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On the Lowest-Winning-Bid and the Highest-Losing-Bid Auctions

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Abstract

Theoretical models of multi-unit, uniform-price auctions assume that the price is given by the highest losing bid. In practice, however, the price is usually given by the lowest winning bid. We derive the equilibrium bidding function of the lowest-winning-bid auction when there are k objects for sale and n bidders with unit demand, and prove that it converges to the bidding function of the highest-losing-bid auction if and only if the number of losers $n-k$ gets large. When the number of losers grows large, the bidding functions converge at a linear rate and the prices in the two auctions converge in probability to the expected value of an object to the marginal winner.

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1 Introduction

Uniform-price auctions have been extensively used for the sale of homogeneous goods in several countries (e.g., in the sale of Treasury bills and electrical power). In these auctions, the price is usually given by the lowest winning bid. Theoretical models of multi-unit, uniform-price auctions, on the other hand, assume that the price is given by the highest losing bid (e.g.,

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Milgrom, 1981, Weber, 1983, Pesendorfer and Swinkels, 1997, 2000, Milgrom and Weber, 2000, Kremer, 2002, Jackson and Kremer, 2004, 2006, and Mezzetti, Pekeč and Tsetlin, 2007). When bidders have unit-demand, highest-losing-bid, multi-unit auctions behave very much like the second-price auction with a single item for sale. In particular, in a symmetric equilibrium, each bidder bids his expected value for an object conditional on being tied with the price setter. This simplicity is the main reason for their use by theorists. A natural question then is: do lowest-winning-bid auctions behave very differently from highest-losing-bid auctions? Do the two auction formats yield similar behavior and prices as the number of bidders increases? An affirmative answer to both questions would provide justification for the theorists' focus on the analytically simpler highest-losing-bid auctions.

First, we derive the equilibrium bidding function of the lowest-winning-bid auction in the general affiliated value model with unit demand introduced by Milgrom and Weber (1982); as far as we know, we are the first to study such auctions. Then we show that the bidding functions of the lowest-winning-bid and the highest-losing-bid auction converge as the number of losing bidders grows large. More precisely, letting n be the number of bidders and k the number of objects sold, we show that the two bidding functions converge if and only if $n - k$ goes to infinity. As $n - k$ grows, the bidding functions converge at a linear rate. We also show that the prices in the two auctions converge in probability when $n - k$ goes to infinity. They converge to the expected value of an object to the marginal winner; hence, the two auctions become perfectly competitive markets as $n - k$ grows.

It is worth to point out two other properties of the lowest-winning bid auction. As it is well known, in the general affiliated model, the second-price auction and its generalization, the highest-losing-bid auction, have a continuum of undominated asymmetric equilibria (Milgrom, 1981, Bikhchandani and Riley, 1991). The first-price auction and, we conjecture, the lowest-winning-bid auction, do not suffer from this problem (McAdams, 2006). Second, it is easy to extend the arguments first introduced by Robinson (1985) for the first and second-price auction, and show that collusive agreements are easier to sustain in the highest-losing-bid than in the lowest-winning-bid auction. In a highest-losing-bid auction, cartel members have no incentive to deviate from an agreement in which only one of the highest value members submits a meaningful bid. On the contrary, cartel members will want to deviate from such an agreement in a lowest-winning bid auction. The robustness to collusion and equilibrium multiplicity may help to explain the prevalence in practice of the lowest-winning-bid auction. That its equilibrium converges to the symmetric equilibrium of the highest-losing-bid auction makes us feel confident that the latter is a good approximation of the uniform auctions used in practice, at least when the number $n-k$ of losing bidders is large.

The paper is organized as follows. The next section introduces the model and derives the bidding function of the k-th price (i.e., the lowest-winning bid) auction for k objects. Section 3 studies the convergence properties of the k-th and $(k+1)$-st price auctions. Section 4 concludes. An appendix contains the proofs omitted from the main text.

2 The Model and Bidding Functions

We consider a sequence of auctions $\{A_r\}_{r=1}^{\infty}$, where the r-th auction has n_r bidders and k_r objects, with $1 \leq k_r < n_r < n_{r+1}$. Each bidder is risk neutral and only demands one object. Bidder i, $i = 1, 2, \ldots, n_r$, observes the realization x_i of a signal X_i. Denote with $s = (x_1, ..., x_{n_r})$ the vector of signal realizations. Let $s \vee s'$ be the component-wise maximum and $s \wedge s'$ be the component-wise minimum of s and s'. Signals are real random variables drawn from a distribution with a joint pdf $f_r(s)$, which satisfies the affiliation property (Milgrom and Weber, 1982):

$$f_r(s \vee s')f_r(s \wedge s') \geq f_r(s)f_r(s') \qquad \text{for all } s \neq s'. \tag{1}$$

The support of f_r is $[\underline{x}, \overline{x}]^{n_r}$, with $-\infty < \underline{x} < \overline{x} < +\infty$.

We make the standard assumption that the random variables $X_1, X_2, \ldots$ are symmetric. More precisely, the infinite sequence $\mathbf{X} = (X_1, X_2, ...)$ is exchangeable; that is, for all finite n the joint distribution of $(X_{\pi_1}, \cdots, X_{\pi_n})$ is the same as that of $(X_1, \cdots, X_n)$ for all permutations π. By de Finetti's exchangeability theorem (e.g., see Kingman, 1978, for a simple exposition) there exists a real random variable ζ with distribution $H(\zeta)$ and a conditional distribution function $G(\cdot|\zeta)$ such that, for all n, the joint distribution of the random variables $X_1, X_2, \cdots, X_n$ is:

$$P(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n) = \int_{-\infty}^{+\infty} G(x_1|\zeta)G(x_2|\zeta)\cdots G(x_n|\zeta)dH(\zeta). \tag{2}$$

We will make the following uniform boundedness assumption. There exists $\eta_0 > 0$ such that, for all x, x', and ζ:[2]

$$\eta_0 < \frac{g(x|\zeta)}{g(x'|\zeta)} < \frac{1}{\eta_0}, \tag{3}$$

where $g(\cdot|\zeta)$ is the density of $G(\cdot|\zeta)$.

The value $V_r^i = u_r(X_i, \{X_j\}_{j \neq i})$ of an object to bidder i is a function of all signals.[3] The function $u_r(\cdot)$ is non-negative, differentiable, strictly increasing in X_i, increasing and symmetric in the other bidders' signals X_j, $j \neq i$.

In studying the symmetric equilibrium bidding function in a given auction, it is useful to take the point of view of one of the bidders, say bidder 1 with signal $X_1 = x$, and to consider the order statistics associated with the signals of all other bidders. We denote with Y_r^j the j-th highest signal of bidders $2, 3, ..., n_r$ (i.e., all bidders except bidder 1). Define

$$v_r^j(x, y) = E\left[V_r^1 | X_1 = x, Y_r^j = y\right]. \tag{4}$$

Affiliation implies that $v_r^j(x, y)$ is increasing in both x and y, and hence differentiable almost everywhere (Milgrom and Weber, 1982, Theorem 5). We also assume that there exist real

[2]This assumption implies that a bidder's signal only conveys a bounded amount of information about the other bidders' signals. Pesendorfer and Swinkels (1997) make a similar assumption in the context of pure common values.

[3]Milgrom and Weber (1982) also allowed the function u_r to depend on other signals which are not observed by the bidders. Since we never use the unobserved signals, we have omitted them. This is with no loss of generality. For example, the case of an unobserved common value corresponds in our model to a function u_r which is symmetric in all signals and equals the expected value of the object conditional on the signals observed by all the bidders.

numbers $a > 0$ and $b < \infty$ such that, for all r:

$$a < \frac{dv_r^{k_r}(x,x)}{dx} < b. \tag{5}$$

In a $(k_r + 1)$-st price (or highest-losing-bid) auction, the k_r bidders with the highest bids win at a price equal to the $(k_r + 1)$-st highest bid. Milgrom (1981) showed that the bidding function in such an auction is $v_r^{k_r}(x,x)$. Bidder 1 bids his expected value of an object conditional on his own signal, $X_1 = x$, and on his signal being just high enough to guarantee winning (i.e., being equal to the k_r-th highest signal of all other bidders).

In a k_r-th price (or lowest-winning-bid) auction, the k_r bidders with the highest bids win an object at a price equal to the k_r-th highest bid. In studying equilibrium of such an auction, it is useful to consider another bidder besides bidder 1, say bidder 2 with signal $X_2 = y$. Denote the signals of bidders $3, \ldots, n_r$, ordered descendingly, by $Z_r^1, \ldots, Z_r^{n_r-2}$. Let $f_r^{X_2}(y|X_1 = x, Z_r^{k_r-1} > y > Z_r^{k_r})$ be the density of X_2 conditional on $X_1 = x$ and $Z_r^{k_r-1} > y > Z_r^{k_r}$; let $F_r^{X_2}(y|X_1 = x, Z_r^{k_r-1} > y > Z_r^{k_r})$ be the corresponding cumulative distribution function.[4] Define the functions

$$Q_r(y,x) = (n_r - k_r)\frac{f_r^{X_2}(y|X_1 = x, Z_r^{k_r-1} > y > Z_r^{k_r})}{F_r^{X_2}(y|X_1 = x, Z_r^{k_r-1} > y > Z_r^{k_r})}, \tag{6}$$

$$L_r(z) = e^{-\int_z^x Q_r(t,t)dt}. \tag{7}$$

The following lemma is proved in the appendix.

Lemma 1 *The increasing symmetric equilibrium of the lowest-winning-bid auction for k_r objects with n_r bidders is:*

$$b_r(x) = v_r^{k_r}(x,x) - \int_{\underline{x}}^{x} L_r(z)dv_r^{k_r}(z,z), \tag{8}$$

where $v_r^{k_r}(\cdot)$ is defined by (4) and $L_r(z)$ is defined by (7).

[4] If $k_r = 1$, we let $Z_r^{k_r-1} = \overline{x}$; in such a case the k_r-th price auction is the first-price auction.

In the lowest-winning-bid auction, a bidder bids his expected value of the object, conditional on being tied with the marginal (or highest bidding) loser, minus a shading factor. This is similar to the equilibrium bidding function of a first-price auction when there is a single object for sale.

3 Convergence

We now study the convergence of the bidding function $b_r(x)$ of the lowest-winning-bid auction to the bidding function $v_r^{k_r}(x,x)$ of the highest-losing-bid auction, as r grows large.

Theorem 1 *The bidding function of the lowest-winning-bid auction, $b_r(x)$ given by (8), converges to the bidding function of the highest-losing-bid auction, $v_r^{k_r}(x,x)$ given by (4), if and only if the number of losing bidders $n_r - k_r$ goes to infinity. When $n_r - k_r$ goes to infinity, $b_r(x)$ converges to $v_r^{k_r}(x,x)$ at a linear rate.*

PROOF. By (8) and (6):

$$\begin{aligned} b_r(x) - v_r^{k_r}(x,x) &= -\int_{\underline{x}}^{x} L_r(z) dv_r^{k_r}(z,z) \\ &= -\int_{\underline{x}}^{x} e^{-\int_z^x Q_r(t,t)dt} dv_r^{k_r}(z,z) \\ &= -\int_{\underline{x}}^{x} e^{-(n_r-k_r)\int_z^x \frac{f_r^{X_2}(t|X_1=t, Z_r^{k_r-1}>t>Z_r^{k_r})}{F_r^{X_2}(t|X_1=t, Z_r^{k_r-1}>t>Z_r^{k_r})}dt} dv_r^{k_r}(z,z). \end{aligned}$$

By the mean value theorem, there exists t' such that:

$$\frac{f_r^{X_2}(t|X_1=t, Z_r^{k_r-1}>t>Z_r^{k_r})}{F_r^{X_2}(t|X_1=t, Z_r^{k_r-1}>t>Z_r^{k_r})} = \frac{f_r^{X_2}(t|X_1=t, Z_r^{k_r-1}>t>Z_r^{k_r})}{(t-\underline{x}) f_r^{X_2}(t'|X_1=t, Z_r^{k_r-1}>t>Z_r^{k_r})}.$$

Since $f_r(x_i, x_{-i}) = \int_{-\infty}^{+\infty} g(x_1|\zeta) g(x_2|\zeta) \cdots g(x_{n_r}|\zeta) dH(\zeta)$, it is simple to show that the boundedness assumption (3) implies that, for all r, x_i, x_i', and $x_{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n_r})$:

$$\eta_0 < \frac{f_r(x_i, x_{-i})}{f_r(x_i', x_{-i})} < \frac{1}{\eta_0}. \tag{9}$$

It follows from (9) that

$$\eta_0 < \frac{f_r^{X_2}(t|X_1 = t, Z_r^{k_r-1} > t > Z_r^{k_r})}{f_r^{X_2}(t'|X_1 = t, Z_r^{k_r-1} > t > Z_r^{k_r})} < \frac{1}{\eta_0},$$

and hence

$$-\int_{\underline{x}}^{x} e^{-(n_r-k_r)\eta_0 \int_z^x \frac{1}{(t-\underline{x})}dt} dv_r^{k_r}(z,z) \leq b_r(x) - v_r^{k_r}(x,x) \leq -\int_{\underline{x}}^{x} e^{-(n_r-k_r)\frac{1}{\eta_0} \int_z^x \frac{1}{(t-\underline{x})}dt} dv_r^{k_r}(z,z), \tag{10}$$

with the inequalities being strict for $x \neq \underline{x}$.

Observe that $-\int_z^x \frac{1}{(t-\underline{x})}dt = \ln \frac{z-\underline{x}}{x-\underline{x}}$; it then follows from (10) that

$$-\int_{\underline{x}}^{x} \left(\frac{z-\underline{x}}{x-\underline{x}}\right)^{(n_r-k_r)\eta_0} \frac{dv_r^{k_r}(z,z)}{dz} dz \leq b_r(x) - v_r^{k_r}(x,x) \leq -\int_{\underline{x}}^{x} \left(\frac{z-\underline{x}}{x-\underline{x}}\right)^{(n_r-k_r)\frac{1}{\eta_0}} \frac{dv_r^{k_r}(z,z)}{dz} dz. \tag{11}$$

Since $\frac{dv_r^{k_r}(z,z)}{dz}$ is uniformly bounded by assumption (5), the left and right hand side of (11) converge linearly to zero if and only if $n_r - k_r$ goes to infinity. This shows that $b_r(x)$ converges to $v_r^{k_r}(x,x)$ if and only if $n_r - k_r$ goes to infinity, and that convergence is at a linear rate. ∎

The intuition behind Theorem 1 is the following. In a k_r-th price auction for k_r objects, a bidder bids the expected value of the object, conditional on his bid being tied with the bid of the marginal loser, minus a shading factor. As the number of losers in the auction increases, the shading factor decreases linearly, reflecting increased competition for the last object. In the limit, the bid in the k_r-th price auction coincides with the bid in the (k_r+1)-st price auction: the expected value of the object conditional on being tied with the marginal loser. In a (k_r+1)-st price auction, the marginal loser is the price setter.

We now show that the prices in the two auctions converge in probability, and they converge to the expected value of an object to the marginal winner. This is because the k_r-th and the (k_r+1)-st order statistic converge in probability as $n_r - k_r$ grows large.

Let X_r^j be the j-th highest signal among all bidders in auction A_r. Consider the marginal winner in auction r, the bidder with the k_r-th highest signal; his expected value for an object conditional on his signal being x is $E\left[v_r^{k_r}(x, X_r^{k_r+1})|X_r^{k_r}=x\right]$.

Theorem 2 *The prices of the lowest-winning-bid auction and the highest-losing-bid auction converge in probability when $n_r - k_r$ goes to infinity; they converge to the expected value of an object to the marginal winner.*

Proof. It suffices to show that the (k_r+1)-st order statistic (i.e., X^{k_r+1}) converges to the k_r-th order statistic in probability when $n_r - k_r$ grows large. To see this note first that the price in a k_r-th price auction is $b_r(X^{k_r})$, while the price in a (k_r+1)-st price auction is $v_r^{k_r}(X_r^{k_r+1}, X_r^{k_r+1})$. By Theorem 1, the prices converge when the order statistics converge. Second, the expected value of an object to the marginal winner with signal x, $E\left[v_r^{k_r}(x, X_r^{k_r+1})|X_r^{k_r}=x\right]$, converges to $v_r^{k_r}(x,x)$ if the order statistics converge.

Fix an arbitrary $\varepsilon > 0$. By (2), the probability that the difference between the k_r-th and the (k_r+1)-st order statistic is more than ε, conditional on the k_r-th order statistic being equal to x, is given by

$$\begin{aligned} P_r(X_r^{k_r} - X_r^{k_r+1} &> \varepsilon|X_r^{k_r}=x) \qquad (12)\\ &= \frac{\int_{-\infty}^{+\infty}\left(\frac{G(x-\varepsilon|\zeta)}{G(x|\zeta)}\right)^{n_r-k_r}(1-G(x|\zeta))^{k_r-1}g(x|\zeta)G(x|\zeta)^{n_r-k_r}dH(\zeta)}{\int_{-\infty}^{+\infty}(1-G(x|\zeta))^{k_r-1}g(x|\zeta)G(x|\zeta)^{n_r-k_r}dH(\zeta)}. \end{aligned}$$

The boundedness assumption (3) implies that there is a real number c such that $0 < c < g(x|\zeta)$ for all x and ζ. This implies that, for all x and ζ:

$$\frac{G(x-\varepsilon|\zeta)}{G(x|\zeta)} \leq \max\left\{\frac{G(x|\zeta)-\varepsilon c}{G(x|\zeta)}, 0\right\} \leq \max\{1-\varepsilon c, 0\}.$$

It follows that

$$P_r(X_r^{k_r} - X_r^{k_r+1} > \varepsilon|X_r^{k_r}=x) \leq (\max\{1-\varepsilon c, 0\})^{n_r-k_r},$$

and hence $P_r(X_r^{k_r} - X_r^{k_r+1} > \varepsilon)$ goes to zero as $n_r - k_r$ goes to infinity. This concludes the proof.[5] ∎

Define an auction as being *competitive* if the price converges to the value of an object to the marginal buyer as the number of losers grows. Theorem 2 shows that the k_r-th and (k_r+1)-st uniform-price auction are competitive.[6]

4 Conclusions

This paper provides a link between the highest-losing-bid auctions, which have been extensively studied by theorists, and the lowest-winning-bid auctions that are used in practice. We have shown that the symmetric equilibrium bidding function of the lowest-winning-bid auction converges to the bidding function of the highest-losing-bid auction if and only if the number of losing bidders gets large. When the number of losers grows large, the two bidding functions converge at a linear rate and prices in the two auctions converge in probability to the willingness to pay of the marginal bidder (his expected value for an object).

In a pure common value model with signals that are independent conditional on the common value, Pesendorfer and Swinkels (1997) showed that the $(k+1)$-st price auction aggregates information (i.e., the price converges to the common value in probability) if and only if the number of objects k and the number of losers $n-k$ go to infinity. The results in this paper, specialized to such a pure common value model, imply that the k-th price auction aggregates information under the same conditions. In particular, if $n-k$ goes to infinity, but

[5] Note that the k_r-th and the (k_r+1)-st order statistic converge even if $n_r - k_r$ does not go to infinity, provided that n_r converges to infinity. If n_r goes to infinity, but $n_r - k_r$ does not, then k_r must converge to infinity. We can then write an expression for $P_r(X_r^{k_r} - X_r^{k_r+1} > \varepsilon | X_r^{k_r+1} = x)$ similar to (12):

$$P_r(X_r^{k_r} - X_r^{k_r+1} > \varepsilon | X_r^{k_r+1} = x) = \frac{\int_{-\infty}^{+\infty} \left(\frac{1-G(x+\varepsilon|\zeta)}{1-G(x|\zeta)}\right)^{k_r} (1-G(x|\zeta))^{k_r} g(x|\zeta) G(x|\zeta)^{n_r-k_r-1} dH(\zeta)}{\int_{-\infty}^{+\infty} (1-G(x|\zeta))^{k_r} g(x|\zeta) G(x|\zeta)^{n_r-k_r-1} dH(\zeta)},$$

which converges to zero uniformly as k_r goes to infinity.

[6] Our definition of a competitive auction is different from the definition in Kremer (2002). In a model with pure common values, he calls an auction competitive if the expected price converges to the expected value of the object. Our definition conforms more closely with the standard definition of a competitive market by economists and applies beyond the common-value setting.

k stays finite, then the expected value of an object to the marginal winner does not converge, in probability, to the object's common value.

Appendix

In this appendix we prove Lemma 1, which gives us the equilibrium bidding function of the k_r-th price auction. We begin with two auxiliary lemmas.

Lemma 2 *$Q_r(y,x)$, defined by (6), is increasing in x.*

PROOF. The proof is essentially the same as the proof of Lemma 1 in Milgrom and Weber (1982). By affiliation, for any $y' < y$ and $x' < x$,

$$\frac{f_r^{X_2}(y|X_1 = x', Z_r^{k_r-1} > y > Z_r^{k_r})}{f_r^{X_2}(y'|X_1 = x', Z_r^{k_r-1} > y > Z_r^{k_r})} \leq \frac{f_r^{X_2}(y|X_1 = x, Z_r^{k_r-1} > y > Z_r^{k_r})}{f_r^{X_2}(y'|X_1 = x, Z_r^{k_r-1} > y > Z_r^{k_r})}.$$

Cross multiplying and integrating with respect to y' over the range $\underline{x} \leq y' < y$ yields the result. ▮

Let $f_r^j(y_j|X_1 = x)$ be the marginal density of Y_r^j conditional on $X_1 = x$.

Lemma 3 *$Q_r(y,x)$, defined by (6), can equivalently be defined as follows:*

$$Q_r(y,x) = \frac{f_r^{k_r}(y|x)}{P_r(Y_r^{k_r} < y < Y_r^{k_r-1}|x)},$$

where $P_r(Y_r^{k_r} < y < Y_r^{k_r-1}|x)$ is the probability that, conditional on $X_1 = x$, $Y_r^{k_r}$ is below y and $Y_r^{k_r-1}$ is above y.

PROOF. Because of the symmetry of the $(n_r - 1)$ signals $X_2, \dots, X_{n_r}$, it is

$$f_r^{k_r}(y|X_1 = x)$$
$$= (n_r - 1)\binom{n_r - 2}{k_r - 1} \int_{\underline{x}}^{y} \dots \int_{\underline{x}}^{y} \left(\int_{y}^{\overline{x}} \dots \int_{y}^{\overline{x}} f_r(y, z_1, \cdots, z_{n_r-2}|X_1 = x) dz_1 \cdots dz_{k_r-1} \right) dz_{k_r} \cdots dz_{n_r-2},$$

and

$$
\begin{aligned}
&P_r(Y_r^{k_r} < y < Y_r^{k_r-1}|X_1 = x) \\
&\quad = \binom{n_r-1}{k_r-1} \int_{\underline{x}}^{y} .. \int_{\underline{x}}^{y} \left(\int_{y}^{\overline{x}} .. \int_{y}^{\overline{x}} f_r(y_1, .., y_{n_r-1}|X_1 = x) dy_1..dy_{k_r-1} \right) dy_{k_r}..dy_{n_r-1} \\
&\quad = \binom{n_r-1}{k_r-1} \int_{\underline{x}}^{y} \int_{\underline{x}}^{y} .. \int_{\underline{x}}^{y} \left(\int_{y}^{\overline{x}} .. \int_{y}^{\overline{x}} f_r(x_2, z_1, .., z_{n_r-2}|X_1 = x) dz_1..dz_{k_r-1} \right) dz_{k_r}..dz_{n_r-2} dx_2.
\end{aligned}
$$

As a result, it is

$$
\begin{aligned}
&\frac{f_r^{k_r}(y|X_1 = x)}{P_r(Y_r^{k_r} < y < Y_r^{k_r-1}|X_1 = x)} \\
&= \frac{(n_r-1)\binom{n_r-2}{k_r-1} \int_{\underline{x}}^{y} ... \int_{\underline{x}}^{y} \left(\int_{y}^{\overline{x}} ... \int_{y}^{\overline{x}} f_r(y, z_1, \cdots, z_{n_r-2}|X_1 = x) dz_1 \cdots dz_{k_r-1} \right) dz_{k_r} \cdots dz_{n_r-2}}{\binom{n_r-1}{k_r-1} \int_{\underline{x}}^{y} \int_{\underline{x}}^{y} .. \int_{\underline{x}}^{y} \left(\int_{y}^{\overline{x}} .. \int_{y}^{\overline{x}} f_r(x_2, z_1, .., z_{n_r-2}|X_1 = x) dz_1..dz_{k_r-1} \right) dz_{k_r}..dz_{n_r-2} dx_2} \\
&= (n_r - k_r) \frac{f_r^{X_2}(y|X_1 = x, Z_r^{k_r-1} > y > Z_r^{k_r})}{F_r^{X_2}(y|X_1 = x, Z_r^{k_r-1} > y > Z_r^{k_r})} \\
&= Q_r(y, x),
\end{aligned}
$$

where the last equality follows from (6). ∎

Consider bidder 1 observing signal x. Bidding according to the function $b^*(\cdot)$ corresponds to a symmetric Nash equilibrium if and only if the expected payoff of the bidder who observes signal x is maximized at $b = b^*(x)$, when all other bidders follow $b^*(\cdot)$.

Define

$$
v_r^{k_r-1,k_r}(x, y_{k_r-1}, y_{k_r}) = E[V_r^1|X_1 = x, Y_r^{k_r-1} = y_{k_r-1}, Y_r^{k_r} = y_{k_r}].
$$

The expected payoff $\Pi(b;x)$ of bidder 1, who observes signal x and bids b, while all other bidders follow $b^*(\cdot)$, is:[7]

$$\begin{aligned}
\Pi(b;x) &= E[\big(V_r^1 - b^*(Y_r^{k_r-1})\big)\, I_{b^*(Y_r^{k_r-1})<b} | X_1 = x] + E[\big(V_r^1 - b\big)\, I_{b^*(Y_r^{k_r})<b<b^*(Y_r^{k_r-1})} | X_1 = x] \\
&= E[E[\big(V_r^1 - b^*(Y_r^{k_r-1})\big)\, I_{b^*(Y_r^{k_r-1})<b} | X_1, Y_r^{k_r-1}] | X_1 = x] \\
&\quad + E[E[\big(V_r^1 - b\big)\, I_{b^*(Y_r^{k_r})<b<b^*(Y_r^{k_r-1})} | X_1, Y_r^{k_r-1}, Y_r^{k_r}] | X_1 = x] \\
&= E[\big(v_r^{k_r-1}(X_1, Y_r^{k_r-1}) - b^*(Y_r^{k_r-1})\big)\, I_{b^*(Y_r^{k_r-1})<b} | X_1 = x] \\
&\quad + E[\big(v_r^{k_r-1,k_r}(X_1, Y_r^{k_r-1}, Y_r^{k_r}) - b\big)\, I_{b^*(Y_r^{k_r})<b<b^*(Y_r^{k_r-1})} | X_1 = x] \\
&= \int_{\underline{x}}^{b^{*^{-1}}(b)} \big(v_r^{k_r-1}(x, y_{k_r-1}) - b^*(y_{k_r-1})\big)\, f_r^{k_r-1}(y_{k_r-1} | X_1 = x) dy_{k_r-1} \\
&\quad + \int_{\underline{x}}^{b^{*^{-1}}(b)} \int_{b^{*^{-1}}(b)}^{\overline{x}} \big(v_r^{k_r-1,k_r}(x, y_{k_r-1}, y_{k_r}) - b\big)\, f_r^{k_r-1,k_r}(y_{k_r-1}, y_{k_r} | X_1 = x) dy_{k_r-1} dy_{k_r},
\end{aligned}$$

where $f_r^{k_r-1,k_r}(y_{k_r-1}, y_{k_r} | X_1 = x)$ is the joint density of Y^{k_r-1} and Y^{k_r} conditional on $X_1 = x$.

Let

$$\begin{aligned}
\Pi_1(b;x) &= \int_{\underline{x}}^{b^{*^{-1}}(b)} \big(v_r^{k_r-1}(x, y_{k_r-1}) - b^*(y_{k_r-1})\big)\, f_r^{k_r-1}(y_{k_r-1} | X_1 = x) dy_{k_r-1}, \\
\Pi_2(b;x) &= \int_{\underline{x}}^{b^{*^{-1}}(b)} \int_{b^{*^{-1}}(b)}^{\overline{x}} \big(v_r^{k_r-1,k_r}(x, y_{k_r-1}, y_{k_r}) - b\big)\, f_r^{k_r-1,k_r}(y_{k_r-1}, y_{k_r} | X_1 = x) dy_{k_r-1} dy_{k_r},
\end{aligned}$$

so that

$$\Pi(b;x) = \Pi_1(b;x) + \Pi_2(b;x).$$

The derivative of $\Pi_1(b;x)$ with respect to b is

$$\Pi_{1b}(b;x) = \frac{1}{b^{*\prime}(b^{*^{-1}}(b))} \left(v_r^{k_r-1}(x, b^{*^{-1}}(b)) - b\right) f_r^{k_r-1}(b^{*^{-1}}(b) | X_1 = x),$$

[7] $I_{b^*(Y^{k_r-1})<b}$ is an indicator function: it equals one if $b^*(Y^{k_r-1}) < b$ and zero otherwise.

and the derivative of $\Pi_2(b;x)$ with respect to b is

$$\begin{aligned}
\Pi_{2b}(b;x) &= \frac{1}{b^{*\prime}(b^{*^{-1}}(b))}\int_{b^{*^{-1}}(b)}^{\overline{x}}\left(v_r^{k_r-1,k_r}(x,y_{k_r-1},b^{*^{-1}}(b))-b\right)f_r^{k_r-1,k_r}(y_{k_r-1},b^{*^{-1}}(b)|X_1=x)dy_{k_r-1} \\
&\quad -\frac{1}{b^{*\prime}(b^{*^{-1}}(b))}\int_{\underline{x}}^{b^{*^{-1}}(b)}\left(v_r^{k_r-1,k_r}(x,b^{*^{-1}}(b),y_{k_r})-b\right)f_r^{k_r-1,k_r}(b^{*^{-1}}(b),y_{k_r}|X_1=x)dy_{k_r} \\
&\quad -\int_{\underline{x}}^{b^{*^{-1}}(b)}\int_{b^{*^{-1}}(b)}^{\overline{x}} f_r^{k_r-1,k_r}(y_{k_r-1},y_{k_r}|X_1=x)dy_{k_r-1}dy_{k_r} \\
&= \frac{1}{b^{*\prime}(b^{*^{-1}}(b))}\left(v_r^{k_r}(x,b^{*^{-1}}(b))-b\right)f_r^{k_r}(b^{*^{-1}}(b)|X_1=x) \\
&\quad -\frac{1}{b^{*\prime}(b^{*^{-1}}(b))}\left(v_r^{k_r-1}(x,b^{*^{-1}}(b))-b\right)f_r^{k_r-1}(b^{*^{-1}}(b)|X_1=x) \\
&\quad -\int_{\underline{x}}^{b^{*^{-1}}(b)}\int_{b^{*^{-1}}(b)}^{\overline{x}} f_r^{k_r-1,k_r}(y_{k_r-1},y_{k_r}|X_1=x)dy_{k_r-1}dy_{k_r}.
\end{aligned}$$

Therefore, the derivative of $\Pi_b(b;x)$ with respect to b is

$$\begin{aligned}
\Pi_b(b;x) &= \frac{\left(v_r^{k_r}(x,b^{*^{-1}}(b))-b\right)f_r^{k_r}(b^{*^{-1}}(b)|X_1=x)}{b^{*\prime}(b^{*^{-1}}(b))} \\
&\quad -\int_{\underline{x}}^{b^{*^{-1}}(b)}\int_{b^{*^{-1}}(b)}^{\overline{x}} f_r^{k_r-1,k_r}(y_{k_r-1},y_{k_r}|X_1=x)dy_{k_r-1}dy_{k_r}.
\end{aligned} \tag{13}$$

Note that the expression $\int_{\underline{x}}^{y}\int_{y}^{\overline{x}} f_r^{k_r-1,k_r}(y_{k_r-1},y_{k_r}|X_1=x)dy_{k_r-1}dy_{k_r}$ is equal to $P_r(Y_r^{k_r}<y<Y_r^{k_r-1}|X_1=x)$, the probability that $Y_r^{k_r}$ is below y and $Y_r^{k_r-1}$ is above y, conditional on $X_1=x$. Therefore, by setting $\Pi_b(b,x)|_{b=b^*(x)}=0$, the differential equation for the candidate for an increasing symmetric equilibrium is

$$\frac{1}{b^{*\prime}(x)}\left(v_r^{k_r}(x,x)-b^*(x)\right)f_r^{k_r}(x|X_1=x)-P_r(Y_r^{k_r}<x<Y_r^{k_r-1}|X_1=x)=0,$$

or

$$b^{*\prime}(x)=\left(v_r^{k_r}(x,x)-b^*(x)\right)\frac{f_r^{k_r}(x|X_1=x)}{P_r(Y_r^{k_r}<x<Y_r^{k_r-1}|x)}. \tag{14}$$

By Lemma 3, we can then write (14) as

$$b^{*\prime}(x) = \left[v_r^{k_r}(x,x) - b^*(x)\right] Q_r(x,x). \tag{15}$$

Using the integrating factor $e^{-\int_x^{\overline{x}} Q_r(t,t)dt}$, and the boundary condition $b^*(\underline{x}) = v_r^{k_r}(\underline{x},\underline{x})$, the solution of this differential equation is:

$$b^*(x) = v_r^{k_r}(\underline{x},\underline{x})e^{-\int_{\underline{x}}^x Q_r(t,t)dt} + \int_{\underline{x}}^x v_r^{k_r}(z,z)Q_r(z,z)e^{-\int_z^x Q_r(t,t)dt}dz.$$

Integrating by parts and using (7) yield (8).

It only remains to show that deviations from (8) are not profitable. From (13), by setting $b^{*^{-1}}(b) = y$, we get

$$\begin{aligned}
\Pi_b(b;x) &= \frac{1}{b^{*\prime}(y)}\left(v_r^{k_r}(x,y) - b^*(y)\right) f_r^{k_r}(y|X_1 = x) - P_r(Y_r^{k_r} < y < Y_r^{k_r-1}|X_1 = x) \\
&= \frac{f_r^{k_r}(y|X_1 = x)}{b^{*\prime}(y)}\left(v_r^{k_r}(x,y) - b^*(y) - \frac{b^{*\prime}(y)}{Q_r(y,x)}\right).
\end{aligned}$$

By Lemma 2, $Q_r(y,x)$ is increasing in x, so $\Pi_b(b;x)$ is positive for $x > y$ and negative for $x < y$, which implies that setting $b = b^*(x) = b_r(x)$ maximizes the expected payoff of bidder 1.

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