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Coupled fixed point theorems in complete metric spaces endowed with a directed graph and application

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Abstract: The purpose of this paper is to present some existence results for coupled fixed point of a (φ, ψ) −contractive condition for mixed monotone operators in metric spaces endowed with a directed graph. Our results generalize the results obtained by Jain et al. in (International Journal of Analysis, Volume 2014, Article ID 586096, 9 pages). Moreover, we have an application to some integral system to support the results.

Keywords: Coupled coincidence point, Coupled fixed point, Edge preserving, Directed graph

MSC: 47H10, 54H25

1 Introduction and preliminaries

The classical Banach's contraction principle (BCP) [1] is a power tool in nonlinear analysis and has been extended and improved by many authors (see [2]-[6]). In 2004, the existence of fixed points for contraction mappings in partially ordered metric spaces has been studied by Ran and Reurings [7], Nietto and Lopez [8]. Extensions and applications of these works appear in (see [9]-[13]).

A concept of coupled fixed point theorem was introduced by Guo and Lakshmikantham [14]. In 2006, Bhaskar and Lakshmikantham [15] introduced the concept of the mixed monotone property as follows.

Definition 1.1 ([15])**.** *Let $(X, \preccurlyeq)$ be a partially ordered set and $F : X^2 \to X$ be a mapping. Then a map F is said to have the mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone non-increasing in y; that is, for any $x, y \in X$,*

$$x_1,\ x_2 \in X, \quad x_1 \preccurlyeq x_2 \ \text{ implies } \quad F(x_1, y) \preccurlyeq F(x_2, y)$$

and

$$y_1,\ y_2 \in X, \quad y_1 \preccurlyeq y_2 \ \text{ implies } \quad F(x, y_1) \succcurlyeq F(x, y_2)\,.$$

Definition 1.2 ([15])**.** *An element $(x, y) \in X^2$ is said to be a coupled fixed point of the mapping $F : X^2 \to X$ if $F(x, y) = x$ and $F(y, x) = y$.*

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Lakshmikantham and Ćirić [16] extended the concept of mixed monotone property to mixed g-monotone property as follows.

Definition 1.3 ([16]). *Let $(X, \preccurlyeq)$ be a partially ordered set and $F : X^2 \to X$ and $g : X \to X$. We say F is called the mixed g-monotone property if for any $x, y \in X$,*

$$x_1, x_2 \in X, \quad gx_1 \preccurlyeq gx_2 \;\; \text{implies} \quad F(x_1, y) \preccurlyeq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad gy_1 \preccurlyeq gy_2 \;\; \text{implies} \quad F(x, y_1) \succcurlyeq F(x, y_2).$$

Definition 1.4 ([16]). *An element $(x, y) \in X^2$ is said to be a coupled coincidence point of the mappings $F : X^2 \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.*

Definition 1.5 ([16]). *An element $(x, y) \in X^2$ is said to be a coupled common fixed point of the mappings $F : X^2 \to X$ and $g : X \to X$ if $F(x, y) = gx = x$ and $F(y, x) = gy = y$.*

Definition 1.6 ([16]). *Let X be a nonempty set and $F : X^2 \to X$ and $g : X \to X$. We say F and g are commutative if $gF(x, y) = F(gx, gy)$ for all $x, y \in X$.*

In 2010, Choudhury and Kundu [17] introduced the notion of compatibility in the context of coupled coincidence point problems as follows.

Definition 1.7 ([17]). *The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be compatible if*

$$\lim_{n\to\infty} d\left(g\left(F(x_n, y_n)\right), F(gx_n, gy_n)\right) = 0 \text{ and } \lim_{n\to\infty} d\left(g\left(F(y_n, x_n)\right), F(gy_n, gx_n)\right) = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y$ with $x, y \in X$.

Definition 1.8 ([18]). *Let Φ denote the class of functions $\varphi : [0, \infty) \to [0, \infty)$ which satisfies the following conditions:*

(φ_1) φ is lower semi-continuous and (strictly) increasing;
(φ_2) $\varphi(t) < t$ for all $t > 0$;
(φ_3) $\varphi(t + s) \leq \varphi(t) + \varphi(s)$ for all $t, s \in [0, \infty)$.
Note that $\lim_{n\to\infty} \varphi(t_n) = 0 \Leftrightarrow \lim_{n\to\infty} t_n = 0$ for $t_n \in [0, \infty)$.
Also, for $\varphi \in \Phi$, Ψ_φ denote all functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:
(ψ_1) $\limsup_{n\to\infty} \psi(t_n) < \varphi(r)$ if $\lim_{n\to\infty} t_n = r > 0$;
(ψ_2) $\lim_{n\to\infty} \psi(t_n) = 0$ if $\lim_{n\to\infty} t_n = 0$ for $t_n \in [0, \infty)$.

Now, we have the following coupled fixed point theorems as the main result of Jain *et al.* in [18].

Theorem 1.9 ([18]). *Let $(X, \leq)$ be a partially ordered set and there is a metric d on X such that (X, d) is a complete metric space. Suppose that $F : X^2 \to X$ is a mapping having the mixed monotone property on X. Assume there exists $\varphi \in \Phi$ and $\psi \in \Psi_\varphi$ such that*

$$\begin{aligned} &\varphi\left(\{d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))\} \times 2^{-1}\right) \\ &\quad \leq \psi\left(\{d(x, u) + d(y, v)\} \times 2^{-1}\right) \end{aligned} \tag{1}$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$.

Suppose that either
(a) F is continuous or;
(b) X has the following properties:
1. *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all n,*

2. *if a non-increasing sequence* $\{y_n\} \to y$, *then* $y \leq y_n$ *for all* n.

If there exist two elements $x_0, y_0 \in X$ *with* $x_0 \leq F(x_0, y_0)$ *and* $y_0 \geq F(y_0, x_0)$. *Then there exist* $x, y \in X$ *such that* $x = F(x, y)$, $y = F(y, x)$ *,that is,* F *has a coupled fixed point in* X.

The fixed point theorem using the context of metric spaces endowed with a graph was initiated by Jachymski [19]. Other results for single valued and multivalued operators in such metric spaces were given by Beg *et al.* [20], Bajor [21], Alfuraid [22, 23], Chifu and Petrusel [24] and Suantai *et al.* [25].

Let (X, d) be a metric space, Δ be a diagonal of X^2, and G be a directed graph with no parallel edges such that the set $V(G)$ of its vertices coincides with X and $\Delta \subseteq E(G)$, where $E(G)$ is the set of the edges of the graph. That is, G is determined by $(V(G), E(G))$. We will use this notation of G throughout this work.

In 2014, Chifu and Petrusel [24] introduced the notion of $G-$continuity for a mapping $F : X^2 \to X$ and the property A as follows.

Definition 1.10 ([24]). *Let* (X, d) *be a complete metric space,* G *be a directed graph, and* $F : X^2 \to X$ *be a mapping. Then*

(i) F *is called* $G-$*continuous if for all* $(x_*, y_*) \in X^2$ *and for any sequence* $\{n_i\}_i \in \mathbb{N}$ *of positive integers such that* $F(x_{n_i}, y_{n_i}) \to x_*$, $F(y_{n_i}, x_{n_i}) \to y_*$ *as* $i \to \infty$ *and* $\left(F(x_{n_i}, y_{n_i}), F(x_{n_i+1}, y_{n_i+1})\right)$, $\left(F(y_{n_i}, x_{n_i}), F(y_{n_i+1}, x_{n_i+1})\right) \in E(G)$, *we have that*

$$F\left(F(x_{n_i}, y_{n_i}), F(y_{n_i}, x_{n_i})\right) \to F(x_*, y_*) \text{ as } i \to \infty$$

and

$$F\left(F(y_{n_i}, x_{n_i}), F(x_{n_i}, y_{n_i})\right) \to F(x_*, y_*) \text{ as } i \to \infty;$$

(ii) (X, d, G) *has property* A *if for any sequence* $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ *with* $x_n \to x$ *as* $n \to \infty$ *and* $(x_n, x_{n+1}) \in E(G)$ *for* $n \in \mathbb{N}$, *then* $(x_n, x) \in E(G)$.

Consider the set $CcFix(F)$ of all coupled coincidence points of mappings $F : X^2 \to X$, $g : X \to X$ and the set $(X^2)_g^F$ as follows:

$$CcFix(F) = \left\{(x, y) \in X^2 : gx = F(x, y) \text{ and } gy = F(y, x)\right\}$$

and

$$\left(X^2\right)_g^F = \left\{(x, y) \in X^2 : (gx, F(x, y)), (gy, F(y, x)) \in E(G)\right\}.$$

In 2015, Suantai *et al.* [25] introduced the concept of $G-$edge preserving and the transitivity property as follows.

Definition 1.11 ([25]). $F : X^2 \to X$, $g : X \to X$ *are* $G-$*edge preserving if*

$$[(gx, gu), (gy, gv) \in E(G)] \Rightarrow [(F(x, y), F(u, v)), (F(y, x), F(v, u)) \in E(G)].$$

Definition 1.12 ([25]). *Let* (X, d) *be a complete metric space, and* $E(G)$ *be the set of the edges of the graph.* $E(G)$ *satisfies the transitivity property if and only if, for all* $x, y, t \in X$,

$$(x, t), (t, y) \in E(G) \to (x, y) \in E(G).$$

The purpose of this paper is to present some existence results for coupled fixed points of $(\varphi, \psi)-$contractive mappings in metric spaces endowed with a directed graph. Our results generalize the results obtained by Jain *et al.* in (International Journal of Analysis, Volume 2014, Article ID 586096, 9 pages). Moreover, we have an application to some integral system to support the results.

2 Main results

Definition 2.1. *Let* (X, d) *be a complete metric space endowed with a directed graph* G. *The mappings* $F : X^2 \to X$, $g : X \to X$ *are called a* $(\varphi, \psi)-$*contractive if:*

1. *F and g is $G-$edge preserving;*
2. *there exists $\varphi \in \Phi$ and $\psi \in \Psi_\varphi$ such that for all $x, y, u, v \in X$ satisfying (gx, gu), $(gy, gv) \in E(G)$,*

$$\begin{aligned}\varphi\left(\{d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))\} \times 2^{-1}\right) \\ \leq \psi\left(\{d(gx, gu) + d(gy, gv)\} \times 2^{-1}\right).\end{aligned} \tag{2}$$

Theorem 2.2. *Let (X, d) be a complete metric space endowed with a directed graph G, and let $F : X^2 \to X$, $g : X \to X$ be a $(\varphi, \psi)-$contractive mapping. Suppose that:*

(i) g is continuous and $g(X)$ is closed;

(ii) $F(X^2) \subseteq g(X)$, and (F, g) is compatible;

(iii)

1. *F is $G-$continuous, or*
2. *the tripled (X, d, G) has a property A;*

(iv) $E(G)$ satisfies the transitivity property.

Under these conditions, $CcFix(F) \neq \emptyset$ iff $(X^2)_g^F \neq \emptyset$.

Proof. Consider $x_0, y_0, t_0, s_0 \in X$ followed by assumptions. Since $F(X^2) \subseteq g(X)$, then there exists $x_1, y_1, t_1, s_1 \in X$ such that $F(x_0, y_0) = gx_1$ and $F(y_0, x_0) = gy_1$, $F(t_0, s_0) = gt_1$ and $F(s_0, t_0) = gs_1$ continuing the procedure above we have the sequence $\{x_n\}$, $\{y_n\}$, $\{t_n\}$, $\{s_n\}$ in X for which

$$\begin{aligned}F(x_n, y_n) = gx_{n+1} \text{ and } F(y_n, x_n) = gy_{n+1}, \\ F(t_n, s_n) = gt_{n+1} \text{ and } F(s_n, t_n) = gs_{n+1}, \text{ for all } n \in \mathbb{N}.\end{aligned} \tag{3}$$

$CcFix(F) \neq \emptyset$. Let $(u, v) \in CcFix(F)$ such that

$$(gu, F(u,v)) = (gu, gu), \ (gv, F(v,u)) = (gv, gv) \in \Delta \subset E(G).$$

Hence, $(gu, F(u,v))$, $(gv, F(v,u)) \in E(G)$. Then we have $(u, v) \in (X^2)_g^F$, thereby $(X^2)_g^F \neq \emptyset$.

Next, assume that $(X^2)_g^F \neq \emptyset$. Let $x_0, y_0 \in X$ such that $(x_0, y_0) \in (X^2)_g^F$. In that case, $(gx_0, F(x_0, y_0))$, $(gy_0, F(y_0, x_0)) \in E(G)$. By (3), we obtain

$$(gx_0, F(x_0, y_0)) = (gx_0, gx_1) \text{ and } (gy_0, F(y_0, x_0)) = (gy_0, gy_1) \in E(G). \tag{4}$$

Since F and g are $G-$edge preserving and by (4), we get

$$(F(x_0, y_0), F(x_1, y_1)) = (gx_1, gx_2) \text{ and } (F(y_0, x_0), F(y_1, x_1)) = (gy_1, gy_2) \in E(G).$$

Continuing this process, we can construct (gx_n, gx_{n+1}) and $(gy_n, gy_{n+1}) \in E(G)$ for each $n \in \mathbb{N}$.

Denote $\kappa_n := (d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})) \times 2^{-1}$ for all $n \in \mathbb{N}$.

Using the $(\varphi, \psi)-$contractive type operator (2) and (3), we get that

$$\begin{aligned}&\varphi\left(\frac{d(gx_{n+1}, gx_{n+2}) + d(gy_{n+1}, gy_{n+2})}{2}\right) \\ &= \varphi\left(\frac{d(F(x_n, y_n), F(x_{n+1}, y_{n+1})) + d(F(y_n, x_n), F(y_{n+1}, x_{n+1}))}{2}\right) \\ &\leq \psi\left(\frac{d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})}{2}\right)\end{aligned} \tag{5}$$

for all $n \in \mathbb{N}$, then we obtain

$$\varphi(\kappa_{n+1}) \leq \psi(\kappa_n) < \varphi(\kappa_n). \tag{6}$$

From (6) and monotonicity of φ, we get that $\{\kappa_n\}$ is a nonnegative decreasing. Then $\kappa_n \to \kappa$ as $n \to \infty$ for some $\kappa \geq 0$. If possible, let $\kappa > 0$. Taking the limit as $n \to \infty$ in (6) and using the properties of φ, ψ, we obtain

$$\varphi(\kappa) \leq \lim \sup_{n \to \infty} \varphi(\kappa_{n+1}) \leq \lim \sup_{n \to \infty} \psi(\kappa_n) < \varphi(\kappa), \tag{7}$$

which is a contradiction. Thus $\kappa = 0$ and then we have

$$\lim_{n\to\infty} \kappa_n = \lim_{n\to\infty} \left(d\left(gx_n, gx_{n+1}\right) + d\left(gy_n, gy_{n+1}\right)\right) \times 2^{-1} = 0. \tag{8}$$

Now we shall prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Let at least one of $\{gx_n\}$ and $\{gy_n\}$ be not Cauchy sequences. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{gx_{n(k)}\}$, $\{gx_{m(k)}\}$ of $\{gx_n\}$ and $\{gy_{n(k)}\}$, $\{gy_{m(k)}\}$ of $\{gy_n\}$ with $n(k) > m(k) \geq k$ such that

$$\xi_k = \left(d\left(gx_{n(k)}, gx_{m(k)}\right) + d\left(gy_{n(k)}, gy_{m(k)}\right)\right) \times 2^{-1} \geq \varepsilon. \tag{9}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a manner that it is the smallest integer for which (9) holds. Then,

$$\left(d\left(gx_{n(k)-1}, gx_{m(k)}\right) + d\left(gy_{n(k)-1}, gy_{m(k)}\right)\right) \times 2^{-1} < \varepsilon. \tag{10}$$

From (9), (10), and triangular inequality, we get

$$\varepsilon \leq \xi_k < \varepsilon + \left(d\left(gx_{n(k)}, gx_{n(k)-1}\right) + d\left(gy_{n(k)}, gy_{n(k)-1}\right)\right) \times 2^{-1}. \tag{11}$$

By using (8) in (11), we obtain

$$\xi_k = \left(d\left(gx_{n(k)}, gx_{m(k)}\right) + d\left(gy_{n(k)}, gy_{m(k)}\right)\right) \times 2^{-1} \to \varepsilon \text{ as } k \to \infty. \tag{12}$$

Again, by the triangle inequality

$$\begin{aligned} \xi_k &= \left(d\left(gx_{n(k)}, gx_{m(k)}\right) + d\left(gy_{n(k)}, gy_{m(k)}\right)\right) \times 2^{-1} \\ &\leq \kappa_{n(k)} + \kappa_{m(k)} + \left(d\left(gx_{n(k)+1}, gx_{m(k)+1}\right) + d\left(gy_{n(k)+1}, gy_{m(k)+1}\right)\right) \times 2^{-1}. \end{aligned}$$

From monotonicity of φ and property (φ_3), we obtain

$$\begin{aligned} \varphi(\xi_k) &\leq \varphi\left(\kappa_{n(k)}\right) + \varphi\left(\kappa_{m(k)}\right) \\ &\quad + \varphi\left(\left(d\left(gx_{n(k)+1}, gx_{m(k)+1}\right) + d\left(gy_{n(k)+1}, gy_{m(k)+1}\right)\right) \times 2^{-1}\right) \\ &\leq \varphi\left(\kappa_{n(k)}\right) + \varphi\left(\kappa_{m(k)}\right) \\ &\quad + \varphi\left(\left(\begin{array}{c} d\left(F\left(x_{n(k)}, y_{n(k)}\right), F\left(x_{m(k)}, y_{m(k)}\right)\right) \\ +d\left(F\left(y_{n(k)}, x_{n(k)}\right), F\left(y_{m(k)}, x_{m(k)}\right)\right) \end{array}\right) \times 2^{-1}\right) \\ &\leq \varphi\left(\kappa_{n(k)}\right) + \varphi\left(\kappa_{m(k)}\right) \\ &\quad + \psi\left(\left(d\left(gx_{n(k)}, gx_{m(k)}\right) + d\left(gy_{n(k)}, gy_{m(k)}\right)\right) \times 2^{-1}\right) \\ &\leq \varphi\left(\kappa_{n(k)}\right) + \varphi\left(\kappa_{m(k)}\right) + \varphi(\xi_k). \end{aligned} \tag{13}$$

As φ is lower semi-continuous by taking limit as $k \to \infty$, we obtain

$$\begin{aligned} \varphi(\varepsilon) &\leq \lim \sup_{k\to\infty} \varphi(\xi_k) \\ &\leq \lim_{k\to\infty} \varphi\left(\kappa_{n(k)}\right) + \lim_{k\to\infty} \varphi\left(\kappa_{m(k)}\right) + \lim \sup_{k\to\infty} \psi(\xi_k) \\ &< \varphi(\varepsilon), \end{aligned} \tag{14}$$

a contradiction. Therefore, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in X. From assumption (i) there exist $u, v \in g(X)$ such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = u$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = v$. Since F and g are compatible mappings, we have

$$\lim_{n\to\infty} d\left(g\left(F\left(x_n, y_n\right)\right), F\left(gx_n, gy_n\right)\right) = 0 = \lim_{n\to\infty} d\left(g\left(F\left(y_n, x_n\right)\right), F\left(gy_n, gx_n\right)\right). \tag{15}$$

Let the assumption (1) hold. For all $n \geq 0$, we have

$$d\left(F\left(gx_n, gy_n\right), gu\right) \leq d\left(F\left(gx_n, gy_n\right), gF\left(x_n, y_n\right)\right) + d\left(gF\left(x_n, y_n\right), gu\right).$$

Letting $n \to \infty$, using (15), by assumption (i) and (iii), we have $d(F(u,v), gu) = 0$, that is, $F(u,v) = gu$. Similarly, we also have $F(v,u) = gv$. Then $CcFix(F) \neq \emptyset$.

Next, we suppose that the assumption (2) holds. Let $u = gx$ and $v = gy$ for some $x, y \in X$. In this way, (gx_n, gx), $(gy_n, gy) \in E(G)$ for each $n \in \mathbb{N}$. Then

$$\begin{aligned} d(gx, F(x,y)) &\leq d(gx, gx_{n+1}) + d(gx_{n+1}, F(x,y)) \\ &= d(gx, gx_{n+1}) + d(F(x_n, y_n), F(x,y)) \\ d(gx, F(x,y)) - d(gx, gx_{n+1}) &\leq d(F(x_n, y_n), F(x,y)) \end{aligned} \tag{16}$$

and

$$\begin{aligned} d(gy, F(y,x)) &\leq d(gy, gy_{n+1}) + d(gy_{n+1}, F(y,x)) \\ &= d(gy, gy_{n+1}) + d(F(y_n, x_n), F(y,x)) \\ d(gy, F(y,x)) - d(gy, gy_{n+1}) &\leq d(F(y_n, x_n), F(y,x)). \end{aligned} \tag{17}$$

Combining (16) and (17), we have

$$\begin{aligned} &\{d(gx, F(x,y)) - d(gx, gx_{n+1}) + d(gy, F(y,x)) - d(gy, gy_{n+1})\} \times 2^{-1} \\ &\leq \{d(F(x_n, y_n), F(x,y)) + d(F(y_n, x_n), F(y,x))\} \times 2^{-1} \end{aligned}$$

by the monotonicity of φ

$$\begin{aligned} &\varphi\left(\{d(gx, F(x,y)) - d(gx, gx_{n+1}) + d(gy, F(y,x)) - d(gy, gy_{n+1})\} \times 2^{-1}\right) \\ &\leq \varphi\left(\{d(F(x_n, y_n), F(x,y)) + d(F(y_n, x_n), F(y,x))\} \times 2^{-1}\right) \\ &\leq \psi\left(\{d(gx_n, gx) + d(gy_n, gy)\} \times 2^{-1}\right). \end{aligned}$$

As φ is lower semi-continuous, letting $k \to \infty$ and by (ψ_2), we obtain $gx = F(x,y)$ and $gy = F(y,x)$. □

Denote by $CmFix(F)$ the set of all common fixed points of mappings $F : X^2 \to X$, $g : X \to X$, that is,

$$CmFix(F) = \left\{(x,y) \in X^2 : x = gx = F(x,y) \text{ and } y = gy = F(y,x)\right\}.$$

Theorem 2.3. *In addition to Theorem 2.2, suppose that*

(v) for any two elements $(x,y), (u,v) \in X^2$, there exists $(t,s) \in X^2$ such that (gx, gt), (gu, gt), (gy, gs), $(gv, gs) \in E(G)$.

Then, $CmFix(F) \neq \emptyset$ iff $(X^2)_g^F \neq \emptyset$.

Proof. Theorem 2.2 implies that there exists $(x,y) \in X^2$ such that $F(x,y) = gx$ and $F(y,x) = gy$. Assume that there exists another $(u,v) \in X^2$ such that $F(u,v) = gu$ and $F(v,u) = gv$. Now, we shall prove that $gu = gx$ and $gv = gy$.

From assumption (v), there exists $(t,s) \in X^2$ such that (gx, gt), (gu, gt), (gy, gs), $(gv, gs) \in E(G)$. By using (3), we define the sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{v_n\}$, $\{t_n\}$ and $\{s_n\}$ in X as follows:

$$\begin{gathered} x = x_0,\ y = y_0,\ u = u_0,\ v = v_0,\ t = t_0,\ s = s_0, \\ F(x_n, y_n) = gx_{n+1} \text{ and } F(y_n, x_n) = gy_{n+1}, \\ F(u_n, v_n) = gu_{n+1} \text{ and } F(v_n, u_n) = gv_{n+1}, \\ F(t_n, s_n) = gt_{n+1} \text{ and } F(s_n, t_n) = gs_{n+1}, \end{gathered} \tag{18}$$

for all $n \in \mathbb{N}$. From the properties of coincidence points, $x = x_n$, $y = y_n$ and $u = u_n$, $v = v_n$, namely,

$$F(x,y) = gx_n,\ F(y,x) = gy_n \text{ and } F(u,v) = gu_n,\ F(v,u) = gv_n$$

for all $n \in \mathbb{N}$. As (gx, gt), $(gy, gs) \in E(G)$, we get (gx, gt_0), $(gy, gs_0) \in E(G)$. Since F and g are $G-$edge preserving, we obtain $(F(x,y), F(t_0, s_0)) = (gx, gt_1)$ and $(F(y,x), F(s_0, t_0)) = (gy, gs_1) \in E(G)$. Similarly, (gx, gt_n) and $(gy, gs_n) \in E(G)$. By (2), we have

$$\varphi\left(\{d(gt_{n+1}, gx) + d(gs_{n+1}, gy)\} \times 2^{-1}\right)$$

$$= \varphi\left(\{d\left(F\left(t_n, s_n\right), F\left(x, y\right)\right) + d\left(F\left(s_n, t_n\right), F\left(y, x\right)\right)\} \times 2^{-1}\right)$$
$$\leq \psi\left(\{d\left(gt_n, gx\right) + d\left(gs_n, gy\right)\} \times 2^{-1}\right). \quad (19)$$

Then, we get $\varphi\left(\rho_{n+1}\right) \leq \psi\left(\rho_n\right)$, where $\rho_n := \{d\left(gt_n, gx\right) + d\left(gs_n, gy\right)\} \times 2^{-1}$. The sequence $\{\rho_n\}$ is a monotonically decreasing sequence of nonnegative real numbers, thus there exists some $\rho \geq 0$ such that $\rho_n \to \rho$ as $n \to \infty$. Now, we shall show that $\rho = 0$. Suppose, to the contrary, that $\rho > 0$. Letting $n \to \infty$ in (19), we have

$$\varphi\left(\rho\right) \leq \lim \sup_{n\to\infty} \varphi\left(\rho_{n+1}\right) \leq \lim \sup_{n\to\infty} \psi\left(\rho_n\right) < \varphi\left(\rho\right),$$

a contradiction. Hence, $\rho = 0$; that is; $\lim_{n\to\infty}\left[\{d\left(gt_n, gx\right) + d\left(gs_n, gy\right)\} \times 2^{-1}\right] = 0$, which implies

$$\lim_{n\to\infty} d\left(gt_n, gx\right) = 0 = \lim_{n\to\infty} d\left(gs_n, gy\right).$$

Similarly, (gt, gu), $(gs, gv) \in E\left(G\right)$, we have

$$\lim_{n\to\infty} d\left(gt_n, gu\right) = 0 = \lim_{n\to\infty} d\left(gs_n, gv\right).$$

From the triangular inequality we obtain

$$\begin{aligned} d\left(gu, gx\right) &\leq d\left(gu, gt_n\right) + d\left(gt_n, gx\right), \\ d\left(gv, gy\right) &\leq d\left(gv, gs_n\right) + d\left(gs_n, gy\right) \end{aligned} \quad (20)$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (20), we obtain that $d\left(gu, gx\right) = 0 = d\left(gv, gy\right)$. Hence, we get $gu = gx$ and $gv = gy$.

The proof of the existence and uniqueness of the coupled common fixed point for our main results can be obtained by using a similar assertion as in Theorem 9 in [18]. □

Remark 2.4. *In this case where $(X, \preccurlyeq)$ is partially ordered complete metric space, taking $E\left(G\right) = \left\{(x, y) \in X^2 : x \preccurlyeq y\right\}$, we obtain Theorem 5 in [18]. By using Remark 6, Remark 7 in [18], we get that our results generalize the results obtained by Bhaskar and Lakshmikantham in Theorem 2.1 of [15], Luong and Thuan in Theorem 2.1 of [26], Berinde in Theorem 3 of [27] and Berinde in Theorem 2 of [28].*

3 Application

In this section, we study the existence of solution of the nonlinear integral equations, as an application of the fixed point theorem proved in Main Results.

Consider the following nonlinear integral equation:

$$\begin{aligned} x\left(t\right) &= q\left(t\right) + \int_0^T A\left(t, s\right) h\left(s, x\left(s\right), y\left(s\right)\right) ds, \\ y\left(t\right) &= q\left(t\right) + \int_0^T A\left(t, s\right) h\left(s, y\left(s\right), x\left(s\right)\right) ds, \end{aligned} \quad (21)$$

where $t \in I = [0, T]$ with $T > 0$.

We considered the space $X := C\left(I, \mathbb{R}^n\right)$. Let $\|x\| = \max_{t\in I} |x\left(t\right)|$, for $x \in X$.

Consider the graph G with partial order relation by

$$x, y \in X, x \leq y \Leftrightarrow x\left(t\right) \leq y\left(t\right) \quad \text{for any } t \in I.$$

Then $(X, \|.\|)$ is a complete metric space endowed with a directed graph G.

Let $E\left(G\right) = \left\{(x, y) \in X^2 : x \leq y\right\}$. Thus $E\left(G\right)$ satisfies the transitivity property, and $(X, \|.\|, G)$ has property A.

We consider the following conditions:

1. $h : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $q : I \to \mathbb{R}^n$ are continuous;

2. there exists a continuous $0 \le \alpha < 1$ such that

$$|h(s,x,y) - h(s,u,v)| \le \alpha(|x-u| + |y-v|) \tag{22}$$

for all $x, y, u, v \in \mathbb{R}^n$ and for all $s \in I$;

3. for all $t, s \in I$, there exists a continuous $A : I \times \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{t \in T} \int_0^T A(t,s)\, ds < 1; \tag{23}$$

4. there exists $(x_0, y_0) \in X^2$ such that

$$\begin{array}{l} x_0(t) \le q(t) + \int_0^T A(t,s)\, h(s, x_0(s), y_0(s))\, ds, \\ y_0(t) \le q(t) + \int_0^T A(t,s)\, h(s, y_0(s), x_0(s))\, ds, \end{array} \tag{24}$$

where $t \in I$.

Theorem 3.1. *Suppose that conditions* $(1) - (4)$ *are satisfied. Then (21) has a unique solution in* $C(I, \mathbb{R}^n)$.

Proof. Let $F : X^2 \to X$, $(x, y) \to F(x, y)$, where

$$F(x,y)(t) = q(t) + \int_0^T A(t,s)\, h(s, x(s), y(s))\, ds, \ \ t \in I, \tag{25}$$

and define $g : X \to X$ by $gx(t) = 2x(t)$. (21) can be stated as

$$x = F(x,y) \text{ and } y = F(y,x). \tag{26}$$

Let $x, y, u, v \in X$ be such that $gx \le gu$ and $gy \le gv$. We get $x \le u$ and $y \le v$ and

$$\begin{aligned} F(x,y)(t) &= q(t) + \int_0^T A(t,s)\, h(s, x(s), y(s))\, ds \\ &\le q(t) + \int_0^T A(t,s)\, h(s, u(s), v(s))\, ds = F(u,v)(t) \text{ for all } t \in I, \end{aligned}$$

$$\begin{aligned} F(y,x)(t) &= q(t) + \int_0^T A(t,s)\, h(s, y(s), x(s))\, ds \\ &\le q(t) + \int_0^T A(t,s)\, h(s, v(s), u(s))\, ds = F(v,u)(t) \text{ for all } t \in I. \end{aligned}$$

Then, F and g are $G-$edge preserving.

From (22) and (23) for all $t \in I$, we have

$$\begin{aligned} &\{|F(x,y)(t) - F(u,v)(t)| + |F(y,x)(t) - F(v,u)(t)|\} \times 2^{-1} \\ &= \left\{\left| q(t) + \int_0^T A(t,s)\, h(s, x(s), y(s))\, ds - q(t) - \int_0^T A(t,s)\, h(s, u(s), v(s))\, ds \right|\right\} \times 2^{-1} \\ &\quad + \left\{\left| q(t) + \int_0^T A(t,s)\, h(s, y(s), x(s))\, ds - q(t) - \int_0^T A(t,s)\, h(s, v(s), u(s))\, ds \right|\right\} \times 2^{-1} \end{aligned}$$

$$\leq \left\{ \int_0^T A(t,s) \left| h(s, x(s), y(s)) - h(s, u(s), v(s)) \right| ds \right\} \times 2^{-1}$$
$$+ \left\{ \int_0^T A(t,s) \left| h(s, y(s), x(s)) - h(s, v(s), u(s)) \right| ds \right\} \times 2^{-1}$$
$$\leq \{\alpha (|x(s) - u(s)| + |y(s) - v(s)|) + \alpha (|u(s) - x(s)| + |v(s) - y(s)|)\} \times 2^{-1}$$
$$\leq \alpha \left(\frac{\|gx - gu\| + \|gy - gv\|}{2} \right).$$

Then, there exists $\varphi(t) = t$ and $\psi(t) = \alpha t$ for $\alpha \in [0, 1)$ such that

$$\varphi \left(\frac{\|F(x,y) - F(u,v)\| + \|F(y,x) - F(v,u)\|}{2} \right) \leq \psi \left(\frac{\|gx - gu\| + \|gy - gv\|}{2} \right).$$

Thus, F and g are a (φ, ψ) −contractive.

By assumption (4) shows that there exists $(x_0, y_0) \in X^2$ such that $gx_0 \leq F(x_0, y_0)$ and $gy_0 \leq F(y_0, x_0)$, which implies that $(X^2)_g^F \neq \emptyset$.

Hence, there exists a coupled common fixed point $(x_*, y_*) \in X^2$ of the mapping F and g, which is the solution of the integral system (21). □

The following illustrative example, considered as $X := C(I, \mathbb{R})$, $h : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $q : I \to \mathbb{R}$, $x, y, u, v \in \mathbb{R}$ and $g : X \to X$ by $gx = x$. Inspired and motivated by Example 3.1 of [29], we present an example of a functional integral equation.

Example 3.2. *Consider the following functional integral equation:*

$$x(t) = \frac{t^2}{1+t^4} + \int_0^1 \frac{\sin s 3^{-s} e^{-s}}{9(t+3)} \left(\frac{|x(s)|}{1+|x(s)|} + \frac{|y(s)|}{1+|y(s)|} \right) ds$$
$$y(t) = \frac{t^2}{1+t^4} + \int_0^1 \frac{\sin s 3^{-s} e^{-s}}{9(t+3)} \left(\frac{|y(s)|}{1+|y(s)|} + \frac{|x(s)|}{1+|x(s)|} \right) ds$$

for $t \in I$. Observe that this equation is a special case of (21) with

$$q(t) = \frac{t^2}{1+t^4},$$
$$A(t,s) = \frac{3^{-s} e^{-s}}{t+3},$$
$$h(s,x,y) = \frac{\sin s}{9} \left(\frac{|x(s)|}{1+|x(s)|} + \frac{|y(s)|}{1+|y(s)|} \right),$$
$$h(s,y,x) = \frac{\sin s}{9} \left(\frac{|y(s)|}{1+|y(s)|} + \frac{|x(s)|}{1+|x(s)|} \right).$$

It also easily seen that these functions are continuous.

For arbitrary $x, y, u, v \in \mathbb{R}$ and for all $s \in I$, we have

$$|h(s,x,y) - h(s,u,v)| = \left| \frac{\sin s}{9} \left(\frac{|x(s)|}{1+|x(s)|} + \frac{|y(s)|}{1+|y(s)|} \right) - \frac{\sin s}{9} \left(\frac{|u(s)|}{1+|u(s)|} + \frac{|v(s)|}{1+|v(s)|} \right) \right|$$
$$\leq \frac{1}{9} (|x-u| + |y-v|).$$

Therefore, the function h satisfies (22) with $\alpha = \frac{1}{9}$.

For all $t, s \in I$, there exists a continuous $A : I \times \mathbb{R} \to \mathbb{R}$ such that

$$\int_0^1 A(t,s) ds = \int_0^1 \frac{3^{-s} e^{-s}}{t+3} ds = -\frac{1}{3} \frac{e^{-1} - 3}{(\ln 3 + 1)(t+3)} = \left(1 - \frac{1}{3e}\right) \frac{1}{(\ln 3 + 1)(t+3)}$$

$$\leq 1 - \frac{1}{3e} \leq \frac{9}{10} < 1.$$

We put $x_0(t) = \frac{5t^2}{7(1+t^4)}$, *we obtain*

$$x_0(t) = \frac{5t^2}{7(1+t^4)} \leq \frac{t^2}{1+t^4} \leq \frac{t^2}{1+t^4} + \int_0^1 \frac{\sin s}{9}\left(\frac{|x(s)|}{1+|x(s)|} + \frac{|y(s)|}{1+|y(s)|}\right) ds$$
$$= q(t) + \int_0^T A(t,s)\, h(s, x_0(s), y_0(s))\, ds.$$

Similarly, we have $y_0(t) \leq q(t) + \int_0^T A(t,s)\, h(s, y_0(s), x_0(s))\, ds$. *This shows that (24) holds.*

Hence the integral equation (21) has a unique solution in X.

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