Research Article/Araştırma Makalesi

POWER SERIES SOLUTION OF NON-LINEAR FIRST ORDER DIFFERENTIAL EQUATION SYSTEMS

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Abstract: In this paper, we use power series method to solve non-linear ordinary differential equations Theoretical considerations has been discussed and some examples were presented to show the ability of the method for non-linear systems of differential equations. We use MAPLE computer algebra systems for numerical calculations (FRANK, 1996).

Key words: Power series, Numerical solution of the ordinary differential equations, MAPLE.

Lineer Olmayan Birinci Mertebeden Differential Denklem Sistemlerinin Kuvvet Serisiyle Çözümü

Özet: Bu makalede, lineer olmayan adi diferansiyel denklemlerin çözümü için kuvvet serisini kullandık. Nümerik yoldan elde edilen sonuçlarla, teorik yoldan elde edilen sonuçlar karşılaştırıldı ve lineer olmayan differansiyel denklem sistemlerinde metodun etkinliğini göstermek için örnekler verildi. Nümerik hesaplamalarda MAPLE bilgisayar cebiri sistemleri kullanıldı (FRANK, 1996).

Anahtar kelimeler: Kuvvet serileri, Adi diferansiyel denklemlerin nümerik çözümü, MAPLE

Introduction

A system of first order differential equations can be considered as:

$$y' = f(x, y), \quad x \in [0, T], \quad |y| < +\infty,$$
$$y(0) = y_0, \tag{1.1}$$

Theoretical solution of the system (1.1) is $y(x)$. Let y_n be an approximation to $y(x_n)$. Here

$$y = [y_1, y_2, \ldots, y_m]^T,$$
$$f = [f_1, f_2, \ldots, f_m]^T,$$

and

$$y_0 = [y_{01}, y_{02}, \ldots, y_{0m}]^T$$

are elements of C^m. where each equation represents the first derivative of one of the unknown functions as a mapping depending on the independent variable x, and m unknown functions $f_1, \ldots, f_m$. Since every ordinary differential equation of order n can be written as a system consisting of n ordinary differential equation of order one, we restrict our study to a system of differential equations of the first order. That is f and y are vector functions for which we assumed sufficient differentiability (BRENAN et all,1989), (HA-

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İRER and WANNER, 1991), (ASCHER and PETZOLD,1998), (AMODIO and MAZZIA, 1997). The solutions of (1.1) can be assumed that

$$y = y_0 + ex \tag{1.2}$$

where e is a vector function which is the same size as y_0. Substitute (1.2) into (1.1) and neglect higher order term. We have the linear equation of e in the form

$$Ae = B \tag{1.3}$$

where A and B is a constant matrixes. Solve this equation of (1.3), the coefficients of x in (1.2) can be determined. Repeating above procedure for higher order terms, we can get the arbitrary order power series of the solutions for (1.1).

The Method

We define another type power series in the form

$$f(x) = f_0 + f_1 x + f_2 x^2 + \cdots + (f_n + p_1 e_1 + \cdots + p_m e_m)x^n \tag{2.1}$$

where $p_1, p_2 \cdots p_m$ are constant. $e_1, e_2 \cdots e_m$ are basis of vector e, m is size of vector e. y is a vector with m elements in (1.2). Every element can be represented by the Power series in (2.1).

$$y_i = y_{i,0} + y_{i,1}x + y_{i,2}x^2 + \cdots + e_i x^n \tag{2.2}$$

Where y_i is i th element of y. Substitute (2.2) into (1.1), we can get the following

$$f_i = (f_{i,n} + p_{i,1}e_1 + \cdots + p_{i,m}e_m)x^{n-j} + O(x^{n-j+1}) \tag{2.3}$$

Where f_i is i th element of $f(y, y', x)$ in (1.1) and j is 0 if $f(y, y', x)$ have y', 1 if don't. From (2.3) and (1.3), we can determine the linear equation in (1.3) as follow

$$A_{i,j} = P_{i,j} \tag{2.4}$$

$$B_i = -f_{i,n} \tag{2.5}$$

Solve this linear equation, we have $e_i \ (i = 1, \cdots, m)$. Substitute e_i into (2.2), we have $y_i \ (i = 1, \cdots, m)$ which are polynomials of degree n. Repeating this procedure from (2.2) to (2.5), we can get the arbitrary order Power series of the solution for differential-algebraic equations in (1.1).

Let step size of x to be h and substitute it into the power series of y and y', we have y and y' at $x = x_0 + h$. If we repeat above procedure, we have numerical solution of differential-algebraic equations in (1.1) (HENRICI,1974),(CORLISS and CHANG, 1982), (HULL et al.,1972). This method has been used to obtained approximate numerical and theoretical solutions of a large class of differential-algebraic equation (ÇELİK et all.,2002),(ÇELİK and BAYRAM, 2003) references therein.

Applications

Example 1. The test problem consider the following differential equation system,

$$
\begin{aligned}
y_1' - 2y_2^2 &= 0 \\
y_2' - e^{-x}y_1 &= 0 \\
y_3' - y_2 - xe^x &= 0
\end{aligned}
\tag{3.1}
$$

with initial condition

$$y_1(0) = 1, \ y_2(0) = 1 \text{ and } y_3(0) = 0.$$

The exact solution is

$$y_1(x) = e^{2x}, \ y_2(x) = e^x \text{ and } y_3(x) = xe^x.$$

From boundary condition, the solutions of (3.1) can be supposed as

$$y_1(x) = y_{0,1} + e_1 x \Rightarrow y_1(x) = 1 + e_1 x$$
$$y_2(x) = y_{0,2} + e_2 x \Rightarrow y_2(x) = 1 + e_2 x \tag{3.2}$$
$$y_3(x) = y_{0,3} + e_3 x \Rightarrow y_3(x) = 0 + e_3 x$$

Substitute (3.2) into (3.1) and neglect higher order terms, We have

$$e_1 - 2 + O(x) = 0$$
$$e_2 - 1 + O(x) = 0 \tag{3.3}$$
$$e3 - 1 + O(x) = 0$$

These formulas correspond to (2.3). The linear equation that corresponds to (2.4) can be given in the following

$$Ae = B \tag{3.4}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

From equation (3.4), We have linear equation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Solve this linear equation. We have

$$e = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

and

$$y_1(x) = 1 + 2x$$
$$y_2(x) = 1 + x \tag{3.5}$$
$$y_3(x) = x$$

From (3.5) the solutions of (3.1) can be supposed as

$$y_1(x) = 1 + 2x + e_1 x^2$$
$$y_2(x) = 1 + x + e_2 x^2 \tag{3.6}$$
$$y_3(x) = x + e_3 x^2$$

In like manners. Substitute (3.6) into (3.1) and neglect higher order terms, we have

$$(2e_1 - 4)x + O(x^2) = 0$$
$$(2e_2 - 1)x + O(x^2) = 0 \tag{3.7}$$
$$(2e_3 - 2)x + O(x^2) = 0$$

where

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

From (3.7) we have linear equation

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

Solve this linear equation, we have

$$e = \begin{bmatrix} 2 \\ \dfrac{1}{2} \\ 1 \end{bmatrix}.$$

Therefore

$$y_1(x) = 1 + 2x + 2x^2$$

$$y_2(x) = 1 + x + \frac{1}{2}x^2 \tag{3.8}$$

$$y_3(x) = x + x^2$$

Repeating above procedure we have

$$y_1(x) \approx 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 \cdots$$

$$y_2(x) \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 \cdots$$

$$y_3 \approx x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \frac{1}{120}x^6 \cdots$$

The power series solution of the given equation (3.1) are coinciding with the exact solutions of (3.1).

Example 2. Let us consider following system of differential equation

$$y_1' - y_3 + \cos(x) = 0, \quad y_1(0) = 1,$$

$$y_2' - y_3 + e^x = 0, \quad y_2(0) = 0, \tag{3.9}$$

$$y_3' - y_1 + \sin(x) = 0, \quad y_3(0) = 2.$$

The analytic solution of the problem is

$$y_1 = e^x, \quad y_2 = \sin(x) \text{ and } y_3 = e^x + \cos(x) \tag{3.10}$$

If we apply the power series method to the given equation system, first six terms of the power series can be obtained as follows.

$$y_1 \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 \cdots$$

$$y_2 \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 \cdots$$

$$y_3 \approx 2 + x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 \cdots$$

The power series solution of the given equation (3.9) are coinciding with the exact solutions of (3.10).

Conclusion

The method has been applied to the solution of systems of non-linear differential equations. A technique has also been considered to avoid the computation of consistent initial conditions. The numerical example has been presented to show that the approach is promising and the research is worth to continue in this direction. Power series method has proposed for solving differential equation systems in this study. This method is very simple an effective for most of differential equations system.

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