Categorical Aspects of Graphs

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At their most basic, graphs consist of a set of vertices and a set of edges. Graphs can be directed (i.e. the edges point from one node to another), or undirected, with no prescribed flow. This section concerns categorical aspects of graphs. We introduce the category of directed graphs and then explore various subcategories, considering graphs with different combinations of properties. Namely, we discuss graphs which are directed, undirected, reflexive, simple, and combinations thereof.

1. The Category of Directed Graphs

Definition 1.1 ($\mathbb{G}$)**.** Define a category $\mathbb{G}$ to be the category

$$V \overset{s}{\underset{t}{\rightrightarrows}} E$$

with two objects V and E and two non-identity morphisms.

Observe that a presheaf F on this category consists of two sets, which we will call G_V and G_E, together with two functions ${s_G}^*, {t_G}^* : G_E \to G_V$, noting that contravariance necessitates reversing the arrows. This structure is precisely a directed graph, where G_V is the set of vertices, G_E the set of edges and s^* and t^* are the source and target functions, respectively. We may thus view a directed graph as a presheaf $F : \mathbb{G}^{\mathsf{op}} \to \mathsf{Set}$, or equivalently an object in the presheaf category $\mathsf{Set}^{\mathbb{G}^{\mathsf{op}}}$.

Definition 1.2 (Category of Directed Graphs; $\mathsf{DirGrph}$)**.** Define the category of directed graphs $\mathsf{DirGrph}$ to be the category $\mathsf{Set}^{\mathbb{G}^{\mathsf{op}}}$. We have already established objects in this category, but it remains to determine the morphisms. Evidently, morphisms of graphs are natural transformations, since the graphs themselves are functors. This means that a morphism $f : G \to H$ of directed graphs

$$G = (G_E, G_V, {s_G}^*, {t_G}^*)$$

and

$$H = (H_E, H_V, {s_H}^*, {t_H}^*)$$

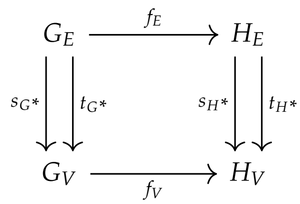
consists of functions $f_V : G_V \to H_V$ and $f_E : G_E \to H_E$ such that

$$f_V \circ {s_G}^* = {s_H}^* \circ f_E$$

and

$$f_V \circ {t_G}^* = {t_H}^* \circ f_E$$

as illustrated by the commutative square below:



2. The Category of Simple Directed Graphs

Having developed the category of directed graphs, we now turn our attention to some of its subcategories, as well as some nice properties that these subcategories have. We begin with the category of simple directed graphs.

Definition 2.1 (Simple Directed Graph)**.** A simple directed graph is a directed graph in which there is at most one edge between any pair of vertices.

Given some graph $G \in \mathsf{DirGrph}$, there is an induced morphism

$$\langle s_G{}^*, t_{G^*} \rangle : G_E \to G_V \times G_V$$

taking an edge $e \in G_E$ to the pair $(s_G{}^*(e), t_G{}^*(e))$. It is easy to see that G is simple precisely when this map is a monomorphism, as each edge has a unique value under $\langle s_G{}^*, t_G{}^* \rangle$ in a simple directed graph.

As simple directed graphs form a subcategory of $\mathsf{DirGrph}$, there is an evident inclusion functor $U : \mathsf{SimpDirGrph} \hookrightarrow \mathsf{DirGrph}$. It is also possible to create a simple directed graph from any directed graph by "collapsing" each edge to one single edge. In fact, we claim that the functor realizing this process defines a left adjoint to the inclusion functor.

Proposition 2.1. The inclusion functor $U : \mathsf{SimpDirGrph} \hookrightarrow \mathsf{DirGrph}$ admits a left adjoint $I : \mathsf{DirGrph} \to \mathsf{SimpDirGrph}$, and thus $\mathsf{SimpDirGrph}$ is a reflective subcategory of $\mathsf{DirGrph}$.

The proof relies on the following lemma.

Lemma 2.1.

(i) Given two graphs G and H with H simple, a map $f : G \to H$ is completely determined by $f_V : G_V \to H_V$.

(ii) A map $\varphi : G_V \to H_V$ is part of a graph map $f = (\varphi, \varphi')$ if and only if for all $v, v' \in G_V$, we have $v \sim_G v' \Rightarrow \varphi(v) \sim_H \varphi(v')$. That is, adjacent vertices are mapped to adjacent vertices.

Proof. The first part of the lemma is clear, as when a graph is simple one has no choice of which edges to map to while respecting adjacency, since simple graphs may have at most one edge between pairs of vertices.

Suppose now that $\varphi : G_V \to H_V$ is part of a graph map $f = (\varphi, \varphi')$, and suppose that $v \sim_G v'$ for vertices $v, v' \in G_V$. Then there is an edge $e \in G_E$ such that ${s_G}^*(e) = v$ and ${t_G}^*(e) = v'$. Observe that by commutativity of φ and φ' with the source and target functions, we have that

$$ {s_H}^*(\varphi'(e)) = \varphi({s_G}^*(e)) = \varphi(v) $$

and

$$ {t_H}^*(\varphi'(e)) = \varphi({t_G}^*(e)) = \varphi(v'). $$

Thus there is an edge between $\varphi(v)$ and $\varphi(v')$, so $\varphi(v) \sim_H \varphi(v')$. Conversely, suppose that for all $v, v' \in G_V$ we have $v \sim_G v' \Rightarrow \varphi(v) \sim_H \varphi(v')$. Take an edge $e \in G_E$ and define $v := {s_G}^*(e)$ and $v' := {t_G}^*(e)$. Thus $v \sim_G v'$, so by assumption we have $\varphi(v) \sim_H \varphi(v')$. We then have an edge $\overline{e} \in H_E$ between $\varphi(v)$ and $\varphi(v')$ with ${s_H}^*(\overline{e}) = \varphi(v)$ and ${t_H}^*(\overline{e}) = \varphi(v')$. Now, define $\varphi'(e) = \overline{e}$. To check the required commutativity, observe that

$$ {s_H}^*(\varphi'(e)) = \varphi(v) = \varphi({s_G}^*(e)) $$

as required, and a similar calculation is made to check commutativity with the target. Thus $f := (\varphi, \varphi')$ forms a graph map. □

Proof of Proposition 2.1. We begin by defining the action of I on the vertex and edge sets of graphs. For $G \in \mathsf{DirGrph}$, define $(IG)_V = G_V$, and define $(IG)_E = \operatorname{Im}(\langle {s_G}^*, {t_G}^* \rangle)$. We claim that I is left adjoint to U; that is, we have the isomorphism

$$ \mathsf{DirGrph}(G, UH) \cong \mathsf{SimpDirGrph}(IG, H). $$

Consider some graph map $f : G \to UH$. As $UH = H$ by definition, we have a map $f : G \to H$. By the first part of Lemma 2.1, f is completely determined by $f_V : G_V \to H_V$. Because $(IG)_V = G_V$, we have $f_V : (IG)_V \to H_V$. By Lemma 2.3(ii), f_V extends to a graph map if $v \sim_{IG} v' \Rightarrow f_V(v) \sim_H f_V(v')$ for all $v, v' \in G_E$. Since $v \sim_{IG} v'$ if and only if $v \sim_G v'$, the result follows by f being a graph map. Thus we obtain a graph map $g : IG \to H$, giving the desired isomorphism. □

We have now established a reflective subcategory of DirGrph, but similar adjunctions exist between subcategories with different graph properties. Next, we turn our focus to reflexive directed graphs.

3. The Category of Reflexive Directed Graphs

We have not yet explored the case in which graphs have loops on each vertex. It turns out that graphs with this property form another subcategory of DirGrph, which also admits some nice properties that we investigate further in this section. Graphs with loops on vertices are called *reflexive* graphs.

Definition 3.1 (Reflexive Directed Graph)**.** A reflexive directed graph is a directed graph in which each vertex has a distinguished loop on it. Equivalently, for each vertex in the graph, there exists a distinguished edge from that vertex to itself.

We build the category of reflexive directed graphs as a presheaf category, but with some additional structure.

Definition 3.2 ($\mathbb{G}_+$)**.** Define a category $\mathbb{G}_+$ to be the category

$$V \underset{t}{\overset{s}{\underset{\longleftarrow r}{\longrightarrow}}} E$$

with two objects and three non-identity morphisms subject to the condition $rs = \mathrm{id}_V = rt$.

In the same spirit as Definition 1.2, we define the category of reflexive directed graphs, $\mathsf{RelfDirGrph}$, to be the category $\mathsf{Set}^{\mathbb{G}_+^{\mathrm{op}}}$. This category is quite similar to the category of directed graphs, with the added morphism r witnessing the loops on each vertex.

Note that there is the evident inclusion functor $i : \mathbb{G} \hookrightarrow \mathbb{G}_+$ which induces a functor $i^* : \mathsf{Set}^{\mathbb{G}_+^{\mathrm{op}}} \to \mathsf{Set}^{\mathbb{G}^{\mathrm{op}}}$ by precomposition:

$$\mathbb{G}^{\mathrm{op}} \xhookrightarrow{i^{\mathrm{op}}} \mathbb{G}_+^{\mathrm{op}} \xrightarrow{G} \mathsf{Set}$$

That is, $i^*G = G \circ i^{\mathrm{op}}$. We claim that this functor has a right adjoint, which takes a directed graph and freely adds loops to each vertex.

Proposition 3.1. The functor $i^* : \mathsf{Set}^{\mathbb{G}_+^{\mathrm{op}}} \to \mathsf{Set}^{\mathbb{G}^{\mathrm{op}}}$ has a right adjoint $i_* : \mathsf{Set}^{\mathbb{G}^{\mathrm{op}}} \to \mathsf{Set}^{\mathbb{G}_+^{\mathrm{op}}}$.

Proof. Define $(i_*G)_V = G_V$ and $(i_*G)_E = G_V \sqcup G_E$ for a graph $G \in \mathsf{DirGrph}$. We define the action of i_* on morphisms as follows. For the source map

$$s_{(i_*G)}{}^* : G_V \sqcup G_E \to G_V$$

define for all $e \in G_V \sqcup G_E$

$$s_{(i_*G)}{}^* = \begin{cases} s_G{}^* & \text{if } e \in G_E \\ e & \text{if } e \in G_V \end{cases}$$

and similarly for the target map. The map $r_{(i_*G)}{}^* : G_V \to G_V \sqcup G_E$ is simply defined by the coproduct inclusion. We wish to establish the isomorphism

$$\mathsf{ReflDirGrph}(G, i_*H) \cong \mathsf{DirGrph}(i^*G, H).$$

Observe that a map $f : G \to i_*H$ of reflexive graphs consists of maps $f_V : G_V \to H_V$ and $f_E : G_V \to H_V \sqcup H_E$. Conversely, a map $g : i^*G \to H$ consists of maps

$$g_V : G_V \to H_V$$

and

$$g_E : G_E \to H_E.$$

It is clear that the vertex maps are in one-to-one correspondence. Notice, however, that the graph G lives in $\mathsf{ReflDirGrph}$, so G is in fact a reflexive graph. Thus a map $g_E : G_E \to H_E$ is precisely a map $f_E : G_E \to H_V \sqcup H_E$ after forgetting about the loops on each vertex. This gives the correspondence between edge maps, completing the isomorphism. □

We continue our study of reflexive directed graphs in the next section, where we investigate simple reflexive directed graphs.

4. The Category of Simple Reflexive Directed Graphs

The final subcategory of $\mathsf{DirGrph}$ to investigate is the category of *simple reflexive directed graphs*, which we denote $\mathsf{SimpReflDirGrph}$. First, observe that $\mathsf{SimpReflDirGrph} \subseteq \mathsf{SimpDirGrph}$, and we have the same pre-composition functor $i^* : \mathsf{SimpReflDirGrph} \to \mathsf{SimpDirGrph}$ as defined in Section 3. It is natural to ask again what kinds of adjunctions this functor admits when we restrict ourselves to simple graphs. If we wish to "reflexify" a graph while keeping it simple, we ultimately have two choices. We may simply remove all vertices which do not have loops, or we may add a loop wherever there was not one, and preserve all other edges. It turns out that these two processes assemble neatly into left and right adjoints to the functor i^*. We first examine the right adjoint, which removes those vertices which do not have loops.

Proposition 4.1. The functor $i^* : \mathsf{SimpReflDirGrph} \to \mathsf{SimpDirGrph}$ admits a right adjoint $i_* : \mathsf{SimpDirGrph} \to \mathsf{SimpReflDirGrph}$.

Proof. Define $(i_*G)_V = \{v \in G_V \mid G_E(v, v) \neq \varnothing\}$. That is, we define $(i_*G)_V$ to be the set of vertices in G_V which have a loop. Now, define $(i_*G)_E(v, v') = G_E(v, v')$ for all $v, v' \in G_V$. The unit of the proposed adjunction is a natural transformation $\eta : \mathrm{id}_{\mathsf{SimpReflDirGrph}} \Rightarrow i_*i^*$. Components are morphisms $\eta_G : G \to i_*i^*G$, and since $i^*G = G$, components are morphisms $\eta_G : G \to i_*G$. But, since G is reflexive already, $i_*G = G$, and we may thus define $\eta_G = \mathrm{id}_G$. Turning to the counit $\epsilon : i^*i_* \Rightarrow \mathrm{id}_{\mathsf{SimpDirGrph}}$, observe that components are morphisms $\epsilon_G : i^*(i_*G) \to G$, or equivalently morphisms $\epsilon_G : i_*G \to G$. Thus, we define ϵ_G to be the embedding of i_*G into G. It is straightforward to check that the unit and counit satisfy the required triangle identities. □

This completes one half of the promise we made at the beginning of the section. We now turn our focus to the left adjoint to i^*, which adds loops rather than removing vertices which do not have them.

Proposition 4.2. The functor $i^* : \mathsf{SimpReflDirGrph} \to \mathsf{SimpDirGrph}$ admits a left adjoint $i_! : \mathsf{SimpDirGrph} \to \mathsf{SimpReflDirGrph}$.

Proof. We define $(i_!G)_V = G_V$ and

$$(i_!G)_E(v, v') = \begin{cases} \{*\} & \text{if } v = v' \text{ or } G_E(v, v') \neq \varnothing \\ \varnothing & \text{if } G_E(v, v') = \varnothing \text{ and } v \neq v'. \end{cases}$$

That is, we preserve existing edges and add loops where there were not loops before. The unit of the intended adjunction is a natural transformation $\eta : \mathrm{id}_{\mathsf{SimpDirGrph}} \Rightarrow i^* i_!$ with components $\eta_G : G \to i^* i_! G$, but since $i^*(i_!G) = i_!G$, we have components $\eta_G : G \to i_!G$. We define η_G to be the embedding of G into $i_!G$. The counit is a natural transformation $\epsilon : i_! i^* \Rightarrow \mathrm{id}_{\mathsf{SimpReflDirGrph}}$, and has components which are morphisms $\epsilon_G : i_! i^*(G) \to G$. Similarly, since $i^* G = G$, we have morphisms $\epsilon_G : i_!G \to G$. Since G was already a simple reflexive graph, $i_!G = G$. Thus we may define $\epsilon_G : G \to G$ to be the identity on G. As in Proposition 4.1, the triangle identities are easily checked. □

Having built our two adjunctions, we have completed our survey of directed graphs. The last uncharted territory is an investigation of undirected graphs, and how our constructions on directed graphs may translate to graphs without direction. The final section will explore these constructions.

5. The Category of Undirected Graphs and Properties

The first question to ask is how to formalize the notion of an undirected graph from a categorical standpoint. Consider the category

$$V \underset{t}{\overset{s}{\rightrightarrows}} E \circlearrowleft \sigma$$

which is the same category we used to build $\mathsf{DirGrph}$, but equipped with an involution $\sigma : E \to E$ such that $\sigma s = t$, $\sigma t = s$ and $\sigma \circ \sigma = \mathrm{id}_E$. In this way, we can view σ as a "swapper" which makes direction ambiguous - applying σ conflates sources and targets, and witnesses our graph becoming "non-directional". Unsurpringly, the category of undirected graphs, Grph, is the presheaf category on this category. Note that we can still use the source and target functions as we did before. For example, the notion of being *simple* is the same: an undirected graph G is simple when the map $\langle s_G{}^*, t_G{}^* \rangle : G_E \to G_V \times G_V$ is a monomorphism. Observe that there is an inclusion of $\mathbb{G}$ into this category.

Another question to ask is how to build reflexive undirected graphs. Consider the category

$$V \begin{array}{c} \xrightarrow{\;\;s\;\;} \\ \xleftarrow{\;\;r\;\;} \\ \xrightarrow[\;\;t\;\;]{} \end{array} E \circlearrowleft \sigma$$

subject to the requirements $rs = \mathrm{id}_V$ and $rt = \mathrm{id}_V$. This is precisely the category $\mathbb{G}_+$ equipped with the involution σ making our graph undirected. As expected, reflexive undirected graphs are presheaves on this category. It is worth observing that this category is actually a full subcategory of Set: there is a fully faithful functor S sending V to the set $\{1\}$, E to the set $\{1, 2\}$, s to the function $1 : \{1\} \to \{1, 2\}$ which picks out 1, and sends t to a similar function $2 : \{1\} \to \{1, 2\}$. The morphism r is sent to the unique map $\{1, 2\} \to \{1\}$, and σ is sent to the function on $\{1, 2\}$ which swaps the argument. There is precisely the same adjunction between $\mathsf{SimpReflGrph}$ and $\mathsf{SimpGrph}$ as there was between $\mathsf{SimpReflDirGrph}$ and $\mathsf{SimpDirGrph}$, as is easily verified by using the same adjoint functors we defined in Section 4.