,\zeta\,;\,0,-1,-\zeta^4,-\zeta^3,-\zeta^2,-\zeta\,|\,\tau) \\ Y(0,1,\zeta,\zeta^2,\zeta^3,\zeta^4\,;\,0,-1,-\zeta,-\zeta^2,-\zeta^3,-\zeta^4\,|\,\tau) \end{pmatrix}, \tag{3.23}$$

$$Y_{\mathbf{3'}}^{(5,2)}(\tau) = \begin{pmatrix} Y_9 \\ Y_{10} \\ Y_{11} \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{\sqrt{2}} Y\left(\sqrt{5},-1,-1,-1,-1,-1;-\sqrt{5},1,1,1,1,1\middle|\tau\right) \\ Y(0,1,\zeta^3,\zeta,\zeta^4,\zeta^2\,;\,0,-1,-\zeta^3,-\zeta,-\zeta^4,-\zeta^2\,|\,\tau) \\ Y(0,1,\zeta^2,\zeta^4,\zeta,\zeta^3\,;\,0,-1,-\zeta^2,-\zeta^4,-\zeta,-\zeta^3\,|\,\tau) \end{pmatrix}, \tag{3.24}$$

where $\zeta \equiv e^{2i\pi/5}$.

Weight-2 multiplets (3.22) to (3.24) admit the following q-expansions:

$$\begin{aligned}
Y_{\mathbf{5}}^{(5,2)}(\tau) &= -2i\pi \begin{pmatrix} -\sqrt{\frac{1}{6}}(1 + 6q_5^5 + 18q_5^{10} + 24q_5^{15} + 42q_5^{20} + \ldots) \\ q_5 + 12q_5^6 + 12q_5^{11} + 31q_5^{16} + \ldots \\ 3q_5^2 + 8q_5^7 + 28q_5^{12} + 18q_5^{17} + 36q_5^{22} + \ldots \\ 4q_5^3 + 15q_5^8 + 14q_5^{13} + 39q_5^{18} + 24q_5^{23} + \ldots \\ 7q_5^4 + 13q_5^9 + 24q_5^{14} + 20q_5^{19} + 60q_5^{24} + \ldots \end{pmatrix}, \\
Y_{\mathbf{3}}^{(5,2)}(\tau) &= 2\sqrt{5}i\pi \begin{pmatrix} -\frac{1}{5\sqrt{2}}(-1 + 30q_5^5 + 20q_5^{10} + 40q_5^{15} + 90q_5^{20} + \ldots) \\ q_5 + 2q_5^6 + 12q_5^{11} + 11q_5^{16} + 12q_5^{21} + \ldots \\ 3q_5^4 + 7q_5^9 + 6q_5^{14} + 20q_5^{19} + 10q_5^{24} + \ldots \end{pmatrix}, \\
Y_{\mathbf{3'}}^{(5,2)}(\tau) &= 2\sqrt{5}i\pi \begin{pmatrix} -\frac{1}{5\sqrt{2}}(1 + 20q_5 + 30q_5^2 + 60q_5^3 + 60q_5^4 + \ldots) \\ q_5^2 + 6q_5^7 + 6q_5^{12} + 16q_5^{17} + 12q_5^{22} + \ldots \\ 2q_5^3 + 5q_5^8 + 12q_5^{13} + 7q^{18} + 22q_5^{23} + \ldots \end{pmatrix},
\end{aligned} \tag{3.25}$$

where $q_5 \equiv \exp(2i\pi\tau/5)$.

Higher weights

$N = 5$ modular forms of higher even weights can be obtained from tensor products of the weight-2 multiplets (3.22) to (3.24). The missing **1** and **4** representations arise at weight 4. Even though one can form 66 products $Y_i Y_j$, the dimension of the space of weight $k = 4$ forms is $5k+1 = 21$. Therefore, there are 45 constraints between the $Y_i Y_j$, which reduce the 66 seemingly independent combinations to 21 truly independent ones. These last combinations arrange themselves into the following multiplets of A_5:

$$
\begin{aligned}
Y^{(5,4)}_{\mathbf{1}} &= Y_1^2 + 2Y_3Y_4 + 2Y_2Y_5 \sim \mathbf{1}\,, \\
Y^{(5,4)}_{\mathbf{3}} &= \begin{pmatrix} -2Y_1Y_6 + \sqrt{3}\,Y_5Y_7 + \sqrt{3}\,Y_2Y_8 \\ \sqrt{3}\,Y_2Y_6 + Y_1Y_7 - \sqrt{6}\,Y_3Y_8 \\ \sqrt{3}\,Y_5Y_6 - \sqrt{6}\,Y_4Y_7 + Y_1Y_8 \end{pmatrix} \sim \mathbf{3}\,, \\
Y^{(5,4)}_{\mathbf{3'}} &= \begin{pmatrix} \sqrt{3}\,Y_1Y_6 + Y_5Y_7 + Y_2Y_8 \\ Y_3Y_6 - \sqrt{2}\,Y_2Y_7 - \sqrt{2}\,Y_4Y_8 \\ Y_4Y_6 - \sqrt{2}\,Y_3Y_7 - \sqrt{2}\,Y_5Y_8 \end{pmatrix} \sim \mathbf{3'}\,, \\
Y^{(5,4)}_{\mathbf{4}} &= \begin{pmatrix} 2Y_4^2 + \sqrt{6}\,Y_1Y_2 - Y_3Y_5 \\ 2Y_2^2 + \sqrt{6}\,Y_1Y_3 - Y_4Y_5 \\ 2Y_5^2 - Y_2Y_3 + \sqrt{6}\,Y_1Y_4 \\ 2Y_3^2 - Y_2Y_4 + \sqrt{6}\,Y_1Y_5 \end{pmatrix} \sim \mathbf{4}\,, \\
Y^{(5,4)}_{\mathbf{5},1} &= \begin{pmatrix} \sqrt{2}\,Y_1^2 + \sqrt{2}\,Y_3Y_4 - 2\sqrt{2}\,Y_2Y_5 \\ \sqrt{3}\,Y_4^2 - 2\sqrt{2}\,Y_1Y_2 \\ \sqrt{2}\,Y_1Y_3 + 2\sqrt{3}\,Y_4Y_5 \\ 2\sqrt{3}\,Y_2Y_3 + \sqrt{2}\,Y_1Y_4 \\ \sqrt{3}\,Y_3^2 - 2\sqrt{2}\,Y_1Y_5 \end{pmatrix} \sim \mathbf{5}\,, \\
Y^{(5,4)}_{\mathbf{5},2} &= \begin{pmatrix} \sqrt{3}\,Y_5Y_7 - \sqrt{3}\,Y_2Y_8 \\ -Y_2Y_6 - \sqrt{3}\,Y_1Y_7 - \sqrt{2}\,Y_3Y_8 \\ -2Y_3Y_6 - \sqrt{2}\,Y_2Y_7 \\ 2Y_4Y_6 + \sqrt{2}\,Y_5Y_8 \\ Y_5Y_6 + \sqrt{2}\,Y_4Y_7 + \sqrt{3}\,Y_1Y_8 \end{pmatrix} \sim \mathbf{5}\,.
\end{aligned}
\tag{3.26}
$$

For the explicit expressions of A_5 modular multiplets with higher weights up to $k = 10$ we refer the reader to appendix C of [P2].

4. Combining CP and modular symmetry [P4, P5]

In models possessing a flavour symmetry G, one can define a *generalised CP transformation* acting on matter fields as

$$\psi_i(x) \xrightarrow{\text{CP}} X_{ij}\,\overline{\psi}_j(x_{\text{P}}), \tag{4.1}$$

with a bar denoting the conjugate field, and where $x = (t, \vec{x})$, $x_{\text{P}} = (t, -\vec{x})$ and X is a unitary matrix acting on flavour space. The form of the matrix X is constrained due to the presence of a flavour symmetry [126, 127]. The key idea is that the theory should be invariant under the sequence of transformations $\text{CP} \to g \in G \to \text{CP}^{-1}$, so the resulting transformation must correspond to another flavour symmetry transformation $g' \in G$. More formally, one can check that CP should act on the flavour group as an *outer automorphism* $u(g) \equiv \text{CP} \circ \gamma \circ \text{CP}^{-1}$. In the case of a linearly realised discrete symmetry group G, this yields a *consistency condition* of the form

$$X\,\rho^*(g)\,X^{-1} = \rho(u(g)) \quad \forall\, g \in G, \tag{4.2}$$

where ρ is the flavour group representation of the matter field ψ.

In this chapter we show how this idea can be extended to the modular invariance framework, following refs. [P4, P5]. Unlike the case of discrete flavour symmetries, field transformation properties under CP are restricted to only two possibilities. The derivation we are going to present is agnostic to the UV completion of the theory and, in particular, the origin of modular symmetry.

4.1. CP transformations

Transformation of the modulus

Let us first apply the consistency condition chain[1]

$$\text{CP} \to \gamma \in \Gamma \to \text{CP}^{-1} = \gamma' \in \Gamma \tag{4.3}$$

[1] It may be possible to generalise the CP transformation such that it can be combined not only with modular but also with other internal symmetries of the theory. We are not going to consider this case here.

to an arbitrary chiral superfield $\psi(x)$ assigned to an irreducible unitary representation $\mathbf{r}$ of Γ'_N, which transforms as $\psi(x) \to X_\mathbf{r}\,\overline{\psi}(x_\mathrm{P})$ under CP:

$$\begin{aligned}\psi(x) &\xrightarrow{\mathrm{CP}} X_\mathbf{r}\,\overline{\psi}(x_\mathrm{P}) \xrightarrow{\gamma} (c\tau^*+d)^{-k}\,X_\mathbf{r}\,\rho^*_\mathbf{r}(\gamma)\,\overline{\psi}(x_\mathrm{P}) \\ &\xrightarrow{\mathrm{CP}^{-1}} (c\tau^*_{\mathrm{CP}^{-1}}+d)^{-k}\,X_\mathbf{r}\,\rho^*_\mathbf{r}(\gamma)X_\mathbf{r}^{-1}\,\psi(x),\end{aligned} \tag{4.4}$$

where $\tau_{\mathrm{CP}^{-1}}$ is the result of applying CP^{-1} to the modulus τ. The resulting transformation should be equivalent to a modular transformation γ' which depends on γ and maps $\psi(x)$ to $(c'\tau+d')^{-k}\rho_\mathbf{r}(\gamma')\psi(x)$. Taking this into account, we get

$$X_\mathbf{r}\,\rho^*_\mathbf{r}(\gamma)X_\mathbf{r}^{-1} = \left(\frac{c'\tau+d'}{c\tau^*_{\mathrm{CP}^{-1}}+d}\right)^{-k}\rho_\mathbf{r}(\gamma'). \tag{4.5}$$

Since the matrices $X_\mathbf{r}$, $\rho_\mathbf{r}(\gamma)$ and $\rho_\mathbf{r}(\gamma')$ are independent of τ, the overall coefficient on the right-hand side has to be a constant:[2]

$$\frac{c'\tau+d'}{c\tau^*_{\mathrm{CP}^{-1}}+d} = \frac{1}{\lambda^*}\,, \tag{4.6}$$

where $\lambda \in \mathbb{C}$, and $|\lambda| = 1$ due to unitarity of $\rho_\mathbf{r}(\gamma)$ and $\rho_\mathbf{r}(\gamma')$. The values of λ, c' and d' depend on γ.

Taking $\gamma = S^{-1}$, so that $c = 1$, $d = 0$, and denoting $c'(S^{-1}) = C$, $d'(S^{-1}) = D$ while keeping henceforth the notation $\lambda(S^{-1}) = \lambda$, we find $\tau = (\lambda\tau^*_{\mathrm{CP}^{-1}} - D)/C$, and consequently,

$$\tau \xrightarrow{\mathrm{CP}^{-1}} \tau_{\mathrm{CP}^{-1}} = \lambda\,(C\tau^*+D)\,, \quad \tau \xrightarrow{\mathrm{CP}} \tau_\mathrm{CP} = \frac{1}{C}\,(\lambda\tau^*-D)\,. \tag{4.7}$$

Let us now act with the chain $\mathrm{CP} \to T \to \mathrm{CP}^{-1}$ on the modulus τ itself:

$$\tau \xrightarrow{\mathrm{CP}} \frac{1}{C}\,(\lambda\tau^*-D) \xrightarrow{T} \frac{1}{C}\,(\lambda\,(\tau^*+1)-D) \xrightarrow{\mathrm{CP}^{-1}} \tau+\frac{\lambda}{C}\,. \tag{4.8}$$

The resulting transformation has to be a modular transformation, therefore $\lambda/C \in \mathbb{Z}$. Since $|\lambda| = 1$, we immediately find $|C| = 1$, $\lambda = \pm 1$. After choosing

[2] Strictly speaking, this is only true for non-zero weights k. We assume that at least one superfield with non-zero modular weight exists in the theory, because otherwise the modulus has no effect on the superfield transformations.

the sign of C as $C = \mp 1$ so that $\mathrm{Im}\,\tau_{\mathrm{CP}} > 0$, the CP transformation rule (4.7) simplifies to

$$\tau \xrightarrow{\mathrm{CP}} n - \tau^*, \tag{4.9}$$

with $n \in \mathbb{Z}$. One can easily check that the chain $\mathrm{CP} \to S \to \mathrm{CP}^{-1} = \gamma'(S)$ (applied to the modulus τ itself) imposes no further restrictions on the form of τ_{CP}. Since S and T generate the entire modular group, we conclude that (4.9) is the most general CP transformation of the modulus τ compatible with the modular symmetry.

It is always possible to redefine the CP transformation in such a way that $n = 0$. Consider the composition $\mathrm{CP}' \equiv T^{-n} \circ \mathrm{CP}$ so that $\tau \to -\tau^*$ under CP′. It is worth noting that this redefinition represents an inner automorphism which does not spoil the form of CP transformation in (4.1). Indeed, chiral superfields transform under CP′ as

$$\psi \xrightarrow{\mathrm{CP}'} \rho_{\mathbf{r}}^{-n}(T)\, X_{\mathbf{r}}\, \overline{\psi}. \tag{4.10}$$

Thus, CP′ has the same properties as the original CP transformation up to a redefinition of $X_{\mathbf{r}}$. Therefore, from now on we will assume without loss of generality that the modulus τ transforms under CP as[3]

$$\tau \xrightarrow{\mathrm{CP}} -\tau^*. \tag{4.11}$$

It obviously follows that τ does not change under the action of CP^2:

$$\tau \xrightarrow{\mathrm{CP}^2} \tau. \tag{4.12}$$

Action on the modular group

Having derived the explicit form of the CP transformation for the modulus τ, we are now in a position to find the action of CP on the modular group Γ as an outer automorphism $u(\gamma) \equiv \mathrm{CP} \circ \gamma \circ \mathrm{CP}^{-1}$. For any modular transformation $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we have

$$\tau \xrightarrow{\mathrm{CP}} -\tau^* \xrightarrow{\gamma} -\frac{a\tau^* + b}{c\tau^* + d} \xrightarrow{\mathrm{CP}^{-1}} \frac{a\tau - b}{-c\tau + d} = u(\gamma)\tau, \tag{4.13}$$

[3]The CP transformation of the modulus derived by us from the requirement of consistency between modular and CP symmetries has appeared in the context of string-inspired models (see, e.g., refs. [80, 128–130]).

which implies

$$u(\gamma) = \sigma(\gamma) \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \in \Gamma, \tag{4.14}$$

where $\sigma(\gamma) = \pm 1$. Note that the signs $\sigma(\gamma)$ are irrelevant in the case of the inhomogeneous modular group $\overline{\Gamma}$ since γ is identified with $-\gamma$, and therefore (4.14) uniquely determines the automorphism $u(\gamma)$. This is not the case for the full modular group Γ, and one has to treat the signs carefully.

Since u is an automorphism, it is sufficient to define its action on the group generators. From (4.14) one has:

$$u(S) = \sigma(S)S^{-1}, \qquad u(T) = \sigma(T)T^{-1}, \qquad u(R) = \sigma(R)R. \tag{4.15}$$

The fact that $u(\gamma)$ is an automorphism implies $u(R) \neq \mathbb{1} = -R$, and so $\sigma(R) = +1$ and $u(R) = +R$. Furthermore, the signs $\sigma(\gamma)$ must be chosen in a way consistent with the group relations in (2.3). In particular, one finds:

$$(ST)^3 = \mathbb{1} \xrightarrow{u} (\sigma(S)\sigma(T))^3 (TS)^{-3} = \mathbb{1}, \tag{4.16}$$

implying that $\sigma(S) = \sigma(T)$, since $(TS)^3 = \mathbb{1}$. Thus, from the outset, two different outer automorphisms may be realised (see also [74, 131]):

$$(\mathrm{CP}_1)\ u\ : \qquad u(S) = S^{-1}, \qquad u(T) = T^{-1}, \qquad u(R) = R, \tag{4.17}$$

$$(\mathrm{CP}_2)\ u'\ : \qquad u'(S) = -S^{-1}, \qquad u'(T) = -T^{-1}, \qquad u'(R) = R. \tag{4.18}$$

CP_1

The first option (4.17), which we call CP_1, corresponds to a trivial sign choice $\sigma(\gamma) = +1$ and therefore admits an explicit formula for generic γ:

$$u : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}. \tag{4.19}$$

This automorphism can be realised as a similarity transformation within $\mathrm{GL}(2, \mathbb{Z})$:

$$u(\gamma) = \mathrm{CP}_1\, \gamma\, \mathrm{CP}_1^{-1} \quad \text{with} \quad \mathrm{CP}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \notin \Gamma. \tag{4.20}$$

Applying the chain $\mathrm{CP}_1 \to \gamma \to \mathrm{CP}_1^{-1}$ to a matter field ψ, which transforms under Γ and CP as in (2.5) and (4.1), one arrives at the *consistency condition* on the matrix X:

$$X\, \rho^*(\gamma)\, X^{-1} = \rho(u(\gamma)) \quad \forall\, \gamma \in \Gamma, \tag{4.21}$$

or, equivalently,

$$X\,\rho^*(S)\,X^{-1} = \rho^\dagger(S), \quad X\,\rho^*(T)\,X^{-1} = \rho^\dagger(T) \tag{4.22}$$

(see also [83]). Note that the consistency condition (4.21) has the same form as in the case of linearly realised discrete flavour symmetries (4.2). However, unlike usual discrete flavour symmetries, modular symmetry restricts the form of the automorphism $u(\gamma)$ as in (4.19). Therefore, for each irrep $\mathbf{r}$, $X_\mathbf{r}$ is fixed up to an overall phase by Schur's lemma.

In a basis where S and T are represented by symmetric matrices, (4.22) is satisfied by the canonical CP transformation $X = \mathbb{1}$. Such basis exists for all irreps of the inhomogeneous finite modular groups Γ_N with $N = 2, 3, 4, 5$ (see [P4] and references therein) and $N = 7$ [52], as well as for all irreps of the homogeneous modular groups Γ'_N with $N = 3$,[4] $N = 4$ (see appendix B.2), and $N = 5$ [108]. This means that CP_1 allows to define a CP transformation consistently and uniquely for all irreps of the aforementioned finite modular groups, hence u acts as a class-inverting automorphism on these groups [132].[5]

The action of CP_1 on matter fields (and the modulus) obeys $\mathrm{CP}_1^2 = \mathbb{1}$, since $\psi_i(x) \to (XX^*)_{ij}\,\psi_j(x)$ under CP_1^2 and $X = \mathbb{1} \implies XX^* = \mathbb{1}$ in the symmetric basis. It further follows that X is symmetric in any representation basis. The modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ is then extended to

$$\begin{aligned}\mathrm{GL}(2,\mathbb{Z}) &\simeq \mathrm{SL}(2,\mathbb{Z}) \rtimes \mathbb{Z}_2^{\mathrm{CP}_1} \\ &= \left\langle S,\, T,\, R,\, \mathrm{CP}_1 \;\middle|\; \begin{array}{l} S^2 = R,\ (ST)^3 = R^2 = \mathrm{CP}_1^2 = \mathbb{1},\ RT = TR, \\ \mathrm{CP}_1\,S\,\mathrm{CP}_1^{-1} = S^{-1},\ \mathrm{CP}_1\,T\,\mathrm{CP}_1^{-1} = T^{-1} \end{array} \right\rangle .\end{aligned} \tag{4.23}$$

CP_2

Let us now discuss the second possibility (4.18) for the modular group outer automorphism, u'. This choice, which we call CP_2, is formally defined by

$$u'(\gamma) = \mathrm{CP}_2\,\gamma\,\mathrm{CP}_2^{-1}, \tag{4.24}$$

[4] One can obtain a symmetric basis for Γ'_3 starting from the one typically considered in the literature [105] and performing a change of basis for all 2-dimensional irreps via the matrix $\mathrm{diag}(e^{-7i\pi/12}, 1)$.

[5] Note however that, at the level of the full modular group, u is not class-inverting. Taking for instance $\gamma = \left(\begin{smallmatrix} 11 & 9 \\ 17 & 14 \end{smallmatrix}\right)$, one can show that $u(\gamma)$ and γ^{-1} are not in the same $\mathrm{SL}(2,\mathbb{Z})$ conjugacy class, via e.g. the LLS invariant of ref. [133].

but cannot be realised as a similarity transformation within $\mathrm{GL}(2,\mathbb{Z})$. It leads to a different consistency condition on the matrix X, namely:

$$X\,\rho^*(\gamma)\,X^{-1} = \sigma(\gamma)^k\,\rho(u'(\gamma)) \quad \forall\,\gamma \in \Gamma, \tag{4.25}$$

or, in terms of the generators S and T,

$$X\,\rho^*(S)\,X^{-1} = (-1)^k\rho(R)\rho^\dagger(S), \quad X\,\rho^*(T)\,X^{-1} = (-1)^k\rho(R)\rho^\dagger(T), \tag{4.26}$$

which are equivalent to (4.25), since $\sigma(\gamma_1)\sigma(\gamma_2) = \sigma(\gamma_1\gamma_2)$.

In practice, the consistency condition (4.26) differs from that of (4.22) and CP_2 differs from CP_1 only when $(-1)^k\,\rho(R) \neq \mathbb{1}$, i.e., whenever the matter field ψ transforms non-trivially under R. For these R-odd fields, however, it is only possible to satisfy the consistency condition if

- characters of T and S vanish, $\chi(S) = \chi(T) = 0$, which follows from (4.26) after taking traces,
- the dimension of the representation of ψ is even, which follows from (4.26) after taking determinants, and
- the level N of the finite group is even, which follows from taking the N-th power of the second relation in (4.26).[6]

This means that, given a finite modular group of level N, CP_2 is incompatible with certain combinations of modular weights and irreps.

In particular, combining the groups Γ'_N with $N = 3, 5, 7$ with CP_2 means that any matter field must be R-even, i.e., satisfy $(-1)^k\,\rho(R) = \mathbb{1}$, and transform canonically under CP, $X_{\mathrm{CP}_2} = \mathbb{1}$, in the symmetric basis. In the case of Γ_2, Γ_4 and Γ'_4 there is the additional option to have R-odd fields, $(-1)^k\,\rho(R) = -\mathbb{1}$, but only for the doublet representations, all of which verify $\chi(S) = \chi(T) = 0$. These fields are constrained to transform under CP with

$$X_{\mathrm{CP}_2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.27}$$

in the symmetric basis. Notice that $\mathrm{CP}_2^2 \neq \mathbb{1}$; instead, the action of CP_2^2 on fields, forms and τ coincides with that of R for these finite groups. Equating these

[6] An associated fact is that $\Gamma(N)$ with $N \geq 2$ is only stable under u' for even N.

two actions, the modular group is in this context minimally extended to the semidirect product[7]

$$\begin{aligned}&\mathrm{SL}(2,\mathbb{Z}) \rtimes \mathbb{Z}_2^{\mathrm{CP}_2\, S} \\ &\quad = \left\langle S,\, T,\, R,\, \mathrm{CP}_2 \;\middle|\; \begin{array}{l} S^2 = \mathrm{CP}_2^2 = R,\ (ST)^3 = R^2 = \mathbb{1},\ RT = TR, \\ \mathrm{CP}_2\, S = S\, \mathrm{CP}_2,\ \mathrm{CP}_2\, T\, \mathrm{CP}_2^{-1} = R\, T^{-1} \end{array} \right\rangle .\end{aligned} \tag{4.28}$$

It should be noted that it is difficult to build phenomenologically viable models of fermion masses and mixing exploiting CP_2 with R-odd fields, as i) the choice of irreps for such fields is quite limited and ii) the R-odd and R-even sectors are segregated by the $\mathbb{Z}_2^R$ symmetry. Taken together, these facts imply the vanishing of some mixing angles or masses in simple models based on the combination of CP_2 with Γ_2, Γ_4, or Γ_4'. Therefore, in what follows we will focus on CP_1, denoting it simply as CP.

Transformation of modular form multiplets

Since modular multiplets $Y(\tau)$ transform under the modular group in essentially the same way as chiral superfields, it is natural to expect that the above discussion holds for modular multiplets as well. In particular, they should transform under CP as $Y \;\longrightarrow\; X_{\mathbf{r}}\, Y^*$. Still, it is instructive to derive their transformation rule explicitly.

Under a modular transformation, $Y(\tau)$ transforms as in (2.14), while under the action of CP one has $Y(\tau) \;\longrightarrow\; Y(-\tau^*)$. One can check that the complex-conjugated CP-transformed multiplets $Y^*(-\tau^*)$ transform almost like the original multiplets $Y(\tau)$ under a modular transformation, namely:

$$\begin{aligned} Y^*(-\tau^*) \xrightarrow{\gamma} Y^*\left(-(\gamma\tau)^*\right) &= Y^*\left(u(\gamma)(-\tau^*)\right) \\ &= (c\tau + d)^k \rho_{\mathbf{r}}^*\left(u(\gamma)\right) Y^*(-\tau^*), \end{aligned} \tag{4.29}$$

for a multiplet $Y(\tau)$ of weight k transforming in the irreducible representation $\mathbf{r}$ of Γ_N'.

Using the consistency condition (4.21), one then sees that it is the object $X_{\mathbf{r}}^T\, Y^*(-\tau^*)$ which transforms like $Y(\tau)$ under a modular transformation:

$$X_{\mathbf{r}}^T Y^*(-\tau^*) \xrightarrow{\gamma} (c\tau + d)^k \rho_{\mathbf{r}}(\gamma)\left[X_{\mathbf{r}}^T Y^*(-\tau^*)\right]. \tag{4.30}$$

[7] The non-trivial automorphism defining this outer semidirect product is $\gamma \mapsto \mathrm{CP}_2\, S\, \gamma\, S^{-1}\, \mathrm{CP}_2^{-1}$.

If there exists a unique modular form multiplet at a certain level N, weight k and representation $\mathbf{r}$, then proportionality follows:

$$Y(\tau) = z\, X_{\mathbf{r}}^T Y^*(-\tau^*), \tag{4.31}$$

with $z \in \mathbb{C}$. This is indeed the case for $2 \leq N \leq 5$ and lowest weight $k = 1$. Since $Y\left(-(-\tau^*)^*\right) = Y(\tau)$, it follows that $X_{\mathbf{r}} X_{\mathbf{r}}^* = |z|^2 \mathbb{1}_{\mathbf{r}}$, implying i) that $z = e^{i\phi}$ is a phase which can be absorbed in the normalisation of $Y(\tau)$, and ii) that $X_{\mathbf{r}}$ must be symmetric in this case, $X_{\mathbf{r}} X_{\mathbf{r}}^* = \mathbb{1}_{\mathbf{r}} \Rightarrow X_{\mathbf{r}} = X_{\mathbf{r}}^T$, independently of the basis. One can then write

$$Y(\tau) \xrightarrow{\text{CP}} Y(-\tau^*) = X_{\mathbf{r}} Y^*(\tau) \tag{4.32}$$

for these multiplets, as anticipated.

As we have seen in section 4.1, in a basis in which the generators S and T of Γ_N' are represented by symmetric matrices, one has $X_{\mathbf{r}} = \mathbb{1}_{\mathbf{r}}$. From (4.31) it follows that $Y(-\tau^*) = e^{i\phi}\, Y^*(\tau)$, the phase ϕ being removable, as commented above. At the q-expansion level this means that, in such a basis, all the expansion coefficients are real up to a common complex phase. This is indeed the case for the lowest-weight modular form multiplets of Γ_N' with $N \leq 5$, as can be explicitly verified from their q-expansions. This is further the case for the higher-weight modular multiplets of these groups in such a basis due to reality of Clebsch-Gordan coefficients.

4.2. CP-invariant theories

Implications for the couplings

We have found so far that a CP transformation consistent with modular symmetry acts on fields and modular form multiplets in the following way:

$$\tau \xrightarrow{\text{CP}} -\tau^*, \quad \psi(x) \xrightarrow{\text{CP}} X_{\mathbf{r}} \overline{\psi}(x_P), \quad Y(\tau) \xrightarrow{\text{CP}} Y(-\tau^*) = X_{\mathbf{r}} Y^*(\tau). \tag{4.33}$$

A SUSY modular-invariant theory is thus CP-conserving if the transformation (4.33) leaves the matter action $\mathcal{S}$ originating from the Lagrangian (2.11) unchanged. In particular, the superpotential W has to transform into its Hermitian conjugate, while the Kähler potential K is allowed to change by a Kähler transformation.

The Kähler potential of (2.15) is clearly invariant under the CP transformation (4.33), since it depends on $|\psi|^2$ and $\mathrm{Im}\,\tau$, both of which remain unchanged

(up to a change $x \to x_{\rm P}$ which does not affect $\mathcal{S}$). On the other hand, the superpotential can be written as a sum of independent terms of the form

$$W \supset \sum_s g_s \left(Y_s(\tau)\psi_1 \dots \psi_n\right)_{\mathbf{1},s} , \tag{4.34}$$

where $Y_s(\tau)$ are modular multiplets of a certain weight and irreducible representation, and g_s are complex coupling constants. Such terms transform non-trivially under CP, which leads to a certain constraint on the couplings g_s.

This can be easily checked for a symmetric basis, as in this basis $X_{\mathbf{r}} = \mathbb{1}_{\mathbf{r}}$ for any representation $\mathbf{r}$, so that one has (assuming proper normalisation of the modular multiplets $Y_s(\tau)$):

$$\begin{aligned} g_s \left(Y_s(\tau)\psi_1 \dots \psi_n\right)_{\mathbf{1},s} &\xrightarrow{\rm CP} g_s \left(Y_s(-\tau^*)\overline{\psi}_1 \dots \overline{\psi}_n\right)_{\mathbf{1},s} \\ = g_s \left(Y_s^*(\tau)\overline{\psi}_1 \dots \overline{\psi}_n\right)_{\mathbf{1},s} &= g_s \overline{\left(Y_s(\tau)\psi_1 \dots \psi_n\right)_{\mathbf{1},s}} , \end{aligned} \tag{4.35}$$

where in the last equality we have used the reality of the Clebsch-Gordan coefficients, which holds for $N \leq 5$. It is now clear that a term in the sum of (4.34) transforms into the Hermitian conjugate of

$$g_s^* \left(Y_s(\tau)\psi_1 \dots \psi_n\right)_{\mathbf{1},s} , \tag{4.36}$$

which should coincide with the original term due to the independence of singlets in (4.34). It now follows that $g_s = g_s^*$, i.e., all coupling constants g_s have to be real to conserve CP.

As a final remark, let us denote by $\tilde{g}_s$ the couplings written for a general basis and arbitrary normalisation of the modular form multiplets. The CP constraint on $\tilde{g}_s$ is then more complicated, since the singlets of different bases coincide only up to normalisation factors, determined by the choice of normalisations of the Clebsch-Gordan coefficients and of the modular form multiplets. Since the normalisation factors can differ between singlets, the corresponding couplings $\tilde{g}_s$ may require non-trivial phases to conserve CP. These phases can be found directly by performing a basis transformation and matching $\tilde{g}_s$ to g_s in the symmetric basis (and with proper modular form multiplet normalisation).

Implications for the mass matrices

As a more concrete example, let us consider the Yukawa coupling term

$$W_L = \sum_s g_s \left(Y_s(\tau)E^c L H_d\right)_{\mathbf{1},s} , \tag{4.37}$$

which gives rise to the charged-lepton mass matrix. Here E^c is a modular symmetry multiplet of SU(2) charged-lepton singlets, L is a modular symmetry multiplet of SU(2) lepton doublets, and H_d is a Higgs doublet which transforms trivially under modular symmetry and whose neutral component acquires a VEV $v_d = \langle H_d^0 \rangle$ after electroweak symmetry breaking.

Expanding the singlets, one gets

$$W_L = \sum_s g_s \lambda^s_{ij}(\tau) E^c_i L_j H_d \equiv \lambda_{ij}(\tau) E^c_i L_j H_d, \tag{4.38}$$

where entries of the matrices $\lambda^s_{ij}(\tau)$ are formed from components of the corresponding modular multiplets $Y_s(\tau)$. In a general basis, superfields transform under CP as

$$E^c \xrightarrow{\text{CP}} X_R^* \, \overline{E^c}\,, \quad L \xrightarrow{\text{CP}} X_L \, \overline{L}\,, \quad H_d \xrightarrow{\text{CP}} \eta_d \, \overline{H_d}\,, \tag{4.39}$$

and we set $\eta_d = 1$ without loss of generality. It follows that

$$W_L \xrightarrow{\text{CP}} \left(X_R^\dagger \, \lambda(-\tau^*) \, X_L \right)_{ij} \overline{E^c_i} \, \overline{L}_j \, \overline{H_d}\,, \tag{4.40}$$

so that CP conservation implies

$$X_R^\dagger \, \lambda(-\tau^*) \, X_L = \lambda^*(\tau). \tag{4.41}$$

The resulting charged-lepton mass matrix $M_e = v_d \lambda^\dagger$ (written in the left-right convention) satisfies

$$X_L^\dagger \, M_e(-\tau^*) \, X_R = M_e^*(\tau), \tag{4.42}$$

which coincides with the corresponding constraint in the case of CP invariance combined with discrete flavour symmetry, apart from the fact that now the mass matrix depends on the modulus τ which also transforms under CP. Similarly, for the neutrino Majorana mass matrix M_ν one has

$$X_L^T M_\nu(-\tau^*) X_L = M_\nu^*(\tau). \tag{4.43}$$

Note that matrix X_L is the same in (4.42) and (4.43) since left-handed charged leptons l_L and left-handed neutrinos ν_{lL} form an electroweak SU(2) doublet L, so they transform uniformly both under CP and modular transformations:

$$X_{l_L} = X_{\nu_{lL}} \equiv X_L, \quad \rho_{l_L}(\gamma) = \rho_{\nu_{lL}}(\gamma) \equiv \rho_L(\gamma), \quad k_{l_L} = k_{\nu_{lL}} \equiv k_L. \tag{4.44}$$

In a symmetric basis $X_L = X_R = \mathbb{1}$, the constraints on the mass matrices simplify to

$$M_e(-\tau^*) = M_e^*(\tau), \quad M_\nu(-\tau^*) = M_\nu^*(\tau), \tag{4.45}$$

which further reduce to reality of the couplings. Namely, for the charged-lepton mass matrix one has

$$\begin{aligned} M_e(-\tau^*) &= v_d \sum_s g_s^* \, (\lambda^s)^\dagger(-\tau^*) = v_d \sum_s g_s^* \, (\lambda^s)^T(\tau), \\ M_e^*(\tau) &= \left(v_d \sum_s g_s^* \, (\lambda^s)^\dagger(\tau) \right)^* = v_d \sum_s g_s \, (\lambda^s)^T(\tau). \end{aligned} \tag{4.46}$$

Clearly, CP invariance requires $g_s = g_s^*$, since $\lambda^s(\tau)$ are linearly independent matrices, which in turn is guaranteed by independence of the singlets.

5. Residual symmetries [P1, P5]

Modular symmetry is spontaneously broken by the VEV of the modulus τ: in fact, there is no value of τ which is left invariant by the modular group action (2.4). However, certain values of τ (called *symmetric points*) break the modular group Γ only partially, with the unbroken generators giving rise to *residual symmetries*. In this chapter we classify the possible residual symmetry groups following refs. [P1, P5]. These symmetries can play an important role in flavour model building, as discussed later in chapters 6, 9 and 10.

5.1. Symmetric points in the fundamental domain

To classify the possible residual symmetries, one first notices that with a proper "gauge choice" τ can always be restricted to the *fundamental domain* $\mathcal{D}$ of the modular group Γ:

$$\mathcal{D} \equiv \left\{\tau \in \mathcal{H} : -\frac{1}{2} \leq \mathrm{Re}\,\tau < \frac{1}{2},\ |\tau| > 1\right\} \cup \left\{\tau \in \mathcal{H} : -\frac{1}{2} < \mathrm{Re}\,\tau \leq 0,\ |\tau| = 1\right\}, \tag{5.1}$$

depicted in fig. 5.1. The fundamental domain describes all possible values of τ up to a modular transformation. Note that, by convention, the right half of the boundary $\partial\mathcal{D}$ is not included into $\mathcal{D}$, as it is related to the left half by suitable modular transformations. Since the underlying theory enjoys the modular symmetry Γ, all the vacua related by modular transformations are physically equivalent. This means that without loss of generality one can assume that $\tau \in \mathcal{D}$.

In the fundamental domain $\mathcal{D}$, there exist only three symmetric points (see also [134]):

- $\tau_C \equiv i$ invariant under S;
- $\tau_L \equiv \omega \equiv e^{2\pi i/3}$ ("the left cusp") invariant under ST;
- $\tau_T \equiv i\infty$ invariant under T.

In addition, the R generator is unbroken for any value of τ.

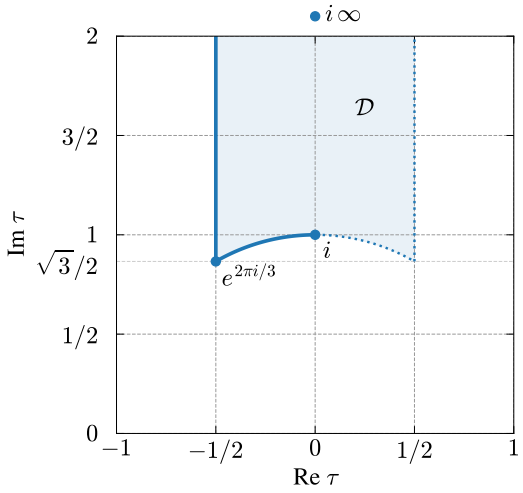

Figure 5.1.: The fundamental domain $\mathcal{D}$ of the modular group Γ, and its three symmetric points $\tau_C = i$, $\tau_L = e^{2\pi i/3}$ and $\tau_T = i\,\infty$.

5.2. CP-conserving points

In a CP-conserving modular-invariant theory both CP and modular symmetry are broken spontaneously by the VEV of the modulus τ. However, there exist certain values of τ which conserve CP. Obviously, this is the case if τ is left invariant by CP:

$$\tau \xrightarrow{\text{CP}} -\tau^* = \tau, \tag{5.2}$$

meaning that τ lies on the imaginary axis, $\text{Re}\,\tau = 0$. In a symmetric basis one then has

$$M_e(\tau) = M_e^*(\tau), \quad M_\nu(\tau) = M_\nu^*(\tau), \tag{5.3}$$

as can be seen from (4.45). The resulting mass matrices are real and the corresponding CPV phases are trivial, such that $\sin\delta = \sin\alpha_{21} = \sin\alpha_{31} = 0$ in the standard parametrisation [135] of the PMNS mixing matrix.

Let us now consider a point $\gamma\tau$ related to a CP-invariant point $\tau = -\tau^*$ by a modular transformation γ. This point is physically equivalent to τ due to modular invariance and therefore it should also be CP-conserving. However, $\gamma\tau$ does not go to itself under CP. Instead, one has

$$\gamma\tau \xrightarrow{\text{CP}} (\gamma\tau)_{\text{CP}} = u(\gamma)\,\tau_{\text{CP}} = u(\gamma)\,\tau = u(\gamma)\gamma^{-1}\gamma\tau, \tag{5.4}$$

so the resulting CP-transformed value $(\gamma\tau)_{\mathrm{CP}}$ is related to the original value $\gamma\tau$ by a modular transformation $u(\gamma)\gamma^{-1}$.

Hence, it is natural to expect that a value of τ conserves CP if it is left invariant by CP *up to a modular transformation*, i.e.,

$$\tau \xrightarrow{\mathrm{CP}} -\tau^* = \gamma\tau \tag{5.5}$$

for some $\gamma \in \Gamma$.[1] Indeed, if we consider a value of τ which satisfies (5.5), then in a symmetric basis we have

$$M_e(\gamma\tau) = M_e^*(\tau), \quad M_\nu(\gamma\tau) = M_\nu^*(\tau), \tag{5.6}$$

from (4.45). Denote by $q(\tau)$ the value of a lepton flavour observable evaluated at τ. Modular invariance requires that $q(\tau) = q(\gamma\tau)$, since τ is physically equivalent to $\gamma\tau$. On the other hand, from (5.6) we find $q(\gamma\tau) = q(\tau)^*$. It follows that the observables evaluated at τ coincide with their complex conjugates, $q(\tau) = q(\tau)^*$, hence CPV phases are trivial (0 or π).

To find all points satisfying (5.5) in the fundamental domain $\mathcal{D}$, let us first notice that its interior, which we denote as $\mathrm{int}(\mathcal{D})$, maps to itself under CP. Apart from that, no two points from $\mathrm{int}(\mathcal{D})$ are related by any non-trivial modular transformation. Therefore, if $\tau \in \mathrm{int}(\mathcal{D})$, then (5.5) reduces to (5.2) and we find again $\mathrm{Re}\,\tau = 0$. The remaining possibility is that τ lies on the boundary of $\mathcal{D}$. Then it is easy to show that it also satisfies (5.5), but with a non-trivial γ. Namely, for the left vertical line we have $\tau = -1/2 + i\,y \;\to\; 1/2 + i\,y = T\tau$ under CP, while for the arc we have $\tau = e^{i\varphi} \;\to\; -e^{-i\varphi} = S\tau$.

To summarise, if a theory is CP-invariant (i.e., its couplings satisfy the constraints discussed in section 4.2), then the CP symmetry is spontaneously broken by any $\tau \in \mathcal{D}$ except for the values lying on the fundamental domain boundary or the imaginary axis:

- $\mathrm{Re}\,\tau = 0$ (the imaginary axis) is invariant under CP;
- $\mathrm{Re}\,\tau = -1/2$ (the left vertical boundary) is invariant under CP T;
- $|\tau| = 1$ (the boundary arc) is invariant under CP S.

Note also that CP always acts on τ as in (4.11), meaning the above statement does not depend on the choice of CP automorphism (CP_1 or CP_2).

[1] A similar condition has been derived in ref. [129] in the context of string theories (in which the CP symmetry represents a discrete gauge symmetry), postulating the action of CP on the compactified directions.

5.3. Classification of the residual symmetry groups

For a given value of τ, the residual symmetry group is simply a group generated by the unbroken transformations subject to relations which can be deduced from (4.23) and (4.28). For instance, the symmetric point $\tau = i$ is invariant under S, R and CP_1 in the case of the full modular group Γ enhanced by CP_1. The corresponding symmetry group is

$$\langle S,\, R,\, \mathrm{CP}_1 \rangle = \left\langle S,\, \mathrm{CP}_1 \,\middle|\, S^4 = \mathbb{1},\, \mathrm{CP}_1^2 = \mathbb{1},\, \mathrm{CP}_1\, S\, \mathrm{CP}_1^{-1} = S^{-1} \right\rangle \simeq D_4\,, \qquad (5.7)$$

where D_4 is the dihedral group of order 8 (the symmetry group of a square). One can find the residual symmetry groups for other values of τ in a similar fashion; we collect the results in table 5.1.

When considering finite modular versions $\Gamma_N^{(\prime)}$ of the modular group, the residual symmetry groups may be reduced, due to the extra relation $T^N = \mathbb{1}$ (recall that for $N > 5$ further constraints are present). For $N \leq 5$, the instances of $\mathbb{Z}^T$ in table 5.1 should be replaced by $\mathbb{Z}_N^T$.

Since every symmetric point outside the fundamental domain $\mathcal{D}$ is physically equivalent to a symmetric point inside $\mathcal{D}$, its residual symmetry group is isomorphic to one of the groups listed in table 5.1.

Table 5.1.: Residual symmetry groups for different values of τ and different choices of the full symmetry group.

Value of τ	Γ	$\overline{\Gamma}$	$\Gamma \rtimes \mathrm{CP}_1$	$\Gamma \rtimes \mathrm{CP}_2$	$\overline{\Gamma} \rtimes \mathrm{CP}$
$\tau = i$	$\mathbb{Z}_4^S$	$\mathbb{Z}_2^S$	$\mathbb{Z}_4^S \rtimes \mathbb{Z}_2^{\mathrm{CP}_1} \simeq D_4$	$\mathbb{Z}_4^S \times \mathbb{Z}_2^{\mathrm{CP}_2 S}$	$\mathbb{Z}_2^S \times \mathbb{Z}_2^{\mathrm{CP}}$
$\tau = e^{2\pi i/3}$	$\mathbb{Z}_3^{ST} \times \mathbb{Z}_2^R$	$\mathbb{Z}_3^{ST}$	$\left(\mathbb{Z}_3^{ST} \rtimes \mathbb{Z}_2^{\mathrm{CP}_1 T}\right) \times \mathbb{Z}_2^R$ $\simeq S_3 \times \mathbb{Z}_2 \simeq D_6$	$\left(\mathbb{Z}_3^{ST} \rtimes \mathbb{Z}_2^{\mathrm{CP}_2 T}\right) \times \mathbb{Z}_2^R$ $\simeq S_3 \times \mathbb{Z}_2 \simeq D_6$	$\mathbb{Z}_3^{ST} \rtimes \mathbb{Z}_2^{\mathrm{CP}\,T} \simeq S_3$
$\tau = i\infty$	$\mathbb{Z}^T \times \mathbb{Z}_2^R$	$\mathbb{Z}^T$	$\left(\mathbb{Z}^T \rtimes \mathbb{Z}_2^{\mathrm{CP}_1}\right) \times \mathbb{Z}_2^R$	$\left(\mathbb{Z}^T \times \mathbb{Z}_2^R\right) \rtimes \mathbb{Z}_2^{\mathrm{CP}_2 T}$	$\mathbb{Z}^T \rtimes \mathbb{Z}_2^{\mathrm{CP}}$
$\mathrm{Re}\,\tau = 0$	$\mathbb{Z}_2^R$	1	$\mathbb{Z}_2^{\mathrm{CP}_1} \times \mathbb{Z}_2^R$	$\mathbb{Z}_4^{\mathrm{CP}_2}$	$\mathbb{Z}_2^{\mathrm{CP}}$
$\lvert\tau\rvert = 1$	$\mathbb{Z}_2^R$	1	$\mathbb{Z}_2^{\mathrm{CP}_1 S} \times \mathbb{Z}_2^R$	$\mathbb{Z}_2^{\mathrm{CP}_2 S} \times \mathbb{Z}_2^R$	$\mathbb{Z}_2^{\mathrm{CP}\,S}$
$\mathrm{Re}\,\tau = -\frac{1}{2}$	$\mathbb{Z}_2^R$	1	$\mathbb{Z}_2^{\mathrm{CP}_1 T} \times \mathbb{Z}_2^R$	$\mathbb{Z}_2^{\mathrm{CP}_2 T} \times \mathbb{Z}_2^R$	$\mathbb{Z}_2^{\mathrm{CP}\,T}$
generic τ	$\mathbb{Z}_2^R$	1	$\mathbb{Z}_2^R$	$\mathbb{Z}_2^R$	1

6. Mass and mixing patterns [P6]

We have seen in the previous chapter that certain residual symmetry groups remain unbroken at symmetric points $\tau = \tau_{\text{sym}}$. For these values of τ, flavour textures can be severely constrained by the residual symmetry group, which may enforce the presence of multiple zero entries in the mass matrices. As τ moves away from its symmetric value, these entries will generically become non-zero.

We show in this chapter that the magnitudes of such (residual-)symmetry-breaking entries are controlled by the size of the departure ϵ from τ_{sym} and by the field transformation properties under the residual symmetry group, which may depend on the modular weights. This idea can be utilized to explain the hierarchical patterns of fermion masses and mixing without fine-tuning of the model parameters.

6.1. Mass hierarchies without fine-tuning

Mass matrices close to symmetric points

Consider a modular-invariant bilinear

$$\psi_i^c \, M(\tau)_{ij} \, \psi_j \,, \tag{6.1}$$

where the superfields ψ and ψ^c transform under the modular group as[1]

$$\begin{aligned} \psi &\xrightarrow{\gamma} (c\tau + d)^{-k} \rho(\gamma) \, \psi, \\ \psi^c &\xrightarrow{\gamma} (c\tau + d)^{-k^c} \rho^c(\gamma) \, \psi^c, \end{aligned} \tag{6.2}$$

so that each $M(\tau)_{ij}$ is a modular form of level N and weight $K \equiv k + k^c$. Modular invariance requires $M(\tau)$ to transform as

$$M(\tau) \xrightarrow{\gamma} M(\gamma\tau) = (c\tau + d)^K \rho^c(\gamma)^* M(\tau) \rho(\gamma)^\dagger. \tag{6.3}$$

[1] Note that in the case of a Dirac bilinear ψ and ψ^c are independent fields, so in general $k^c \neq k$ and $\rho^c \neq \rho, \rho^*$.

Taking τ to be close to the symmetric point, and setting γ to the residual symmetry generator, one can use this transformation rule to constrain the form of the mass matrix $M(\tau)$. We consider each of the three symmetric points in turn.

$\tau_{\text{sym}} = i\infty$

The representation basis for the group generators S and T typically found in the literature is the T-diagonal basis, in which $\rho^{(c)}(T) = \text{diag}\left(\rho_i^{(c)}\right)$. This basis is particularly useful for the analysis of models where τ is "close" to $\tau_{\text{sym}} = i\infty$, i.e., models with large $\text{Im}\,\tau$. By setting $\gamma = T$ in (6.3), one finds

$$M_{ij}(T\tau) = \left(\rho_i^c \rho_j\right)^* M_{ij}(\tau). \tag{6.4}$$

It is convenient to treat the M_{ij} as a function of $q_N \equiv \exp(2\pi i\tau/N)$, so that

$$\epsilon \equiv |q_N| = e^{-2\pi\,\text{Im}\,\tau/N} \tag{6.5}$$

parametrises the deviation of τ from the symmetric point. Note that the entries $M_{ij}(q_N)$ depend analytically on q_N and that $q_N \longrightarrow \zeta_N q_N$ under T, with $\zeta_N \equiv \exp(2\pi i/N)$. Thus, in terms of q_N, (6.4) reads

$$M_{ij}(\zeta_N q_N) = (\rho_i^c \rho_j)^* M_{ij}(q_N). \tag{6.6}$$

Expanding M_{ij} in powers of q_N as $M_{ij}(q_N) = a_0 + a_1\, q_N + a_2\, q_N^2 + \ldots$, one finds

$$\zeta_N^n\, a_n = (\rho_i^c \rho_j)^*\, a_n, \tag{6.7}$$

which means that a_n can only be non-zero for values of n such that $(\rho_i^c \rho_j)^* = \zeta_N^n$.

It is clear that in the symmetric limit $q_N \longrightarrow 0$ the entry $M_{ij} = a_0$ is only allowed to be non-zero if $\rho_i^c \rho_j = 1$. More generally, if $(\rho_i^c \rho_j)^* = \zeta^l$ with $0 \leq l < N$,

$$M_{ij}(q_N) = a_l\, q_N^l + a_{N+l}\, q_N^{N+l} + a_{2N+l}\, q_N^{2N+l} + \ldots \tag{6.8}$$

in the vicinity of the symmetric point. It crucially follows that the entry M_{ij} is expected to be $O(\epsilon^l)$ whenever $\text{Im}\,\tau$ is large. The power l only depends on how the representations of ψ and ψ^c decompose under the residual symmetry group $\mathbb{Z}_N^T$. This point will be made explicit in the next section.

$\tau_{\text{sym}} = i$

For the analysis of models where τ is in the vicinity of $\tau_{\text{sym}} = i$, it is convenient to switch to the basis where the S generator is represented by a diagonal matrix.

In this S-diagonal basis, one has $\rho^{(c)}(S) = \text{diag}\left(\rho_i^{(c)}\right)$.[2] It is useful to define and work with

$$\tilde{\rho}_i^{(c)} \equiv i^{k^{(c)}} \rho_i^{(c)}, \tag{6.9}$$

which not only simplify the algebra, but also correspond to representations of the residual symmetry group, see (6.23). By setting $\gamma = S$ in (6.3), one finds

$$M_{ij}(S\tau) = (-i\tau)^K \left(\tilde{\rho}_i^c \tilde{\rho}_j\right)^* M_{ij}(\tau). \tag{6.10}$$

We now treat the M_{ij} as functions of

$$s \equiv \frac{\tau - i}{\tau + i}, \tag{6.11}$$

so that, in this context, $\epsilon \equiv |s|$ parametrises the deviation of τ from the symmetric point. Note that the entries $M_{ij}(s)$ depend analytically on s and that $s \xrightarrow{S} -s$. Thus, in terms of s, (6.10) reads

$$M_{ij}(-s) = \left(\frac{1+s}{1-s}\right)^K (\tilde{\rho}_i^c \tilde{\rho}_j)^* M_{ij}(s) \;\Rightarrow\; \tilde{M}_{ij}(-s) = (\tilde{\rho}_i^c \tilde{\rho}_j)^* \tilde{M}_{ij}(s), \tag{6.12}$$

where we have introduced $\tilde{M}_{ij}(s) \equiv (1-s)^{-K} M_{ij}(s)$. Expanding $\tilde{M}_{ij}$ in powers of s as $\tilde{M}_{ij}(s) = a_0 + a_1 s + a_2 s^2 + \ldots$, one obtains

$$(-1)^n a_n = (\tilde{\rho}_i^c \tilde{\rho}_j)^* a_n. \tag{6.13}$$

It should be clear from (6.13) that for $\tau \simeq i$ the mass matrix entry $M_{ij} \sim \tilde{M}_{ij}$ is only allowed to be $O(1)$ when $\tilde{\rho}_i^c \tilde{\rho}_j = 1$. If instead $\tilde{\rho}_i^c \tilde{\rho}_j = -1$, the entry $M_{ij} \sim \tilde{M}_{ij}$ is expected to be $O(\epsilon)$, with $\epsilon = |s|$. Note that, unlike the case $\tau_{\text{sym}} = i\infty$, the relevant factors $\tilde{\rho}_i^{(c)}$ depend on the weights $k^{(c)}$ via (6.9).

$\tau_{\text{sym}} = \omega$

Finally, for the analysis of models where τ is in the vicinity of $\tau_{\text{sym}} = \omega$, we consider the basis where the product ST is represented by a diagonal matrix. In this ST-diagonal basis where $\rho^{(c)}(ST) = \text{diag}\left(\rho_i^{(c)}\right)$, it is useful to define

$$\tilde{\rho}_i^{(c)} \equiv \omega^{k^{(c)}} \rho_i^{(c)}, \tag{6.14}$$

which are representations under the residual symmetry group, see (6.24). By setting $\gamma = ST$ in (6.3), one finds

$$M_{ij}(ST\tau) = [-\omega(\tau+1)]^K \left(\tilde{\rho}_i^c \tilde{\rho}_j\right)^* M_{ij}(\tau). \tag{6.15}$$

[2] Although we make use of the same notation, the $\rho_i^{(c)}$ depend on the basis under consideration.

It is now convenient to treat the M_{ij} as functions of

$$u \equiv \frac{\tau - \omega}{\tau - \omega^2}, \tag{6.16}$$

so that, in this context, $\epsilon \equiv |u|$ parametrises the deviation of τ from the symmetric point. Note that the entries $M_{ij}(u)$ depend analytically on u and that $u \to \omega^2 u$ under ST. Thus, in terms of u, (6.15) reads

$$M_{ij}(\omega^2 u) = \left(\frac{1-\omega^2 u}{1-u}\right)^K (\tilde{\rho}_i^c \tilde{\rho}_j)^* M_{ij}(u) \;\Rightarrow\; \tilde{M}_{ij}(\omega^2 u) = (\tilde{\rho}_i^c \tilde{\rho}_j)^* \tilde{M}_{ij}(u), \tag{6.17}$$

where $\tilde{M}_{ij}(u) \equiv (1-u)^{-K} M_{ij}(u)$. Expanding $\tilde{M}_{ij}$ in powers of u as $\tilde{M}_{ij}(u) = a_0 + a_1\, u + a_2\, u^2 + \ldots$, one obtains

$$\omega^{2n}\, a_n = (\tilde{\rho}_i^c \tilde{\rho}_j)^*\, a_n. \tag{6.18}$$

It follows that for $\tau \simeq \omega$ the mass matrix entry $M_{ij} \sim \tilde{M}_{ij}$ is only allowed to be $O(1)$ when $\tilde{\rho}_i^c \tilde{\rho}_j = 1$. More generally, if $\tilde{\rho}_i^c \tilde{\rho}_j = \omega^l$ with $l = 0, 1, 2$, then the entry $M_{ij} \sim \tilde{M}_{ij}$ is expected to be $O(\epsilon^l)$ in the vicinity of $\tau = \omega$, with $\epsilon = |u|$. Like in the case $\tau_{\rm sym} = i$, the factors $\tilde{\rho}_i^{(c)}$ depend on the weights $k^{(c)}$, see (6.14).

Decomposition under residual symmetries

We have just shown that, as τ departs from a symmetric value $\tau_{\rm sym}$, the entries of fermion mass matrices are of $O(\epsilon^l)$, where ϵ parametrises the deviation of τ from $\tau_{\rm sym}$. The powers l are extracted from products of factors which, in this section, are shown to correspond to representations of the residual symmetry group. One can systematically identify these residual symmetry representations for the different possible choices of Γ'_N representations of matter fields. This knowledge will later be exploited to construct hierarchical mass matrices via controlled corrections to entries which are zero in the symmetric limit.

We start by noting that matter fields ψ furnish "weighted" representations $(\mathbf{r}, k)$ of the finite modular group Γ'_N. Whenever a residual symmetry is preserved by the value of τ, matter fields decompose into unitary representations of the residual symmetry group. Modulo a possible $\mathbb{Z}_2^R$ factor,[3] these groups are the cyclic groups $\mathbb{Z}_N^T$, $\mathbb{Z}_4^S$, and $\mathbb{Z}_3^{ST}$ (see chapter 5). A cyclic group $\mathbb{Z}_n \equiv \langle a \mid a^n = 1\rangle$ has n inequivalent 1-dimensional irreps $\mathbf{1}_k$, where $k = 0, \ldots, n-1$ is sometimes

[3] See the discussion in appendix C.

referred to as a charge. The group generator a is represented by one of the n-th roots of unity,

$$\mathbf{1}_k \ : \quad \rho(a) = \exp\left(2\pi i \frac{k}{n}\right). \tag{6.19}$$

For odd n, the only real irrep of $\mathbb{Z}_n$ is the trivial one, 1_0 (the reality of an irrep is indicated by removing the boldface). For even n, there is one more real irrep, $1_{n/2}$. All other irreps are complex, and split into pairs of conjugated irreps: $(\mathbf{1}_k)^* = \mathbf{1}_{n-k}$.

To illustrate the aforementioned decomposition of representations at symmetric points, we take as an example a $(\mathbf{3}, k)$ triplet ψ of S_4'. It transforms under the unbroken $\gamma = ST$ at $\tau = \omega$ as

$$\psi_i \xrightarrow{ST} (-\omega - 1)^{-k} \rho_{\mathbf{3}}(ST)_{ij}\, \psi_j = \omega^k \rho_{\mathbf{3}}(ST)_{ij}\, \psi_j \,. \tag{6.20}$$

One can check that the eigenvalues of $\rho_{\mathbf{3}}(ST)$ are 1, ω and ω^2, and so in a suitable (ST-diagonal) basis the transformation rule explicitly reads

$$\psi \xrightarrow{ST} \omega^k \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \psi = \begin{pmatrix} \omega^k & 0 & 0 \\ 0 & \omega^{k+1} & 0 \\ 0 & 0 & \omega^{k+2} \end{pmatrix} \psi, \tag{6.21}$$

which means that ψ decomposes as $\psi \rightsquigarrow \mathbf{1}_k \oplus \mathbf{1}_{k+1} \oplus \mathbf{1}_{k+2}$ under the residual $\mathbb{Z}_3^{ST}$.

One can find the residual symmetry representations for any other "weighted" multiplet of a finite modular group in a similar fashion. For a given level N, the decompositions of fields under a certain residual symmetry group only depend on the pair $(\mathbf{r}, k)$. In general:

- At $\tau = i\infty$, $\psi \sim (\mathbf{r}, k)$ transforms under the unbroken $\gamma = T$ as

  $$\psi_i \xrightarrow{T} \rho_{\mathbf{r}}(T)_{ij}\, \psi_j = \rho_i\, \psi_i \,, \tag{6.22}$$

  where for the last equality we have assumed to be in a T-diagonal basis (no sum over i). The phase factors ρ_i correspond to the $\mathbb{Z}_N^T$ irreps into which ψ decomposes. It follows that each ρ_i is a power of $\zeta = \exp(2\pi i/N)$, depending on $\mathbf{r}$ but not on k.

- At $\tau = i$, $\psi \sim (\mathbf{r}, k)$ transforms under the unbroken $\gamma = S$ as

  $$\psi_i \xrightarrow{S} (-i)^{-k} \rho_{\mathbf{r}}(S)_{ij}\, \psi_j = i^k \rho_i\, \psi_i \,, \tag{6.23}$$

where for the last equality we have assumed to be in an S-diagonal basis (no sum over i). The phase factors $\tilde{\rho}_i = i^k \rho_i$ correspond to the $\mathbb{Z}_4^S$ irreps into which ψ decomposes. It follows that each $\tilde{\rho}_i$ is a power of i which depends both on $\mathbf{r}$ and on k (mod 4).

- At $\tau = \omega$, $\psi \sim (\mathbf{r}, k)$ transforms under the unbroken $\gamma = ST$ as

$$\psi_i \xrightarrow{ST} (-\omega - 1)^{-k} \rho_{\mathbf{r}}(ST)_{ij}\, \psi_j = \omega^k \rho_i\, \psi_i\,, \tag{6.24}$$

where for the last equality we have assumed to be in an ST-diagonal basis (no sum over i), as in the explicit example of (6.21). The phase factors $\tilde{\rho}_i = \omega^k \rho_i$ correspond to the $\mathbb{Z}_3^{ST}$ irreps into which ψ decomposes. It follows that each $\tilde{\rho}_i$ is a power of ω which depends both on $\mathbf{r}$ and on k (mod 3).

After identifying the $\tilde{(\rho)}_i$ and $\tilde{(\rho)}_i^c$ factors for the fields ψ and ψ^c entering a bilinear (equivalently, their irrep decompositions), one can apply the results of the previous section to determine the structure of a mass matrix in the vicinity of a symmetric point in terms of powers of ϵ, in the appropriate basis. It follows from the above that, in the analysis with large $\mathrm{Im}\,\tau$, the product $(\rho_i^c \rho_j)^*$ matches some power ζ_N^l with $0 \leq l < N$, while in the analysis corresponding to $\tau \simeq \omega$ one necessarily has $\tilde{\rho}_i^c \tilde{\rho}_j = \omega^l$ with $l = 0, 1, 2$. These were tacitly taken as the most general possibilities in section 6.1.

The same reasoning implies that, in the $\tau \simeq i$ context, $\tilde{\rho}_i^c \tilde{\rho}_j$ is some integer power i^l, with $l = 0, 1, 2, 3$. It turns out that only two out of the four possibilities are viable, namely $l = 0, 2$ so that $\tilde{\rho}_i^c \tilde{\rho}_j = \pm 1$, as considered in section 6.1. This is due to the fact that $M(\tau)_{ij}$ is R-even and thus the fields ψ_i^c and ψ_j need to carry the same R-parity (see also appendix C).

We list in tables C.1 to C.4 of appendix C the decompositions of the weighted representations of Γ'_N ($N \leq 5$) under the three residual symmetry groups.

Hierarchical structures

We are in a position to use the results found so far and construct hierarchical mass matrices in the vicinity of a symmetric point. We have seen that in an appropriate basis $M(\tau(\epsilon))_{ij} \sim \mathcal{O}(\epsilon^l)$. For each (i, j) pair, the power l can be obtained from the residual symmetry group decompositions of tables C.1 to C.4.

Note that a modular-symmetric mass matrix $M(\tau(\epsilon))$ depends analytically on the small real parameter ϵ, defined in section 6.1 for each symmetric point.

Physical masses are the singular values of $M(\tau)$ and are also analytic functions of ϵ.[4] After the modular symmetry breaking, the leading superpotential contribution to each fermion mass is thus expected to be proportional to a power of ϵ which depends on the hierarchical structure of the entries of M. To find out which, one can make use of the following set of relations, valid for any $n \times n$ complex matrix M [137]:

$$\sum_{i_1<\ldots<i_p} m_{i_1}^2 \ldots m_{i_p}^2 = \sum \left|\det M_{p\times p}\right|^2 , \tag{6.25}$$

where $p = 1, \ldots, n$ is fixed, m_i are the singular values of M, and the sum on the right-hand side goes over all possible $p \times p$ submatrices $M_{p\times p}$ of M. In the particular case of $n = 3$, we denote the masses by (m_1, m_2, m_3) such that their leading terms are respectively of order $(\epsilon^{d_1}, \epsilon^{d_2}, \epsilon^{d_3})$ with $d_1 \geq d_2 \geq d_3 \geq 0$. Then,

$$\begin{aligned} m_3^2 &\sim \sum_{i,j} |M_{ij}|^2 = \mathrm{Tr}\, M^\dagger M, \\ m_2^2 m_3^2 &\sim \sum |\det M_{2\times 2}|^2 \quad \Rightarrow \quad m_2^2 \sim \frac{\sum |\det M_{2\times 2}|^2}{\mathrm{Tr}\, M^\dagger M}, \\ m_1^2 m_2^2 m_3^2 &= |\det M|^2 \qquad\quad \Rightarrow \quad m_1^2 \sim \frac{|\det M|^2}{\sum |\det M_{2\times 2}|^2}, \end{aligned} \tag{6.26}$$

where $\sim$ refers to power counting in ϵ and not necessarily to a reliable approximation. Note that so far the considered mass spectrum is generic. This is to be contrasted with the special case of a *hierarchical* 3×3 mass matrix, for which $d_1 > d_2 > d_3 \geq 0$ and thus $m_1 \ll m_2 \ll m_3$. In this case, (6.26) turns into useful approximation,

$$\begin{aligned} m_3^2 &\simeq \sum_i m_i^2 = \mathrm{Tr}\, M^\dagger M, \\ m_2^2 m_3^2 &\simeq \sum_{i<j} m_i^2 m_j^2 = \frac{1}{2}\left((\mathrm{Tr}\, M^\dagger M)^2 - \mathrm{Tr}(M^\dagger M)^2\right), \end{aligned} \tag{6.27}$$

and lead to reliable expressions for m_3, m_2 and $m_1 = |\det M|/(m_2 m_3)$.

As an example of application of our results, consider a model at level $N = 5$ with τ having a large imaginary part and with matter fields in weighted

[4] More precisely, the elements of the unordered tuple of non-zero singular values are absolute values of analytic functions of ϵ, see theorem 4.3.17 in ref. [136].

representations $\psi \sim (\mathbf{3}, k)$ and $\psi^c \sim (\mathbf{3}', k^c)$. From table C.4 one sees that $\psi \rightsquigarrow 1_0 \oplus \mathbf{1}_1 \oplus \mathbf{1}_4$ and $\psi^c \rightsquigarrow 1_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_3$ under the residual group at the symmetric point $\tau_{\text{sym}} = i\infty$. One can then identify $\rho_i = \text{diag}(1, \zeta, \zeta^4)$ and $\rho_i^c = \text{diag}(1, \zeta^2, \zeta^3)$, with $\zeta = \exp(2\pi i/5)$, which allows for the structure

$$M(\tau(\epsilon)) \sim \begin{pmatrix} 1 & \epsilon^4 & \epsilon \\ \epsilon^3 & \epsilon^2 & \epsilon^4 \\ \epsilon^2 & \epsilon & \epsilon^3 \end{pmatrix}, \quad \text{with } \epsilon = e^{-2\pi \,\text{Im}\,\tau/5}. \tag{6.28}$$

Resorting to (6.26), one finds that spectrum is hierarchical, with $(m_3, m_2, m_1) \sim (1, \epsilon, \epsilon^4)$.

Note that to have a non-zero mass matrix one needs the sum $K = k + k^c$ to be even (in this case), since matter fields furnish unhatted representations of the finite modular group and should carry the same R-parity (see appendix C). Furthermore, in order to obtain the full structure of (6.28) and the expected hierarchical spectrum, K must be large enough that sufficient modular forms contribute to $M(\tau)$. For instance, for $K = 2$ the superpotential may turn out to include a unique contribution:

$$W \supset \alpha \left(Y_{\mathbf{5}}^{(5,2)}(\tau)\psi^c\psi\right)_{\mathbf{1}} \Rightarrow M(\tau) = \alpha \begin{pmatrix} \sqrt{3}Y_1 & Y_5 & Y_2 \\ Y_4 & -\sqrt{2}Y_3 & -\sqrt{2}Y_5 \\ Y_3 & -\sqrt{2}Y_2 & -\sqrt{2}Y_4 \end{pmatrix}_{Y_{\mathbf{5}}^{(5,2)}}, \tag{6.29}$$

where α is the coupling constant, and the rightmost matrix subscript indicates the multiplet to which the Y_i components belong. We have considered the T-diagonal basis for A_5'. One can see from (3.25) that, at leading order in $\epsilon = |q_5|$, the components of $Y_{\mathbf{5}}^{(5,2)}(\tau)$ read $(Y_1, Y_2, Y_3, Y_4, Y_5) \propto \left(-1/\sqrt{6},\, q_5,\, 3q_5^2,\, 4q_5^3,\, 7q_5^4\right)$. The power structure matches that of (6.28) and naively this corresponds to the desired $(1, \epsilon, \epsilon^4)$ spectrum. Upon closer inspection, however, one realises that the determinant of M vanishes identically for *any* value of τ,

$$\det M \propto \sqrt{6}\,Y_1Y_3Y_4 - Y_2^2Y_4 + Y_2\left(Y_3^2 - \sqrt{6}\,Y_1Y_5\right) + Y_5\left(Y_4^2 - Y_3Y_5\right) = 0\,, \tag{6.30}$$

meaning that at least one fermion is massless. In the vicinity of $\tau_{\text{sym}} = i\infty$, we have $(m_3, m_2, m_1) \sim (1, \epsilon, 0)$. This issue is resolved already at weight $K = 4$, for which the modular multiplets $Y_{\mathbf{4}}^{(5,4)}$, $Y_{\mathbf{5},1}^{(5,4)}$, and $Y_{\mathbf{5},2}^{(5,4)}$ are available. In this case the spectrum follows a $(1, \epsilon, \epsilon^4)$ pattern, without a massless fermion.

Let us pause and describe our philosophy going forward. We are interested in identifying 3×3 hierarchical mass matrices where the hierarchical pattern is a result of the proximity of the modulus to a point of residual symmetry and no massless fermions are present in the spectrum.

We assume to be dealing with bilinears of the type (6.1) and consider all possible 3-dimensional representations for the fields ψ and ψ^c. While the representations $\mathbf{r}$ and $\mathbf{r}^c$ are in general reducible, we focus on the case where the same weight is shared between the irreps into which they decompose.[5]

Thus, in our search, we take $\mathbf{r}^{(c)}$ to be either irreducible or a direct sum of irreps sharing the same $\rho(R)$. While it is possible for $\mathbf{r}^{(c)}$ to be a direct sum of hatted and unhatted representations, the requirement of a common weight $k^{(c)}$ would result in the co-existence of R-odd and R-even fields within $\psi^{(c)}$. The fact that $M(\tau)$ is R-even would then imply the isolation of these sectors by the $\mathbb{Z}_2^R$ symmetry and the vanishing of some mixing angles.

Finally, it is straightforward to recognise that if all mass matrix entries are either $O(1)$ or $O(\epsilon)$, then leading contributions to the masses themselves are not expected to be smaller than $O(\epsilon)$, unless one resorts to cancellations. Therefore, for $\tau \simeq i$ one cannot produce the desired hierarchical patterns solely as a consequence of the smallness of ϵ.

The result of our analysis is given in tables D.1 to D.4 of appendix D. These tables summarise, for each of the levels $N \leq 5$, the patterns which may arise in the vicinity of the two potentially viable symmetric points, $\tau_{\text{sym}} = \omega$ and $i\infty$, for all $(\mathbf{r}, \mathbf{r}^c)$ pairs of 3-dimensional representations and all weights $k^{(c)}$. One finds that it is only possible to obtain *hierarchical* spectra for a small list of representation pairs, the most promising of which are collected here, in table 6.1.

We have excluded from this summary table reducible representations made up of three copies of the same singlet, since in those cases at least three independent modular multiplets of the same type must contribute to the mass matrix to avoid a massless fermion, and the number of superpotential parameters is unappealingly high. Still, such cases may result in other interesting hierarchical patterns such as $(1, \epsilon^2, \epsilon^3)$ and $(\epsilon, \epsilon^2, \epsilon^3)$ and can be found in the tables of appendix D.

[5] The freedoms in choosing i) the normalisations of modular multiplets and ii) the normalisations of Clebsch-Gordan coefficients introduce ambiguities in the identification of hierarchies. In the interest of minimizing their impact, when possible we make use of modular form multiplets obtained from tensor products of a single $k = 1$ multiplet with itself, via canonically normalised Clebsch-Gordan coefficients. Using this procedure, one expects that relative normalisations of modular multiplets cannot be responsible for hierarchies, at least within the same weight k.

Table 6.1.: Hierarchical mass patterns which can be realised in the vicinity of symmetric points. These patterns are unaffected by the exchange $\mathbf{r} \leftrightarrow \mathbf{r}^c$ and may only be viable for certain weights (see appendix D). Subscripts run over irreps of a certain dimension, and $\mathbf{1}'''_a = \mathbf{1}_a$ for $N = 3$, while $\mathbf{1}''_a = \mathbf{1}_a$ for $N = 4$. Primes in parenthesis are uncorrelated.

N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r} \otimes \mathbf{r}^c$
2	S_3	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{2} \oplus \mathbf{1}^{(\prime)}] \otimes [\mathbf{1} \oplus \mathbf{1}^{(\prime)} \oplus \mathbf{1}']$
3	A'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{1}_a \oplus \mathbf{1}_a \oplus \mathbf{1}'_a] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}''_b]$
			$\tau \simeq i\infty$	$[\mathbf{1}_a \oplus \mathbf{1}_a \oplus \mathbf{1}'_a] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}''_b]$ with $\mathbf{1}_a \neq (\mathbf{1}_b)^*$
4	S'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{3}_a$, or $\mathbf{2} \oplus \mathbf{1}^{(\prime)}$, or $\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}^{(\prime)}]$ $\otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}'_b]$
		$(1, \epsilon, \epsilon^3)$	$\tau \simeq i\infty$	$\mathbf{3} \otimes [\mathbf{2} \oplus \mathbf{1}$, or $\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}']$, $\mathbf{3}' \otimes [\mathbf{2} \oplus \mathbf{1}'$, or $\mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}']$, $\hat{\mathbf{3}}' \otimes [\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$, or $\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}']$, $\hat{\mathbf{3}} \otimes [\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$, or $\hat{\mathbf{1}} \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}']$
5	A'_5	$(1, \epsilon, \epsilon^4)$	$\tau \simeq i\infty$	$\mathbf{3} \otimes \mathbf{3}'$

6.2. Large mixing angles without fine-tuning

We have seen in the previous sections that a slightly broken residual modular symmetry allows to accommodate hierarchical charged-lepton masses without fine-tuning of the corresponding couplings. However, the resulting models are still subject to fine-tuning in the neutrino sector, since residual symmetries typically constrain not only the charged-lepton masses, but also the form of the PMNS matrix by forcing some of its entries to be zeros. This raises the question of whether it is possible to have a PMNS matrix which is close to the observed one even in the symmetric limit, i.e., such that either none of its entries vanish, or only the (13) entry vanishes as $\epsilon \to 0$.

This possibility has been investigated in ref. [138] for arbitrary flavour symmetry groups. In particular, this analysis directly applies to the case of the flavour symmetry being a residual modular symmetry. One of the main conclusions

of ref. [138] is that only a limited number of flavour symmetry representation choices for L and E^c give rise to a PMNS matrix which is viable in the symmetric limit (as defined above). Most notably, there are only two such cases consistent with hierarchical charged-lepton masses:

1. $L \rightsquigarrow 1 \oplus 1 \oplus 1$, $E^c \rightsquigarrow 1 \oplus r$, where 1 is some real singlet of the flavour symmetry, and r is some (possibly reducible) representation such that $r \not\supset 1$;

2. $L \rightsquigarrow \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}^*$, $E^c \rightsquigarrow \mathbf{1}^* \oplus r$, where $\mathbf{1}$ is some complex singlet of the flavour symmetry, $\mathbf{1}^*$ is its conjugate, and r is some (possibly reducible) representation such that $r \not\supset \mathbf{1}, \mathbf{1}^*$.

The above original result makes use of the assumption that one charged-lepton mass and at least one neutrino mass does not vanish in the symmetric limit. However, one can also deduce from the analysis performed in [138] that the PMNS matrix is generically unconstrained in the symmetric limit when the opposite is true. Therefore, we extend the list of viable cases with the following two:

3. all charged-lepton masses vanish in the symmetric limit, i.e., the corresponding hierarchical pattern involves only positive powers of ϵ, e.g. $(\epsilon, \epsilon^2, \epsilon^3)$;

4. all light neutrino masses vanish in the symmetric limit, i.e., L decomposes into three (possibly identical) complex singlets none of which are conjugated to each other.

It follows that a modular-symmetric model of lepton flavour with hierarchical charged-lepton masses may be free of fine-tuning if it satisfies any of the properties 1–4. Applying this filter to the promising hierarchical cases of table 6.1, one is left with the representation pairs listed here, in table 6.2. In this summary table, we have once again disregarded reducible representations made up of three copies of the same singlet. We proceed by constructing such a model in chapter 10.

As a final remark, we note that the argument of ref. [138] is only valid in the case when the flavour symmetry analysis can be applied directly to the light neutrino mass matrix. In our setup, this corresponds to the situation when light neutrino masses arise either directly from a modular-invariant Weinberg

Table 6.2.: Hierarchical charged-lepton mass patterns which may be realised in the vicinity of symmetric points without fine-tuned mixing (PMNS close to the observed one in the symmetric limit). The property which is satisfied (from 1–4, see text) is given in the last column and may depend on the weights k and k^c. The case $N = 3$ with $\tau \simeq \omega$ is the only one in the table for which $\mathbf{r}_{E^c} \leftrightarrow \mathbf{r}_L$ may be required, and for which not all $k^{(c)}$ choices are viable. For other notation, see the caption of table 6.1.

N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r}_{E^c} \otimes \mathbf{r}_L$	Case
2	S_3	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{2} \oplus \mathbf{1}^{(\prime)}] \otimes [\mathbf{1} \oplus \mathbf{1}^{(\prime)} \oplus \mathbf{1}']$	1 or 4
3	A'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{1}_a \oplus \mathbf{1}_a \oplus \mathbf{1}'_a] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}''_b]$	2
			$\tau \simeq i\infty$	$[\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}'] \otimes [\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}']$, $[\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}''] \otimes [\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'']$	2
4	S'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{3}_a$, or $\mathbf{2} \oplus \mathbf{1}^{(\prime)}$, or $\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}^{(\prime)}] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}'_b]$	1 or 4
5	A'_5	–	–	–	–

operator, or via a type-I seesaw UV completion such that none of the gauge-singlet neutrinos N^c becomes massless in the symmetric limit (so that they can be integrated out). This is the case for the model described later in chapter 10.

Part III.

Phenomenology

7. General setup [P1]

In this part of the thesis we describe several viable models of lepton flavour based on the theoretical developments of part II. We start our discussion with the general model-building setup. In section 7.1 we describe the field content of our models, transformation properties of these fields and the origin of lepton masses. Next, in section 7.2 we discuss potential sources of corrections and show that they are negligible in a sizeable region of the parameter space. Finally, in section 7.3 we describe our numerical procedure which we use to fit the models to the experimental data and to explore the viable regions in the parameter space. These methods will be employed to analyse the models presented later in chapters 8 to 10.

7.1. Field content

We consider models of lepton masses and mixings in an MSSM-like (minimal supersymmetric Standard Model) setting. The relevant matter superfields are collected in table 7.1. For simplicity, we take the Higgs fields H_u and H_d to be singlets under the modular group. Charged-lepton masses are obtained from their Yukawa interactions,

$$W \supset \sum_s \alpha_s \left(Y_{\mathbf{r}_s}^{(N,k_Y)}(\tau)\, E^c\, L\right)_{\mathbf{1},s} H_d, \tag{7.1}$$

where L and E^c denote the lepton doublet and the charged-lepton singlet superfields with weights k_L and k_E, respectively. Neutrino masses are generated

Table 7.1.: Chiral matter fields and their transformation properties under the electroweak gauge group and the finite modular group.

	E^c	N^c	L	H_d	H_u
$\mathrm{SU(2)_L \times U(1)_Y}$	$(\mathbf{1}, +1)$	$(\mathbf{1}, 0)$	$(\mathbf{2}, -1/2)$	$(\mathbf{2}, -1/2)$	$(\mathbf{2}, +1/2)$
Γ'_N	ρ_E	ρ_N	ρ_L	$\mathbf{1}$	$\mathbf{1}$
k_I	k_E	k_N	k_L	0	0

either by the Weinberg operator,

$$W \supset \frac{1}{\Lambda} \sum_s g_s \left(Y_{\mathbf{r}_s}^{(N,k_W)}(\tau)\, L^2 \right)_{\mathbf{1},s} H_u^2, \tag{7.2}$$

or within a type-I seesaw UV completion,

$$W \supset \sum_s g_s \left(Y_{\mathbf{r}_s}^{(N,k_\mathcal{Y})}(\tau)\, N^c\, L \right)_{\mathbf{1},s} H_u \,+\, \sum_s \Lambda_s \left(Y_{\mathbf{r}_s}^{(N,k_M)}(\tau)\, (N^c)^2 \right)_{\mathbf{1},s}, \tag{7.3}$$

where at least 2 neutrino gauge-singlet superfields N^c of weight k_N are present in the model. To compensate the modular weights of field monomials, the modular forms entering the Weinberg and Majorana terms need to have weights $k_W = 2k_L$ and $k_M = 2k_N$, while those in Yukawa terms need instead $k_Y = k_L + k_E$ and $k_\mathcal{Y} = k_L + k_N$.

The relevant superpotentials can be cast in the form

$$W \,=\, \lambda_{ij}\, E_i^c\, L_j\, H_d + \begin{cases} \dfrac{1}{2}\, c_{ij}\, L_i\, L_j\, H_u^2 & \text{(Weinberg)} \\ \mathcal{Y}_{ij}\, N_i^c\, L_j + \dfrac{1}{2}\, (M_N)_{ij}\, N_i^c\, N_j^c & \text{(Seesaw)} \end{cases} . \tag{7.4}$$

After electroweak symmetry breaking, with $\langle H_u \rangle = (0, v_u)^T$ and $\langle H_d \rangle = (v_d, 0)^T$, these result in the Lagrangian mass terms for leptons

$$\mathcal{L} \,\supset\, -\big(M_e\big)_{ij}\, \overline{e_{iL}}\, e_{jR} - \frac{1}{2}\, \big(M_\nu\big)_{ij}\, \overline{\nu_{iR}^c}\, \nu_{jL} + \text{h.c.}, \tag{7.5}$$

which have been written in terms of four-spinors. Here, $M_e = v_d \lambda^\dagger$, while

$$M_\nu \,=\, \begin{cases} v_u^2\, c & \text{(Weinberg)} \\ -v_u^2\, \mathcal{Y}^T\, M_N^{-1}\, \mathcal{Y} & \text{(Seesaw)} \end{cases} . \tag{7.6}$$

7.2. Potential sources of corrections

Predictions extracted from modular-invariant theories of flavour correspond to some high energy scale Λ. Therefore, they may strongly depend on the SUSY-breaking effects, and the RG (renormalisation group) running. These effects were analysed in detail in ref. [14].

As far as SUSY-breaking effects are concerned, it was demonstrated in ref. [14] that corrections to masses and mixing which may not be absorbed in a redefinition of superpotential parameters can still be made negligible, provided one realises a sufficient separation between i) the scale M of communication of SUSY-breaking effects to the visible sector and ii) the characteristic scale $m_{\rm SUSY} \sim F/M$ of the soft terms, with F being the spurion VEV assumed to parameterise the breaking of SUSY. Asking for such a gap does not hinder dramatically the choice of possible values for $m_{\rm SUSY}$.

RG effects on neutrino mixing parameters strongly depend on i) $\tan\beta \equiv v_u/v_d$ and ii) the absolute neutrino mass scale $m_{\rm min}$. The effects generically become larger when either $\tan\beta$ or $m_{\rm min}$ are increased (see, e.g., [139]). Furthermore, for the IO (inverted ordering) neutrino mass spectrum, these effects can be sizeable even for $m_{\rm min} \to 0$, since in this case the one-loop β-functions for θ_{12} and δ are enhanced by $\Delta m^2_{23}/\Delta m^2_{21}$ independently of $m_{\rm min}$ (see table 2 in [139]).

It has been found in ref. [14] that for a model predicting the NO (normal ordering) spectrum of neutrino masses with $m_{\rm min} \approx 0.01$ eV, the RG effects on the predictions of the neutrino parameters are negligible even for relatively large value of $\tan\beta = 25$. All three models considered in this thesis lead to the NO spectrum and are allowed to have $m_{\rm min} \lesssim 0.01$ eV. Thus, in what follows we assume to be in a regime in which RG corrections to masses and mixings are negligible.

In general, one should also take into account threshold corrections. They depend on the specific SUSY spectrum and, as argued in ref. [14], can be rendered unimportant. This naturally happens if $\tan\beta$ is small.

7.3. Numerical procedure

As we shall see, each of the investigated models depends on a set of dimensionless parameters

$$p_i = (\tau,\ \beta/\alpha,\ \gamma/\alpha,\ g'/g,\ \ldots,\ \Lambda'/\Lambda,\ \ldots), \tag{7.7}$$

which determine dimensionless observables (mass ratios, mixing angles and phases), and two overall mass scales: one for the charged-lepton mass matrix M_e and one for the neutrino mass matrix M_ν. Phenomenologically viable models are those that lead to values of observables which are in close agreement with

Table 7.2.: Best-fit values and 1σ ranges for neutrino oscillation parameters, obtained from the global analysis of ref. [140], and for charged-lepton mass ratios, given at the scale 2×10^{16} GeV with the $\tan\beta$ averaging described in [12], obtained from ref. [141]. The parameters entering the definition of r are $\delta m^2 \equiv m_2^2 - m_1^2$ and $\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$.

Observable	Best fit value and 1σ range	
m_e/m_μ	0.0048 ± 0.0002	
m_μ/m_τ	0.0565 ± 0.0045	
	NO	IO
$\delta m^2/(10^{-5}\ \mathrm{eV}^2)$	$7.34^{+0.17}_{-0.14}$	
$\lvert\Delta m^2\rvert/(10^{-3}\ \mathrm{eV}^2)$	$2.485^{+0.029}_{-0.032}$	$2.465^{+0.030}_{-0.031}$
$r \equiv \delta m^2/\lvert\Delta m^2\rvert$	0.0295 ± 0.0008	0.0298 ± 0.0008
$\sin^2\theta_{12}$	$0.305^{+0.014}_{-0.013}$	$0.303^{+0.014}_{-0.013}$
$\sin^2\theta_{13}$	$0.0222^{+0.0006}_{-0.0008}$	$0.0223^{+0.0007}_{-0.0006}$
$\sin^2\theta_{23}$	$0.545^{+0.020}_{-0.047}$	$0.551^{+0.016}_{-0.034}$
δ/π	$1.28^{+0.38}_{-0.18}$	$1.52^{+0.13}_{-0.15}$

the experimental results summarised in table 7.2.[1]

As a measure of goodness of fit, we use the sum of one-dimensional $\Delta\chi_j^2$ functions

$$\Delta\chi^2(p_i) = \sum_{j=1}^{6} \Delta\chi_j^2(p_i), \tag{7.8}$$

for six accurately known dimensionless[2] observable quantities

$$q_j = (m_e/m_\mu,\ m_\mu/m_\tau,\ r,\ \sin^2\theta_{12},\ \sin^2\theta_{13}, \sin^2\theta_{23}). \tag{7.9}$$

In (7.8) we have assumed approximate independence of the fitted quantities (observables). In what follows, we define $N\sigma \equiv \sqrt{\Delta\chi^2}$. For $\sin^2\theta_{ij}$, we make use

[1] The atmospheric mass-squared difference is $\Delta m_{31}^2 = \Delta m^2 + \delta m^2/2$ for the NO spectrum of light neutrino masses and $\Delta m_{32}^2 = \Delta m^2 - \delta m^2/2$ for the IO spectrum.

[2] If a model successfully reproduces dimensionless observables, the overall mass scales can be easily recovered by fitting them to the charged lepton masses m_e, m_μ, m_τ, and the neutrino mass-squared differences δm^2 and $|\Delta m^2|$.

of the one-dimensional projections $\Delta\chi^2_j$ ($j = 4, 5, 6$) from ref. [140],[3] whereas for the remaining quantities we employ the Gaussian approximation:

$$\Delta\chi^2_j(p_i) = \left(\frac{q_j(p_i) - q_{j,\text{best-fit}}}{\sigma_j}\right)^2, \quad j = 1, 2, 3. \tag{7.10}$$

Our goal is to explore phenomenologically viable regions in the parameter space, i.e.,

$$\{p_i : \; l(p_i) \leq l_{\text{max}}\}\,, \tag{7.11}$$

where $l(p_i)$ is the "loss" objective function, which we define as $l(p_i) \equiv N\sigma(p_i) \equiv \sqrt{\Delta\chi^2(p_i)}$, and l_{max} is the threshold, which we set to 3, so that it corresponds to compatibility with the observed data at 3σ confidence level.

We decompose this problem into two parts: first, we find local minima $p_i^{(1)}$, $p_i^{(2)}$, ... of $l(p_i)$, and then we explore connected regions around the minima $p_i^{(n)}$ that satisfy the constraint $l\,(p_i) \leq l_{\text{max}}$.

To find local minima of $l(p_i)$, we use the following algorithm:

1. Pick parameters p_i at random until we find a "good enough" point such that $l(p_i) < l_{0.01}$. The threshold $l_{0.01}$ is a 0.01 quantile of the $l(p_i)$ distribution, i.e., it is chosen in such a way that we accept roughly 1% points. We use this preliminary step to filter out unpromising points which are very far from the regions of interest. Note that typically $l_{0.01} > l_{\text{max}}$, i.e., the regions of interest cover only a tiny fraction of the parameter space, so this step is needed to speed up the computation.

2. Run a conventional gradient-based local minimisation algorithm for the objective function $l(p_i)$ starting from this point. If the resulting local minimum satisfies the constraint $l \leq l_{\text{max}}$, then add it to a set of viable minima.

3. Repeat steps 1 and 2 until we stop finding any new viable minima.

At this point, we have a set of distinct viable minima, so for each of them we have to explore the viable region around them. A simple approach to the problem is to vary parameters p_i individually until the objective function $l(p_i)$ increases to l_{max}. It corresponds to approximation of the viable region with a parallelepiped. A more sophisticated approach is to approximate the viable

[3] These one-dimensional $\Delta\chi^2_j$ ($j = 4, 5, 6$) projections were kindly shared with us by the authors of ref. [140], and they are represented in figure 1 of this reference.

region with an ellipsoid by expanding $l(p_i)$ around the minimum up to the second order. However, neither of these approaches work well in our setting due to peculiar shapes of viable regions. Typically, only a small part of a viable region can be approximated with a parallelepiped or an ellipsoid, therefore such approximations lead to a significant underestimation of the full viable parameter space.

Instead, we explore a viable region with a random walk process known as the Metropolis algorithm. The algorithm mimics the Brownian motion of a probe particle in a potential. The procedure is as follows:

1. Define a "potential"

$$V(p_i) = \begin{cases} l(p_i) & \text{if } l(p_i) \le l'_{\max}, \\ +\infty & \text{otherwise.} \end{cases} \tag{7.12}$$

   We set $l'_{\max} = 5 > l_{\max}$ in order to make the boundary $l(p_i) = l_{\max}$ clearly visible in the plots.

2. Start a sequence with any point $p_i^{(0)}$ from the viable region, e.g., the local minimum found previously.

3. At iteration t, generate a candidate point p'_i from a Gaussian distribution centred at $p_i^{(t)}$ with covariance $\Sigma = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_6^2)$, where σ_i are "step sizes" along different axes, which have to be tuned.

4. Accept the candidate point with probability

$$\alpha = \min\left[1, \exp\left(\left(V(p_i^{(t)}) - V(p'_i)\right)/T\right)\right], \tag{7.13}$$

   where T is the "temperature" to be tuned.

5. Repeat steps 3 and 4 until the region is fully explored.

One can show that the resulting sequence is distributed according to the Boltzmann (Gibbs) distribution $P(p_i) \propto \exp\left(-V(p_i)/T\right)$, which explains our choice of the potential $V(p_i)$.

8. Predictive model based on S_4 [P1, P4]

As a first example of a viable modular-invariant model of lepton flavour, we consider a model based on $\Gamma_4 \simeq S_4$ modular group with type-I seesaw. Models of this type were studied in ref. [P1], and several viable choices depending on 8 parameters were identified. In this chapter we present one of these models, which was shown to remain viable [P4] after imposing the CP symmetry (see chapter 4), thus reducing the number of parameters from 8 to 7. More recently, this model was found among the 9 predictive S_4 modular-invariant models in the comprehensive study [25] (labelled as L2 therein).

8.1. Model description

We choose the modular group representations and weights $(\mathbf{r}, k)$ of the relevant fields as

$$\rho_L \sim (\mathbf{3}, 2), \quad \rho_E \sim (\mathbf{1'}, 0) \oplus (\mathbf{1}, 2) \oplus (\mathbf{1'}, 2), \quad \rho_N \sim (\mathbf{3'}, 0), \tag{8.1}$$

which leads to the superpotential of the form

$$\begin{aligned} W = {} & \alpha \left(E_1^c L\, Y_{\mathbf{3'}}^{(4,2)}\right)_{\mathbf{1}} H_d + \beta \left(E_2^c L\, Y_{\mathbf{3}}^{(4,4)}\right)_{\mathbf{1}} H_d + \gamma \left(E_3^c L\, Y_{\mathbf{3'}}^{(4,4)}\right)_{\mathbf{1}} H_d \\ & + g \left(N^c L\, Y_{\mathbf{2}}^{(4,2)}\right)_{\mathbf{1}} H_u + g' \left(N^c L\, Y_{\mathbf{3'}}^{(4,2)}\right)_{\mathbf{1}} H_u + \Lambda \left(N^c N^c\right)_{\mathbf{1}}, \end{aligned} \tag{8.2}$$

where the multiplets of modular forms $Y_{\mathbf{r}}^{(4,2)}$ and $Y_{\mathbf{r}}^{(4,4)}$ have been constructed in section 3.1 (for the group theory of S_4, see appendix B.2). Here no sums are implied, since each singlet is unique, and the coefficients $(\alpha, \beta, \gamma) = (\alpha_1, \alpha_2, \alpha_3)$, g and Λ are real without loss of generality, as the corresponding phases can be absorbed into the fields E_1^c, E_2^c, E_3^c, L and N^c, respectively. Therefore, the only complex parameter of the theory is g'/g. If a symmetric basis is used and the modular form multiplets are properly normalised, then CP is conserved whenever

$$\mathrm{Im}\,(g'/g) = 0. \tag{8.3}$$

The basis used in ref. [P1] is not symmetric. One can check that it can be related to the symmetric basis here considered by the following transformation matrices $U_{\mathbf{r}}$:

$$
\begin{aligned}
&U_{\mathbf{1}} = U_{\mathbf{1}'} = 1, \qquad U_{\mathbf{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \\
&U_{\mathbf{3}} = U_{\mathbf{3}'} = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -e^{-i\pi/4} & 0 \\ 0 & 0 & -e^{i\pi/4} \end{pmatrix} \begin{pmatrix} 2 & 2 & 2 \\ -2 & 1+\sqrt{3} & 1-\sqrt{3} \\ -2 & 1-\sqrt{3} & 1+\sqrt{3} \end{pmatrix}.
\end{aligned} \tag{8.4}
$$

By direct comparison of the singlets $(N^c L\, Y^{(4,2)}_{\mathbf{2}})_{\mathbf{1}}$ and $(N^c L\, Y^{(4,2)}_{\mathbf{3}'})_{\mathbf{1}}$ written in different bases, and taking into account an extra factor of i arising from the normalisation of the modular form multiplets used in ref. [P1], we find that also in this basis CP invariance results in the condition (8.3). In what follows, we report the parameter values in the basis of ref. [P1].

8.2. Numerical result

Through numerical search, we find only one viable region of the parameter space consistent with the condition (8.3). We report the corresponding best-fit values and the confidence intervals of the parameters and observables in table 8.1.

This minimal CP-invariant model, predicting 12 observables, is characterised by 7 parameters: the 6 real parameters $v_d\,\alpha$, β/α, γ/α, $v_u^2\, g^2/\Lambda$, g'/g, $\mathrm{Im}\,\tau$ and the phase $\mathrm{Re}\,\tau$.[1] The three real parameters $v_d\,\alpha$, β/α and γ/α are fixed by fitting the three charged-lepton masses. The remaining three real parameters $v_u^2\, g^2/\Lambda$, g'/g, $\mathrm{Im}\,\tau$ and the phase $\mathrm{Re}\,\tau$ describe the nine neutrino observables: three neutrino masses m_1, m_2, m_3; three neutrino mixing angles θ_{12}, θ_{13}, θ_{23}; and three CPV phases δ, α_{21}, α_{31}, where α_{21} and α_{31} are the Majorana phases [142].

The model predicts the type of spectrum neutrino masses obey, or the neutrino mass ordering; the value of the lightest neutrino mass, and thus the sum of the three neutrino masses; the values of the Dirac and the Majorana CPV phases δ and α_{21}, α_{31}, and thus the magnitude of CP violation effects in neutrino oscillations [142–144] and of the neutrinoless double beta decay effective Majorana mass $|\langle m \rangle|$ (see, e.g., [10, 145]). These predictions will be tested in ongoing and future planned neutrino experiments (see, e.g., [10]). In addition,

[1] $\mathrm{Re}\,\tau$ should be treated as a phase since the dependence of Yukawa couplings and fermion mass matrices on τ arises through powers of $q_4 \equiv \exp(2\pi i\tau/4)$.

Table 8.1.: Best-fit values along with 3σ ranges of the parameters and observables in the predictive CP-invariant modular S_4 model. CP symmetry is spontaneously broken by the VEV of the modulus τ.

	Best-fit value	3σ range
$\mathrm{Re}\,\tau$	± 0.09922	$\pm(0.09371 - 0.1049)$
$\mathrm{Im}\,\tau$	1.016	$1.014 - 1.018$
β/α	9.348	$7.845 - 12.25$
γ/α	0.002203	$0.001954 - 0.00246$
g'/g	-0.02093	$-(0.01682 - 0.02528)$
$v_d\,\alpha$, MeV	53.61	
$v_u^2\,g^2/\Lambda$, eV	0.0135	
m_e/m_μ	0.004796	$0.004251 - 0.005351$
m_μ/m_τ	0.05756	$0.04399 - 0.06861$
r	0.02981	$0.02769 - 0.03212$
δm^2, 10^{-5} eV^2	7.326	$6.953 - 7.694$
$\lvert\Delta m^2\rvert$, 10^{-3} eV^2	2.457	$2.396 - 2.511$
$\sin^2\theta_{12}$	0.305	$0.2687 - 0.3427$
$\sin^2\theta_{13}$	0.02136	$0.0192 - 0.02372$
$\sin^2\theta_{23}$	0.4862	$0.484 - 0.4882$
m_1, eV	0.01211	$0.01185 - 0.01236$
m_2, eV	0.01483	$0.01473 - 0.01493$
m_3, eV	0.05139	$0.05074 - 0.05195$
$\sum_i m_i$, eV	0.07833	$0.07734 - 0.07921$
$\lvert\langle m\rangle\rvert$, eV	0.01201	$0.01178 - 0.01221$
δ/π	± 1.641	$\pm(1.627 - 1.656)$
α_{21}/π	± 0.3464	$\pm(0.324 - 0.3713)$
α_{31}/π	± 1.254	$\pm(1.229 - 1.283)$
$N\sigma$	1.012	

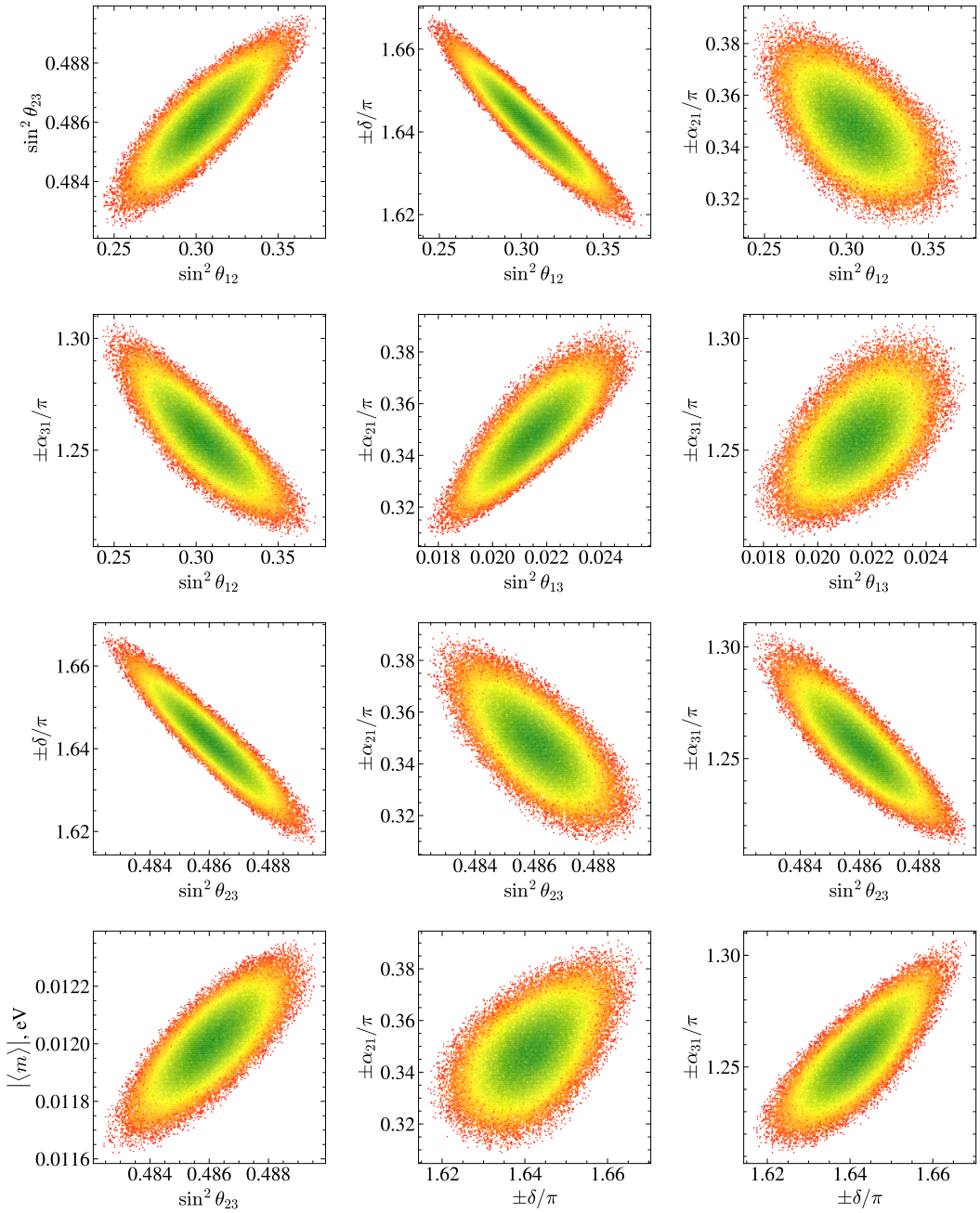

Figure 8.1.: Correlations between pairs of observables in the predictive CP-invariant modular S_4 model.

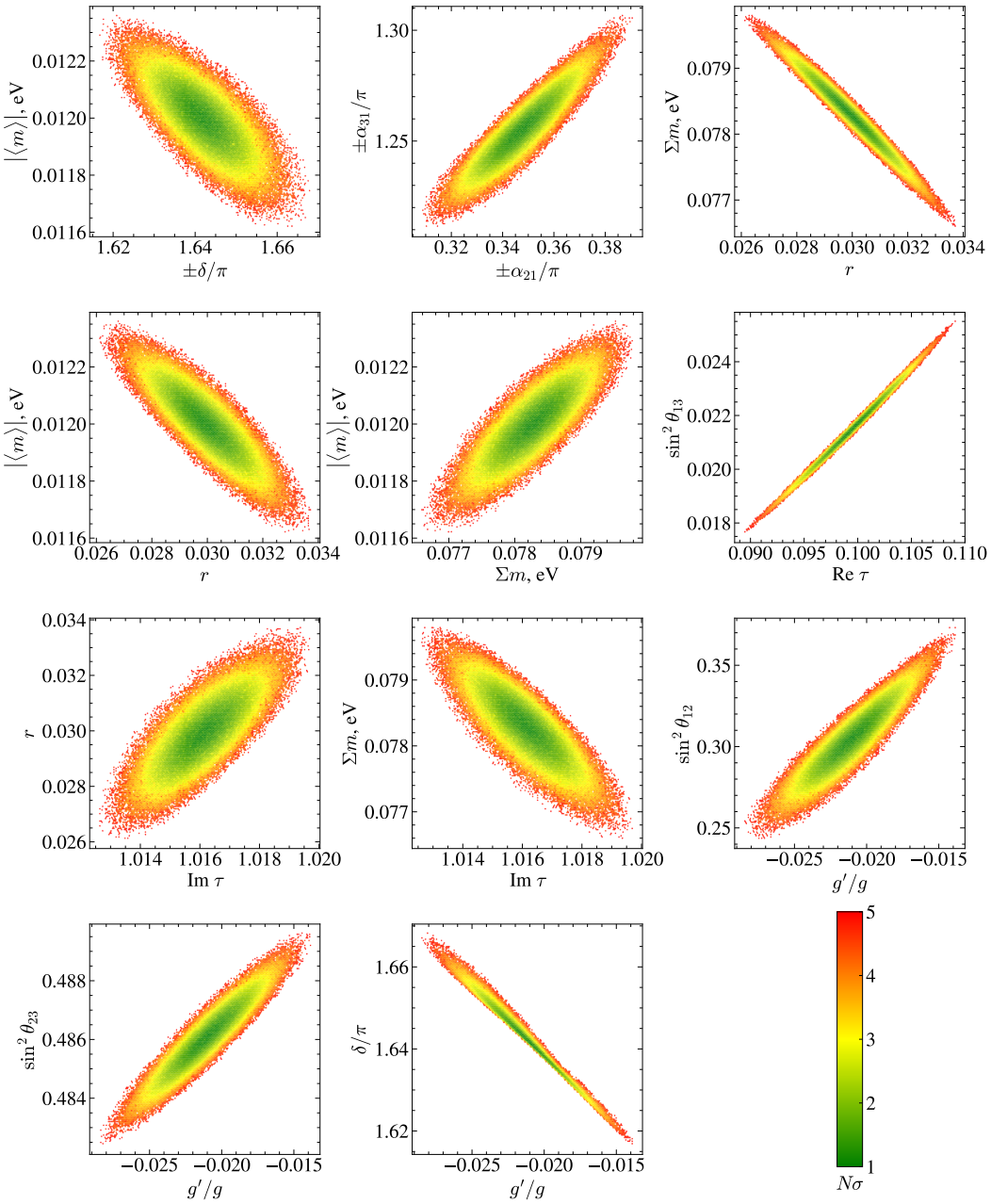

Figure 8.2.: Correlations between pairs of observables (continued from fig. 8.1) and between observables and parameters.

since all observables are functions of the VEV of the modulus τ, the model also predicts unique correlations between the different observables. Some of these correlations are illustrated in figs. 8.1 and 8.2.

We note also that this model has more predictive power than the original model from ref. [P1], which is described by the same parameters and an additional phase $\arg(g'/g)$. In fact, the correlations between $\sin^2\theta_{23}$, the neutrino masses and the CPV phases, which were present in the original model, now reduce to accurate predictions of these observables at a few percent level. Apart from that, many correlations between pairs of observables and between observables and parameters arise. As we have already indicated, some of these correlations are reported in figs. 8.1 and 8.2.

We also check numerically that CP invariance is restored for the CP-conserving values of τ derived in section 5.2. To achieve this, we vary the value of τ while keeping all other parameters fixed to their best-fit values, and present the resulting $\sin^2\delta(\tau)$, $\sin^2\alpha_{21}(\tau)$ and $\sin^2\alpha_{31}(\tau)$ as heatmap plots in the τ plane in fig. 8.3. Notice that this variation is done for illustrative purposes only, as it spoils the values of the remaining observables. Those are in agreement with experimental data only in a small region of the τ plane. The sine-squared of a phase measures the strength of CPV, with the value of 0 (shown with green colour) corresponding to no CPV and the value of 1 (shown with red colour) corresponding to maximal CPV. As anticipated, both the boundary of $\mathcal{D}$ and the imaginary axis conserve CP, appearing in green colour in fig. 8.3. However, even a small departure from a CP-conserving value of τ can lead to large CPV due to strong dependence of the observables on τ. This is noticeably the case in the vicinity of the boundary of the fundamental domain.

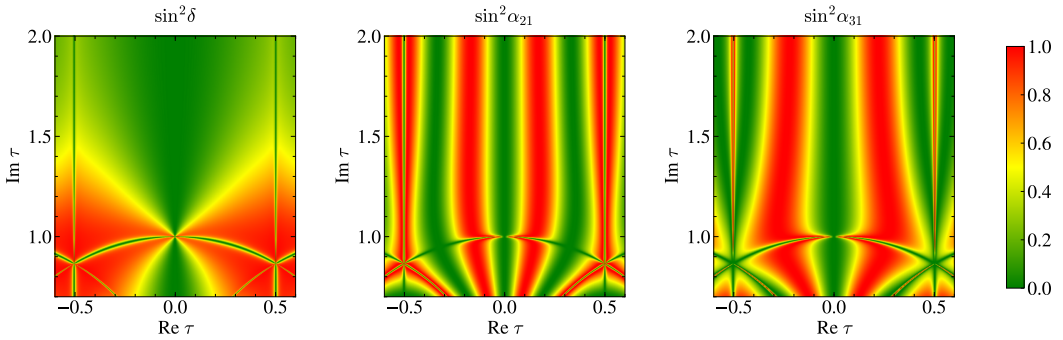

Figure 8.3.: CP violation strength, measured as sine-squared of the CPV phases, for different values of the modulus τ. The boundary of the fundamental domain $\mathcal{D}$ defined in (5.1) and the imaginary axis $\mathrm{Re}\,\tau = 0$ conserve CP. The ranges of $\mathrm{Re}\,\tau$ and $\mathrm{Im}\,\tau$ are chosen to extend slightly beyond the fundamental domain $\mathcal{D}$ to make its boundary clearly visible.

9. Trimaximal mixing from modular A_4 group [P3]

In this chapter we investigate the possibility to construct a modular-invariant model of lepton flavour with an unbroken residual symmetry (see chapter 5), following ref. [P3]. Residual symmetries can be used to explain in a natural way the observed neutrino mixing pattern in the context of usual non-Abelian discrete flavour symmetries.

By considering modular-invariant mass matrices of $\Gamma_3 \simeq A_4$, we implement this idea in the context of the modular symmetry approach to flavour. More specifically, we construct a model with an unbroken $\mathbb{Z}_3^T$ or $\mathbb{Z}_3^{ST}$ group in the charged-lepton sector, and an unbroken $\mathbb{Z}_2^S$ group in the neutrino sector, so that the neutrino mixing matrix has the trimaximal form. This model also gives rise to leptonic sum rules, which have been studied more recently in ref. [23].

9.1. Modular forms of level 3

Weight-2 modular forms of $\Gamma_3 \simeq A_4$ have been explicitly constructed in [12] (for the group theory of A_4, see appendix B.1). They furnish a triplet representation of A_4, and can be expressed in terms of the Dedekind eta function $\eta(\tau)$ (see appendix A) and its derivative:

$$Y_{\mathbf{3}}^{(3,2)}(\tau) = \frac{i}{2\pi}\begin{pmatrix} \dfrac{\eta'(\tau/3)}{\eta(\tau/3)} + \dfrac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \dfrac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \dfrac{27\eta'(3\tau)}{\eta(3\tau)} \\ -2\left(\dfrac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2\dfrac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega\ \dfrac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right) \\ -2\left(\dfrac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega\ \dfrac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2\dfrac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right) \end{pmatrix} \tag{9.1}$$

with the following q-expansions in $q_3 \equiv \exp(2i\pi\tau/3)$:

$$Y_{\mathbf{3}}^{(3,2)}(\tau) = \begin{pmatrix} Y_1^{(3,2)}(\tau) \\ Y_2^{(3,2)}(\tau) \\ Y_3^{(3,2)}(\tau) \end{pmatrix} = \begin{pmatrix} 1 + 12q_3^3 + 36q_3^6 + 12q_3^9 + \dots \\ -6(q_3 + 7q_3^4 + 8q_3^7 + \dots) \\ -18(q_3^2 + 2q_3^5 + 5q_3^8 + \dots) \end{pmatrix}. \tag{9.2}$$

Table 9.1.: Modular forms of weights $k = 2, 4$ up to a common prefactor $\left(Y_1^{(3,2)}\right)^{k/2}$ and the magnitude of $Y_1^{(3,2)}$ at the symmetric points.

	weight 2	weight 4		
τ	$\mathbf{3}$	$\mathbf{3}$	$\{\mathbf{1},\ \mathbf{1'}\}$	$Y_1^{(3,2)}$
τ_L	$(1, \omega, -\frac{1}{2}\omega^2)$	$3(1, -\frac{1}{2}\omega, \omega^2)$	$\{0,\ \frac{9}{4}\omega\}$	$0.95\ldots$
τ_C	$(1, 1-\sqrt{3}, -2+\sqrt{3})$	$(1,1,1)$	$\{6\sqrt{3}-9,\ 9-6\sqrt{3}\}$	$1.02\ldots$
τ_T	$(1,0,0)$	$(1,0,0)$	$\{1,\ 0\}$	1

Higher-weight modular forms can be obtained as tensor products of $Y_{\mathbf{3}}^{(3,2)}$. In particular, at weight 4, we find one triplet $\mathbf{3}$ and two singlets $\mathbf{1}$, $\mathbf{1'}$:

$$\begin{aligned} Y_{\mathbf{3}}^{(3,4)} &= \frac{2}{3}\begin{pmatrix} (Y_1^{(3,2)})^2 - Y_2^{(3,2)}Y_3^{(3,2)} \\ (Y_3^{(3,2)})^2 - Y_1^{(3,2)}Y_2^{(3,2)} \\ (Y_2^{(3,2)})^2 - Y_1^{(3,2)}Y_3^{(3,2)} \end{pmatrix}, \\ Y_{\mathbf{1}}^{(3,4)} &= (Y_1^{(3,2)})^2 + 2Y_2^{(3,2)}Y_3^{(3,2)}, \\ Y_{\mathbf{1'}}^{(3,4)} &= (Y_3^{(3,2)})^2 + 2Y_1^{(3,2)}Y_2^{(3,2)}. \end{aligned} \tag{9.3}$$

At the symmetric points τ_L, τ_C and τ_T, modular forms take specific values which we collect in table 9.1 for weights 2 and 4.

9.2. Lepton mass matrices at the symmetric points

We assume that the neutrino masses originate from the Weinberg operator, and assign the modular group representations and weights as

$$\rho_L \sim (\mathbf{3}, k_L), \quad \rho_E \sim (\mathbf{1}, k_E) \oplus (\mathbf{1'}, k_E) \oplus (\mathbf{1''}, k_E). \tag{9.4}$$

Restricting ourselves to modular forms of weights 2 and 4, we obtain the superpotential of the form

$$\begin{aligned} W = {} & \alpha \left(E_1^c L\, Y_{\mathbf{3}}^{(3,k_Y)}\right)_{\mathbf{1}} H_d + \beta \left(E_2^c L\, Y_{\mathbf{3}}^{(3,k_Y)}\right)_{\mathbf{1}} H_d + \gamma \left(E_3^c L\, Y_{\mathbf{3}}^{(3,k_Y)}\right)_{\mathbf{1}} H_d \\ & + \frac{1}{\Lambda}\sum_s g_s \left(L\, L\, Y_{\mathbf{r}_s}^{(3,k_W)}\right)_{\mathbf{1}} H_u^2, \end{aligned} \tag{9.5}$$

where the sum in the second line includes one term (with $\mathbf{r}_s = \mathbf{3}$) if $k_W = 2$, and three terms (with $\mathbf{r}_s = \mathbf{3}, \mathbf{1}, \mathbf{1'}$) if $k_W = 4$.

Charged leptons

The superpotential (9.5) yields a mass matrix of charged leptons, which can be written in terms of the A_4 modular-form triplet with weight $k_Y = 2$ or 4:

$$M_E = v_d \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{\mathbf{3}} Y_{\mathbf{3}}^{(3,k_Y)} \,. \tag{9.6}$$

Without loss of generality the coefficients α, β, and γ can be made real positive by rephasing the right-handed charged lepton fields.

We will discuss next the charged-lepton mass matrix at the symmetric points in the case of $k_Y = 2$. At $\tau = \tau_L$, the matrix $M_E^\dagger M_E$, which is relevant for the left-handed mixing, is given by:

$$\begin{aligned} M_E^\dagger M_E = {} & \frac{9}{4} v_d^2 \, (Y_1^{(3,2)})^2 \times \\ & \begin{pmatrix} \alpha^2 + \beta^2 + \frac{1}{4}\gamma^2 & -\frac{\omega^2}{2}\alpha^2 + \omega^2\beta^2 - \frac{\omega^2}{2}\gamma^2 & \omega\alpha^2 - \frac{\omega}{2}\beta^2 - \frac{\omega}{2}\gamma^2 \\ * & \frac{1}{4}\alpha^2 + \beta^2 + \gamma^2 & -\frac{\omega^2}{2}\alpha^2 - \frac{\omega^2}{2}\beta^2 + \omega^2\gamma^2 \\ * & * & \alpha^2 + \frac{1}{4}\beta^2 + \gamma^2 \end{pmatrix}, \end{aligned} \tag{9.7}$$

where through asterisks we omit some entries of a Hermitian matrix. It is easily noticed that this matrix commutes with ST, which is guaranteed by the residual symmetry $\mathbb{Z}_3^{ST}$ at $\tau = \tau_L$, where

$$ST = \frac{1}{3} \begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2 & -\omega & 2\omega^2 \\ 2 & 2\omega & -\omega^2 \end{pmatrix}. \tag{9.8}$$

Both matrices $M_E^\dagger M_E$ and ST are diagonalised by the unitary matrix

$$U_E \equiv TS = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2\omega & -\omega & 2\omega \\ 2\omega^2 & 2\omega^2 & -\omega^2 \end{pmatrix}, \tag{9.9}$$

so that

$$\begin{aligned} U_E^\dagger \, ST \, U_E &= T = \mathrm{diag}\,(1, \omega, \omega^2), \\ U_E^\dagger \, M_E^\dagger M_E \, U_E &= \frac{9}{4} v_d^2 \, (Y_1^{(3,2)})^2 \, \mathrm{diag}(\gamma^2, \alpha^2, \beta^2). \end{aligned} \tag{9.10}$$

Note that U_E is independent of the parameters α, β, γ.

At $\tau = \tau_C$, the determinant of M_E vanishes. Therefore, this mass matrix leads to a massless charged lepton, and thus cannot be used for model building.

Finally, at $\tau = \tau_T$ we obtain a real diagonal matrix:

$$M_E = v_d\, Y_1^{(3,2)} \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}. \tag{9.11}$$

Let us now briefly discuss the case of $k_Y = 4$. First note that only the modular-form triplet $Y_{\mathbf{3}}^{(3,4)}$ contributes to the charged-lepton mass matrix. As seen from table 9.1, this triplet coincides with the weight-2 triplet at $\tau = \tau_L$ up to a permutation, complex conjugation, and a prefactor. In fact one can check that this choice yields the same numerical results as $k_Y = 2$, therefore we do not consider this case in what follows. Next, at $\tau = \tau_C$, the charged-lepton mass matrix is of rank one, i.e., two massless charged leptons appear since the modular-form triplet is proportional to $(1, 1, 1)$. Finally, at $\tau = \tau_T$, the charged-lepton mass matrix is proportional to the diagonal matrix given in (9.11) since the weight-4 triplet coincides with the weight-2 triplet up to a prefactor.

Neutrinos

The triplet modular form contribution to the neutrino mass matrix is given by:

$$M_\nu = \frac{v_u^2}{\Lambda} \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix}_{Y_{\mathbf{3}}^{(3,k_W)}} . \tag{9.12}$$

In the case $k_W = 2$ it is easily checked that two lightest neutrino masses are degenerate at $\tau = \tau_L$, while the determinant of M_ν vanishes at $\tau = \tau_C$. In the latter case one neutrino is massless and two neutrino masses are degenerate. The two lightest neutrino masses are degenerate also at $\tau = \tau_T$. In fact, these degeneracies of neutrino masses still hold even if we use the seesaw mechanism by introducing the three right-handed neutrino fields as A_4 triplet. Thus, a realistic neutrino mass matrix is not obtained as far as we take weight 2 modular forms at $\tau = \tau_L, \tau_C, \tau_T$.

Let us now consider the case $k_W = 4$. At $\tau = \tau_L$, the modular-form triplet contribution to the neutrino mass matrix is similar to the case of weight 2, where two neutrino masses are degenerate. There is an additional contribution from the modular-form singlet $\mathbf{1}'$, which however does not resolve the degeneracy. It

is easily noticed that two neutrino masses are degenerate also at $\tau = \tau_T$ since $Y_{\mathbf{3}}^{(3,4)}(\tau_T) \propto (1,0,0)$. An additional $Y_{\mathbf{1}}^{(3,4)}$ contribution does not change this situation. Finally, at $\tau = \tau_C$, contributions from all three multiplets $Y_{\mathbf{3}}^{(4,3)}$, $Y_{\mathbf{1}}^{(4,3)}$, $Y_{\mathbf{1}'}^{(4,3)}$ are present, and the neutrino masses are non-degenerate.

To summarise, the charged-lepton mass matrix could be consistent with the observed masses at $\tau = \tau_L, \tau = \tau_T$ for both cases $k_Y = 2, 4$. On the other hand, the neutrino Majorana mass matrix is consistent with the observed masses only at $\tau = \tau_C$ for the case $k_W = 4$. There is no common symmetric value of τ, which leads to charged lepton and neutrino masses that are consistent with the data.

9.3. Model with two moduli

The mass matrices described in the previous section are inconsistent with the observed masses if taken at the same symmetric point. Therefore, we consider instead the case of having two moduli in the theory: one τ^ℓ, responsible via its VEV for the breaking of the modular A_4 symmetry in the charged-lepton sector, and another one τ^ν, breaking the modular symmetry in the neutrino sector. This possibility can be realised within the formalism of multiple modular symmetries [70, 71].

We present next our setup. For the charged lepton mass matrix, we take weight-2 modular forms at $\tau^\ell = \tau_T$ (Case I) or at $\tau^\ell = \tau_L$ (Case II). At the same time we use weight-4 modular forms at $\tau^\nu = \tau_C$ for constructing the neutrino Majorana mass term. In order for the modular weight in the superpotential to vanish, we assign the matter field weights as $k_L = 2$, $k_E = 0$. Note that $k_L = 2$ is common in both τ^ℓ and τ^ν modular spaces.

Then, the charged lepton mass matrix is obtained by using as input the expressions for the weight-2 modular forms given in table 9.1. At τ_T, it is a diagonal matrix:

$$M_E = v_d\, Y_1^{(3,2)} \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} : \quad \text{Case I.} \tag{9.13}$$

At $\tau = \tau_L$, the charged lepton mass matrix has the form:

$$M_E = v_d\, Y_1^{(3,2)} \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & \omega^2 & -\frac{1}{2}\omega \\ -\frac{1}{2}\omega & 1 & \omega^2 \\ \omega^2 & -\frac{1}{2}\omega & 1 \end{pmatrix} : \quad \text{Case II.} \tag{9.14}$$

The matrix $M_E^\dagger M_E$, which is relevant for the calculation of the left-handed mixing, is given in (9.7).

The neutrino mass matrix represents a sum of the contributions of modular-form multiplets **3**, **1** and **1′**, with the terms involving the two singlet modular forms entering the sum with arbitrary complex coefficients A and B:

$$M_\nu = \frac{v_u^2}{\Lambda}\,(Y_1^{(3,2)})^2 \left\{ \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \left[A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - B \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right] \right\}. \quad (9.15)$$

The two models with charged-lepton mass matrix M_E specified in (9.13) and (9.14) and neutrino mass matrix M_ν given in (9.15), as we will show, lead to the same phenomenology.

Neutrino mixing

In case I, only the neutrino mass matrix contributes to the PMNS matrix $U_{\rm PMNS}$ since the charged lepton mass matrix is diagonal. The neutrino mass matrix in this case leads to the so called TM_2 mixing form of the PMNS matrix [146, 147] where the second column of $U_{\rm PMNS}$ is trimaximal:

$$U_{\rm PMNS}^{\rm I} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & e^{i\phi}\sin\theta \\ 0 & 1 & 0 \\ -e^{-i\phi}\sin\theta & 0 & \cos\theta \end{pmatrix} \mathrm{P}. \quad (9.16)$$

Here θ and ϕ are arbitrary mixing angle and phase, respectively, and P is a diagonal phase matrix containing contributions to the Majorana phases of $U_{\rm PMNS}$. Employing the standard parametrisation of $U_{\rm PMNS}$ (see, e.g., [10]), it is possible to show that the trimaximal mixing pattern leads to the following relation between the reactor angle θ_{13} and θ, between the atmospheric neutrino mixing angle θ_{23} and θ_{13} and θ, and sum rules for the solar neutrino mixing angle θ_{12} and for the Dirac phase δ [146, 147] (see also [9, 148]):

$$\sin^2\theta_{13} = \frac{2}{3}\sin^2\theta, \quad (9.17)$$

$$\sin^2\theta_{12} = \frac{1}{3\,\cos^2\theta_{13}}, \quad (9.18)$$

$$\sin^2\theta_{23} = \frac{1}{2} + \frac{s_{13}}{2}\frac{\sqrt{2-3s_{13}^2}}{1-s_{13}^2}\cos\phi, \quad (9.19)$$

$$\cos\delta = \frac{\cos 2\theta_{23}\ \cos 2\theta_{13}}{\sin 2\theta_{23}\ \sin\theta_{13}\ (2-3\sin^2\theta_{13})^{\frac{1}{2}}}\,. \tag{9.20}$$

Using the 3σ allowed range of $\sin^2\theta_{13}$ from [149] and (9.17) we get the following constraints on $\sin\theta$:

$$0.17 \lesssim |\sin\theta| \lesssim 0.19. \tag{9.21}$$

To leading order in s_{13} we obtain from (9.19):

$$\begin{gathered}\frac{1}{2} - \frac{s_{13}}{\sqrt{2}} \lesssim \sin^2\theta_{23} \lesssim \frac{1}{2} + \frac{s_{13}}{\sqrt{2}}\,,\ \text{or} \\ 0.391(0.390) \lesssim \sin^2\theta_{23} \lesssim 0.609\ (0.611),\end{gathered} \tag{9.22}$$

where the numerical values correspond to the maximal allowed value of $\sin^2\theta_{13}$ at 3σ confidence level for NO (IO) neutrino mass spectrum [149]. The interval of possible values of $\sin^2\theta_{23}$ in (9.22) is somewhat wider that the 3σ ranges of experimentally allowed values of $\sin^2\theta_{23}$ for NO and IO spectra given in [149]. Using the 3σ allowed ranges of $\sin^2\theta_{23}$ and $\sin^2\theta_{13}$ for NO (IO) spectra from [149] and (9.19) we also get:

$$-\ 0.640(-0.508) \lesssim \cos\phi \leq 1. \tag{9.23}$$

The phase ϕ is related to the Dirac phase δ [9]:

$$\sin 2\theta_{23}\sin\delta = \sin\phi. \tag{9.24}$$

The Majorana phase $\alpha_{31}/2$ of the standard parametrisation of $U_{\rm PMNS}$ [10] receives contributions from the phase ϕ via [9]

$$\frac{\alpha_{31}}{2} = \frac{\xi_{31}}{2} + \alpha_2 + \alpha_3, \tag{9.25}$$

where the phase ξ_{31} will be specified later,

$$\alpha_2 = \arg\left(-\frac{c}{\sqrt{2}} - \frac{s}{\sqrt{6}}\,e^{i\phi}\right), \quad \alpha_3 = \arg\left(\frac{c}{\sqrt{2}} - \frac{s}{\sqrt{6}}\,e^{i\phi}\right), \tag{9.26}$$

so that

$$\sin\alpha_2 = -\,\frac{s}{\sqrt{6}}\,\frac{\sin\phi}{s_{23}\,c_{13}} = -\,\tan\theta_{13}\ \cos\theta_{23}\ \sin\delta, \tag{9.27}$$

$$\sin\alpha_3 = -\,\frac{s}{\sqrt{6}}\,\frac{\sin\phi}{c_{23}\,c_{13}} = -\,\tan\theta_{13}\ \sin\theta_{23}\ \sin\delta. \tag{9.28}$$

We also have [9]:

$$\sin(\phi - \alpha_2 - \alpha_3) = -\sin\delta. \tag{9.29}$$

For further discussion of phenomenology of the neutrino trimaximal mixing (9.16), see, e.g., [9, 150–152].

In case II, the contribution of the rotation of the charged-lepton sector is added to the trimaximal mixing, which is derived from the neutrino mass matrix in (9.15). The mixing matrix in the charged-lepton sector is the matrix U_E in (9.10). The PMNS matrix is given by:

$$U^{\mathrm{II}}_{\mathrm{PMNS}} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2\omega & -\omega & 2\omega \\ 2\omega^2 & 2\omega^2 & -\omega^2 \end{pmatrix}^{\dagger} \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & e^{i\phi}\sin\theta \\ 0 & 1 & 0 \\ -e^{-i\phi}\sin\theta & 0 & \cos\theta \end{pmatrix} \mathrm{P}. \tag{9.30}$$

It is straightforward to check that after a substitution $\theta \to \theta - \pi/2$, $\phi \to -\phi$, the PMNS matrix (9.30) can be rewritten as

$$U^{\mathrm{II}}_{\mathrm{PMNS}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & e^{i\pi/3} & 0 \\ 0 & 0 & e^{-i\pi/3} \end{pmatrix} U^{\mathrm{I}}_{\mathrm{PMNS}} \begin{pmatrix} e^{i(\phi-\pi/2)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i(\phi+\pi/2)} \end{pmatrix}. \tag{9.31}$$

The leftmost phase matrix does not contribute to the mixing, since its effect can be absorbed into the charged-lepton field phases. The rightmost phase matrix contributes only to the Majorana phases, therefore the numerical predictions in this case are the same as in Case I, apart possibly from the corresponding shift of the Majorana phases. However, as can be shown analytically, and we have confirmed numerically, also the predictions for the Majorana phases in Case II coincide with the predictions in Case I.

Neutrino masses and Majorana phases

It follows from (9.15) that the neutrino mass matrix M_ν is a linear combination of three basis matrices:

$$M_1 = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{9.32}$$

To diagonalise M_ν, it is convenient to rewrite it in a different basis:

$$
\begin{aligned}
M_1' &= \frac{1}{\sqrt{3}}\,(M_2 + 2M_3) = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \\
M_2' &= M_2 + \frac{1}{3}M_1 = \frac{1}{3}\begin{pmatrix} 5 & -1 & -1 \\ -1 & 2 & 2 \\ -1 & 2 & 2 \end{pmatrix}, \\
M_3' &= M_2 - \frac{1}{3}M_1 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 4 \\ 1 & 4 & -2 \end{pmatrix},
\end{aligned}
\tag{9.33}
$$

so that $M_\nu = c\left(M_1' + aM_2' + bM_3'\right)$, where a and b are arbitrary complex coefficients and c is the overall scale factor which can be rendered real positive. M_ν is diagonalised by a unitary matrix U_ν° of the following form:

$$
U_\nu^\circ = V_{\text{TBM}}\, U_{13}(\theta, \phi),
\tag{9.34}
$$

so that

$$
\begin{aligned}
M_\nu &= (U_\nu^\circ)^*\, M_\nu^{\text{diag}}\, (U_\nu^\circ)^\dagger, \\
M_\nu^{\text{diag}} &= \text{diag}\left(m_1 e^{-i2\phi_1}, m_2 e^{-i2\phi_2}, m_3 e^{-i2\phi_3}\right),
\end{aligned}
\tag{9.35}
$$

where $m_i e^{-i2\phi_i}$ are complex eigenvalues and $m_i \geq 0$ are the neutrino masses.[1] Extracting the phases ϕ_i from M_ν^{diag}, we find:

$$
\begin{aligned}
M_\nu^{\text{diag}} &= e^{-i2\phi_1}\, \text{P}^*\, \text{diag}\,(m_1, m_2, m_3)\; \text{P}^*, \\
\text{P} &= \text{diag}\left(1, e^{i(\phi_2 - \phi_1)}, e^{i(\phi_3 - \phi_1)}\right),
\end{aligned}
\tag{9.36}
$$

where the phases $(\phi_2 - \phi_1)$ and $(\phi_3 - \phi_1)$ contribute to the Majorana phases $\alpha_{21}/2$ and $\alpha_{31}/2$ of the standard parametrisation of the PMNS matrix [10]. Thus, the PMNS matrix has the form:

$$
U_{\text{PMNS}} = U_\nu^\circ\, \text{P} = e^{-i2\phi_1}\, V_{\text{TBM}}\, U_{13}(\theta, \phi)\, \text{P},
\tag{9.37}
$$

[1] In general, the standard labelling of the neutrino masses [10] corresponds to some permutation of the neutrino mass matrix eigenvalues, which affects the order of the PMNS matrix columns. However, the only non-trivial permutation of the TM_2 matrix columns consistent with the experimental data is (321), which is equivalent to a shift $\theta \to \theta - \pi/2$ up to an unphysical overall column sign. Hence, we can assume that the order of neutrino mass matrix eigenvalues coincides with the standard labelling without loss of generality.

where the common phase factor $e^{-i2\phi_1}$ is unphysical. The phase $\xi_{31}/2$ in (9.25) can be identified now with $(\phi_3 - \phi_1)$: $\xi_{31}/2 = \phi_3 - \phi_1$. Thus, the Majorana phases $\alpha_{21}/2$ and $\alpha_{31}/2$ are given by:

$$\frac{\alpha_{21}}{2} = \phi_2 - \phi_1, \quad \frac{\alpha_{31}}{2} = \phi_3 - \phi_1 + \alpha_2 + \alpha_3. \tag{9.38}$$

The complex rotation parameters θ and ϕ are fixed by a choice of a and b, which we will now show explicitly. We find by direct calculation that

$$\begin{aligned}
U_\nu^{\circ\,T} M_1' U_\nu^\circ &= \begin{pmatrix} -e^{-i\phi}\sin 2\theta & 0 & \cos 2\theta \\ 0 & \sqrt{3} & 0 \\ \cos 2\theta & 0 & e^{i\phi}\sin 2\theta \end{pmatrix}, \\
U_\nu^{\circ\,T} M_2' U_\nu^\circ &= \begin{pmatrix} 2\cos^2\theta & 0 & e^{i\phi}\sin 2\theta \\ 0 & 1 & 0 \\ e^{i\phi}\sin 2\theta & 0 & 2e^{2i\phi}\sin^2\theta \end{pmatrix}, \\
U_\nu^{\circ\,T} M_3' U_\nu^\circ &= \begin{pmatrix} -2e^{-2i\phi}\sin^2\theta & 0 & e^{-i\phi}\sin 2\theta \\ 0 & 1 & 0 \\ e^{-i\phi}\sin 2\theta & 0 & -2\cos^2\theta \end{pmatrix}.
\end{aligned} \tag{9.39}$$

Thus, the neutrino mass matrix M_ν is diagonalised when the corresponding linear combination of the off-diagonal entries vanishes, which leads to

$$\cos 2\theta + ae^{i\phi}\sin 2\theta + be^{-i\phi}\sin 2\theta = 0 \;\Leftrightarrow\; ae^{i\phi} + be^{-i\phi} = -\cot 2\theta. \tag{9.40}$$

The above condition is equivalent to:

$$e^{i\phi} = \pm\frac{a^* - b}{|a^* - b|}, \quad \cot 2\theta = \mp\frac{|a|^2 - |b|^2}{|a^* - b|}. \tag{9.41}$$

It proves convenient to introduce the complex parameter

$$z = ae^{i\phi} - be^{-i\phi} = \pm\frac{|a|^2 + |b|^2 - 2ab}{|a^* - b|}. \tag{9.42}$$

(θ, ϕ, z) is a reparametrisation of (a, b) defined by (9.41) and (9.42). The inverse parameter transformation is given by

$$\begin{aligned}
a &= \frac{e^{-i\phi}}{2}\left(z - \cot 2\theta\right), \\
b &= \frac{e^{i\phi}}{2}\left(-z - \cot 2\theta\right).
\end{aligned} \tag{9.43}$$

The neutrino mass matrix eigenvalues are the corresponding linear combinations of the diagonal entries in (9.39):

$$\begin{aligned} m_1 e^{-i(2\phi_1-\phi)} &= c\left(z - \frac{1}{\sin 2\theta}\right), \\ m_2 e^{-i2\phi_2} &= c\left(\sqrt{3} - iz\sin\phi - \cot 2\theta\cos\phi\right), \\ m_3 e^{-i(2\phi_3+\phi)} &= c\left(z + \frac{1}{\sin 2\theta}\right). \end{aligned} \tag{9.44}$$

Fitting the mass-squared differences to experimentally observed values, we find the following constraint on z in terms of θ, ϕ and $r \equiv \Delta m^2_{21}/\Delta m^2_{31}$:

$$|z - z_0|^2 = R^2, \quad \text{sign}\,(\text{Re}\, z) = \pm\,\text{sign}\,(\sin 2\theta)\,, \tag{9.45}$$

where the plus (minus) sign corresponds to NO (IO) spectrum of neutrino masses, and

$$\begin{aligned} z_0(\theta,\phi,r) &= \frac{1-2r}{\cos^2\phi\sin 2\theta} + \tan\phi\left(\frac{\sqrt{3}}{\cos\phi} - \cot 2\theta\right) i, \\ R^2(\theta,\phi,r) &= \left[\left(\sqrt{3} - \cot 2\theta\cos\phi\right)^2 + \frac{(1-2r)^2 - \cos^2\phi}{\sin^2 2\theta}\right] \Big/ \cos^4\phi. \end{aligned} \tag{9.46}$$

Since θ and r are tightly constrained by the experimental data, the set of phenomenologically viable models is effectively described by two angles ϕ and ψ, with the latter being the angle parameter on the circle (9.45), i.e., $z = z_0 + R\,e^{i\psi}$. Scanning through ϕ and ψ numerically, we find that to each set of the experimentally allowed values of the mixing angles and the mass-squared differences corresponds a range of models (parameterised by ψ) with different values of the neutrino masses and the Majorana phases.

We report the numerical results in the case of NO spectrum in fig. 9.1. The allowed range of the sum of neutrino masses depends on the value of $\sin^2\theta_{23}$. The lower bound slightly decreases from 0.097 eV to 0.074 eV as $\sin^2\theta_{23}$ runs through its 3σ confidence interval of $[0.46, 0.58]$. On the other hand, the upper bound is highly dependent on the value of $\sin^2\theta_{23}$, and tends to infinity as $\sin^2\theta_{23}$ approaches 0.5, which corresponds to $\delta = \phi = 3\pi/2$. This means that at this point the sum of neutrino masses is allowed to take any value greater than its lower bound of 0.093 eV.

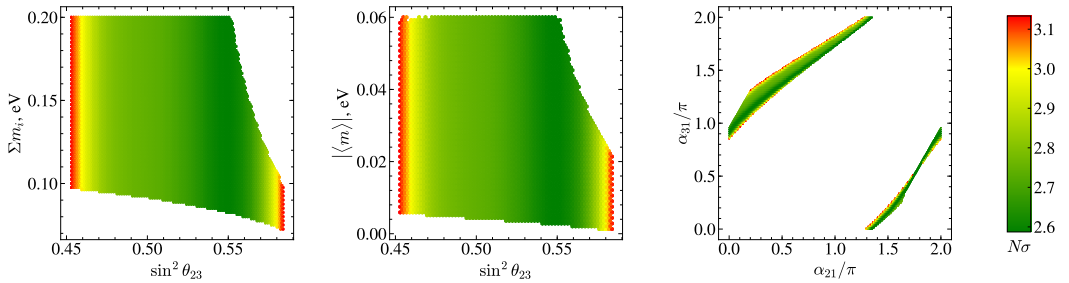

Figure 9.1.: Correlations between $\sin^2\theta_{23}$ and the sum of neutrino masses $\sum m_i$, between $\sin^2\theta_{23}$ and the effective Majorana mass $|\langle m\rangle|$, and between the Majorana phases α_{31} and α_{21} in the case of NO neutrino mass spectrum. See text for further details.

The dependence of the effective Majorana mass $|\langle m\rangle|$ on $\sin^2\theta_{23}$ is qualitatively similar to that of the sum of neutrino masses. The maximal value of $|\langle m\rangle| \cong 0.059$ eV is practically independent of $\sin^2\theta_{23}$ for $0.46 \leq \sin^2\theta_{23} \leq 0.55$. The lower bound of $|\langle m\rangle|$ varies from 0.0015 eV to 0.0059 eV for $\sin^2\theta_{23}$ in its 3σ range. However, for values of $\sin^2\theta_{23}$ from its 3σ range, $0.46 \leq \sin^2\theta_{23} \leq 0.58$, $|\langle m\rangle|$ can have values in the interval $[0.0059, 0.059]$ eV (see fig. 9.1). Most (if not all) of these values may be probed in the future neutrinoless double beta decay experiments.

There is also a strong correlation between the Majorana phases. The set of best-fit models corresponds to $\phi = 1.664\pi$ and leads to the following values of observables:

$$\begin{aligned}
\sin^2\theta_{12} &= 0.3406, & m_1 &= 0.0143 - 0.0612 \text{ eV},\\
\sin^2\theta_{13} &= 0.02125, & m_2 &= 0.0166 - 0.0618 \text{ eV},\\
\sin^2\theta_{23} &= 0.5511, & m_3 &= 0.0519 - 0.079 \text{ eV},\\
\delta m^2 &= 7.34\cdot 10^{-5} \text{ eV}^2, & \textstyle\sum_i m_i &= 0.0828 - 0.2019 \text{ eV},\\
\Delta m^2 &= 2.455\cdot 10^{-3} \text{ eV}^2, & |\langle m\rangle| &= 0.0029 - 0.0589 \text{ eV},\\
r &= 0.0299, & \delta/\pi &= 1.339,
\end{aligned} \tag{9.47}$$

consistent with the experimental data at 2.59σ confidence level.

Similar analysis can be performed in the case of IO neutrino mass spectrum. However, in that case the minimal value of the sum of the three neutrino masses is 0.63 eV, and we do not analyse this case further.

10. S_4' model free from fine-tuning [P6]

We have seen in chapter 6 that hierarchical patterns of fermion masses can be explained by a small deviation of the modulus VEV from one of the symmetric points. In the final chapter of the thesis we realise this idea by considering the most "structured" cases within the surviving lepton flavour models of table 6.2. These arise at level $N = 4$ in the vicinity of $\tau = \omega$ and correspond to E^c and L being a triplet and a direct sum of three singlets of the finite modular group S_4', respectively. The expected charged-lepton mass pattern is $(m_\tau, m_\mu, m_e) \sim (1, \epsilon, \epsilon^2)$.

We have performed a systematic scan restricting ourselves to promising models involving the minimal number of effective parameters (9, including $\mathrm{Re}\,\tau$ and $\mathrm{Im}\,\tau$). Once again, models predicting a massless electron are rejected, while the N^c furnish a complete irrep of dimension 2 or 3 (N^c are present since Weinberg models require more parameters). Out of 48 models, we have identified a model which is viable and not fine-tuned in the regime of interest. For this model,

$$\rho_L \sim (\hat{\mathbf{1}}, 2) \oplus (\hat{\mathbf{1}}, 2) \oplus (\hat{\mathbf{1}}', 2), \quad \rho_E \sim (\hat{\mathbf{3}}, 4), \quad \rho_N \sim (\mathbf{3}', 1). \tag{10.1}$$

The corresponding superpotential reads:

$$\begin{aligned} W = &\left[\alpha_1 \left(Y^{(4,6)}_{\mathbf{3}',1} E^c L_1\right)_{\mathbf{1}} + \alpha_2 \left(Y^{(4,6)}_{\mathbf{3}',2} E^c L_1\right)_{\mathbf{1}}\right. \\ &\left. + \alpha_3 \left(Y^{(4,6)}_{\mathbf{3}',1} E^c L_2\right)_{\mathbf{1}} + \alpha_4 \left(Y^{(4,6)}_{\mathbf{3}',2} E^c L_2\right)_{\mathbf{1}} + \alpha_5 \left(Y^{(4,6)}_{\mathbf{3}} E^c L_3\right)_{\mathbf{1}}\right] H_d \\ &+ \left[g_1 \left(Y^{(4,3)}_{\hat{\mathbf{3}}} N^c L_1\right)_{\mathbf{1}} + g_2 \left(Y^{(4,3)}_{\hat{\mathbf{3}}} N^c L_2\right)_{\mathbf{1}} + g_3 \left(Y^{(4,3)}_{\hat{\mathbf{3}}'} N^c L_3\right)_{\mathbf{1}}\right] H_u \\ &+ \Lambda \left(Y^{(4,2)}_{\mathbf{2}} (N^c)^2\right)_{\mathbf{1}}. \end{aligned} \tag{10.2}$$

Since L_1 and L_2 are indistinguishable, one of the constants α_i, with $i = 1, \ldots, 4$, is effectively not an independent parameter and can be set to zero by a suitable rotation without loss of generality. We choose to set $\alpha_2 = 0$.

At leading order in the small parameter $|\epsilon|$, with $\epsilon \equiv 1 - \frac{1+\sqrt{3}}{1-i}\frac{\varepsilon}{\theta}$ and $|\epsilon| \simeq$

$2.8\left|\frac{\tau-\omega}{\tau-\omega^2}\right|$ in the context of this chapter,[1] the charged-lepton mass matrix reads

$$M_e^\dagger \simeq -\frac{3(\sqrt{3}-1)^6}{\sqrt{13}} v_d \alpha_1 \theta^{12} \begin{pmatrix} 1 & \tilde{\alpha}_3 + \frac{\sqrt{13}}{2}\tilde{\alpha}_4 & \frac{i\sqrt{39}}{2}\tilde{\alpha}_5 \\ \sqrt{3}\,\epsilon & \sqrt{3}\left(\tilde{\alpha}_3 - \frac{\sqrt{13}}{2}\tilde{\alpha}_4\right)\epsilon & \frac{i\sqrt{13}}{2}\tilde{\alpha}_5\,\epsilon \\ \frac{5}{2}\epsilon^2 & \frac{1}{4}\left(10\tilde{\alpha}_3 + \sqrt{13}\tilde{\alpha}_4\right)\epsilon^2 & -\frac{5i\sqrt{13}}{4\sqrt{3}}\tilde{\alpha}_5\,\epsilon^2 \end{pmatrix}, \tag{10.3}$$

while the charged-lepton mass ratios are given by

$$\begin{aligned} \frac{m_e}{m_\mu} &\simeq 2\,\frac{|\tilde{\alpha}_4\tilde{\alpha}_5|\sqrt{4+\left(2\tilde{\alpha}_3+\sqrt{13}\tilde{\alpha}_4\right)^2+39\tilde{\alpha}_5^2}}{3\tilde{\alpha}_4^2+\left[1+\left(\tilde{\alpha}_3-\sqrt{13}\tilde{\alpha}_4\right)^2\right]\tilde{\alpha}_5^2}\,|\epsilon|, \\ \frac{m_\mu}{m_\tau} &\simeq 4\sqrt{13}\,\frac{\sqrt{3\tilde{\alpha}_4^2+\left[1+\left(\tilde{\alpha}_3-\sqrt{13}\tilde{\alpha}_4\right)^2\right]\tilde{\alpha}_5^2}}{4+\left(2\tilde{\alpha}_3+\sqrt{13}\tilde{\alpha}_4\right)^2+39\tilde{\alpha}_5^2}\,|\epsilon|, \end{aligned} \tag{10.4}$$

with $\tilde{\alpha}_i \equiv \alpha_i/\alpha_1$, $i = 3, 4, 5$. As a measure of fine-tuning in the charged-lepton sector, we use the Barbieri-Giudice measure [153]

$$\max(\mathrm{BG}) \equiv \max\left|\partial \ln(\text{mass ratio})/\partial \ln \tilde{\alpha}_{3,4,5}\right|. \tag{10.5}$$

An observable O is typically considered fine-tuned with respect to some parameter p if $\mathrm{BG} \equiv |\partial \ln O/\partial \ln p| \gtrsim 10$ [153]. With respect to charged-lepton mass ratios, the model best-fit point is found to correspond to $\max(\mathrm{BG}) \simeq 0.85$.

Up to an overall normalisation $\mathcal{K}$, the light neutrino mass matrix is instead given by:

$$M_\nu \simeq \mathcal{K}\epsilon \begin{pmatrix} 0 & 0 & \tilde{g}_3 \\ 0 & 0 & \tilde{g}_2\tilde{g}_3 \\ \tilde{g}_3 & \tilde{g}_2\tilde{g}_3 & 2i\sqrt{\frac{2}{3}}\tilde{g}_3^2 \end{pmatrix} \tag{10.6}$$

at leading order in $|\epsilon|$, where $\tilde{g}_i \equiv g_i/g_1$, $i = 2, 3$. Note that the smallness of $|\epsilon|$ does not constrain the M_ν contribution to the mixing matrix, which depends only on the couplings g_i, and large mixing angles are allowed.

[1] The definition of ϵ is motivated by the fact that $\varepsilon/\theta = (1-i)/(1+\sqrt{3})$ at $\tau = \omega$, see (3.5).

From the form of M_ν it is clear that, in the limit of unbroken SUSY, there is a massless neutrino, even though N^c is a triplet. This follows from the modular-symmetric superpotential, which implies the proportionality of the first two columns of $\mathcal{Y}$, reducing its rank and therefore the rank of M_ν. The neutrino masses thus read

$$m_1 = 0\,, \quad m_{2,3} \simeq \sqrt{\frac{2}{3}}\,\mathcal{K}\,\tilde{g}_3^2 \left(\sqrt{1+\frac{3(1+\tilde{g}_2^2)}{2\tilde{g}_3^2}} \mp 1\right)|\epsilon|, \tag{10.7}$$

and imply the ϵ-independent prediction

$$r = \frac{m_2^2 - m_1^2}{m_3^2 - (m_1^2+m_2^2)/2} \simeq \frac{6 + 6\tilde{g}_2^2 + 8\tilde{g}_3^2 - 4\sqrt{6\big(1+\tilde{g}_2^2\big)+4\tilde{g}_3^2}\,|\tilde{g}_3|}{3 + 3\tilde{g}_2^2 + 4\tilde{g}_3^2 + 6\sqrt{6\big(1+\tilde{g}_2^2\big)+4\tilde{g}_3^2}\,|\tilde{g}_3|}\,, \tag{10.8}$$

which, by taking into account the 1σ range for r in table 7.2, isolates a viable region in the plane of coupling constants. At the model best-fit point, the Barbieri-Giudice measure $\max\{|\partial \ln r/\partial \ln \tilde{g}_2|, |\partial \ln r/\partial \ln \tilde{g}_3|\}$ has an acceptable value of 2.9. Additionally, the 3σ ranges for $\tilde{g}_{2,3}$ are not especially narrow.

The result of the fit of this S_4' model is summarised in table 10.1. The viable region in the τ plane corresponds to a neutrino spectrum with NO and is located very close to $\tau_{\rm sym} = \omega$, as can be seen from fig. 10.1. The annular form of the region is explained by the fact that the phase of $(\tau - \omega)$ has no effect on the observables, as it enters only through ϵ and its effects are suppressed by the smallness of $|\epsilon|$. Therefore, in the regime $\tau \simeq \omega$ this model is effectively described by 8 rather than 9 parameters.

To summarise, in the vicinity of the symmetric point, i.e., for small $|\epsilon|$, this model can naturally lead to the observed charged-lepton mass hierarchies, see (10.4). The neutrino mass-squared difference ratio r is, in this region, insensitive to ϵ and depends only on the two ratios $\tilde{g}_{2,3}$ of neutrino couplings, see (10.8). Furthermore, it is not especially sensitive to these couplings. Finally, since light neutrino masses vanish in the symmetric point, the symmetric limit allows for a generic mixing matrix (case 4 of section 6.2). Therefore, the fit is not expected to be tuned in a way that compensates some "wrong PMNS" symmetric prediction.

In fact, we have numerically verified that sending $\tau \to \omega$ ($\epsilon \to 0$) has almost no effect on the values of mixing angles. This can be understood by considering, in turn, each of the contributions to the mixing matrix. The rotation to the mass

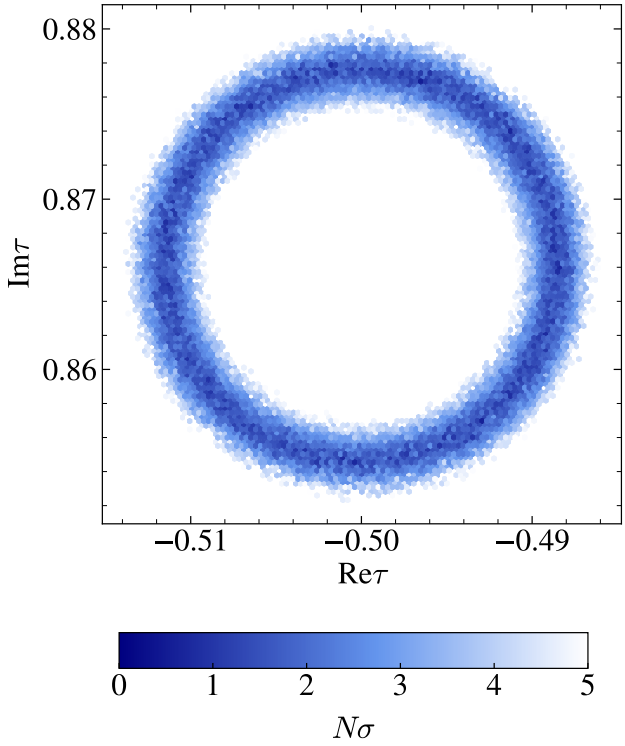

Figure 10.1.: Allowed region in the τ plane for the S_4' model free from fine-tuning. Points outside the fundamental domain, while redundant, are kept for illustrative purposes.

basis in the neutrino sector, on the one hand, is independent of ϵ in the region of interest, see (10.6), and thus has a well-defined limit as $\epsilon \to 0$ (it is unchanged) even though light neutrinos become massless. This rotation depends only on the ratios $\tilde{g}_{2,3}$ of neutrino couplings. On the other hand, one can check that the charged-lepton rotation arising from the diagonalisation of $M_e M_e^\dagger$, with $M_e^\dagger$ given in (10.3), also has a well-defined limit as $\epsilon \to 0$ even though two of the three charged leptons become massless. This limiting form closely matches the rotation obtained for finite, non-zero ϵ, and depends only on the ratios $\tilde{\alpha}_{3,4,5}$ of charged-lepton couplings.

Table 10.1.: Best-fit values along with 3σ ranges of the parameters and observables in the S_4' model free from fine-tuning.

	Best-fit value	3σ range
$\mathrm{Re}\,\tau$	-0.496	$-(0.487 - 0.513)$
$\mathrm{Im}\,\tau$	0.877	$0.853 - 0.879$
α_3/α_1	2.45	$2.03 - 2.90$
α_4/α_1	-2.37	$-(2.01 - 2.67)$
α_5/α_1	1.01	$0.95 - 1.07$
g_2/g_1	1.5	$1.36 - 1.65$
g_3/g_1	2.22	$2.07 - 2.39$
$v_d\,\alpha_1$, GeV	4.61	$3.28 - 5.93$
$v_u^2\,g_1^2/\Lambda$, eV	0.268	$0.205 - 0.325$
$\epsilon(\tau)$	0.0186	$0.0163 - 0.0214$
CL mass pattern	$(1, \epsilon, \epsilon^2)$	
max(BG)	0.848	
m_e/m_μ	0.00475	$0.00423 - 0.000535$
m_μ/m_τ	0.0556	$0.0440 - 0.0691$
r	0.0298	$0.0275 - 0.0317$
δm^2, 10^{-5} eV^2	7.38	$6.94 - 7.73$
$\lvert\Delta m^2\rvert$, 10^{-3} eV^2	2.48	$2.44 - 2.53$
$\sin^2\theta_{12}$	0.304	$0.268 - 0.343$
$\sin^2\theta_{13}$	0.0221	$0.0202 - 0.0240$
$\sin^2\theta_{23}$	0.539	$0.440 - 0.591$
m_1, eV	0	
m_2, eV	0.0086	$0.0083 - 0.0088$
m_3, eV	0.0502	$0.0497 - 0.0506$
$\Sigma_i m_i$, eV	0.0588	$0.0585 - 0.0590$
$\lvert\langle m\rangle\rvert$, eV	0.00144	$0.00111 - 0.00179$
δ/π	$1 \pm O(10^{-6})$	
α_{21}/π	0	
α_{31}/π	$1 \pm O(10^{-5})$	
$N\sigma$	0.563	

Summary and conclusions

In the present thesis we have developed various aspects of the modular symmetry approach to lepton flavour, which remains one of the outstanding fundamental problems in particle physics.

As seen in chapter 2, the field couplings are significantly constrained within this approach, so that the fermion mass matrices are typically expressed in terms of a small number of complex constants g_s and certain functions (modular forms) of a single VEV of a complex scalar field τ (the modulus). In order to construct such mass matrices explicitly in the case of the finite modular groups A_5 and S_4', we have derived in chapter 3 the modular forms furnishing representations of the corresponding groups (see also appendix B) in terms of the Dedekind eta function and the Jacobi theta functions (see also appendix A). We have further shown in chapter 4 that the complex constants g_s can be made real by extending modular symmetry with a CP transformation, thus increasing predictivity of the models. This construction allows also for the attractive possibility of CP violation and the flavour symmetry breaking having a common origin — the VEV of the modulus τ.

By fitting modular-invariant lepton mass matrices to the observed data as explained in chapter 7, one can obtain testable predictions for the neutrino mass ordering, the lightest neutrino mass, the Dirac and the Majorana CPV phases, and the neutrinoless double beta decay effective Majorana mass. An example of such model of lepton flavour has been presented in chapter 8.

However, such "black-box" fitting procedure does not naturally explain the observed flavour patterns and, in particular, the strong hierarchy of fermion masses. Instead, in the overwhelming majority of modular-invariant quark and lepton flavour models available in the literature, they are obtained by fine-tuning the constants g_s. We have shown that it is possible to overcome this problem by making use of the residual symmetries, which exist for certain values of the modulus τ (symmetric points). We have classified these symmetries in chapter 5 (see also appendix C). We have also described two possible constructions which utilise residual symmetries and allow to explain the observed lepton flavour structures.

One of these constructions, presented in chapter 9, assumes that the modu-

lus τ^ℓ entering the charged-lepton mass matrix is different from the modulus τ^ν entering the neutrino mass matrix, $\tau^\ell \neq \tau^\nu$. By setting the moduli VEVs to different symmetric values, one obtains two distinct residual symmetries in the charged-lepton sector and in the neutrino sector. This leads to a specific neutrino mixing pattern, e.g. trimaximal mixing, as in the context of conventional non-Abelian discrete symmetries.

Another construction is based on the analysis of hierarchical mass matrices in the vicinity of the symmetric points. This analysis, performed in chapter 6, shows that a suitable choice of field representations naturally leads to hierarchical charged-lepton masses without forcing the neutrino mixing angles to be small. We have presented a viable model which realises this idea in chapter 10. We have also derived all possible hierarchical patterns of fermion masses that can be obtained in the vicinity of the symmetric points $\tau = i$, $\tau = i\infty$ and $\tau = \omega$ for all 3-dimensional (possibly reducible) fermion representations of the finite modular groups S_3, A_4', S_4' and A_5' (see appendix D).

To summarise, modular invariance provides a predictive framework possessing rich mathematical structure suitable for the treatment of the flavour problem, and thus having the strong potential to explain the observed flavour patterns, both in the lepton and quark sector of the Standard Model. We are encouraged by the results obtained so far, and we believe that the future is bright for this field as many fundamental questions of this promising novel approach still remain unanswered.

Appendix

A. Dedekind eta and Jacobi theta

The Dedekind eta function is a holomorphic function defined in the complex upper half-plane as

$$\eta(\tau) \equiv q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \tag{A.1}$$

where $q \equiv e^{2\pi i \tau}$ and $\mathrm{Im}\,\tau > 0$. In this work, fractional powers $q^{1/n}$, n being a non-zero integer, should be read as $e^{2\pi i \tau/n}$.

The Jacobi theta functions $\Theta_i(z,\tau)$, $i = 1, \ldots, 4$, (see e.g. [154]) are special functions of two complex variables. We are primarily interested in $\Theta_3(z,\tau)$ defined as[1]

$$\Theta_3(z,\tau) \equiv \sum_{k\in\mathbb{Z}} q^{\frac{k^2}{2}} e^{2\pi i k z}, \tag{A.2}$$

and in the so-called theta constants $\Theta_i(\tau) \equiv \Theta_i(0,\tau)$ which are functions of one complex variable defined in the upper half-plane by

$$\begin{aligned} \Theta_2(\tau) &\equiv \sum_{k\in\mathbb{Z}} q^{\frac{1}{2}\left(k+\frac{1}{2}\right)^2}, \\ \Theta_3(\tau) &\equiv \sum_{k\in\mathbb{Z}} q^{\frac{k^2}{2}}, \\ \Theta_4(\tau) &\equiv \sum_{k\in\mathbb{Z}} (-1)^k q^{\frac{k^2}{2}} \end{aligned} \tag{A.3}$$

(the first theta constant, $\Theta_1(\tau)$, is identically zero). The theta constants transform under the generators of the modular group as

$$\begin{aligned} &\Theta_2(\tau) \xrightarrow{T} e^{\pi i/4}\Theta_2(\tau), && \Theta_2(\tau) \xrightarrow{S} \sqrt{-i\tau}\,\Theta_4(\tau), \\ &\Theta_3(\tau) \xrightarrow{T} \Theta_4(\tau), && \Theta_3(\tau) \xrightarrow{S} \sqrt{-i\tau}\,\Theta_3(\tau), \\ &\Theta_4(\tau) \xrightarrow{T} \Theta_3(\tau), && \Theta_4(\tau) \xrightarrow{S} \sqrt{-i\tau}\,\Theta_2(\tau). \end{aligned} \tag{A.4}$$

Note that in the S transformation the principal value of the square root is assumed.

[1] In the notation of ref. [154] $q \equiv e^{\pi i \tau}$, which corresponds to $q^{1/2}$ in our notation.

Apart from the power series expansions (A.3), the theta constants admit the following infinite product representations:

$$\begin{aligned}\Theta_2(\tau) &= 2\,q^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q^n)\,(1+q^n)^2\,,\\ \Theta_3(\tau) &= \prod_{n=1}^{\infty}(1-q^n)\left(1+q^{n-\frac{1}{2}}\right)^2,\\ \Theta_4(\tau) &= \prod_{n=1}^{\infty}(1-q^n)\left(1-q^{n-\frac{1}{2}}\right)^2.\end{aligned} \tag{A.5}$$

By comparing the product expansions (A.5) with the definition of the Dedekind eta function (A.1), one can relate the theta constants to the Dedekind eta as

$$\Theta_2(\tau) = \frac{2\eta^2(2\tau)}{\eta(\tau)}, \quad \Theta_3(\tau) = \frac{\eta^5(\tau)}{\eta^2\left(\frac{\tau}{2}\right)\eta^2(2\tau)}. \tag{A.6}$$

Finally, using the power series expansions (A.3) one can prove a useful identity:

$$\Theta_3(2\tau) = \frac{1}{2}\left[\Theta_3\left(\frac{\tau}{2}\right) + \Theta_4\left(\frac{\tau}{2}\right)\right]. \tag{A.7}$$

B. Finite modular group theory

B.1. $\Gamma_3 \simeq A_4$

A_4 is the group of even permutations of four objects. It contains $4!/2 = 12$ elements and admits four irreducible representations, namely $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$, and $\mathbf{3}$ (see, e.g., [6]). It can be generated by two elements S and T satisfying

$$S^2 = (ST)^3 = T^3 = \mathbb{1}. \tag{B.1}$$

We collect the representations matrices for the A_4 generators in our working basis in table B.1. The non-trivial Clebsch-Gordan coefficients in the above basis are collected in table B.2.

Table B.1.: Representation matrices for the group generators of different A_4 irreps $\mathbf{r}$.

$\mathbf{r}$	$\rho_{\mathbf{r}}(S)$	$\rho_{\mathbf{r}}(T)$
$\mathbf{1}$	1	1
$\mathbf{1}'$	1	ω
$\mathbf{1}'$	1	ω^2
$\mathbf{3}$	$\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$

B.2. $\Gamma'_4 \simeq S'_4$

The homogeneous finite modular group $S'_4 \equiv \mathrm{SL}(2, \mathbb{Z}_4)$ can be defined by three generators S, T and R satisfying the relations:

$$S^2 = R, \quad T^4 = (ST)^3 = R^2 = \mathbb{1}, \quad TR = RT. \tag{B.2}$$

Table B.2.: Decomposition of all non-trivial tensor products of A_4 irreps, and corresponding Clebsch-Gordan coefficients. Entries of each multiplet entering the tensor product are denoted by α_i and β_i.

Product	Clebsch-Gordan coefficients
$\mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}''$ $\mathbf{1}' \otimes \mathbf{1}'' = \mathbf{1}$ $\mathbf{1}'' \otimes \mathbf{1}'' = \mathbf{1}'$	$\alpha_1\beta_1$
$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'' \oplus \mathbf{3} \oplus \mathbf{3}$	$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$ $\oplus\ \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_3\beta_3$ $\oplus\ \alpha_1\beta_3 + \alpha_2\beta_2 + \alpha_3\beta_1$ $\oplus\ \frac{1}{3}\begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 \\ 2\alpha_2\beta_2 - \alpha_1\beta_3 - \alpha_3\beta_1 \end{pmatrix}$ $\oplus\ \frac{1}{2}\begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix}$

It is a group of 48 elements (twice as many as S_4), with group ID `(48,30)` in the computer algebra system GAP [123, 124]. It admits 10 irreps: 4 one-dimensional, 2 two-dimensional, and 4 three-dimensional, which we denote by

$$\mathbf{1},\ \hat{\mathbf{1}},\ \mathbf{1}',\ \hat{\mathbf{1}}',\ \mathbf{2},\ \hat{\mathbf{2}},\ \mathbf{3},\ \hat{\mathbf{3}},\ \mathbf{3}',\ \hat{\mathbf{3}}'. \tag{B.3}$$

The notation has been chosen such that irreps without a hat have a direct correspondence with S_4 irreps, whereas hatted irreps are novel and specific to S_4'. In fact, for the hatless irreps, the new generator R is represented by the identity matrix and the construction effectively reduces to that of $S_4 \simeq S_4' \,/\, \{R = \mathbb{1}\}$. We also note that the hatless irreps are real, while the hatted irreps are complex except for $\hat{\mathbf{2}}$ which is pseudoreal.

The 48 elements of S_4' are organised into 10 conjugacy classes. The character table is given in table B.3 and shows at least one representative element for each class.

In table B.4, we summarise the working basis for the representation matrices of the group generators S, T and R. We present the non-trivial Clebsch-Gordan coefficients in the above basis in tables B.5 to B.8.

Table B.3.: Character table for S_4', obtained via the GAP `Irr()` function. nC_k denotes a conjugacy class of n elements of order k.

	Rep. element(s)	$\mathbf{1}$	$\hat{\mathbf{1}}$	$\mathbf{1'}$	$\hat{\mathbf{1}}'$	$\mathbf{2}$	$\hat{\mathbf{2}}$	$\mathbf{3}$	$\hat{\mathbf{3}}$	$\mathbf{3'}$	$\hat{\mathbf{3}}'$
$1C_1$	$\mathbb{1}$	1	1	1	1	2	2	3	3	3	3
$1C_2$	R	1	−1	1	−1	2	−2	3	−3	3	−3
$3C_2$	T^2	1	−1	1	−1	2	−2	−1	1	−1	1
$3\hat{C}_2$	RT^2	1	1	1	1	2	2	−1	−1	−1	−1
$6C_4$	S	1	i	−1	$-i$	0	0	1	i	−1	$-i$
$6\hat{C}_4$	$RS = S^{-1}$	1	$-i$	−1	i	0	0	1	$-i$	−1	i
$6C_4'$	T	1	$-i$	−1	i	0	0	−1	i	1	$-i$
$6\hat{C}_4'$	RT, T^{-1}	1	i	−1	$-i$	0	0	−1	$-i$	1	i
$8C_3$	ST	1	1	1	1	−1	−1	0	0	0	0
$8C_6$	RST	1	−1	1	−1	−1	1	0	0	0	0

B.3. $\Gamma_5 \simeq A_5$

A_5 is the group of even permutations of five objects. It contains $5!/2 = 60$ elements and admits five irreducible representations, namely **1**, **3**, **3′**, **4** and **5** (see, e.g., [6]). It can be generated by two elements S and T satisfying

$$S^2 = (ST)^3 = T^5 = \mathbb{1}. \tag{B.4}$$

We will employ the group theoretical results of ref. [155], using in particular the explicit basis for the A_5 generators summarised in table B.9. The non-trivial Clebsch-Gordan coefficients in the above basis are collected in tables B.10 to B.15.

Table B.4.: Representation matrices for the group generators of different S_4' irreps $\mathbf{r}$.

$\mathbf{r}$	$\rho_{\mathbf{r}}(S)$	$\rho_{\mathbf{r}}(T)$	$\rho_{\mathbf{r}}(R)$
$\mathbf{1}$	1	1	1
$\hat{\mathbf{1}}$	i	$-i$	-1
$\mathbf{1}'$	-1	-1	1
$\hat{\mathbf{1}}'$	$-i$	i	-1
$\mathbf{2}$	$\frac{1}{2}\begin{pmatrix}-1 & \sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$
$\hat{\mathbf{2}}$	$\frac{i}{2}\begin{pmatrix}-1 & \sqrt{3}\\ \sqrt{3} & 1\end{pmatrix}$	$\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}$	$-\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$
$\mathbf{3}$	$-\frac{1}{2}\begin{pmatrix}0 & \sqrt{2} & \sqrt{2}\\ \sqrt{2} & -1 & 1\\ \sqrt{2} & 1 & -1\end{pmatrix}$	$\begin{pmatrix}-1 & 0 & 0\\ 0 & -i & 0\\ 0 & 0 & i\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$
$\hat{\mathbf{3}}$	$-\frac{i}{2}\begin{pmatrix}0 & \sqrt{2} & \sqrt{2}\\ \sqrt{2} & -1 & 1\\ \sqrt{2} & 1 & -1\end{pmatrix}$	$\begin{pmatrix}i & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}$	$-\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$
$\mathbf{3}'$	$\frac{1}{2}\begin{pmatrix}0 & \sqrt{2} & \sqrt{2}\\ \sqrt{2} & -1 & 1\\ \sqrt{2} & 1 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\ 0 & i & 0\\ 0 & 0 & -i\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$
$\hat{\mathbf{3}}'$	$\frac{i}{2}\begin{pmatrix}0 & \sqrt{2} & \sqrt{2}\\ \sqrt{2} & -1 & 1\\ \sqrt{2} & 1 & -1\end{pmatrix}$	$\begin{pmatrix}-i & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}$	$-\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

Table B.5.: Decomposition of all non-trivial tensor products involving 1-dimensional S_4' irreps, and corresponding Clebsch-Gordan coefficients. Entries of each multiplet entering the tensor product are denoted by α_i and β_i.

Product	Clebsch-Gordan coefficients
$\mathbf{1'} \otimes \mathbf{1'} = \mathbf{1}$ $\mathbf{1'} \otimes \hat{\mathbf{1}} = \hat{\mathbf{1}}'$ $\mathbf{1'} \otimes \hat{\mathbf{1}}' = \hat{\mathbf{1}}$ $\hat{\mathbf{1}} \otimes \hat{\mathbf{1}} = \mathbf{1'}$ $\hat{\mathbf{1}} \otimes \hat{\mathbf{1}}' = \mathbf{1}$ $\hat{\mathbf{1}}' \otimes \hat{\mathbf{1}}' = \mathbf{1'}$	$\alpha_1\beta_1$
$\mathbf{1'} \otimes \mathbf{2} = \mathbf{2}$ $\hat{\mathbf{1}}' \otimes \mathbf{2} = \hat{\mathbf{2}}$ $\mathbf{1'} \otimes \hat{\mathbf{2}} = \hat{\mathbf{2}}$ $\hat{\mathbf{1}} \otimes \hat{\mathbf{2}} = \mathbf{2}$	$\alpha_1 \begin{pmatrix} \beta_2 \\ -\beta_1 \end{pmatrix}$
$\hat{\mathbf{1}} \otimes \mathbf{2} = \hat{\mathbf{2}}$ $\hat{\mathbf{1}}' \otimes \hat{\mathbf{2}} = \mathbf{2}$	$\alpha_1 \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$
$\mathbf{1'} \otimes \mathbf{3} = \mathbf{3'}$ $\hat{\mathbf{1}} \otimes \mathbf{3} = \hat{\mathbf{3}}$ $\hat{\mathbf{1}}' \otimes \mathbf{3} = \hat{\mathbf{3}}'$ $\mathbf{1'} \otimes \mathbf{3'} = \mathbf{3}$ $\hat{\mathbf{1}} \otimes \mathbf{3'} = \hat{\mathbf{3}}'$ $\hat{\mathbf{1}}' \otimes \mathbf{3'} = \hat{\mathbf{3}}$ $\mathbf{1'} \otimes \hat{\mathbf{3}} = \hat{\mathbf{3}}'$ $\hat{\mathbf{1}} \otimes \hat{\mathbf{3}} = \mathbf{3'}$ $\hat{\mathbf{1}}' \otimes \hat{\mathbf{3}} = \mathbf{3}$ $\mathbf{1'} \otimes \hat{\mathbf{3}}' = \hat{\mathbf{3}}$ $\hat{\mathbf{1}} \otimes \hat{\mathbf{3}}' = \mathbf{3}$ $\hat{\mathbf{1}}' \otimes \hat{\mathbf{3}}' = \mathbf{3'}$	$\alpha_1 \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$

Table B.6.: Decomposition of tensor products involving two 2-dimensional S_4' irreps, and corresponding Clebsch-Gordan coefficients. Note that the order is important to match the left and right columns.

Product	Clebsch-Gordan coefficients
$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{2}$ $\mathbf{2} \otimes \hat{\mathbf{2}} = \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{2}}$	$\frac{1}{\sqrt{2}}(\alpha_1\beta_1 + \alpha_2\beta_2)$ $\oplus \frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \alpha_2\beta_1)$ $\oplus \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha_2\beta_2 - \alpha_1\beta_1 \\ \alpha_1\beta_2 + \alpha_2\beta_1\end{pmatrix}$
$\hat{\mathbf{2}} \otimes \hat{\mathbf{2}} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{2}$	$\frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \alpha_2\beta_1)$ $\oplus \frac{1}{\sqrt{2}}(\alpha_1\beta_1 + \alpha_2\beta_2)$ $\oplus \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha_1\beta_2 + \alpha_2\beta_1 \\ \alpha_1\beta_1 - \alpha_2\beta_2\end{pmatrix}$

Table B.7.: The same as in table B.6, but for products involving a 2-dimensional and a 3-dimensional irrep.

Product	Clebsch-Gordan coefficients
$\mathbf{2} \otimes \mathbf{3} = \mathbf{3} \oplus \mathbf{3}'$ $\mathbf{2} \otimes \hat{\mathbf{3}} = \hat{\mathbf{3}} \oplus \hat{\mathbf{3}}'$ $\hat{\mathbf{2}} \otimes \mathbf{3} = \hat{\mathbf{3}} \oplus \hat{\mathbf{3}}'$ $\hat{\mathbf{2}} \otimes \hat{\mathbf{3}}' = \mathbf{3} \oplus \mathbf{3}'$	$\begin{pmatrix} \alpha_1 \beta_1 \\ \left(\sqrt{3}/2\right) \alpha_2 \beta_3 - (1/2)\, \alpha_1 \beta_2 \\ \left(\sqrt{3}/2\right) \alpha_2 \beta_2 - (1/2)\, \alpha_1 \beta_3 \end{pmatrix}$ $\oplus \begin{pmatrix} -\alpha_2 \beta_1 \\ \left(\sqrt{3}/2\right) \alpha_1 \beta_3 + (1/2)\, \alpha_2 \beta_2 \\ \left(\sqrt{3}/2\right) \alpha_1 \beta_2 + (1/2)\, \alpha_2 \beta_3 \end{pmatrix}$
$\mathbf{2} \otimes \mathbf{3}' = \mathbf{3} \oplus \mathbf{3}'$ $\mathbf{2} \otimes \hat{\mathbf{3}}' = \hat{\mathbf{3}} \oplus \hat{\mathbf{3}}'$ $\hat{\mathbf{2}} \otimes \mathbf{3}' = \hat{\mathbf{3}} \oplus \hat{\mathbf{3}}'$ $\hat{\mathbf{2}} \otimes \hat{\mathbf{3}} = \mathbf{3} \oplus \mathbf{3}'$	$\begin{pmatrix} -\alpha_2 \beta_1 \\ \left(\sqrt{3}/2\right) \alpha_1 \beta_3 + (1/2)\, \alpha_2 \beta_2 \\ \left(\sqrt{3}/2\right) \alpha_1 \beta_2 + (1/2)\, \alpha_2 \beta_3 \end{pmatrix}$ $\oplus \begin{pmatrix} \alpha_1 \beta_1 \\ \left(\sqrt{3}/2\right) \alpha_2 \beta_3 - (1/2)\, \alpha_1 \beta_2 \\ \left(\sqrt{3}/2\right) \alpha_2 \beta_2 - (1/2)\, \alpha_1 \beta_3 \end{pmatrix}$

Table B.8.: The same as in table B.6, but for products involving two 3-dimensional irreps.

Product	Clebsch-Gordan coefficients
$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3}'$ $\mathbf{3} \otimes \hat{\mathbf{3}} = \hat{\mathbf{1}} \oplus \hat{\mathbf{2}} \oplus \hat{\mathbf{3}} \oplus \hat{\mathbf{3}}'$ $\mathbf{3}' \otimes \mathbf{3}' = \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3}'$ $\mathbf{3}' \otimes \hat{\mathbf{3}}' = \hat{\mathbf{1}} \oplus \hat{\mathbf{2}} \oplus \hat{\mathbf{3}} \oplus \hat{\mathbf{3}}'$ $\hat{\mathbf{3}} \otimes \hat{\mathbf{3}}' = \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3}'$	$\frac{1}{\sqrt{3}}(\alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2)$ $\oplus \frac{1}{\sqrt{2}}\begin{pmatrix}(2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2)/\sqrt{3}\\ \alpha_2\beta_2 + \alpha_3\beta_3\end{pmatrix}$ $\oplus \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha_3\beta_3 - \alpha_2\beta_2\\ \alpha_1\beta_3 + \alpha_3\beta_1\\ -\alpha_1\beta_2 - \alpha_2\beta_1\end{pmatrix}$ $\oplus \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha_3\beta_2 - \alpha_2\beta_3\\ \alpha_2\beta_1 - \alpha_1\beta_2\\ \alpha_1\beta_3 - \alpha_3\beta_1\end{pmatrix}$
$\mathbf{3} \otimes \mathbf{3}' = \mathbf{1}' \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3}'$ $\mathbf{3} \otimes \hat{\mathbf{3}}' = \hat{\mathbf{1}}' \oplus \hat{\mathbf{2}} \oplus \hat{\mathbf{3}} \oplus \hat{\mathbf{3}}'$ $\mathbf{3}' \otimes \hat{\mathbf{3}} = \hat{\mathbf{1}}' \oplus \hat{\mathbf{2}} \oplus \hat{\mathbf{3}} \oplus \hat{\mathbf{3}}'$ $\hat{\mathbf{3}} \otimes \hat{\mathbf{3}} = \mathbf{1}' \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3}'$ $\hat{\mathbf{3}}' \otimes \hat{\mathbf{3}}' = \mathbf{1}' \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3}'$	$\frac{1}{\sqrt{3}}(\alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2)$ $\oplus \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha_2\beta_2 + \alpha_3\beta_3\\ (-2\alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2)/\sqrt{3}\end{pmatrix}$ $\oplus \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha_3\beta_2 - \alpha_2\beta_3\\ \alpha_2\beta_1 - \alpha_1\beta_2\\ \alpha_1\beta_3 - \alpha_3\beta_1\end{pmatrix}$ $\oplus \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha_3\beta_3 - \alpha_2\beta_2\\ \alpha_1\beta_3 + \alpha_3\beta_1\\ -\alpha_1\beta_2 - \alpha_2\beta_1\end{pmatrix}$

Table B.9.: Representation matrices for the group generators of different A_5 irreps $\mathbf{r}$. Here $\zeta \equiv e^{2\pi i/5}$ and $\varphi \equiv (1+\sqrt{5})/2$.

$\mathbf{r}$	$\rho_{\mathbf{r}}(S)$	$\rho_{\mathbf{r}}(T)$
$\mathbf{1}$	1	1
$\mathbf{3}$	$\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\varphi & 1/\varphi \\ -\sqrt{2} & 1/\varphi & -\varphi \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^4 \end{pmatrix}$
$\mathbf{3'}$	$\frac{1}{\sqrt{5}}\begin{pmatrix} -1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1/\varphi & \varphi \\ \sqrt{2} & \varphi & -1/\varphi \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^3 \end{pmatrix}$
$\mathbf{4}$	$\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 1/\varphi & \varphi & -1 \\ 1/\varphi & -1 & 1 & \varphi \\ \varphi & 1 & -1 & 1/\varphi \\ -1 & \varphi & 1/\varphi & 1 \end{pmatrix}$	$\begin{pmatrix} \zeta & 0 & 0 & 0 \\ 0 & \zeta^2 & 0 & 0 \\ 0 & 0 & \zeta^3 & 0 \\ 0 & 0 & 0 & \zeta^4 \end{pmatrix}$
$\mathbf{5}$	$\frac{1}{5}\begin{pmatrix} -1 & \sqrt{6} & \sqrt{6} & \sqrt{6} & \sqrt{6} \\ \sqrt{6} & 1/\varphi^2 & -2\varphi & 2/\varphi & \varphi^2 \\ \sqrt{6} & -2\varphi & \varphi^2 & 1/\varphi^2 & 2/\varphi \\ \sqrt{6} & 2/\varphi & 1/\varphi^2 & \varphi^2 & -2\varphi \\ \sqrt{6} & \varphi^2 & 2/\varphi & -2\varphi & 1/\varphi^2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 & 0 \\ 0 & 0 & \zeta^2 & 0 & 0 \\ 0 & 0 & 0 & \zeta^3 & 0 \\ 0 & 0 & 0 & 0 & \zeta^4 \end{pmatrix}$

Table B.10.: Decomposition of all non-trivial tensor products of A_5 irreps, and corresponding Clebsch-Gordan coefficients. Entries of each multiplet entering the tensor product are denoted by α_i and β_i.

Product	Clebsch-Gordan coefficients
$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$	$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$ $\oplus \begin{pmatrix} \alpha_2\,\beta_3 - \alpha_3\,\beta_2 \\ \alpha_1\,\beta_2 - \alpha_2\,\beta_1 \\ \alpha_3\,\beta_1 - \alpha_1\,\beta_3 \end{pmatrix}$ $\oplus \begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ -\sqrt{3}\,\alpha_1\beta_2 - \sqrt{3}\,\alpha_2\beta_1 \\ \sqrt{6}\,\alpha_2\beta_2 \\ \sqrt{6}\,\alpha_3\beta_3 \\ -\sqrt{3}\,\alpha_1\beta_3 - \sqrt{3}\,\alpha_3\beta_1 \end{pmatrix}$
$\mathbf{3}' \otimes \mathbf{3}' = \mathbf{1} \oplus \mathbf{3}' \oplus \mathbf{5}$	$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$ $\oplus \begin{pmatrix} \alpha_2\,\beta_3 - \alpha_3\,\beta_2 \\ \alpha_1\,\beta_2 - \alpha_2\,\beta_1 \\ \alpha_3\,\beta_1 - \alpha_1\,\beta_3 \end{pmatrix}$ $\oplus \begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ \sqrt{6}\,\alpha_3\beta_3 \\ -\sqrt{3}\,\alpha_1\beta_2 - \sqrt{3}\,\alpha_2\beta_1 \\ -\sqrt{3}\,\alpha_1\beta_3 - \sqrt{3}\,\alpha_3\beta_1 \\ \sqrt{6}\,\alpha_2\beta_2 \end{pmatrix}$
$\mathbf{3} \otimes \mathbf{3}' = \mathbf{4} \oplus \mathbf{5}$	$\begin{pmatrix} \sqrt{2}\,\alpha_2\beta_1 + \alpha_3\beta_2 \\ -\sqrt{2}\,\alpha_1\beta_2 - \alpha_3\beta_3 \\ -\sqrt{2}\,\alpha_1\beta_3 - \alpha_2\beta_2 \\ \sqrt{2}\,\alpha_3\beta_1 + \alpha_2\beta_3 \end{pmatrix}$ $\oplus \begin{pmatrix} \sqrt{3}\,\alpha_1\beta_1 \\ \alpha_2\beta_1 - \sqrt{2}\,\alpha_3\beta_2 \\ \alpha_1\beta_2 - \sqrt{2}\,\alpha_3\beta_3 \\ \alpha_1\beta_3 - \sqrt{2}\,\alpha_2\beta_2 \\ \alpha_3\beta_1 - \sqrt{2}\,\alpha_2\beta_3 \end{pmatrix}$

Table B.11.: Decomposition of all non-trivial tensor products of A_5 irreps, and corresponding Clebsch-Gordan coefficients (continued).

Product	Clebsch-Gordan coefficients
$\mathbf{3} \otimes \mathbf{4} = \mathbf{3'} \oplus \mathbf{4} \oplus \mathbf{5}$	$\begin{pmatrix} -\sqrt{2}\,\alpha_2\beta_4 - \sqrt{2}\,\alpha_3\beta_1 \\ \sqrt{2}\,\alpha_1\beta_2 - \alpha_2\beta_1 + \alpha_3\beta_3 \\ \sqrt{2}\,\alpha_1\beta_3 + \alpha_2\beta_2 - \alpha_3\beta_4 \end{pmatrix}$ $\oplus \begin{pmatrix} \alpha_1\beta_1 - \sqrt{2}\,\alpha_3\beta_2 \\ -\alpha_1\beta_2 - \sqrt{2}\,\alpha_2\beta_1 \\ \alpha_1\beta_3 + \sqrt{2}\,\alpha_3\beta_4 \\ -\alpha_1\beta_4 + \sqrt{2}\,\alpha_2\beta_3 \end{pmatrix}$ $\oplus \begin{pmatrix} \sqrt{6}\,\alpha_2\beta_4 - \sqrt{6}\,\alpha_3\beta_1 \\ 2\sqrt{2}\,\alpha_1\beta_1 + 2\alpha_3\beta_2 \\ -\sqrt{2}\,\alpha_1\beta_2 + \alpha_2\beta_1 + 3\alpha_3\beta_3 \\ \sqrt{2}\,\alpha_1\beta_3 - 3\alpha_2\beta_2 - \alpha_3\beta_4 \\ -2\sqrt{2}\,\alpha_1\beta_4 - 2\alpha_2\beta_3 \end{pmatrix}$
$\mathbf{3'} \otimes \mathbf{4} = \mathbf{3} \oplus \mathbf{4} \oplus \mathbf{5}$	$\begin{pmatrix} -\sqrt{2}\,\alpha_2\beta_3 - \sqrt{2}\,\alpha_3\beta_2 \\ \sqrt{2}\,\alpha_1\beta_1 + \alpha_2\beta_4 - \alpha_3\beta_3 \\ \sqrt{2}\,\alpha_1\beta_4 - \alpha_2\beta_2 + \alpha_3\beta_1 \end{pmatrix}$ $\oplus \begin{pmatrix} \alpha_1\beta_1 + \sqrt{2}\,\alpha_3\beta_3 \\ \alpha_1\beta_2 - \sqrt{2}\,\alpha_3\beta_4 \\ -\alpha_1\beta_3 + \sqrt{2}\,\alpha_2\beta_1 \\ -\alpha_1\beta_4 - \sqrt{2}\,\alpha_2\beta_2 \end{pmatrix}$ $\oplus \begin{pmatrix} \sqrt{6}\,\alpha_2\beta_3 - \sqrt{6}\,\alpha_3\beta_2 \\ \sqrt{2}\,\alpha_1\beta_1 - 3\alpha_2\beta_4 - \alpha_3\beta_3 \\ 2\sqrt{2}\,\alpha_1\beta_2 + 2\alpha_3\beta_4 \\ -2\sqrt{2}\,\alpha_1\beta_3 - 2\alpha_2\beta_1 \\ -\sqrt{2}\,\alpha_1\beta_4 + \alpha_2\beta_2 + 3\alpha_3\beta_1 \end{pmatrix}$

Table B.12.: Decomposition of all non-trivial tensor products of A_5 irreps, and corresponding Clebsch-Gordan coefficients (continued).

Product	Clebsch-Gordan coefficients
$\mathbf{3}\otimes\mathbf{5}=\mathbf{3}\oplus\mathbf{3'}\oplus\mathbf{4}\oplus\mathbf{5}$	$\begin{pmatrix} -2\alpha_1\beta_1+\sqrt{3}\,\alpha_2\beta_5+\sqrt{3}\,\alpha_3\beta_2 \\ \sqrt{3}\,\alpha_1\beta_2+\alpha_2\beta_1-\sqrt{6}\,\alpha_3\beta_3 \\ \sqrt{3}\,\alpha_1\beta_5-\sqrt{6}\,\alpha_2\beta_4+\alpha_3\beta_1 \end{pmatrix}$ $\oplus\begin{pmatrix} \sqrt{3}\,\alpha_1\beta_1+\alpha_2\beta_5+\alpha_3\beta_2 \\ \alpha_1\beta_3-\sqrt{2}\,\alpha_2\beta_2-\sqrt{2}\,\alpha_3\beta_4 \\ \alpha_1\beta_4-\sqrt{2}\,\alpha_2\beta_3-\sqrt{2}\,\alpha_3\beta_5 \end{pmatrix}$ $\oplus\begin{pmatrix} 2\sqrt{2}\,\alpha_1\beta_2-\sqrt{6}\,\alpha_2\beta_1+\alpha_3\beta_3 \\ -\sqrt{2}\,\alpha_1\beta_3+2\alpha_2\beta_2-3\alpha_3\beta_4 \\ \sqrt{2}\,\alpha_1\beta_4+3\alpha_2\beta_3-2\alpha_3\beta_5 \\ -2\sqrt{2}\,\alpha_1\beta_5-\alpha_2\beta_4+\sqrt{6}\,\alpha_3\beta_1 \end{pmatrix}$ $\oplus\begin{pmatrix} \sqrt{3}\,\alpha_2\beta_5-\sqrt{3}\,\alpha_3\beta_2 \\ -\alpha_1\beta_2-\sqrt{3}\,\alpha_2\beta_1-\sqrt{2}\,\alpha_3\beta_3 \\ -2\alpha_1\beta_3-\sqrt{2}\,\alpha_2\beta_2 \\ 2\alpha_1\beta_4+\sqrt{2}\,\alpha_3\beta_5 \\ \alpha_1\beta_5+\sqrt{2}\,\alpha_2\beta_4+\sqrt{3}\,\alpha_3\beta_1 \end{pmatrix}$
$\mathbf{3'}\otimes\mathbf{5}=\mathbf{3}\oplus\mathbf{3'}\oplus\mathbf{4}\oplus\mathbf{5}$	$\begin{pmatrix} \sqrt{3}\,\alpha_1\beta_1+\alpha_2\beta_4+\alpha_3\beta_3 \\ \alpha_1\beta_2-\sqrt{2}\,\alpha_2\beta_5-\sqrt{2}\,\alpha_3\beta_4 \\ \alpha_1\beta_5-\sqrt{2}\,\alpha_2\beta_3-\sqrt{2}\,\alpha_3\beta_2 \end{pmatrix}$ $\oplus\begin{pmatrix} -2\alpha_1\beta_1+\sqrt{3}\,\alpha_2\beta_4+\sqrt{3}\,\alpha_3\beta_3 \\ \sqrt{3}\,\alpha_1\beta_3+\alpha_2\beta_1-\sqrt{6}\,\alpha_3\beta_5 \\ \sqrt{3}\,\alpha_1\beta_4-\sqrt{6}\,\alpha_2\beta_2+\alpha_3\beta_1 \end{pmatrix}$ $\oplus\begin{pmatrix} \sqrt{2}\,\alpha_1\beta_2+3\alpha_2\beta_5-2\alpha_3\beta_4 \\ 2\sqrt{2}\,\alpha_1\beta_3-\sqrt{6}\,\alpha_2\beta_1+\alpha_3\beta_5 \\ -2\sqrt{2}\,\alpha_1\beta_4-\alpha_2\beta_2+\sqrt{6}\,\alpha_3\beta_1 \\ -\sqrt{2}\,\alpha_1\beta_5+2\alpha_2\beta_3-3\alpha_3\beta_2 \end{pmatrix}$ $\oplus\begin{pmatrix} \sqrt{3}\,\alpha_2\beta_4-\sqrt{3}\alpha_3\beta_3 \\ 2\alpha_1\beta_2+\sqrt{2}\,\alpha_3\beta_4 \\ -\alpha_1\beta_3-\sqrt{3}\,\alpha_2\beta_1-\sqrt{2}\,\alpha_3\beta_5 \\ \alpha_1\beta_4+\sqrt{2}\,\alpha_2\beta_2+\sqrt{3}\,\alpha_3\beta_1 \\ -2\alpha_1\beta_5-\sqrt{2}\,\alpha_2\beta_3 \end{pmatrix}$

Table B.13.: Decomposition of all non-trivial tensor products of A_5 irreps, and corresponding Clebsch-Gordan coefficients (continued).

Product	Clebsch-Gordan coefficients
$\mathbf{4} \otimes \mathbf{4} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{4} \oplus \mathbf{5}$	$\alpha_1\beta_4 + \alpha_2\beta_3 + \alpha_3\beta_2 + \alpha_4\beta_1$ $\oplus \begin{pmatrix} -\alpha_1\beta_4 + \alpha_2\beta_3 - \alpha_3\beta_2 + \alpha_4\beta_1 \\ \sqrt{2}\,\alpha_2\beta_4 - \sqrt{2}\,\alpha_4\beta_2 \\ \sqrt{2}\,\alpha_1\beta_3 - \sqrt{2}\,\alpha_3\beta_1 \end{pmatrix}$ $\oplus \begin{pmatrix} \alpha_1\beta_4 + \alpha_2\beta_3 - \alpha_3\beta_2 - \alpha_4\beta_1 \\ \sqrt{2}\,\alpha_3\beta_4 - \sqrt{2}\,\alpha_4\beta_3 \\ \sqrt{2}\,\alpha_1\beta_2 - \sqrt{2}\,\alpha_2\beta_1 \end{pmatrix}$ $\oplus \begin{pmatrix} \alpha_2\beta_4 + \alpha_3\beta_3 + \alpha_4\beta_2 \\ \alpha_1\beta_1 + \alpha_3\beta_4 + \alpha_4\beta_3 \\ \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_4\beta_4 \\ \alpha_1\beta_3 + \alpha_2\beta_2 + \alpha_3\beta_1 \end{pmatrix}$ $\oplus \begin{pmatrix} \sqrt{3}\,\alpha_1\beta_4 - \sqrt{3}\,\alpha_2\beta_3 - \sqrt{3}\,\alpha_3\beta_2 + \sqrt{3}\,\alpha_4\beta_1 \\ -\sqrt{2}\,\alpha_2\beta_4 + 2\sqrt{2}\,\alpha_3\beta_3 - \sqrt{2}\,\alpha_4\beta_2 \\ -2\sqrt{2}\,\alpha_1\beta_1 + \sqrt{2}\,\alpha_3\beta_4 + \sqrt{2}\,\alpha_4\beta_3 \\ \sqrt{2}\,\alpha_1\beta_2 + \sqrt{2}\,\alpha_2\beta_1 - 2\sqrt{2}\,\alpha_4\beta_4 \\ -\sqrt{2}\,\alpha_1\beta_3 + 2\sqrt{2}\,\alpha_2\beta_2 - \sqrt{2}\,\alpha_3\beta_1 \end{pmatrix}$

Table B.14.: Decomposition of all non-trivial tensor products of A_5 irreps, and corresponding Clebsch-Gordan coefficients (continued).

Product	Clebsch-Gordan coefficients
$\mathbf{4} \otimes \mathbf{5} =$ $\mathbf{3} \oplus \mathbf{3'} \oplus \mathbf{4}$ $\oplus\, \mathbf{5}_1 \oplus \mathbf{5}_2$	$\begin{pmatrix} 2\sqrt{2}\,\alpha_1\beta_5 - \sqrt{2}\,\alpha_2\beta_4 + \sqrt{2}\,\alpha_3\beta_3 - 2\sqrt{2}\,\alpha_4\beta_2 \\ -\sqrt{6}\,\alpha_1\beta_1 + 2\alpha_2\beta_5 + 3\alpha_3\beta_4 - \alpha_4\beta_3 \\ \alpha_1\beta_4 - 3\alpha_2\beta_3 - 2\alpha_3\beta_2 + \sqrt{6}\,\alpha_4\beta_1 \end{pmatrix}$ $\oplus \begin{pmatrix} \sqrt{2}\,\alpha_1\beta_5 + 2\sqrt{2}\,\alpha_2\beta_4 - 2\sqrt{2}\,\alpha_3\beta_3 - \sqrt{2}\,\alpha_4\beta_2 \\ 3\alpha_1\beta_2 - \sqrt{6}\,\alpha_2\beta_1 - \alpha_3\beta_5 + 2\alpha_4\beta_4 \\ -2\alpha_1\beta_3 + \alpha_2\beta_2 + \sqrt{6}\,\alpha_3\beta_1 - 3\alpha_4\beta_5 \end{pmatrix}$ $\oplus \begin{pmatrix} \sqrt{3}\,\alpha_1\beta_1 - \sqrt{2}\,\alpha_2\beta_5 + \sqrt{2}\,\alpha_3\beta_4 - 2\sqrt{2}\,\alpha_4\beta_3 \\ -\sqrt{2}\,\alpha_1\beta_2 - \sqrt{3}\,\alpha_2\beta_1 + 2\sqrt{2}\,\alpha_3\beta_5 + \sqrt{2}\,\alpha_4\beta_4 \\ \sqrt{2}\,\alpha_1\beta_3 + 2\sqrt{2}\,\alpha_2\beta_2 - \sqrt{3}\,\alpha_3\beta_1 - \sqrt{2}\,\alpha_4\beta_5 \\ -2\sqrt{2}\,\alpha_1\beta_4 + \sqrt{2}\,\alpha_2\beta_3 - \sqrt{2}\,\alpha_3\beta_2 + \sqrt{3}\,\alpha_4\beta_1 \end{pmatrix}$ $\oplus \begin{pmatrix} \sqrt{2}\,\alpha_1\beta_5 - \sqrt{2}\,\alpha_2\beta_4 - \sqrt{2}\,\alpha_3\beta_3 + \sqrt{2}\,\alpha_4\beta_2 \\ -\sqrt{2}\,\alpha_1\beta_1 - \sqrt{3}\,\alpha_3\beta_4 - \sqrt{3}\,\alpha_4\beta_3 \\ \sqrt{3}\,\alpha_1\beta_2 + \sqrt{2}\,\alpha_2\beta_1 + \sqrt{3}\,\alpha_3\beta_5 \\ \sqrt{3}\,\alpha_2\beta_2 + \sqrt{2}\,\alpha_3\beta_1 + \sqrt{3}\,\alpha_4\beta_5 \\ -\sqrt{3}\,\alpha_1\beta_4 - \sqrt{3}\,\alpha_2\beta_3 - \sqrt{2}\,\alpha_4\beta_1 \end{pmatrix}$ $\oplus \begin{pmatrix} 2\alpha_1\beta_5 + 4\alpha_2\beta_4 + 4\alpha_3\beta_3 + 2\alpha_4\beta_2 \\ 4\alpha_1\beta_1 + 2\sqrt{6}\,\alpha_2\beta_5 \\ -\sqrt{6}\,\alpha_1\beta_2 + 2\alpha_2\beta_1 - \sqrt{6}\,\alpha_3\beta_5 + 2\sqrt{6}\,\alpha_4\beta_4 \\ 2\sqrt{6}\,\alpha_1\beta_3 - \sqrt{6}\,\alpha_2\beta_2 + 2\alpha_3\beta_1 - \sqrt{6}\,\alpha_4\beta_5 \\ 2\sqrt{6}\,\alpha_3\beta_2 + 4\alpha_4\beta_1 \end{pmatrix}$

Table B.15.: Decomposition of all non-trivial tensor products of A_5 irreps, and corresponding Clebsch-Gordan coefficients (continued).

Product	Clebsch-Gordan coefficients
$\mathbf{5} \otimes \mathbf{5} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{4}_1 \oplus \mathbf{4}_2 \oplus \mathbf{5}_1 \oplus \mathbf{5}_2$	$\alpha_1\beta_1 + \alpha_2\beta_5 + \alpha_3\beta_4 + \alpha_4\beta_3 + \alpha_5\beta_2$ $\oplus \begin{pmatrix} \alpha_2\beta_5 + 2\alpha_3\beta_4 - 2\alpha_4\beta_3 - \alpha_5\beta_2 \\ -\sqrt{3}\,\alpha_1\beta_2 + \sqrt{3}\,\alpha_2\beta_1 + \sqrt{2}\,\alpha_3\beta_5 - \sqrt{2}\,\alpha_5\beta_3 \\ \sqrt{3}\,\alpha_1\beta_5 + \sqrt{2}\,\alpha_2\beta_4 - \sqrt{2}\,\alpha_4\beta_2 - \sqrt{3}\,\alpha_5\beta_1 \end{pmatrix}$ $\oplus \begin{pmatrix} 2\alpha_2\beta_5 - \alpha_3\beta_4 + \alpha_4\beta_3 - 2\alpha_5\beta_2 \\ \sqrt{3}\,\alpha_1\beta_3 - \sqrt{3}\,\alpha_3\beta_1 + \sqrt{2}\,\alpha_4\beta_5 - \sqrt{2}\,\alpha_5\beta_4 \\ -\sqrt{3}\,\alpha_1\beta_4 + \sqrt{2}\,\alpha_2\beta_3 - \sqrt{2}\,\alpha_3\beta_2 + \sqrt{3}\,\alpha_4\beta_1 \end{pmatrix}$ $\oplus \begin{pmatrix} 3\sqrt{2}\,\alpha_1\beta_2 + 3\sqrt{2}\,\alpha_2\beta_1 - \sqrt{3}\,\alpha_3\beta_5 + 4\sqrt{3}\,\alpha_4\beta_4 - \sqrt{3}\,\alpha_5\beta_3 \\ 3\sqrt{2}\,\alpha_1\beta_3 + 4\sqrt{3}\,\alpha_2\beta_2 + 3\sqrt{2}\,\alpha_3\beta_1 - \sqrt{3}\,\alpha_4\beta_5 - \sqrt{3}\,\alpha_5\beta_4 \\ 3\sqrt{2}\,\alpha_1\beta_4 - \sqrt{3}\,\alpha_2\beta_3 - \sqrt{3}\,\alpha_3\beta_2 + 3\sqrt{2}\,\alpha_4\beta_1 + 4\sqrt{3}\,\alpha_5\beta_5 \\ 3\sqrt{2}\,\alpha_1\beta_5 - \sqrt{3}\,\alpha_2\beta_4 + 4\sqrt{3}\,\alpha_3\beta_3 - \sqrt{3}\,\alpha_4\beta_2 + 3\sqrt{2}\,\alpha_5\beta_1 \end{pmatrix}$ $\oplus \begin{pmatrix} \sqrt{2}\,\alpha_1\beta_2 - \sqrt{2}\,\alpha_2\beta_1 + \sqrt{3}\,\alpha_3\beta_5 - \sqrt{3}\,\alpha_5\beta_3 \\ -\sqrt{2}\,\alpha_1\beta_3 + \sqrt{2}\,\alpha_3\beta_1 + \sqrt{3}\,\alpha_4\beta_5 - \sqrt{3}\,\alpha_5\beta_4 \\ -\sqrt{2}\,\alpha_1\beta_4 - \sqrt{3}\,\alpha_2\beta_3 + \sqrt{3}\,\alpha_3\beta_2 + \sqrt{2}\,\alpha_4\beta_1 \\ \sqrt{2}\,\alpha_1\beta_5 - \sqrt{3}\,\alpha_2\beta_4 + \sqrt{3}\,\alpha_4\beta_2 - \sqrt{2}\,\alpha_5\beta_1 \end{pmatrix}$ $\oplus \begin{pmatrix} 2\alpha_1\beta_1 + \alpha_2\beta_5 - 2\alpha_3\beta_4 - 2\alpha_4\beta_3 + \alpha_5\beta_2 \\ \alpha_1\beta_2 + \alpha_2\beta_1 + \sqrt{6}\,\alpha_3\beta_5 + \sqrt{6}\,\alpha_5\beta_3 \\ -2\alpha_1\beta_3 + \sqrt{6}\,\alpha_2\beta_2 - 2\alpha_3\beta_1 \\ -2\alpha_1\beta_4 - 2\alpha_4\beta_1 + \sqrt{6}\,\alpha_5\beta_5 \\ \alpha_1\beta_5 + \sqrt{6}\,\alpha_2\beta_4 + \sqrt{6}\,\alpha_4\beta_2 + \alpha_5\beta_1 \end{pmatrix}$ $\oplus \begin{pmatrix} 2\alpha_1\beta_1 - 2\alpha_2\beta_5 + \alpha_3\beta_4 + \alpha_4\beta_3 - 2\alpha_5\beta_2 \\ -2\alpha_1\beta_2 - 2\alpha_2\beta_1 + \sqrt{6}\,\alpha_4\beta_4 \\ \alpha_1\beta_3 + \alpha_3\beta_1 + \sqrt{6}\,\alpha_4\beta_5 + \sqrt{6}\,\alpha_5\beta_4 \\ \alpha_1\beta_4 + \sqrt{6}\,\alpha_2\beta_3 + \sqrt{6}\,\alpha_3\beta_2 + \alpha_4\beta_1 \\ -2\alpha_1\beta_5 + \sqrt{6}\,\alpha_3\beta_3 - 2\alpha_5\beta_1 \end{pmatrix}$

C. Residual group decompositions

The multiplets of Γ'_N are "weighted", i.e., are described by a pair $(\mathbf{r}, k)$. At a symmetric point these multiplets decompose into 1-dimensional representations of the corresponding residual symmetry group. In this appendix we present the decompositions of Γ'_N multiplets ($N \leq 5$) under the three residual groups of interest (tables C.1 to C.4). As seen in chapter 5, these are $\mathbb{Z}_4^S$, $\mathbb{Z}_3^{ST} \times \mathbb{Z}_2^R$ and $\mathbb{Z}_N^T \times \mathbb{Z}_2^R$.

Before proceeding, let us comment on the $\mathbb{Z}_2^R$ factors in $\mathbb{Z}_3^{ST} \times \mathbb{Z}_2^R$ and in $\mathbb{Z}_N^T \times \mathbb{Z}_2^R$. While kept as part of the residual symmetry group definition in this appendix, they have been omitted in the main text of section 6.1. To understand why they can be ignored without loss of generality, note that a direct product $\mathbb{Z}_n \times \mathbb{Z}_2 \equiv \langle a, b \,|\, a^n = b^2 = 1,\ ab = ba \rangle$ has $2n$ irreps $\mathbf{1}_k^{\pm}$, $k = 0, \ldots, n-1$, which are simply given as products of the $\mathbb{Z}_n$ and $\mathbb{Z}_2$ irreps:

$$\mathbf{1}_k^{\pm} \ : \quad \rho(a) = \exp\left(2\pi i \frac{k}{n}\right), \quad \rho(b) = \pm 1. \tag{C.1}$$

In this notation, 1_0^+ is the trivial irrep. The representation under $\mathbb{Z}_2$ is just a sign and does not affect the reality/complexity of a representation. Hence real irreps are 1_0^+, 1_0^- and, for even n, $1_{n/2}^+$, $1_{n/2}^-$ (one also has $(\mathbf{1}_k^{\pm})^* = \mathbf{1}_{n-k}^{\pm}$). Since $M(\tau)$ in the bilinear of (6.1) is a function of τ alone, it is R-even. The fields ψ and ψ^c are then constrained to carry the same R-parity, i.e., transform with the same sign under $\mathbb{Z}_2^R$. Fields in unhatted representations $\mathbf{r}$ — for which $\rho_{\mathbf{r}}(R) = \mathbb{1}$ — are even (odd) under $\mathbb{Z}_2^R$ if k is even (odd), while the opposite happens for hatted representations. Keeping this in mind, one can omit the $\mathbb{Z}_2^R$ factor and ignore the superscript signs in the following tables.

Finally, notice that a $\mathbb{Z}_2^R$ factor is hidden in the residual $\mathbb{Z}_4^S$, as $S^2 = R$. Fields transforming under $\mathbb{Z}_4^S$ as 1_0 or 1_2 are R-even while fields transforming as $\mathbf{1}_1$ or $\mathbf{1}_3$ are R-odd. Requiring that ψ and ψ^c carry the same R-parity implies that one effectively works with $\mathbb{Z}_4^S/\mathbb{Z}_2^R \simeq \mathbb{Z}_2$, which is why it is generic to consider $\tilde{\rho}_i^c \tilde{\rho}_j = \pm 1$ in section 6.1.

Table C.1.: Decompositions of "weighted" $(\mathbf{r}, k)$ multiplets of $\Gamma_2' \simeq S_3$ under the residual symmetry groups. Irrep subscripts should be understood modulo n, where $n = 4, 3$ in the first and second columns, respectively. Upper (lower) signs correspond to even (odd) values of k.

$\mathbf{r}$	$\mathbb{Z}_4^S\,(\tau = i)$	$\mathbb{Z}_3^{ST} \times \mathbb{Z}_2^R\,(\tau = \omega)$	$\mathbb{Z}_2^T \times \mathbb{Z}_2^R\,(\tau = i\infty)$
$\mathbf{1}$	$\mathbf{1}_k$	$\mathbf{1}_k^{\pm}$	$1_0^{\pm}$
$\mathbf{1}'$	$\mathbf{1}_{k+2}$	$\mathbf{1}_k^{\pm}$	$1_1^{\pm}$
$\mathbf{2}$	$\mathbf{1}_k \oplus \mathbf{1}_{k+2}$	$\mathbf{1}_{k-1}^{\pm} \oplus \mathbf{1}_{k+1}^{\pm}$	$1_0^{\pm} \oplus 1_1^{\pm}$

Table C.2.: Decompositions of "weighted" $(\mathbf{r}, k)$ multiplets of $\Gamma_3' \simeq A_4' = T'$ under the residual symmetry groups. Irrep subscripts should be understood modulo n, where $n = 4, 3$ in the first and second columns, respectively. Upper (lower) signs correspond to even (odd) values of k.

$\mathbf{r}$	$\mathbb{Z}_4^S\,(\tau = i)$	$\mathbb{Z}_3^{ST} \times \mathbb{Z}_2^R\,(\tau = \omega)$	$\mathbb{Z}_3^T \times \mathbb{Z}_2^R\,(\tau = i\infty)$
$\mathbf{1}$	$\mathbf{1}_k$	$\mathbf{1}_k^{\pm}$	$1_0^{\pm}$
$\mathbf{1}'$	$\mathbf{1}_k$	$\mathbf{1}_{k+1}^{\pm}$	$\mathbf{1}_1^{\pm}$
$\mathbf{1}''$	$\mathbf{1}_k$	$\mathbf{1}_{k+2}^{\pm}$	$\mathbf{1}_2^{\pm}$
$\hat{\mathbf{2}}$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_k^{\mp} \oplus \mathbf{1}_{k+1}^{\mp}$	$1_0^{\mp} \oplus \mathbf{1}_1^{\mp}$
$\hat{\mathbf{2}}'$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$\mathbf{1}_1^{\mp} \oplus \mathbf{1}_2^{\mp}$
$\hat{\mathbf{2}}''$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_k^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$1_0^{\mp} \oplus \mathbf{1}_2^{\mp}$
$\mathbf{3}$	$\mathbf{1}_k \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2}$	$\mathbf{1}_k^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$1_0^{\pm} \oplus \mathbf{1}_1^{\pm} \oplus \mathbf{1}_2^{\pm}$

Table C.3.: Decompositions of "weighted" $(\mathbf{r}, k)$ multiplets of $\Gamma_4' \simeq S_4' = \mathrm{SL}(2, \mathbb{Z}_4)$ under the residual symmetry groups. Irrep subscripts should be understood modulo n, where $n = 4, 3$ in the first and second columns, respectively. Upper (lower) signs correspond to even (odd) values of k.

$\mathbf{r}$	$\mathbb{Z}_4^S\,(\tau = i)$	$\mathbb{Z}_3^{ST} \times \mathbb{Z}_2^R\,(\tau = \omega)$	$\mathbb{Z}_4^T \times \mathbb{Z}_2^R\,(\tau = i\infty)$
$\mathbf{1}$	$\mathbf{1}_k$	$\mathbf{1}_k^{\pm}$	$1_0^{\pm}$
$\hat{\mathbf{1}}$	$\mathbf{1}_{k+1}$	$\mathbf{1}_k^{\mp}$	$\mathbf{1}_3^{\mp}$
$\mathbf{1}'$	$\mathbf{1}_{k+2}$	$\mathbf{1}_k^{\pm}$	$1_2^{\pm}$
$\hat{\mathbf{1}}'$	$\mathbf{1}_{k+3}$	$\mathbf{1}_k^{\mp}$	$\mathbf{1}_1^{\mp}$
$\mathbf{2}$	$\mathbf{1}_{k+2} \oplus \mathbf{1}_k$	$\mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$1_0^{\pm} \oplus 1_2^{\pm}$
$\hat{\mathbf{2}}$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$\mathbf{1}_1^{\mp} \oplus \mathbf{1}_3^{\mp}$
$\mathbf{3}$	$\mathbf{1}_{k+2} \oplus \mathbf{1}_k \oplus \mathbf{1}_k$	$\mathbf{1}_k^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$\mathbf{1}_1^{\pm} \oplus 1_2^{\pm} \oplus \mathbf{1}_3^{\pm}$
$\hat{\mathbf{3}}$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_k^{\mp} \oplus \mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$1_0^{\mp} \oplus \mathbf{1}_1^{\mp} \oplus 1_2^{\mp}$
$\mathbf{3}'$	$\mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_k$	$\mathbf{1}_k^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$1_0^{\pm} \oplus \mathbf{1}_1^{\pm} \oplus \mathbf{1}_3^{\pm}$
$\hat{\mathbf{3}}'$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_k^{\mp} \oplus \mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$1_0^{\mp} \oplus 1_2^{\mp} \oplus \mathbf{1}_3^{\mp}$

Table C.4.: Decompositions of "weighted" $(\mathbf{r}, k)$ multiplets of $\Gamma_5' \simeq A_5' = \mathrm{SL}(2, \mathbb{Z}_5)$ under the residual symmetry groups. Irrep subscripts should be understood modulo n, where $n = 4, 3$ in the first and second columns, respectively. Upper (lower) signs correspond to even (odd) values of k.

$\mathbf{r}$	$\mathbb{Z}_4^S\,(\tau = i)$	$\mathbb{Z}_3^{ST} \times \mathbb{Z}_2^R\,(\tau = \omega)$	$\mathbb{Z}_5^T \times \mathbb{Z}_2^R\,(\tau = i\infty)$
$\mathbf{1}$	$\mathbf{1}_k$	$\mathbf{1}_k^{\pm}$	$1_0^{\pm}$
$\hat{\mathbf{2}}$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$\mathbf{1}_2^{\mp} \oplus \mathbf{1}_3^{\mp}$
$\hat{\mathbf{2}}'$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$\mathbf{1}_1^{\mp} \oplus \mathbf{1}_4^{\mp}$
$\mathbf{3}$	$\mathbf{1}_k \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2}$	$\mathbf{1}_k^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$1_0^{\pm} \oplus \mathbf{1}_1^{\pm} \oplus \mathbf{1}_4^{\pm}$
$\mathbf{3}'$	$\mathbf{1}_k \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2}$	$\mathbf{1}_k^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$1_0^{\pm} \oplus \mathbf{1}_2^{\pm} \oplus \mathbf{1}_3^{\pm}$
$\mathbf{4}$	$\mathbf{1}_k \oplus \mathbf{1}_k \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2}$	$\mathbf{1}_k^{\pm} \oplus \mathbf{1}_k^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$\mathbf{1}_1^{\pm} \oplus \mathbf{1}_2^{\pm} \oplus \mathbf{1}_3^{\pm} \oplus \mathbf{1}_4^{\pm}$
$\hat{\mathbf{4}}$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_k^{\mp} \oplus \mathbf{1}_k^{\mp} \oplus \mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$\mathbf{1}_1^{\mp} \oplus \mathbf{1}_2^{\mp} \oplus \mathbf{1}_3^{\mp} \oplus \mathbf{1}_4^{\mp}$
$\mathbf{5}$	$\mathbf{1}_k \oplus \mathbf{1}_k \oplus \mathbf{1}_k \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2}$	$\mathbf{1}_k^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$1_0^{\pm} \oplus \mathbf{1}_1^{\pm} \oplus \mathbf{1}_2^{\pm} \oplus \mathbf{1}_3^{\pm} \oplus \mathbf{1}_4^{\pm}$
$\hat{\mathbf{6}}$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+1} \oplus \mathbf{1}_{k+1}$ $\oplus\, \mathbf{1}_{k+3} \oplus \mathbf{1}_{k+3} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_k^{\mp} \oplus \mathbf{1}_k^{\mp} \oplus \mathbf{1}_{k+1}^{\mp}$ $\oplus\, \mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$1_0^{\mp} \oplus 1_0^{\mp} \oplus \mathbf{1}_1^{\mp}$ $\oplus\, \mathbf{1}_2^{\mp} \oplus \mathbf{1}_3^{\mp} \oplus \mathbf{1}_4^{\mp}$

D. Possible hierarchical patterns

In this appendix we list the hierarchical patterns which may arise in the vicinities of the two symmetric points of interest (see main text). We consider in turn the finite modular groups $\Gamma'_2 \simeq S_3$, $\Gamma'_3 \simeq A'_4 = T'$, $\Gamma'_4 \simeq S'_4 = \mathrm{SL}(2,\mathbb{Z}_4)$, and $\Gamma'_5 \simeq A'_5 = \mathrm{SL}(2,\mathbb{Z}_5)$ (tables D.1 to D.4). We have focused on 3-dimensional (possibly reducible) representations $(\mathbf{r},\mathbf{r}^c)$ entering the bilinear (6.1). Dependence on the weights $k^{(c)}$ may only arise for $\tau \sim \omega$ and through the combination $K = k + k^c$, modulo 3. One can see from tables C.1 to C.4 that if one of the 3d multiplets (say ψ) entering the bilinear is not a sum of 3 singlets, then its decomposition under the $\mathbb{Z}_3^{ST}$ residual symmetry includes all possible singlets, 1_0, $\mathbf{1}_1$, and $\mathbf{1}_2$, independently of the weight k. In such cases, hierarchies are independent of weights since a change in k^c can be absorbed by a change in k in their sum.

Note that for $N = 2$ the residual symmetry group at $\tau_{\mathrm{sym}} = i\infty$ is $\mathbb{Z}_2^T$. Mass matrix entries are then expected to be either $O(1)$ or $O(\epsilon)$ and, as was the case for $\tau \simeq i$, one cannot obtain the sought-after hierarchical patterns from the smallness of ϵ alone. As such, only $\tau \simeq \omega$ is considered in table D.1.

Table D.1.: Leading-order mass spectra patterns of bilinears $\psi^c\psi$ in the vicinity of the symmetric point ω, for 3d multiplets $\psi \sim (\mathbf{r},k)$ and $\psi^c \sim (\mathbf{r}^c,k^c)$ of the finite modular group $\Gamma'_2 \simeq S_3$. Spectra are insensitive to transposition, i.e., to the exchange $\psi \leftrightarrow \psi^c$. Congruence relations for $k + k^c$ are modulo 3.

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$		
		$k + k^c \equiv 0$	$k + k^c \equiv 1$	$k + k^c \equiv 2$
$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{2} \oplus \mathbf{1}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{2} \oplus \mathbf{1}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
$\mathbf{2} \oplus \mathbf{1}'$	$\mathbf{2} \oplus \mathbf{1}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}'$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}'$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}'$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$

Table D.1.: (cont.)

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$		
		$k+k^c \equiv 0$	$k+k^c \equiv 1$	$k+k^c \equiv 2$
$\mathbf{2} \oplus \mathbf{1}'$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$

Table D.2.: Leading-order mass spectra patterns of bilinears $\psi^c\psi$ in the vicinity of the symmetric points ω and $i\infty$, for 3d multiplets $\psi \sim (\mathbf{r}, k)$ and $\psi^c \sim (\mathbf{r}^c, k^c)$ of the finite modular group $\Gamma_3' \simeq A_4' = T'$. Spectra are insensitive to transposition, i.e., to the exchange $\psi \leftrightarrow \psi^c$. Congruence relations for $k + k^c$ are modulo 3.

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$			$\tau \simeq i\infty$
		$k+k^c \equiv 0$	$k+k^c \equiv 1$	$k+k^c \equiv 2$	
$\mathbf{3}$	$\mathbf{3}$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$
$\mathbf{3}$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$
$\mathbf{3}$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$
$\mathbf{3}$	$\mathbf{1}'' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$
$\mathbf{3}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$
$\mathbf{3}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$
$\mathbf{3}$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$
$\mathbf{3}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}'$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$
$\mathbf{3}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{3}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{3}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}''$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$
$\mathbf{1}'' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$

Table D.2.: (cont.)

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$ $k+k^c \equiv 0$	$\tau \simeq \omega$ $k+k^c \equiv 1$	$\tau \simeq \omega$ $k+k^c \equiv 2$	$\tau \simeq i\infty$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$
$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$
$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$
$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}'$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, 1)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, 1, 1)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, 1, 1)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$	$(1, 1, \epsilon)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, \epsilon^2, \epsilon^2)$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}'$	$(1, \epsilon, \epsilon^2)$	$(1, 1, 1)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{1}'' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, \epsilon^2)$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$
$\mathbf{1}'' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, 1, 1)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{1}'' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, 1, 1)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{1}'' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}'$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$	$(1, \epsilon, \epsilon)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$	$(1, \epsilon, \epsilon)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}$	$(1, 1, 1)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, 1, 1)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, \epsilon, \epsilon^2)$	$(1, 1, 1)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}'$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}''$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}$	$(1, \epsilon^2, \epsilon^2)$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$
$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$
$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}'$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, 1, 1)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, 1, \epsilon)$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$	$(1, 1, \epsilon)$
$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}'$	$(1, 1, 1)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, 1, 1)$
$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}'$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}'$	$(1, 1, \epsilon^2)$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$	$(1, 1, \epsilon^2)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, \epsilon^2)$	$(\epsilon, \epsilon^2, \epsilon^2)$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon^2)$	$(1, 1, \epsilon)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon^2)$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$	$(\epsilon, \epsilon^2, \epsilon^2)$	$(1, \epsilon, \epsilon)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(\epsilon, \epsilon^2, \epsilon^2)$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$	$(\epsilon, \epsilon^2, \epsilon^2)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}'$	$(\epsilon, \epsilon, \epsilon^2)$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon^2)$
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(\epsilon, \epsilon^2, \epsilon^2)$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$	$(\epsilon, \epsilon^2, \epsilon^2)$

Table D.2.: (cont.)

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$			$\tau \simeq i\infty$
		$k+k^c \equiv 0$	$k+k^c \equiv 1$	$k+k^c \equiv 2$	
$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}''$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$	$(\epsilon, \epsilon^2, \epsilon^2)$	$(1, \epsilon, \epsilon)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\mathbf{1}'' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon^2)$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$
$\mathbf{1}'' \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}''$	$(\epsilon, \epsilon, \epsilon^2)$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon^2)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(\epsilon, \epsilon, \epsilon^2)$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon^2)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}''$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon^2)$	$(1, 1, \epsilon)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}$	$(1, 1, \epsilon^2)$	$(\epsilon, \epsilon^2, \epsilon^2)$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$
$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}''$	$(\epsilon, \epsilon^2, \epsilon^2)$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$	$(\epsilon, \epsilon^2, \epsilon^2)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$	$(\epsilon, \epsilon^2, \epsilon^2)$	$(1, \epsilon, \epsilon)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}'$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon^2)$	$(1, 1, \epsilon)$
$\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}''$	$(1, 1, \epsilon^2)$	$(\epsilon, \epsilon^2, \epsilon^2)$	$(1, \epsilon, \epsilon)$	$(1, 1, \epsilon^2)$
$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}'$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}''$	$(1, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon^2)$	$(1, 1, \epsilon)$	$(1, \epsilon^2, \epsilon^2)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}''$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}''$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$
$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}''$	$\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}''$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$

Table D.3.: Leading-order mass spectra patterns of bilinears $\psi^c\psi$ in the vicinity of the symmetric points ω and $i\infty$, for 3d multiplets $\psi \sim (\mathbf{r}, k)$ and $\psi^c \sim (\mathbf{r}^c, k^c)$ of the finite modular group $\Gamma_4' \simeq S_4' = \mathrm{SL}(2, \mathbb{Z}_4)$. Spectra are insensitive to transposition, i.e., to the exchange $\psi \leftrightarrow \psi^c$. Congruence relations for $k + k^c$ are modulo 3.

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$			$\tau \simeq i\infty$
		$k+k^c \equiv 0$	$k+k^c \equiv 1$	$k+k^c \equiv 2$	
$\mathbf{3}$	$\mathbf{3}$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$
$\mathbf{3}$	$\mathbf{3}'$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, \epsilon^2)$
$\mathbf{3}$	$\hat{\mathbf{3}}$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, \epsilon^3)$
$\mathbf{3}$	$\hat{\mathbf{3}}'$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, \epsilon)$
$\mathbf{3}'$	$\mathbf{3}'$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$
$\mathbf{3}'$	$\hat{\mathbf{3}}'$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, \epsilon^3)$
$\hat{\mathbf{3}}$	$\mathbf{3}'$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, \epsilon)$
$\hat{\mathbf{3}}$	$\hat{\mathbf{3}}$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, \epsilon^2)$

Table D.3.: (cont.)

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$ $k+k^c \equiv 0$	$\tau \simeq \omega$ $k+k^c \equiv 1$	$\tau \simeq \omega$ $k+k^c \equiv 2$	$\tau \simeq i\infty$
$\hat{\mathbf{3}}$	$\hat{\mathbf{3}}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
$\hat{\mathbf{3}}'$	$\hat{\mathbf{3}}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon^2)$
$\mathbf{3}$	$\mathbf{2}\oplus\mathbf{1}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,\epsilon,\epsilon^3)$
$\mathbf{3}$	$\mathbf{2}\oplus\mathbf{1}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,\epsilon,\epsilon)$
$\mathbf{3}$	$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon^3)$
$\mathbf{3}$	$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon)$
$\mathbf{3}'$	$\mathbf{2}\oplus\mathbf{1}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,\epsilon,\epsilon)$
$\mathbf{3}'$	$\mathbf{2}\oplus\mathbf{1}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,\epsilon,\epsilon^3)$
$\mathbf{3}'$	$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon)$
$\mathbf{3}'$	$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon^3)$
$\hat{\mathbf{3}}$	$\mathbf{2}\oplus\mathbf{1}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon^3)$
$\hat{\mathbf{3}}$	$\mathbf{2}\oplus\mathbf{1}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon)$
$\hat{\mathbf{3}}$	$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,\epsilon,\epsilon)$
$\hat{\mathbf{3}}$	$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,\epsilon,\epsilon^3)$
$\hat{\mathbf{3}}'$	$\mathbf{2}\oplus\mathbf{1}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon)$
$\hat{\mathbf{3}}'$	$\mathbf{2}\oplus\mathbf{1}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon^3)$
$\hat{\mathbf{3}}'$	$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,\epsilon,\epsilon^3)$
$\hat{\mathbf{3}}'$	$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,\epsilon,\epsilon)$
$\mathbf{2}\oplus\mathbf{1}$	$\mathbf{2}\oplus\mathbf{1}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
$\mathbf{2}\oplus\mathbf{1}$	$\mathbf{2}\oplus\mathbf{1}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,\epsilon^2)$
$\mathbf{2}\oplus\mathbf{1}$	$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(\epsilon,\epsilon,\epsilon)$
$\mathbf{2}\oplus\mathbf{1}$	$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(\epsilon,\epsilon,\epsilon^3)$
$\mathbf{2}\oplus\mathbf{1}'$	$\mathbf{2}\oplus\mathbf{1}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
$\mathbf{2}\oplus\mathbf{1}'$	$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(\epsilon,\epsilon,\epsilon^3)$
$\mathbf{2}\oplus\mathbf{1}'$	$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(\epsilon,\epsilon,\epsilon)$
$\mathbf{3}$	$\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^3)$
$\mathbf{3}$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon)$
$\mathbf{3}$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon^3)$
$\mathbf{3}$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon)$
$\mathbf{3}'$	$\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon)$
$\mathbf{3}'$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^3)$
$\mathbf{3}'$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon)$
$\mathbf{3}'$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon^3)$
$\mathbf{3}$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon^2,\epsilon^3)$
$\mathbf{3}$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^3)$

Table D.3.: (cont.)

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$ $k+k^c \equiv 0$	$\tau \simeq \omega$ $k+k^c \equiv 1$	$\tau \simeq \omega$ $k+k^c \equiv 2$	$\tau \simeq i\infty$
$\mathbf{3}$	$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon^2, \epsilon^3)$
$\mathbf{3}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{3}'$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^3)$
$\mathbf{3}'$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(\epsilon, \epsilon^2, \epsilon^3)$
$\mathbf{3}'$	$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\mathbf{3}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon^2, \epsilon^3)$
$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, \epsilon^2)$
$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$
$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$	$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, \epsilon^2)$
$\hat{\mathbf{3}}$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, 1, \epsilon^3)$
$\hat{\mathbf{3}}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, 1, \epsilon)$
$\hat{\mathbf{3}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon)$
$\hat{\mathbf{3}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^3)$
$\hat{\mathbf{3}}'$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, 1, \epsilon)$
$\hat{\mathbf{3}}'$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, 1, \epsilon^3)$
$\hat{\mathbf{3}}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^3)$
$\hat{\mathbf{3}}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon)$
$\hat{\mathbf{3}}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon^2, \epsilon^3)$
$\hat{\mathbf{3}}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\hat{\mathbf{3}}$	$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^3)$
$\hat{\mathbf{3}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(\epsilon, \epsilon^2, \epsilon^3)$
$\hat{\mathbf{3}}'$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$
$\hat{\mathbf{3}}'$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon^2, \epsilon^3)$
$\hat{\mathbf{3}}'$	$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(\epsilon, \epsilon^2, \epsilon^3)$
$\hat{\mathbf{3}}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^3)$
$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, 1, 1)$
$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, 1, \epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{2} \oplus \mathbf{1}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon^3)$
$\mathbf{2} \oplus \mathbf{1}'$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, 1, \epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}'$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, 1, 1)$
$\mathbf{2} \oplus \mathbf{1}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon^3)$
$\mathbf{2} \oplus \mathbf{1}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$
$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, 1, \epsilon^2)$
$\mathbf{2} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon^2, \epsilon^2)$

Table D.3.: (cont.)

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$ $k+k^c \equiv 0$	$\tau \simeq \omega$ $k+k^c \equiv 1$	$\tau \simeq \omega$ $k+k^c \equiv 2$	$\tau \simeq i\infty$
$\mathbf{2}\oplus\mathbf{1}$	$\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon^3)$
$\mathbf{2}\oplus\mathbf{1}$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon^3,\epsilon^3)$
$\mathbf{2}\oplus\mathbf{1}'$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon^2,\epsilon^2)$
$\mathbf{2}\oplus\mathbf{1}'$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon^2)$
$\mathbf{2}\oplus\mathbf{1}'$	$\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon^3,\epsilon^3)$
$\mathbf{2}\oplus\mathbf{1}'$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon^3)$
$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}$	$\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$
$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon^3)$
$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon^2)$
$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,1)$
$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}'$	$\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon^3)$
$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}'$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$
$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}'$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,1)$
$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}'$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon^2)$
$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon^3)$
$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon^3,\epsilon^3)$
$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}$	$\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon^2,\epsilon^2)$
$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon^2)$
$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}'$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon^3,\epsilon^3)$
$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}'$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon^3)$
$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}'$	$\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,1,\epsilon^2)$
$\hat{\mathbf{2}}\oplus\hat{\mathbf{1}}'$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon^2,\epsilon^2)$
$\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,1)$
$\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,\epsilon^2)$
$\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1}$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon,\epsilon,\epsilon)$
$\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1}$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon,\epsilon,\epsilon^3)$
$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,1)$
$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon,\epsilon,\epsilon^3)$
$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon,\epsilon,\epsilon)$
$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,\epsilon^2)$
$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,\epsilon^2,\epsilon^2)$
$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}\oplus\hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon,\epsilon,\epsilon^3)$
$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon,\epsilon^3,\epsilon^3)$
$\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}'\oplus\mathbf{1}'\oplus\mathbf{1}'$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,\epsilon^2,\epsilon^2)$
$\mathbf{1}'\oplus\mathbf{1}\oplus\mathbf{1}$	$\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'\oplus\hat{\mathbf{1}}'$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon,\epsilon^3,\epsilon^3)$

Table D.3.: (cont.)

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$ $k+k^c \equiv 0$	$\tau \simeq \omega$ $k+k^c \equiv 1$	$\tau \simeq \omega$ $k+k^c \equiv 2$	$\tau \simeq i\infty$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,\epsilon^2)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon,\epsilon,\epsilon^3)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon,\epsilon^3,\epsilon^3)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon,\epsilon,\epsilon^3)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,1)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon,\epsilon,\epsilon)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon^3,\epsilon^3,\epsilon^3)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,1)$
$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon,\epsilon,\epsilon)$
$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,\epsilon^2)$
$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,1)$
$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,\epsilon^2)$
$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon,\epsilon,\epsilon^3)$
$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon,\epsilon^3,\epsilon^3)$
$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,\epsilon^2,\epsilon^2)$
$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,\epsilon^2)$
$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,\epsilon^2)$
$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,\epsilon^2,\epsilon^2)$
$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon^3,\epsilon^3,\epsilon^3)$
$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$
$\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,1)$
$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$\hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}'$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$

Table D.4.: Leading-order mass spectra patterns of bilinears $\psi^c\psi$ in the vicinity of the symmetric points ω and $i\infty$, for 3d multiplets $\psi \sim (\mathbf{r}, k)$ and $\psi^c \sim (\mathbf{r}^c, k^c)$ of the finite modular group $\Gamma_5' \simeq A_5' = \mathrm{SL}(2,\mathbb{Z}_5)$. Spectra are insensitive to transposition, i.e., to the exchange $\psi \leftrightarrow \psi^c$. Congruence relations for $k+k^c$ are modulo 3.

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$ $k+k^c \equiv 0$	$\tau \simeq \omega$ $k+k^c \equiv 1$	$\tau \simeq \omega$ $k+k^c \equiv 2$	$\tau \simeq i\infty$
$\mathbf{3}$	$\mathbf{3}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
$\mathbf{3}$	$\mathbf{3}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,\epsilon,\epsilon^4)$

Table D.4.: (cont.)

$\mathbf{r}$	$\mathbf{r}^c$	$\tau \simeq \omega$			$\tau \simeq i\infty$
		$k + k^c \equiv 0$	$k + k^c \equiv 1$	$k + k^c \equiv 2$	
$\mathbf{3}'$	$\mathbf{3}'$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$
$\mathbf{3}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^4)$
$\mathbf{3}'$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon^2, \epsilon^3)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$

Index

automorphy factor 8

fundamental domain 32

homogeneous finite modular group 8

inhomogeneous finite modular group 8

inhomogeneous modular group 8

level 8

metaplectic group 14

modular form 10

modular group 7

modulus 7

multiplier system 9

principal congruence subgroup 8

residual symmetry 32

symmetric point 32

weight 8

Bibliography

[1] F. Feruglio, "Pieces of the Flavour Puzzle", Eur. Phys. J. C **75**, 373 (2015), arXiv:1503.04071 [hep-ph].

[2] C. D. Froggatt and H. B. Nielsen, "Hierarchy of Quark Masses, Cabibbo Angles and CP Violation", Nucl. Phys. B **147**, 277 (1979).

[3] S. Weinberg, "Baryon and Lepton Nonconserving Processes", Phys. Rev. Lett. **43**, 1566 (1979).

[4] S. T. Petcov, "Neutrino Theory", PoS **ICHEP2018**, 699 (2019).

[5] G. Altarelli and F. Feruglio, "Discrete Flavor Symmetries and Models of Neutrino Mixing", Rev. Mod. Phys. **82**, 2701 (2010), arXiv:1002.0211 [hep-ph].

[6] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada, and M. Tanimoto, "Non-Abelian Discrete Symmetries in Particle Physics", Prog. Theor. Phys. Suppl. **183**, 1 (2010), arXiv:1003.3552 [hep-th].

[7] S. F. King, A. Merle, S. Morisi, Y. Shimizu, and M. Tanimoto, "Neutrino Mass and Mixing: from Theory to Experiment", New J. Phys. **16**, 045018 (2014), arXiv:1402.4271 [hep-ph].

[8] M. Tanimoto, "Neutrinos and flavor symmetries", AIP Conf. Proc. **1666**, edited by E. Kearns, 120002 (2015).

[9] S. T. Petcov, "Discrete Flavour Symmetries, Neutrino Mixing and Leptonic CP Violation", Eur. Phys. J. C **78**, 709 (2018), arXiv:1711.10806 [hep-ph].

[10] M. Tanabashi et al. (Particle Data Group), "Review of Particle Physics", Phys. Rev. D **98**, 030001 (2018).

[11] Y. Fukuda et al. (Super-Kamiokande), "Evidence for oscillation of atmospheric neutrinos", Phys. Rev. Lett. **81**, 1562 (1998), arXiv:hep-ex/9807003.

[12] F. Feruglio, "Are neutrino masses modular forms?", in *From My Vast Repertoire ...: Guido Altarelli's Legacy*, edited by A. Levy, S. Forte, and G. Ridolfi (2019), arXiv:1706.08749 [hep-ph].

[13] R. de Adelhart Toorop, F. Feruglio, and C. Hagedorn, "Finite Modular Groups and Lepton Mixing", Nucl. Phys. B **858**, 437 (2012), arXiv:1112.1340 [hep-ph].

[14] J. C. Criado and F. Feruglio, "Modular Invariance Faces Precision Neutrino Data", SciPost Phys. **5**, 042 (2018), arXiv:1807.01125 [hep-ph].

[15] T. Kobayashi, K. Tanaka, and T. H. Tatsuishi, "Neutrino mixing from finite modular groups", Phys. Rev. D **98**, 016004 (2018), arXiv:1803.10391 [hep-ph].

[16] J. T. Penedo and S. T. Petcov, "Lepton Masses and Mixing from Modular S_4 Symmetry", Nucl. Phys. B **939**, 292 (2019), arXiv:1806.11040 [hep-ph].

[17] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, and T. H. Tatsuishi, "New A_4 lepton flavor model from S_4 modular symmetry", JHEP **02**, 097 (2020), arXiv:1907.09141 [hep-ph].

[18] H. Okada and Y. Orikasa, "Neutrino mass model with a modular S_4 symmetry", (2019), arXiv:1908.08409 [hep-ph].

[19] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, and T. H. Tatsuishi, "A_4 lepton flavor model and modulus stabilization from S_4 modular symmetry", Phys. Rev. D **100**, [Erratum: Phys.Rev.D 101, 039904 (2020)], 115045 (2019), arXiv:1909.05139 [hep-ph].

[20] G.-J. Ding, S. F. King, X.-G. Liu, and J.-N. Lu, "Modular S_4 and A_4 symmetries and their fixed points: new predictive examples of lepton mixing", JHEP **12**, 030 (2019), arXiv:1910.03460 [hep-ph].

[21] X. Wang and S. Zhou, "The minimal seesaw model with a modular S_4 symmetry", JHEP **05**, 017 (2020), arXiv:1910.09473 [hep-ph].

[22] X. Wang, "Dirac neutrino mass models with a modular S_4 symmetry", Nucl. Phys. B **962**, 115247 (2021), arXiv:2007.05913 [hep-ph].

[23] J. Gehrlein and M. Spinrath, "Leptonic Sum Rules from Flavour Models with Modular Symmetries", JHEP **03**, 177 (2021), arXiv:2012.04131 [hep-ph].

[24] X. Zhang and S. Zhou, "Inverse Seesaw Model with a Modular S_4 Symmetry: Lepton Flavor Mixing and Warm Dark Matter", (2021), arXiv:2106.03433 [hep-ph].

[25] B.-Y. Qu, X.-G. Liu, P.-T. Chen, and G.-J. Ding, "Flavor mixing and CP violation from the interplay of S_4 modular group and gCP", (2021), arXiv:2106.11659 [hep-ph].

[26] G.-J. Ding, S. F. King, and X.-G. Liu, "Neutrino mass and mixing with A_5 modular symmetry", Phys. Rev. D **100**, 115005 (2019), arXiv:1903.12588 [hep-ph].

[27] T. Kobayashi, N. Omoto, Y. Shimizu, K. Takagi, M. Tanimoto, and T. H. Tatsuishi, "Modular A_4 invariance and neutrino mixing", JHEP **11**, 196 (2018), arXiv:1808.03012 [hep-ph].

[28] T. Nomura and H. Okada, "A modular A_4 symmetric model of dark matter and neutrino", Phys. Lett. B **797**, 134799 (2019), arXiv:1904.03937 [hep-ph].

[29] T. Nomura and H. Okada, "A two loop induced neutrino mass model with modular A_4 symmetry", Nucl. Phys. B **966**, 115372 (2021), arXiv:1906.03927 [hep-ph].

[30] G.-J. Ding, S. F. King, and X.-G. Liu, "Modular A_4 symmetry models of neutrinos and charged leptons", JHEP **09**, 074 (2019), arXiv:1907.11714 [hep-ph].

[31] H. Okada and Y. Orikasa, "A radiative seesaw model in modular A_4 symmetry", (2019), arXiv:1907.13520 [hep-ph].

[32] T. Nomura, H. Okada, and O. Popov, "A modular A_4 symmetric scotogenic model", Phys. Lett. B **803**, 135294 (2020), arXiv:1908.07457 [hep-ph].

[33] T. Asaka, Y. Heo, T. H. Tatsuishi, and T. Yoshida, "Modular A_4 invariance and leptogenesis", JHEP **01**, 144 (2020), arXiv:1909.06520 [hep-ph].

[34] D. Zhang, "A modular A_4 symmetry realization of two-zero textures of the Majorana neutrino mass matrix", Nucl. Phys. B **952**, 114935 (2020), arXiv:1910.07869 [hep-ph].

[35] T. Nomura, H. Okada, and S. Patra, "An inverse seesaw model with A_4-modular symmetry", Nucl. Phys. B **967**, 115395 (2021), arXiv:1912.00379 [hep-ph].

[36] T. Kobayashi, T. Nomura, and T. Shimomura, "Type II seesaw models with modular A_4 symmetry", Phys. Rev. D **102**, 035019 (2020), arXiv:1912.00637 [hep-ph].

[37] X. Wang, "Lepton flavor mixing and CP violation in the minimal type-(I+II) seesaw model with a modular A_4 symmetry", Nucl. Phys. B **957**, 115105 (2020), arXiv:1912.13284 [hep-ph].

[38] M. Abbas, "Modular A_4 Invariance Model for Lepton Masses and Mixing", Phys. Atom. Nucl. **83**, 764 (2020).

[39] H. Okada and Y. Shoji, "A radiative seesaw model with three Higgs doublets in modular A_4 symmetry", Nucl. Phys. B **961**, 115216 (2020), arXiv:2003.13219 [hep-ph].

[40] G.-J. Ding and F. Feruglio, "Testing Moduli and Flavon Dynamics with Neutrino Oscillations", JHEP **06**, 134 (2020), arXiv:2003.13448 [hep-ph].

[41] M. K. Behera, S. Mishra, S. Singirala, and R. Mohanta, "Implications of A_4 modular symmetry on Neutrino mass, Mixing and Leptogenesis with Linear Seesaw", (2020), arXiv:2007.00545 [hep-ph].

[42] T. Nomura and H. Okada, "A linear seesaw model with A_4-modular flavor and local $U(1)_{B-L}$ symmetries", (2020), arXiv:2007.04801 [hep-ph].

[43] T. Nomura and H. Okada, "Modular A_4 symmetric inverse seesaw model with $SU(2)_L$ multiplet fields", (2020), arXiv:2007.15459 [hep-ph].

[44] M. K. Behera, S. Singirala, S. Mishra, and R. Mohanta, "A modular A_4 symmetric Scotogenic model for Neutrino mass and Dark Matter", (2020), arXiv:2009.01806 [hep-ph].

[45] T. Asaka, Y. Heo, and T. Yoshida, "Lepton flavor model with modular A_4 symmetry in large volume limit", Phys. Lett. B **811**, 135956 (2020), arXiv:2009.12120 [hep-ph].

[46] K. I. Nagao and H. Okada, "Lepton sector in modular A_4 and gauged $U(1)_R$ symmetry", (2020), arXiv:2010.03348 [hep-ph].

[47] P. T. P. Hutauruk, D. W. Kang, J. Kim, and H. Okada, "Muon g-2 explanation in a modular A_4 symmetry", (2020), arXiv:2012.11156 [hep-ph].

[48] H. Okada, Y. Shimizu, M. Tanimoto, and T. Yoshida, "Modulus τ linking leptonic CP violation to baryon asymmetry in A_4 modular invariant flavor model", JHEP **07**, 184 (2021), arXiv:2105.14292 [hep-ph].

[49] I. de Medeiros Varzielas and J. Lourenço, "Two A4 modular symmetries for Tri-Maximal 2 mixing", (2021), arXiv:2107.04042 [hep-ph].

[50] H. Okada and Y. Orikasa, "Modular S_3 symmetric radiative seesaw model", Phys. Rev. D **100**, 115037 (2019), arXiv:1907.04716 [hep-ph].

[51] S. Mishra, "Neutrino mixing and Leptogenesis with modular S_3 symmetry in the framework of type III seesaw", (2020), arXiv:2008.02095 [hep-ph].

[52] G.-J. Ding, S. F. King, C.-C. Li, and Y.-L. Zhou, "Modular Invariant Models of Leptons at Level 7", JHEP **08**, 164 (2020), arXiv:2004.12662 [hep-ph].

[53] H. Okada and M. Tanimoto, "CP violation of quarks in A_4 modular invariance", Phys. Lett. B **791**, 54 (2019), arXiv:1812.09677 [hep-ph].

[54] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, T. H. Tatsuishi, and H. Uchida, "Finite modular subgroups for fermion mass matrices and baryon/lepton number violation", Phys. Lett. B **794**, 114 (2019), arXiv:1812.11072 [hep-ph].

[55] H. Okada and M. Tanimoto, "Towards unification of quark and lepton flavors in A_4 modular invariance", Eur. Phys. J. C **81**, 52 (2021), arXiv:1905.13421 [hep-ph].

[56] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, and T. H. Tatsuishi, "Modular S_3-invariant flavor model in SU(5) grand unified theory", PTEP **2020**, 053B05 (2020), arXiv:1906.10341 [hep-ph].

[57] J.-N. Lu, X.-G. Liu, and G.-J. Ding, "Modular symmetry origin of texture zeros and quark lepton unification", Phys. Rev. D **101**, 115020 (2020), arXiv:1912.07573 [hep-ph].

[58] M. Abbas, "Fermion masses and mixing in modular A4 Symmetry", Phys. Rev. D **103**, 056016 (2021), arXiv:2002.01929 [hep-ph].

[59] H. Okada and M. Tanimoto, "Quark and lepton flavors with common modulus τ in A_4 modular symmetry", (2020), arXiv:2005.00775 [hep-ph].

[60] X. Du and F. Wang, "SUSY breaking constraints on modular flavor S_3 invariant SU(5) GUT model", JHEP **02**, 221 (2021), arXiv:2012.01397 [hep-ph].

[61] Y. Zhao and H.-H. Zhang, "Adjoint SU(5) GUT model with modular S_4 symmetry", JHEP **03**, 002 (2021), arXiv:2101.02266 [hep-ph].

[62] P. Chen, G.-J. Ding, and S. F. King, "SU(5) GUTs with A_4 modular symmetry", JHEP **04**, 239 (2021), arXiv:2101.12724 [hep-ph].

[63] S. F. King and Y.-L. Zhou, "Twin modular S_4 with SU(5) GUT", JHEP **04**, 291 (2021), arXiv:2103.02633 [hep-ph].

[64] G.-J. Ding, S. F. King, and C.-Y. Yao, "Modular $S_4 \times SU(5)$ GUT", (2021), arXiv:2103.16311 [hep-ph].

[65] T. Nomura, H. Okada, and Y. Orikasa, "Quark and lepton flavor model with leptoquarks in a modular A_4 symmetry", (2021), arXiv:2106.12375 [hep-ph].

[66] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, T. H. Tatsuishi, and H. Uchida, "*CP* violation in modular invariant flavor models", Phys. Rev. D **101**, 055046 (2020), arXiv:1910.11553 [hep-ph].

[67] H. Okada and M. Tanimoto, "Spontaneous CP violation by modulus τ in A_4 model of lepton flavors", JHEP **03**, 010 (2021), arXiv:2012.01688 [hep-ph].

[68] C.-Y. Yao, J.-N. Lu, and G.-J. Ding, "Modular Invariant A_4 Models for Quarks and Leptons with Generalized CP Symmetry", JHEP **05**, 102 (2021), arXiv:2012.13390 [hep-ph].

[69] X. Wang and S. Zhou, "Explicit Perturbations to the Stabilizer $\tau = \mathrm{i}$ of Modular A_5' Symmetry and Leptonic CP Violation", JHEP **07**, 093 (2021), arXiv:2102.04358 [hep-ph].

[70] I. de Medeiros Varzielas, S. F. King, and Y.-L. Zhou, "Multiple modular symmetries as the origin of flavor", Phys. Rev. D **101**, 055033 (2020), arXiv:1906.02208 [hep-ph].

[71] S. F. King and Y.-L. Zhou, "Trimaximal TM_1 mixing with two modular S_4 groups", Phys. Rev. D **101**, 015001 (2020), arXiv:1908.02770 [hep-ph].

[72] I. de Medeiros Varzielas, M. Levy, and Y.-L. Zhou, "Symmetries and stabilisers in modular invariant flavour models", JHEP **11**, 085 (2020), arXiv:2008.05329 [hep-ph].

[73] G.-J. Ding, F. Feruglio, and X.-G. Liu, "Automorphic Forms and Fermion Masses", JHEP **01**, 037 (2021), arXiv:2010.07952 [hep-th].

[74] G.-J. Ding, F. Feruglio, and X.-G. Liu, "CP Symmetry and Symplectic Modular Invariance", SciPost Phys. **10**, 133 (2021), arXiv:2102.06716 [hep-ph].

[75] T. Kobayashi, T. Shimomura, and M. Tanimoto, "Soft supersymmetry breaking terms and lepton flavor violations in modular flavor models", Phys. Lett. B **819**, 136452 (2021), arXiv:2102.10425 [hep-ph].

[76] M. Tanimoto and K. Yamamoto, "Electron EDM arising from modulus τ in the supersymmetric modular invariant flavor models", (2021), arXiv:2106.10919 [hep-ph].

[77] T. Kobayashi, S. Nagamoto, S. Takada, S. Tamba, and T. H. Tatsuishi, "Modular symmetry and non-Abelian discrete flavor symmetries in string compactification", Phys. Rev. D **97**, 116002 (2018), arXiv:1804.06644 [hep-th].

[78] T. Kobayashi and S. Tamba, "Modular forms of finite modular subgroups from magnetized D-brane models", Phys. Rev. D **99**, 046001 (2019), arXiv:1811.11384 [hep-th].

[79] F. J. de Anda, S. F. King, and E. Perdomo, "$SU(5)$ grand unified theory with A_4 modular symmetry", Phys. Rev. D **101**, 015028 (2020), arXiv:1812.05620 [hep-ph].

[80] A. Baur, H. P. Nilles, A. Trautner, and P. K. S. Vaudrevange, "Unification of Flavor, CP, and Modular Symmetries", Phys. Lett. B **795**, 7 (2019), arXiv:1901.03251 [hep-th].

[81] Y. Kariyazono, T. Kobayashi, S. Takada, S. Tamba, and H. Uchida, "Modular symmetry anomaly in magnetic flux compactification", Phys. Rev. D **100**, 045014 (2019), arXiv:1904.07546 [hep-th].

[82] A. Baur, H. P. Nilles, A. Trautner, and P. K. S. Vaudrevange, "A String Theory of Flavor and $\mathscr{CP}$", Nucl. Phys. B **947**, 114737 (2019), arXiv:1908.00805 [hep-th].

[83] H. P. Nilles, S. Ramos-Sánchez, and P. K. S. Vaudrevange, "Eclectic Flavor Groups", JHEP **02**, 045 (2020), arXiv:2001.01736 [hep-ph].

[84] T. Kobayashi and H. Otsuka, "Classification of discrete modular symmetries in Type IIB flux vacua", Phys. Rev. D **101**, 106017 (2020), arXiv:2001.07972 [hep-th].

[85] H. Abe, T. Kobayashi, S. Uemura, and J. Yamamoto, "Loop Fayet-Iliopoulos terms in T^2/Z_2 models: Instability and moduli stabilization", Phys. Rev. D **102**, 045005 (2020), arXiv:2003.03512 [hep-th].

[86] H. Ohki, S. Uemura, and R. Watanabe, "Modular flavor symmetry on a magnetized torus", Phys. Rev. D **102**, 085008 (2020), arXiv:2003.04174 [hep-th].

[87] T. Kobayashi and H. Otsuka, "Challenge for spontaneous *CP* violation in Type IIB orientifolds with fluxes", Phys. Rev. D **102**, 026004 (2020), arXiv:2004.04518 [hep-th].

[88] H. P. Nilles, S. Ramos-Sanchez, and P. K. S. Vaudrevange, "Lessons from eclectic flavor symmetries", Nucl. Phys. B **957**, 115098 (2020), arXiv:2004.05200 [hep-ph].

[89] S. Kikuchi, T. Kobayashi, S. Takada, T. H. Tatsuishi, and H. Uchida, "Revisiting modular symmetry in magnetized torus and orbifold compactifications", Phys. Rev. D **102**, 105010 (2020), arXiv:2005.12642 [hep-th].

[90] H. P. Nilles, S. Ramos-Sánchez, and P. K. S. Vaudrevange, "Eclectic flavor scheme from ten-dimensional string theory – I. Basic results", Phys. Lett. B **808**, 135615 (2020), arXiv:2006.03059 [hep-th].

[91] S. Kikuchi, T. Kobayashi, H. Otsuka, S. Takada, and H. Uchida, "Modular symmetry by orbifolding magnetized $T^2 \times T^2$: realization of double cover of Γ_N", JHEP **11**, 101 (2020), arXiv:2007.06188 [hep-th].

[92] A. Baur, M. Kade, H. P. Nilles, S. Ramos-Sanchez, and P. K. S. Vaudrevange, "The eclectic flavor symmetry of the $\mathbb{Z}_2$ orbifold", JHEP **02**, 018 (2021), arXiv:2008.07534 [hep-th].

[93] K. Ishiguro, T. Kobayashi, and H. Otsuka, "Spontaneous CP violation and symplectic modular symmetry in Calabi-Yau compactifications", (2020), arXiv:2010.10782 [hep-th].

[94] H. P. Nilles, S. Ramos-Sánchez, and P. K. S. Vaudrevange, "Eclectic flavor scheme from ten-dimensional string theory - II detailed technical analysis", Nucl. Phys. B **966**, 115367 (2021), arXiv:2010.13798 [hep-th].

[95] K. Ishiguro, T. Kobayashi, and H. Otsuka, "Landscape of Modular Symmetric Flavor Models", JHEP **03**, 161 (2021), arXiv:2011.09154 [hep-ph].

[96] K. Hoshiya, S. Kikuchi, T. Kobayashi, Y. Ogawa, and H. Uchida, "Classification of three-generation models by orbifolding magnetized $T^2 \times T^2$", PTEP **2021**, 033B05 (2021), arXiv:2012.00751 [hep-th].

[97] A. Baur, M. Kade, H. P. Nilles, S. Ramos-Sanchez, and P. K. S. Vaudrevange, "Siegel modular flavor group and CP from string theory", Phys. Lett. B **816**, 136176 (2021), arXiv:2012.09586 [hep-th].

[98] S. Kikuchi, T. Kobayashi, and H. Uchida, "Modular flavor symmetries of three-generation modes on magnetized toroidal orbifolds", (2021), arXiv:2101.00826 [hep-th].

[99] Y. Almumin, M.-C. Chen, V. Knapp-Pérez, S. Ramos-Sánchez, M. Ratz, and S. Shukla, "Metaplectic Flavor Symmetries from Magnetized Tori", JHEP **05**, 078 (2021), arXiv:2102.11286 [hep-th].

[100] K. Hoshiya, S. Kikuchi, T. Kobayashi, K. Nasu, H. Uchida, and S. Uemura, "Majorana neutrino masses by D-brane instanton effects in magnetized orbifold models", (2021), arXiv:2103.07147 [hep-th].

[101] A. Baur, M. Kade, H. P. Nilles, S. Ramos-Sanchez, and P. K. S. Vaudrevange, "Completing the eclectic flavor scheme of the $\mathbb{Z}_2$ orbifold", JHEP **06**, 110 (2021), arXiv:2104.03981 [hep-th].

[102] H. P. Nilles, S. Ramos-Sanchez, A. Trautner, and P. K. S. Vaudrevange, "Orbifolds from $\mathbf{Sp(4,\mathbb{Z})}$ and their modular symmetries", (2021), arXiv:2105.08078 [hep-th].

[103] K. Ishiguro, T. Kobayashi, and H. Otsuka, "Symplectic modular symmetry in heterotic string vacua: flavor, CP, and *R*-symmetries", (2021), arXiv:2107.00487 [hep-th].

[104] S. Ferrara, D. Lust, and S. Theisen, "Target Space Modular Invariance and Low-Energy Couplings in Orbifold Compactifications", Phys. Lett. B **233**, 147 (1989).

[105] X.-G. Liu and G.-J. Ding, "Neutrino Masses and Mixing from Double Covering of Finite Modular Groups", JHEP **08**, 134 (2019), arXiv:1907.01488 [hep-ph].

[106] X.-G. Liu, C.-Y. Yao, and G.-J. Ding, "Modular invariant quark and lepton models in double covering of S_4 modular group", Phys. Rev. D **103**, 056013 (2021), arXiv:2006.10722 [hep-ph].

[107] X. Wang, B. Yu, and S. Zhou, "Double covering of the modular A_5 group and lepton flavor mixing in the minimal seesaw model", Phys. Rev. D **103**, 076005 (2021), arXiv:2010.10159 [hep-ph].

[108] C.-Y. Yao, X.-G. Liu, and G.-J. Ding, "Fermion masses and mixing from the double cover and metaplectic cover of the A_5 modular group", Phys. Rev. D **103**, 095013 (2021), arXiv:2011.03501 [hep-ph].

[109] C.-C. Li, X.-G. Liu, and G.-J. Ding, "Modular symmetry at level 6 and a new route towards finite modular groups", (2021), arXiv:2108.02181 [hep-ph].

[110] F. Feruglio, V. Gherardi, A. Romanino, and A. Titov, "Modular invariant dynamics and fermion mass hierarchies around $\tau = i$", JHEP **05**, 242 (2021), arXiv:2101.08718 [hep-ph].

[111] H. Okada and M. Tanimoto, "Modular invariant flavor model of A_4 and hierarchical structures at nearby fixed points", Phys. Rev. D **103**, 015005 (2021), arXiv:2009.14242 [hep-ph].

[112] J. C. Criado, F. Feruglio, and S. J. D. King, "Modular Invariant Models of Lepton Masses at Levels 4 and 5", JHEP **02**, 001 (2020), arXiv:1908.11867 [hep-ph].

[113] S. J. D. King and S. F. King, "Fermion mass hierarchies from modular symmetry", JHEP **09**, 043 (2020), arXiv:2002.00969 [hep-ph].

[114] H. Kuranaga, H. Ohki, and S. Uemura, "Modular origin of mass hierarchy: Froggatt-Nielsen like mechanism", JHEP **07**, 068 (2021), arXiv:2105.06237 [hep-ph].

[115] S. Ferrara, D. Lust, A. D. Shapere, and S. Theisen, "Modular Invariance in Supersymmetric Field Theories", Phys. Lett. B **225**, 363 (1989).

[116] L. J. Dixon, V. Kaplunovsky, and J. Louis, "On Effective Field Theories Describing (2,2) Vacua of the Heterotic String", Nucl. Phys. B **329**, 27 (1990).

[117] L. E. Ibanez and D. Lust, "Duality anomaly cancellation, minimal string unification and the effective low-energy Lagrangian of 4-D strings", Nucl. Phys. B **382**, 305 (1992), arXiv:hep-th/9202046.

[118] Y. Olgun-Trejo and S. Ramos-Sánchez, "Kähler potential of heterotic orbifolds with multiple Kähler moduli", J. Phys. Conf. Ser. **912**, edited by I. Bautista, E. de la Cruz Burelo, A. Fernández-Téllez, G. López-Castro, M. Rodrguez-Cahuantzi, and P. Roig, 012029 (2017), arXiv:1707.09966 [hep-th].

[119] X.-G. Liu, C.-Y. Yao, B.-Y. Qu, and G.-J. Ding, "Half-integral weight modular forms and application to neutrino mass models", Phys. Rev. D **102**, 115035 (2020), arXiv:2007.13706 [hep-ph].

[120] M.-C. Chen, S. Ramos-Sánchez, and M. Ratz, "A note on the predictions of models with modular flavor symmetries", Phys. Lett. B **801**, 135153 (2020), arXiv:1909.06910 [hep-ph].

[121] M.-C. Chen, V. Knapp-Perez, M. Ramos-Hamud, S. Ramos-Sanchez, M. Ratz, and S. Shukla, "Quasi-Eclectic Modular Flavor Symmetries", (2021), arXiv:2108.02240 [hep-ph].

[122] D. Schultz, "Notes on Modular Forms", https://faculty.math.illinois.edu/~schult25/ModFormNotes.pdf (2015).

[123] *GAP – Groups, Algorithms, and Programming, Version 4.11.1*, https://www.gap-system.org, The GAP Group (2021).

[124] H. U. Besche, B. Eick, and E. O'Brien, *SmallGrp – a GAP package, Version 1.4.2*, https://gap-packages.github.io/smallgrp (2020).

[125] T. Mano, "Differential relations for modular forms of level five", J. Math. Kyoto Univ. **42**, 41 (2002).

[126] F. Feruglio, C. Hagedorn, and R. Ziegler, "Lepton Mixing Parameters from Discrete and CP Symmetries", JHEP **07**, 027 (2013), arXiv:1211.5560 [hep-ph].

[127] M. Holthausen, M. Lindner, and M. A. Schmidt, "CP and Discrete Flavour Symmetries", JHEP **04**, 122 (2013), arXiv:1211.6953 [hep-ph].

[128] B. S. Acharya, D. Bailin, A. Love, W. A. Sabra, and S. Thomas, "Spontaneous breaking of CP symmetry by orbifold moduli", Phys. Lett. B **357**, [Erratum: Phys.Lett.B 407, 451–451 (1997)], 387 (1995), arXiv:hep-th/9506143.

[129] T. Dent, "CP violation and modular symmetries", Phys. Rev. D **64**, 056005 (2001), arXiv:hep-ph/0105285.

[130] J. Giedt, "CP violation and moduli stabilization in heterotic models", Mod. Phys. Lett. A **17**, 1465 (2002), arXiv:hep-ph/0204017.

[131] L. K. Hua and I. Reiner, "Automorphisms of the Unimodular Group", T. Am. Math. Soc. **71**, 331 (1951).

[132] M.-C. Chen, M. Fallbacher, K. T. Mahanthappa, M. Ratz, and A. Trautner, "CP Violation from Finite Groups", Nucl. Phys. B **883**, 267 (2014), arXiv:1402.0507 [hep-ph].

[133] O. Karpenkov, *Geometry of continued fractions*, Algorithms and Computation in Mathematics (Springer, 2013), p. 405.

[134] J.-P. Serre, *A Course in Arithmetic*, Graduate Texts in Mathematics (Springer, 1973).

[135] K. Nakamura and S. T. Petcov, "Neutrino Masses, Mixing, and Oscillations*, in M. Tanabashi et al. (Particle Data Group),* Review of Particle Physics", Phys. Rev. **D98**, and 2019 update, 030001 (2018).

[136] D. Hinrichsen and A. J. Pritchard, *Mathematical systems theory i, Modelling, state space analysis, stability and robustness*, Vol. 48, Texts in Applied Mathematics (Springer-Verlag Berlin Heidelberg, 2005), p. 804.

[137] D. Marzocca and A. Romanino, "Stable fermion mass matrices and the charged lepton contribution to neutrino mixing", JHEP **11**, 159 (2014), arXiv:1409.3760 [hep-ph].

[138] Y. Reyimuaji and A. Romanino, "Can an unbroken flavour symmetry provide an approximate description of lepton masses and mixing?", JHEP **03**, 067 (2018), arXiv:1801.10530 [hep-ph].

[139] S. Antusch, J. Kersten, M. Lindner, and M. Ratz, "Running neutrino masses, mixings and CP phases: Analytical results and phenomenological consequences", Nucl. Phys. B **674**, 401 (2003), arXiv:hep-ph/0305273.

[140] F. Capozzi, E. Di Valentino, E. Lisi, A. Marrone, A. Melchiorri, and A. Palazzo, "Global constraints on absolute neutrino masses and their ordering", Phys. Rev. D **95**, [Addendum: Phys.Rev.D 101, 116013 (2020)], 096014 (2017), arXiv:2003.08511 [hep-ph].

[141] G. Ross and M. Serna, "Unification and fermion mass structure", Phys. Lett. B **664**, 97 (2008), arXiv:0704.1248 [hep-ph].

[142] S. M. Bilenky, J. Hosek, and S. T. Petcov, "On Oscillations of Neutrinos with Dirac and Majorana Masses", Phys. Lett. B **94**, 495 (1980).

[143] N. Cabibbo, "Time Reversal Violation in Neutrino Oscillation", Phys. Lett. B **72**, 333 (1978).

[144] P. I. Krastev and S. T. Petcov, "Resonance Amplification and t Violation Effects in Three Neutrino Oscillations in the Earth", Phys. Lett. B **205**, 84 (1988).

[145] S. M. Bilenky and S. T. Petcov, "Massive Neutrinos and Neutrino Oscillations", Rev. Mod. Phys. **59**, [Erratum: Rev.Mod.Phys. 61, 169 (1989), Erratum: Rev.Mod.Phys. 60, 575–575 (1988)], 671 (1987).

[146] W. Grimus and L. Lavoura, "A Model for trimaximal lepton mixing", JHEP **09**, 106 (2008), arXiv:0809.0226 [hep-ph].

[147] C. H. Albright, A. Dueck, and W. Rodejohann, "Possible Alternatives to Tri-bimaximal Mixing", Eur. Phys. J. C **70**, 1099 (2010), arXiv:1004.2798 [hep-ph].

[148] I. Girardi, S. T. Petcov, A. J. Stuart, and A. V. Titov, "Leptonic Dirac CP Violation Predictions from Residual Discrete Symmetries", Nucl. Phys. B **902**, 1 (2016), arXiv:1509.02502 [hep-ph].

[149] F. Capozzi, E. Lisi, A. Marrone, and A. Palazzo, "Current unknowns in the three neutrino framework", Prog. Part. Nucl. Phys. **102**, 48 (2018), arXiv:1804.09678 [hep-ph].

[150] Y. Shimizu, M. Tanimoto, and A. Watanabe, "Breaking Tri-bimaximal Mixing and Large θ_{13}", Prog. Theor. Phys. **126**, 81 (2011), arXiv:1105.2929 [hep-ph].

[151] I. Girardi, S. T. Petcov, and A. V. Titov, "Predictions for the Majorana CP Violation Phases in the Neutrino Mixing Matrix and Neutrinoless Double Beta Decay", Nucl. Phys. B **911**, 754 (2016), arXiv:1605.04172 [hep-ph].

[152] Y. Shimizu, M. Tanimoto, and K. Yamamoto, "Predicting CP violation in Deviation from Tri-bimaximal mixing of Neutrinos", Mod. Phys. Lett. A **30**, 1550002 (2015), arXiv:1405.1521 [hep-ph].

[153] R. Barbieri and G. F. Giudice, "Upper Bounds on Supersymmetric Particle Masses", Nucl. Phys. B **306**, 63 (1988).

[154] S. Kharchev and A. Zabrodin, "Theta vocabulary I", J. Geom. Phys. **94**, 19 (2015), arXiv:1502.04603.

[155] G.-J. Ding, L. L. Everett, and A. J. Stuart, "Golden Ratio Neutrino Mixing and A_5 Flavor Symmetry", Nucl. Phys. B **857**, 219 (2012), arXiv:1110.1688 [hep-ph].