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Centralizers of abelian Hamiltonian actions on rational ruled surfaces

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Abstract

In this thesis, we compute the homotopy type of the group of equivariant symplectomorphisms of $S^2 \times S^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ under the presence of Hamiltonian group actions of either S^1 or finite cyclic groups. For Hamiltonian circle actions, we prove that the centralizers are homotopy equivalent to either a torus, or to the homotopy pushout of two tori depending on whether the circle action extends to a single toric action or to exactly two non-equivalent toric actions. We can show that the same holds for the centralizers of most finite cyclic groups in the Hamiltonian group $\mathrm{Ham}(M)$. Our results rely on J-holomorphic techniques, on Delzant's classification of toric actions, on Karshon's classification of Hamiltonian circle actions on 4-manifolds, and on the Chen-Wilczyński smooth classification of $\mathbb{Z}_n$-actions on Hirzebruch surfaces.

Keywords: Homotopy type of Symplectomorphism group, pseudo-holomorphic curves, symplectic rational ruled manifolds, centraliser, abelian Hamiltonian actions.

Summary for lay audience

The study of symplectic manifolds is motivated by classical mechanics. Consider a physical system such as a simple pendulum, or a spring with a mass attached. Associated to such a system is a space called the phase space which encapsulates every possible state that the system can attain. Such a space comes naturally equipped with a non-degenerate two form called a symplectic form and the time evolution of a particle corresponds to flowing along the symplectic gradient of the Hamiltonian of the system.

By Darboux's theorem all symplectic manifolds are locally alike and hence there are no local invariants to distinguish symplectic manifolds. Global invariants of symplectic manifolds can be obtained by investigating the homotopy type of mapping spaces (such as symplectomorphism groups or symplectic embedding spaces) related to the symplectic structure. In his seminal paper [21], M. Gromov provided one such invariant called the Gromov width. This paper also shows that studying the topology of mapping spaces such as the space of symplectic embeddings of a ball into a symplectic manifold, or similarly, the group of self maps that preserve the symplectic structure, gives us key symplectic insights about the symplectic manifold.

In general, investigating symplectomorphism groups or embedding spaces are very hard problems. However, in dimension 4, due to certain special features of J-holomorphic curves we have more tools at our disposal to understand such questions of a global nature.

It is very natural for physical systems to have symmetries. These symmetries of a system correspond to Hamiltonian groups actions on the phase spaces. In this setting, we are interested in the time evolution of particles that preserve these symmetries (or group actions). The maps that preserve these symmetries are called equivariant symplectomorphims. These symmetries can be continuous symmetries like the action of circle S^1 or discrete symmetries like the n-th roots of unity.

In this thesis we combine the theory of holomorphic curves as in [3], [4] and [36] together with moment map techniques as in [25] to study the topology of spaces of all equivariant symplectomorphisms of $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ and $S^2 \times S^2$ endowed with Hamiltonian actions of either the circle or a finite cyclic group.

To Amma and Appa

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Chapter 1

Introduction

In the works of [2], [3], [4], [36], [5] the homotopical properties of the group of symplectomorphisms of $\mathbb{C}P^2$, $S^2 \times S^2$ and their symplectic blow-ups were studied. Given any Hamiltonian group action of a group G, it is very natural to ask what the homotopical properties of the centralizer of G inside the Hamiltonian group of these manifolds are.

In Proposition 3.21 in [35] it was shown that

Theorem 1.0.1. *Let (M, ω) be a compact symplectic manifold. Given an effective toric action $\rho : \mathbb{T}^n \hookrightarrow \mathrm{Symp}\,(M^{2n}, \omega)$ with moment map $\mu : M \to \mathrm{t} \simeq \mathbb{R}^n$. Let $\mathrm{Symp}^{\mathbb{T}^n}(M, \omega)$ denote the centralizer of $\mathbb{T}^n$ in $\mathrm{Symp}(M, \omega)$. Then the centralizer $\mathrm{Symp}^{\mathbb{T}^n}(M, \omega)$ is equal to the group of all symplectomorphisms ϕ that preserve the moment map, that is, such that $\mu \circ \phi = \mu$. Moreover, $\mathrm{Symp}^{\mathbb{T}^n}(M, \omega)$ is connected and $\mathrm{Symp}^{\mathbb{T}^n}(M, \omega) \subset \mathrm{Ham}(M, \omega)$.*

Using this and the fact that for toric actions, the level sets of the momentum map are orbits for the toric action, one can derive that $\mathrm{Symp}^{\mathbb{T}^n}(M, \omega)$ is homotopic to $\mathbb{T}^n$. Further for the manifolds $(S^2 \times S^2, \omega_\lambda)$ and $(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$, where λ is a parameter that determines the symplectic form, it is possible to obtain the same result using the pseudo-holomorphic curve techniques. A key point to notice is that the homotopy type of the centraliser for a toric action is independent of the action.

The next most natural case is of S^1-actions on $(S^2 \times S^2, \omega_\lambda)$ and $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)$. The situation becomes more complex than in the toric case, primarily because the level sets of the momentum map are no longer single orbits but rather unions of orbits: an S^1 equivariant symplectomorphism can swap orbits whilst preserving the level sets.

In this thesis we use both pseudo-holomorphic curve techniques and moment map techniques to determine the homotopy type of equivariant symplectomorphisms of $S^2 \times S^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ under the presence of circle actions. The advantage of using pseudo-holomorphic curve techniques is the that the proofs generalise under the presence of any compact abelian group actions.

The thesis is structures as follows:

In Chapter 2, we present the background material for both moment map techniques and pseudo-holomorphic curves techniques that we use in the thesis.

The crux of the thesis lies in Chapters 3, 4 and 5. We adapt the framework of [4] to study symplectomorphism groups in the presence of an S^1 action. In particular we show that the space of invariant almost complex structures $\mathcal{J}^{S^1}_{\omega_\lambda}$ decomposes into disjoint strata, each of them being homotopy equivalent to an orbit of the equivariant symplectomorphism group with stabilizer being homotopy equivalent to S^1 equivariant Kähler isometries (Theorems 3.3.15 and 3.3.23). In Chapter 3, we use Karshon's classification of circle action on 4-manifolds [25] to investigate into how many invariant strata the space of invariant almost complex structures $\mathcal{J}^{S^1}_{\omega_\lambda}$ decomposes. We prove that $\mathcal{J}^{S^1}_{\omega_\lambda}$ decomposes into either one or two strata, and that the later case occurs only for an exceptional family of circle actions on $S^2 \times S^2$ (Theorems 3.1.9 and 3.1.5).

In Chapter 4, using techniques similar to the ones developed in [5] we obtain the homotopy type of $\mathrm{Symp}^{S^1}(S^2 \times S^2, \omega_\lambda)$ for all Hamiltonian circle actions on $(S^2 \times S^2, \omega_\lambda)$. We notice that in most cases the homotopy type of $\mathrm{Symp}^{S^1}(S^2 \times S^2, \omega_\lambda)$ is the same as that of the space of S^1 equivariant Kähler isometries. But for the exceptional family circle of actions on $S^2 \times S^2$ for which $\mathcal{J}^{S^1}_{\omega_\lambda}$ has two invariant strata, we see that the homotopy type of $\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$ undergoes a "phase transition" for a particular value of λ. The homotopy types changes from being one of a finite dimensional Lie group to one of an infinite-dimensional space (Theorem 4.3.1).

In Chapter 5, we prove S^1 equivariant analogues of some key lemmas involving deformation theory as in [3]. We use these techniques to prove that for the exceptional family of circle actions, the stratum with positive codimension in $\mathcal{J}^{S^1}_{\omega_\lambda}$ is always of codimension two.

In Chapter 6, we carry out a similar analysis on the manifold $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ and obtain the homotopy type of $\mathrm{Symp}^{S^1}(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$ for all Hamiltonian circle actions on $(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$.

We explore the homotopy type of the equivariant symplectomorphisms under the presence of a finite cyclic group in Chapter 7 of the thesis. We define a Hamiltonian action of a finite group G on (M, ω) to be a morphism of G into the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M, \omega)$. The list of all finite groups that admit a Hamiltonian action on $(S^2 \times S^2, \omega_\lambda)$ or $(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$ is given in [9]. In particular, they prove the following two theorems.

Theorem 1.0.2. *Let F be a finite group that acts effectively and symplectically on the product $(S^2 \times S^2, \omega_\lambda)$.*

- *If $\lambda \neq 1$, F is isomorphic to a subgroup of $G_1 \times G_2$ for some finite subgroups G_1, G_2 of* $\mathrm{SO}(3)$.

- *If* $\lambda = 1$, *F is isomorphic to a subgroup of* $G_1 \times G_2$ *for some finite subgroups* G_1, G_2 *of* $\mathrm{SO}(3)$, *or F belongs to an exact sequence*

$$1 \to H \times H \to F \to \mathbb{Z}/2 \to 1$$

for some finite subgroup H of $\mathrm{SO}(3)$.

For the non-trivial bundle $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, the list is even simpler.

Theorem 1.0.3. *A finite group F acts effectively and symplectically on the non-trivial bundle* $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)$ *if and only if F is isomorphic to a finite subgroup of* $U(2)$.

Hence the only finite abelian groups with Hamiltonian actions on $(S^2 \times S^2, \omega_\lambda)$ with $\lambda > 1$ are of the form $\mathbb{Z}_n$ or $\mathbb{Z}_n \times \mathbb{Z}_m$. In Chapter 7, we explore the homotopy type of the equivariant symplectomorphism groups of $(S^2 \times S^2, \omega_\lambda)$ and $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)$ under the presence of Hamiltonian $\mathbb{Z}_n$ actions. Unlike the S^1 case, Hamiltonian finite group actions do not admit momentum maps. Hence, we extract information about the equivariant symplectomorphism group by using pseudo-holomorphic curve techniques. Most of the techniques we use in the S^1 case go through mutatis mutandis in the $\mathbb{Z}_n$ case as well, but unlike in the S^1 case, we do not have a classification of $\mathbb{Z}_n$ actions on $S^2 \times S^2$ (and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$) up to $\mathbb{Z}_n$ equivariant symplectomorphisms. We can still use the Chen-Wilczyński classification of $\mathbb{Z}_n$-actions up to oriented diffeomorphisms given in [12] and [40] to obtain the homotopy type of $\mathbb{Z}_n$ equivariant symplectomorphisms of $(S^2 \times S^2, \omega_\lambda)$ and $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)$ for a large class of Hamiltonian $\mathbb{Z}_n$ actions.

Finally, in chapter 8 we outline potential research directions that emerge from the thesis.

Chapter 2

Preliminaries

2.1 Hamiltonian actions

Definition 2.1.1. *Let G be a Lie group acting symplectically on the symplectic manifold (M, ω). Let $\mathfrak{g}$ denote the Lie algebra of G and $\mathfrak{g}^*$ be it's dual. Given $Y \in \mathfrak{g}$, we denote by $\overline{Y}$ the fundamental vector field associated to Y. We say that the action is Hamiltonian if it satisfies the following conditions*

1. *There exists a moment map $\mu : M \to \mathfrak{g}^*$ such that $d\mu_p(X_p)(Y) = \omega(X, \overline{Y})$ for all $X \in T_pM$ and $Y \in \mathfrak{g}$.*

2. *This map μ is equivariant with respect to the G action on M and the coadjoint action on $\mathfrak{g}^*$*

Definition 2.1.2. *A smooth action of a finite group G on (M, ω) is called Hamiltonian, if there exists a group homomorphism $\rho : G \to \mathrm{Ham}(M, \omega)$ where $\mathrm{Ham}(M, \omega)$ denotes the group of Hamiltonian diffeomorphisms of (M, ω).*

We then have the following theorems

Theorem 2.1.3. *(Atiyah-Guillemin-Sternberg) Let (M, ω) be a symplectic manifold with a Hamiltonian action of a torus $\mathbb{T}^d$ on M. Then the image of the moment map μ is a convex polytope of $\mathfrak{t}^*$ whose vertices are images of the fixed points of the torus action.*

We call a Hamiltonian torus action toric if the torus acting is half the dimension of the manifold M. When a manifold admits a toric action we have the following theorem.

Theorem 2.1.4. *(Delzant [15]) Let $\mathbb{T}^n \times M^{2n} \to M^{2n}$ be a toric action on a $2n$-dimensional symplectic manifold (M^{2n}, ω), with momentum map μ. Then the moment polytope $\mu(M)$ determines the Hamiltonian space up to $\mathbb{T}^n$-equivariant symplectomorphisms.*

Y. Karshon proved in [25] an analogous equivariant classification for S^1 action on 4-dimensional symplectic manifolds (M, ω) in which the moment map image is replaced by labelled graphs. More precisely, given any Hamiltonian S^1 action on a 4-manifold M, one can associate a labelled graph to the action as follows:

- Each component of the fixed point set corresponds to a unique vertex of the graph.
- Each vertex is labeled by the value of the moment map on the corresponding fixed point component. If an extremal vertex corresponds to a symplectic surface S, two additional labels are attached: the genus of that surface, and its normalized symplectic area.
- Two vertices are connected by an edge if and only if the corresponding isolated fixed points are connected by a $\mathbb{Z}_k$-sphere i.e by a S^1 invariant sphere on which the S^1 acts by a global stabilizer $\mathbb{Z}_k$.
- Each edge is labelled by the isotropy weight k of the corresponding $\mathbb{Z}_k$ sphere.

Just as Delzant polytopes classify toric actions up to symplectomorphisms, labelled graphs classify Hamiltonian S^1 actions.

Theorem 2.1.5. *(Karshon [25]) The labelled graph determines the Hamiltonian circle action and the manifold M up to S^1-equivariant symplectomorphisms.*

In particular, the classification tells that that it is not important to keep track of the spheres with trivial isotropy and hence, these spheres do not appear in the labelled graphs.

2.1.1 Torus actions on $S^2 \times S^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$

We would like to use the above theorems to understand all the possible Hamiltonian circle actions on $S^2 \times S^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ and their relations to toric actions. To this end, we first recall the Lalonde-McDuff classification of symplectic forms on $S^2 \times S^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. We equip $S^2 \times S^2$ with the product symplectic form $\omega_\lambda = \lambda\sigma \oplus \sigma$, where $\lambda > 0$ and σ is the standard area form on S^2 normalized such that $\sigma(S^2) = 1$. If we think of $S^2 \times S^2$ as a trivial fiber bundle, ω_λ gives area 1 to the fibers, while the area of horizontal sections is λ. Similarly, if we view $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ as the non-trivial S^2 bundle over S^2, we define an analogous form ω_λ which gives area 1 to the fibers and area $\lambda - 1$ to symplectic sections of self-intersection -1, that is, to sections homologous to the exceptional divisor. From an homological point of view, if F denotes the homology class of a fiber in either $S^2 \times S^2$ or $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, if B denotes the class of a section of self-intersection 0 in $S^2 \times S^2$, if E denotes the class of the exceptional divisor in $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, and if L denotes the class of a line in $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, then $[\omega_\lambda]F = 1$, $[\omega_\lambda]B = \lambda$, $[\omega_\lambda]L = \lambda$ and $[\omega_\lambda]E = \lambda - 1$. We can now state the Lalonde-McDuff classification theorem.

Theorem 2.1.6 (Lalonde-McDuff [29], Theorem 1.1)**.** *Any symplectic form on $S^2 \times S^2$ or $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ is diffeomorphic to a constant multiple of ω_λ with $\lambda \geqslant 1$. Moreover, any two cohomologous forms are diffeomorphic.*

If we consider any Hamiltonian circle action on $S^2 \times S^2$ or $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, the first important result is an extension theorem due to Y. Karshon:

Theorem 2.1.7 (Karshon [26], Theorem 1)**.** *Any symplectic S^1 action on $(S^2 \times S^2, \omega_\lambda)$ and $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)$ extends to an Hamiltonian toric action.*

To characterise all possible S^1 action on $(S^2 \times S^2, \omega_\lambda)$ and $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)$, we need to first understand what the possible inequivalent toric actions on these spaces are. In order to determine this we recall how Hirzebruch surfaces are defined.

We define the Hirzebruch surface W_m as the complex submanifold of $\mathbb{C}P^1 \times \mathbb{C}P^2$ satisfying the equation

$$W_m := \left\{ ([x_1, x_2], [y_1, y_2, y_3]) \in \mathbb{C}P^1 \times \mathbb{C}P^2 \mid x_1^m y_2 - x_2^m y_1 = 0 \right\}$$

The projection map $\mathbb{C}P^1 \times \mathbb{C}P^2 \longrightarrow \mathbb{C}P^1$ gives W_m the structure of a $\mathbb{C}P^1$ bundle over $\mathbb{C}P^1$ which is diffeomorphic to $S^2 \times S^2$ if m is even and diffeomorphic to the non-trivial S^2 bundle over S^2 i.e $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ if m is odd. Thus, for each m we have an integrable complex structure J_m induced on $S^2 \times S^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. We can endow $\mathbb{C}P^1 \times \mathbb{C}P^2$ with the symplectic form $(\lambda - \frac{m}{2})\sigma_1 \oplus \sigma_2$, where σ_1 and σ_2 are the standard Fubini-Study forms on $\mathbb{C}P^1$ and $\mathbb{C}P^2$ respectively and restricting this symplectic form to W_m makes it a symplectic manifold. We can analogously define the form $(\lambda - \frac{m+1}{2})\sigma_1 \oplus \sigma_2$ when m is odd. With these choices of symplectic forms, W_m is symplectomorphic to $(S^2 \times S^2, \omega_\lambda)$ if m is even and $(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$ when m is odd.

The torus $\mathbb{T}^2$ acts on $\mathbb{C}P^1 \times \mathbb{C}P^2$ by setting

$$(u, v) \cdot ([x_1, x_2], [y_1, y_2, y_3]) = ([ux_1, x_2], [u^m y_1, y_2, vy_3])$$

This action leaves W_m invariant and preserves both the complex and the symplectic structures. Its restriction to W_m defines a toric action that we denote $\mathbb{T}^2_m$. Its momentum map is

$$\mu\left(([x_1, x_2], [y_1, y_2, y_3])\right) = \left(\left(\frac{(\lambda - \frac{m}{2})|x_1|^2}{|x_1|^2 + |x_2|^2} + m \frac{|y_1|^2}{|y_1|^2 + |y_2|^2 + |y_3|^2} \right), \frac{|y_3|^2}{|y_1|^2 + |y_2|^2 + |y_3|^2} \right)$$

When m is even, the image of the moment map is the polytope of Figure 2.1

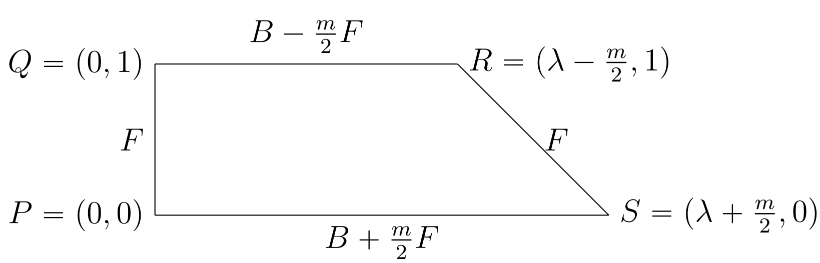


Figure 2.1: Even Hirzebruch surface

The labels above the edges in the above picture refer to the homology classes of the $\mathbb{T}^2$ invariant spheres in $S^2 \times S^2$. With our normalization, we have $\omega_\lambda(B) = \lambda$ and $\omega_\lambda(F) = 1$. Also the vertices P,Q,R,S are the fixed points for the torus action.

Similarly, when m is odd, we have the following momentum map image

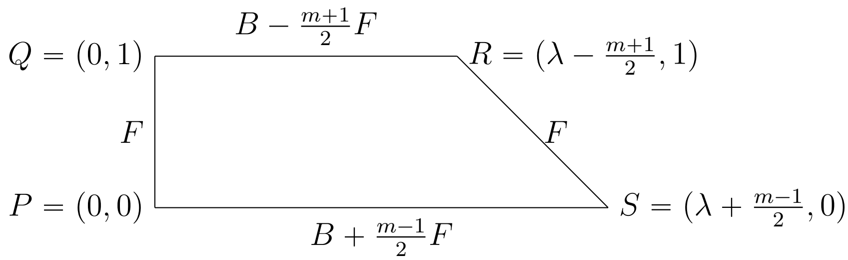


Figure 2.2: Odd Hirzebruch surface

where B now refers to the homology class of a line L in $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ and F refers to the class $L - E$ where L is the class of the line and E is the class of the exceptional divisor.

We define the zero-section s_0 to be

$$s_0 : \mathbb{C}P^1 \to W_m$$
$$[x_1; x_2] \mapsto \{[x_1, x_2], [0; 0; 1]\}$$

and the section at infinity s_∞ to be

$$s_\infty : \mathbb{C}P^1 \to W_m$$
$$[x_1; x_2] \mapsto \{[x_1, x_2], [x_1^m; x_2^m; 0]\}$$

The curve s_0 has homology class $B - \frac{m}{2}$ in $S^2 \times S^2$ and $B - \frac{m+1}{2}$ in $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. Similarly, the section s_∞ has homology class $B + \frac{m}{2}F$ in $S^2 \times S^2$ and $B - \frac{m+1}{2}F$ in $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. Finally, the homology class F can be represented by a fixed fibre such as $\{[1, 0], [y_1, 0, y_3]\}$.

Since the action of $\mathbb{T}^2_m$ is holomorphic with respect to the complex structure J_m, there always exists holomorphic curves coming from the sections s_0 and s_∞ in class $B - \frac{m}{2}$ and

$B + \frac{m}{2}F$ in $S^2 \times S^2$ (and analogously in classes $B - \frac{m+1}{2}$ and $B - \frac{m+1}{2}F$ in $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$). It follows from Delzant's classification that any toric action on $S^2 \times S^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ is an action of the above form. In particular, we have the following lemma.

Lemma 2.1.8. *Up to equivalence, the toric action $\mathbb{T}^2_m$ is characterised by the existence of an invariant, embedded, symplectic sphere C_m in class $B - \frac{m}{2}F$ with self intersection $-m$.* □

Lemma 2.1.9. *Write $\lambda \geqslant 1$ as $\lambda = \ell + \delta$ with ℓ an integer and $0 < \delta \leqslant 1$. Then, up to symplectomorphisms and reparametrizations,*

- *there are exactly $\ell + 1$ inequivalent toric actions on $(S^2 \times S^2, \omega_\lambda)$ given by the even Hirzebruch actions $\mathbb{T}^2_{2k}$ with $0 \leqslant k \leqslant \ell$, and*
- *there are exactly ℓ inequivalent toric actions on $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)$ given by the odd Hirzebruch actions $\mathbb{T}^2_{2k+1}$ with $0 \leqslant k \leqslant \ell - 1$.*

Proof. Write $m = 2k$ or $m = 2k + 1$ with $k \geqslant 0$. As seen above, it follows from Delzant's classification that any toric action on $S^2 \times S^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ is $\mathbb{T}^2$-equivariantly symplectomorphic to one of the actions $\mathbb{T}^2_m$. As there always exists a J_m-holomorphic curve C_m in the class $B - kF$ on $S^2 \times S^2$ which is T_m invariant (or equivalently a curve C_m in the class $L - (k+1)F$ in $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$), and as this curve must always have positive area, that is,

$$0 < \omega_\lambda(B - kF) = \lambda - k = \ell + \delta - k \quad \text{or} \quad 0 < \omega_\lambda(L - (k+1)F) = \lambda - (k+1)F = \ell + \delta - (k+1)$$

the result follows. □

By Theorem 2.1.7 we know that every symplectic S^1 action on $S^2 \times S^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ extends to an action of a torus $\mathbb{T}^2_m$. Equivalently, circle actions are given by embeddings

$$\begin{aligned} S^1 &\hookrightarrow \mathbb{T}^2_m \\ t &\mapsto (t^a, t^b) \end{aligned}$$

Consequently, any such Hamiltonian circle action corresponds to a unique triple of numbers $(a, b; m) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_{\geqslant 0}$. Since we are only interested in effective actions (i.e actions with no global stabilizer), this translates numerically into the condition $\gcd(a, b) = 1$. We shall always assume this unless otherwise stated.

Definition 2.1.10. *We shall say a circle action $S^1(a, b; m)$ extends to a toric action $\mathbb{T}^2_{m'}$ if it is S^1-equivariantly symplectomorphic to a circle action of the form $S^1(a', b'; m')$.*

Note that Theorem 2.1.7 does not give us how many tori a given symplectic circle action on $S^2 \times S^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ extends to. We explore this question in the next chapter.

We shall now explore how the graphs for the circle actions $S^1(a, b; m)$ on W_m look like. But before we do that we need to recall a few facts.

By the slice theorem applied to the fixed points, there exists a neighbourhood of p which is equivariantly diffeomorphic with a neighbourhood of the origin in $\mathbb{R}^4$ with the torus acting on $\mathbb{R}^4$ via real linear transformations. Fixing a $\mathbb{T}^2$ invariant compatible almost complex structure J, the torus action on the tangent space at p acts as a subgroup of the standard $U(2)$ action on $\mathbb{C}^2$. By Schur's Lemma, any unitary $\mathbb{T}^2$-representation splits into a sum of 1-dimensional representations. Hence in a local model the action looks like

$$(u, v) \cdot z_1 = u^{\alpha_1^1} v^{\alpha_2^1} z_1$$
$$(u, v) \cdot z_2 = u^{\alpha_1^2} v^{\alpha_2^2} z_2$$

where $u, v \in S^1$ and $z_1, z_2 \in \mathbb{C}^2$ are eigen vectors. The vector $((\alpha_1^1, \alpha_1^2), (\alpha_2^1, \alpha_2^2))$ in $\mathbb{Z}^2 \times \mathbb{Z}^2$ are called the weights of the action at the fixed point. Note that the weights are only well defined up to change in order of the tuples. To see that the weights up to change in order of tuples are independent of the choice of $\mathbb{T}^2$ invariant almost complex structure J, we note that the weights define a continuous map from the space of $\mathbb{T}^2$ invariant compatible almost complex structures to the space of unordered integer tuples $\left((\mathbb{Z}\times\mathbb{Z})\times(\mathbb{Z}\times\mathbb{Z})\right)/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on the $(\mathbb{Z}\times\mathbb{Z})\times(\mathbb{Z}\times\mathbb{Z})$ as follows:

$$\mathbb{Z}_2\times(\mathbb{Z}\times\mathbb{Z})\times(\mathbb{Z}\times\mathbb{Z})\to(\mathbb{Z}\times\mathbb{Z})\times(\mathbb{Z}\times\mathbb{Z})$$
$$\left(-1,\left((\alpha_1^1,\alpha_1^2),(\alpha_2^1,\alpha_2^2)\right)\right)\to\left((\alpha_1^2,\alpha_1^1),(\alpha_2^2,\alpha_2^1)\right)$$

As the space of $\mathbb{T}^2$ invariant compatible almost complex structures is contractible, the weights up to change in order of tuples are independent of the choice of J. If the point p had weights (α_1^1,α_2^1) and (α_1^2,α_2^2) for the T_m action, then for the restricted $S^1(a,b;m)$ action the weights up to change in order of the tuples are given by

$$\left(a\alpha_1^1+b\alpha_2^1, a\alpha_1^2+b\alpha_2^2\right)$$

We would now like to understand how the weight at a fixed point transforms under the action of an equivariant symplectomorphism.

Lemma 2.1.11. *Let (M,ω) be a symplectic manifold with a S^1 Hamiltonian action with momentum map μ. Then $\phi\in\mathrm{Symp}^{S^1}(S^2\times S^2,\omega_\lambda)$ iff $\mu\circ\phi=\mu$ and $\phi\in\mathrm{Symp}(M,\omega)$.*

Proof. $(\Leftarrow)$ Let $X\in\mathbb{R}$ (where we think of $\mathbb{R}$ as the lie algebra of S^1) and let $\underline{X}$ denote the fundamental vector field associated to X. Since ϕ is a symplectomorphism preserving μ, we have $\omega(d\phi^{-1}(\underline{X}),Y)=\phi^*\omega(\underline{X},d\phi(Y))=\omega(\underline{X},d\phi(Y))=d\mu(d\phi(Y))=d\mu(Y)=\omega(X,Y)$ for any vector field Y, which implies $d\phi(\underline{X})=\underline{X}$ for all $X\in\mathbb{R}$. Consequently, ϕ commutes with the action.

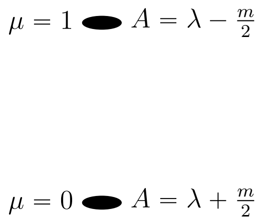
$(\Rightarrow)$ Let $\rho:S^1\to\mathrm{Symp}(M,\omega)$ denote the action. Then $\phi^{-1}\circ\rho\circ\phi=\rho$ for all $\phi\in\mathrm{Symp}^{S^1}(S^2\times S^2,\omega_\lambda)$. But if the action ρ is generated by a Hamiltonian H_t and has momentum map μ then $\phi^{-1}\circ\rho\circ\phi$ is generated by $H_t\circ\phi^{-1}$ and has momentum map $\mu\circ\phi^{-1}$. But as the two actions are the same implies $\mu\circ\phi^{-1}=\mu+C$ for some constant C. As $S^2\times S^2$ is compact we can choose this constant to be 0. $\square$

Corollary 2.1.12. *Let ϕ be an S^1-equivariant symplectomorphism. Then ϕ acts on the fixed point set preserving the weights of the fixed points up to change of order of each tuple.*

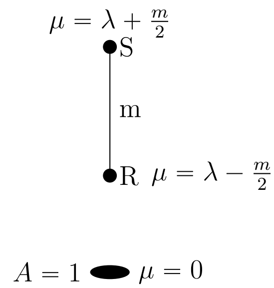
Proof. Firstly we note that as the space of S^1-invariant compatible almost structures $\mathcal{J}_{\omega_\lambda}^{S^1}$ is contractible, the weights at a fixed point are independent of the choice of invariant almost complex structure used to calculate them. Let p_0 be a fixed point of the S^1 action, and choose an arbitrary $J \in \mathcal{J}_{\omega_\lambda}^{S^1}$ to calculate the weights of the S^1 action at p_0. Choose $\phi_* J$ to calculate the weights at $\phi(p_0)$. As ϕ is by definition holomorphic with respect to the chosen almost complex structures, ϕ preserves the weights of the fixed points up to change of order of each tuple □

Remark 2.1.13. *One can also prove the above theorem using Lemma 2.1.11 as follows. Any S^1 equivariant symplectomorphism ϕ takes one fixed point to another and, by Lemma 2.1.11, preserves the momentum map. By the local normal form theorem (Proposition I.2.1 in [6]), the momentum map is determined by the weights in a neighbourhood of a fixed point. If the weights at p are w_1 and w_2, then the momentum map is locally given by $w_1|z_1|^2 + w_2|z_2|^2$. This implies that the weights at a fixed point p and $\phi(p)$ have to be the same up to change of order of each tuple.*

From this and Table 2.1.1, we can construct the graphs of all $S^1(a, b; m)$ action on $(S^2 \times S^2, \omega_\lambda)$. We present a few of them below. The following graphs are for circles of the form $S^1(a, b; m)$ with $a > 0$, $b > 0$ and m is even. The other cases are similar. When the value for one of the labels on the edge is 1, then invariant sphere is free and we omit that edge in the graphs.

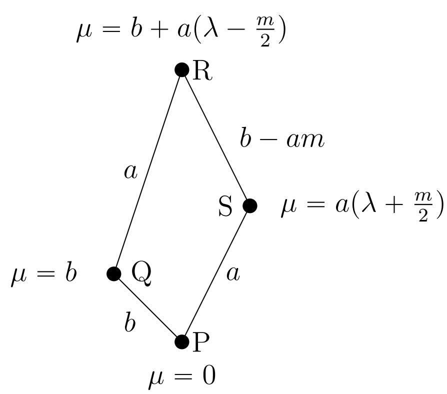


(a) When $(a, b) = (0, 1)$

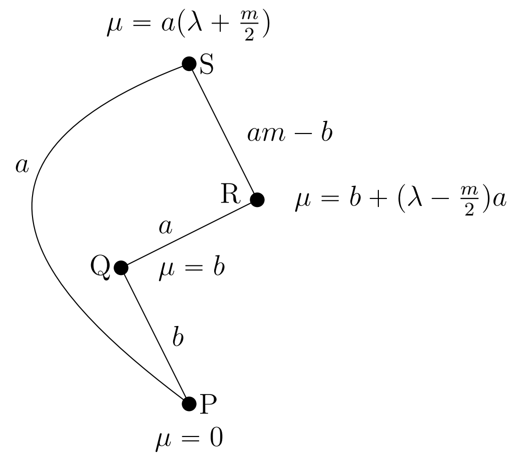


(b) When $(a, b) = (1, 0)$

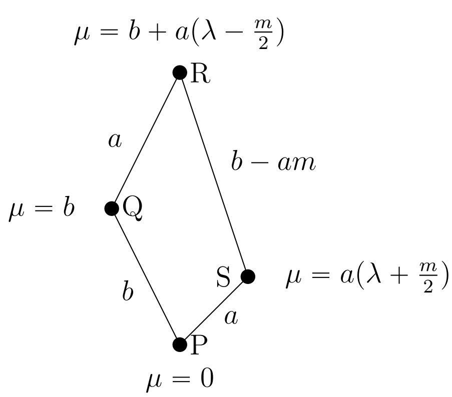
Figure 2.3: Graphs for circle actions with embedded surfaces in the fixed points set



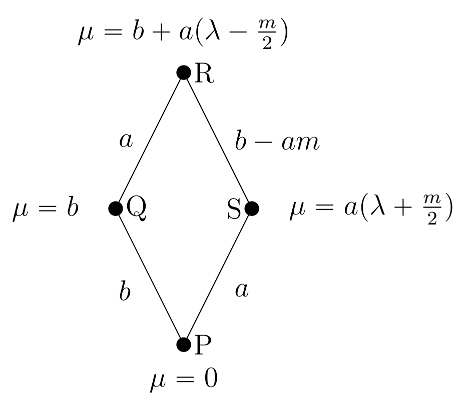
(a) When $b > am$ and $a(\lambda + \frac{m}{2}) > b$



(b) When $am > b$



(c) When $b > am$ and $a(\lambda + \frac{m}{2}) < b$



(d) When $b > am$ and $a(\lambda + \frac{m}{2}) = b$

Figure 2.4: Graphs for circle actions with no fixed surfaces

Here the labels μ represents the value of the momentum map and A the area of the fixed surface. All fixed surfaces have genus 0. For the above torus action on the Hirzebruch surfaces, the isotropy weights at the fixed points are given in the following table:

Weights at vertices		
Vertex	Weights for $\mathbb{T}^2_m$ action	Weights for the $S^1(a,b;m)$ action
P	$((1,0),(0,1))$	(a,b)
Q	$((1,0),(0,-1))$	$(a,-b)$
R	$((-1,m),(0,-1))$	$(-a,am-b)$
S	$((-1,-m),(0,1))$	$(-a,-am+b)$

In turns, the weights at fixed points put strong restrictions on the graphs associated to the circle actions.

Remark 2.1.14. *The graph for the circle action $S^1(-1,-m;m)$ is given by*

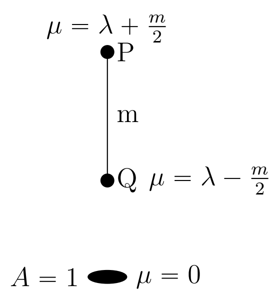


Figure 2.5: When $(a,b) = (-1,-m)$

From the above graphs we notice that the action $S^1(1,0;m)$ is S^1-equivariantly symplectomorphic to $S^1(-1,-m;m)$. Similarly $S^1(-1,0;m)$ is S^1-equivariantly symplectomorphic to $S^1(1,m;m)$.

2.2 *J*-Holomorphic Preliminaries

In order to investigate the homotopy type of the group of equivariant symplectomorphisms we shall use the theory of *J*-holomorphic curves. Before we begin, we shall recall a few facts about holomorphic curves and the space of compatible almost complex structures in $(S^2 \times S^2, \omega_\lambda)$ and $(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$. We present most of our results for $(S^2 \times S^2, \omega_\lambda)$, the case for $(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$ is analogous.

Definition 2.2.1 (Compatible almost complex structures)**.** *An almost complex structure J on a symplectic manifold (M, ω) is said to be compatible with ω if $\omega(u, Ju) > 0$ and $\omega(Ju, Jv) = \omega(u, v)$ for all non-zero $u, v \in T_*M$.*

Lemma 2.2.2. *The space $\mathcal{J}_\omega = \mathcal{J}(M, \omega)$ of all compatible almost complex structures on a symplectic manifold (M, ω) is non-empty and contractible.*

Definition 2.2.3. *J-holomorphic spheres: Let (M, ω) be a symplectic manifold endowed with a compatible almost complex structure J. A rational J-holomorphic map, also called a parametrized J-holomorphic sphere, is a C^∞ map*

$$u : (S^2, j) \longrightarrow (M, \omega, J)$$

satisfying the Cauchy-Riemann equation

$$\bar{\partial}_J(u) = \frac{1}{2}(du \circ j - J \circ du) = 0$$

where j is the usual complex structure on the sphere. The image of a J-holomorphic rational map is called a rational J-holomorphic curve or simply a J-curve.

Remark 2.2.4. *A J-holomorphic map defines an integral homology class $[u] := u_*[S^2] \in H_2(M, \mathbb{Z})$.*

Definition 2.2.5 (Multi-covered and simple maps)**.** *We say that a J-holomorphic map $u : \mathbb{C}P^1 \longrightarrow (M, J)$ is multi-covered if $u = u' \circ f$, where $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ is a holomorphic map of degree greater than 1 and where $u' : \mathbb{C}P^1 \to (M, J)$ is a J-holomorphic map. We call a J-holomorphic map simple if it is not multi-covered.*

Remark 2.2.6. *We usually assume that a J-holomorphic map is somewhere injective, meaning that $\exists z \in S^2$ such that $du_z \neq 0$ and $u^{-1}u(z) = z$. In particular, somewhere injective maps do not factor through multiple covers $h : S^2 \to S^2$.*

Definition 2.2.7 (Moduli spaces of J-holomorphic maps or curves)**.** *Let (M, ω) be a symplectic manifold and let $J \in \mathcal{J}_\omega$. Given $A \in H_2(M, \mathbb{Z})$ we denote by $\widetilde{\mathcal{M}}(A, J)$ the space of all J-holomorphic, somewhere injective maps representing the homology class A. The Mobius group $G = \mathrm{PSL}(2, \mathbb{C})$ acts freely on this space by reparametrization and the quotient space $\mathcal{M}(A, J) := \widetilde{\mathcal{M}}(A, J)/G$ is called the moduli space of (unparametrised) J-curves in class A.*

In dimension 4, the geometric properties of J-holomorphic curves are, to a large extend, controlled by homological data. As a result, many properties of complex algebraic curves in complex algebraic surfaces extend to J-holomorphic curves in 4-dimensional symplectic manifolds. Below we list some key properties of J-holomorphic curves we will be relying on.

Theorem 2.2.8 (Positivity)**.** *Let (M, ω) be a 4-dimensional symplectic manifold. If a homology class $A \in H_2(M, \mathbb{Z})$ is represented by a nonconstant J-curves for some $J \in \mathcal{J}_\omega$ then $\omega(A) > 0$.*

The following facts rely on well-known results about J-holomorphic curves in symplectic 4-manifolds that we briefly recall for convenience. The proofs can be found in [31], [24] and [21].

Theorem 2.2.9 (Fredholm property and automatic regularity)**.** *.Let (M, ω) be a 4-dimensional symplectic manifold. Then the universal moduli space*

$$\widetilde{\mathcal{M}}(A, \mathcal{J}_\omega) := \bigcup_{J \in \mathcal{J}_\omega} \widetilde{\mathcal{M}}(A, J)$$

with C^l-topology ($l \geqslant 2$) is a smooth Banach manifold and the projection map

$$\pi_A : \widetilde{\mathcal{M}}(A, \mathcal{J}_\omega) \longrightarrow \mathcal{J}_\omega$$

is a Fredholm map of index $2(c_1(A) + 2)$ where $c_1 \in H^2(M, \mathbb{Z})$ is the first chern class of (TM, J) (note that the Chern class is independent of choice of $J \in \mathcal{J}_\omega$). An almost complex structure is said to be regular for the class A if it is a regular value for the *projection π_A. If this is the case then the moduli spaces $\widetilde{\mathcal{M}}(A, J)$ and $\mathcal{M}(A, J)$ are smooth manifolds of dimensions $2(c_1(A) + 2)$ and $2(c_1(A) - 1)$ respectively. The set of regular values $\mathcal{J} \in \mathcal{J}_\omega$ is a subset of second category and is denoted by $\mathcal{J}_\omega^{reg}(A)$. If $J \in J_\omega$ is integrable and S is an embedded J-holomorphic sphere with self-intersection number $[S] \cdot [S] \geqslant -1$, then J is regular for the class $[S]$. In dimension 4, the same conclusion holds without the integrability assumption.*

Definition 2.2.10 (Cusp Curves)**.** *Let (M, ω) be a symplectic manifold. Let $J \in J_\omega$. A J-holomorphic cusp curve C is a connected finite union of J-holomorphic curves*

$$C = C_1 \cup C_2 \ldots \cup C_k$$

where $C_i = u_i(\mathbb{C}P^1)$ and $u_i : \mathbb{C}P^1 \to (M, J)$ is a (possibly multi-covered) J-holomorphic map.

Theorem 2.2.11 (Gromov's compactness theorem)**.** *Let (M, ω) be a compact symplectic manifold. Let $J_n \in \mathcal{J}_\omega$ be a sequence converging to J in the C^∞ topology and let S_i be J_i-holomorphic spheres of bounded symplectic area $\omega(S_i)$. Then there is a subsequence of the S_i which converges weakly to a J-holomorphic curve or cusp-curve S. In particular if all the S_i's belong to the class A, then S also belongs the class A, and any cusp curve defines a homological decomposition of $A = \sum_i A_i$ such that $\omega(A_i) > 0$.*

Theorem 2.2.12 (Positivity of intersections)**.** *Let $J \in \mathcal{J}_{\omega_\lambda}$ and A, B be two distinct J-holomorphic curves in a 4-dimensional manifold. Then they intersect at only finitely many points and each point contributes positively to the intersection multiplicity $[A] \cdot [B]$. Moreover, $[A] \cdot [B] = 1$ iff the curves intersect transversally at exactly one point, while $[A] \cdot [B] = 0$ iff the curves are disjoint.*

As a corollary of Positivity of intersections we have the following result under the presence of a group action.

Corollary 2.2.13. *Let (M, ω) be a symplectic 4-manifold and let G be a compact Lie group acting symplectically on M. Suppose that G acts trivially on homology. Let $\mathcal{J}_G$*

denote the space of ω tame (or compatible) almost complex structures and let C be a J holomorphic curve for some $J \in \mathcal{J}^G$. Then,

1. *if C has negative self intersection, then $g \cdot C = C$ for all $g \in G$.*

2. *If C has zero self intersection, then $g \cdot C = C$ or $g \cdot C \cap C = \emptyset$ for all $g \in G$.*

Theorem 2.2.14 (Adjunction formula)**.** *Let $u : (S^2, j) \longrightarrow (M^4, J)$ be a somewhere injective J-holomorphic map representing the homology class A in a 4-dimensional manifold. Define the virtual genus of A as*

$$g_v(A) = 1 + \frac{1}{2}([A] \cdot [A] - c_1(A))$$

where $c_1(A) = \langle c_1(TM, J), A\rangle$. Then $g_v(A) \geqslant 0$ with equality if, and only if, the map u is an embedding.

2.3 J-holomorphic spheres in $S^2 \times S^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$

For most symplectic 4-manifolds equipped with a generic compatible almost complex structure J, there are relatively few J-holomorphic spheres. But for symplectic 4-manifolds with $b_2^+ = 1$, like $\mathbb{C}P^2$, $S^2 \times S^2$ and their k-fold blow-ups, the spaces of J-holomorphic spheres have a very rich structure. For $S^2 \times S^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ it is possible to study the J-holomorphic spheres in detail. We will show how the existence of certain J-holomorphic spheres induces a natural partition of the space $\mathcal{J}_\omega$. We present the analysis for $(S^2 \times S^2, \omega_\lambda)$. Using similar techniques as outlined below we can prove analogous results for $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)$ as well. Hence we present the theorems without any proof in the case of $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)$.

Recall that we defined the homology classes $B = [S^2 \times \{*\}]$ and $F = [\{*\} \times S^2]$. With our normalization, we have $\omega_\lambda(B) = \lambda$ and $\omega_\lambda(F) = 1$. For $S^2 \times S^2$ it is easily seen that the class $[B]$ and $[F]$ generate the homology.

Proposition 2.3.1. *Let $A = a[B]+b[F] \in H^2(S^2 \times S^2, \mathbb{Z})$ be represented by a somewhere injective J-holomorphic sphere for some $J \in \mathcal{J}_\omega$. Then exactly one of the following is true*

1. $a, b \geqslant 2$,
2. $a = 1$ *and* $b > -\lambda$,
3. $b = 1$ *and* $a \geqslant 0$.

Proof. By the adjunction formula we have that

$$0 \leqslant g_v(A) = 1 + \frac{1}{2}(A \cdot A - c_1(A)) = 1 + \frac{1}{2}(2ab - a - b) = (a-1)(b-1)$$

Also, as A is J-holomorphic, we must have $\int_A \omega = \lambda a + b > 0$. Putting these conditions together yields the required result. □

Corollary 2.3.2. *Let $\lambda = l + \delta$ where $l \in \mathbb{N}$ and $0 < \delta \leqslant 1$. Then we have*

1. *Any J-holomorphic representative of the class F is a simple curve.*
2. *The only J-holomorphic decomposition of the class B are of the form $B = (B - kF) + kF$, where $0 \leqslant k \leqslant \ell$. In this decomposition, the J-holomorphic representative of the class $(B - kF)$ is an embedded sphere, while the class kF may be represented by a collection of (possibly multiply covered) spheres representing multiples of the class F.*

Proof. Suppose $[F]$ decomposes as $\sum_i m_i[C_i]$ where C_i are somewhere injective J-curves. Let $[C_i] = a_i[B] + b_i[F]$. Suppose there exists i such that $a_i \neq 0$, Without loss of generality, we can assume $a_i \geqslant 0$ and hence we must have a $j \neq i$ such that $a_j \leqslant 0$. But that is impossible due to positivity as in Proposition 2.3.1.

Suppose $[B] = \sum_i m_i[C_i]$ where C_i are somewhere injective J-curves and let $[C_i] = a_i[B]+b_i[F]$ as above. Suppose there exists i such that $b_i \neq 0$, Without loss of generality, we can assume $b_i \geqslant 0$ and hence we must have a $j \neq i$ such that $b_j \leqslant 0$. By proposition 2.3.1, we have that $0 \geqslant b_j > -\lambda$. So we have $[B] = [B - b_j F] + \sum_{i \neq j} m_i[C_i]$. But then $\sum_{i \neq j} m_i[C_i]$ must be equal to $b_j[F]$. But as $[F]$ is indecomposable we have that the only decomposition for $b_j[F] = \sum_i q_i[F]$ where $\sum_i q_i = b_j$. Thus the only J-holomorphic decompositions of the class B are of the form $B = (B-kF)+kF$. Finally, the adjunction formula 2.2.14 implies that the $(B - kF)$ representative must be embedded. □

Proposition 2.3.3. *Let F be the class of a fiber. Then the moduli space of J-holomorphic maps $\widetilde{\mathcal{M}}(F, J)$ is either empty or a smooth manifold of dimension 8. The moduli space $\mathcal{M}(F, J)$ of J-holomorphic curves is always compact.*

Proof. As $c_1(F) = 2 \geqslant 1$, automatic regularity in dimension 4 (Theorem 2.2.9) implies that, for all $J \in \mathcal{J}_\omega$, $\widetilde{\mathcal{M}}(F, J)$ is a (possibly empty) smooth manifold. The expected dimension can then be calculated using the index theorem. In general the dimension of the moduli space $\widetilde{\mathcal{M}}(A, J)$ for generic J is given by the formula $(n - 3)2 + 2c_1(A) + 6$, where n is half the dimension of the ambient manifold. Plugging in the numbers in our case we get $\dim \widetilde{\mathcal{M}}(F, J) = 8$ whenever it is non-empty. Thus we only need to show that $\mathcal{M}(F, J) = \widetilde{\mathcal{M}}(F, J)/G$ is compact.

Let $u_n \in \widetilde{\mathcal{M}}(F, J)$ be a sequence of curves. By Gromov compactness we know that there is a subsequence of u_n (which we again denote by u_n itself for brevity) that either converges to a cusp curve or to a $C^\infty J$-curve. Suppose it converges to a cusp curve. Then the class F would decompose as $F = A_1 + \ldots A_n$, $n \geqslant 2$, which is impossible by Theorem 2.3.2. Hence the sequence u_n converges to an honest J-curve in $\mathcal{M}(F, J)$, proving compactness. Note that the curve that u_n converges to is only defined up to reparametrization and hence is a well defined element of $\mathcal{M}(F, J)$. □

Proposition 2.3.4. *Given a point $p \in (S^2 \times S^2, \omega_\lambda)$ and any almost complex structure $J \in \mathcal{J}_{\omega_\lambda}$ such that $\widetilde{\mathcal{M}}(F, J) \neq \varnothing$, there exists a unique unparametrised J-curve in the class F passing through p.*

Proof. We will show that the the evaluation map

$$\begin{aligned} \mathrm{ev}_{(F,J)} : \widetilde{\mathcal{M}}(F,J) \times_{\mathrm{PSL}(2;\mathbb{C})} S^2 &\to S^2 \times S^2 \\ (u,z) &\mapsto u(z) \end{aligned}$$

is a diffeomorphism.

The evaluation map is injective: Suppose $\mathrm{ev}_{(F,J)}(u,z) = \mathrm{ev}_{(F,J)}(u',z')$ then the images of u and u' intersect at a point.As both u and u' represent the class F, and as $F \cdot F = 0$, by Positivity of intersections (Theorem 2.2.8) the images of u and u' must coincide i.e there exists an element $\phi \in \mathrm{PSL}(2,C)$ such that $u\prime = u \circ \phi$.

Further it can be shown through explicit calculations (see page 312 in [31]) that $D\mathrm{ev}_J$ is surjective at all points and that ev_J is a proper map.

Also we note that the

- dimension of $\widetilde{\mathcal{M}}(F,J) = 8$ (As $\widetilde{\mathcal{M}}(F,J) \neq \varnothing$)
- dimension of $\mathrm{PSL}(2,C) = 6$
- dimension of $S^2 = 2$

Thus we have $\widetilde{\mathcal{M}}(F,J) \times_{\mathrm{PSL}(2;\mathbb{C})} S^2 = 8+2-6 = 4 =$ dimension of $S^2 \times S^2$. Hence we have that ev_J is smooth proper submersion between 2 manifolds of the same dimension. By Ehresmann's fibration theorem we have that ev_J is a diffeomorphism, which concludes the proof of the proposition. □

Proposition 2.3.5. *The moduli space $\widetilde{\mathcal{M}}(F,J) \neq \varnothing$ for all $J \in J_\omega$. In particular, for every compatible almost complex structure $J \in \mathcal{J}_{\omega_\lambda}$, and for any given point $p \in S^2 \times S^2$, there is a unique embedded J-holomorphic sphere representing the class F passing through p.*

Proof. Let $J_0 = j_0 \times j_0$ be the standard split complex structure on $S^2 \times S^2$ and let $J_1 \in J_\omega$ be an arbitrary compatible almost structure. Consider a path J_t from J_0 to J_1.

Let $S = \{t \in [0,1] \mid \widetilde{\mathcal{M}}(F, J_t) \neq \varnothing\}$. To show S is open in $[0,1]$, we proceed as follows. Suppose there exist t_0 such that $J_{t_0} \in S$. By automatic regularity (Theorem 2.2.9), as $F \cdot F = 0 > -1$ we can conclude that there exists a open neighbourhood N around t_0 such that for all $t \in N$, $\widetilde{\mathcal{M}}(F, J_t) \neq \varnothing$.

Next we show S is a closed set in $[0,1]$. Consider a sequence $t_n \longrightarrow t$, $t_n \in S$. This implies that there exist J_t-holomorphic curves u_t representing the homology class F. As J_{t_n} converges to J_t Gromov compactness theorem (Theorem 2.2.11) implies that there is a subsequence(which we denote again by u_n for brevity) that converges either to a cusp curve or to an honest J-holomorphic curve which we denote by u. However, by Theorem 2.3.2 we conclude that it cannot converge to a cusp curve. Consequently the limit curve u is an honest J_t holomorphic curve showing that $\mathcal{M}(F, J_t)$ is non-empty and hence S is closed.

As $[0,1]$ is connected and as $0 \in S$ we conclude that $S = [0,1]$, thus proving the theorem. The uniqueness follows from Positivity of intersection (Theorem 2.2.8). □

We also have an analogous result for $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)$.

Theorem 2.3.6. *Given any point $p \in \mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, and any $J \in \mathcal{J}^{S^1}_{\omega_\lambda}$, there is a J-holomorphic curve in the class F passing through p.*

The following Theorem due to Abreu and McDuff [4] tells us about the decomposition of the space of compatible almost complex structures on $(S^2 \times S^2, \omega_\lambda)$ into finitely many strata.

Theorem 2.3.7. *Let $\mathcal{J}_{\omega_\lambda}$ denote the space of all compatible almost complex structures (not necessarily invariant) for the form ω_λ, on $S^2 \times S^2$ then the space $\mathcal{J}_{\omega_\lambda}$ admits a finite decomposition into disjoint Fréchet manifolds of finite codimensions*

$$\mathcal{J}_{\omega_\lambda} = U_0 \sqcup U_2 \sqcup U_4 \ldots \sqcup U_{2n}$$

where $2n = \lceil 2\lambda \rceil - 1$ *and* $\lceil \lambda \rceil$ *is the unique integer* l *such that* $l < \lambda \leqslant l+1$ *and where*

$$U_k := \left\{ J \in \mathcal{J}_{\omega_\lambda} \mid \left(B - \frac{k}{2} F \right) \in H_2(S^2 \times S^2, \mathbb{Z}) \text{ is represented by a } J\text{-holomorphic sphere} \right\}$$

A completely analogous description holds true for $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

Theorem 2.3.8. *Let* $\mathcal{J}_{\omega_\lambda}$ *denote the space of all compatible almost complex structures (not necessarily invariant) for the form* ω_λ*, then the space* $\mathcal{J}_{\omega_\lambda}$ *admits a finite decomposition into disjoint Fréchet manifolds of finite codimensions*

$$\mathcal{J}_{\omega_\lambda} = U_1 \sqcup U_3 \sqcup U_5 \ldots \sqcup U_{2n-1}$$

where $2n = \lceil 2\lambda \rceil - 1$*, and* $\lceil \lambda \rceil$ *is the unique integer* l *such that* $l < \lambda \leqslant l+1$ *and where*

$$U_k := \left\{ J \in \mathcal{J}_{\omega_\lambda} \mid \left(B - \frac{k+1}{2} F \right) \in H_2(S^2 \times S^2, \mathbb{Z}) \text{ is represented by a } J\text{-holomorphic sphere} \right\}$$

Remark 2.3.9. *We label the strata in* $S^2 \times S^2$ *and* $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$*, by the homological self-intersection of the classes* $B - kF$ *or* $L - (k+1)F$*.*

Remark 2.3.10. *We note that for both* $S^2 \times S^2$ *and* $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$*, there is a canonical integrable almost complex structure* J_m *in the strata* U_m *coming from realizing* $S^2 \times S^2$ *and* $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ *as the* m^{th}*- Hirzebruch surface* W_m *of section 1.2. Further recall that associated to each* J_m *we have a unique* J_m*-holomorphic Hamiltonian toric action* $\mathbb{T}_m$*. Thus the set of possible equivalence classes of toric actions (up to* $\mathbb{T}^2$ *equivariant symplectomorphisms) on* $S^2 \times S^2$ *and* $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ *are in one-to-one correspondence with the strata in the decomposition of* $\mathcal{J}_{\omega_\lambda}$*. This fact will be crucial in our later analysis of centralizer subgroups.*

Theorem 2.3.11 (see [31])**.** *Consider* $M = (S^2 \times S^2, \omega_\lambda)$ *and the classes* $B = S^2 \times pt$ *and* $F = pt \times S^2$*. Then for any* $J \in U_0$ *the map*

$$\begin{aligned} \Psi : \widetilde{\mathcal{M}}(B,J)/\operatorname{PSL}(2,\mathbb{C}) \times \widetilde{\mathcal{M}}(F,J)/\operatorname{PSL}(2,\mathbb{C}) &\longrightarrow S^2 \times S^2 \\ ([u],[v]) &\longmapsto \operatorname{im}[u] \cap \operatorname{im}[v] \end{aligned}$$

is a diffeomorphism.

Proof. By proposition 2.3.5, we know that for every $J \in U_0$, and for any point $p \in S^2 \times S^2$, there exist a unique J-holomorphic sphere passing through p in each of the classes B and F. Hence, the map Ψ is bijective and its inverse can be written as

$$\Psi^{-1} = (\pi_B \times \pi_F) \circ (\mathrm{ev}_{(B,J)} \times \mathrm{ev}_{(F,J)})^{-1} \circ \Delta$$

where

$$\pi_B : \widetilde{\mathcal{M}}(B,J) \times_{\mathrm{PSL}(2;\mathbb{C})} S^2 \longrightarrow \widetilde{\mathcal{M}}(B,J)/\,\mathrm{PSL}(2;\mathbb{C})$$

is the usual projection map (and similarly for π_F), and where $\Delta : M \longrightarrow M \times M$ is the diagonal inclusion. We can compute the differential explicitly to conclude, using the inverse function theorem, that the map Ψ is indeed a diffeomorphism. □

Remark 2.3.12. *For $J \in U_0$, let u_w denote the J-curve in the class B through $(0,w) \in (S^2 \times S^2, \omega_\lambda)$ and v_z denote the J-curve in the class F through $(z,0)$ and let $G =$* $\mathrm{PSL}(2, C)$*. It can be shown that the map*

$$\begin{aligned} S^2 &\to \widetilde{\mathcal{M}}(B,J)/G \\ w &\mapsto u_w \end{aligned}$$

is a diffeomorphism (and similarly for $\widetilde{\mathcal{M}}(F,J)/G$). Thus we can show that the map

$$\begin{aligned} \tilde{\Psi} : S^2 \times S^2 &\longrightarrow S^2 \times S^2 \\ (z,w) &\longmapsto u_w \cap v_z \end{aligned}$$

is a diffeomorphism.

Chapter 3

Action of $\mathrm{Symp}^{S^1}(S^2 \times S^2, \omega_\lambda)$ on $\mathcal{J}_{\omega_\lambda}^{S^1}$

In this chapter we show that the space $\mathcal{J}_{\omega_\lambda}^{S^1}$ can be decomposed into strata each of which being homotopy equivalent to a homogeneous space under the action of the equivariant symplectomorphism group. In the following sections, apart from a few general observations, we shall only deal with the case of the product $(S^2 \times S^2, \omega_\lambda)$. The case of the non-trivial bundle $(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$ is postponed to Chapter 6.

3.1 Intersection of $\mathcal{J}_{\omega_\lambda}^{S^1}$ with the strata

In this section we fix a circle action on $(S^2 \times S^2, \omega_\lambda)$, and use Karshon's classification of circle actions to determine which strata that the space of invariant almost complex structures $\mathcal{J}_{\omega_\lambda}^{S^1}$ intersects.

In what follows, we will use the following simple observation several times. Let (M, ω) be a simply connected symplectic 4-manifold. There is a left-exact sequence

$$1 \to \mathrm{Symp}_h(M, \omega) \to \mathrm{Symp}(M, \omega) \to \mathrm{Aut}_{c_1,\omega}\left(H_2(M, \mathbb{Z})\right) \tag{3.1}$$

where $\mathrm{Symp}_h(M, \omega)$ is the subgroup of symplectomorphisms acting trivially on homology, and where $\mathrm{Aut}_{c_1,\omega_\lambda}\left(H_2(M, \mathbb{Z})\right)$ is the group of automorphisms of $H_2(M, \mathbb{Z})$ that preserve the intersection product and the Poincaré duals of the cohomology classes $c_1(TM)$ and

$[\omega_\lambda]$. This later group is the identity for $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)$ with $\lambda \geqslant 1$ and for $(S^2\times S^2,\omega_\lambda)$ with $\lambda > 1$. In the case of $(S^2\times S^2,\omega_\lambda)$ with $\lambda = 1$, the group $\mathrm{Aut}_{c_1,\omega_\lambda}(H_2(M,\mathbb{Z}))$ is equal to $\mathbb{Z}_2$ and is generated by the symplectomorphism that swaps the two S^2 factors. Consequently, for any symplectically ruled rational surface, the above sequence is also right-exact and splits.

Lemma 3.1.1. *We have the following equalities among symplectomorphism groups:*

- $\mathrm{Symp}(S^2\times S^2,\omega_\lambda) = \mathrm{Symp}_h(S^2\times S^2,\omega_\lambda) \ltimes \mathbb{Z}_2$ *when* $\lambda = 1$,
- $\mathrm{Symp}(S^2\times S^2,\omega_\lambda) = \mathrm{Symp}_h(S^2\times S^2,\omega_\lambda)$ *when* $\lambda > 1$, *and*
- $\mathrm{Symp}_h(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda) = \mathrm{Symp}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)$ *for all* $\lambda \geqslant 1$.

□

3.1.1 The case $\mathcal{J}_{\omega_\lambda}^{S^1}$ intersects only one strata

Proposition 3.1.2. *Suppose* $\lambda = 1$. *Then the space* $\mathcal{J}_{\omega_1}$ *of compatible, almost-complex structures on* $(S^2\times S^2,\omega_1)$ *is made of only one stratum. In particular, any Hamiltonian circle action extends to the toric action* $\mathbb{T}_0^2$ *and the subspace* $\mathcal{J}_{\omega_\lambda}^{S^1}$ *of* S^1 *invariant almost-complex structures is contractible.*

Proof. This follows directly from Theorem 2.3.7 and Remark 2.3.10. □

We use the following lemmas in order to calculate the self intersection and area of invariant spheres for a given S^1 action.

Lemma 3.1.3 (Lemma 5.4 in [25])**.** *Let* S^1 *act on* S^2 *by rotating it* k *times while fixing the north and south poles. Suppose that the action lifts to a complex line bundle* E *over* S^2. *Then* S^1 *acts linearly on the fibers over the north and south poles; let* m *and* n *be the weights for these actions. Then*

$$m - n = -ek$$

where e *is the self intersection of the zero section.*

Lemma 3.1.4. *Let S^1 act on S^2 by rotating it k times while fixing the north and south poles. Let $\mu(n)$ denote the value of the momentum map at the north pole and $\mu(s)$ the value of the momentum map at the south pole. Then the symplectic area of the S^2 is given by $\frac{\mu(n)-\mu(s)}{k}$.*

Hence given the weights of the circle action $S^1(a, b; m)$ we can calculate the self-intersection and symplectic area of the invariant spheres that appear in the graph associated to the action. As any homology class in $(S^2 \times S^2, \omega_\lambda)$ and $(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$ is determined by their self intersection and area the formulae in Lemmas 3.1.3 and 3.1.4 determine the homology class of the invariant curve.

Theorem 3.1.5. *Suppose $\lambda > 1$. Consider an $S^1(a, b; m)$ action. Under the following numerical conditions on a, b, m, λ, the space $\mathcal{J}_{\omega_\lambda}^{S^1}$ intersects only the stratum U_m:*

- *when $a \neq \pm 1$;*
- *when $a = \pm 1$ and $b \in \{0, \pm m\}$;*
- *when $a = \pm 1$ and $2\lambda \leqslant |2b - m|$.*

Proof. As $\lambda > 1$, given an $S^1(a, b; m)$ action, it suffices to understand the self intersection and symplectic area – hence the homology class – of the S^1 invariant symplectic spheres in order to characterise which strata $\mathcal{J}_{\omega_\lambda}^{S^1}$ intersects. There are two subcases to consider.

Case1: The circle action has no fixed surfaces: Under the condition that there are no fixed surfaces, the three families of circle actions considered in the theorem correspond to the graphs in Figure 2.4. If the action has only isolated fixed points, then any S^1 invariant sphere passes through two of the four fixed points P,Q,R,S. Thus from the calculation of weights at the 4 fixed points given in Table 2.1.1 and the formula in Lemma 3.1.3 we can determine the self intersection of all possible S^1 invariant curves. In particular, when $m \neq 0$, we conclude that the above S^1 actions admit a unique invariant sphere C with negative self-intersection $-m$. Also, if $m \neq 0$, by the same reasoning we can see that it is not possible to have an invariant curve in the homology class B in these graphs, thus showing that $\mathcal{J}^{S^1}_{\omega_\lambda}$ doesn't intersect the stratum to U_0. Finally, if $m = 0$, we see that there is no invariant curve with negative self intersection. Therefore, the only stratum $\mathcal{J}^{S^1}_{\omega_\lambda}$ intersects is U_m.

Case2: If the circle action has a fixed surface: This corresponds to the circle actions with graphs in Figure 2.3. As the action is effective, the weights at any point on the fixed surface are $(0, \pm 1)$. Thus we can argue as in case 1 to conclude the proof of the theorem. $\square$

3.1.2 The case when $\mathcal{J}^{S^1}_{\omega_\lambda}$ intersects two strata

The only cases left to investigate are the circle actions of type $S^1(a, b; m)$ where the parameters a, b, and λ satisfy

- (i) $a = 1$, $b \neq \{0, m\}$, and $2\lambda > |2b - m|$; or
- (ii) $a = -1$, $b \neq \{0, -m\}$, and $2\lambda > |2b + m|$.

Let us first investigate the case (i) in the subcase $b > m$. Under these conditions on a, b, m and λ, the graph for the circle action $S^1(1, b; m)$ is given in Figure 2.4 (A) in which the edges labelled "a" are removed. Similarly, when $b > m$, we see that the graph associated to the circle action $S^1(1, b; 2b - m) \subset \mathbb{T}^2_{2b-m}$ is given in Figure 2.4 (B), with the two edges labelled "a" removed. In both cases, we obtain the labelled graph of Figure 3.1 below, proving that the actions $S^1(1, b; m)$ and $S^1(1, b; 2b-m)$ are equivariantly symplectomorphic. Consequently, when $a = 1$, $b \neq \{0, m\}$, $2\lambda > |2b - m|$, and $b > m$, the circle action $S^1(a, b; m)$ admits two distinct toric extensions, namely $\mathbb{T}^2_m$ and $\mathbb{T}^2_{2b-m}$.

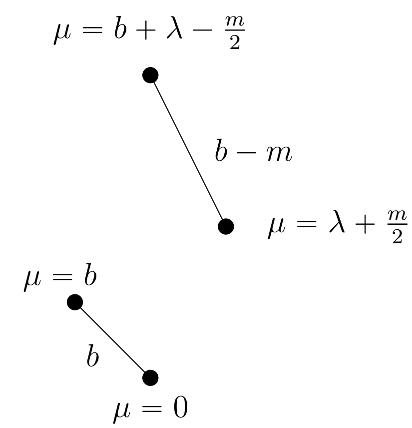


Figure 3.1: Graph for the circle action $S^1(1, b; m)$ with $b \neq \{0, m\}$

In the two other subcases $b < 0$ or $0 < b < m$, a similar argument can be used to show the existence of two toric extensions in case (i). We summarise the results in the following theorems.

Theorem 3.1.6. *Consider the S^1 actions $S^1(1, b; m)$ on $(S^2 \times S^2, \omega_\lambda)$ and suppose $2\lambda > |m - 2b|$. Then under the following numerical conditions on b and m, the $S^1(1, b; m)$ action extends to the toric action $\mathbb{T}^2_{|m-2b|}$ and is equivariantly symplectomorphic to the following subcircle in $\mathbb{T}^2_{|m-2b|}$:*

1. *if $b > 0$ and $b > m$, then $S^1(1, b; m)$ is equivalent to $S^1(1, b; |m - 2b|)$;*
2. *if $b > 0$, $m > b$, and $2b - m < 0$, then $S^1(1, b; m)$ is equivalent to $S^1(1, -b; |m - 2b|)$;*
3. *if $b > 0$, $m > b$, and $2b - m > 0$, then $S^1(1, b; m)$ is equivalent to $S^1(1, b; |m - 2b|)$;*
4. *finally, if $b < 0$, then $S^1(1, b; m)$ is equivalent to $S^1(1, -b; |m - 2b|)$.*

A completely similar discussion applies in the case (ii), namely, when $a = -1$, $b \neq \{0, -m\}$, and $2\lambda > |2b + m|$. The details are left to the reader.

Theorem 3.1.7. *Consider the S^1 actions $S^1(-1, b; m)$ on $(S^2 \times S^2, \omega_\lambda)$ and suppose $2\lambda > |m + 2b|$. Then under the following numerical conditions on b and m, the $S^1(-1, b; m)$ action extends to the toric action $\mathbb{T}^2_{|m+2b|}$ and is equivariantly symplectomorphic to the following subcircle in $\mathbb{T}^2_{|m+2b|}$:*

1. *if* $b < 0$ *and* $m > -2b$*, then* $S^1(-1, b; m)$ *is equivalent to* $S^1(-1, -b; |m + 2b|)$*;*

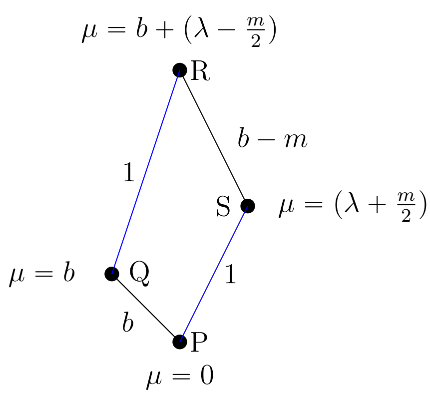
2. *if* $b < 0$*,* $m > -b$*, and* $-2b > m$*, then* $S^1(-1, b; m)$ *is equivalent to* $S^1(-1, b; |m + 2b|)$*;*

3. *if* $b < 0$ *and* $-b > m$*, then* $S^1(-1, b; m)$ *is equivalent to* $S^1(-1, b; |m + 2b|)$*;*

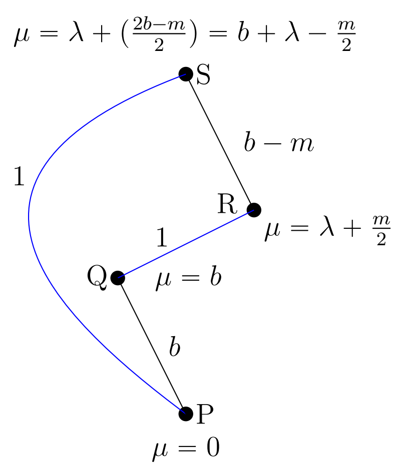
4. *if* $b > 0$*, then* $S^1(-1, b; m)$ *is equivalent to* $S^1(-1, -b; |m + 2b|)$*.*

□

Example 3.1.8. *Consider the circle actions of the form* $S^1(1, b; m)$ *with* $b > m$*. Under these conditions, there are no surfaces fixed under the* S^1 *action. We shall try to see the homology classes of the invariant spheres between the isolated fixed points P,Q,R,S in this case. The graph for the circle action* $S^1(1, b; m)$ *with* $b > m$ *can be obtained by setting the edges labelled "a" in Figure 2.4(A). As* $b > m$ *is equivalent to the condition* $2b - m > b$*, the graph of the circle action* $S^1(1, b; 2b - m)$ *can be obtained by setting the the edges labelled "a" in in Figure 2.4(B). In order to highlight the different configurations of the invariant sphere for the two actions, we present the graphs along with the free invariant spheres for the circle action* $S^1(1, b; m)$ *and* $S^1(1, b; 2b - m)$ *respectively below.*



(a) $S^1(1, b; m)$ with $b > m$



(b) $S^1(1, b; 2b - m)$ with $b > m$

and the weights at the four fixed points are given by

Vertex	*Weights for the $S^1(1, b; m)$ action*
P	$(1, b)$
Q	$(1, -b)$
R	$(-1, m - b)$
S	$(-1, -m + b)$

By Theorem 3.1.6 we know that the two actions are equivariantly symplectomorphic. Let us denote the symplectomorphism by ϕ. By Corollary 2.1.12 and the above table we see that the equivariant symplectomorphism ϕ satisfies $\phi(P) = P$, $\phi(Q) = Q$, $\phi(R) = S$ and $\phi(S) = R$. Hence the invariant sphere between Q and R in Figure 3.2b is taken to an invariant sphere between Q and S in Figure 3.2a.

Using the formulae in Theorems 3.1.3 and 3.1.4, we can calculate the homology class of invariant curves that pass through the fixed points P, Q, R and S.

In particular we have

Pair of vertices	*Self intersection of the curve that passes through the given vertices*	*Area of the curve*	*Homology class of curve*
(P,Q)	0	*1*	F
(Q,R)	$-m$	$\lambda - \frac{m}{2}$	$B - \frac{m}{2}F$
(R,S)	0	*1*	F
(S,P)	m	$\lambda + \frac{m}{2}$	$B + \frac{m}{2}F$
(P,R)	$-(2b - m)$	$\lambda - \frac{2b-m}{2}$	$B - \left(\frac{2b-m}{2}\right) F$
(Q,S)	$2b - m$	$\lambda + \frac{2b-m}{2}$	$B + \left(\frac{2b-m}{2}\right) F$

We shall now use the above theorems to understand how many strata the space of $S^1(\pm 1, b, m)$ compatible almost complex structures $\mathcal{J}^{S^1}_{\omega_\lambda}$ intersects.

Fix an action of the form $S^1(\pm 1, b; m)$. Assume $2\lambda > |2b - m|$ if the action is of the form $S^1(1, b; m)$ or $2\lambda > |2b + m|$ if the action is of the form $S^1(-1, b; m)$. We know by Theorem 3.1.6 that the $S^1(1,b,m)$ action is equivariantly symplectomorphic to $S^1(1,b',m')$ (where $b' = \pm b$ and $m' \in \{|2b-m|,|2b+m|\}$ depending on the numerical condition as given in Theorems 3.1.6 and 3.1.7). Let J_m and $J_{m'}$ be the standard integrable complex structures on $S^2 \times S^2$ (as introduced in Chapter 1) which are equivariant under the $S^1(1,b;m)$ and $S^1(1,b';m')$ action. Let $(S^2 \times S^2)_m$ denote $S^2 \times S^2$ endowed with the action $S^1(1,b;m)$ and similarly let $(S^2 \times S^2)_{m'}$ denote $S^2 \times S^2$ endowed with the action $S^1(1,b',m')$. Then Theorem 3.1.6 tell us that there is a S^1 equivariant symplectomorphism $\phi : ((S^2 \times S^2)_m, \omega_\lambda) \to ((S^2 \times S^2)_{m'}, \omega_\lambda)$.

Theorem 3.1.9. *Consider the $S^1(\pm 1,b,m)$ action on $(S^2 \times S^2, \omega_\lambda)$, then $\mathcal{J}_{\omega_\lambda}^{S^1}$ intersects exactly two strata. More precisely,*

1. *if $a = 1$, $b \neq \{0,m\}$, $\lambda > 1$, and $2\lambda > |2b-m|$, then the space of $S^1(1,b;m)$-equivariant almost complex structures $\mathcal{J}_{\omega_\lambda}^{S^1}$ intersects the two strata U_m and $U_{|m-2b|}$.*
2. *If $a = -1$,$b \neq \{0,-m\}$, $\lambda > 1$, and $2\lambda > |2b+m|$, then the space of $S^1(-1,b;m)$-equivariant almost complex structures $\mathcal{J}_{\omega_\lambda}^{S^1}$ intersects the two strata U_m and $U_{|m+2b|}$.*

Proof. We shall present the proof only in case one, the proof in the second case being similar. Consider the almost complex structure $\phi^* J_{2b-m} := \phi_* \circ J_{|2b-m|} \circ \phi_*^{-1}$, where $\phi : ((S^2 \times S^2)_m, \omega_\lambda) \to ((S^2 \times S^2)_{|2b-m|}, \omega_\lambda)$ is the equivariant symplectomorphism as defined above. We can check that $\phi^* J_{2b-m}$ is invariant with respect to the $S^1(1,b;m)$ action.

Let $\overline{D}_{|2b-m|}$ be the standard $J_{|2b-m|}$-holomorphic curve in the class $B - \frac{|2b-m|}{2}F$ which is invariant under the $S^1(1,b';|2b-m|)$ action. By Lemma 3.1.1, when $\lambda > 1$, the group $\mathrm{Symp}_h(S^2 \times S^2, \omega_\lambda)$ is equal to $\mathrm{Symp}(S^2 \times S^2, \omega_\lambda)$. Thus the symplectomorphism ϕ preserves homology. In particular we have that $\phi^{-1}(\overline{D}_{|2b-m|})$ is holomorphic with respect to the integrable complex structure $\phi^* J_{|2b-m|}$ and invariant under the $S^1(1,b;m)$ action. Finally as ϕ preserves homology we have that $\phi^{-1}(\overline{D}_{|2b-m|})$ is also in the homology class $B - \frac{|2b-m|}{2}F$. This shows that $\phi^* J_{|2b-m|} \in \mathcal{J}_{\omega_\lambda}^{S^1} \cap U_{|2b-m|}$. Finally we note that

$J_m \in \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_m$, thus proving the $\mathcal{J}^{S^1}_{\omega_\lambda}$ intersects the strata U_m and $U_{|2b-m|}$.

We now prove that these are the only strata that $\mathcal{J}^{S^1}_{\omega_\lambda}$ intersects. Suppose $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_k \neq \varnothing$ for $k \notin \{m, |2b-m|\}$, then there exists an $S^1(1, b; m)$ invariant curve in the class $B - \frac{k}{2}F$. Note that any invariant sphere must pass through two of the four fixed points P,Q,R and S. Further, by Theorem 3.1.3, the calculation of weights and the moment map values at the fixed points determine the homology class of the invariant curve passing through the fixed points. Arguing as in Example 3.1.8 we conclude that the homology class of an invariant curve connecting any two of the 4 fixed points must be either $B - \frac{m}{2}F$, $B - \frac{|m-2b|}{2}F$, F, $B + \frac{m}{2}F$, or $B + \frac{|m+2b|}{2}F$. Hence we can conclude the such an invariant curve in the class $B - \frac{k}{2}F$ for $k \notin \{m, |2b-m|\}$ cannot exist. This completes the proof.

□

3.2 Symplectic actions of compact abelian groups on $\mathbb{R}^4$

In order to study the action of the the equivariant symplectomorphism group $\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$ on each invariant stratum $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_k$, we will need to understand the equivariant topology of linearised symplectic actions. In this section, we consider an arbitrary compact abelian group G. The following two theorems were proven by W.Chen in the manuscript [11]. As this paper was never published, we shall reproduce their proof here.

Let G be an abelian group acting effectively and symplectically on $\mathbb{C}^2 = \mathbb{R}^4$ with the symplectic form $\omega_0 := dx_1 \wedge dy_1 + (dx_2 \wedge dy_2)$. We say it acts linearly if it acts as a subgroup of $\mathrm{U}(2) \subset \mathrm{Sp}(4)$. As G is abelian, the representation $G \hookrightarrow \mathrm{U}(2)$ decomposes into irreducible complex 1 dimensional representations.

Theorem 3.2.1. *Let G be a compact abelian group acting linearly on $(\mathbb{R}^4, \omega_0)$. Suppose V is a G invariant compact star shaped neighbourhood of* 0. *Let $f : \mathbb{R}^4 \backslash V \to \mathbb{R}^4$ be a*

G-equivariant symplectic embedding which is the identity near infinity. Then, for every G-invariant neighbourhood W of V, there exists a G-equivariant symplectomorphism $g : \mathbb{R}^4 \to \mathbb{R}^4$ such that $g|_{\mathbb{R}^4\backslash W} = f$.

Proof. As $0 \in int(V)$ and as f is the identity near infinity, there exist $T > 0$ such that $f(Tx) = Tx$ for all $x \in \mathbb{R}^4\backslash V$. Define $f_t(x) = \frac{f(tx)}{t}$ for $1 \leqslant t \leqslant T$. Then we observe that as the G action is linear, f_t is equivariant for all $t \in [1, T]$, $f_1 = f$, $f_T = id$ and $f_t^*\omega_0 = \omega_0$ for all t. Thus there are compact sets $V_t = f_t(V)$ and open neighbourhoods $W_t = f_t(W)$ of V_t such that the restriction $f_t : \mathbb{R}^4\backslash V \to \mathbb{R}^4\backslash V_t$ and $f_t : \mathbb{R}^4\backslash W \to \mathbb{R}^4\backslash W_t$ are diffeomorphic. As G acts linearly, each of the sets W_t and V_t are G-invariant.

Define X_t as the vector field that satisfies $\frac{d}{dt}f_t = X_t \circ f_t$ and consider the one form $\alpha_t = i_{X_t}\omega_0$. As f_t is G equivariant, both X_t and α_t are G-equivariant. Let $H_t : \mathbb{R}^4\backslash V_t \to \mathbb{R}$ be a one parameter family of Hamiltonians that are G-invariant and that satisfy $\alpha_t = dH_t$. Note that as f_t is the identity near infinity, this implies that H_t is constant near infinity and we can take this constant to be 0.

Finally we can take a family of G-invariant bump functions $\rho_t : \mathbb{R}^4 \to [0, 1]$ such that $\rho_t \equiv 0$ in a neighbourhood of V_t and $\rho_t \equiv 1$ on $\mathbb{R}^4\backslash W_t$. Then the Hamiltonian $\rho_t H_t : \mathbb{R}^4 \to \mathbb{R}$ is defined on the whole $\mathbb{R}^4$ and is also G-invariant. The Hamiltonian isotopy g_t generated by $\rho_t H_t$ is G equivariant for all $1 \leqslant t \leqslant T$ and satisfies the properties $g_T = \text{id}$ and $g_1|_{\mathbb{R}^4\backslash W} = f$. Thus g_1 is the required symplectomorphism g in the statement of the theorem. □

Theorem 3.2.2. *Let (V, ω) be a compact star shaped open neighbourhood of the origin and let ω be such that $\omega = \omega_0$ near the boundary of V. Let G be a compact abelian group acting on (V, ω) via symplectomorphisms that are linear near the boundary of V. Then the G action is conjugate to a linear symplectic action of G on (V, ω_0) by a diffeomorphism Φ which is the identity near the boundary and which satisfies $\Phi^*\omega = \omega_0$.*

Proof. Identify $\mathbb{R}^4$ with $\mathbb{C}^2$. The linear action near ∂V extends to a unitary action on $\mathbb{R}^4$. As G is abelian this linear action splits into two eigenspaces namely $\mathbb{C} \oplus \mathbb{C}$. Fix the above decomposition of $\mathbb{C}^2$. Then we can compactify each eigenspace $\mathbb{C}$ to an S^2 and hence this G action extends to a symplectic action $S^2 \times S^2$ with respect to the form $\tilde{\omega}$ induced by ω. Similarly lets denote by $\tilde{\omega}_0$ the form induced on $S^2 \times S^2$ by ω_0. $S^2 \times S^2$ comes equipped with two actions of G namely $\rho : G \hookrightarrow \mathrm{Symp}(S^2 \times S^2, \tilde{\omega})$ which is the action coming from extending the G action on $V$and $\rho_{\mathrm{lin}} : G \hookrightarrow \mathrm{Symp}(S^2 \times S^2, \tilde{\omega})$ which is the extension of the linear G action on $\mathbb{C}^2$ to $S^2 \times S^2$. Note that there exists a star shaped subset $V_1 \subset V$ such that both the action ρ and ρ_{lin} agree on $\mathbb{C}^2 \backslash V_1$.

The point at infinity $p := (\infty, \infty)$ is a fixed point for both the action ρ and ρ_{lin}. We then fix a $\tilde{\omega}$ compatible G-invariant almost complex structure on $S^2 \times S^2$ such that J is the standard almost complex structure in a neighborhood $X_\epsilon := (S^2 \times D_\epsilon) \cup (S^2 \times D_\epsilon)$ of the wedge $(S^2 \times \{\infty\}) \vee (\{\infty\} \times S^2) \subset S^2 \times S^2$ (where D_ϵ is thought of as a small ball around p of radius epsilon). As $(S^2 \times \{\infty\})$ and $(\{\infty\} \times S^2)$ are holomorphic spheres for the induced J representing the classes B and F respectively, we conclude that there exists foliations $\mathcal{B}_J$ and $\mathcal{F}_J$ by embedded J-holomorphic spheres in the classes B and F. Given any $q = (z, w) \in S^2 \times S^2$, let u_w denote the unique curve in the class B passing through $(0, w)$ and, similarly, let v_z be the curve in class F passing through $(z, 0)$. We can define a diffeomorphism (see Chapter 9 [31] for more details as to why the map is a diffeomorphism) of $S^2 \times S^2$ by setting

$$\begin{aligned} \Psi_J : S^2 \times S^2 &\longrightarrow S^2 \times S^2 \\ (z, w) &\longmapsto u_w \cap v_z \end{aligned}$$

This Ψ_J is in fact an equivariant diffeomorphism of $S^2 \times S^2$ where the action on the domain in the linear action given by ρ_{lin} and the action on the target $S^2 \times S^2$ is the action given by ρ. Moreover, as J is the standard complex structure in a neighbourhood X_ϵ of the wedge, Ψ_J is the identity near the base point p. We modify Ψ_J as follows in order to make it the identity in the neighbourhood X_ϵ. As J is the standard complex structure on X_ϵ we have that

$$\Psi_J(z,w) = \begin{cases} (z, \phi_2(z,w)) & \text{if } z \in D_\epsilon \subset S^2 \times S^2 \\ (\phi_1(z,w), w) & \text{if } w \in D_\epsilon \subset S^2 \times S^2 \end{cases}$$

where $\phi_1(z,0) = z$ and $\phi_2(0,w) = w$ for all $z, w \in S^2$.

Choose a G equivariant (for the G action on $\{\infty\} \times S^2$)) smooth map $\beta_1 : S^2 \longrightarrow S^2$ such that $\beta_1(z) = z$ for all $z \in D_\epsilon$ and $\beta_1 = \infty$ in a neighbourhood D_δ contained in D_ϵ and such that $\det(\beta_{1,*}(z)) \geqslant 0 \ \forall \ z \in S^2$. Similarly define a G equivariant map(for the G action on $S^2 \times \{\infty\}$) $\beta_2 : S^2 \to S^2$ satisfying analogous conditions as β_1. Then we define the modified Ψ by setting

$$\Psi'_J(z,w) = \begin{cases} \Psi_J & \text{if } z \in (S^2 \times S^2) \backslash X_\epsilon \\ (z, (\phi_2(\beta_1(z), w)) & \text{if } z \in D_\epsilon \\ (\phi_1(z, \beta_2(w)), w) & \text{if } w \in D_\epsilon \end{cases}$$

This modification makes Ψ'_J identity in a smaller neighbourhood $X_\delta := .(S^2 \times D_\delta) \cup (S^2 \times D_\delta)$ $(\delta < \epsilon)$. The submanifolds $\{z\} \times S^2$ and $S^2 \times \{w\}$ for all $z, w \in S^2$ are symplectic for the form ${\Psi'_J}^*\tilde{\omega}$ and hence $\tilde{\omega}_0 \wedge \Psi'^*_J\tilde{\omega} > 0$. Thus the path $\omega_t := t\tilde{\omega} + (1-t)\Psi'^*_J\tilde{\omega}$ is a path of non-degenerate symplectic forms for all $t \in [0,1]$.

We use a equivariant Moser isotopy to get an equivariant diffeomorphism α of $S^2 \times S^2$ such that $\alpha^*{\Psi'_J}^*\tilde{\omega} = \tilde{\omega}_0$. Further, as $\Psi'^*_J\tilde{\omega} = \tilde{\omega}_0$ on X_δ we have α restricted to X_δ to be the identity. We define $\tilde{\Psi}_J := \Psi'_J \circ \alpha$.

The restriction of $\tilde{\Psi}_J : \mathbb{C}^2 \to \mathbb{C}^2$ gives us a map which is G-equivariant with respect to the action ρ_{lin} on the domain $\mathbb{C}^2$ and ρ on the target $\mathbb{C}^2$. As noted before there exists a star shaped subset $V_1 \subset V$ such that both the action ρ and ρ_{lin} agree on $\mathbb{C}^2 \backslash V_1$. We can choose V_1 such that $0 \in int(V_1)$. We now apply Theorem 3.2.1 to the map $f := \tilde{\Psi}_J^{-1}|_{\mathbb{C}^2 \setminus V_1}$ and we choose W in Theorem 3.2.1 to be a G-invariant open subset of V which contains V_1. Let $g : \mathbb{C}^2 \to \mathbb{C}^2$ be an equivariant symplectomorphism as in Theorem 3.2.1 such that $g|_{\mathbb{R}^4 \setminus W} = f$, then the map $\Phi := \left(\tilde{\Psi}_J \circ g\right)|_V : V \to V$ is identity near the boundary and satisfies $\Phi^*\omega = \omega_0$ and is G-equivariant where the action of G on the domain of ϕ is the linear action ρ_{lin} while on the range of Φ it is the G action ρ on V that we started out with. Thus Φ is the required equivariant symplectomorphism that linearizes the given G action and takes the form ω to ω_0. $\square$

Any unitary representation of a compact abelian group G on $\mathbb{C}^2$ induces a splitting into eigenspaces $\mathbb{C}^2 = \mathbb{C} \cdot V_1 \bigoplus \mathbb{C} \cdot V_2$. Consider a polydisk $D^2 \times D^2$ such that each D^2 is contained in $\mathbb{C} \cdot V_i$. Consider the symplectic form ω on $\mathbb{R}^4$ given such that outside of some smaller polydisk of the form $D_r \times D_r \subset D^2 \times D^2$ for some radius r, $\omega = \omega_0$.

Theorem 3.2.3. *Let G be an abelian group. Let ω be a symplectic form on $D^2 \times D^2$ which is equal to ω_0 near the boundary. Let G act symplectically on $(D^2 \times D^2, \omega)$ and suppose the action is linear near the boundary. Then the group $\mathrm{Symp}^G_c(D^2 \times D^2, \omega)$ of equivariant symplectomorphisms that are equal to the identity near the boundary of $D^2 \times D^2$ is non-empty and contractible.*

Proof. As the G action outside of $D_r \times D_r \subset \mathbb{R}^4$ is linear, we can extend this G action to the whole of $\mathbb{R}^4$. Identify $\mathbb{R}^4$ with $\mathbb{C}^2$ and as G is abelian the linear action splits into 2 eigenspaces namely $\mathbb{C} \oplus \mathbb{C}$. Fix the above decomposition of $\mathbb{C}^2$. Then we can compactify each eigenspace $\mathbb{C}$ to an S^2 and hence this G action extends to a symplectic action $S^2 \times S^2$ with respect to the form $\tilde{\omega}$ induced by ω.

By Theorem 3.2.2 we can conjugate our G action by a symplectomorphism which is identity near the boundary to get a linear G action on the whole of V. As any two conjugate topological subgroups are homeomorphic, we shall just study the homotopy type of the compactly supported equivariant symplectomorphism group $\mathrm{Symp}^G_{c,\text{lin}}(D^2 \times D^2, \omega)$

for the linear G action on V.

Let $\mathcal{J}^G_\omega$ be the non-empty and contractible space of all equivariant almost complex structures on $D^2 \times D^2$ that are compatible with ω and are the standard split almost complex structure J_0 outside of $D_r \times D_r$. As they are the standard almost complex structures outside of a neighbourhood these almost complex structure extend to $S^2 \times S^2$ and are compatible with $\tilde{\omega}_0$. Further once we pick a base point $p = (\infty, \infty) \in S^2 \times S^2$ and identify $D^2 \times D^2$ with the complement of a standard neighborhood $X_\epsilon := (S^2 \times D_\epsilon) \cup (S^2 \times D_\epsilon)$ of the wedge $(S^2 \times \{\infty\}) \vee (\{\infty\} \times S^2) \subset S^2 \times S^2$ (note that the wedge point (∞, ∞) is a fixed point for the extended action of G on $S^2 \times S^2$), then any element $J \in \mathcal{J}^G_\sigma$ extends to a equivariant almost complex structure of $S^2 \times S^2$ which is the standard product complex structure on $S^2 \times S^2$ on a neighbourhood X_ϵ of the wedge $(S^2 \times \{y\}) \vee (\{x\} \times S^2) \subset S^2 \times S^2$. Conversely any such equivariant almost complex structure compatible with $\tilde{\omega}$ that is standard in some neighbourhood of the wedge $(S^2 \times \{\infty\}) \vee (\{\infty\} \times S^2) \subset S^2 \times S^2$ gives us an element of $\mathcal{J}^G_\omega$.

In order to show that $\mathrm{Symp}^G_{c,\mathrm{lin}}(D^2 \times D^2, \omega)$ is contractible, we shall prove that it is homotopy equivalent to the contractible space $\mathcal{J}^G_\omega$.

Define the map $\tilde{\Psi}_J$ as in the proof of Theorem 3.2.2. Thus we have a map

$$\begin{aligned} \tau : \mathcal{J}_\omega &\longrightarrow \mathrm{Symp}^G_{c,\mathrm{lin}}(D^2 \times D^2, \omega) \\ J &\longmapsto \tilde{\Psi}_J \end{aligned}$$

To prove that τ is a homotopy equivalence we construct a homotopy inverse as follows.

$$\begin{aligned} \beta : \mathrm{Symp}^G_{c,\mathrm{lin}}(D^2 \times D^2, \omega) &\longrightarrow \mathcal{J}_\omega \\ \phi &\longmapsto \phi_* J_0 \end{aligned}$$

By construction we see that $\tau(\beta(\phi)) = \mathrm{id}$ and the other direction is homotopic to the identity as J_ω is contractible. $\square$

We shall repeatedly use the following theorem in our analysis of the homotopy type of the equivariant symplectomorphism groups of $S^2 \times S^2$.

Theorem 3.2.4. *(Equivariant Gromov Theorem) Let (V, ω) be an compact star shaped symplectic manifold of $\mathbb{R}^4$ such that $0 \in int(V)$ and let ω be such that $\omega = \omega_0$ near the boundary of V. Let G be an abelian group that acts symplectically and linearly near the boundary and send the boundary to itself, then the space of equivariant symplectomorphisms that act as identity near the boundary (denoted by $\mathrm{Symp}^G_c(V, \omega)$) is non-empty and contractible.*

Proof. By Theorem 3.2.2 we can conjugate our G action by a symplectomorphism which is identity near the boundary to get a linear G action on the whole of V and such that it takes the form ω to ω_0. As the homotopy type of the two conjugate equivariant symplectomorphism group is the same (they are in fact homeomorphic), we shall just study the homotopy type of the compactly supported equivariant symplectomorphism group for the linear G action on (V, ω_0). We denote this group by $\mathrm{Symp}^G_{c,\mathrm{lin}}(V, \omega_0)$.

Choose real numbers $r > 0$ and $T > 1$, $D_r \times D_r$ is a polydisk of radius r, such that $\frac{1}{T}V \subset D_r \times D_r \subset int(V)$, and consider the family of maps $F_t : \mathrm{Symp}^G_{c,\mathrm{lin}}(V, \omega_0) \to \mathrm{Symp}^G_{c,\mathrm{lin}}(V, \omega_0)$ for $1 \leqslant t \leqslant T$ defined by $F_t(\phi)(x) = \frac{\phi(tx)}{t}$ for all $x \in V$.

Then we have that F_t is G equivariant for all $1 \leqslant t \leqslant T$, $F_1(\phi) = \phi$ for all $\phi \in \mathrm{Symp}^G_{c,\mathrm{lin}}(V, \omega_0)$, $F_t(id) = id$ for all t, and $F_T\left(\mathrm{Symp}^G_{c,\mathrm{lin}}(V, \omega_0)\right) \subset \mathrm{Symp}^G_{c,\mathrm{lin}}(D_r \times D_r, \omega)$.

The proof of Theorem 3.2.3 tells us that the inclusion $i : \mathrm{Symp}^G_{c,\mathrm{lin}}(D_r \times D_r, \omega) \hookrightarrow \mathrm{Symp}^G_{c,\mathrm{lin}}(V, \omega_0)$ is contractible. Hence we can fix a contraction α_t for $T \leqslant t \leqslant T+1$ such that $\alpha_T = i$ and $\alpha_{T+1}(\phi) = id$ for all $\phi \in \mathrm{Symp}^G_{c,\mathrm{lin}}(D_r \times D_r, \omega)$. Then the concatenation

$$\tilde{F}_t := \begin{cases} F_t \quad 1 \leqslant t \leqslant T \\ F_T \circ \alpha_t \quad T \leqslant t \leqslant T+1 \end{cases}$$

gives us a retraction of $\mathrm{Symp}^G_{c,\mathrm{lin}}(V, \omega_0)$ to id and hence $\mathrm{Symp}^G_c(V, \omega)$ is contractible. $\square$

3.3 Homotopical description of $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_k$

We now consider the action of the group of equivariant symplectomorphisms on the contractible space $\mathcal{J}^{S^1}_{\omega_\lambda}$ of invariant, compatible, almost-complex structures, and we investigate the orbit-type stratification of this action up to homotopy. Recall that, by the Lalonde-McDuff classification Theorem 2.1.6, it is sufficient to consider $\lambda \geqslant 1$.

3.3.1 Notation

We shall use the following notation in the rest of the document. Let M denote the manifolds $S^2 \times S^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Let G be a compact abelian group acting symplectically on (M, ω_λ). Let p_0 be a fixed point for the group action. Given a G invariant symplectic curve C, and a ω_λ orthogonal G invariant sphere $\overline{F}$ in the homology class F that intersects C at a point p_0, we define the following spaces:

- $N(C)$:= The symplectic normal bundle to a symplectic submanifold C.
- $\mathrm{Symp}^G_h(M, \omega_\lambda)$:= The group of G equivariant symplectomorphisms on (M, ω_λ) that acts trivially on homology.
- $\mathrm{Stab}^G(C)$:= The group of all $\phi \in \mathrm{Symp}^G_h(M, \omega_\lambda)$ such that $\phi(C) = C$, that is, such that ϕ stabilises C but does not necessarily act as the identity on C.
- $\mathrm{Fix}^G(C)$:= The group of all $\phi \in \mathrm{Symp}^G_h(M, \omega_\lambda)$ such that $\phi|_C = id$, that is, such that fixes C pointwise.
- $\mathrm{Fix}^G(N(C))$:= The group of all $\phi \in \mathrm{Symp}^G_h(M, \omega_\lambda)$ such that $\phi|_C = \mathrm{id}$ and $d\phi|_{N(C)} : N(C) \to N(C)$ is the identity on $N(C)$.
- $\mathrm{Gauge}^G(N(C))$:= The group of G-equivariant symplectic gauge automorphisms of the symplectic normal bundle of C.

- $\mathrm{Gauge}^G(N(C \vee \overline{F}))$:= The group of G-equivariant symplectic gauge automorphism of the symplectic normal bundle of the crossing divisor $C \vee \overline{F}$ that are identity in a neighbourhood of the wedge point.

- $\mathcal{S}^G_K$:= The space of unparametrized G-invariant symplectic embedded spheres in the homology class K.

- $\mathcal{S}^G_{K,p_0}$:= The space of unparametrized G-invariant symplectic embedded spheres in the homology class K passsing through p_0.

- $\mathcal{J}^G_{\omega_\lambda}(C)$:= The space of G-equivariant ω_λ compatible almost complex structures s.t the curve C is holomorphic.

- $\mathrm{Symp}^G(C)$:= The space pf all G-equivariant symplectomorphisms of the curve C.

- $\mathrm{Fix}^G(N(C \vee \overline{F}))$:= The space of all G-equivariant symplectomorphisms that are the identity in the neighbourhood of $C \vee \overline{F}$.

- $\mathrm{Symp}^{S^1}(\overline{F}, N(p_0))$:= equivariant symplectomorphism of the sphere $\overline{F}$ that are the identity in an open set of $\overline{F}$ around p_0.

- $\overline{\mathcal{S}^G_{F,p_0}}$:= The space of unparametrized G-invariant symplectic spheres in the homology class F that are equal to a fixed curve $\overline{F}$ in a neighbourhood of p_0.

- $\mathrm{Symp}^G_{p_0,h}(M, \omega_\lambda)$:= The group of all $\phi \in \mathrm{Symp}^G_h(M, \omega_\lambda)$ fixing p_0.

- $\mathrm{Stab}^G_{p_0}(C)$:= The group of all $\phi \in \mathrm{Stab}^G(C)$ such that $\phi(p_0) = p_0$.

All the above spaces are equipped with the C^∞ topology.

3.3.2 Case 1: $\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$ action on $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{2k}$ with $k \neq 0$

Let D_{2k} denote the homology class $B - kF$ and let $\mathcal{S}^{S^1}_{D_{2k}}$ denote the space of all S^1 invariant symplectic embedded spheres in the class D_{2k}. The $2k$ in the notation stands for the self-intersection of the curve i.e $(B - kF) \cdot (B - kF) = -2k$. Further we shall assume that $\mathcal{S}_{D_{2k}}^{S^1}$ is non-empty. We already addressed the question as to when these spaces are non-empty in Theorems 3.1.5 and 3.1.9.

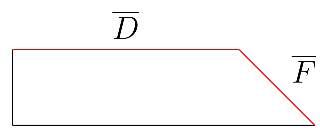
We fix a S^1 action $S^1(a,b;m)$ on $(S^2 \times S^2, \omega_\lambda)$ where $\lambda > 1$. Let $\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)$ denote the space of S^1 equivariant symplectomorphisms of $(S^2 \times S^2, \omega_\lambda)$. We first show that $\mathcal{S}_{D_{2k}}^{S^1}$ is a homogeneous space.

Lemma 3.3.1. $\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)$ *acts transitively on* $\mathcal{S}_{D_{2k}}^{S^1}$.

Proof. Given a S^1-invariant sphere $C \in \mathcal{S}_{D_{2k}}^{S^1}$, choose an invariant almost complex structure $J \in \mathcal{J}_c^{S^1}$ for which C J-holomorphic. Let p_0 be a fixed point for the symplectic S^1 action on C. The existence of such a fixed point is guaranteed by the fact that any symplectic S^1 action on S^2 has a fixed point. Let A be a J holomorphic curves in the class F passing through p_0. As p_0 is a fixed point we have that A is S^1 invariant. Using Lemma B.0.2 we can equivariantly isotope A to A' such that C and A' are ω_0 orthogonal. Now using the symplectic neighbourhood theorem we can get a neighbourhood V of $C \cup A'$ and a diffeomorphism $\alpha : S^2 \times S^2 \to S^2 \times S^2$ such that $\alpha|_V$ is a symplectomorphism of V onto an open neighbourhood V' of $\overline{D} \cup \overline{F}$, the standard configuration of curves in the class D_{2k} and F in W_{2k}. This configuration is outlined by red in the moment map image below. Note that the support of α can be taken to be in an arbitrary small neighbourhood of the configuration $C \cup A'$. Also we observe that the complement of V' is diffeomorphic to a star shaped set in $\mathbb{R}^4$. Take a smaller subset $U \subset V$ in $S^2 \times S^2$ such that $S^2 \times S^2 \backslash U$ is diffeomorphic to a star shaped set W in $\mathbb{R}^4$. Then we have that $\alpha^*(\omega_\lambda)|_{S^2 \times S^2 \backslash U}$ is a symplectic form which is equal to the standard symplectic form ω_0 near the boundary of W. By Theorem 3.2.4 we get that that there is an equivariant symplectomorphism γ such that γ is equivariant, equal to the identity near the boundary and satisfies $\gamma^*\alpha^*(\omega_\lambda) = \omega_\lambda$. As γ is the identity near the boundary of W it extends to $S^2 \times S^2$ and if we take our map α to have support in U, then the maps α and γ patch together to give us a map $\tilde{\gamma} : S^2 \times S^2 \to S^2 \times S^2$ defined by setting

$$\tilde{\gamma} = \begin{cases} \gamma \text{ on } S^2 \times S^2 \backslash V \\ \alpha \text{ on } V \end{cases}$$

By construction, $\tilde{\gamma}$ takes C to $\overline{D}$ and $\gamma \in \mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$.



$\square$

Remark 3.3.2. *We note that the above proof can be carried out depending continuously on a finite number of parameters whose values run through a compact set.*

The homotopy type of $\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$ is related to the strata $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{2k}$ through the following sequence of fibrations. We use the symbol "$\simeq$" to mean "weakly homotopy equivalent" throughout the rest of the document. In the fibrations below, we use the notation established in Section 3.3.1.

$$\mathrm{Stab}^{S^1}(\overline{D}) \longrightarrow \mathrm{Symp}^{S^1}_{h,p_0}(S^2 \times S^2, \omega_\lambda) \twoheadrightarrow \mathcal{S}^{S^1}_{D_{2k}} \xrightarrow{\simeq} \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{2k}$$

$$\mathrm{Fix}^{S^1}(\overline{D}) \longrightarrow \mathrm{Stab}^{S^1}(\overline{D}) \twoheadrightarrow \mathrm{Symp}^{S^1}(\overline{D}) \xrightarrow{\simeq} S^1 \text{ or } \mathrm{SO}(3)$$

$$\mathrm{Fix}^{S^1}(N(\overline{D})) \longrightarrow \mathrm{Fix}^{S^1}(\overline{D}) \twoheadrightarrow \mathrm{Gauge}^{S^1}(N(\overline{D})) \xrightarrow{\simeq} S^1$$

$$\mathrm{Stab}^{S^1}(\overline{F}) \cap \mathrm{Fix}^{S^1}(N(\overline{D})) \longrightarrow \mathrm{Fix}^{S^1}(N(\overline{D})) \twoheadrightarrow \overline{\mathcal{S}^{S^1}_{F,p_0}} \xrightarrow{\simeq} \mathcal{J}^{S^1}(\overline{D}) \simeq \{*\}$$

$$\mathrm{Fix}^{S^1}(\overline{F}) \longrightarrow \mathrm{Stab}^{S^1}(\overline{F}) \cap \mathrm{Fix}^{S^1}(N(\overline{D})) \twoheadrightarrow \mathrm{Symp}^{S^1}(\overline{F}, N(p_0)) \xrightarrow{\simeq} \{*\}$$

$$\{*\} \xleftarrow{\simeq} \mathrm{Fix}^{S^1}(N(\overline{D} \vee \overline{F})) \longrightarrow \mathrm{Fix}^{S^1}(\overline{F}) \twoheadrightarrow \mathrm{Gauge}^{S^1}(N(\overline{D} \vee \overline{F})) \xrightarrow{\simeq} \{*\}$$

Here $\overline{F}$ and $\overline{D}$ are the ω_λ-orthogonally intersecting invariant curves in the $2k^{\text{th}}$-Hirzebruch surface W_{2k} and whose momentum map images are depicted below in red. We note that

$\overline{F}$ is the only fiber the given curve $\overline{D} \in \mathcal{S}_{D_{2k}}^{S^1}$ intersecting ω_λ-orthogonally at p_0. In the second fibration, the group $\mathrm{Symp}^{S^1}(\overline{D})$ is homotopy equivalent to SO(3) when the S^1 action fixes the curve $\overline{D}$ pointwise. Otherwise, it is homotopy equivalent to S^1.

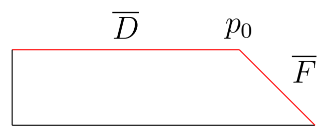


Figure 3.3: The isolated fixed point p_0

Assuming the homotopy equivalence in the first fibration, we immediately get

Theorem 3.3.3. *Consider the $S^1(a,b;m)$ action on $(S^2 \times S^2, \omega_\lambda)$ with $\lambda > 1$. If $\mathcal{J}_{\omega_\lambda}^{S^1} \cap U_{2k} \neq \phi$, then* $\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)/\mathrm{Stab}^{S^1}(\overline{D}) \simeq \mathcal{J}_{\omega_\lambda}^{S^1} \cap U_{2k}$.

Furthermore, tracking down the various homotopy equivalences in the other fibrations, we will prove that the equivariant stabilizer of the curve $\overline{D}$, namely $\mathrm{Stab}^{S^1}(\overline{D})$, is homotopy equivalent to the equivariant stabilizer of the corresponding complex structure under the natural action of $\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)$. More precisely,

- $\mathrm{Stab}^{S^1}(\overline{D}) \simeq \mathbb{T}_{2k}^2$ when $(a,b) \neq (0, \pm 1)$;
- $\mathrm{Stab}^{S^1}(\overline{D}) \simeq SO(3) \times S^1$ when $(a,b) = (0, \pm 1)$.

We shall now justify each of the homotopy equivalences in the above fibrations.

Lemma 3.3.4. *Let $\overline{D}$ be a fixed symplectic sphere in the class $B - kF$, then the evaluation map*

$$\begin{aligned} \theta : \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda) &\twoheadrightarrow \mathcal{S}_{D_{2k}}^{S^1} \\ \phi &\mapsto \phi(\overline{D}) \end{aligned}$$

is a Serre fibration with fibre over $\overline{D}$ given by

$$\mathrm{Stab}(\overline{D}) := \left\{ \phi \in \mathrm{Symp}^{S^1}(S^2 \times S^2, \omega_\lambda) \mid \phi(\overline{D}) = \overline{D} \right\}$$

Proof. In order to show that $\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda) \twoheadrightarrow \mathcal{S}^{S^1}_{D_{2k}}$ is a Serre fibration, we need to show that given a map γ from a n dimensional disk D^n to $\mathcal{S}^{S^1}_{D_{2k}}$, then γ lifts to $\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$.

$$\begin{tikzcd} & \mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda) \arrow[d, "\theta"] \\ D^n \arrow[ur, dashed, "\exists\ \overline{\gamma}"] \arrow[r, "\gamma"] & \mathcal{S}^{S^1}_{D_{2k}} \end{tikzcd}$$

This follows from noting that the proof of Lemma 3.3.1 goes through for compact families of curves as in Remark 3.3.2. $\square$

Remark 3.3.5. *As both* $\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$ *and* $\mathcal{S}^{S^1}_{D_{2k}}$ *can be shown to be CW-complexes, we see from Theorem 1 in [38] (with proof corrected in [10]), that a Serre fibration in which the total space and base space are both CW complexes is necessarily a Hurewicz fibration. Thus the map* $\theta : \mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda) \twoheadrightarrow \mathcal{S}^{S^1}_{D_{2k}}$ *is in fact a Hurewicz fibration and hence the fibre over any arbitrary* $D \in \mathcal{S}^{S^1}_{D_{2k}}$ *is homotopy equivalent to* $\mathrm{Stab}(\overline{D})$.

Now that we know that the action map

$$\mathrm{Stab}^{S^1}(\overline{D}) \to \mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda) \twoheadrightarrow \mathcal{S}^{S^1}_{D_{2k}}$$

is a fibration, we show that $\mathcal{S}^{S^1}_{D_{2k}}$ is weakly homotopic to $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{2k}$.

Lemma 3.3.6. *The natural map* $\alpha : \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{2k} \to \mathcal{S}^{S^1}_{D_{2k}}$ *defined by sending an almost complex structure* $J \in \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{2k}$ *to the unique* J*- holomorphic curve in class* D_{2k} *is a weak homotopic equivalence.*

Proof. To show that α is a weak homotopy equivalence, we first show that the map is Serre fibration. To do so, consider an arbitrary element $D \in \mathcal{S}^{S^1}_{D_{2k}}$. As in the proof of Lemma 3.3.4, it suffices to show that given a map γ from a n-dimensional disk D^n, such that $\gamma(0) = D$, there exists a lift γ' to $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{2k}$.

$$\begin{tikzcd} & \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{2k} \arrow[d, "\alpha"] \\ D^n \arrow[ur, dashed, "\exists\ \gamma'"] \arrow[r, "\gamma"] & \mathcal{S}^{S^1}_{D_{2k}} \end{tikzcd}$$

Once again by Lemma 3.3.1 and Remark 3.3.2, we have that there exists a lift $\overline{\gamma} : D^n \to \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)$ of γ. Pick an element $J \in \alpha^{-1}(D)$ and define $\gamma'(s) := \overline{\gamma}^* J$. This defines a lift γ' of γ. Hence α is a fibration, with fibre begin contractible. Thus we get the required result that $\mathcal{J}_{\omega_\lambda}^{S^1} \cap U_{2k} \simeq \mathcal{S}_{D_{2k}}^{S^1}$. □

Lemma 3.3.7. *The restriction map* $\mathrm{Stab}^{S^1}(\overline{D}) \twoheadrightarrow \mathrm{Symp}^{S^1}(\overline{D})$ *is a fibration.*

Proof. To show that the restriction map is a fibration we use Theorem C.0.9, in which we set $X = \mathrm{Symp}^{S^1}(\overline{D})$, $G = \mathrm{Stab}^{S^1}(\overline{D})$ and the action is given by

$$G \times X \to X$$
$$(\phi, \psi) \to \phi|_{\overline{D}} \circ \psi$$

Hence in order to show that the restriction map $r : \mathrm{Stab}^{S^1}(\overline{D}) \to \mathrm{Symp}^{S^1}(\overline{D})$ is a fibration, we only need to show that the action described above admits local cross sections. Suppose we only show that a neighbourhood of identity admits local cross sections and that $\mathrm{Stab}^{S^1}(\overline{D})$ acts transitively on $\mathrm{Symp}^{S^1}(\overline{D})$ this would suffice to show that r is a fibration as by Theorem C.0.9, its a local fibration near the identity and the map r is equivariant with respect to the action of $\mathrm{Stab}^{S^1}(\overline{D})$, thus completing the proof.

Consider the identity $id \in \mathrm{Symp}^{S^1}(\overline{D})$. As Let $\alpha : N(\overline{D}) \to U$ be a equivariant diffeomorphism between the symplectic normal bundle $N(\overline{D})$ and a neighbourhood U of $\overline{D}$. As $\mathrm{Symp}^{S^1}(\overline{D})$ is locally contractible (This can be seen for example by noticing that the proof of Prop 3.3.14 in [32] can be made equivariant) there is a neighbourhood V of id, and fix a retraction β_t of the neighbourhood V onto the identity. Hence given any $\psi \in \mathrm{Symp}^{S^1}(\overline{D})$, we get a one parameter family $\beta_t(\psi)$ of symplectomorphisms. As $\pi_1(\overline{D}) = 0$, $\beta_t(\psi)$ is Hamiltonian and is generated by a function H_t. Let $\pi : N(\overline{D}) \to \overline{D}$ be the projection of the normal bundle. Define $\tilde{H}_t := \alpha \circ \pi^* H_t$. Thus $\tilde{H}_t$ defines an invariant function on a U. Fix an invariant bump function ρ with support in U and is 1 in a small neighbourhood around $\overline{D}$, then $\rho \tilde{H}_t$ is an invariant function and the corresponding symplectomorphism it generates $\tilde{\psi}$ belong to $\mathrm{Stab}^{S^1}(\overline{D})$ and extends ψ. Note that if we fix the neighbourhood U, the bump function and the retraction of the neighbourhood in $\mathrm{Symp}^{S^1}(\overline{D})$ then this procedure gives us a lift of ψ near id in $\mathrm{Symp}^{S^1}(\overline{D})$ to $\mathrm{Stab}^{S^1}(\overline{D})$. By Theorem C.0.9 this shows r is fibration. □

The above proof also shows that the action is transitive. As given $\gamma \in \mathrm{Symp}^{S^1}(\overline{D})$ we give a procedure to construct an element $\tilde{\gamma} \in \mathrm{Stab}^{S^1}(\overline{D})$ such that the restriction to $\overline{D}$ is γ, thus showing the action is transitive.

Lemma 3.3.8. $\mathrm{Symp}^{S^1}(\overline{D})$ *is homotopic to* $\mathrm{SO}(3)$ *for the circle action* $S^1(0, \pm 1, m)$ *and* $\mathrm{Symp}^{S^1}(\overline{D})$ *is homotopic to* S^1 *for all other circle actions.*

Proof. Consider the circle action induced by $\overline{D}$. The action $S^1(0, \pm 1, m)$ fixes $\overline{D}$ pointwise. Hence $\mathrm{Symp}^{S^1}(\overline{D}) = \mathrm{Symp}(\overline{D})$. By Smale's theorem we know that $\mathrm{Symp}(\overline{D})$ is homotopy equivalent to $\mathrm{SO}(3)$.

For all other actions, we have the following two subcases. Consider an S^1 action not of the form $S^1(0, \pm 1, m)$. Assume that the action is effective. Let $\mu : \overline{D} \to T^*S^1$ be it's momentum map. Then as explained in the proof of Proposition 3.21 in [35] we have that $\mathrm{Symp}^{S^1}(\overline{D}) \simeq C^\infty(\mu(\overline{D}), S^1)$, where $C^\infty(\mu(\overline{D}, S^1)$ denotes the space of smooth maps from the image of the momentum map to S^1. The image of the momentum map is an interval. We shall now argue that $C^\infty(\mu(\overline{D}), S^1) \simeq S^1$.

Fix a point $v \in \mu(\overline{D})$, let P denote that space of smooth maps from the image of the momentum map to S^1 and finally let P_v denote the space of space of smooth maps from the image of the momentum map to S^1 that send v to $1 \in S^1$. Then we have a fibration.

$$P_v \to P \xrightarrow{ev} S^1$$

$$f \mapsto f(v)$$

The evaluation map $ev : P \to S^1$ is a surjective fibration and the fibre over $1 \in S^1$ is P_v. Finally as $\mu(\overline{D})$ is contractible this implies that P_v is contractible, thus giving us the required statement.

Finally if the induced symplectic S^1 action on $\overline{D}$ has $\mathbb{Z}_k$ stabilizer, the action of $S^1/\mathbb{Z}_k \cong S^1$, is effective and the space of symplectomorphisms equivariant with respect to this quotient effective action is the same as space of symplectomorphisms equivariant with respect to the non-effective S^1 action. Thus the homotopy type of $\mathrm{Symp}^{S^1}(\overline{D}) \simeq S^1$. $\square$

Remark 3.3.9. *We note an alternate way to prove Lemma 3.3.8 is to mimick the proof of Lemma A.0.7.*

Lemma 3.3.10. *The map*

$$\alpha : \mathrm{Fix}^{S^1}(\overline{D}) \twoheadrightarrow \mathrm{Gauge}^{S^1}(N(\overline{D}))$$

$$\phi \mapsto d\phi|_{N(\overline{D})}$$

is a Serre fibration with fibre homotopic to $\mathrm{Fix}^{S^1}(N(\overline{D}))$*. The base space* $\mathrm{Gauge}^{S^1}(N(\overline{D}))$ *is homotopy equivalent to* S^1*.*

Proof. The fact that $\mathrm{Gauge}^{S^1}(N(\overline{D})) \simeq S^1$ is explained in Appendix A. Thus we only need to prove that the restriction of the derivative indeed is a fibration and the fibre is homotopic to $\mathrm{Fix}^{S^1}(N(\overline{D}))$.

Consider the action

$$\mathrm{Fix}^{S^1}(\overline{D}) \times \mathrm{Gauge}^{S^1}(N(\overline{D})) \to \mathrm{Gauge}^{S^1}(N(\overline{D}))$$

$$(\phi, \psi) \to d\phi|_{N(D)} \circ \psi$$

Again by Theorem C.0.9 it suffices to show that the there is a local section to above action. Such a local section is produced by Lemma C.0.5.

The fibre is a priori given by all equivariant symplectomorphisms that act as identity on the normal bundle of $\overline{D}$. The claim is that this is in fact homotopy equivalent to the space $\mathrm{Fix}^{S^1}(N(\overline{D}))$. This follows from lemma C.0.7. $\square$

Lemma 3.3.11. *The map*

$$\mathrm{Fix}^{S^1}(N(\overline{D})) \to \overline{\mathcal{S}^{S^1}_{F,p_0}}$$

$$\phi \mapsto \phi(\overline{F})$$

is a fibration and $\overline{\mathcal{S}^{S^1}_{F,p_0}} \simeq \mathcal{J}^{S^1}_{\omega_\lambda}(\overline{D}) \simeq \{*\}$

Proof. The proof for this is similar to Lemma 3.3.1. Given a $F' \in \overline{\mathcal{S}^{S^1}_{F,p_0}}$, by Equivariant Symplectic Neighbourhood Theorem there is a equivariant diffeomorphism of $S^2 \times S^2$ such that α is a symplectomorphism in a neighbourhood of $\overline{D} \cup \overline{F}$ to another neighbourhood of $\overline{D} \cup \overline{F}$. Then by Theorem 3.2.4 we again have that $\mathrm{Fix}^{S^1}(N(\overline{D}))$ acts transitively on $\overline{\mathcal{S}^{S^1}_{F,p_0}}$, and thus the action map induces a fibration.

Proof that $\mathcal{J}^{S^1}_{\omega_\lambda}(\overline{D}) \simeq \{*\}$: This follows from the equivariant version of the standard proof of considering the homeomorphic space of equivariant compatible metrics and noting that this space of metrics is contractible.

Proof that $\overline{\mathcal{S}^{S^1}_{F,p_0}} \simeq \mathcal{J}^{S^1}_{\omega_\lambda}(\overline{D}) \simeq \{*\}$: Let $\mathcal{S}^{\perp}_{F,p_0}$ denote the space of all S^1 invariant symplectically embedded spheres S in class F such that $S \cap \overline{D} = p_0$ and S and $\overline{D}$ intersect ω_λ-orthogonally at p_0. Let $\mathcal{S}^{\pitchfork}_{F,p_0}$ denote the space of all S^1 invariant symplectically embedded spheres S in class F such that S is transverse to $\overline{D}$. By Theorem B.0.2 we have that $\mathcal{S}^{\pitchfork}_{F,p_0} \simeq \mathcal{S}^{\perp}_{F,p_0}$. Further, by Lemma B.0.1 we see that there exists a $J \in \mathcal{J}^{S^1}_{\omega_\lambda}(\overline{D})$ such that the configuration $S \vee \overline{D}$ is J-holomorphic. We now have the following fibration

$$\mathcal{J}^{S^1}_{\omega_\lambda}(\overline{D}) \twoheadrightarrow \mathcal{S}^{\pitchfork}_{F,p_0} \simeq \mathcal{S}^{\perp}_{F,p_0}$$

Where the map $\gamma : \mathcal{J}^{S^1}_{\omega_\lambda}(\overline{D}) \to \mathcal{S}^{\pitchfork}_{F,p_0}$ is just sending $J \in \mathcal{J}^{S^1}_{\omega_\lambda}(\overline{D})$ to the corresponding curve in class F passing through p_0. Now we show that γ is a homotopy equivalence. To do that we consider the following commutative diagram

$$\begin{tikzcd} T \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & \mathcal{S}^{\pitchfork}_{F,p_0} \\ \mathcal{J}^{S^1}_{\omega_\lambda}(\overline{D}) \arrow[ur, "\gamma"] & \end{tikzcd}$$

Where $T := \left\{(A, J) \in \mathcal{S}^{\pitchfork}_{F,p_0} \times \mathcal{J}^{S^1}_{\omega_\lambda}(\overline{D}) \mid A \text{ is J-holomorphic}\right\}$. Both the maps π_1 and π_2 are fibrations (this can be argued as in Lemma 3.3.4) with contractible fibres. As the diagram commutes, the map γ must be a homotopy equivalence. □

Finally we have,

Lemma 3.3.12. *The inclusion $i : \overline{\mathcal{S}^{S^1}_{F,p_0}} \hookrightarrow \mathcal{S}^{\perp}_{F,p_0}$ is a weak homotopy equivalence*

Proof. Let $\mathrm{Sp}(4)^{S^1} \cdot T_{p_0}(\overline{F})$ denote the orbit of $T_{p_0}(\overline{F})$ in $T_{p_0}(S^2 \times S^2)$ under action of $\mathrm{Sp}(4)^{S^1}$ where $\mathrm{Sp}^{S^1}(4)$ the centralizer of $S^1 \subset \mathrm{Sp}(4)$. For the $S^1(a, b; m)$ action such that $a \neq b$ then we have the following fibration,

$$\overline{\mathcal{S}^{S^1}_{F,p_0}} \to \mathcal{S}^{\perp}_{F,p_0} \twoheadrightarrow \mathrm{Sp}(4)^{S^1} \cdot T_{p_0}(\overline{F})$$
$$S \mapsto T_{p_0}S$$

This is a fibration by Theorem B.0.2. We would be done in this case if we can show that $\mathrm{Sp}(4)^{S^1} \cdot T_{p_0}(\overline{F})$ is contractible. We note that $\mathrm{Sp}^{S^1}(4)$ acts transitively on $\mathrm{Sp}(4)^{S^1} \cdot T_{p_0}(\overline{F})$ with stabilizer $\mathrm{Sp}^{S^1}(2) \times \mathrm{Sp}^{S^1}(2)$.

As the $S^1(a, b)$ action is also holomorphic, by the equivariant version of usual proof of $\mathrm{Sp}(2n)$ retracting to $U(n)$ we can show that $\mathrm{Sp}^{S^1}(2n)$ retracts to $U^{S^1}(n)$. Hence we can show that

$$\mathrm{Sp}(4)^{S^1} \cdot T_{p_0}(\overline{F}) \cong \frac{\mathrm{Sp}(4)^{S^1}}{\mathrm{Sp}^{S^1}(2) \times \mathrm{Sp}^{S^1}(2)} \simeq \frac{U^{S^1}(2)}{U(1) \times U(1)}$$

As we considered the case when $a \neq b$ we have that $U^{S^1}(2) = \mathbb{T}^2$, thus $\mathrm{Sp}(4)^{S^1} \cdot T_{p_0}(\overline{F}) \simeq \{*\}$.

The other case when we have $a = b$ (as we assumed that a, b were co-prime it implies that $a, b \in \{1, -1\}$), then we have a similar fibration to the one above (now we need to remove $T_{p_0}(\overline{D})$ from the orbit as $a = b$ there exists an element of $\mathrm{Sp}(4)^{S^1}$ that takes $T_{p_0}(\overline{F})$ to $T_{p_0}(\overline{D})$)

$$\overline{\mathcal{S}^{S^1}_{F,p_0}} \longrightarrow \mathcal{S}^{\perp}_{F,p_0} \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Sp}(4)^{S^1} \cdot T_{p_0}(\overline{F}) \backslash T_{p_0}(\overline{D})$$

□

We again show that $\mathrm{Sp}(4)^{S^1} \cdot T_{p_0}(\overline{F}) \setminus T_{p_0}(\overline{D})$ is contractible. To do that we note as above that

$$\mathrm{Sp}(4)^{S^1} \cdot T_{p_0}(\overline{F}) \simeq \frac{U^{S^1}(2)}{U(1) \times U(1)}$$

In this case we note that $U^{S^1}(2) = U(2)$ and thus we have that $\mathrm{Sp}(4)^{S^1} \cdot T_{p_0}(\overline{F}) \cong \frac{U(2)}{U(1)\times U(1)} \cong \mathbb{C}P^1$. And hence $\mathrm{Sp}(4)^{S^1} \cdot T_{p_0}(\overline{F}) \setminus T_{p_0}(\overline{D}) \simeq \{*\}$.

Lemma 3.3.13.

$$\mathrm{Stab}^{S^1}(\overline{F}) \cap \mathrm{Fix}^{S^1}(N(\overline{D})) \to \mathrm{Symp}^{S^1}(\overline{F}, N(p_0))$$

$$\phi \mapsto \phi|_{\overline{F}}$$

is a fibration and $\mathrm{Symp}^{S^1}(\overline{F}, N(p_0)) \simeq \{*\}$

Proof. The fact that this is a fibration follows from applying the proof of Lemma 3.3.7 mutatis mutandis. To proof that $\mathrm{Symp}^{S^1}(\overline{F}, N(p_0)) \simeq \{*\}$, we note that again similar to Lemma 3.3.7 $\mathrm{Symp}^{S^1}(\overline{F}, N(p_0))$ is homotopy equivalent to maps from the interval $[0,1]$ to S^1 that is identity near 0. The space of such maps is contractible thus giving the result. □

Lemma 3.3.14.

$$\mathrm{Fix}^{S^1}(\overline{F}) \to \mathrm{Gauge}^{S^1}(N(\overline{D} \vee \overline{F}))$$

$$\phi \mapsto d\phi|_{N(\overline{D}\vee\overline{F})}$$

is a fibration and $\mathrm{Gauge}^{S^1}(N(\overline{D} \vee \overline{F})) \simeq \{*\}$ *and the fibre* $\mathrm{Fix}^{S^1}(N(\overline{D} \vee \overline{F})) \simeq \{*\}$

Proof. The proof that this is a fibration is similar to the proof of Lemma 3.3.10. The fact that $\mathrm{Gauge}^{S^1}(N(\overline{D}\vee\overline{F})) \simeq \{*\}$ follows from by Lemma A.0.4. The fact that $\mathrm{Fix}_{S^1}(N(\overline{D}\vee\overline{F})) \simeq \{*\}$ follows from the Equivariant Gromov Theorem 3.2.4. □

Putting all the fibrations together gives the following theorem.

Theorem 3.3.15. *Consider the $S^1(a,b;m)$ action on $(S^2 \times S^2, \omega_\lambda)$ with $\lambda > 1$. If $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{2k} \neq \phi$, then we have the following homotopy equivalences:*

1. *when $(a,b) \neq (0,\pm1)$, we have $\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)/\mathbb{T}^2_{2k} \simeq \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{2k}$;*

2. *when $(a,b) = (0,\pm1)$, we have $\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)/(SO(3) \times S^1) \simeq \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{2k}$.*

Proof. When $(a,b;m) \neq (0,\pm1;0)$ we have a commutative diagram of fibrations

$$\begin{tikzcd}
\mathrm{Fix}^{S^1}(\overline{D}) \arrow[r] & \mathrm{Stab}^{S^1}(\overline{D}) \arrow[r, two heads] & \mathrm{Symp}^{S^1}(\overline{D}) \\
S^1 \arrow[r] \arrow[u, hook] & \mathbb{T}^2_{2k} \arrow[r] \arrow[u, hook] & S^1 \arrow[u, hook]
\end{tikzcd}$$

while in the case $(a,b) = (0,\pm1)$, we have the diagram

$$\begin{tikzcd}
\mathrm{Fix}^{S^1}(\overline{D}) \arrow[r] & \mathrm{Stab}^{S^1}(\overline{D}) \arrow[r, two heads] & \mathrm{Symp}^{S^1}(\overline{D}) \\
S^1 \arrow[r] \arrow[u, hook] & S^1 \times SO(3) \arrow[r] \arrow[u, hook] & SO(3) \arrow[u, hook]
\end{tikzcd}$$

In both the diagrams the leftmost and the rightmost arrows are homotopy equivalences. As the diagram commutes, the 5 lemma implies that the middle inclusion $\mathbb{T}^2 \hookrightarrow \mathrm{Stab}^{S^1}(\overline{D})$ or $(S^1 \times SO(3)) \hookrightarrow \mathrm{Stab}^{S^1}(\overline{D})$ are also homotopy equivalences. This gives us the required result. $\square$

Remark 3.3.16. *Let J_{2k} be the standard complex structure on W_{2k}. We note that for the action $S^1(0,\pm1,;m)$ the stabiliser of J_{2k} under the natural action of $\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$ on $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{2k}$ is the group of Kähler isometries $S^1 \times \mathrm{SO}(3)$. For all other circle actions $S^1(a,b;m)$ with $(a,b) \neq (0,\pm1)$, the stabiliser of J_{2k} is the maximal torus $\mathbb{T}^2_{2k} \subset S^1 \times \mathrm{SO}(3)$.*

3.3.3 Case 2: $\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$ action on $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$

In order to describe the action of $\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$ on the open stratum $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$, we need to modify slightly the setting introduced in the previous section. The main difference comes from the fact that for an almost-complex structure $J \in \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$, there is no invariant curve with negative self-intersection representing a class $B - kF$, $k \geqslant 1$. Instead, each such J determines a regular 2-dimensional foliation of J-holomorphic curves in the class B. Consequently, there is no natural map between the stratum $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$ and the space $\mathcal{S}^{S^1}_B$ of invariant curves in the class B. However, once we choose a fixed point p_0, given any $J \in \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$, there is a unique invariant J-holomorphic curve in the class B passing through p_0(Theorem 2.3.11). This defines a map $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0 \to \mathcal{S}^{S^1}_{B,p_0}$ that can be used to prove that the space $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$ is homotopy equivalent to an orbit of $\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$. To do so, because the fixed point p_0 is not unique, we must also investigate how the group $\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$ acts on the fixed point set of the circle action. This is done in Lemma 3.3.18. Before we proceed to prove this lemma we first describe the action of $\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$ on $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$. Note that by Theorems 3.1.5, 3.1.6 and 3.1.7, the space $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$ is non-empty only for the following circle actions:

- $S^1(a, b; 0)$, or
- $S^1(1, b; m)$ with $|2b - m| = 0$ and $2\lambda > |2b - m|$, or
- $S^1(-1, b; m)$ with $|2b + m| = 0$ and $2\lambda > |2b + m|$.

Secondly, we observe that all these actions have at least one isolated fixed point <u>except</u> the actions of the forms

- $S^1(\pm 1, 0; 0)$ and
- $S^1(0, \pm 1; 0)$

Actions with an isolated fixed point

We now consider actions $S^1(a, b; m)$ with an isolated fixed point p_0. We can choose p_0 to correspond to the vertex R in the Hirzerbruch surface W_m shown in Figure 2.1. Given $J \in \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$, there is a unique J-holomorphic curve $B_{p_0,J}$ in class B that passes through p_0. Because p_0 is fixed, J is invariant, and $B \cdot B = 0$, positivity of intersection implies that $B_{p_0,J}$ is S^1-invariant. We thus get a well-defined map

$$\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0 \to \mathcal{S}^{S^1}_{B,p_0}$$

where $\mathcal{S}^{S^1}_{B,p_0}$ denotes the space of invariant, embedded, symplectic spheres representing the class B and containing the point p_0.

Lemma 3.3.17. *Consider any $S^1(a,b;m)$ action on $(S^2 \times S^2, \omega_\lambda)$. Let p_0 and p_1 be two fixed points such that there exists an invariant fibre $\{*\} \times S^2$ passing through them . Then there exists no S^1 invariant curve in the class $B - kF$ for $k \geqslant 0$ passing through p_0 and p_1.*

Proof. Suppose not, let $\overline{D_{2k}}$ be a S^1 invariant curve in the class $B - kF$ with $k \geqslant 0$ passing through p_0 and p_1. Then the projection onto the first factor

$$\pi_1 : \overline{D_{2k}} \to S^2 \times \{0\} \subset S^2 \times S^2$$

is surjective. Hence the curve $\overline{D_{2k}}$ passes through a third fixed point p_2. As the symplectic S^1 action on $\overline{D_{2k}}$ has three fixed points, it has to fix $\overline{D_{2k}}$ pointwise. This is a contradiction as all fixed surfaces for S^1 actions must be either a maximum or minimum for the momentum map, but the fixed points p_2, p_1 and p_0 cannot have the same momentum map value. $\square$

Lemma 3.3.18. *Let $S^1(a,b;m)$ be a circle action for which the space $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$ is non-empty. Assume there is an isolated fixed point p_0 corresponding to the vertex R in Figure 2.1. Then any equivariant symplectomorphism that preserves homology $\phi \in \mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$ fixes p_0.*

Proof. <u>Case 1: $\lambda > 1$:</u> By Lemma 2.1.11 and Corollary 2.1.12 any such ϕ must preserve the momentum values and the weights of the fixed points (up to change of order of the tuples). These weights are given in Table 2.1.1 and the momentum map values are given in the graphs 2.3 and 2.4. The two conditions on the circle action imply that either $m = 0$, $|2b - m| = 0$, or $|2b + m| = 0$. It is now easy to see that under any of these three numerical conditions, the weights and momentum map values at R differ from the weights at all other fixed points. The result follows.

Case 2: $\lambda = 1$: If the actions are not of the form $S^1(1,1;0)$ or $S^1(-1,-1;0)$ with $\lambda = 1$, then an argument similar to Case 1 holds. The only case left are the actions of the form $S^1(1,1;0)$ or $S^1(-1,-1;0)$ with $\lambda = 1$. In this case, the homology classes F, B have the same area and the fixed points R and Q have the same weights (up to change of order of tuples) and the same momentum map values. We again argue by contradiction in this case. Let $\overline{B}$ denote a fixed curve in class B passing through R and P. Suppose $\phi \in \mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$ doesn't fix the point $p_0 = R$. Then ϕ has to take the point R to the point Q. Further by Lemma 2.1.11, ϕ fixes the maximum and minimum and hence $\phi(P) = P$. As ϕ preserves homology, the curve $\phi(\overline{B})$ has homology class B and has as to pass through Q and P which contradicts Lemma 3.3.17. □

Let $J_0 \in U_0$ be the complex structure of the Hirzebruch surface W_0 and let B_{p_0} be the unique J_0-holomorphic curve containing p_0 and representing the homology class B.

Corollary 3.3.19. *Let $S^1(a,b;m)$ be a circle action with an isolated fixed point and for which the structure $J_0 \in U_0$ is invariant. Then the group $\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$ acts transitively on the space $\mathcal{S}^{S^1}_{B,p_0}$, and the action map*

$$\begin{aligned} \mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda) &\twoheadrightarrow \mathcal{S}^{S^1}_{B,p_0} \\ \phi &\mapsto \phi(B_{p_0}) \end{aligned}$$

is a Serre fibration.

Proof. Since any element of $\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$ fixes p_0, it follows that this group acts on $B_{p_0,J}$. The transitivity of the action and the fact that the action defines a fibration follow from the exact same arguments as in the proof of Lemma 3.3.1 and Lemma 3.3.4. □

As before, we can now show that the stratum $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$ is homotopy equivalent to a space of invariant curves.

Lemma 3.3.20. *The natural map* $\alpha : \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0 \to \mathcal{S}^{S^1}_{B,p_0}$ *defined by sending an almost complex structure* $J \in \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$ *to the unique* J*-holomorphic curve in class* B *passing through* p_0 *is a weak homotopic equivalence.*

Proof. The argument is identical to the proof of Lemma 3.3.6 □

From now on, we can determine the homotopy type of $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$ by going through a similar sequence of fibrations and homotopy equivalences as in Section 3.3.2, namely,

$$\mathrm{Stab}^{S^1}(B_{p_0}) \to \mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda) \twoheadrightarrow \mathcal{S}^{S^1}_{B,p_0} \xrightarrow{\simeq} \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$$

$$\mathrm{Fix}^{S^1}(B_{p_0}) \to \mathrm{Stab}^{S^1}_{p_0}(B_{p_0}) \twoheadrightarrow \mathrm{Symp}^{S^1}(B_{p_0}) \xrightarrow{\simeq} S^1$$

$$\mathrm{Fix}^{S^1}(N(B_{p_0})) \to \mathrm{Fix}^{S^1}(B_{p_0}) \twoheadrightarrow \mathrm{Gauge}^{S^1}(N(B_{p_0})) \xrightarrow{\simeq} S^1$$

$$\mathrm{Stab}^{S^1}(\overline{F}) \cap \mathrm{Fix}^{S^1}(N(B_{p_0})) \to \mathrm{Fix}^{S^1}(N(B_{p_0})) \twoheadrightarrow \overline{\mathcal{S}^{S^1}_{F,p_0}} \xrightarrow{\simeq} \mathcal{J}^{S^1}(B_{p_0})$$

$$\mathrm{Fix}^{S^1}(\overline{F}) \to \mathrm{Stab}^{S^1}(\overline{F}) \cap \mathrm{Fix}^{S^1}(N(B_{p_0})) \twoheadrightarrow \mathrm{Symp}^{S^1}(\overline{F}, N(p_0)) \xrightarrow{\simeq} \{*\}$$

$$\{*\} \xleftarrow{\simeq} \mathrm{Fix}^{S^1}(N(B_{p_0} \vee \overline{F})) \to \mathrm{Fix}^{S^1}(\overline{F}) \twoheadrightarrow \mathrm{Gauge}^{S^1}(N(B_{p_0} \vee \overline{F})) \xrightarrow{\simeq} \{*\}$$

where $\overline{\mathcal{S}^{S^1}_{F,p_0}}$ denotes the space of all symplectically embedded curve in the class F that pass through p_0 and agree with a standard curve F_{p_0} in a neighbourhood of p_0. The proofs that these maps are fibrations, and the proofs of the homotopy equivalences are exactly the same as before. Consequently, we obtain the following homotopical description of $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$.

Theorem 3.3.21. *Consider one of the following circle actions on* $(S^2 \times S^2, \omega_\lambda)$

- $S^1(a, b; 0)$ *with* $(a, b) \neq (\pm 1, 0)$ *and* $(a, b) \neq (0, \pm 1)$, *or*
- $S^1(1, b; m)$ *with* $|2b - m| = 0$ *and* $2\lambda > |2b - m|$, *or*
- $S^1(-1, b; m)$ *with* $|2b + m| = 0$ *and* $2\lambda > |2b + m|$.

Then the stratum $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$ *is non-empty and*

$$\mathrm{Symp}^{S^1}_{h,p_0}(S^2 \times S^2, \omega_\lambda)/\mathbb{T}^2_0 \simeq \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$$

□

Actions without isolated fixed points

We now turn our attention to the action of $\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$ on the stratum $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$ when the circle action is either

1. $S^1(\pm 1, 0; 0)$ or
2. $S^1(0, \pm 1; 0)$.

These actions has no isolated fixed points and the associated graphs are of the form

(a) Subcase 1: $S^1(\pm 1, 0; 0)$ (b) Subcase 2: $S^1(0, \pm 1; 0)$

where μ denotes the value of the momentum map and A denotes the area of the fixed surface. We notice that there are pointwise fixed curves in the class F for the circle action $S^1(\pm 1, 0; 0)$ and pointwise fixed curves in class B for the action $S^1(0, \pm 1; 0)$. We denote the fixed surface which is a minimum for the momentum map as F_{min}, B_{min} respectively and the maximum by F_{max}, B_{max}.

Consider the action $S^1(0, \pm 1; 0)$. By Lemma 2.1.11 we note that any $\phi \in \mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$ must send B_{max} to itself. Then, given $p_0 \in B_{max}$, we define the following sequence of fibrations and homotopy equivalences:

$$\mathrm{Fix}^{S^1}(B_{max}) \longrightarrow \mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda) \twoheadrightarrow \mathrm{Symp}(B_{max}) \xrightarrow{\simeq} SO(3)$$

$$\mathrm{Stab}^{S^1}(F_{p_0}) \longrightarrow \mathrm{Fix}^{S^1}(B_{max}) \twoheadrightarrow \overline{\mathcal{S}^{S^1}_{F,p_0}} \xrightarrow{\simeq} \mathcal{J}^{S^1}_{\omega_\lambda} \simeq \{*\}$$

$$\mathrm{Fix}^{S^1}(F_{p_0}) \longrightarrow \mathrm{Stab}^{S^1}(F_{p_0}) \twoheadrightarrow \mathrm{Symp}^{S^1}(F_{p_0}) \xrightarrow{\simeq} S^1$$

$$\{*\} \xleftarrow{\simeq} \mathrm{Fix}^{S^1}(N(B_{max} \vee F_{p_0})) \longrightarrow \mathrm{Fix}^{S^1}(F_{p_0}) \twoheadrightarrow \mathrm{Gauge}^{S^1}(N(B_{max} \vee F_{p_0})) \xrightarrow{\simeq} \{*\}$$

For the other circle action $S^1(\pm 1, 0; 0)$, we obtain a similar sequence of fibrations and homotopy equivalences in which $B_{\max}$ is replaced by the curve $F_{\max}$. As before, putting all the homotopy equivalences together, we obtain the following theorem:

Theorem 3.3.22. *Consider the following two circle actions on* $(S^2 \times S^2, \omega_\lambda)$

- $S^1(\pm 1, 0; 0)$ *or*
- $S^1(0, \pm 1; 0)$

Then there is a homotopy equivalence

$$\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)/(S^1 \times \mathrm{SO}(3)) \simeq \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$$

□

For convenience, we collect together the two main results of this section in the theorem below.

Theorem 3.3.23. *Consider the action* $S^1(a, b; m)$ *on* $(S^2 \times S^2, \omega_\lambda)$ *such that one of the following hold:*

- $S^1(a, b; 0)$ *with* $(a, b) \neq (\pm 1, 0)$ *and* $(a, b) \neq (0, \pm 1)$, *or*
- $S^1(1, b; m)$ *with* $|2b - m| = 0$ *and* $2\lambda > |2b - m|$, *or*
- $S^1(-1, b; m)$ *with* $|2b + m| = 0$ *and* $2\lambda > |2b + m|$.

Then the stratum $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$ *is non-empty and*

$$\mathrm{Symp}^{S^1}_{h,p_0}(S^2 \times S^2, \omega_\lambda)/\mathbb{T}^2_0 \simeq \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_0$$

If instead the $S^1(a, b; m)$ *action satisfies*

- $(a, b; m) = (\pm 1, 0; 0)$ *or*
- $(a, b; m) = (0, \pm 1; 0)$

then we have that

$$\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)/(S^1 \times \mathrm{SO}(3)) \simeq \mathcal{J}_{\omega_\lambda}^{S^1} \cap U_0$$

and $\mathcal{J}_{\omega_\lambda}^{S^1}$ *intersects only the strata* U_0.

$\square$

Chapter 4

The homotopy type of the symplectic centralisers of $S^1(a, b; m)$

Given any Hamiltonian circle action on $(S^2 \times S^2, \omega_\lambda)$, the two Theorems 3.1.5 and 3.1.9 give us a complete understanding of which strata the space $\mathcal{J}_{\omega_\lambda}^{S^1}$ intersects. Together with Theorems 3.3.15, and 3.3.23 describing the strata as homogeneous spaces, this allows us to compute the homotopy type of the group of equivariant symplectomorphisms.

4.1 When $\mathcal{J}_{\omega_\lambda}^{S^1}$ is homotopy equivalent to a single symplectic orbit

Theorem 4.1.1. *Consider the circle action $S^1(a, b; m)$ on $(S^2 \times S^2, \omega_\lambda)$. Under the following numerical conditions on a, b, m, λ, the homotopy type of $\mathrm{Symp}^{S^1}(S^2 \times S^2, \omega_\lambda)$ is given by the table below.*

S^1 *action* $(a,b;m)$	λ	*Number of strata* $\mathcal{J}_{\omega_\lambda}^{S^1}$ *intersects*	*Homotopy type of* $\mathrm{Symp}^{S^1}(S^2\times S^2)$
$(0,\pm1;m)\quad m\neq 0$	$\lambda>1$	*1*	$S^1\times SO(3)$
$(0,\pm1;0)$ *or* $(\pm1,0;0)$	$\lambda=1$ $\lambda>1$	*1* *1*	$S^1\times SO(3)$ $S^1\times SO(3)$
$(\pm1,\pm1,0)$	$\lambda=1$	*1*	$\mathbb{T}^2\times\mathbb{Z}_2$
$(\pm1,0;m)$ $m\neq 0$	$\lambda>1$	*1*	$\mathbb{T}^2$
$(\pm1,\pm m;m)$ $m\neq 0$	$\lambda>1$	*1*	$\mathbb{T}^2$
$(1,b;m)$ $b\neq\{m,0\}$	$\lvert 2b-m\rvert\geqslant 2\lambda\geqslant 1$	*1*	$\mathbb{T}^2$
$(-1,b;m), b\neq\{-m,0\}$	$\lvert 2b+m\rvert\geqslant 2\lambda\geqslant 1$	*1*	$\mathbb{T}^2$
All other values of $(a,b;m)$ *except* $(\pm1,b;m)$	$\forall\lambda$	*1*	$\mathbb{T}^2$

Proof. By Theorem 3.1.5, in each of the above $S^1(a,b;m)$ actions, the space of S^1 invariant compatible almost complex structures $\mathcal{J}_{\omega_\lambda}^{S^1}$ intersects only the stratum U_m. Consequently,

$$\mathrm{Symp}_h^{S^1}(S^2\times S^2,\omega_\lambda)/\,\mathrm{Stab}(J_m)\simeq\mathcal{J}_{\omega_\lambda}^{S^1}\cap U_m=\mathcal{J}_{\omega_\lambda}^{S^1}\simeq\{*\}$$

where $\mathrm{Stab}(J_m)$ denotes the stabiliser of the standard complex structure $J_m\in U_m$. Thus, for all the actions in the table, we have that $\mathrm{Symp}_h^{S^1}(S^2\times S^2,\omega_\lambda)\simeq\mathrm{Stab}(J_m)$. For the S^1 action given by the triples $(0,\pm1,m)$, $(\pm1,0,0)$ or the circle action $S^1(0,\pm1,0)$ when $\lambda=1$, Theorems 3.3.15 and 3.3.23 imply that $\mathrm{Stab}(J_m)\simeq S^1\times\mathrm{SO}(3)$. For all other S^1 actions in the table, the stabilizers are homotopy equivalent to $\mathbb{T}^2$.

We now show how to recover the homotopy type of the full group $\mathrm{Symp}^{S^1}(S^2\times S^2,\omega_\lambda)$ from the homotopy type of the subgroup $\mathrm{Symp}_h^{S^1}(S^2\times S^2,\omega_\lambda)$. When $\lambda>1$, we have the equality $\mathrm{Symp}^{S^1}(S^2\times S^2,\omega_\lambda)=\mathrm{Symp}_h^{S^1}(S^2\times S^2,\omega_\lambda)$ as stated in Lemma 3.1.1.

When $\lambda=1$ and $a\neq b$, there exists standard $S^1(a,b;m)$ invariant curves in classes

B and F such that the isotropy weight of the action on the curve in class B is a and the isotropy weight of the S^1 action on the curve in the class F is b. Hence, as ϕ is an equivariant symplectomorphism, Lemma 3.1.1 implies that must have $\phi_*[F] = [F]$ and $\phi_*[B] = [B]$. Consequently, $\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda) = \mathrm{Symp}^{S^1}(S^2 \times S^2, \omega_\lambda)$.

In the special case when $\lambda = 1$ and $a = b = \pm 1$, then we have an equivariant version of the exact sequence 3.1

$$1 \longrightarrow \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda) \longrightarrow \mathrm{Symp}^{S^1}(S^2 \times S^2, \omega_\lambda) \longrightarrow \mathrm{Aut}_{c_1,\omega_\lambda}(H^2(S^2 \times S^2)) \longrightarrow 1$$

where $\mathrm{Aut}_{c_1,\omega_\lambda}(H^2(S^2 \times S^2)) \cong \mathbb{Z}_2$. The map

$$\phi : S^2 \times S^2 \to S^2 \times S^2$$
$$(z, w) \mapsto (w, z)$$

is a S^1 equivariant symplectomorphism (for the action $S^1(1, 1, 0)$ or $(-1, -1, 0)$) and gives a section from $\mathbb{Z}_2$ to $\mathrm{Aut}_{c_1,\omega_\lambda}(H^2(S^2 \times S^2))$. Thus we have $\mathrm{Symp}^{S^1}(S^2 \times S^2, \omega_\lambda) \cong \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda) \rtimes \mathbb{Z}_2$. As the semidirect product of two topological groups is homotopy equivalent to the direct product of the groups, we have that $\mathrm{Symp}^{S^1}(S^2 \times S^2, \omega_\lambda) \cong \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda) \rtimes \mathbb{Z}_2 \simeq \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda) \times \mathbb{Z}_2 \simeq \mathbb{T}^2 \times \mathbb{Z}_2$. This completes the proof. $\square$

4.2 When $\mathcal{J}_{\omega_\lambda}^{S^1}$ is homotopy equivalent to the union of two symplectic orbits

Theorem 4.1.1 gives the homotopy type of the group of equivariant symplectomorphisms for all circle actions apart from the following two families of actions:

- (i) $a = 1$, $b \neq \{0, m\}$, and $2\lambda > |2b - m|$; or
- (ii) $a = -1$, $b \neq \{0, -m\}$, and $2\lambda > |2b + m|$.

For convenience, we will write m' for either $|2b-m|$ or $|2b+m|$ depending on which of the above families we consider. Up to swapping m and m', we will also assume $m' > m$. The goal of this section is to show that the symplectic stabilizers of any of these circle actions is homotopy equivalent to the homotopy colimit of the two tori $\mathbb{T}^2_m$ and $\mathbb{T}^2_{m'}$ it extends to.

Before delving into the technicalities, it may be useful to outline the proof, which is an adaptation of the Anjos-Granja argument used in [5] to compute the homotopy type of the full group of symplectomorphisms of $S^2 \times S^2$ for $1 < \lambda \leqslant 2$. The first step is to show that the two inclusions

$$\mathbb{T}^2_m \hookrightarrow \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda) \quad \text{and} \quad \mathbb{T}^2_{m'} \hookrightarrow \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)$$

induce injective maps in homology. By the Leray-Hirsch theorem, it follows that the cohomology modules of the total space of the fibrations

$$\mathbb{T}^2_m \to \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda) \to \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)/\mathbb{T}^2_m \simeq \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_m$$

$$\mathbb{T}^2_{m'} \to \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda) \to \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)/\mathbb{T}^2_{m'} \simeq \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{m'}$$

split (with coefficients in an arbitrary field k). Using the fact that the contractible space of invariant compatible almost-complex structures decomposes as the disjoint union

$$\mathcal{J}^{S^1}_{\omega_\lambda} = (\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_m) \sqcup (\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{m'})$$

the rank of $H^i(\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda); k)$ can be computed inductively from Alexander-Eells duality. We then compute the cohomology algebra and the Pontryagin algebra of the homotopy colimit (or homotopy pushout)

$$P = \mathrm{hocolim}(\mathbb{T}^2_m \leftarrow S^1(a,b,m) \to \mathbb{T}^2_{m'})$$

and use this to show that the natural map

$$\Upsilon : P \to \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)$$

is a homotopy equivalence in the category of topological groups. We further prove that P is weakly homotopy equivalent, as a space, to the product $\Omega S^3 \times S^1 \times S^1 \times S^1$.

4.2.1 Homological injectivity

We first show that the two inclusions $\mathbb{T}^2_m \hookrightarrow \mathrm{Symp}^{S^1}_h(S^2\times S^2,\omega_\lambda)$ and $\mathbb{T}^2_{m'} \hookrightarrow \mathrm{Symp}^{S^1}_h(S^2\times S^2,\omega_\lambda)$ induce injective maps in homology. As the argument does not depend on m, we shall only provide the details for the inclusion $\mathbb{T}^2_m \hookrightarrow \mathrm{Symp}^{S^1}_h(S^2\times S^2,\omega_\lambda)$.

Fix a homology preserving diffeomorphism $\phi_m : (W_m, *) \to (S^2 \times S^2, *)$ where the base point $\{*\}$ is the $S^1(1, b, m)$ fixed point $([0, 1][0, 0, 1])$ in W_m. Let $\mathcal{E}(S^2 \times S^2, *)$ and $\mathcal{E}(W_m, \{*\})$ denote the space of orientation preserving homotopy, pointed, self-equivalences of $(S^2 \times S^2, *)$ and $(W_m, *)$. The homology preserving diffeomorphism ϕ_m induces a homeomorphism between $\mathcal{E}(S^2 \times S^2, *) \cong \mathcal{E}(W_m, *)$. Further define $\mathcal{E}(S^2, *)$ to be the space of all orientation preserving homotopy self-equivalences of the sphere preserving a base point $\{*\}$.

We now observe that for the above two families of circle actions (i) and (ii), the same argument as in Lemma 3.3.18 shows that any $\phi \in \mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$ fixes the base point $\{*\}$.

Now, recall that the zero section s_0 of W_m is given by

$$\begin{aligned} s_0 : S^2 &\to W_m \\ [z_0, z_1] &\mapsto ([z_0, z_1], [0, 0, 1]) \end{aligned}$$

and the projection to the first factor is

$$\begin{aligned} \pi_1 : W_m &\to S^2 \\ ([z_0, z_1], [w_0, w_1, w_2]) &\mapsto [z_0, z_1] \end{aligned}$$

We define a continuous map $h_1 : \mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda) \to \mathcal{E}(S^2, *)$ by setting

$$\begin{aligned} h_1 : \mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda) &\to \mathcal{E}\left(S^2, *\right) \\ \psi &\mapsto \psi_1 := \pi_1 \circ \psi \circ s_0 \end{aligned}$$

Similarly, using the inclusion of S^2 as the fiber

$$f : S^2 \to W_m$$
$$[z_0, z_1] \mapsto ([0, 1], [0, z_0, z_1])$$

and the projection to the second factor $\pi_2 : S^2 \times S^2 \to S^2$, we can define a map

$$h_2 : \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda) \to \mathcal{E}\left(S^2, *\right)$$
$$\psi \mapsto \psi_2 := \pi_2 \circ \psi \circ \phi_m \circ f$$

where $\phi_m : W_m \to S^2 \times S^2$ is our fixed identification of W_m with $S^2 \times S^2$. We thus get a continuous map

$$h : \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda) \to \mathcal{E}(S^2, *) \times \mathcal{E}(S^2, *)$$
$$\psi \mapsto (h_1(\psi), h_2(\psi))$$

Lemma 4.2.1. *The inclusion $i_m : \mathbb{T}^2_m \hookrightarrow \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)$ induces a map which is injective in homology with coefficients in any field k.*

Proof. As $\mathbb{T}^2$ is connected, $i_m : H_0(\mathbb{T}^2_m; k) \to H_0(\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda); k)$ is injective. To show that the inclusion map induces an injection at the H_1 level, we consider the composition $\alpha : \mathbb{T}^2_m \to \mathcal{E}(S^2, *) \times \mathcal{E}(S^2, *)$ given by

$$\mathbb{T}^2_m \hookrightarrow \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda) \xrightarrow{h} \mathcal{E}(S^2, \{*\}) \times \mathcal{E}(S^2, *)$$

and show that α induces a map which is injective in homology.

We claim that $H_1(\mathcal{E}(S^2, *); \mathbb{Z}) \simeq \mathbb{Z}$. Indeed, the standard action of SO(3) on S^2 gives rise to a diagram of fibrations

$$\begin{tikzcd}
\mathcal{E}(S^2, *) \arrow[r] & \mathcal{E}(S^2) \arrow[r, two heads, "\mathrm{ev}"] & S^2 \\
S^1 = \mathrm{SO}(2) \arrow[r] \arrow[u, hook] & \mathrm{SO}(3) \arrow[r, two heads, "\mathrm{ev}"] \arrow[u, hook] & S^2 \arrow[u, equal]
\end{tikzcd}$$

where the maps ev are evaluations at the base point $\{*\}$. This induces a long exact ladder of homotopy groups

$$\begin{tikzcd}
\cdots \arrow[r] & \cancelto{\mathbb{Z}}{\pi_2(S^2)} \arrow[r] \arrow[d, equal] & \pi_1(\mathcal{E}(S^2,*)) \arrow[r] & \pi_1(SO(3)) \times \cancelto{0}{\pi_1(\Omega)} \arrow[r] & \cancelto{0}{\pi_1(S^2)} \arrow[d, equal] \\
\cdots \arrow[r] & \mathbb{Z} \arrow[r] & \cancelto{\mathbb{Z}}{\pi_1(S^1)} \arrow[r] \arrow[u, "\beta"] & \pi_1(SO(3)) \arrow[r] \arrow[u] & \cancelto{0}{\pi_1(S^2)}
\end{tikzcd}$$

where we have used the fact, proven by Hansen in [34], that $\mathcal{E}(S^2) \simeq SO(3) \times \widetilde{\Omega^2}$, where $\widetilde{\Omega^2}$ denotes the universal covering space for the connected component of the double loop space of S^2 containing the constant based map, and where the SO(3) component is just the inclusion. Consequently, $\pi_1(\widetilde{\Omega^2}) = 0$ and the map $\pi_1(SO(3)) \to \pi_1(SO(3)) \times \pi_1(\widetilde{\Omega^2})$ is an isomorphism. From the commutativity of the middle square, it follows that $\beta : \pi_1(S^1) \to \pi_1(\mathcal{E}(S^2,*))$ is also an isomorphism. As the spaces we consider are topological groups, π_1 is abelian and hence $\pi_1 = H_1$, proving the claim.

Now, the classes a, b, of the subcircles $(0,1)$ and $(1,0)$ form a basis for $H_1(\mathbb{T}^2_m;k)$. We claim that $\alpha_*[0,1]$ and $\alpha_*[1,0]$ generate a subgroup of rank 2. To see this, let write α^1_* and α^2_* for the components of α_*. Then, $\alpha^1_*(0,1) = 0$ as the circle $(0,1)$ fixes the zero section $([x_1,x_2],[0,0,1]) \subset W_m$ pointwise, while $\alpha^2_*[0,1] \neq 0$ by the reasoning in the previous paragraph. Similarly, $\alpha^1_*[1,0] \neq 0$ and $\alpha^2_*[1,0] = 0$, proving our claim. We conclude that α is injective on $H_1(\mathbb{T}^2_m;k)$.

Finally, to show that i_* is injective on $H_2(\mathbb{T}^2_m;k)$, we will prove the dual statement, namely, that the map $i^* : H^2(\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda);k) \to H^2(\mathbb{T}^2_m;k)$ is surjective. A generator of $H^2(\mathbb{T}^2_m;k) \cong k$ is given by $a \cup b$. Because i_* is injective at the H_1 level, $i^* : H^1(\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda);k) \to H^1(\mathbb{T}^2;k)$ is surjective, hence there exists elements a', $b' \in H^1(\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda);k)$ such that $i^*(a') = a$ and $i^*(b') = b$. Since $i^*(a') \cup i^*(b') = a \cup b$, it follows that $i^* : H^2(\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda);k) \to H^2(\mathbb{T}^2_m;k)$ is surjective. $\square$

4.2.2 Cohomology module of the centralizer of $S^1(\pm 1, b; m)$

We are now ready to compute the cohomology module of the centralizer of $S^1(\pm 1, b; m)$ with coefficients in in a field k. By duality, this is equivalent to determining the homology module.

Recall that the contractible space of invariant compatible almost-complex structures $\mathcal{J}^{S^1}_{\omega_\lambda}$ decomposes as the disjoint union

$$\mathcal{J}^{S^1}_{\omega_\lambda} = (\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_m) \sqcup (\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{m'}) =: U^{S^1}_m \sqcup U^{S^1}_{m'}$$

where, for convenience, we set $U^{S^1}_m = \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_m$ and $U^{S^1}_{m'} = \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{m'}$. We will show in Chapter 5 the following two important facts:

- the strata $U^{S^1}_m$ and $U^{S^1}_{m'}$ are submanifolds of $\mathcal{J}^{S^1}_{\omega_\lambda}$ (see Corollary 5.2.6), and
- the stratum $U^{S^1}_m$ is open in $\mathcal{J}^{S^1}_{\omega_\lambda}$, while $U^{S^1}_{m'}$ is of codimension 2 (see Theorem 5.3.1).

In particular, it follows that $U^{S^1}_m = \mathcal{J}^{S^1}_{\omega_\lambda} - U^{S^1}_{m'}$ is connected. As explained in Appendix D, Proposition D.0.5, the Alexander-Eells duality induces an isomorphism of homology groups

$$\lambda_* : H_p(U^{S^1}_{m'}; k) \to H_{p+1}(U^{S^1}_m; k) \tag{4.1}$$

Now recall that we also have fibrations

$$\begin{aligned} \mathbb{T}^2_m \to \mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda) &\xrightarrow{p_m} \mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)/\mathbb{T}^2_m \simeq U^{S^1}_m \\ \mathbb{T}^2_{m'} \to \mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda) &\xrightarrow{p_{m'}} \mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)/\mathbb{T}^2_{m'} \simeq U^{S^1}_{m'} \end{aligned} \tag{4.2}$$

From the first fibration, the connectedness of the open stratum $U^{S^1}_m$ implies that the group $\mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$ is connected. In turns, the second fibration implies that the codimension 2 stratum $U^{S^1}_{m'}$ is also connected. Because the two inclusions

$$\mathbb{T}^2_m \hookrightarrow \mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda) \quad \text{and} \quad \mathbb{T}^2_{m'} \hookrightarrow \mathrm{Symp}^{S^1}_h(S^2 \times S^2, \omega_\lambda)$$

induce surjective maps in cohomology, the Leray-Hirsch theorem implies that the cohomology module of $\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)$ splits as

$$H^*(\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda), k) \cong H^*(U_m^{S^1}; k) \otimes H^*(\mathbb{T}_m^2; k) \tag{4.3}$$
$$H^*(\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda), k) \cong H^*(U_{m'}^{S^1}; k) \otimes H^*(\mathbb{T}_{m'}^2; k)$$

By duality, we have corresponding splittings in homology, namely,

$$H_*(\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda), k) \cong H_*(U_m^{S^1}; k) \otimes H_*(\mathbb{T}_m^2; k) \tag{4.4}$$
$$H_*(\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda), k) \cong H_*(U_{m'}^{S^1}; k) \otimes H_*(\mathbb{T}_{m'}^2; k)$$

It follows that

$$H_p(U_m; k) \simeq H_p(U_{m'}; k) \text{ for all } p \geqslant 0$$

Together with the Alexander-Eells isomorphism (4.1) and the connectedness of $U_{m'}$, this implies that

$$H_p(U_m; k) \simeq k \text{ for all } p \geqslant 0$$

Using the splitting 4.4 and dualizing, we can finally compute the cohomology module of $\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)$.

Theorem 4.2.2. *Consider any of the following circle actions:*

- *(i)* $a = 1$, $b \neq \{0, m\}$, *and* $2\lambda > |2b - m|$; *or*
- *(ii)* $a = -1$, $b \neq \{0, -m\}$, *and* $2\lambda > |2b + m|$.

Then, the cohomology groups of the symplectic centralizer are

$$H^p\left(\mathrm{Symp}^{S^1}(S^2 \times S^2, \omega_\lambda); k\right) \simeq \begin{cases} k^4 & p \geqslant 2 \\ k^3 & p = 1 \\ k & p = 0 \end{cases}$$

for any field k. *In particular, the topological group* $\mathrm{Symp}^{S^1}(S^2 \times S^2, \omega_\lambda)$ *is of finite type.*

4.2.3 The homotopy colimit of $T_m \leftarrow S^1(\pm 1, b; m) \rightarrow T_{m'}$

As explained in Theorems 3.1.6 and 3.1.7, the circle actions $S^1(\pm 1, b; m)$ we are considering in this section extend to exactly two toric actions $\mathbb{T}^2_m$ and $\mathbb{T}^2_{m'}$. Geometrically, this means that the two tori $\mathbb{T}^2_m$ and $\mathbb{T}^2_{m'}$ intersect in $\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)$ along the circle $S^1(\pm 1, b; m)$ and, in particular, that we have two inclusions of Lie groups

$$\begin{tikzcd} S^1 \arrow[r, "(1,b')"] \arrow[d, "(1,b)"'] & T^2_{m'} \\ T^2_m & \end{tikzcd}$$

In this section we consider the homotopy colimit – or homotopy pushout – of these two inclusions, namely,

$$P := \mathrm{hocolim}(T_m \leftarrow S^1(1, b; m) \rightarrow T_{m'})$$

This pushout is to be understood in the category of topological groups. As we will show later, the topological group P turns out to be a model for the homotopy type of the centralizer $\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)$.

The Pontryagin algebra of the pushout

In what follows, all k algebras are graded, and the commutator of two elements is given by

$$[a, b] = ab - (-1)^{|a| \cdot |b|} ba$$

For any field k, and for any abelian group A, the Pontryagin algebra $H_*(A; k)$ is isomorphic to the cohomology algebra $H^*(A; k)$. It follows that $H_*(S^1)$ is isomorphic to $\Lambda(t)$, where t is of degree 1. Similarly, the Pontryagin algebra $H_*(T^2; k)$ is isomorphic to the to an exterior algebra $\Lambda(z_1, z_2)$ generated by two elements of degree one. The pushout diagram of topological groups

$$\begin{tikzcd} S^1 \arrow[r, "(1,b')"] \arrow[d, "(1,b)"'] & T^2_{m'} \arrow[d] \\ T^2_m \arrow[r] & P \end{tikzcd}$$

is homologically free (see Definition 3.1 in [5]). As before, P denotes the pushout in the category of topological groups. By Theorem 3.8 of [5], the Pontryagin algebra of P is the pushout of k algebras

$$\begin{array}{ccc} H_*(S^1;k) & \xrightarrow{H(1,b')} & H_*(T^2_{m'};k) \\ {\scriptstyle H(1,b)}\downarrow & & \downarrow \\ H_*(T^2_m;k) & \longrightarrow & H_*(P;k) \end{array}$$

which is isomorphic to

$$\begin{array}{ccc} \Lambda(t) & \xrightarrow{(1,b')} & \Lambda(y_1,y_2) \\ {\scriptstyle (1,b)}\downarrow & & \downarrow \\ \Lambda(x_1,x_2) & \longrightarrow & P^{alg}_* \end{array}$$

where $P^{alg}_* \simeq H_*(P;k)$. By the description of the pushout of k algebras as amalgamated products (See [5] for more details), the k algebra P^{alg}_* can be identified with equivalence classes of finite linear combinations of words in the letters $\{x_1,x_2,y_1,y_2\}$ under the relations $x_ix_i = 0$, $y_iy_i = 0$, $[x_1,x_2] = 0$, $[y_1,y_2] = 0$, and $x_1 + bx_2 = y_1 + b'y_2$. From the last equality, we can write $y_1 = (x_1 + bx_2) - b'y_2$, which means that we can choose, as generators, the elements

$$\{t = x_1 + bx_2,\ x_2,\ y_2\}$$

with the relations $t^2 = x_2^2 = y_2^2 = 0$, $[t,x_2] = [t,y_2] = 0$. The remaining commutator $w = [x_2,y_2]$ is nonzero and commutes with t, x_2 and y_2. It follows that any word in t, x_2, y_2 is equivalent to a linear combination of words of the form

$$w^\alpha x_2^\beta y_2^\gamma t^\delta$$

with $\alpha \in \mathbb{N} \cup \{0\}$, and $\beta,\gamma,\delta \in \{0,1\}$. Hence, there is an isomorphism of graded algebras

$$P^{alg}_* \cong \frac{F(x_2,y_2)}{\langle x_2^2, y_2^2\rangle} \otimes \Lambda(t)$$

where $F(x_2, y_2)$ denotes the free graded algebra over k generated by the elements x_2 and y_2, and where x_2, y_2, t are of degree one. In particular,

$$P_p^{alg} \simeq \begin{cases} k & p = 0 \\ k^3 & p = 1 \\ k^4 & p \geqslant 2 \end{cases}$$

and the words $w^\alpha x_2^\beta y_2^\gamma t^\delta$ form an additive basis of the homology module P_*^{alg}.

By duality, the cohomology modules $P^{alg,*}$ are $P^{alg,0} \simeq k$, $P^{alg,1} \simeq k^3$, and $P^{alg,p} \simeq k^4$ for all $p \geqslant 2$. The algebra structure of $P^{alg,*}$ can be determined as follows. Let $\hat{t}$, $\hat{x}_2$, and $\hat{y}_2$ be the duals of the generators of degree 1, and let $\hat{w}$ be the dual of the generator $w = [x_2, y_2]$ of degree 2.

Let us now recall the Hopf-Borel theorem (see [30] Theorem 6.36).

Theorem 4.2.3. *(Hopf-Borel) Let k be a field of characteristic p where p may be zero or a prime. A connected Hopf algebra H over k is said to be monogenic if H is generated as an algebra by 1 and one homogeneous element x of degree strictly greater than 0. If H is a monogenic Hopf algebra, then*

1. *if $p \neq 2$ and degree x is odd, then $H \cong \Lambda(x)$,*
2. *if $p \neq 2$ and degree x is even, then $H \cong k[x]/\langle x^s \rangle$ where s is a power of p or is infinite i.e $H \cong k[x]$,*
3. *if $p = 2$, then $H \cong k[x]/\langle x^s \rangle$ where s is a power of 2 or is infinite.*

As $P^{alg,*}$ is an associative, graded commutative Hopf algebra of finite type, the Hopf-Borel theorem implies that $P^{alg,*}$ is a tensor product of monogenic Hopf algebras. For a field k of characteristic p different from 2, including $p = 0$, $P^{alg,*}$ contains a subalgebra of the form

$$A^* = \Lambda(\hat{t}, \hat{x}_2, \hat{y}_2) \otimes k[\hat{w}]/\langle \hat{w}^s \rangle$$

where s is a power of p or is infinite. Suppose $s = p^n \geqslant 3$ is finite. Then, the rank of A^i would coincide with the rank of $P^{alg,i}$ up to degree $i = 2s - 1$, and we would have $A^i = 0$ for $i \geqslant 2s$. Therefore, we would need 4 more generators of degree $2s$ to account for the rank of $P^{alg,2s}$, and their pairwise products would imply that $\mathrm{rk}\, P^{alg,4s} > 4$. This contradiction shows that s must be infinite and that the rank of A^i equals the rank of $P^{alg,i}$ for all $i \geqslant 0$. Consequently, for a field k of characteristic $p \neq 2$, the k-algebra $P^{alg,*}$ is isomorphic to

$$P^{alg,*} \cong \Lambda(\hat{t}, \hat{x}_2, \hat{y}_2) \otimes S(\hat{w})$$

In characteristic $p = 2$, $P^{alg,*}$ is the tensor product of truncated polynomial algebras $k[z_i]/z_i^{s_i}$ where s_i is a power of 2. As before, it contains a subalgebra of the form

$$A^* = k[\hat{t}, \hat{x}_2, \hat{y}_2]/\langle \hat{t}^2, \hat{x}_2^2, \hat{y}_2^2 \rangle \otimes k[\hat{w}]/\langle \hat{w}^s \rangle$$

Again, assuming s is finite forces the existence of 4 new generators in degree $2s$ whose products would yield too many generators in degree $4s$. Therefore, in characteristic $p = 2$, the cohomology algebra of P is isomorphic to

$$P^{alg,*} \cong k[\hat{t}, \hat{x}_2, \hat{y}_2]/\langle \hat{t}^2, \hat{x}_2^2, \hat{y}_2^2 \rangle \otimes k[\hat{w}]$$

In characteristic zero, the computation of the cohomology ring yields the minimal model of $H^*(P) \otimes \mathbb{Q}$. As P is a H-space, it is a nilpotent space (see Exercise 1.13 in [19]), so that the main theorem of dgc rational homotopy theory applies (see [19], Theorem 2.50) namely, the dimension $\pi_p(P) \otimes \mathbb{Q}$ for $p \geqslant 2$ is equal to the number of generators of degree p in the minimal model. For $p = 1$, as P is a topological group, the dimension of $\pi_1(P) \otimes \mathbb{Q}$ is same as the rank of $H_1(P, \mathbb{Q})$. Consequently,

$$\pi_p(P) \otimes \mathbb{Q} \simeq \begin{cases} \mathbb{Q} & p = 0 \\ \mathbb{Q}^3 & p = 1 \\ \mathbb{Q} & p = 2 \\ 0 & p \geqslant 3 \end{cases}$$

The homotopy type of P

We want to better understand the homotopy type of the space P. To this end, consider the embeddings

$$\begin{aligned} f_m : T^2_m &\to S^1 \times S^1 \times S^1 \\ (x_1, x_2) &\mapsto (x_1, x_2, b' x_1) \end{aligned} \tag{4.5}$$

$$\begin{aligned} f_{m'} : T^2_{m'} &\to S^1 \times S^1 \times S^1 \\ (y_1, y_2) &\mapsto (y_1, b y_1, y_2) \end{aligned} \tag{4.6}$$

The universal property of pushouts implies that there is a unique map $f_P : P \to S^1 \times S^1 \times S^1$ making the following diagram commutative

$$\begin{tikzcd}
BS^1 \arrow[r, "B(1{,}b')"] \arrow[d, "B(1{,}b)"'] & BT^2_{m'} \arrow[d] \arrow[ddr, bend left, "Bf_{m'}"] & \\
BT^2_m \arrow[r] \arrow[drr, bend right, "Bf_m"'] & BP \arrow[dr, dotted, "Bf_P"] & \\
 & & BS^1 \times BS^1 \times BS^1
\end{tikzcd}$$

By Theorem 3.9 of [5], the homotopy fiber of Bf_P is the pushout of the homotopy fibers of the other maps in the diagram. To determine this fiber, we first replace the maps in the diagram of groups by homotopy equivalent fibrations

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, "(1{,}1{,}a_1)"'] & \mathbb{Z} \times \mathbb{Z} \arrow[l, "a_2"'] \arrow[r, "a_1"] \arrow[d, "(1{,}a_1{,}a_2)"] & \mathbb{Z} \arrow[d, "(1{,}1{,}a_1)"] \\
T^2_m \times \mathbb{R} \arrow[d, "(a_1{,}a_2{,}b'a_1e(a_3))"'] & S^1 \times \mathbb{R} \times \mathbb{R} \arrow[l, "(a_1{,}ba_1e(a_2){,}a_3)"'] \arrow[r, "(a_1{,}b'a_1e(a_3){,}a_2)"] \arrow[d, "(a_1{,}ba_1e(a_2){,}b'a_1e(a_3))"] & T^2_{m'} \times \mathbb{R} \arrow[d, "(a_1{,}ba_1e(a_3){,}a_2)"] \\
S^1 \times S^1 \times S^1 \arrow[r, "="] & S^1 \times S^1 \times S^1 & S^1 \times S^1 \times S^1 \arrow[l, "="']
\end{tikzcd}$$

where a_i denote the i^{th} coordinate function and $e(a_j) = e^{2\pi i a_j}$. Applying the classifying space functor, this gives

$$\begin{tikzcd}
S^1 \arrow[d] & S^1 \times S^1 \arrow[l, "\mathrm{pr}_2"'] \arrow[r, "\mathrm{pr}_1"] \arrow[d] & S^1 \arrow[d] \\
BT^2_m \arrow[d] & BS^1 \arrow[l] \arrow[r] \arrow[d] & BT^2_{m'} \arrow[d] \\
BS^1 \times BS^1 \times BS^1 \arrow[r, "="] & BS^1 \times BS^1 \times BS^1 & BS^1 \times BS^1 \times BS^1 \arrow[l, "="']
\end{tikzcd}$$

which shows that the homotopy fiber of the canonical map $BP \to BS^1 \times BS^1 \times BS^1$ is homotopy equivalent to

$$\operatorname{hocolim}\{S^1 \xleftarrow{\mathrm{pr}_2} S^1 \times S^1 \xrightarrow{\mathrm{pr}_1} S^1\} \simeq S^1 * S^1 \simeq S^3$$

Consequently, BP is the total space of a fibration

$$S^3 \to BP \to BS^1 \times BS^1 \times BS^1$$

that, after looping, becomes

$$\begin{tikzcd}
 & T^2_{m'} \arrow[d, "j_{m'}"'] \arrow[dr, "f_{m'}=(a_1, ba_1, a_2)"] & \\
\Omega S^3 \arrow[r] & P \arrow[r, "f_P"] & S^1 \times S^1 \times S^1 \\
 & T^2_m \arrow[u, "j_m"] \arrow[ur, "f_m=(a_1, a_2, b' a_1)"'] &
\end{tikzcd}$$

The map f_P admits a section given by

$$s(a_1, a_2, a_3) = j_{m'}(a_1, b'^{-1} a_3) j_m(1, b^{-1} a_1^{-1} a_2)$$

It follows that, as a space, P is weakly homotopically equivalent to the product

$$P \simeq \Omega S^3 \times S^1 \times S^1 \times S^1$$

which is consistent with the algebraic computations of the previous section.

4.2.4 Homotopy type of $S^1(\pm 1, b; m)$ equivariant symplectomorphisms

We are now able to determine the homotopy type of the group $\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)$ for the circle actions

- $S^1(1, b, m)$ when $2\lambda > |2b - m|$, and
- $S^1(-1, b, m)$ when $2\lambda > |2b + m|$.

Since the arguments are identical in the two cases, we will only discuss the first one. Again, in order to keep the notation simple, we write $\mathbb{T}^2_m$ and $\mathbb{T}^2_{m'}$ for the two tori the circle extends to, assuming $m' > m$, and we write $(1, b) : S^1 \to \mathbb{T}^2_m$ and $(1, b') : S^1 \to \mathbb{T}^2_{m'}$ for the two inclusions.

From the universal property of pushouts, there is a canonical map

$$\Upsilon : P_*^{alg} \to H_*(\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda); k)$$

making the following diagram commutative

$$\begin{tikzcd}
\Lambda(t) \arrow[r, "{(1,b')}"] \arrow[d, "{(1,b)}"'] & \Lambda(y_1, y_2) \arrow[d] \arrow[ddr, bend left, "i_{m'}"] & \\
\Lambda(x_1, x_2) \arrow[r] \arrow[drr, bend right, "i_m"'] & P_*^{alg} \arrow[dr, dotted, "\Upsilon"] & \\
& & H_*(\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda); k)
\end{tikzcd}$$

Proposition 4.2.4. *For every field k, the map $\Upsilon : P_*^{alg} \to H_*(\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda); k)$ is an isomorphism of k-algebras.*

Proof. By definition, the map Υ is an homomorphism of k-algebras. Since $P_i^{alg} \cong H_i(\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda); k)$ for each i, it is sufficient to show that Υ is surjective.

Let R be the image of Υ. Since the maps i_m and $i_{m'}$ are injective, R is the subring generated by the classes t, x_2, y_2 viewed as elements in $H_*(\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda); k)$. Consider the two fibrations induced by the action maps

$$\mathbb{T}^2_m \to \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda) \xrightarrow{p_m} \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)/\mathbb{T}^2_m \simeq U_m^{S^1}$$
$$\mathbb{T}^2_{m'} \to \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda) \xrightarrow{p_{m'}} \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)/\mathbb{T}^2_{m'} \simeq U_{m'}^{S^1}$$

Observe that $p_m(t) = 0$, $p_m(x_2) = 0$, $p_{m'}(t) = 0$, and $p_{m'}(y_2) = 0$. Now suppose there is an element $z \in H_*(\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda); k)$, not in R, and of minimal degree d. Since

$$\begin{aligned} &H_d(\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda); k) \cong \\ &\quad H_d(U_m^{S^1}; k) \otimes H_0(T^2_m; k) \ \oplus\ H_{d-1}(U_m^{S^1}; k) \otimes H_1(T^2_m; k) \ \oplus\ H_{d-2}(U_m^{S^1}; k) \otimes H_2(T^2_m; k) \end{aligned} \tag{4.7}$$

we would have a decomposition

$$z = c_1 \otimes \mathbf{1} \ \oplus\ c_t \otimes t \ \oplus\ c_{x_2} \otimes x_2 + c_T \otimes [T^2_m]$$

with at least one coefficient c_j which is not a polynomial in the classes $p_m(w)$ and $p_m(y_2)$. Let c_ℓ be such coefficient of minimal degree $d-2 \leqslant \ell \leqslant d$. The inverse of the Alexander-Eells isomorphism of Proposition D.0.5

$$\lambda_*^{-1} : H_{p+1}(U_m^{S^1}) \to H_p(U_{m'}^{S^1})$$

would map c_ℓ to a class $c'_{\ell-1} \in H_{\ell-1}(U_{m'}^{S^1}; k)$. This class could not be a polynomial in $p_{m'}(w)$ and $p_{m'}(x_2)$ since, otherwise,

$$c_\ell = \lambda_*(c'_{\ell-1}) = p_m\big([y_2 \otimes c'_{\ell-1}]\big)$$

would be a polynomial in the classes $p_m(w)$ and $p_m(y_2)$. In turn, this class $c'_{\ell-1}$ would have to be the image of some element in $H_{\ell-1}(\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda); k)$ not in R, contradicting the minimality of z. $\square$

Corollary 4.2.5. *The map* $\Upsilon : P_*^{alg} \to H_*(\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda); \mathbb{Z})$ *is an isomorphism of Pontryagin algebras over the ring of integers.*

Proof. This follows from the well known fact that a map induces isomorphisms on homology with $\mathbb{Z}$ coefficients iff it induces isomorphisms on homology with $\mathbb{Q}$ and $\mathbb{Z}_p$ coefficients for all primes p, see [22], Corollary 3A.7 (b). □

Theorem 4.2.6. *The map* $\Upsilon : P \to \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)$ *is an homotopy equivalence.*

Proof. The map Υ is a homology equivalence on integral homology. Because P and $\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)$ are topological groups, it follows that it is a weak equivalence, see [16], Example 4.2. Because both spaces are homotopy equivalent to CW-complexes, this weak equivalence is a homotopy equivalence. See [22], Proposition 4.74. □

4.3 Centralizers of Hamiltonian S^1 actions on $S^2 \times S^2$

We summarise all the results we have obtained in this chapter in the following theorem.

Theorem 4.3.1. *Consider any Hamiltonian circle action* $S^1(a, b; m)$ *on* $(S^2 \times S^2, \omega_\lambda)$*. The homotopy type of the symplectic stabilizer* $\mathrm{Symp}^{S^1}(S^2 \times S^2, \omega_\lambda)$ *is given in the table below:*

Values of $(a,b;m)$	λ	*Number of strata* $\mathcal{J}^{S^1}_{\omega_\lambda,l}$ *intersects*	*Homotopy type of* $\mathrm{Symp}^{S^1}(S^2 \times S^2, \omega_\lambda)$
$(0, \pm 1; m),\ m \neq 0$	$\lambda > 1$	*1*	$S^1 \times SO(3)$
$(0, \pm 1; 0)$ *or* $(\pm 1, 0; 0)$	$\lambda = 1$	*1*	$S^1 \times SO(3)$
	$\lambda > 1$	*1*	$S^1 \times SO(3)$
$(\pm 1, \pm 1; 0)$	$\lambda = 1$	*1*	$\mathbb{T}^2 \times \mathbb{Z}_2$
$(\pm 1, 0; m), m \neq 0$	$\lambda > 1$	*1*	$\mathbb{T}^2$
$(\pm 1, \pm m; m), m \neq 0$	$\lambda > 1$	*1*	$\mathbb{T}^2$
$(1, b; m), b \neq \{m, 0\}$	$\lvert 2b - m \rvert \geqslant 2\lambda \geqslant 1$	*1*	$\mathbb{T}^2$
	$2\lambda > \lvert 2b - m \rvert \geqslant 0$	*2*	$\Omega S^3 \times S^1 \times S^1 \times S^1$
$(-1, b; m), b \neq \{-m, 0\}$	$\lvert 2b + m \rvert \geqslant 2\lambda \geqslant 1$	*1*	$\mathbb{T}^2$
	$2\lambda > \lvert 2b + m \rvert \geqslant 0$	*2*	$\Omega S^3 \times S^1 \times S^1 \times S^1$
All other values of $(a,b;m)$	$\forall \lambda$	*1*	$\mathbb{T}^2$

where ΩS^3 *denotes the based loop space of* S^3. □

Chapter 5

Calculating the codimension

As seen in the previous chapter we calculate the homotopy type of the group of $S^1(\pm 1, b; m)$ equivariant symplectomorphisms assuming that the codimension of the the invariant strata $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{m'}$ in $\mathcal{J}^{S^1}_{\omega_\lambda}$ was 2. In this chapter, we first prove using deformation theory that the invariant strata $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{m'}$ is a submanifold of $\mathcal{J}^{S^1}_{\omega_\lambda}$ and then calculate the dimension of normal bundle of this submanifold to obtain the codimension.

We mimic the techniques in [3] in the equivariant setting. Fix a Kähler 4-manifold (M, ω, J) and an S^1 action on (M, ω, J) such that $g^*\omega = \omega$ and $g^*J = J$ $\forall g \in S^1$. We work in this generality and note that our required manifolds $S^2 \times S^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ along with the circle action $S^1(a, b; m)$ satisfies the given conditions. Throughout the section we note that the holomorphic S^1 action on the base manifold M induces a natural action on the various tensor spaces such as $T^{1,0}M$, $\Omega^{0,k}_J(M, TM)$ etc considered below. We write $(T^{1,0}M)^{S^1}$, $\left(\Omega^{0,k}_J(M, TM)\right)^{S^1}$ etc to denote the S^1 invariant elements of these tensor spaces.

5.1 Space of invariant complex structures

Define $\mathcal{J}_l^{S^1}$ to be the space of S^1 invariant almost complex structures of M with regularity C^l endowed with C^l topology. We briefly outline the procedure to put a Banach manifold structure on $\mathcal{J}_l^{S^1}$ as follows. Fix a $J \in \mathcal{J}_l^{S^1}$ and consider the sub-bundle $h_l^{S^1}(TM, J)$ of $\mathrm{End}_l^{S^1}(TM)$ of invariant endomorphisms of the tangent bundle of regularity C^l defined by

$$h_l^{S^1}(TM, J) = \left\{ A \in \mathrm{End}_l^{S^1}(TM) \mid AJ + JA = 0 \right\}$$

Consider the map $\phi_J : h_l^{S^1}(TM, J) \to \mathcal{J}_l^{S^1}$ given by

$$\phi_J(A) = Je^A$$

As shown in [14], the map ϕ_J is a global homeomorphism sending C^k endomorphisms $(k \geqslant l)$ in $h_l^{S^1}(TM, J)$ to C^k almost complex structures. Consequently, we use ϕ_J to define charts at J on $\mathcal{J}_l^{S^1}$ (See [14] for more details).

Let $I_l^{S^1}$ denote the space of integrable almost complex structures of M with regularity C^l. We first show that $I_l^{S^1}$is a Banach submanifold of $\mathcal{J}_l^{S^1}$. Let $N_J(X, Y) = [X, Y] + J([JX, Y] + [X, JY]) - [JX, JY]$ denote the Nijenhuis tensor with respect to J. We construct a vector bundle over $\mathcal{J}_l^{S^1}$ with fibres $\Omega_{l-1}^{0,2}(M, TM)^{S^1}$ of S^1- invariant $(0, 2)$ forms with regularity C^{l-1} and take values in the holomorphic tangent bundle TM. We construct of this bundle with fibre $\Omega_{l-1}^{0,2}(M, TM)^{S^1}$ as a sub-bundle of the trivial bundle $\mathcal{J}_l^{S^1} \times \left(\Omega_{l-1}^{2}(M, TM \otimes \mathbb{C})\right)^{S^1}$. It is indeed easy to check that the Nijenhuis tensor takes elements in $\mathcal{J}_l^{S^1}$ to elements in $\Omega_{l-1}^{0,2}(M, TM)^{S^1}$. This is because

$$\begin{aligned} g \cdot N_J(X, Y) &:= g \cdot [X, Y] + g \cdot J([JX, Y] + [X, JY]) - g \cdot [JX, JY] \\ &= g_*[g_*^{-1}X, g_*^{-1}Y] + g_*J\left([Jg_*^{-1}X, g_*^{-1}Y] + [g_*^{-1}X, Jg_*^{-1}Y]\right) - g_* \cdot [Jg_*^{-1}X, Jg_*^{-1}Y] \\ &= [X, Y] + J([JX, Y] + [X, JY]) - [JX, JY] \end{aligned}$$

Where the last equality follows from the fact that $g_*[X,Y] = [g_*X, g_*Y]$ and noting that J is equivariant. Thus we have a well defined map

$$N : \mathcal{J}_l^{S^1} \to \Omega^{0,2}_{l-1}(M,TM)^{S^1}$$
$$J \mapsto N_J$$

If we show that the derivative of this map is surjective then indeed we would have that I_l is a Banach submanifold.

As shown in Appendix A of [3], we can define $\overline{\partial}_J : \Omega^{0,1}_{J,l}(M,TM) \to \Omega^{0,2}_{J,l-1}(M,TM)$ for any almost complex structure J. Further let $\nabla N(J)$ denote the following composition.

$$\begin{tikzcd} \left(\Omega^{0,1}_{J,l}(M,TM)\right)^{S^1} \arrow[r, "dN_J"] \arrow[rr, bend right, "\nabla N_J"'] & \left(\Omega^{2}_{l-1}(M,TM\otimes\mathbb{C})\right)^{S^1} \arrow[r, "\pi"] & \Omega^{0,2}_{l-1}(M,TM)^{S^1} \end{tikzcd}$$

where π is the canonical projection of $\Omega^2_{l-1}(M,TM\otimes\mathbb{C})^{S^1} \cong \Omega^{2,0}_{l-1}(M,TM)^{S^1} \oplus \Omega^{1,1}_{l-1}(M,TM)^{S^1} \oplus \Omega^{0,2}_{l-1}(M,TM)^{S^1}$ onto the last summand. Then,

Theorem 5.1.1. $\nabla N(J) = -2J\overline{\partial}_J$

Proof. Check Appendix A in [3]. □

In particular to show that $\nabla N(J)$ is surjective, we require that $\overline{\partial}_J : \Omega^{0,1}_l(M,TM)^{S^1} \to \Omega^{0,2}_{l-1}(M,TM)^{S^1}$ is surjective. This is trivially true whenever the manifold as M is 4 dimensional and $H^{0,2}_J(M,TM) = 0$. Thus we have the following theorem.

Theorem 5.1.2. *Suppose M is a 4- manifold with $H^{0,2}_J(M,TM) = 0$ with $J \in I_l^{S^1}$,then the space $I_l^{S^1}$ is a Banach submanifold of $\mathcal{J}_l^{S^1}$ in a neighbourhood of J with tangent space at J identified with* $\ker\overline{\partial}_J : \Omega^{0,1}_l(M,TM)^{S^1} \to \Omega^{0,2}_{l-1}(M,TM)^{S^1}$ *or equivalently*

$$T_J I_l \cong \left(\operatorname{im}\overline{\partial}_J : \left(\Omega^{0,0}_l(M,TM)\right)^{S^1} \to \Omega^{0,1}_l(M,TM)^{S^1}\right) \oplus \left(H^{0,1}(M,TM)\right)^{S^1}$$

Let $\mathcal{J}^{S^1}_{\omega,l}$ denote the space of all S^1 equivariant <u>compatible</u> almost complex structures of regularity C^l. Our next goal is to show for under some cohomological restrictions, that the space of equivariant integrable compatible almost complex structures of regularity C^l denoted by $I^{S^1}_{\omega,l}$ is a Banach submanifold of $\mathcal{J}^{S^1}_{\omega,l}$. We first note that Given $J \in \mathcal{J}^{S^1}_{\omega,l}$, that the equivariant metric $h_J(\cdot,\cdot) := \omega(\cdot, J\cdot) - i\omega(\cdot,\cdot)$ induced by the pair (ω, J) identifies $T_J\mathcal{J}^{S^1}_l = \Omega^{0,1}_l(M,TM)^{S^1}$ with the space $(T^{0,2})^{S^1} := (\Omega^{0,2}(M))^{S^1} \otimes (\Omega^{0,2}(M))^{S^1}$ of complex equivariant $(0,2)$-tensors via the map

$$\theta : \left(T^{0,2}\right)^{S^1} \to \Omega^{0,1}_l(M,TM)^{S^1}$$
$$A \mapsto \theta(A) := h_J(A\cdot,\cdot)$$

Let us denote by $S\Omega^{0,1}_l(M,TM)^{S^1}$ the tangent space of $T_J\mathcal{J}^{S^1}_{\omega,l} \subset T_J\mathcal{J}^{S^1}_l$ of all equivariant compatible almost complex structures. More explicitly, the tangent space consists of elements $A \in \Omega^{0,1}_l(M,TM)^{S^1}$ such that $AJ + JA = 0$ and $\omega(A\cdot,\cdot) = -\omega(\cdot, A\cdot)$. Under the above identification, we can check that $S\Omega^{0,1}_l(M,TM)^{S^1}$ gets mapped to the space of symmetric S^1 invariant $(0,2)$-tensors which we denote by $(S^{0,2})^{S^1}$.

Further we may identify the quotient with the space of invariant $(0,2)$ forms on M as follows.

$$T_J\mathcal{J}^{S^1}_l / T_J\mathcal{J}^{S^1}_{\omega_\lambda,l} = \Omega^{0,1}_l(M,TM)^{S^1} / S\Omega^{0,1}_l(M,TM)^{S^1} \cong T^{0,2}_J(M) / \left(S^{0,2}\right)^{S^1} = \left(\Omega^{0,2}_J(M)\right)^{S^1}$$

As before, the Nijenhuis tensor defines a map

$$N : \mathcal{J}^{S^1}_{\omega,l} \to \Omega^{0,2}_{l-1}(M,TM)^{S^1}$$

whose kernel is precisely the submanifold $I^{S^1}_{\omega,l}$, It would once again suffice to show that the derivative ∇N is surjective. As we know that $\nabla N(J) = -2J\overline{\partial}_J$, we would need to show that $\overline{\partial} : S\Omega^{0,1}_l(M,TM)^{S^1} \to \Omega^{0,2}_{l-1}(M,TM)^{S^1}$ is surjective. As M is a 4-manifold, all forms in $\Omega^{0,2}_{l-1}(M,TM)^{S^1}$ are in fact closed, hence to show that the restriction of $\overline{\partial}$ to

$S\Omega_l^{0,1}(M,TM)^{S^1}$ is surjective, it would suffice to show that the group

$$\begin{aligned} SH_J^{0,2}(TM)^{S^1} &:= \frac{\ker\overline{\partial} : \Omega_{l-1}^{0,2}(M,TM)^{S^1} \to \left(\Omega_{l-2}^{0,3}(M,TM)\right)^{S^1}}{\operatorname{im}\overline{\partial} : S\Omega_l^{0,1}(M,TM)^{S^1} \to \Omega_{l-1}^{0,2}(M,TM)^{S^1}} \\ &= \frac{\Omega_{l-1}^{0,2}(M,TM)^{S^1}}{\operatorname{im}\overline{\partial} : S\Omega_l^{0,1}(M,TM)^{S^1} \to \Omega_{l-1}^{0,2}(M,TM)^{S^1}} \end{aligned}$$

is in fact 0.

As the above condition is not easy to check directly, we would like to get a simpler condition on the manifold that would guarantee that $SH_J^{0,2}(TM)^{S^1}$ is indeed 0. In order to do that, we consider the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & S\Omega_l^{0,1}(M,TM)^{S^1} \arrow[r, "\overline{\partial}"] \arrow[d] & \Omega_{l-1}^{0,2}(M,TM)^{S^1} \arrow[r] \arrow[d] & 0 \\ \Omega_{l+1}^{0,0}(M,TM)^{S^1} \arrow[r, "\overline{\partial}"] \arrow[d, "\alpha"] & \Omega_l^{0,1}(M,TM)^{S^1} \arrow[r, "\overline{\partial}"] \arrow[d] & \Omega_{l-1}^{0,2}(M,TM)^{S^1} \arrow[r] \arrow[d] & 0 \\ \Omega_{l+1}^{0,1}(M)^{S^1} \arrow[r, "\overline{\partial}"] & \Omega_l^{0,2}(M)^{S^1} \arrow[r, "\overline{\partial}"] & 0 \arrow[r] & 0 \end{tikzcd}$$

where the map $S\Omega_l^{0,1}(M,TM)^{S^1} \to \Omega_l^{0,1}(M,TM)^{S^1}$ is just the inclusion and the map $\Omega_l^{0,1}(TM)^{S^1} \to \Omega_l^{0,2}(M)^{S^1}$ is the quotient $\Omega_l^{0,1}(TM)^{S^1} \to \Omega_l^{0,1}(TM)^{S^1}/S\Omega_l^{0,1}(M,TM)^{S^1}$ followed by identifying $\Omega_l^{0,1}(TM)^{S^1}/S\Omega_l^{0,1}(M,TM)^{S^1} \cong \Omega_{l-1}^{0,2}(M,TM)^{S^1}$. We refer the reader to page 548 of [3] for more details about how this identification is made. Finally the map α is defined as follows

$$\begin{aligned} \alpha : \Omega_{l+1}^{0,0}(M,TM)^{S^1} &\to \Omega_{l+1}^{0,1}(M)^{S^1} \\ X &\mapsto \alpha(X)(Y) := \omega(X,JY) - i\omega(X,Y) \end{aligned}$$

where $J \in I_{\omega,l}^{S^1}$ and $X, Y \in \Omega_{l+1}^{0,0}(M,TM)^{S^1}$. We refer the reader to Appendix B in [3] for the proof of commutativity of the diagram. We note that the proof for the equivariant case follows mutatis mutandis from the proof in Appendix B of [3] by observing that the $\overline{\partial}$ operator is equivariant and hence takes invariant elements to invariant ones. Thus the above diagram gives rise to a long exact sequence is cohomology

$$\begin{aligned} 0 \longrightarrow \left(H^0_J(TM)\right)^{S^1} \longrightarrow cl\Omega^{0,1}_J(M)^{S^l} \xrightarrow{\delta} clS\Omega^{0,1}_J(M,TM)^{S^1} \xrightarrow{q} H^{0,1}_J(TM)^{S^1} \longrightarrow \\ \longrightarrow H^{0,2}_J(M)^{S^1} \longrightarrow SH^{0,2}_J(TM)^{S^1} \longrightarrow H^{0,2}_J(TM)^{S^1} \longrightarrow 0 \end{aligned} \tag{5.1}$$

where $cl\Omega^{0,1}_J(M)^{S^l}$ denotes the kernel of $\overline{\partial}$ in $\left(\Omega^{0,1}_J(M)\right)^{S^l}$ and similarly $clS\Omega^{0,1}_J(M,TM)^{S^1}$ is the kernel of $\overline{\partial} : S\Omega^{0,1}_l(M,TM)^{S^1} \to \Omega^{0,2}_{l-1}(M,TM)^{S^1}$. Thus if we had a 4-manifold M with an S^1 invariant compatible integrable almost complex structure J such that $H^{0,2}_J(M) = 0$ and $H^{0,2}_J(TM) = 0$, noting that $\overline{\partial}$ takes S^1 invariant elements to S^1 invariant elements we can conclude that $H^{0,2}_J(M)^{S^1} = 0$ and $H^{0,2}_J(M,TM)^{S^1} = 0$. Further, it follows from equation 5.1 for such a manifold (M,ω,J) as above, that $SH^{0,2}_J(TM)^{S^1} = 0$ and hence $I^{S^1}_{\omega,l}$ would indeed be a manifold in a neighbourhood of such a J. Thus $H^{0,2}_J(M) = 0$ and $H^{0,2}(TM) = 0$ gives us a simpler condition for when $I^{S^1}_{\omega,l}$ would indeed be a manifold in a neighbourhood of J as required.

Additionally the averaging operator commutes with the $\overline{\partial}$ operator, $H^{0,2}_J(M) = 0$ implies that $H^{0,2}_J(M)^{S^1} = 0$. This tells us that $q : clS\Omega^{0,1}_J(M,TM)^{S^1} \to H^{0,1}_J(TM)^{S^1}$ is surjective and hence by the first isomorphism theorem we have $\frac{clS\Omega^{0,1}_J(M,TM)^{S^1}}{ker\ q: clS\Omega^{0,1}_J(M,TM)^{S^1} \to H^{0,1}_J(TM)^{S^1}}$ is isomorphic to $H^{0,1}_J(TM)^{S^1}$. Then the above long exact sequence gives us

$$\frac{clS\Omega^{0,1}_J(M,TM)^{S^1}}{ker\ q} \cong \frac{clS\Omega^{0,1}_J(M,TM)^{S^1}}{\operatorname{im}\delta} \cong H^{0,1}_J(M,TM)^{S^1}$$

Putting all this together we have that

Theorem 5.1.3. *If (M,ω,J) is a Kähler 4-manifold such that the groups $H^{0,2}_J(M) = 0$ and $H^{0,2}_J(TM) = 0$, then $I^{S^1}_{\omega,l}$ is a Banach submanifold of $\mathcal{J}^{S^1}_{\omega,l}$ in a neighbourhood of J with tangent space at $J \in I^{S^1}_{\omega,l}$ identified with*

$$T_J I^{S^1}_{\omega,l} = clS\Omega^{0,1}_J(M,TM)^{S^1} = \ker\overline{\partial} : S\Omega^{0,1}_l(M,TM)^{S^1} \longrightarrow \Omega^{0,2}_{l-1}(M,TM)^{S^1}$$

or equivalently

$$T_J I^{S^1}_{\omega,l} \cong \operatorname{im}\delta \bigoplus H^{0,1}_J(TM)^{S^1}$$

Proposition 5.1.4. *The conditions $H^{0,2}_J(M) = 0$ and $H^{0,2}_J(M,TM) = 0$ are satisfied for all the Hirzebruch surfaces.*

Proof. To prove $H^{0,2}_J(M, TM) = 0$ for all Hirzebruch surfaces see the computation in Example 6.2 b) pg 312 in [27]. To prove, $H^{0,2}_J(M) = 0$ we note that the the rank of $H^{0,2}_J(M) = 0$ (usually called the geometric genus p_g) is a birational invariant. As all Hirzebruch surfaces are birationally equivalent, the result follows from the computation on pg 220 in [27]. $\square$

Finally we would like to show that the strata $U_{m,l} \cap \mathcal{J}^{S^1}_{\omega,l}$ is a Banach submanifold of $\mathcal{J}^{S^1}_{\omega,l}$. The most naive method to try to prove this would be to consider the universal moduli space $\mathcal{M}(B - \frac{m}{2}F, \mathcal{J}_{\omega_\lambda})$ of curves in the class $B - \frac{m}{2}F$ and try to prove that $\mathcal{J}^{S^1}_{\omega_\lambda,l}$ intersects the image of $\mathcal{M}(B - \frac{m}{2}F, \mathcal{J}_{\omega_\lambda})$ under the projection map to the space of all compatible almost complex structures of regularity C^l.

$$\begin{tikzcd} & \mathcal{M}(B - \frac{m}{2}F, \mathcal{J}_{\omega_\lambda}) \arrow[d, "\pi"] \\ \mathcal{J}^{S^1}_{\omega_\lambda,l} \arrow[r, "i"] & \mathcal{J}_{\omega_\lambda} \end{tikzcd}$$

However, this approach is flawed as the two maps are never transversal. The alternative method is to try to define an equivariant universal moduli space $\mathcal{M}^{S^1}(B - \frac{m}{2}F, \mathcal{J}^{S^1}_{\omega_\lambda,l})$ and argue that the image under the projection to $\mathcal{J}^{S^1}_{\omega_\lambda,l}$ is a Banach submanifold of $\mathcal{J}^{S^1}_{\omega_\lambda,l}$. This is the plan of action we implement in the following section.

5.2 Construction of Equivariant moduli spaces

In this section we construct the moduli space of S^1 invariant J-holomorphic maps into $S^2 \times S^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Once again we shall present the analysis and note that all the arguments go through even in the case when the target manifold is $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Recall that J_k was the standard complex structure on the k^{th} Hizerbruch surface. As seen in Chapter 2 there is a standard J_k-holomorphic curve $\overline{D}$ in $S^2 \times S^2$ in the homology class $B - \frac{k}{2}F$. Consider the $S^1(a, b; m)$ action on $(S^2 \times S^2, \omega_\lambda)$. From the graph for the circle action $S^1(a, b; m)$ we see that S^1 acts on $\overline{D}$ in a non-effective manner with global stabilizer $\mathbb{Z}_a$. The following theorem is useful in our analysis.

Lemma 5.2.1. *Consider the $S^1(a,b;m)$ action on $(S^2 \times S^2, \omega_\lambda)$ or $(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$. Let S be any $S^1(a,b;m)$-invariant symplectic embedded sphere in the same homology class $B - \frac{k}{2}F$ with $k > 0$. Then the S^1 action on S has global stabilizer isomorphic to $\mathbb{Z}_a$.*

Proof. This follows 3.1.3 from noting that any other S^1 invariant curve passes through the same set of fixed points as $\overline{D}$, and hence by 3.1.3 again we that the global stabilizer is the same. □

Thus we can fix an action on a base sphere, namely the standard S^1 action that agrees with the action of S^1 on $\overline{D}$ and consider the moduli space of all equivariant maps $u : (S^2, j_o) \to (S^2 \times S^2, J)$ for some $J \in \mathcal{J}^{S^1}_{\omega_\lambda, l}$. We define $\mathcal{M}^{S^1}(B - kF, \mathcal{J}^{S^1}_{\omega_\lambda, l})$ as follows

$$\mathcal{M}^{S^1}(B - kF, \mathcal{J}^{S^1}_{\omega_\lambda, l}) := \{(u, J) \mid \text{u is equivariant, somewhere injective, J-holomorphic and represents the class } B - kF\}$$

Remark 5.2.2. *As we are only interested in the case when $k > 0$, the curves in class $B - \frac{k}{2}F$ have negative self intersection and the adjunction formula tells us that these curves are embedded. Thus all somewhere injective curves in class $B - \frac{k}{2}F$ for $k > 0$ are embedded.*

As in the non-equivariant case we now wish to prove that this moduli space is a smooth Banach manifold.

To prove this we recall the set up. The set up is analogous to the non-equivariant set up as in [31]. We have a bundle

$$\begin{array}{c} \mathscr{E}^{S^1}_{k-1,p} \\ \big\downarrow{\scriptstyle \pi} \\ B^{S^1}_{k,p} \times \mathcal{J}^{S^1}_l \end{array}$$

where $\mathscr{E}^{S^1}$ is a vector bundle with fibre over (u, J) consisting of S^1 invariant elements in $\Gamma(S^2, \Omega^{0,1}_J(S^2, u^*T(S^2 \times S^2))$, $B^{S^1}_{k,p} := \{u \in \left(W^{k,p}(S^2, S^2 \times S^2)\right)^{S^1} \mid [u] = B - kF\}$ and $W^{k,p}(S^2, S^2 \times S^2))^{S^1}$ denotes the space ofequivariant maps of Sobolev regularity $W^{k,p}$

from S^2 to $S^2 \times S^2$.

We would like to show that the section $\overline{\partial}_J^{-1} := (u, \overline{\partial}_J u) : B^{k,p} \times \mathcal{J}_l^{S^1} \to \mathscr{E}_{k-1,p}^{S^1}$ where $\overline{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j_{S^2})$ is transversal to the zero section. Note that $\overline{\partial}_J^{-1}(0) = \mathcal{M}^{S^1}(B - kF, \mathcal{J}_{\omega_\lambda,l}^{S^1})$, thus giving it a smooth structure as in the non-equivariant case (McDuff- Salamon Lemma 3.2.1)

In order to show trasnversality we project the map to the fibre and show that the derivative is surjective at (u, J) when u is a simple equivariant curve. (Check Lemma 3.2.1 in [31] for more details) i.e

$$D\mathscr{F}_{u,J}^{S^1} : \mathscr{W}^{k,p}(S^2, u^*T(S^2 \times S^2))^{S^1} \times C^l(S^2 \times S^2, \mathrm{End}(TS^2 \times S^2, J, \omega)^{S^1} \\ \to \mathscr{W}^{k,p}(S^2, \Omega_J^{0,1}(S^2, u^*T(S^2 \times S^2)))^{S^1}$$

But by Lemma 3.2.1 in [31] we know that the linearized derivative $D\mathscr{F}_{u,J}$ in the non-equivariant case

$$D\mathscr{F}_{u,J} : \mathscr{W}^{k,p}(S^2, u^*T(S^2 \times S^2)) \times C^l(S^2 \times S^2, End(TS^2 \times S^2, J, \omega) \\ \to \mathscr{W}^{k,p}\left((S^2, \Omega_J^{0,1}\left(S^2, u^*T(S^2 \times S^2)\right)\right)$$

is surjective.

As $J \in \mathcal{J}_{\omega_\lambda}^{S^1}$, the $\bar{\partial}_J$ operator commutes with the averaging operator with respect to the S^1 action. Averaging the above non-equivariant derivative $D\mathscr{F}_{u,J}$ by the S^1 action would prove that $D\mathscr{F}_{u,J}^{S^1}$ is surjective. Hence we can conclude the following theorem.

Theorem 5.2.3. *$\mathcal{M}^{S^1}(B - kF, \mathcal{J}_{\omega_\lambda,l}^{S^1})$ is a smooth Banach manifold.*

□

Also we have the projection map

$$\begin{array}{c} \mathcal{M}^{S^1}(B - kF, \mathcal{J}_{\omega_\lambda,l}^{S^1}) \\ \downarrow \pi \\ \mathcal{J}_{\omega_\lambda,l}^{S^1} \end{array}$$

To conclude that the image of π is a submanifold of $\mathcal{J}^{S^1}_{\omega_\lambda,l}$ we need the following theorem whose proof can be found in [1].

Theorem 5.2.4. *(Theorem 3.5.18 in [1]) If there is a smooth map $f : M \to N$ where M, N are Banach manifolds such that*

1. $\ker Tf$ *is a sub-bundle of* TM
2. *For each* $m \in M$, $f_*(T_m M)$ *is closed and splits in* $T_{f(m)}N$
3. f *is open or closed onto it's image*

Then $f(M)$ is a smooth Banach submanifold of N

A map that satisfies the above conditions is called a sub-immersion.

Lemma 5.2.5. *The projection map $\pi : \mathcal{M}^{S^1}(B - kF, \mathcal{J}^{S^1}_{\omega_\lambda,l}) \to \mathcal{J}^{S^1}_{\omega_\lambda,l}$ is a sub-immersion.*

Proof. Note that the $\ker d\pi$ is of constant rank and is the tangent space to the reparametrization group $\mathbb{C}^*$ is which freely on $\mathcal{M}^{S^1}(B - kF, \mathcal{J}^{S^1}_{\omega_\lambda,l})$. Hence $\ker d\pi$ is a sub-bundle of $T\mathcal{M}^{S^1}(B - kF, \mathcal{J}^{S^1}_{\omega_\lambda,l})$.

Now we show that the image of $d\pi$ is closed $T_J\mathcal{J}^{S^1}_{\omega_\lambda,l}$. Note first that $T_J\mathcal{J}^{S^1}_{\omega_\lambda,l} = S\Omega^{0,1}_J(M,TM)^{S^1}$, hence $\pi_*\left(T_{(u,J)}\mathcal{M}^{S^1}\left(B - kF, \mathcal{J}^{S^1}_{\omega_\lambda,l}\right)\right)$ as a subspace of $S\Omega^{0,1}_J(M,TM)^{S^1}$ can be described as follows:

$$\pi_* T_{(u,J)}\mathcal{M}^{S^1}\left(B - kF, \mathcal{J}^{S^1}_{\omega_\lambda,l}\right) =$$
$$= \left\{\alpha \in S\Omega^{0,1}_J(M,TM)^{S^1} \mid [\alpha \circ du \circ j_{S^2}] = 0 \in H^{0,1}_J(S^2, u^*TM)^{S^1}\right\} \quad (5.2)$$

(This follows from noting that the proof of proposition 2.8 in [3] goes through under the presence of a compact group action.) Let $\gamma_n \in \pi_*\left(T_{(u,J)}\mathcal{M}^{S^1}\left(B - kF, \mathcal{J}^{S^1}_{\omega_\lambda,l}\right)\right)$ i.e $\gamma_n \in \Omega^{0,1}(M,TM)^{S^1} = T\mathcal{J}^{S^1}_{\omega_\lambda,l}$ and satisfies $[\gamma_n \circ du \circ J_{S^2}] = 0 \in H^{0,1}_J(S^2, u^*TM)^{S^1}$. Further assume the sequence γ_n converges to γ in $\Omega^{0,1}(M,TM)^{S^1}$. Then $[\gamma \circ du \circ j_{S^2}] = 0 \in$

$H^{0,1}_{j_{S^2}}(S^2, u^*TM)^{S^1}$, thus showing image of $d\pi$ is closed $T\mathcal{J}^{S^1}_{\omega_\lambda,l}$.

Next to that the image of $d\pi$ splits in $T\mathcal{J}^{S^1}_{\omega_\lambda,l}$, we proceed as follows. We firstly show that the codimension of the image of $d\pi$ in $T\mathcal{J}^{S^1}_{\omega_\lambda,l}$ is finite and hence it follows that the image of $d\pi$ splits. Consider the map

$$\begin{aligned} L : S\Omega^{0,1}_J(M, TM)^{S^1} &\to H^{0,1}_{j_{S^2}}\left(S^2, u^*TM\right)^{S^1} \\ \alpha &\mapsto [\alpha \circ du \circ j_{S^2}] \end{aligned}$$

By equation 5.2 we see that the kernel of this map is precisely the image of the map $d\pi$. As $H^{0,1}_{j_{S^2}}\left(S^2, u^*TM\right)^{S^1}$ is finite dimensional it follows that the codimension of $d\pi$ is finite and hence the image of $d\pi$ in $T\mathcal{J}^{S^1}_{\omega_\lambda,l}$ splits.

Finally to show that π is open onto it's image we note that,

$$\begin{tikzcd} \mathcal{M}^{S^1}(B-kF, \mathcal{J}^{S^1}_{\omega_\lambda,l}) \arrow[d, "q"] \arrow[dr, "\pi"] & \\ \mathcal{M}^{S^1}(B-kF, \mathcal{J}^{S^1}_{\omega_\lambda,l})/\mathbb{C}^* \arrow[r, "h", "\cong"'] & \text{im } \pi \end{tikzcd}$$

where the map $h : \mathcal{M}^{S^1}(B-kF, \mathcal{J}^{S^1}_{\omega_\lambda,l})/\mathbb{C}^* \to \text{im}\pi$ is a homeomorphism. As q is a quotient map for a group action, we have that q is an open map and as $\pi = h \circ q$, we have the π too is an open map, thus showing that π satisfies all the conditions in the lemma and hence π is a sub-immersion. $\square$

Corollary 5.2.6. *$U_{2k,l} \cap \mathcal{J}^{S^1}_{\omega_\lambda,l}$ is a Banach submanifold of $\mathcal{J}^{S^1}_{\omega_\lambda,l}$.*

Proof. Follows from Lemma 5.2.5 and noting that the image of π is $U_{2k,l} \cap \mathcal{J}^{S^1}_{\omega_\lambda,l}$. $\square$

Finally we would like to understand what the normal bundle to $U_{2k,l} \cap \mathcal{J}^{S^1}_{\omega_\lambda,l}$ looks like (when $k > 0$). To do this consider the following

$$\begin{tikzcd} & \mathcal{M}^{S^1}(B-kF, \mathcal{J}^{S^1}_{\omega_\lambda,l}) \arrow[d, "\pi"] \\ I^{S^1}_{\omega,l} \arrow[r, "i"] & \mathcal{J}^{S^1}_{\omega_\lambda,l} \end{tikzcd}$$

We would like to arrive at a condition when these two spaces intersect transversely and thus we can use this transverse intersection to get a description of the normal bundle of $U_{2k,l} \cap \mathcal{J}^{S^1}_{\omega_\lambda,l}$.

Lemma 5.2.7. *For any of the Hirzebruch surfaces $(S^2 \times S^2, \omega_\lambda, J_m)$ or $(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda, J_m)$ we have that $u^* : H^{0,1}_{J_m}(M, TM)^{S^1} \to H^{0,1}_{j_{S^2}}(S^2, u^*TM)^{S^1}$ is an isomorphism.*

Proof. From Proposition 3.4 in [3] we know that $u^* : H^{0,1}_{J_m}(M, TM) \to H^{0,1}_{j_{S^2}}(S^2, u^*TM)$ is an isomorphism. As u is equivariant this indeed gives us that

$$u^* : H^{0,1}_{J_m}(M, TM)^{S^1} \longrightarrow H^{0,1}_{j_{S^2}}\left(S^2, u^*TM\right)^{S^1}$$

is also an isomorphism. □

Lemma 5.2.8. *Let (M, ω_λ) denote either $(S^2 \times S^2, \omega_\lambda)$ or $(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$. Further let $i : I^{S^1}_{\omega,l} \hookrightarrow \mathcal{J}^{S^1}_{\omega_\lambda,l}$ denote the inclusion and $\pi : \mathcal{M}^{S^1}(B - kF, \mathcal{J}^{S^1}_{\omega_\lambda,l}) \to \mathcal{J}^{S^1}_{\omega_\lambda,l}$ denote the projection. Then we have that*

- $i \pitchfork \pi$
- *the infinitesimal complement(i.e the fibre of the normal bundle) of $U_{m,l} \cap \mathcal{J}^{S^1}_{\omega_\lambda,l}$ at $J_m \in I^{S^1}_{\omega,l}$ can be identified with $H^{0,1}_{J_m}(M, TM)^{S^1}$.*

Proof. Recall by Theorem 5.1.3, that the tangent space of $I^{S^1}_{\omega,l}$ was given by

$$T_J I^{S^1}_{\omega,l} = clS\Omega^{0,1}_J(M, TM)^{S^1} := \ker \overline{\partial} : S\Omega^{0,1}_l(M, TM)^{S^1} \longrightarrow \Omega^{0,2}_{l-1}(M, TM)^{S^1}$$

Let $\gamma \in T_J \mathcal{J}^{S^1}_{\omega_\lambda,l} = \left(S\Omega^{0,1}(M, TM)\right)^{S^1}$ and define $[\gamma \circ du \circ j_{S^2}] := \eta \in H^{0,1}_{j_{S^2}}(S^2, u^*TM)^{S^1}$. To show that $i \pitchfork \pi$, we need to produce $\beta \in T_{J_m} I^{S^1}_{\omega,l} = clS\Omega^{0,1}_{J_m}(M, TM)^{S^1}$ such that $[(\gamma - \beta) \circ du \circ j_{S^2}] = 0$. To do so, we consider the following commutative diagram.

$$\begin{tikzcd}
H^{0,1}_{J_m}(M,TM)^{S^1} \arrow[rrr, "u^*"] \arrow[ddd, "J_m^*"'] & & & H^{0,1}_{j_{S^2}}(S^2,u^*TM)^{S^1} \arrow[ddd, "j_{S^2}^*"] \\
& {[\alpha]} \arrow[r, mapsto] \arrow[d, mapsto] & {[\alpha\circ du]} \arrow[d, mapsto] & \\
& {[\alpha\circ J_m]} \arrow[r, mapsto] & {[\alpha\circ du\circ j_{S^2}] = [\alpha\circ J_m\circ du]} & \\
H^{0,1}_{J_m}(M,TM)^{S^1} \arrow[rrr, "u^*"'] & & & H^{0,1}_{j_{S^2}}(S^2,u^*TM)^{S^1}
\end{tikzcd}$$

where all the maps u^*,J^* and $j^*_{S^2}$ are isomorphisms. Further we have the equality $[\alpha \circ du \circ j_{S^2}] = [\alpha \circ J \circ du]$ as u is j_{S^2}-J_m holomorphic. As we know that $H^{0,2}_{J_m}(M)^{S^1} = 0$, from the long exact sequence equation 5.1 we see that the quotient map

$$clS\Omega^{0,1}_{J_m}(M,TM)^{S^1} \to H^{0,1}_{J_m}(M,TM)^{S^1}$$

is surjective. As both u^* and J^* are isomorphisms, there exists $\beta \in clS\Omega^{0,1}_{J_m}(M,TM)^{S^1} = T_J I^{S^1}_{\omega_\lambda,l}$ such that $[\beta \circ J \circ du] = [\beta \circ du \circ j_{S^2}] = \eta := [\gamma \circ du \circ j_{S^2}]$. Hence we indeed have $[(\gamma - \beta) \circ du \circ j_{S^2}] = 0$ as required.

We now show that the fibre of the normal bundle of $U_{m,l} \cap \mathcal{J}^{S^1}_{\omega_\lambda,l}$ at $J_m \in I^{S^1}_{\omega,l}$ can be identified with $H^{0,1}_{J_m}(M,TM)^{S^1}$. As seen in the proof of Lemma 5.2.5, we know that there is a map

$$L : S\Omega^{0,1}_J(M,TM)^{S^1} \to H^{0,1}_J\left(S^2, u^*TM\right)^{S^1}$$
$$\alpha \mapsto [\alpha \circ du \circ j_{S^2}]$$

As the quotient map $clS\Omega^{0,1}_{J_m}(M,TM)^{S^1} \to H^{0,1}_{J_m}(M,TM)^{S^1}$ is surjective and the maps u^* and j_{S^2} are isomorphisms, we have that the map L is surjective. As the kernel of L is the image of $d\pi$, the cokernel can be identified with $H^{0,1}_{j_{S^2}}\left(S^2, u^*TM\right)^{S^1} \cong H^{0,1}_{J_m}(M,TM)^{S^1}$ Hence the the fibre of the normal bundle of $U_{m,l} \cap \mathcal{J}^{S^1}_{\omega_\lambda,l}$ at $J_m \in I^{S^1}_{\omega,l}$ can be identified with $H^{0,1}_{J_m}(M,TM)^{S^1}$.

$\square$

Hence to calculate the codimension of $U_{2k,l} \cap \mathcal{J}^{S^1}_{\omega_\lambda,l}$, we only need to calculate the dimension of $H^{0,1}_{J_m}(M,TM)^{S^1}$. We present that calculation in the next section.

5.3 Isotropy representations

By Theorem 4.2 in [3], the action of the isometry group $K(2n) \simeq S^1 \times \mathrm{SO}(3)$ on the space $H^{0,1}_J(M,TM)$ of infinitesimal deformations is isomorphic to $\mathrm{Det} \otimes \mathrm{Sym}^{2n-2}$, where Det is the standard action of $S^1 = U(1)$ on $\mathbb{C}^2$, and where $\mathrm{Sym}(\mathbb{C}^2)$ is the representation $\mathscr{W}_{n-1}$ of SO(3) induced by the $(2n-2)$-fold symmetric product of the standard representation of SU(2) on $\mathbb{C}^2$.

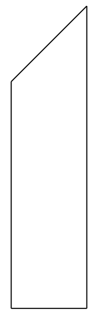
5.3.1 Hirzebruch surfaces and their isometry groups

In this section we mostly try to follow the same notation as in [3]. The reader may note that in our case $2n = m$.

Following [3], we construct the Hirzebruch surface $\mathbb{F}_{2n}$ by Kähler reduction of $\mathbb{C}^4$ under the action of the torus T^2_{2n} defined by

$$(s,t) \cdot z = (s^{2n} t z_1, t z_2, s z_3, s z_4)$$

The moment map is $\phi(z) = (2n|z_1|^2 + |z_3|^2 + |z_4|^2, |z_1|^2 + |z_2|^2)$ and the reduced manifold at level $(\lambda + n, 1)$ is symplectomorphic to $(S^2 \times S^2, \omega_\lambda)$ and biholomorphic to the Hirzebruch surface $\mathbb{F}_{2n}$. In this model, the projection to the base is given by $[(z_1,\ldots,z_4)] \mapsto [z_3 : z_4]$, the zero section is $[w_0 : w_1] \mapsto [(w_0^{2n}, 0, w_0, w_1)]$, and a fiber is $[w_0 : w_1] \mapsto [(w_0 w_1^{2n}, w_0 w_1, 0, w_1)]$. The torus $T^2(2n) = T^4/T^2_{2n}$ acts on $\mathbb{F}_{2n}$. This torus is generated by the elements $[(1, e^{it}, 1, 1)]$ and $[(1, 1, e^{is}, 1)]$, and its moment map is $[(z_1, z_2, z_3, z_4)] \mapsto (|z_2|^2, |z_3|^2)$. The moment polytope $\Delta(2n)$ is the convex hull of the vertices $(0,0)$, $(1,0)$, $(1, \lambda + n)$, and $(0, \lambda - n)$.



The isometry group of $\mathbb{F}_{2n}$ is

$$K(2n) = Z_{\mathrm{U}(4)}(T^2_{2n})/T^2_{2n} = (T^2 \times \mathrm{U}(2))/T^2_{2n} \simeq S^1 \times \mathrm{PU}(2) \simeq S^1 \times \mathrm{SO}(3)$$

where the middle isomorphism is given by

$$[(s,t),A] \mapsto (s^{-1}t \det A^n, [A])$$

Under this isomorphism, an element $[(1,a,b,1)]$ of the torus $T(2n)$ is taken to

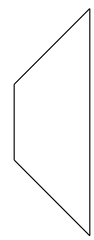
$$\left(ab^n, \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} \right) = \left(b^n a, \begin{bmatrix} b^{1/2} & 0 \\ 0 & b^{-1/2} \end{bmatrix} \right)$$

Consequently, at the Lie algebra level of the maximal tori, the map identifying the maximal torus of $K(2n)$ whose lie algebra is denoted by $\mathfrak{t}^2(2n)$ with the maximal torus $S^1 \times SO(2) \subset S^1 \times SO(3)$ whose lie algebra is denoted by $\mathfrak{t}^2$ (where $SO(2)$ is identified with the rotations around the z-axis) is given by

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

The moment polytope associated to the maximal torus $T^2 \subset K(2n)$ is thus the balanced polytope obtained from $\Delta(2n)$ by applying the inverse transpose $\begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}$.

Thus the moment polytope associated to the maximal torus $T^2 \subset K(2n)$ has the following shape



5.3.2 Even isotropy representations

Let J_m be the standard S^1 invariant integrable almost complex structure in the strata U_m, coming from the Hirzebruch surface W_m. The action of the isometry group $K(2n) \simeq S^1 \times \mathrm{SO}(3)$ on the space $H^{0,1}_{J_m}(S^2 \times S^2, T(S^2 \times S^2) \cong \mathbb{C}^{m-1}$ (see [27] Example 6.2(b)(4), p.309 for more details about how the isomorphism is obtained) of infinitesimal deformations is isomorphic to $\mathrm{Det} \otimes \mathrm{Sym}^{2n-2}$, where Det is the standard action of $S^1 = U(1)$ on $\mathbb{C}^2$, and where $\mathrm{Sym}(\mathbb{C}^2)$ is the representation $\mathscr{W}_{n-1}$ of SO(3) induced by the $(2n-2)$-fold symmetric product of the standard representation of SU(2) on $\mathbb{C}^2$ (see Theorem 4.2 in[3]). We shall denote this $(2n-2)$-fold symmetric product of the standard representation of SU(2) on $\mathbb{C}^2$ as $\mathscr{V}_{2n-2}$. See [8] for more details about the representation theory of SO(3) and SU(2)

The circle of $\mathrm{SO}(3) = \mathrm{PU}(2) = \mathrm{U}(2)/\Delta(S^1)$

$$R(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix}, \ t \in [0, 2\pi)$$

lifts to

$$e(t/2) := \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix} \in \mathrm{SU}(2)$$

so that the character of $\mathscr{W}_{n-1}$ at $R(t)$ is given by (see [8] p.88)

$$\chi_{\mathscr{W}_{n-1}}(R(t)) = \chi_{\mathscr{V}_{2n-2}}(e(t/2)) = \sum_{k=1-n}^{n-1} e^{ikt} = \sum_{k=0}^{2n-2} e^{i(n-1-k)t} = \frac{e^{int} - e^{-int}}{e^{-it} - e^{-it}}$$

The character of Det is,

$$\chi_{\mathrm{Det}}(R(t)) = e^{it}$$

For $p.q \geqslant 0$, we have the orthogonality relations

$$\langle e^{ipt}, e^{iqt} \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{-ipt} e^{iqt}\, dt = \delta_{p,q}$$

In general, given a representation V of a compact group G, the dimension of the invariant subspace V^G is given by (see [8] Thm. 4.11)

$$\dim V^G = \int \chi(g)\, dg$$

For a, b coprime, consider the embedding $S^1 \to S^1(a, b, 2n) \subset K(2n)$ defined by

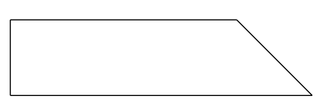
$$\begin{aligned} S^1 &\hookrightarrow S^1 \times \mathrm{SO}(3) = K(2n) \\ t &\mapsto (at, R(bt)) \end{aligned}$$

Then, the $S^1(a, b, 2n)$-invariant subspace of $\mathrm{Det} \otimes \mathrm{Sym}^{2n-2}(\mathbb{C}^2)$ has dimension

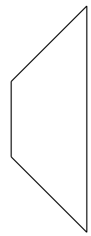
$$\begin{aligned} d_{a,b,2n} &= \frac{1}{2\pi} \sum_{k=1-n}^{n-1} \int_0^{2\pi} e^{iat} e^{ikbt}\, dt = \frac{1}{2\pi} \sum_{k=1-n}^{n-1} \int_0^{2\pi} e^{i(a+bk)t}\, dt \\ &= \begin{cases} 1 & \text{if } a + bk = 0 \text{ for some } k \in \{1-n, \ldots, n-1\} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Note that the above codimension calculation was with respect to the basis of the maximal torus in $K(2n)$. Hence to calculate the codimension for the $S^1(1, b, m) \subset \mathbb{T}^2_m$ as in our case, we need to transform the basis by multiplication by the matrix $\begin{pmatrix} \frac{m}{2} & -1 \\ 1 & 0 \end{pmatrix}$.

Thus it takes the vector $\begin{pmatrix} 1 \\ b \end{pmatrix}$ in the basis for the standard moment polytope



to the vector $\begin{pmatrix} \frac{m}{2} - b \\ 1 \end{pmatrix}$ in the basis for the balanced polytope (for which we did the above calculations).



Therefore the codimension of $S^1(1,b;m)$ is given by the number of $k \in \{1-\frac{m}{2}, \cdots, \frac{m}{2}-1\}$ such that $(\frac{m}{2}-b)+k=0$. Relabelling k' as $\frac{m}{2}+k$, we have that the codimension is given by the number of $k' \in \{1,\cdots,m-1\}$ such that $k'=b$.

Theorem 5.3.1. *Given the circle action $S^1(1,b,m)$ with $2\lambda > |2b-m|$ and $b \neq \{0,m\}$, the complex codimension of the stratum $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_m$ in $\mathcal{J}^{S^1}_{\omega_\lambda}$ in given by the number of $k \in \{1,\cdots,m-1\}$ such that $k=b$.*
Similarly for the action $S^1(-1,b,m)$ with $2\lambda > |2b+m|$ and $b \neq \{0,-m\}$, the complex codimension of of the stratum $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_m$ in $\mathcal{J}^{S^1}_{\omega_\lambda}$ in given by the number of $k \in \{1,\cdots,m-1\}$ such that $k=-b$.

Corollary 5.3.2. *For the circle actions*

- *(i) $a=1$, $b \neq \{0,m\}$, and $2\lambda > |2b-m|$; or*
- *(ii) $a=-1$, $b \neq \{0,-m\}$, and $2\lambda > |2b+m|$.*

The complex codimension of the stratum $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_m$ in $\mathcal{J}^{S^1}_{\omega_\lambda}$is either 0 or 1.

Proof. Follows from the calculation and discussion above. □

Alternative calculation of the codimension

As explained above, the action of $K(2n)$ on $H^{0,1}_{J_m}(S^2 \times S^2, T^{1.0}_{J_m}(S^2 \times S^2)) \cong \mathbb{C}^{m-1}$ is isomorphic to $\mathrm{Det} \otimes \mathrm{Sym}^{2n-2}$.Hence to calculate the the codimension we only need to calculate the dimension of the invariant subspace of $H^{0,1}(S^2 \times S^2, T^{1.0}_{J_m}(S^2 \times S^2)) \cong \mathbb{C}^{m-1}$ under this action. To do so we note that a basis of Sym^{2n-2} is given by the homogeneous polynomials $P_k = z_1^{2n-2-k} z_2^k$ for $k \in \{0,\ldots,2n-2\}$. The action of $R(t)$ on P_k is

$$R(t) \cdot P_k = e(t/2) \cdot P_k = e^{i\left(2n-2-2k\right)t/2} P_k = e^{it(n-1-k)} P_k$$

so that the action of $(e^{is}, R(t)) \subset S^1 \times \mathrm{SO}(3)$ on P_k is

$$(e^{is}, R(t) \cdot P_k = e^{i\left(s+t(n-1-k)\right)} P_k$$

Each P_k generates an eigenspace for the action of the maximal torus $T(2n)$. In particular, the circle $S^1(a, b; 2n)$ acts trivially on P_k if, and only if,

$$a + b(n-1-k) = (a, b) \cdot (1, n-1-k) = 0$$

for $k \in \{0, 2n-2\}$. Equivalently, we must have

$$a + bk = (a, b) \cdot (1, k) = 0$$

for $k \in \{1-n, \ldots, n-1\}$

Hence the dimension of the invariant subspace is given by the number of $k \in \{1-n, \ldots, n-1\}$ such that $a + bk = 0$ as derived in the previous section.

As an aside we also note that, The generator $(1, 1-k)$ (in the balanced basis of $K(2n)$) of $T(2k)/S^1(a, b; 2n)$ then acts on the eigenspace $\langle P_k \rangle$ with weight

$$(1, 1-k) \cdot (1, k) = 1$$

which shows that the action is effective and does not depend on a, b, or $m = 2n$.

Remark 5.3.3. *In the beginning of the section, we only show that the space $\mathcal{J}^{S^1}_{\omega_\lambda, l} \cap U_{2k,l}$ was a Banach submanifold. But in order to obtain the topology of the space of $\mathrm{Symp}^{S^1}(S^2 \times S^2, \omega_\lambda)$ with $\mathbb{C}^\infty$-topology, we require that the space $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{2k}$ with the C^∞ topology is a Fréchet manifold and that the codimension of $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{2k}$ in $\mathcal{J}^{S^1}_{\omega_\lambda}$ is given by the same formula as in Theorem 5.3.1. As this discrepancy exists in the literature even in the non-equivariant case, and as a resolution of this issue is well beyond the scope of the thesis we do not attempt to resolve this here.*

Chapter 6

Odd Hirzebruch surfaces

6.1 Homotopy type of $\mathrm{Symp}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)$

We now compute the centralisers for the S^1 actions on the odd Hirzebruch surfaces. The theory is extremely analogous to the even Hirzebruch case i.e $S^2 \times S^2$, hence we shall only point out the key differences. Most of the setup was discussed in the preliminaries but we shall repeat them in the beginning of this section for the purposes of continuity of exposition.

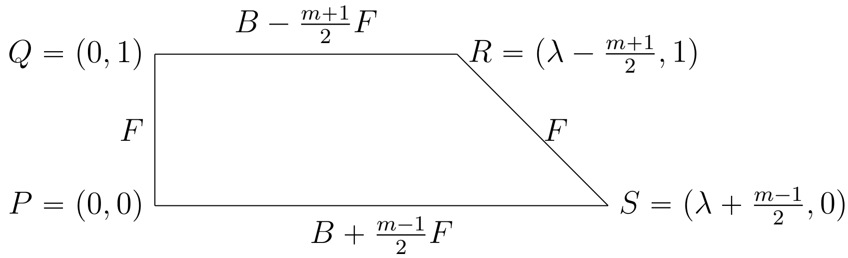
As noted before we have that the odd Hirzebruch surface W_m (where m is odd) is defined as a complex submanifold of $\mathbb{C}P^1 \times \mathbb{C}P^2$ defined by setting

$$W_m := \left\{([x_1, x_2], [y_1, y_2, y_3]) \in \mathbb{C}P^1 \times \mathbb{C}P^2 \mid {x^m}_1 y_2 - x_2 {y^m}_1 = 0\right\}$$

This manifold is diffeomorphic to $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. The Torus $\mathbb{T}^2$ acts on $\mathbb{C}P^1 \times \mathbb{C}P^2$ in the following manner.

$$(u, v) \cdot ([x_1, x_2], [y_1, y_2, y_3]) = ([ux_1, x_2], [u^m y_1, y_2, vy_3])$$

(again with m being odd) and the momentum map image looks like



where B now refers to the homology class of a line L in $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ and F refers to the class $L - E$ where L is the class of the line and E is the class of the exceptional divisor. There is a canonical form which we also call ω_λ on $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, which has weight λ on B and 1 on F. As before all symplectic S^1 action on $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ extend to toric actions. Hence only need to consider sub-circles of the above family of torus actions. The graphs for the different circles are analogous to the $S^2 \times S^2$ case, the only difference being in the momentum map labels. Again as before we have the following stratification of the space of compatible almost complex structures

Theorem 6.1.1. *Let $\mathcal{J}_{\omega_\lambda}$ denote the space of all compatible almost complex structures (not necessarily invariant) for the form ω_λ, then the space $\mathcal{J}_{\omega_\lambda}$ admits a finite decomposition into disjoint Fréchet manifolds of finite codimensions*

$$\mathcal{J}_{\omega_\lambda} = U_1 \sqcup U_3 \sqcup U_5 \ldots \sqcup U_{2n+1}$$

where $n = \lfloor \lambda \rfloor$ is the unique integer such that $n \leqslant \lambda < 2n+1$ and where

$$U_k := \left\{ J \in \mathcal{J}_{\omega_\lambda} \mid (B - \frac{k-1}{2}F) \in H_2(S^2 \times S^2, \mathbb{Z}) \text{ is represented by a } J\text{-holomorphic sphere} \right\}$$

Using similar notation to the discussion in the $S^2 \times S^2$ case, we have the following fibrations.

$$\mathrm{Stab}^{S^1}(\overline{D}) \longrightarrow \mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda) \twoheadrightarrow \mathcal{S}^{S^1}_{D_{2k+1}} \xrightarrow{\simeq} J^{S^1} \cap U_{2k+1}$$

$$\mathrm{Fix}^{S^1}(\overline{D}) \longrightarrow \mathrm{Stab}^{S^1}(\overline{D}) \twoheadrightarrow \mathrm{Symp}^{S^1}(\overline{D}) \xrightarrow{\simeq} S^1 \text{ or } SO(3)$$

$$\mathrm{Fix}^{S^1}(N(\overline{D})) \longrightarrow \mathrm{Fix}^{S^1}(\overline{D}) \twoheadrightarrow \mathrm{Gauge}^{S^1}(N(\overline{D})) \xrightarrow{\simeq} S^1$$

$$\mathrm{Stab}^{S^1}(\overline{F}) \cap \mathrm{Fix}^{S^1}(N(\overline{D})) \longrightarrow \mathrm{Fix}^{S^1}(N(\overline{D})) \twoheadrightarrow \overline{\mathcal{S}^{S^1}_{F,p_0}} \xrightarrow{\simeq} \mathcal{J}^{S^1}(\overline{D}) \simeq \{*\}$$

$$\mathrm{Fix}^{S^1}(\overline{F}) \longrightarrow \mathrm{Stab}^{S^1}(\overline{F}) \cap \mathrm{Fix}^{S^1}(N(\overline{D})) \twoheadrightarrow \mathrm{Symp}^{S^1}(\overline{F}, N(p_0)) \xrightarrow{\simeq} \{*\}$$

$$\{*\} \xleftarrow{\simeq} \mathrm{Fix}^{S^1}(N(\overline{D} \vee \overline{F})) \longrightarrow \mathrm{Fix}^{S^1}(\overline{F}) \twoheadrightarrow \mathrm{Gauge}^{S^1}(N(\overline{D} \vee \overline{F})) \xrightarrow{\simeq} \{*\}$$

Thus we have that when the $S^1(a,b) \subset \mathbb{T}^2_m$ action where $(a,b) \neq (0, \pm 1)$

$$\begin{tikzcd}
\mathrm{Fix}^{S^1}(\overline{D}) \arrow[r] & \mathrm{Stab}^{S^1}(\overline{D}) \arrow[r, two heads] & \mathrm{Symp}^{S^1}(\overline{D}) \\
S^1 \arrow[r] \arrow[u, hook] & \mathbb{T}^2_{2k+1} \arrow[r] \arrow[u, hook] & S^1 \arrow[u, hook]
\end{tikzcd}$$

When $(a,b) = (0, \pm 1)$ we have

$$\begin{tikzcd}
\mathrm{Fix}^{S^1}(\overline{D}) \arrow[r] & \mathrm{Stab}^{S^1}(\overline{D}) \arrow[r, two heads] & \mathrm{Symp}^{S^1}(\overline{D}) \\
S^1 \arrow[r] \arrow[u, hook] & U(2) \arrow[r] \arrow[u, hook] & SO(3) \arrow[u, hook]
\end{tikzcd}$$

Where both the leftmost and the rightmost arrow's are homotopy equivalences from the fibrations set up above. As the diagram above commutes, the left and right most arrows being homotopy equivalences would imply via the 5 lemma that the middle inclusion $\mathbb{T}^2 \hookrightarrow \mathrm{Stab}^{S^1}(\overline{D})$ or $U(2) \hookrightarrow \mathrm{Stab}^{S^1}(\overline{D})$ are also homotopy equivalences. Thus for the action $S^1(a,b;m)$ we have the following cases. If $(a,b) \neq (0,\pm 1)$ then

$$\mathrm{Symp}^{S^1}_h(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)/\mathbb{T}^2_{2k+1} \simeq \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{2k+1}$$

and if $(a,b) = (0, \pm 1)$ then

$$\mathrm{Symp}^{S^1}_h(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)/U(2) \simeq \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{2k+1}$$

.

As before, we have the following theorem.

Theorem 6.1.2. *Consider the circle action $S^1(a, b; m)$ on $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)$, then we have the following cases:*

1. *if $a = 1$, $b \neq \{0, m\}$, $\lambda > 1$, and $2\lambda > |2b - m|$, then the space of $S^1(1, b; m)$-equivariant almost complex structures $\mathcal{J}^{S^1}_{\omega_\lambda}$ intersects the two strata U_m and $U_{|m-2b|}$.*
2. *If $a = -1$,$b \neq \{0, -m\}$, $\lambda > 1$, and $2\lambda > |2b + m|$, then the space of $S^1(-1, b; m)$-equivariant almost complex structures $\mathcal{J}^{S^1}_{\omega_\lambda}$ intersects the two strata U_m and $U_{|m+2b|}$.*
3. *for all other cases $\mathcal{J}^{S^1}_{\omega_\lambda}$ intersects only one strata U_m.*

As before, we use the intersection with strata to conclude the following theorem.

Theorem 6.1.3. *Consider the circle action $S^1(a, b; m)$ on $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)$. Under the following numerical conditions on a, b, m, λ, the homotopy type of $\mathrm{Symp}^{S^1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)$ is given by the table below.*

Values of $(a, b; m)$	λ	*Number of strata* $\mathcal{J}^{S^1}_{\omega_\lambda,l}$ *intersects*	*Homotopy type of* $\mathrm{Symp}^{S^1}(S^2 \times S^2, \omega_\lambda)$
$(0, \pm 1; m)$, $m \neq 0$	$\lambda > 1$	*1*	$U(2)$
$(0, \pm 1; 0)$ *or* $(\pm 1, 0; 0)$	$\lambda = 1$	*1*	$U(2)$
	$\lambda > 1$	*1*	$U(2)$
$(\pm 1, 0; m), m \neq 0$	$\lambda > 1$	*1*	$\mathbb{T}^2$
$(\pm 1, \pm m; m), m \neq 0$	$\lambda > 1$	*1*	$\mathbb{T}^2$
$(1, b; m), b \neq \{m, 0\}$	$\|2b - m\| \geqslant 2\lambda \geqslant 1$	*1*	$\mathbb{T}^2$
$(-1, b; m), b \neq \{-m, 0\}$	$\|2b + m\| \geqslant 2\lambda \geqslant 1$	*1*	$\mathbb{T}^2$
All other values of $(a, b; m)$ *except* $(\pm 1, b; m)$	$\forall \lambda$	*1*	$\mathbb{T}^2$

□

Theorem 6.1.3 gives the homotopy type of the group of equivariant symplectomorphisms for all circle actions apart from the following two families of actions:

- (i) $a = 1$, $b \neq \{0, m\}$, and $2\lambda > |2b - m|$; or
- (ii) $a = -1$, $b \neq \{0, -m\}$, and $2\lambda > |2b + m|$.

For the above family of actions, we compute as in Chapter 4, the dimension of the vector spaces $H^p(\mathrm{Symp}^{S^1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda),k)$ for any field k, . Firstly, we show that the map $\mathbb{T}^2_m \hookrightarrow \mathrm{Symp}^{S^1}_h(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)$ induces a map that is injective in homology. Before we embark on the proof of this claim we set up the following notation.

Fix a curve $\overline{F}$ in the homology class F and passing through the fixed points Q and P in figure 6.1. Let $\mathcal{S}^{S^1}_{F,Q}$ denote the space of S^1-invariant curves in the class F passing through Q (defined in figure 6.1) and let $\mathrm{Symp}^{S^1}_h(S^2\times S^2,\overline{F},\omega_\lambda)$ denote the space of S^1-equivariant symplectomorphisms that pointwise fix the curve $\overline{F}$.

Without loss of generality, assume that $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{|m-2b|}$ is the strata of positive codimension in $\mathcal{J}^{S^1}_{\omega_\lambda}$. As $\mathcal{J}^{S^1}_{\omega_\lambda}$ is contractible, $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_m = \mathcal{J}^{S^1}_{\omega_\lambda} - \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{|m-2b|}$, and the real codimension of $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{|m-2b|}$ in $\mathcal{J}^{S^1}_{\omega_\lambda}$ is 2 (See Corollary 6.2.3), it follows that $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_m$ is connected. Further we have that $\mathrm{Symp}^{S^1}_h(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda) \simeq \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_m/\mathbb{T}^2_m$ is connected. As the fixed points for the $S^1(\pm 1, b, m)$ actions are isolated and as $\mathrm{Symp}^{S^1}_h(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)$ is connected, any element $\phi \in \mathrm{Symp}^{S^1}_h(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)$ takes a fixed point for the S^1 action to itself. Thus the action of $\mathrm{Symp}^{S^1}_h(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)$ on $\mathcal{S}^{S^1}_{F,Q}$ is well defined.

Lemma 6.1.4. *The inclusion* $i : \mathrm{Symp}^{S^1}_h(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\overline{F},\omega_\lambda) \hookrightarrow \mathrm{Symp}^{S^1}_h(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)$ *is a homotopy equivalence.*

Proof. Consider the fibration

$$\mathrm{Symp}^{S^1}_h(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\overline{F},\omega_\lambda) \hookrightarrow \mathrm{Symp}^{S^1}_h(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda) \twoheadrightarrow \mathcal{S}^{S^1}_{F,Q}$$

To show that the action $\mathrm{Symp}^{S^1}_h(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)$ on $\mathcal{S}^{S^1}_{F,Q}$ is transitive we note that given $F' \in \mathcal{S}^{S^1}_{F,Q}$, there exists a $J' \in \mathcal{J}^{S^1}_{\omega_\lambda}$ such that F' is J'-holomorphic. As $\mathcal{J}^{S^1}_{\omega_\lambda}$ is connected, consider a path J_t such that $J_0 = J'$ and $J_1 = J_m$ where J_m is the standard complex structure on the m^{th}-Hirzebruch surface for which the curve $\overline{F}$ is holomorphic.

By Theorem 2.3.5, for every J_t we have a family of curves F_t (with $F_0 = F'$ and $F_1 = \overline{F}$) in class F passing through Q and this curve is S^1 invariant as J_t are S^1 invariant. By Lemma C.0.8 we have a one parameter family of Hamiltonian symplectomorphisms $\phi_t \in \mathrm{Symp}_h^{S^1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)$ such that $\phi_t(F_0) = F_t$ for all t.

Thus it suffices to show that $\mathcal{S}_{F,Q}^{S^1}$ is contractible to complete the proof. To do this note that,

$$\mathcal{J}_{\omega_\lambda}^{S^1}(\overline{F}) \to \mathcal{J}_{\omega_\lambda}^{S^1} \to \mathcal{S}_{F,Q}^{S^1}$$

where $\mathcal{J}_{\omega_\lambda}^{S^1}(\overline{F})$ denotes the space of S^1 invariant almost complex structures for which the curve $\overline{F}$ is J-holomorphic. As both $\mathcal{J}_{\omega_\lambda}^{S^1}(\overline{F})$ and $\mathcal{J}_{\omega_\lambda}^{S^1}$ are contractible, $\mathcal{S}_{F,Q}^{S^1}$ is contractible as well completing the proof. $\square$

Define the following projections just as in the exposition above Theorem 4.2.1. In our case we take the point $\{*\}$ to be the point Q in Figure 6.1.

$$\begin{aligned} s_0 : S^2 &\to W_m \\ [z_0, z_1] &\mapsto ([z_0, z_1], [0, 0, 1]) \end{aligned}$$

and the projection to the first factor of $\mathbb{C}P^1 \times \mathbb{C}P^2$ is

$$\begin{aligned} \pi_1 : W_m &\to S^2 \\ ([z_0, z_1], [w_0, w_1, w_2]) &\mapsto [z_0, z_1] \end{aligned}$$

We define a continuous map $h_1 : \mathrm{Symp}_h^{S^1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda) \to \mathcal{E}(S^2, *)$ by setting

$$\begin{aligned} h_1 : \mathrm{Symp}_h^{S^1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda) &\to \mathcal{E}\left(S^2, *\right) \\ \psi &\mapsto \psi_1 := \pi_1 \circ \psi \circ s_0 \end{aligned}$$

Further define the restriction map $r : \mathrm{Symp}_h^{S^1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \overline{F}, \omega_\lambda) \to \mathcal{E}(S^2, *)$ by just restricting $\phi \in \mathrm{Symp}_h^{S^1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \overline{F}, \omega_\lambda)$ to the fibre $\overline{F}$.

Thus we have a well defined map

$$h : \mathrm{Symp}_h^{S^1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\overline{F},\omega_\lambda) \to \mathcal{E}\left(S^2,*\right)\times\mathcal{E}\left(S^2,*\right)$$
$$\phi \mapsto (h_1(\phi), r(\phi))$$

Theorem 6.1.5. *The inclusion map $i : \mathbb{T}^2_m \hookrightarrow \mathrm{Symp}_h^{S^1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)$ induces a map that is injective in homology.*

Proof. By Lemma 6.1.4, it suffices to prove that the inclusion $i : \mathbb{T}^2_m \hookrightarrow \mathrm{Symp}_h^{S^1}(S^2\times S^2,\overline{F},\omega_\lambda)$ induces a map that is injective in homology.

Composing with h we have an inclusion of $h\circ i : \mathbb{T}^2_m \hookrightarrow \mathcal{E}(S^2,*)\times\mathcal{E}(S^2,*)$ and it suffices to show that this map induces a map that is injective in homology. The proof of this claim in analogous to the proof of Theorem 4.2.1.

Remark 6.1.6. *The same proof as above also shows that for the family of circle actions given by $S^1(1,b,m)$ with $2\lambda > |m-2b|$, the inclusion $\mathbb{T}^2_{|m-2b|}$ into $\mathrm{Symp}_h^{S^1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)$ also induces a map that is injective in homology and similarly for the $S^1(-1,b;m)$ actions with $2\lambda > |m+2b|$, the inclusion $\mathbb{T}^2_{|m+2b|}$ into $\mathrm{Symp}_h^{S^1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)$ also induces a map that is injective in homology.*

□

As in the $S^2\times S^2$ case, we have that $i : \mathbb{T}^2_m \hookrightarrow \mathrm{Symp}^{S^1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)$ induces a map which is injective in homology, From our discussion above we also had that $\mathrm{Symp}_h^{S^1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)/\mathbb{T}^2_m \simeq \mathcal{J}^{S^1}\cap U_m$ and $\mathrm{Symp}_h^{S^1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)/\mathbb{T}^2_{|m-2b|} \simeq \mathcal{J}^{S^1}\cap U_{|m-2b|}$.Let $\mathcal{J}^{S^1}\cap U_m := P$ and $\mathcal{J}^{S^1}\cap U_{|m-2b|};= Q$, as $i : \mathbb{T}^2_m \hookrightarrow \mathrm{Symp}^{S^1}(S^2\times S^2,\omega_\lambda)$ induces a map which is injective in homology, further by Leray-Hirsch theorem we have that

$$H^*(\mathrm{Symp}_h^{S^1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)) \cong H^*(P)\bigotimes H^*(\mathbb{T}^2)$$
$$H^*(\mathrm{Symp}_h^{S^1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)) \cong H^*(Q)\bigotimes H^*(\mathbb{T}^2)$$

As before we need to compute the codimension of the strata $\mathcal{J}^{S^1}_{\omega_\lambda,l} \cap U_{|m-2b|}$in $\mathcal{J}^{S^1}_{\omega_\lambda,l}$. The computation in the section below shows it to be 2 (see Corollary 6.2.3) .

Thus we have the following theorem on the ranks of the homology groups of the space of equivariant symplectomorphisms.

Theorem 6.1.7. *Consider the following circle actions on* $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$

- *(i)* $a = 1$, $b \neq \{0, m\}$, *and* $2\lambda > |2b - m|$; *or*
- *(ii)* $a = -1$, $b \neq \{0, -m\}$, *and* $2\lambda > |2b + m|$.

Then we have

$$H^p\left(\mathrm{Symp}^{S^1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda), k\right) = \begin{cases} k^4 & p \geqslant 2 \\ k^3 & p = 1 \\ k & p = 0 \end{cases}$$

for any field k.

As the proof of Theorem 4.2.6 holds verbatim for the $S^1(a, b; m)$ actions on $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ satisfying the conditions

- (i) $a = 1$, $b \neq \{0, m\}$, and $2\lambda > |2b - m|$; or
- (ii) $a = -1$, $b \neq \{0, -m\}$, and $2\lambda > |2b + m|$.

The above results give us the homotopy type of the centralizer for all circle actions on $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ which we summarise in the table below.

Theorem 6.1.8. *For the* S^1 *action given by the integers* $(a, b; m)$, *acting on* $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)$, *we have the following cases*

Values of $(a,b;m)$	λ	*Number of strata* $\mathcal{J}^{S^1}_{\omega_\lambda,l}$ *intersects*	*Homotopy type of* $\mathrm{Symp}^{S^1}(S^2\times S^2,\omega_\lambda)$
$(0,\pm1;m)$, $m\neq 0$	$\lambda>1$	*1*	$U(2)$
$(0,\pm1;0)$ *or* $(\pm1,0;0)$	$\lambda=1$	*1*	$U(2)$
	$\lambda>1$	*1*	$U(2)$
$(\pm1,0;m), m\neq 0$	$\lambda>1$	*1*	$\mathbb{T}^2$
$(\pm1,\pm m;m), m\neq 0$	$\lambda>1$	*1*	$\mathbb{T}^2$
$(1,b;m), b\neq\{m,0\}$	$\lvert 2b-m\rvert\geqslant 2\lambda\geqslant 1$	*1*	$\mathbb{T}^2$
	$2\lambda>\lvert 2b-m\rvert\geqslant 0$	*2*	$\Omega S^3\times S^1\times S^1\times S^1$
$(-1,b;m), b\neq\{-m,0\}$	$\lvert 2b+m\rvert\geqslant 2\lambda\geqslant 1$	*1*	$\mathbb{T}^2$
	$2\lambda>\lvert 2b+m\rvert\geqslant 0$	*2*	$\Omega S^3\times S^1\times S^1\times S^1$
All other values of $(a,b;m)$	$\forall\lambda$	*1*	$\mathbb{T}^2$

6.2 Isometry groups of odd Hirzebruch surfaces

In this section we calculate the codimension of the smaller strata in $\mathcal{J}^{S^1}_{\omega_\lambda}$. In order to retain the notation as in [3], we use the convention $m=2n+1$. The Hirzebruch surface $\mathbb{F}_{2n+1}$ is obtained by Kähler reduction of $\mathbb{C}^4$ under the action of the torus T^2_{2n+1} defined by

$$(s,t)\cdot z=(s^{2n+1}tz_1,tz_2,sz_3,sz_4)$$

The moment map is $\phi(z)=((2n+1)|z_1|^2+|z_3|^2+|z_4|^2,|z_1|^2+|z_2|^2)$ and the reduced manifold at level $(\lambda+n,1)$ is symplectomorphic to $(\mathbb{C}P^2\#\overline{\mathbb{C}}P^2,\omega_\lambda)$ and biholomorphic to the Hirzebruch surface $\mathbb{F}_{2n+1}$. In this model, the projection to the base is given by $[(z_1,\dots,z_4)]\mapsto[z_3:z_4]$, the zero section is $[w_0:w_1]\mapsto[(w_0^{2n+1},0,w_0,w_1)]$, and a fiber is $[w_0:w_1]\mapsto[(w_0w_1^{2n+1},w_0w_1,0,w_1)]$. The torus $T^2(2n+1)=T^4/T^2_{2n+1}$ acts on $\mathbb{F}_{2n+1}$. This torus is generated by the elements $[(1,e^{it},1,1)]$ and $[(1,1,e^{is},1)]$, and its moment map is $[(z_1, z_2, z_3, z_4)] \mapsto (|z_2|^2, |z_3|^2)$. The moment polytope $\Delta(2n+1)$ is the convex hull of the vertices $(0,0)$, $(1,0)$, $(1, \lambda+n)$, and $(0, \lambda-n-1)$.

The isometry group of $\mathbb{F}_{2n+1}$ is

$$K(2n+1) = Z_{\mathrm{U}(4)}(T^2_{2n+1})/T^2_{2n+1} = (T^2 \times \mathrm{U}(2))/T^2_{2n+1} \simeq \mathrm{U}(2)$$

where the last isomorphism is given by

$$[(s,t), A] \mapsto (s^{-1} t \det A^n) A$$

Under this isomorphism, an element $[(1, a, b, 1)]$ of the torus $T(2n+1)$ is taken to

$$ab^n \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix}$$

Consequently, at the Lie algebra level of the maximal tori, the map $\mathfrak{t}^2(2n+1) \to \mathfrak{t}^2$ is given by

$$\begin{pmatrix} 1 & n+1 \\ 1 & n \end{pmatrix}$$

The moment polytope associated to the maximal torus $T^2 \subset K(2n+1)$ is thus the balanced polytope obtained from $\Delta(2n+1)$ by applying the inverse transpose $\begin{pmatrix} -n & 1 \\ n+1 & -1 \end{pmatrix}$.

6.2.1 Odd isotropy representations

The action of the isometry group $K(2n+1) \simeq \mathrm{U}(2)$ on the space $H^{0,1}_J(M, TM)$ of infinitesimal deformations is isomorphic to $\mathrm{Det}^{1-n} \otimes \mathrm{Sym}^{2n-1}$, where Det is the determinant representation of $\mathrm{U}(2)$ on $\mathbb{C}$, and where $\mathrm{Sym}^k(\mathbb{C}^2)$ is the k-fold symmetric product of the standard representation of $\mathrm{U}(2)$ on $\mathbb{C}^2$. Using the double covering $S^1 \times \mathrm{SU}(2) \to \mathrm{U}(2)$, we see that irreducible representations of $\mathrm{U}(2)$ correspond to irreducible representations of $S^1 \times \mathrm{SU}(2)$ for which $(-1, -\mathrm{id})$ acts trivially. If A_m denotes the representation $t \cdot z = t^m z$ of S^1 on $\mathbb{C}$, and if V_n is the n-fold symmetric product of the defining representation of

$\mathrm{SU}(2)$ on $\mathbb{C}^2$, then the irreducible representations of $\mathrm{U}(2)$ are $A_m \otimes V_n$ with $m+n$ even. In this notation, we have the identifications $\mathrm{Det} = A_2$, while $\mathrm{Sym} = A_1 \otimes V_1$. Consequently, $\mathrm{Det}^{1-n} \otimes \mathrm{Sym}^{2n-1} = A_1 \otimes V_{2n-1}$ whose character is given by

$$\chi\left(z \otimes e(t)\right) = \chi\left(z \otimes \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}\right) = z \sum_{k=0}^{2n-1} e^{i(2n-1-2k)t}$$

With respect to the double covering $S^1 \times \mathrm{SU}(2) \to \mathrm{U}(2)$, the maximal torus $T^2 \subset \mathrm{U}(2)$ of diagonal matrices $D_{s,t} := \begin{pmatrix} e^{is} & 0 \\ 0 & e^{it} \end{pmatrix}$ lifts to

$$\left(|D_{s,t}|^{1/2}, \frac{D_{s,t}}{|D_{s,t}|^{1/2}}\right) = \left(e^{i(s+t)/2}, \begin{pmatrix} e^{i(s-t)/2} & 0 \\ 0 & e^{i(t-s)/2} \end{pmatrix}\right)$$

so that the character of $\mathrm{Det}^{1-n} \otimes \mathrm{Sym}^{2n-1}$ at $D_{s,t}$ is given by

$$\chi(D_{s,t}) = e^{i(s+t)/2} \sum_{k=0}^{2n-1} e^{i(2n-2k-1)(s-t)/2}$$

Remark 6.2.1. *More directly, the character of the* $(2n-1)$*-fold symmetric product of the standard representation of* $\mathrm{U}(2)$ *on* $\mathbb{C}^2$ *is*

$$\chi(D_{s,t}) = \sum_{k=0}^{2n-1} e^{i[s(2n-1-k)+tk]}$$

while the character of Det^{1-n} *is*

$$e^{i(s+t)(1-n)}$$

which gives the same result as above.

For a, b coprime, consider the embedding $S^1 \to S^1(a, b, 2n+1) \subset K(2n+1) = \mathrm{U}(2)$ defined by

$$S^1 \hookrightarrow \mathrm{U}(2)$$

$$t \mapsto D_{at,bt}$$

Then, the $S^1(a,b,2n+1)$-invariant subspace of $\mathrm{Det}^{1-n}\otimes\mathrm{Sym}^{2n-1}$ has dimension

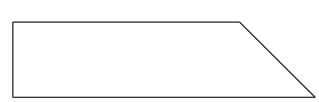
$$\begin{aligned} d_{a,b,2n+1} &= \int_{S^1} \chi(D_{at,bt})\,dt \\ &= \frac{1}{2\pi}\int_0^1 e^{i(a+b)t/2}\sum_{k=0}^{2n-1} e^{i(2n-2k-1)(a-b)t/2}\,dt \\ &= \frac{1}{2\pi}\sum_{k=0}^{2n-1}\int_0^1 e^{it[(n-k)(a-b)+b]}\,dt \\ &= \#\big\{k\in\{0,\dots,2n-1\}\mid (n-k)(a-b)+b=0\big\} \end{aligned}$$

Assuming $ab\neq 0$, we have

$$(n-k)(a-b)+b=0 \iff \frac{a}{b}=\frac{n-k-1}{n-k}$$

which implies $a=n-k-1$ and $b=n-k$. Consequently, for a given n, there is at most one solution $k\in\{0,\dots,2n-1\}$.

Note that just as in the $S^2\times S^2$ case, the above codimension calculation was with respect to the basis of the maximal torus in $K(2n+1)$. Hence to calculate the codimension for the $S^1(1,b,m)\subset\mathbb{T}^2_m$ as in our case, we need to transform the basis by multiplication by the matrix $\begin{pmatrix}\frac{m-1}{2}+1 & -1\\ \frac{m-1}{2} & -1\end{pmatrix}$. Thus it takes the vector $\begin{pmatrix}1\\ b\end{pmatrix}$ in the basis for the standard moment polytope



to the vector $\begin{pmatrix}\frac{m-1}{2}-b\\ \frac{m-1}{2}-b\end{pmatrix}$ in the basis for the balanced polytope. Hence the a and b in the formula above need to be replace by $\frac{m-1}{2}-b$ and $\frac{m-1}{2}-b$ to get the correct codimension for the $S^1(1,b,m)$ action.

Thus we have the following theorem.

Theorem 6.2.2. *Given the circle action $S^1(1,b,m)$ on $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)$ with $2\lambda > |2b-m|$ and $b \neq \{0,m\}$, the complex codimension of of the strata $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_m$ in $\mathcal{J}^{S^1}_{\omega_\lambda}$ in given by the number of $k \in \{1,\cdots,m-1\}$ such that $k=b$.*

Similarly for the action $S^1(-1,b,m)$ with $2\lambda > |2b+m|$ and $b \neq \{0,-m\}$, the complex codimension of the strata $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_m$ in $\mathcal{J}^{S^1}_{\omega_\lambda}$ in given by the number of $k \in \{1,\cdots,m-1\}$ such that $k=-b$.

□

Corollary 6.2.3. *For the circle actions*

- *(i) $a=1$, $b\neq\{0,m\}$, and $2\lambda > |2b-m|$; or*
- *(ii) $a=-1$, $b\neq\{0,-m\}$, and $2\lambda > |2b+m|$.*

The complex codimension of the stratum $\mathcal{J}^{S^1}_{\omega_\lambda} \cap U_m$ in $\mathcal{J}^{S^1}_{\omega_\lambda}$ is either 0 or 1.

□

Alternative calculation for the codimension

As explained above, the action of $K(2n+1)$ on $H^{0,1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, T^{1,0}_{J_m}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2})) \cong \mathbb{C}^{m-1}$ is isomorphic to $\mathrm{Det}^{1-n}\otimes\mathrm{Sym}^{2n-1}$. Hence to calculate the the codimension we only need to calculate the dimension of the invariant subspace of the vector space $H^{0,1}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, T^{1,0}_{J_m}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2})) \cong \mathbb{C}^{m-1}$ under the $S^1(1,b;m)$ action. To do so we note that a basis of Sym^{2n-1} is given by the homogeneous polynomials $P_k = z_1^{2n-1-k}z_2^k$ for $k\in\{0,\ldots,2n-1\}$. The action of $D_{s,t}$ on P_k is

$$D_{s,t}\cdot P_k = e^{i\left((s+t)(1-n)+s(2n-1-k)+tk\right)}P_k$$

so that each P_k generates an eigenspace for the action of the maximal torus $T(2n+1)$ generated by $D_{s,t}$. In particular, the circle $S^1(a,b;2n+1)$ acts trivially on P_k if, and only if,

$$(a-b)(n-k)+b = (a,b)\cdot(n-k,k-n+1) = 0$$

Thus the codimension (in the balanced basis of the maximal torus of K(2n+1) is given by the number of $k \in \{0, \ldots, 2n-1\}$ such that $(a-b)(n-k)+b = (a,b)\cdot(n-k, k-n+1) = 0$. Just as proved above, we can transform using a change of basis to get the result for the standard moment polytope.

As an aside we also note that generator $(1,1)$ (in the balanced basis of $K(2n+1)$) of $T(2k+1)/S^1(a,b;2n+1)$ then acts on the eigenspace $\langle P_k \rangle$ with weight

$$(1,1)\cdot(n-k, k-n+1) = 1$$

which shows that this action is effective and does not depend on a, b, or $m = 2n+1$.

Chapter 7

Centralizers of finite cyclic groups

Having established the homotopy type of the S^1 equivariant symplectomorphisms, we would like to do a similar analysis for finite cyclic groups acting via Hamiltonian diffeomorphisms. The two key differences between $\mathbb{Z}_n$ and S^1 actions are that we no longer have a momentum map associated to a $\mathbb{Z}_n$ action, and we do not have a classification of $\mathbb{Z}_n$ actions on $S^2 \times S^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ up to equivariant symplectomorphisms. Consequently, we have to modify our approach substantially. In particular, we replace the space of invariant and compatible almost-complex structures $\mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n}$ with the subspace $I_{\omega_\lambda}^{\mathbb{Z}_n}$ of <u>integrable</u>, compatible, and invariant complex structures, and a large part of our work consists in showing that the action of the centralizer on $I_{\omega_\lambda}^{\mathbb{Z}_n}$ is homotopically equivalent to its action on $\mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n}$. The restriction to <u>integrable</u> structures allows to use the classification of complex structures on $S^2 \times S^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ together with the Chen-Wilczyński classification of $\mathbb{Z}_n$ actions up to oriented diffeomorphisms, to partially make up for the lack of a proper classification of Hamiltonian $\mathbb{Z}_n$ actions, and to determine which stratum U_{2k} the space $I_{\omega_\lambda}^{\mathbb{Z}_n}$ intersects. This is enough to deal with most Hamiltonian $\mathbb{Z}_n$ actions, leaving open the cases of actions satisfying some specific numerical conditions.

7.1 Symplectic actions of finite abelian groups

As shown in [9], when $\lambda > 1$ the only finite abelian groups that have an Hamiltonian action on $(S^2 \times S^2, \omega_\lambda)$ are abstractly isomorphic to finite subgroups of $\mathrm{SO}(3) \times \mathrm{SO}(3)$ (see Theorem 1.0.2 in the present document). In particular this means that the only finite abelian group that have Hamiltonian actions on $(S^2 \times S^2, \omega_\lambda)$ when $\lambda > 1$ are groups of the form $\mathbb{Z}_n$ and $\mathbb{Z}_n \times \mathbb{Z}_m$. Similary, we have that all finite groups that admit a Hamiltonian action on $(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$ are abstractly isomorphic to subgroups of $U(2)$ (see Theorem 1.0.3 in the present document).

We shall say an action $\mathbb{Z}_n(a, b; r)$ <u>extends</u> to a toric action $\mathbb{T}^2_{r'}$ if it is $\mathbb{Z}_n$-equivariantly symplectomorphic to an action of a finite subgroup of $\mathbb{T}^2_{r'}$. Similarly we say a $\mathbb{Z}_n$ action <u>extends</u> to a circle $S^1(a', b'; r')$ if it is $\mathbb{Z}_n$-equivariantly symplectomorphic to a finite subgroup $\mathbb{Z}_n$ of $S^1(a', b'; r')$.

As shown in [13], we know that every Hamiltonian $\mathbb{Z}_n$ action on $(S^2 \times S^2, \omega_\lambda)$ and $(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$ extends to Hamiltonian S^1 actions. Consequently, each triple of numbers $(a, b; r)$ determines a single Hamiltonian $\mathbb{Z}_n$ action on $S^2 \times S^2$ or $(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$ up to a possible reparametrization of $\mathbb{Z}_n \subset S^1(a, b; r)$ and, conversely, each such reparametrization class of Hamiltonian $\mathbb{Z}_n$ action is given by triples, one for each possible S^1 the $\mathbb{Z}_n$ action extends to. We note that if $a \equiv a' \pmod{n}$, $b \equiv b' \pmod{n}$ then $\mathbb{Z}_n(a, b; r)$ and $\mathbb{Z}_n(a', b', r')$ denote the same action. As we are only interested in effective $\mathbb{Z}_n$ actions, we only consider pairs of values $a, b \in \{0, \cdots, n-1\}$ with $\gcd(a, b) = 1$.

7.2 $\mathbb{Z}_n(a,b;r)$ actions on $(S^2\times S^2,\omega_\lambda)$ with $\lambda>1$ and $r\neq 0$.

In order to simplify the discussion, we first consider $\mathbb{Z}_n$ actions on the product $(S^2\times S^2,\omega_\lambda)$. For technical reasons, we further restrict ourselves to $\mathbb{Z}_n$ actions for which $\mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n}$ does not intersect the open stratum U_0, see Remark 7.2.2 at the end of this section. In particular, we assume $\lambda>1$ and we only consider triples $(a,b;r)$ with $r\neq 0$. We saw in Chapter 3 that for such a triple $(a,b;r)$, there is a unique J_r-holomorphic curve in the class $B-\frac{r}{2}F$ which is invariant under the $S^1(a,b;r)$ action. In particular, together with positivity of intersection (Theorem 2.2.12) this implies that there is a unique J_r-holomorphic curve in the class $B-\frac{r}{2}F$ which is invariant under the $\mathbb{Z}_n(a,b;r)$ action. As Theorem 3.2.4 holds for finite abelian groups, the arguments used to establish the existence of the fibrations and homotopy equivalences associated to the action of $\mathrm{Symp}_h^{S^1}(S^2\times S^2,\omega_\lambda)$ on invariant curves and on invariant almost-complex structures work mutatis mutandis for $\mathbb{Z}_n$ actions. Consequently, assuming $\mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n}\cap U_{2k}$ is nonempty for some $\underline{2k>0}$, we have the following sequence of fibrations and homotopy equivalences:

$$\mathrm{Stab}^{\mathbb{Z}_n}(\overline{D})\to\mathrm{Symp}_h^{\mathbb{Z}_n}(S^2\times S^2,\omega_\lambda)\twoheadrightarrow\mathcal{S}_{D_{2k}}^{\mathbb{Z}_n}\xrightarrow{\simeq}\mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n}\cap U_{2k}$$

$$\mathrm{Fix}^{\mathbb{Z}_n}(\overline{D})\to\mathrm{Stab}^{\mathbb{Z}_n}(\overline{D})\twoheadrightarrow\mathrm{Symp}^{\mathbb{Z}_n}(\overline{D})\xrightarrow{\simeq}\mathbb{Z}_n\ \text{or}\ \mathrm{SO}(3)$$

$$\mathrm{Fix}^{\mathbb{Z}_n}(N(\overline{D}))\to\mathrm{Fix}^{\mathbb{Z}_n}(\overline{D})\twoheadrightarrow\mathrm{Gauge}^{\mathbb{Z}_n}(N(\overline{D}))\xrightarrow{\simeq}\mathbb{Z}_n$$

$$\mathrm{Stab}^{\mathbb{Z}_n}(\overline{F})\cap\mathrm{Fix}^{\mathbb{Z}_n}(N(\overline{D}))\to\mathrm{Fix}^{\mathbb{Z}_n}(N(\overline{D}))\twoheadrightarrow\overline{\mathcal{S}_{F,p_0}^{\mathbb{Z}_n}}\xrightarrow{\simeq}\mathcal{J}^{\mathbb{Z}_n}(\overline{D})\simeq\{*\}$$

$$\mathrm{Fix}^{\mathbb{Z}_n}(\overline{F})\to\mathrm{Stab}^{\mathbb{Z}_n}(\overline{F})\cap\mathrm{Fix}^{\mathbb{Z}_n}(N(\overline{D}))\twoheadrightarrow\mathrm{Symp}^{\mathbb{Z}_n}(\overline{F},N(p_0))\xrightarrow{\simeq}\{*\}$$

$$\{*\}\xleftarrow{\simeq}\mathrm{Fix}^{\mathbb{Z}_n}(N(\overline{D}\vee\overline{F}))\to\mathrm{Fix}^{\mathbb{Z}_n}(\overline{F})\twoheadrightarrow\mathrm{Gauge}^{\mathbb{Z}_n}(N(\overline{D}\vee\overline{F}))\xrightarrow{\simeq}\{*\}$$

As before, the homotopy type of $\mathrm{Symp}^{\mathbb{Z}_n}(\overline{D})$ depends on whether the unique J_r-holomorphic curve in the class $B-\frac{r}{2}F$ is pointwise fixed under the $\mathbb{Z}_n$ action or not. The homotopy type is SO(3) if it pointwise fixed and is S^1 otherwise.

Putting all the homotopy equivalences together, we again have that, in all cases where there is no symplectic sphere in class $B - \frac{r}{2}F$ pointwise fixed under the $\mathbb{Z}_n$ action that

$$\mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda)/\mathbb{T}^2_{2k} \simeq \mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_{2k}$$

and in all other cases where there exists a symplectic sphere in class $B - \frac{r}{2}F$ pointwise fixed under the $\mathbb{Z}_n$ action, we have

$$\mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda)/(S^1 \times SO(3)) \simeq \mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_{2k}$$

Keeping track of the homotopy equivalences, we obtain (as in Remark 3.3.16) the following Theorem.

Theorem 7.2.1. *Given a symplectic $\mathbb{Z}_n(a, b; r)$ action on $(S^2 \times S^2, \omega_\lambda)$, and assuming $\mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_{2k}$ is nonempty for some $\underline{2k > 0}$, the map*

$$\begin{aligned} \mathrm{ev}_{J_r} : \mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda)/\,\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J_{2k}) &\to U_{2k} \cap \mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n} \\ \varphi &\mapsto \left(\varphi^{-1}\right)^* J_{2k} \end{aligned}$$

is a homotopy equivalence. Here J_{2k} denotes the standard integrable almost complex structure on the Hirzebruch surface W_{2k} and $\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J_{2k})$ denotes the space of $\mathbb{Z}_n$-equivariant Kähler isometries of the space $(S^2 \times S^2, \omega_\lambda, J_{2k})$.

Remark 7.2.2. *Throughout this chapter, we assume $\mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n}$ does not intersect the strata U_0. We put this restriction to avoid the analysis of the $\mathbb{Z}_n$ action on the fixed point set. In the case of circle actions, this analysis was done in two steps: when there is an isolated fixed point p_0, and when there were only fixed surfaces. In particular, we explicitly used momentum map arguments to show that for the S^1 actions for which the space of invariant almost complex structures $\mathcal{J}_{\omega_\lambda,l}^{S^1}$ intersected the stratum U_0, every equivariant symplectomorphism fixed the isolated fixed p_0 (see Lemma 3.3.18). As $\mathbb{Z}_n$ actions don't admit a momentum map, the proof of this lemma does not generalise readily to $\mathbb{Z}_n$ actions. However, we may still use the techniques in subsection 3.3.3 to obtain results about the group $\mathrm{Symp}_{h,p_0}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda)$ of $\mathbb{Z}_n$ equivariant symplectomorphims that* *leave the fixed point p_0 invariant. More explicitly, for a $\mathbb{Z}_n(a,b;r)$ action such that the following conditions are satisfied*

- $(a,b) \neq (\pm 1, 0)$ *or* $(0, \pm 1)$
- $\mathcal{J}^{\mathbb{Z}_n}_{\omega_\lambda} \cap U_0$ *in non-empty*

we can still show that

$$\mathrm{Symp}^{\mathbb{Z}_n}_{h,p_0}(S^2 \times S^2, \omega_\lambda)/\mathbb{T}^2_0 \simeq \mathcal{J}^{\mathbb{Z}_n}_{\omega_\lambda} \cap U_0.$$

Then, the homotopy type of $\mathrm{Symp}^{\mathbb{Z}_n}_h(S^2 \times S^2, \omega_\lambda)$ *can be recovered by a careful analysis of its action on the fixed point set of* $\mathbb{Z}_n(a,b;r)$*. We leave this to future work.*

7.3 Compatible complex structures

In this section, our goal is to show that the action of the centralizer $\mathrm{Symp}^{\mathbb{Z}_n}_h(S^2 \times S^2, \omega_\lambda)$ on the space of invariant, integrable, and compatible complex structures $I^{\mathbb{Z}_n}_{\omega_\lambda}$ is homotopically equivalent to its action on $\mathcal{J}^{\mathbb{Z}_n}_{\omega_\lambda}$. To achieve this, we follow the approach of M. Abreu, G. Granja, and N. Kitchloo and reprove Proposition 2.5 and Corollary 2.6 in [3] under the presence of a group action.

7.3.1 Classification of complex structures on ruled surfaces

We first recall the classification of complex structure on the product $S^2 \times S^2$ and on the non-trivial bundle $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. For a more complete exposition, we invite the reader to consult the paper [37] and the references therein. We also recall some classical facts about the automorphisms groups of these complex structure, see Gauduchon [20].

Theorem 7.3.1 (Classification of complex structures on $S^2 \times S^2$)**.** *Pick an orientation on $S^2 \times S^2$. Let J be a complex structure on $S^2 \times S^2$ compatible with the given orientation.*

1. *There exists a orientation preserving diffeomorphism that takes J to exactly one of the standard even Hirzebruch structures J_{2k}, $2k \geqslant 0$. The complex structures diffeomorphic to J_{2k} are characterized by the existence of a complex ruling $\mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^1$ that admits a holomorphic section of self-intersection $-2k$. In particular, if the homology class of the J-fibers coincides with the class of the J_{2k}-fibers, then the diffeomorphism ϕ acts trivially on homology.*

2. $\mathrm{Hol}(J_0) \simeq \mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C}) \times \mathbb{Z}_2$ *where the $\mathbb{Z}_2$ factor is generated by swapping the two S^2 factors. Consequently, the maximal compact subgroup of the identity component of* $\mathrm{Hol}(J_{2k})$ *is isomorphic to* $\mathrm{SO}(3) \times \mathrm{SO}(3)$.

3. *For $2k \geqslant 2$,* $\mathrm{Hol}(J_{2k}) \simeq \mathrm{GL}(2,\mathbb{C})/\mu_{2k} \ltimes \mathbb{C}^{2k}$, *where μ_{2k} is the subgroup of diagonal matrices $\{\xi \cdot \mathrm{id} \mid \xi \in \mathbb{Z}_{2k}\}$. In particular, the group* $\mathrm{Hol}(J_{2k})$ *is connected, and its maximal compact subgroup is isomorphic to* $\mathrm{SO}(3) \times S^1$. □

Theorem 7.3.2 (Classification of complex structures on $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$)**.** *Let J be a complex structure on $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ compatible with the given orientation.*

1. *There exists a diffeomorphism acting trivially on homology that takes J to exactly one of the standard odd Hirzebruch structures J_{2k+1}, $2k+1 \geqslant 1$. The complex structures diffeomorphic to $J_{2k}+1$ are characterized by the existence of a complex ruling $\mathbb{C}P^1 \to \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ that admits a holomorphic section of self-intersection $-(2k+1)$.*

2. *For all $2k+1 \geqslant 1$,* $\mathrm{Hol}(J_{2k+1}) \simeq \mathrm{GL}(2,\mathbb{C})/\mu_{2k+1} \ltimes \mathbb{C}^{2k+1}$, *where μ_{2k+1} is the subgroup of diagonal matrices $\{\xi \cdot \mathrm{id} \mid \xi \in \mathbb{Z}_{2k+1}\}$. In particular, the group* $\mathrm{Hol}(J_{2k+1})$ *is connected, and its maximal compact subgroup is isomorphic to* $\mathrm{U}(2)$. □

Corollary 7.3.3. *For any $k \geqslant 1$, the group* $\mathrm{Hol}(J_k)$ *of complex automorphisms of the Hirzebruch complex structure J_k acts trivially on homology.* □

7.3.2 The action of $\mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda)$ on $I_{\omega_\lambda}^{\mathbb{Z}_n}$

Define $I_{\omega_\lambda}^{\mathbb{Z}_n}$ to be the space of integrable ω_λ-compatible almost complex structures on $S^2 \times S^2$ invariant under the $\mathbb{Z}_n$ action. Fix a $J^0 \in I_{\omega_\lambda}^{\mathbb{Z}_n}$. We also define the following spaces and groups:

- $\mathrm{Diff}_{[\omega_\lambda]}^{\mathbb{Z}_n}(S^2 \times S^2)$:= The group of $\mathbb{Z}_n$-equivariant diffeomorphisms ϕ such that ϕ preserves the cohomology class of ω_λ

- $\mathrm{Diff}_0^{\mathbb{Z}_n}(S^2 \times S^2)$:= The identity component of the group of $\mathbb{Z}_n$-equivariant diffeomorphisms of $S^2 \times S^2$.

- $\mathcal{I}_{J^0}^{\mathbb{Z}_n}$: = The space of all $J \in I_{\omega_\lambda}^{\mathbb{Z}_n}$, such that $J^0 = \phi^* J$ for some $\phi \in \mathrm{Diff}_{[\omega_\lambda]}^{\mathbb{Z}_n}(S^2 \times S^2)$, that is, the intersection of the $\mathrm{Diff}_{[\omega_\lambda]}^{\mathbb{Z}_n}(S^2 \times S^2)$ orbit of J^0 with $I_{\omega_\lambda}^{\mathbb{Z}_n}$.

- $\Omega_{J^0}^{\mathbb{Z}_n}$:= The space of all $\mathbb{Z}_n$ invariant symplectic forms η in the same cohomology class of ω_λ and such that η is compatible with J^0.

- $\mathrm{Hol}_{[\omega_\lambda]}^{\mathbb{Z}_n}(S^2 \times S^2, J^0)$:= The space of $\mathbb{Z}_n$-equivariant complex automorphisms of $(S^2 \times S^2, J^0)$ that preserve the cohomology class ω_λ.

- $\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J^0)$:= The space of $\mathbb{Z}_n$-equivariant Kähler isometries of $(S^2 \times S^2, \omega_\lambda, J^0)$.

We now prove equivariant versions of Proposition 2.5 and Corollary 2.6 in [3] in order to show that the orbit intersection

$$\mathcal{I}_{J^0}^{\mathbb{Z}_n} = \left(\mathrm{Diff}_{[\omega_\lambda]}^{\mathbb{Z}_n}(S^2 \times S^2) \cdot J^0\right) \cap I_{\omega_\lambda}^{\mathbb{Z}_n}$$

is homotopically equivalent to the homogeneous space

$$\mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \times \{\omega_\lambda\} / \mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J^0).$$

Consider the map

$$\Psi : \Omega_{J^0}^{\mathbb{Z}_n} \to \mathrm{Diff}_0^{\mathbb{Z}_n}(S^2 \times S^2)$$

$$\eta \mapsto \Psi(\eta)$$

where $\Psi(\eta)$ is defined as follows. Let $\psi_t \in \mathrm{Diff}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2)$ denote an isotopy satisfying $\psi_t^*\left((1-t)\omega_\lambda + t\eta)\right) = \omega_\lambda$. We can pick ψ_t canonically by choosing a canonical primitive in $\mathbb{Z}_n$-equivariant Moser's method using Hodge theory for the canonical $\mathbb{Z}_n$-equivariant metric coming from (ω_λ, J^0). Then we define $\Psi(\eta)$ to be ϕ_1. In particular, $\Psi(\eta)^*(\eta) = \phi_1 * \eta = \omega_\lambda$.

Theorem 7.3.4. *The map*

$$\mu : \mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \times \Omega^{\mathbb{Z}_n}_{J^0} \to \mathcal{I}^{\mathbb{Z}_n}_{J^0}$$
$$(\phi, \eta) \mapsto \left(\phi^{-1}\right)^* \circ (\Psi(\eta))^* J^0$$

is a principal $\mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J^0)$ *bundle and the fibre over* J, $\mu^{-1}(J)$ *can be identified with* $\mu^{-1}(J) \cong \left\{\varphi \in \mathrm{Diff}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2) \mid J_0 = \varphi^* J\right\}$.

Proof. We first show that the map μ is surjective. Suppose $J \in \mathcal{I}^{\mathbb{Z}_n}_{J^0}$, we need to produce $(\phi, \eta) \in \mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \times \Omega^{\mathbb{Z}_n}_{J^0}$ such that $\mu(\varphi, \eta) = J$. By the definition of $\mathcal{I}^{\mathbb{Z}_n}_{J^0}$ there exists $\phi \in \mathrm{Diff}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2)$ such that $\phi^* J := \phi_*^{-1} J \phi_* = J^0$. Consider $\eta = \phi^*\omega_\lambda$, then for all $v, w \in T_p(S^2 \times S^2)$ we have

$$\eta(J^0 v, J^0 w) = \phi^*\omega_\lambda(J^0 v, J^0 w) = \omega_\lambda\left(\phi_*(J^0 v), \phi_*(J^0 w)\right) =$$
$$= \omega_\lambda\left(J(\phi_*(v)), J(\phi_*(w))\right) = \phi^*\omega_\lambda = \eta$$

where $\omega_\lambda\left(J(\phi_*(v)), J(\phi_*(w))\right) = \phi^*\omega_\lambda$ because J is compatible with ω_λ, and where $\omega_\lambda\left(\phi_*(J^0 v), \phi_*(J^0 w)\right) = \omega_\lambda\left(J(\phi_*(v)), J(\phi_*(w))\right)$ because $\phi^* J = J^0$.

Similarly,

$$\eta(v, J^0(v)) = \omega_\lambda\left(\phi_*(v), \phi_*(J^0 v)\right) = \omega_\lambda\left(\phi_*(v), J(\phi_* v)\right) > 0$$

showing that η is compatible with J^0. As $\phi \in \mathrm{Diff}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2)$ we have that $[\eta] = [\omega_\lambda] \in H^2(S^2 \times S^2, \mathbb{R})$. Hence η belongs to $\Omega^{\mathbb{Z}_n}_{J^0}$.

Moreover, as $(\phi\circ\Psi(\eta))^*\omega_\lambda = (\Psi(\eta))^*\eta = \omega_\lambda$, we have that $\phi\circ\Psi(\eta)$ belongs to $\mathrm{Symp}^{\mathbb{Z}_n}(S^2\times S^2,\omega_\lambda)$. Thus we have that

$$\mu(\phi\circ\Psi(\eta),\eta) = \left(\Psi(\eta)^{-1}\circ\phi^{-1}\right)^*\circ\Psi(\eta)^*J^0 = \left(\phi^{-1}\right)^*J^0 = J$$

thus showing μ is surjective.

Next we investigate what the fibres of μ look like, and why they are free orbits of a $\mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2\times S^2,J^0)$ action. Given $J\in\mathcal{I}^{\mathbb{Z}_n}_{J^0}$, and an element $(\phi,\eta)\in\mu^{-1}(J)$, consider $\varphi=\phi\circ(\Psi(\eta))^{-1}$. Then $\varphi\in\mathrm{Diff}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2\times S^2)$ and as $\mu(\phi,\eta) = (\phi^{-1})^*\Psi(\eta)^*J^0 = J$, it implies that $J^0=\varphi^*J$.

Conversely, given $\varphi\in\mathrm{Diff}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2\times S^2)$ such that $J^0=\varphi^*J$, consider $\eta:=\varphi^*\omega_\lambda$. Then we have that $(\varphi\circ\Psi(\eta),\eta)$ belongs to $\mu^{-1}(J)$. Hence

$$\begin{aligned}\rho:\mu^{-1}(J) &\to \left\{\varphi\in\mathrm{Diff}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2\times S^2)\mid J_0=\varphi^*J\right\}\\ (\phi,\eta)&\mapsto\phi\circ(\Psi(\eta))^{-1}\end{aligned}$$

is an homeomorphism. We define a right $\mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2\times S^2,J^0)$ action on $\mathrm{Symp}^{\mathbb{Z}_n}(S^2\times S^2,\omega_\lambda)\times\Omega^{\mathbb{Z}_n}_{J^0}$ by

$$(\phi,\eta)\cdot\varphi = \left(\phi\Psi(\eta)^{-1}\varphi\Psi\left(\varphi^*(\eta)\right),\varphi^*(\eta)\right)$$

for $\varphi\in\mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2\times S^2,J^0)$. A quick check shows us that if (ϕ,η) satisfies $\mu(\phi,\eta)=J$, then $\mu((\phi,\eta)\cdot\varphi)=J$, thus verifying that the action defined above preserves the fibres of μ.

We now check that the action is free on the fibres of μ. Consider the action of $\mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2\times S^2,J^0)$ on $\mathrm{Diff}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2\times S^2)$ defined by

$$\alpha\cdot\varphi=\alpha\circ\varphi$$

for all $\alpha \in \mathrm{Diff}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2)$ and $\varphi \in \mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J^0)$. It is a simple check to see that the identification of $\mu^{-1}(J)$ with $\left\{\varphi \in \mathrm{Diff}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2) \mid J_0 = \varphi^* J\right\}$ given by

$$\begin{aligned} \rho : \mu^{-1}(J) &\to \left\{\varphi \in \mathrm{Diff}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2) \mid J_0 = \varphi^* J\right\} \\ (\phi, \eta) &\mapsto \phi \circ (\Psi(\eta))^{-1} \end{aligned}$$

is equivariant under this action on $\mathrm{Diff}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2)$ and the action on $\mu^{-1}(J)$ defined above. As the action of $\mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J^0)$ on $\mathrm{Diff}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2)$ is free this implies that the action of $\mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J^0)$ on $\mu^{-1}(J)$ is also free.

Finally, to show that this is a principal bundle, we need to check that the map satisfy local triviality. This follows by producing a local section and invoking Theorem C.0.9. As we already used similar arguments several times, we leave the details to the reader. □

Remark 7.3.5. *Note that when μ is restricted to* $\mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \times \{\omega_\lambda\}$ *is just the map given by*

$$\begin{aligned} \mu : \mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \times \{\omega_\lambda\} &\to \mathcal{I}^{\mathbb{Z}_n}_{J^0} \\ (\phi, \omega_\lambda) &\mapsto \left(\phi^{-1}\right)^* J^0 \end{aligned}$$

Corollary 7.3.6. *If $J^0 \in I^{\mathbb{Z}_n}_{\omega_\lambda}$ is such that* $\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J^0) \hookrightarrow \mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J^0)$ *is a weak homotopy equivalence, then the inclusion of* $\mathrm{Symp}^{\mathbb{Z}_n}$ *orbit of J^0 in $\mathcal{I}^{\mathbb{Z}_n}_{J^0}$,i.e*

$$\begin{aligned} \mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \times \{\omega_\lambda\}/\,\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J^0) &\hookrightarrow \mathcal{I}^{\mathbb{Z}_n}_{J^0} \\ (\phi, \omega_\lambda) &\mapsto \left(\phi^{-1}\right)^* J^0 \end{aligned}$$

is also a weak homotopy equivalence.

Proof. From Theorem 7.3.4, we have that

$$\mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J^0) \cong \mu^{-1}(J^0) \hookrightarrow \mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \times \Omega^{\mathbb{Z}_n}_{J^0} \xrightarrow{\mu} \mathcal{I}^{\mathbb{Z}_n}_{J^0}$$

Moreover we note that if $\eta_1, \eta_2 \in \Omega^{\mathbb{Z}_n}_{J^0}$, then $(1-t)\eta_1 + t\eta_2$ is also in $\Omega^{\mathbb{Z}_n}_{J^0}$, thus giving us that the space $\Omega^{\mathbb{Z}_n}_{J^0}$ is contractible. In other words we have that the inclusion $\{\omega_\lambda\} \hookrightarrow \Omega^{\mathbb{Z}_n}_{J^0}$ is a homotopy equivalence and hence $\mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \times \{\omega_\lambda\} \simeq \mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \times \Omega^{\mathbb{Z}_n}_{J^0}$.

Quotienting the above fibration by $\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J^0)$ we have that

$$\mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J^0)/\,\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J^0) \hookrightarrow \mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \times \Omega^{\mathbb{Z}_n}_{J^0}/\,\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J^0) \xrightarrow{\bar{\mu}} \mathcal{I}^{\mathbb{Z}_n}_{J^0}$$

where $\bar{\mu}$ is just the map induced on the quotient space by μ. From the assumption that $\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J^0) \hookrightarrow \mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J^0)$ is a weak homotopy equivalence, we get that $\mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J^0)/\,\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J^0)$ is weakly contractible. Putting all this together we get that $\mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \times \Omega^{\mathbb{Z}_n}_{J^0}/\,\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J^0) \hookrightarrow \mathcal{I}^{\mathbb{Z}_n}_{J^0}$ is a weak homotopy equivalence.

Finally, as $\mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \times \{\omega_\lambda\}/\,\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J^0) \simeq \mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \times \Omega^{\mathbb{Z}_n}_{J^0}/\,\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J^0)$ we get the required result. □

Let $\mathrm{Hol}^{\mathbb{Z}_n}(S^2 \times S^2, J_k)$ denote the space of all holomorphic $\mathbb{Z}_n$-equivariant automorphisms of $(S^2 \times S^2, J_k)$.

Lemma 7.3.7. *For all canonical integrable almost complex structures $J_k \in I^{\mathbb{Z}_n}_{\omega_\lambda} \cap U_k$, with $k > 0$ the space $\mathrm{Hol}^{\mathbb{Z}_n}(S^2 \times S^2, J_k)$ is equal to $\mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J_k)$.*

Proof. When $k > 0$, $\mathrm{Hol}^{\mathbb{Z}_n}(S^2 \times S^2, J_k)$ is connected and hence preserves cohomology (See Corollary 7.3.3). □

Theorem 7.3.8. *Fix a $\mathbb{Z}_n(a, b; r)$ action on $(S^2 \times S^2, \omega_\lambda)$. For all canonical integrable almost complex structures $J_k \in I^{\mathbb{Z}_n}_{\omega_\lambda} \cap U_k$, the inclusion map*

$$\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J_k) \hookrightarrow \mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J_k)$$

is a weak homotopy equivalence.

Proof. By Theorem 7.3.1 when $k > 0$, we have that the group of holomorphic automorphims of $(S^2 \times S^2, J_k)$ is given by $\mathrm{GL}(2, \mathbb{C})/\mu_k \ltimes H^0(\mathbb{C}P^1, \mathcal{O}(-k))$. Under the presence of a $\mathbb{Z}_n$ action which is holomorphic with respect to J_k, we can similarly show that $\mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J_k) = \mathrm{Hol}^{\mathbb{Z}_n}(S^2 \times S^2, J_k) = \left(\mathrm{GL}(2,\mathbb{C})/\mu_k \ltimes H^0(\mathbb{C}P^1, \mathcal{O}(-k))\right)^{\mathbb{Z}_n}$, where $\left(\mathrm{GL}(2,\mathbb{C})/\mu_k \ltimes H^0(\mathbb{C}P^1, \mathcal{O}(-k))\right)^{\mathbb{Z}_n}$ denotes the centraliser of $\mathbb{Z}_n$ in $\mathrm{GL}(2,\mathbb{C})/\mu_k \ltimes H^0(\mathbb{C}P^1, \mathcal{O}(-k))$. Further, we see that that $\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J_k)$ is a maximal compact subgroup of $\left(\mathrm{GL}(2,\mathbb{C})/\mu_k \ltimes H^0(\mathbb{C}P^1, \mathcal{O}(-k))\right)^{\mathbb{Z}_n}$ thus proving the result.

Although not necessary for our purposes we also show that the theorem holds even when $k = 0$. To show that $\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J_k) \hookrightarrow \mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J_k)$ is a homotopy equivalence when $k = 0$, we have proceed as follows. If $\lambda > 1$, then any $\phi \in \mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J_0)$ has to send the foliation by J_0-holomorphic curves in the class F to itself and similarly send the foliation J_0-holomorphic curves in the class B to itself. Hence we have that $\mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J_k) \cong \mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C})$. Denote the centralizer of $\mathbb{Z}_n$ in $\mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C})$ by $(\mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C}))^{\mathbb{Z}_n}$. Then we have that $\mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J_0) \cong (\mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C}))^{\mathbb{Z}_n}$ and $\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J_0) \hookrightarrow (\mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C}))^{\mathbb{Z}_n}$ is a maximal compact subgroup and hence the inclusion is a homotopy equivalence.

Finally in the case, when $k = 0$ and $\lambda = 1$ then for the $\mathbb{Z}_n(a, b; 0)$ action we have

$$\mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J_0) \cong (\mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C}) \ltimes \mathbb{Z}_2)^{\mathbb{Z}_n}.$$

Again $\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J_0) \hookrightarrow \mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J_0)$ is a maximal compact subgroup and hence the inclusion is a homotopy equivalence. □

Corollary 7.3.9. *For any J^0 in $I^{\mathbb{Z}_n}_{\omega_\lambda}$ there is an homotopy equivalence*

$$\mathcal{I}^{\mathbb{Z}_n}_{J^0} = \left(\mathrm{Diff}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2) \cdot J^0\right) \cap I^{\mathbb{Z}_n}_{\omega_\lambda} \simeq \mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \times \{\omega_\lambda\}/\,\mathrm{Isom}^{\mathbb{Z}_n}(\omega_\lambda, J^0).$$

□

We now relate the orbit intersection $\mathcal{I}^{\mathbb{Z}_n}_{J_k}$ with the stratum $I^{\mathbb{Z}_n}_{\omega_\lambda} \cap U_k$.

Lemma 7.3.10. *Fix a $\mathbb{Z}_n(a, b; r)$ action on $(S^2 \times S^2, \omega_\lambda)$. Then, for any for $k > 0$, we have the equality $\mathcal{I}^{\mathbb{Z}_n}_{J_k} = I^{\mathbb{Z}_n}_{\omega_\lambda} \cap U_k$.*

Proof. We see from the definition that $\mathcal{I}_{J_k}^{\mathbb{Z}_n} \subset I_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_k$. To show the reverse inclusion, we need to produce a $\phi \in \mathrm{Diff}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2)$ such that $\phi^* J = J_k$ for all $J \in I_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_k$. We do that as follows.

Let $J \in I_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_k$, by the classification of complex structures on $S^2 \times S^2$, and by the argument above Theorem 3.3 in [3], we know that there exists a $\phi \in \mathrm{Diff}_{[\omega_\lambda]}(S^2 \times S^2)$ such that $\phi^* J = J_k$. Then we have the following diffeomorphism between $\mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J)$ and $\mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J_k)$.

$$\begin{aligned} \rho_\phi : \mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J) &\to \mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J_k) \\ \psi &\mapsto \phi \circ \psi \circ \phi^{-1} \end{aligned}$$

As J is equivariant with respect to the $Z_n(a, b; r)$ action, we denote by $i_J(\mathbb{Z}_n)$ the embedding of $\mathbb{Z}_n$ into $\mathrm{Hol}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2, J)$ induced by the action $Z_n(a, b; r)$. Similarly we denote by $i_{J_k}(\mathbb{Z}_n(a, b; r))$ the embedding given by the action of $\mathbb{Z}_n$ into $\mathrm{Hol}_{[\omega_\lambda]}(S^2 \times S^2, J_k)$. Then $(\rho_\phi)\,(i_J(\mathbb{Z}_n))$ defines a $\mathbb{Z}_n$ action holomorphic with respect to J_k. We note that although the fixed point set may apriori be different for the $(\rho_\phi)\,(i_J(\mathbb{Z}_n))$ and the $i_{J_k}(\mathbb{Z}_n(a, b; r))$ action, the weights at the fixed points of the $(\rho_\phi)\,(i_J(\mathbb{Z}_n))$ are the same as the weights for the action $i_{J_k}(\mathbb{Z}_n(a, b; r))$. By Theorem 4.1 in [40], we see that there exist a $\psi \in \mathrm{Hol}_{[\omega_\lambda]}(S^2 \times S^2, J_k)$ such that $\psi \circ (\rho_\phi)\,(i_J(\mathbb{Z}_n))\psi^{-1} = i_{J_k}(\mathbb{Z}_n(a, b; r))$.Thus $\phi \circ \psi \in \mathrm{Diff}^{\mathbb{Z}_n}_{[\omega_\lambda]}(S^2 \times S^2)$ and $(\phi \circ \psi)^* J = \psi^* \circ \phi^* J = \psi^* J_k = J_k$, thus proving the theorem. $\square$

Remark 7.3.11. *Apriori Theorem 4.1 in [40] only guarantees the existence of a* $\psi \in \mathrm{Hol}(S^2 \times S^2, J_k)$ *such that* $\psi \circ (\rho_\phi)\,(i_J(\mathbb{Z}_n))\psi^{-1} = i_{J_k}(\mathbb{Z}_n(a, b; r))$. *As* $k > 0$, *we use Lemma 7.3.7 to show that* ψ *is in fact in the group* $\mathrm{Hol}_{[\omega_\lambda]}(S^2 \times S^2, J_k)$

We can finally conclude that the action of $\mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda)$ on the space of invariant, integrable structures $I_{\omega_\lambda}^{\mathbb{Z}_n}$ is homotopically equivalent to its action on the space $\mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n}$ of invariant almost-complex structures.

Theorem 7.3.12. *Fix a $\mathbb{Z}_n(a,b;r)$ action on $(S^2\times S^2,\omega_\lambda)$, then, for all for $2k>0$,*

$$I^{\mathbb{Z}_n}_{\omega_\lambda}\cap U_k \simeq \mathcal{J}^{\mathbb{Z}_n}_{\omega_\lambda}\cap U_k.$$

Proof. We shall prove something much stronger. By Theorem 7.2.1, Corollary 7.3.6 and Theorem 7.3.10 we have that both ev_{J_k} and the map $\mathrm{ev}_{J_k}\circ i_1$ in the diagram below are homotopy equivalences.

$$\mathrm{Symp}^{\mathbb{Z}_n}(S^2\times S^2,\omega_\lambda)/\,\mathrm{Isom}^{Z_n}(\omega_\lambda,J_k)\xhookrightarrow{\mathrm{ev}_{J_k}} I^{\mathbb{Z}_n}_{\omega_\lambda}\cap U_k \xhookrightarrow{i_1} \mathcal{J}^{\mathbb{Z}_n}_{\omega_\lambda}\cap U_k$$
$$\phi\mapsto\left(\phi^{-1}\right)^* J_k$$

It follows that i_1 is a homotopy equivalence as required. □

7.3.3 A characterization of the intersection $I^{\mathbb{Z}_n}_{\omega_\lambda}\cap U_{2k}$

We now take advantage of the classification of complex structures to give a criterion as to when $I^{\mathbb{Z}_n}_{\omega_\lambda}$ intersects the strata U_{2k}.

Theorem 7.3.13. *Let $\mathbb{Z}_n(a,b;r)$ be a symplectic action on $(S^2\times S^2,\omega_\lambda)$ with $\lambda>1$. Then the space of $\mathbb{Z}_n$-equivariant complex structures $I^{\mathbb{Z}_n}_{\omega_\lambda}$ intersects the strata $U_{r'}$ iff $\mathbb{Z}_n(a,b;r)$ is equivariantly symplectomorphic to a $\mathbb{Z}_n(a',b';r')$ action that acts as a subgroup of the torus action $\mathbb{T}^2_{r'}$.*

Proof. ($\Leftarrow$)Let $\mathbb{Z}_n$ be symplectomorphic to a $\mathbb{Z}_n(a',b';r')$ action via the symplectomorphism ϕ. Note that the standard almost complex structure $J_{r'}\in U_{r'}$ is invariant under the $T_{r'}$ action and hence in particular invariant under the $\mathbb{Z}_n(a',b';r')$ action. Thus $\phi^*J_{r'}\in I^{\mathbb{Z}_n}_{\omega_\lambda}\cap U_{r'}$.

($\Rightarrow$) Let $J\in I^{\mathbb{Z}_n}_{\omega_\lambda}\cap U_{r'}$. Let $\mathrm{Isom}(\omega_\lambda,J)$ denote the group of Kähler isometries of $(S^2\times S^2,J,\omega_\lambda)$ and let $\mathrm{Hol}(J)$ denote the space of holomorphic automorphisms of the complex structure J. As $J\in I^{\mathbb{Z}_n}_{\omega_\lambda}$, the $\mathbb{Z}_n$ action induces a map, $\rho:\mathbb{Z}_n\hookrightarrow\mathrm{Hol}(J)$. As J is compatible with ω_λ, and the action is also symplectic, the image of ρ in fact lands in $\mathrm{Isom}(\omega_\lambda, J)$. By the classification of complex structures on $S^2 \times S^2$, there exists a diffeomorphism ϕ acting trivially on homology that takes J onto $J_{r'}$. This diffeomorphism takes ω_λ to $\omega' := (\phi^{-1})^* \omega_\lambda$, and the $\mathbb{Z}_n$ action ρ to another action ρ' that is, by construction, Kähler with respect to the pair $(\omega', J_{r'})$ and such that the cohomology class $[\omega'] = [\omega_\lambda]$. Apriori, there is no reason for the action ρ' to extend to a ω' symplectic $\mathbb{T}^2$ action. However, $\mathrm{Hol}(J_{r'})$ being connected, the subgroup $\mathbb{Z}_n \subset \mathrm{Hol}(J_{r'})$ extends to a maximal compact subgroup K which is conjugated to the maximal compact subgroup $\mathrm{SO}(3) \times S^1 = \mathrm{Isom}(\omega_\lambda, J_{r'})$ via some $h \in \mathrm{Hol}(J_{r'})$. Note that although h need not preserve the form ω', however it preserves the cohomology class of ω'. Hence, we have that $C_h \circ C_\phi(\mathrm{Isom}(\omega_\lambda, J)) = \mathrm{Isom}(\omega_\lambda, J_{r'})$ where C_ϕ and C_h denote conjugation by ϕ and h respectively. As a result we have that the diffeomorphism $h \circ \phi$ takes the $\mathbb{Z}_n$ action ρ to another action ρ'' that is Kähler with respect to the standard pair $(\omega_\lambda, J_{r'})$. This action ρ'' does extends to a $\mathbb{T}^2$ action α which is Kähler with respect to the standard pair $(\omega_\lambda, J_{r'})$. Take the triple $(\omega_\lambda, J_{r'}, \alpha)$ back to (ω''', J, α') using the inverse composition $(h \circ \phi)^{-1}$. Note that $\alpha' : \mathbb{T}^2 \to \mathrm{Isom}(\omega''', J) \subset \mathrm{Hol}(J)$ is a toric action with respect to ω''' that extends ρ. The forms ω_λ and ω''' are cohomologous, and there is an α'-invariant curve in class $B - \frac{r'}{2}F$. By the classification of toric actions, the α' action is equivariantly symplectomorphic to the standard $\mathbb{T}^2_{r'}$ toric action via some diffeomorphism $\psi \in \mathrm{Diff}_h(S^2 \times S^2)$ (but this diffeomorphism does not intertwine the complex structures). Hence $\psi \circ \alpha' : \mathrm{Hol}(J) \supset \mathbb{T}^2 \to \mathbb{T}^2_{r'}$ defines a $\mathbb{Z}_n(a', b'; r')$ action that is $\mathbb{Z}_n$-equivariantly symplectomorphic to $\mathbb{Z}_n(a, b; r)$ as required. $\square$

7.4 Toric extensions of cyclic actions and homotopy type of centralizers

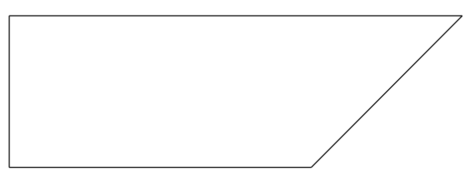
From Theorem 7.3.13, we infer that in order to understand which strata $I_{\omega_\lambda}^{\mathbb{Z}_n}$ intersects, we need to understand which tori $\mathbb{T}^2_{r'}$ the $\mathbb{Z}_n(a, b; r)$ action extends to. In the circle action case, we used the Karshon classification of S^1 actions up to equivariant symplectomorphims to answer the above question (See Theorem 3.1.9). But no such classification up to equivariant symplectomorphism exists for $\mathbb{Z}_n$ actions. However, we do have a classification up to $\mathbb{Z}_n$-equivariant diffeomorphisms due to W. Chen [12] and D. Wilczyński [40]. We shall use this classification to determine the homotopy type of $\mathbb{Z}_n$-equivariant symplectomorphisms for a subfamily of $\mathbb{Z}_n$ actions. We present this classification up to $\mathbb{Z}_n$-equivariant diffeomorphisms in the following paragraph.

7.4.1 The Chen–Wilczyński classification

Consider two Hirzerbruch surfaces W_r and $W_{r'}$ endowed with smooth $\mathbb{Z}_n$ action $\mathbb{Z}_n(a, b; r)$ and $\mathbb{Z}_n(a', b', r')$ respectively. Denote by $Z_n(a, b, -r)$ the action which is the restriction of the sub-circle $S^1(a, b)$ in the torus T_{-r} where T_{-r} the denotes the following torus action on $(S^2 \times S^2, \omega_\lambda)$

$$(u, v) \cdot ([x_1, x_2], [y_1, y_2, y_3]) = \left([ux_1, x_2], \left[u^{-r}y_1, y_2, vy_3\right]\right)$$

whose moment map looks like



As described in the works of W. Chen [12] and D. Wilczyński [40], one can establish the existence of six types of diffeomorphisms $c_1, \cdots, c_6$ which give a $\mathbb{Z}_n$ equivariant diffeomorphism between the Hirzerbruch surfaces W_r and $W_{r'}$ with the respective actions, provided the triples (a, b, r) and (a', b', r') satisfy the following conditions:

- Type c_1: When $a' = -a, b' = -b$ and $r' = r$
- Type c_2: When $a' = -a, b' = -b + ra$ and $r' = r$
- Type c_3: When $a' = a, b' = -b$ and $r' = -r$

- Type c_4: When $r' = r = 0$, $a' = b$, and $b' = a$
- Type c_5: When $a' = a$, $b' = b$, and $r' \equiv r$ (mod 2n)
- Type c_6: When $a' = a$, $b' = b$, and $r'a' \equiv 2b - ra$ (mod 2n).

Remark 7.4.1. *We note of the above types of diffeomorphisms only c_5 and c_6 are between cyclic groups in different torus actions $\mathbb{T}^2_r$ and $\mathbb{T}^2_{r'}$. The equivariant diffeomorphisms of type c_1, c_2, c_3, c_4 are between sub-circles in the same torus $\mathbb{T}^2_r$ (up to reparametrization of the torus $\mathbb{T}^2_r$).*

We call the above equivariant diffeomorphism standard of type $c_1, \cdots, c_6$. One of the main results of [12] is the following theorem.

Theorem 7.4.2. *(Chen [12]) Two $\mathbb{Z}_n$-Hirzebruch surfaces are orientation-preserving equivariantly diffeomorphic iff there is a composition of standard equivariant diffeomorphisms between them.*

7.4.2 Consequences of the classification

We shall now use the above results to obtain the homotopy type of $\mathbb{Z}_n$ equivariant symplectomorphisms for a fixed $\mathbb{Z}_n(a, b; r)$ action on $(S^2 \times S^2, \omega_\lambda)$ when $\lambda > 1$ and $\gcd(a, n) \neq 1$. In order to do this we need the following lemma from [13]

Lemma 7.4.3. *If G is any compact group and J is any G-invariant almost complex structure. Suppose S is the connected component of the fixed point set of a non-trivial subgroup H $\{id\} \neq H \subset G$. Then S is J-holomorphic.*

Proof. To show S is J-holomorphic we need to show that for any vector $v \in T_xS$ $Jv \in T_xS$. As S is pointwise fixed by a non-trivial subgroup H, all tangent vectors $v \in T_xS$ are characterised by the property that $dh \cdot v = v$ for all $h \in H$. Thus in order to show that that $Jv \in T_xS$, it suffices to prove that $dh \cdot Jv = Jv$ for all $h \in H$. But this immediately follows from the equivariant of J as

$$dh \cdot Jv = J(dh \cdot v) = Jv$$

$\square$

Theorem 7.4.4. *Under the numerical conditions $2\lambda > r > 1$ and $\gcd(a,n) \neq 1$, the finite cyclic group $\mathbb{Z}_n(a,b;r) \subset S^1(a,b;r)$ action can only extend to circles $S^1(a',b';r')$ with $r' = r$.*

Proof. Suppose $\mathbb{Z}_n(a,b;r)$ extends to another circle $S^1(a',b';r')$ with $r' \neq r$. From the Chen-Wilczyński classification we know that if this happened then either $a = a'$ or $a = -a'$. This implies $\gcd(a,n) = \gcd(a',n) \neq 1$ and hence $\mathbb{Z}_n \cap \mathbb{Z}_a = \mathbb{Z}_n \cap \mathbb{Z}_{a'} \neq \{\mathrm{id}\}$. We know that there are two invariant spheres S_r and $S_{r'}$ in the homology classes $B - \frac{r}{2}F$ and $B - \frac{r'}{2}F$ which are holomorphic for the integrable complex structures J_r and $J_{r'}$ respectively. As $\mathbb{Z}_n(a,b;r) \subset S^1(a,b;r)$, we further know by looking at the graph associated to the action that S_r is invariant under the $S^1(a,b;r)$ action and the global stabilizer for the $S^1(a,b;r)$ action on S_r is $\mathbb{Z}_a$. Similarly $S_{r'}$ is invariant under the $S^1(a',b';r')$ action and has global stabilizer $\mathbb{Z}_{a'}$. As $\mathbb{Z}_n \cap \mathbb{Z}_a = \mathbb{Z}_n \cap \mathbb{Z}_{a'} \neq \{\mathrm{id}\}$, we know that both S_r and $S_{r'}$ are pointwise fixed by the non-trivial subgroups $\mathbb{Z}_n \cap \mathbb{Z}_a$ and $\mathbb{Z}_n \cap \mathbb{Z}_{a'}$ respectively. By Lemma 7.4.3 for any $\mathbb{Z}_n$ invariant J both S_r and $S_{r'}$ are J-holomorphic. This is a contradiction by positivity of intersections (Theorem 2.2.12) as $r' \neq r$. $\square$

Theorem 7.4.5. *For finite cyclic groups $\mathbb{Z}_n(a,b;r) \subset S^1(a,b;r)$ actions such that $2\lambda > r > 1$ $\gcd(a,n) \neq 1$, $\mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) = \mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \simeq \mathbb{T}^2_r$*

Proof. As $\lambda > 1$, we can to conclude that $\mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) = \mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda)$. By Theorem 7.4.4, we see that under the above conditions for the $\mathbb{Z}_n(a,b;r)$ action, it extends to only one torus $\mathbb{T}^2_r$ and thus we have our result. $\square$

Although the proof of Theorem 7.4.4 works for $\gcd(a,n) \neq 1$ and $2\lambda > r > 1$. Using the classification of $\mathbb{Z}_n$ action up to diffeomorphism we can obtain a partial result when $a = 1$, for the actions $\mathbb{Z}_n(1,b;r)$ such that $n > 2\lambda > r > 1$ and $n > 2\lambda > |r - 2b| > 1$.

Before we embark on these results we recall a few things about the classification up to $\mathbb{Z}_n$-equivariant diffeomorphisms. As noted in the remark above Theorem 7.4.2, the only types of $\mathbb{Z}_n$-equivariant diffeomorphisms between finite cyclic subgroups of different tori are diffeomorphisms of type c_5 and c_6. Similarly we note that the only diffeomorphism that changes the parametrization of the torus $\mathbb{T}^2_r$ is c_3.

As explained in Theorem 7.3.13, we know that if the given $\mathbb{Z}_n(a,b;r) \subset \mathbb{T}^2_r$ action is symplectomorphic to an $\mathbb{Z}_n(a',b';r')$ action which is a subgroup of the torus $\mathbb{T}^2_{r'}$, then the space of $\mathbb{Z}_n(a,b;r)$ invariant almost complex structures $I^{\mathbb{Z}_n}_{\omega_\lambda}$ intersects the strata U_r and $U_{r'}$. Conversely, if $I^{\mathbb{Z}_n}_{\omega_\lambda}$ intersected U_r and $U_{r'}$, then we would be able to realise $\mathbb{Z}_n(a',b';r')$ from $\mathbb{Z}_n(a,b;r)$ via a composition of diffeomorphisms $c_1, \cdots, c_6$. Further if $r \neq r'$, then at least one $\mathbb{Z}_n$ equivariant diffeomorphism of type c_5 or c_6 feature in this composition of diffeomorphisms. In the theorems that follow we shall use numerical conditions on r, n, a and b to rule out the different solutions to r' that satisfy the modular equations in c_5 and c_6 and hence glean information about which strata $I^{\mathbb{Z}_n}_{\omega_\lambda}$ intersects.

Remark 7.4.6. *Note that c_5 and c_6 only tell us that the $\mathbb{Z}_n(a,b;r)$ and $\mathbb{Z}_n(a',b';r')$ are only equivariantly diffeomorphic (and not necessarily symplectomorphic). Thus a solutions r' that satisfy the modular equations in c_5 and c_6 doesn't necessarily tell us the $I^{\mathbb{Z}_n}_{\omega_\lambda}$ intersect $U_{r'}$. Thus the Chen-Wilczyński classification only gives us obstructions to which strata $U_\prime$, $I^{\mathbb{Z}_n}_{\omega_\lambda}$ intersects.*

Lemma 7.4.7. *Consider the following families of Hamiltonian $\mathbb{Z}_n(a,b;r)$ actions on $(S^2 \times S^2, \omega_\lambda)$ with $r \neq 0$ and $\lambda > 1$*

- *(i) $a = 1$, $b \neq \{0, r\}$, $n > 2\lambda$ and $n > 2\lambda > |r - 2b| > 1$, or*
- *(ii) $a = -1$, $b \neq \{0, -r\}$, $n > 2\lambda$, and $2\lambda > |2b + r| > 1$.*

Then in case (i) the only strata that $I^{\mathbb{Z}_n}_{\omega_\lambda}$ intersects are U_r and $U_{|r-2b|}$ and in case (ii) the only trata that $I^{\mathbb{Z}_n}_{\omega_\lambda}$ intersects are U_r and $U_{|r+2b|}$.

Proof. We shall only prove the theorem for case (i). The proof for case (ii) works similarly. Recall that if $I^{\mathbb{Z}_n}_{\omega_\lambda}$ intersects the strata $U_{r'}$ then $r' < 2\lambda$. Now we shall use numerical conditions on r, n, a and b to rule out the different solutions to r' that satisfy the modular equations in c_5 and c_6.

Using the fact that $n > 2\lambda > r > 1$ we note that the only solutions for r' such that $|r'| < 2\lambda$ and r' satisfies the modular equation $r' \equiv r \pmod{2n}$ is just the trivial solution $r = r'$. Hence under these numerical conditions any word generated by a composition of c_3 and c_5 only give a diffeomorphism between $Z_n(1, b; r)$ and another $\mathbb{Z}_n$ action within the same torus $\mathbb{T}^2_r$.

Similarly as $n > 2\lambda > |r - 2b| > 1$ using triangle inequality we note that the only solution to the modular equation $r'a' \equiv 2b - ra \pmod{2n}$ and satisfies the equation $|r'| < 2\lambda$ is again the trivial solution $r' = 2b - r$. Hence under these numerical conditions any word generated by a composition of c_3 and c_6 only give a diffeomorphism between $Z_n(1, b; r)$ and $\mathbb{Z}_n(1, \pm b, |r - 2b|)$ action within the torus $\mathbb{T}^2_{|r-2b|}$.

These two inequalities tell us the only candidates for the strata $U_{r'}$ that $I^{\mathbb{Z}_n}_{\omega_\lambda}$ can intersect is U_r and $U_{|r-2b|}$. As we started off with a $\mathbb{Z}_n(a, b; r) \subset \mathbb{T}^2_r$ action, we know that $I^{\mathbb{Z}_n}_{\omega_\lambda} \cap U_r \neq \phi$. Apriori the above argument doesn't tell us that the intersection $I^{\mathbb{Z}_n}_{\omega_\lambda} \cap U_{|r-2b|}$ is nonempty. This is because the two $\mathbb{Z}_n$ actions inside T_r and $T_{|r-2b|}$ are apriori only equivariantly diffeomorphic and not $\mathbb{Z}_n$-equivariantly symplectomorphic.

To see that they are in fact equivariantly symplectomorphic, we note that $\mathbb{Z}_n(1, b; r) \subset S^1(1, b; r)$ and by Karshon classification of S^1 actions we know that when $2\lambda \geqslant |r - 2b|$ then $S^1(1, b; r)$ is S^1 equivariantly symplectomorphic (and thus $\mathbb{Z}_n$ equivariantly symplectomorphic) to $S^1(1, b'; |r - 2b|) \subset \mathbb{T}^2_{|r-2b|}$ (where b' is either equal to b or $-b$ as explained in Theorem 3.1.6). As all the other diffeomorphisms $c_1, \cdots, c_4$ are between $\mathbb{Z}_n$ actions inside torus actions T_r, we have the only strata $I_{\omega_\lambda}^{\mathbb{Z}_n}$ intersects are U_r and $U_{|r-2b|}$ thus completing the proof. □

We use the above result to derive a result about $\mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda)$ for specific $\mathbb{Z}_n$ actions.

Theorem 7.4.8. *Consider a $\mathbb{Z}_n(a,b;r)$ action on $S^2 \times S^2$ which satisfies the following numerical conditions $n > 2\lambda > r > 1$, $\gcd(a,b) = 1$ and $r \neq 0$. Then we have the following cases:*

1. *If $(a,b) \in \{(\pm 1, 0), (\pm 1, \pm r)\}$ then $\mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \simeq \mathbb{T}^2$.*
2. *if $(a,b) = (0, \pm 1)$ then, $\mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \simeq S^1 \times SO(3)$.*
3. *Further if we have one of the following additional conditions*
   - *Either $a = 1$, $b \neq \{0, r\}$ and $2\lambda > |r - 2b| > 1$ or*
   - *$a = -1$, $b \neq \{0, -r\}$ and $2\lambda > |r + 2b| > 1$*

   then we have the following result:

   In the first case $I_{\omega_\lambda}^{\mathbb{Z}_n}$ intersects exactly 2 strata U_r and $U_{|r-2b|}$ and in the second it intersects the 2 strata U_r and $U_{|r+2b|}$ as before. WLOG if we assume that $I_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_r$ is the strata of positive codimension in $I_{\omega_\lambda}^{\mathbb{Z}_n}$. Then the complex codimension of $I_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_r$ in $I_{\omega_\lambda}^{\mathbb{Z}_n}$ is given by the number of $k \in \{1, \cdots, r-1\}$ such that $k \equiv b \pmod{n}$.

Proof. Case 1: If $(a,b) = \{(\pm 1, 0), (\pm 1, \pm r)\}$, $r \neq 0$.

Suppose $I_{\omega_\lambda}^{\mathbb{Z}_n}$ intersected U_r and $U_{r'}$. Using the fact that $n > 2\lambda > r > 1$ and $2\lambda > r'$, we see that the only solution for r' in the modular equations in Chen-Wilczyński classification is $r' = r$. Hence the only stratum that $I_{\omega_\lambda}^{\mathbb{Z}_n}$ intersects is U_r, and thus we have

$$\mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda)/\mathbb{T}_r^2 \simeq I_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_r = I_{\omega_\lambda}^{\mathbb{Z}_n} \simeq \{*\}$$

Giving us the required result.

Case 2: If $(a, b) = (0, \pm 1)$.

We see from the calculation of local isotropy weights at the fixed points that there is a fixed sphere in the class $B - \frac{r}{2}F$. As above, we can argue that the solution for r' in the modular equations in Chen-Wilczyński classification is $r' = r$ and hence $\mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \simeq S^1 \times SO(3)$

Case 3: If $(a, b; r) = (\pm 1, b; r)$ and $2\lambda > |r - 2b| > 1$.

This follows from 7.4.7.The codimension of the smaller strata can be calculated analogous to the S^1 case and the precise calculation is explained below in the next section. The case $a = -1$, $b \neq \{0, -r\}$ and $2\lambda > |r + 2b| > 1$ is similar to case 3. $\square$

7.5 Codimension calculation

Firstly, we observe that Lemma 3.1.3 holds verbatim for $\mathbb{Z}_n$ actions. Further, for the $\mathbb{Z}_n$ actions $\mathbb{Z}_n(\pm 1, b; r)$ with $b \neq \{0, \pm r\}$, we can see using Lemma 3.1.3 for $\mathbb{Z}_n$ actions that the only fixed points that admit a curve with self intersection $-r$ are the fixed points Q and R in figure 2.1. Hence Lemma 5.2.1 holds for $\mathbb{Z}_n$ actions.

Lemma 7.5.1. *Consider the $\mathbb{Z}_n(a, b; m)$ action on $(S^2 \times S^2, \omega_\lambda)$ or $(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$. Let S be any $\mathbb{Z}_n(a, b; m)$-invariant symplectic embedded sphere in the same homology class $B - \frac{k}{2}F$ with $k > 0$. Then the $\mathbb{Z}_n$ action on S has global stabilizer isomorphic to $\mathbb{Z}_a$.*

Proof. The proof follows exactly as in Lemma 5.2.1. $\square$

Hence the moduli spaces

$$\mathcal{M}^{\mathbb{Z}_n}(B-kF,\mathcal{J}^{S^1}_{\omega_\lambda,l}) := \{(u,J) \mid \text{u is } \mathbb{Z}_n\text{-equivariant, somewhere injective,}$$
$$J\text{-holomorphic, and represents the class } B-kF\}$$

for $k>0$ are well defined. Arguing as in Theorem 5.1.3 we can conclude that $I^{\mathbb{Z}_n}_{\omega_\lambda,l}$ of C^l integrable equivariant almost complex structures is a Banach manifold. Further an analogous argument to Theorem 5.2.8 gives us that $I^{\mathbb{Z}_n}_{\omega_\lambda,l}$ intersects the strata $U_{k,l}$ transversally. Hence $I^{\mathbb{Z}_n}_{\omega_\lambda,l}\cap U_{k,l}$ is a Banach submanifold of $I^{\mathbb{Z}_n}_{\omega_\lambda,l}$. One can argue that $I^{\mathbb{Z}_n}_{\omega_\lambda,l}$ equipped with the C^l-topology is homotopy equivalent to $I^{\mathbb{Z}_n}_{\omega_\lambda}$ equipped with the C^∞ topology. As explained before, the normal bundle to the strata $U_m\cap\mathcal{J}^{\mathbb{Z}_n}_{\omega_\lambda}$ at an equivariant integrable almost complex structure J_m can be identified with $H^{0,1}_{J_m}(S^2\times S^2, T(S^2\times S^2))^{\mathbb{Z}_n}$. It can be shown that the normal to $U_m\cap\mathcal{J}^{\mathbb{Z}_n}_{\omega_\lambda}$ at J_m is given by normal to $U_m\cap\mathcal{J}^{\mathbb{Z}_n}_{\omega_\lambda}$ intersected with $I^{\mathbb{Z}_n}_{\omega_\lambda}$. Thus we see that the normal to $U_m\cap\mathcal{J}^{\mathbb{Z}_n}_{\omega_\lambda}$ at J_m can also be identified with $H^{0,1}_{J_m}(S^2\times S^2, T(S^2\times S^2))^{\mathbb{Z}_n}$. Further we note that, $H^{0,1}_{J_m}(S^2\times S^2, T(S^2\times S^2))\cong\mathbb{C}^{m-1}$. Hence to calculate the codimension we need to calculate the dimension of the subspace of invariant elements in $H^{0,1}_{J_m}(S^2\times S^2, T(S^2\times S^2))$ under the action of $\mathbb{Z}_n\subset S^1\times SO(3)$. By [3], we know the action of the Kähler isometry group $K(r)=S^1\times SO(3)$ on infinitesimal deformations is isomorphic to $\mathrm{Det}\otimes\mathrm{Sym}^{r-2}$. A basis of Sym^{r-2} is given by the homogeneous polynomials $P_k=z_1^{r-2-k}z_2^k$ for $k\in\{0,\dots,r-2\}$. The action of $R(t)$ on P_k is

$$R(t)\cdot P_k = e(t/2)\cdot P_k = e^{i\left(r-2-2k\right)t/2}P_k = e^{it(\frac{r}{2}-1-k)}P_k$$

so that the action of $S^1=(e^{is},R(t))\subset S^1\times\mathrm{SO}(3)$ on P_k is

$$(e^{is},R(t))\cdot P_k = e^{i\left(s+t(\frac{r}{2}-1-k)\right)}P_k$$

Each P_k generates an eigenspace for the action of the maximal torus $T(r)$. In particular, the finite group $\mathbb{Z}_q(a,b;r)\subset S^1(a,b;r)$ acts trivially on P_k if, and only if,

$$b+a(\frac{r}{2}-1-k) = (a,b)\cdot(\frac{r}{2}-1-k,1) = 0 \ (mod\ n)$$

for $k \in \{0, r-2\}$. Equivalently, we must have

$$ak + b = (a, b) \cdot (k, 1) \equiv 0 \ (mod\ n)$$

for $k \in \{1 - \frac{r}{2}, \ldots, \frac{r}{2} - 1\}$

Note that as explained in the case of codimension calculation for the S^1 case, the above calculation was done with respect to the basis of the maximal torus in $K(2n)$. Hence to calculate the codimension for the $\mathbb{Z}_n(a, b; r) \subset S^1(1, b, r) \subset \mathbb{T}^2_m$ as in our case, we need to transform the basis by multiplication by the matrix $\begin{pmatrix} \frac{r}{2} & -1 \\ 1 & 0 \end{pmatrix}$. Thus it takes the vector $\begin{pmatrix} 1 \\ b \end{pmatrix}$ in the basis for the standard moment polytope to the vector $\begin{pmatrix} \frac{r}{2} - b \\ 1 \end{pmatrix}$ in the basis for the balanced polytope. Therefore after transforming into the right basis we get the following theorem.

Theorem 7.5.2. *Given the action $\mathbb{Z}_n(1, b; r)$ on $(S^2 \times S^2, \omega_\lambda)$ with $b \neq \{0, r\}$, $\gcd(a, b) = 1$ and λ satisfying $n > 2\lambda > r > 1$, and $2\lambda > |r - 2b| > 1$, the complex codimension of of the strata $I^{\mathbb{Z}_n}_{\omega_\lambda} \cap U_r$ in $I^{\mathbb{Z}_n}_{\omega_\lambda}$ in given by the number of $k \in \{1, \cdots, r-1\}$ such that $k \equiv b$ (mod n).*

Similarly for the action $\mathbb{Z}_n(-1, b; r)$ with $b \neq \{0, -r\}$,$\gcd(a, b) = 1$ and λ satisfying $n > 2\lambda > r > 1$, and $2\lambda > |r + 2b| > 1$ the complex codimension of of the strata $I^{\mathbb{Z}_n}_{\omega_\lambda} \cap U_r$ in $I^{\mathbb{Z}_n}_{\omega_\lambda}$ is given by the number of $k \in \{1, \cdots, r-1\}$ such that $k \equiv -b$ (mod n).

□

Corollary 7.5.3. *For the family of $\mathbb{Z}_n$ actions on $(S^2 \times S^2, \omega_\lambda)$,*

- *$\mathbb{Z}_n(1, b; r)$ on $(S^2 \times S^2, \omega_\lambda)$ with $b \neq \{0, r\}$, $\gcd(a, b) = 1$ and λ satisfying $n > 2\lambda > r > 1$, and $2\lambda > |r - 2b| > 1$ or*

- *$\mathbb{Z}_n(-1, b; r)$ with $b \neq \{0, -r\}$,$\gcd(a, b) = 1$ and λ satisfying $n > 2\lambda > r > 1$, and $2\lambda > |r + 2b| > 1$*

the complex codimension of $I^{\mathbb{Z}_n}_{\omega_\lambda} \cap U_r$ in $I^{\mathbb{Z}_n}_{\omega_\lambda}$ is either 0 or 1.

□

7.6 Homotopy type of centralizers of $\mathbb{Z}_n(\pm 1, b; r)$

Finally, the proof of Theorem 4.2.1 also gives us that

Theorem 7.6.1. *Consider a $\mathbb{Z}_n(a, b; r)$ action on $(S^2 \times S^2, \omega_\lambda)$ with $r \neq 0$ and $\lambda > 1$. Then,*

1. *If $a = 1$, $b \neq \{0, r\}$, $n > 2\lambda > 1$, and $2\lambda > |2b - r| > 1$, the inclusion $i : \mathbb{T}^2_m, \mathbb{T}^2_{|m-2b|} \hookrightarrow \mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda)$ induces a map which is injective in homology with coefficients in any field k.*

2. *If $a = -1$, $b \neq \{0, -r\}$, $n > 2\lambda > 1$, and $2\lambda > |2b + r| > 1$, the inclusion $i : \mathbb{T}^2_r, \mathbb{T}^2_{|r+2b|} \hookrightarrow \mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda)$ induces a map which is injective in homology (with coefficients in any field k).*

As a consequence of the above theorem and Leray-Hirsch Theorem we have

Corollary 7.6.2. $H^*(\mathrm{Symp}^{\mathbb{Z}_n}_h(S^2 \times S^2, \omega_\lambda), \mathbb{R}) \cong H^*(I^{\mathbb{Z}_n}_{\omega_\lambda} \cap U_r, \mathbb{R}) \bigotimes H^*(\mathbb{T}^2, \mathbb{R})$

Also, we have that

Theorem 7.6.3. *For the following family of $\mathbb{Z}_n(a, b; r)$ symplectic actions on $(S^2 \times S^2, \omega_\lambda)$*

- *$\mathbb{Z}_n(1, b; r)$ on $(S^2 \times S^2, \omega_\lambda)$ with $b \neq \{0, r\}$, $\gcd(a, b) = 1$ and λ satisfying $n > 2\lambda > r > 1$, and $2\lambda > |r - 2b| > 1$ or*

- *$\mathbb{Z}_n(-1, b; r)$ with $b \neq \{0, -r\}$,$\gcd(a, b) = 1$ and λ satisfying $n > 2\lambda > r > 1$, and $2\lambda > |r + 2b| > 1$*

the space $I^{\mathbb{Z}_n}_{\omega_\lambda}$ is contractible.

Proof. By Theorem 7.4.8, we have that $I_{\omega_\lambda}^{\mathbb{Z}_n} = \left(I_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_r\right) \sqcup \left(I_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_{|2b-r|}\right)$. Further by Theorem 7.3.12, we have that the inclusions $I_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_r \hookrightarrow \mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_r$ and $I_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_{|r-2b|} \hookrightarrow \mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_{|r-2b|}$ are homotopy equivalences. As $\mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n} = \left(\mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_r\right) \sqcup \left(\mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_{|2b-r|}\right)$ is contractible, we have the required result. □

Using the calculation of the codimension, Corollary 7.6.2 and Theorem 7.6.3 we figure out the cohomology $H^*(\mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda), \mathbb{R})$. Using theorem 4.2.1 and techniques used in the proof of theorem 4.2.2 we get the following theorem.

Theorem 7.6.4. *Consider the following $\mathbb{Z}_n$ actions on $S^2 \times S^2$.*

- *(i) $a = 1$, $b \neq \{0, r\}$, $n > 2\lambda > 1$ and $2\lambda > |2b - r| > 1$; or*
- *(ii) $a = -1$, $b \neq \{0, -r\}$, $n > 2\lambda > 1$ and $2\lambda > |2b + r| > 1$.*

From Theorem 7.4.8 we see that $I_{\omega_\lambda}^{\mathbb{Z}_n}$ intersects 2 strata. Without loss of generality let $I_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_r$ be the strata with positive codimension in $I_{\omega_\lambda}^{\mathbb{Z}_n}$. Then we have that,

$$H^p(I_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_r, k) = \begin{cases} k & p \geqslant 1 \\ k & p = 0 \\ 0 & \text{otherwise} \end{cases}$$

with coefficients in a field k. Further, the cohomology of $\mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda)$ is isomorphic to

$$H^*\left(\mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda), k\right) \cong H^*(I_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_r, k) \bigotimes H^*(\mathbb{T}^2, k).$$

More explicitly, the ranks of the cohomology groups are given by

$$H^p\left(\mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda), k\right) = \begin{cases} k^4 & p \geqslant 2 \\ k^3 & p = 1 \\ k & p = 0 \end{cases}$$

□

Remark 7.6.5. *As $\lambda > 1$ it can be argued as in Theorem 4.1.1 that $\mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) = \mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda)$ and hence we get the ranks of the cohomology of the entire symplectomorphism group.*

The exact same argument as in Theorem 4.2.6 gives us the following theorem.

Theorem 7.6.6. *Consider the following $\mathbb{Z}_n$ actions on $S^2 \times S^2$.*

- *(i) $a = 1$, $b \neq \{0, r\}$, $n > 2\lambda > 1$ and $2\lambda > |2b - r| > 1$; or*
- *(ii) $a = -1$, $b \neq \{0, -r\}$, $n > 2\lambda > 1$ and $2\lambda > |2b + r| > 1$.*

Then $\mathrm{Symp}^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \simeq \Omega S^3 \times S^1 \times S^1 \times S^1$ where ΩS^3 denotes the based loop space of S^3.

□

7.7 $\mathbb{Z}_n$ actions on $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)$ when $\lambda > 1$

In this section we only present the statements of our results on the homotopy type of $\mathrm{Symp}_{\mathbb{Z}_n}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)$. The proofs follow from simple modifications of the arguments given in Chapter 5 and Section 6.1. We also note that Theorems 7.4.2, lemma 7.4.7 all hold even in the case when r is odd.

Fix a $\mathbb{Z}_n(a, b; r)$ action on $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)$. By [13] we know that every such S^1 action extends to a circle action and hence to a $\mathbb{T}^2$ action. Just as in Chapter 6 we have the following fibration.

$$\mathrm{Stab}^{\mathbb{Z}_n}(\overline{D}) \to \mathrm{Symp}_h^{\mathbb{Z}_n}(S^2 \times S^2, \omega_\lambda) \twoheadrightarrow \mathcal{S}_{D_{2k}}^{\mathbb{Z}_n} \xrightarrow{\simeq} \mathcal{J}_{\omega_\lambda}^{\mathbb{Z}_n} \cap U_{2k}$$

$$\mathrm{Fix}^{\mathbb{Z}_n}(\overline{D}) \to \mathrm{Stab}^{\mathbb{Z}_n}(\overline{D}) \twoheadrightarrow \mathrm{Symp}^{\mathbb{Z}_n}(\overline{D}) \xrightarrow{\simeq} \mathbb{Z}_n \text{ or } SO(3)$$

$$\mathrm{Fix}^{\mathbb{Z}_n}(N(\overline{D})) \to \mathrm{Fix}^{\mathbb{Z}_n}(\overline{D}) \twoheadrightarrow \mathrm{Gauge}^{\mathbb{Z}_n}(N(\overline{D})) \xrightarrow{\simeq} \mathbb{Z}_n$$

$$\mathrm{Stab}^{\mathbb{Z}_n}(\overline{F}) \cap \mathrm{Fix}^{\mathbb{Z}_n}(N(\overline{D})) \to \mathrm{Fix}^{\mathbb{Z}_n}(N(\overline{D})) \twoheadrightarrow \overline{\mathcal{S}_{F,p_0}^{\mathbb{Z}_n}} \xrightarrow{\simeq} \mathcal{J}^{\mathbb{Z}_n}(\overline{D}) \simeq \{*\}$$

$$\mathrm{Fix}^{\mathbb{Z}_n}(\overline{F}) \to \mathrm{Stab}^{\mathbb{Z}_n}(\overline{F}) \cap \mathrm{Fix}^{\mathbb{Z}_n}(N(\overline{D})) \twoheadrightarrow \mathrm{Symp}^{\mathbb{Z}_n}(\overline{F}, N(p_0)) \xrightarrow{\simeq} \{*\}$$

$$\{*\} \xleftarrow{\simeq} \mathrm{Fix}^{\mathbb{Z}_n}(N(\overline{D} \vee \overline{F})) \to \mathrm{Fix}^{\mathbb{Z}_n}(\overline{F}) \twoheadrightarrow \mathrm{Gauge}^{\mathbb{Z}_n}(N(\overline{D} \vee \overline{F})) \xrightarrow{\simeq} \{*\}$$

As in Chapter 6, we can deduce from the above fibrations that for the action $\mathbb{Z}_n(a, b; r)$, such that $(a, b) \neq (0, \pm 1)$ then

$$\mathrm{Symp}_h^{\mathbb{Z}_n}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)/\mathbb{T}^2_{2k+1} \simeq \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{2k+1}$$

and for the $\mathbb{Z}_n(a, b; r)$ such that $(a, b) = (0, \pm 1)$ we have

$$\mathrm{Symp}_h^{\mathbb{Z}_n}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)/U(2) \simeq \mathcal{J}^{S^1}_{\omega_\lambda} \cap U_{2k+1}$$

.

Arguing as in Theorem 7.3.13 and Theorem 7.6.3, we have the following 2 theorems.

Theorem 7.7.1. *Fix a $\mathbb{Z}_n(a, b; k)$ action on $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)$, then the space $I^{\mathbb{Z}_n}_{\omega_\lambda}$ is contractible.*

Theorem 7.7.2. *Let $\mathbb{Z}_n(a, b; r)$ be a symplectic action on $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)$. Then the space of $\mathbb{Z}_n$-equivariant complex structures $I^{\mathbb{Z}_n}_{\omega_\lambda}$ intersects the strata $U_{r'}$ iff $\mathbb{Z}_n$ is symplectomorphic to a $\mathbb{Z}_n(a', b'; r')$ action that acts as a subgroup of the torus action $\mathbb{T}^2_{r'}$.*

Further by analogous arguments as in theorems 7.4.8 and 7.4.7, we obtain the following result

Theorem 7.7.3. *Consider a $\mathbb{Z}_n(a, b; r)$ action on $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda)$ for which $n > 2\lambda > r \geqslant 1$ and $\gcd(a, b) = 1$. Then*

1. *If $(a, b) = \{(\pm 1, 0), (\pm 1, \pm r)\}$ then $\mathrm{Symp}_h^{\mathbb{Z}_n}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda) \simeq \mathbb{T}^2$*

2. *if $(a, b) = (0, \pm 1)$ then, $\mathrm{Symp}_h^{\mathbb{Z}_n}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \omega_\lambda) \simeq U(2)$*

3. *Further if we have one of the following additional conditions*

    - *Either $a = 1$, $b \neq \{0, r\}$ and $2\lambda > |r - 2b| > 1$ or*

- *$a=-1$, $b\neq\{0,-r\}$ and $2\lambda>|r+2b|>1$*

then we have the following result:

- *If $a=1$ and $2\lambda>|r-2b|$, $I^{\mathbb{Z}_n}_{\omega_\lambda}$ intersects exactly 2 strata U_r and $U_{|r-2b|}$.Further, if we assume that $I^{\mathbb{Z}_n}_{\omega_\lambda}\cap U_r$ is the strata with positive codimension in $I^{\mathbb{Z}_n}_{\omega_\lambda}$. Then the complex codimension of $I^{\mathbb{Z}_n}_{\omega_\lambda}\cap U_r$ in $I^{\mathbb{Z}_n}_{\omega_\lambda}$ is given by the number of $k\in\{1,\cdots,r-1\}$ such that $k\equiv b$ (mod n).*
- *If $a=-1$ and $2\lambda>|r+2b|$ $I^{\mathbb{Z}_n}_{\omega_\lambda}$ intersects the 2 strata U_r and $U_{|r+2b|}$ as before. Further, if we assume that $I^{\mathbb{Z}_n}_{\omega_\lambda}\cap U_r$ is the strata with positive codimension in $I^{\mathbb{Z}_n}_{\omega_\lambda}$. Then the complex codimension of $I^{\mathbb{Z}_n}_{\omega_\lambda}\cap U_r$ in $I^{\mathbb{Z}_n}_{\omega_\lambda}$ is given by the number of $k\in\{1,\cdots,r-1\}$ such that $k\equiv -b$ (mod n).*

Proof. The follows verbatim from the proof of 7.4.8. The only difference is the calculation on the codimension which we present below.

The action of $K(2n+1)$ on infinitesimal deformations is isomorphic to $\mathrm{Det}^{1-n}\otimes\mathrm{Sym}^{2n-1}$. A basis of Sym^{2n-1} is given by the homogeneous polynomials $P_k=z_1^{2n-1-k}z_2^k$ for $k\in\{0,\ldots,2n-1\}$. The action of $D_{s,t}$ on P_k is

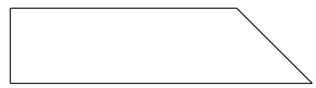
$$D_{s,t}\cdot P_k=e^{i\left((s+t)(1-n)+s(2n-1-k)+tk\right)}P_k$$

so that each P_k generates an eigenspace for the action of the maximal torus $T(2n+1)$ generated by $D_{s,t}$. In particular, the c $S^1(a,b;2n+1)$ acts trivially on P_k if, and only if,

$$(a-b)(n-k)+b=(a,b)\cdot(n-k,k-n+1)\equiv 0\ (mod\ n)$$

Thus the codimension (in the balanced basis of the maximal torus of K(2n+1) is given by the number of $k\in\{0,\ldots,2n-1\}$ such that $(a-b)(n-k)+b=(a,b)\cdot(n-k,k-n+1)\equiv 0\ (mod\ n)$.

Hence to calculate the codimension for the $\mathbb{Z}_n(1,b,r) \subset \mathbb{T}^2_r$ as in our case, we need to transform the basis by multiplication by the matrix $\begin{pmatrix} \frac{r-1}{2}+1 & -1 \\ \frac{r-1}{2} & -1 \end{pmatrix}$. Thus it takes the vector $\begin{pmatrix} 1 \\ b \end{pmatrix}$ in the basis for the standard moment polytope



to the vector $\begin{pmatrix} \frac{r-1}{2}-b \\ \frac{r-1}{2}-b \end{pmatrix}$ in the basis for the balanced polytope. Hence the a and b in the formula above need to be replace by $\frac{r-1}{2}-b$ and $\frac{r-1}{2}-b$ to get the correct codimension for the $\mathbb{Z}_n(1,b,r)$ action.

After making this substitution we get the required result that the codimension of stratum(which we assumed to be $I^{\mathbb{Z}_n}_{\omega_\lambda} \cap U_{|r-2b|}$ is given by the number of $k \in \{1, \cdots, |r-2b|-1\}$ such that $k-b \equiv 0$ (mod n). □

As before, we have

Theorem 7.7.4. *For the following family of $\mathbb{Z}_n$ actions we have*

- *(i) $a=1$, $b \neq \{0,r\}$, $n > 2\lambda > 1$, $\gcd(a,b)=1$ and $2\lambda > |2b-r|$; The inclusion maps $i : \mathbb{T}^2_r, \mathbb{T}^2_{|r-2b|} \hookrightarrow \mathrm{Symp}^{\mathbb{Z}_n}(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$ induces a map which is injective in homology (with coefficients in a field k). or*
- *(ii) $a=-1$, $b \neq \{0,-r\}$, $n > 2\lambda > 1$, $\gcd(a,b)=1$ and $2\lambda > |2b+r|$. The inclusion maps $i : \mathbb{T}^2_r, \mathbb{T}^2_{|r+2b|} \hookrightarrow \mathrm{Symp}^{\mathbb{Z}_n}(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$ induces a map which is injective in homology (with coefficients in a field k).*

Proof. The proof is similar to the proof of Theorem 6.1.4. □

and

Theorem 7.7.5. *Consider the following families of $\mathbb{Z}_n$ actions on $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)$ for which $n>2\lambda>1$, $\gcd(a,b)=1$*

- *(i) $a=1$, $b\neq\{0,r\}$, and $2\lambda>|2b-r|$; or*
- *(ii) $a=-1$, $b\neq\{0,-r\}$, and $2\lambda>|2b+r|$.*

From Theorem 7.4.8 we see that $I^{\mathbb{Z}_n}_{\omega_\lambda}$ intersects two strata U_r and $U_{|r-2b|}$. Without loss of generality let $I^{\mathbb{Z}_n}_{\omega_\lambda}\cap U_r$ be the strata with positive codimension in $I^{\mathbb{Z}_n}_{\omega_\lambda}$. Then,

$$H^p(I^{\mathbb{Z}_n}_{\omega_\lambda}\cap U_r,k)=\begin{cases} k & p\geqslant 1\\ k & p=0\\ 0 & \text{otherwise}\end{cases}$$

and the cohomology $H^*(\mathrm{Symp}^{\mathbb{Z}_n}_h(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda),k)\cong H^*(I^{\mathbb{Z}_n}_{\omega_\lambda}\cap U_r,k)\bigotimes H^*(\mathbb{T}^2,k)$. More concretely,

$$H^p(\mathrm{Symp}^{\mathbb{Z}_n}_h(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda),k)=\begin{cases} k^4 & p\geqslant 2\\ k^3 & p=1\\ k & p=0\end{cases}$$

for any field k.

Remark 7.7.6. *As $\lambda>1$ it can be argued as in Theorem 4.1.1 that*

$$\mathrm{Symp}^{\mathbb{Z}_n}_h(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)=\mathrm{Symp}^{\mathbb{Z}_n}(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)$$

and hence we get the ranks of the cohomology of the entire symplectomorphism group.

Arguing similar to Theorem 4.2.6 we have,

Theorem 7.7.7. *Consider the following families of $\mathbb{Z}_n$ actions on $(\mathbb{C}P^2\#\overline{\mathbb{C}P^2},\omega_\lambda)$ for which $n>2\lambda>1$, $\gcd(a,b)=1$*

- *(i) $a=1$, $b\neq\{0,r\}$, and $2\lambda>|2b-r|$; or*

- *(ii)* $a = -1$, $b \neq \{0, -r\}$, *and* $2\lambda > |2b + r|$.

Then $\mathrm{Symp}^{\mathbb{Z}_n}(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda) \simeq \Omega S^3 \times S^1 \times S^1 \ \times S^1$ *where* ΩS^3 *denotes the based loop space of* S^3.

Finally as lemma 7.4.3 and lemma 7.4.4 hold verbatim for $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, we can conclude the following theorem.

Theorem 7.7.8. *For finite cyclic groups* $\mathbb{Z}_n(a, b; r) \subset S^1(a, b; r)$ *actions such that* $2\lambda > r > 1$ *and* $gcd(a, n) \neq 1$, $\mathrm{Symp}_h^{\mathbb{Z}_n}(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda) = \mathrm{Symp}^{\mathbb{Z}_n}(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda) \simeq \mathbb{T}^2_r$.

$\square$

Chapter 8

Conclusion

The thesis suggests many future directions for possible research. We outline a few of related interesting questions that arise related to the work in the thesis.

8.1 Finite group action

As seen in Chapter 7, we are currently unable to understand the homotopy type of the $\mathbb{Z}_n$ -equivariant symplectomorphisms of the Hirzerbruch surfaces for some $\mathbb{Z}_n$ actions on $(S^2 \times S^2, \omega_\lambda)$ and $(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$. One of the key hurdles to determining the homotopy type is that we currently don't understand the classification of of $\mathbb{Z}_n$ actions on Hirzerbruch surfaces up to $\mathbb{Z}_n$-equivariant symplectomorphisms. Two key questions in this regard are the following

Question 1. *Given two $\mathbb{Z}_n$ actions $(a, b; r)$ and (a', b', r') on $S^2 \times S^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, when are they equivariantly symplectomorphic ?*

Question 2. *Given a $\mathbb{Z}_n$ hamiltonian action $(a, b; r)$ on $S^2 \times S^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ which tori T_m does it extend to ?*

Answering these questions would help in understanding which strata the space of $\mathbb{Z}_n$ invariant almost complex structures intersect and hence the analysis we did in the S^1

case would go through and provide us with an understanding of homotopy type of the centraliser for the $\mathbb{Z}_n$ action in all cases. The proof of this is by no means easy, and might involve getting into analysis of orbifold J-holomorphic curves as in [12].

Another important discussion to be had for compact abelian groups in the following

Question 3. *What can we say about the homotopy type of the normaliser $N(G)$ of a compact abelian group G in $\mathrm{Symp}(S^2 \times S^2)$ and $\mathrm{Symp}(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})$? What about the Weyl group $W(G) = N(G)/\mathrm{Symp}_G(S^2 \times S^2)$?*

As shown in [35], the answer in the toric case has been established and $W(\mathbb{T}^n)$ is always a finite group.

8.2 Alternate proof for S^1 actions

As mentioned in Theorem 1.0.1, we can use only moment map techniques to determine the centraliser for toric actions. We established an analogous result (Theorem 2.1.11) in the case of S^1 actions on 4-manifolds as well. As the graphs associated to the S^1 action contain all the information about the action, it should theoretically be possible to read the homotopy type of the S^1 equivariant symplectomorphims directly from the graphs. Another phenomenon that hints at this is the fact that the various changes in the homotopy type of the $S^1(\pm 1, b; m)$-equivariant symplectomorphism group (as in Theorem 4.3.1) when λ changes correspond precisely to whether the edges in the associated graph overlap or not (as discussed in remark **??**). Thus it would be an interesting endeavour to explore a proof of Theorem 4.1.1 using only momentum map techniques.

This has the added advantage of not only working for our specific case of S^1 actions on rational ruled surfaces, but should be applicable to all 4-manifolds with Hamiltonian S^1 actions.

8.3 Non-abelian group actions

By Theorems 1.0.2 and 1.0.3 there are important classes of non-abelian finite groups that act on Hirzerbruch surfaces. Hence it is natural to study the centralisers of these non-abelian subgroups of the symplectomorphism group. However, in order for the arguments presented in the thesis to work for the case of non-abelian group actions, we would need to answer the following question.

Question 4. *Let $(B(r), \omega)$ be the standard ball of radius r around the origin in $\mathbb{R}^4$ and let ω be such that $\omega = \omega_0$ near the boundary of $B(r)$. Let G be an non-abelian group that acts symplectically and linearly as a subgroup of $U(2)$ near the boundary and sends the boundary to itself. Is the space of equivariant symplectomorphisms that act as identity near the boundary $\mathrm{Symp}_c^G(B(r), \omega)$ non-empty and contractible?*

The proof of Theorem 3.2.4 would not help us in answering the above question as it is essential for our analysis that the groups be abelian in order to guarantee simultaneous diagonalization of the representation $G \to U(2)$. But the reliance on the group being abelian seems to be a superficial artifact of Gromov's proof of compactifying $\mathbb{C} \oplus \mathbb{C}$ to $S^2 \times S^2$. Hence we would need to find other ways of reproving Gromov's theorem without compactifying to $S^2 \times S^2$ which might make it easier to generalise under the presence of a non-abelian finite group action.

A positive answer to the above question would open up the door to prove theorems analogous to Theorem 3.3.15 for non-abelian group actions and figure out which strata the invariant almost complex structures intersect in order to understand the homotopy type of the symplectomorphism group for these actions.

8.4 Embedding spaces

Let $\mathrm{Emb}(B^{2n}(r), M)$ denote the space of symplectic embeddings of the ball $B^{2n}(r)$ into the $2n$-dimensional manifold (M, ω), endowed with the C^∞ topology and let

$$\Im\mathrm{Emb}(B^{2n}(r), M) := \mathrm{Emb}(B^{2n}(r), M)/\operatorname{Symp}(B^{2n}(r))$$

denote the space of unparameterized symplectic balls in (M, ω). Fix an embedding $\iota : B(r) \to M$, by the symplectic isotopy theorem of balls, the natural action of the identity component of the symplectomorphism group $\mathrm{Symp}_0(M, \omega)$ on the component of $\Im\mathrm{Emb}(B^{2n}(r), M)$ containing ι defines the following fibration

$$\mathrm{Stab}(\iota) \longrightarrow \mathrm{Symp}_0(M, \omega) \longrightarrow \Im\mathrm{Emb}_\iota(B^{2n}(r), M)$$

where the fiber over an embedding $\iota : B^{2n}(r) \to M$ is the subgroup of symplectomorphisms sending $\iota(B^{2n}(r))$ to itself. Thus, the homotopy type of $\mathrm{Emb}(B^{2n}(r), M)$ is intimately connected to the homotopy type of symplectomorphism groups.

Having determined the homotopy type of the S^1 and $\mathbb{Z}_n$-equivariant symplectomorphisms for $(S^2 \times S^2, \omega_\lambda)$ and $(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$ we can now try to address the related question on equivariant embeddings. Let G act linearly on the ball of radius r $B^4(r)$ in $\mathbb{R}^4$ and let $\mathrm{Symp}^G(B^4(r))$ denotes the G-equivariant symplectomorphisms of $B^4(r)$.

Question 5. *Given a compact group G acting symplectically on the $(S^2 \times S^2, \omega_\lambda)$ (and analogously for $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$). Let S be a connected component of the fixed point set. Let us denoted by Emb^G_S is the space of G-equivariant symplectic embeddings $i : B^4(r) \hookrightarrow S^2 \times S^2$ such that $i(0) \in S$ connected. Is $\Im\mathrm{Emb}^G_S := \mathrm{Emb}^G_S / \mathrm{Symp}^G(B^4(r))$ connected?*

This question was answered in [33] for toric actions. It was shown that,

Theorem 8.4.1. *(Pelayo, [33]) For every symplectic-toric $2n$ manifold M with a toric action of $\mathbb{T}^n$, there is an associated $\mathbb{Z}$-valued non-increasing step function $k_M : \mathbb{R}_{\geqslant 0} \to [0, n!\chi(M)]$ such that for each $r \geqslant 0$ the space of equivariant symplectic embeddings from* *the* $2n$ *dimensional ball* $\mathbb{B}_r^{2n}$ *into* M *is homotopically equivalent to a disjoint union of* $k_M(r)$ *subspaces, each of which is homeomorphic to the* n*-torus* $\mathbb{T}^n$*.*

In particular this implies that the space of equivariant embeddings "centered" at a fixed point of the $\mathbb{T}^n$ action up to reparametrization is connected and is homotopic to $\mathbb{T}^n$. However, no such theorems have so far been established for S^1 or $\mathbb{Z}_n$ actions on 4-manifolds. We hope to build on the work in the thesis and try to understand these space of equivariant embeddings for $(S^2 \times S^2, \omega_\lambda)$ and $(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega_\lambda)$.

Appendix A

Equivariant Gauge Groups

In this section we basically show how to calculate the homotopy type of the equivariant gauge groups that arise in lemma 3.3.10 and lemma 3.3.14.

Let P be a principle G- bundle over B, where G is an abelian group that acts on the right. Let H be a lie group that act on the base space B and this action lifts to an action on the bundle P. We shall denote this action by a left action. Note that H need not act effectively on the space B.

Let $\mathrm{Gauge}^H(P)$ denote all equivariant (with respect to the action of H) bundle automorphisms i.e equivariant maps u such that the following diagram commutes.

$$\begin{tikzcd} P \arrow[r, "u"] \arrow[d, "\pi"] & P \arrow[dl, "\pi"] \\ B & \end{tikzcd}$$

Given $u \in \mathrm{Gauge}^H(P)$, define the map

$$\phi_u : P \to G$$
$$x \mapsto \phi_u(x)$$

where $\phi_u(x)$ is defined such that

$$x \cdot \phi_u(x) = u(x)$$

Let us now see how the map ϕ_u behaves under the left action of H. We have

$$u(h \cdot x) = h \cdot x \cdot \phi_u(h \cdot x)$$

But we already have that

$$u(h \cdot x) = h \cdot u(x) = h \cdot x \cdot \phi_u(x)$$

Putting the equalities together and noticing that the G -action is free we get that

$$\phi_u(h \cdot x) = \phi_u(x)$$

That is that the map ϕ_u is invariant under the action of H.

Also from the definition we can see that

$$\phi_u(x \cdot g) = g^{-1} \cdot \phi_u(x) \cdot g = \phi_u(x)$$

where the last equality follows from G being abelian.

Denote by $\mathrm{Maps}_{H,G}(P, G)$ the space of all H and G-invariant smooth functions from P to G. Note that because these maps are G invariant this space is the same as H-invariant maps from B to G which we shall denote by $\mathrm{Inv}(B, G)$ Then we now have an map

$$\rho : \mathrm{Gauge}^H(P) \to \mathrm{Maps}_{H,G}(P, G)$$
$$u \mapsto \phi_u$$

is an an homeomorphism for C^∞-topology with the inverse being constructed using the definition $x \cdot \phi_u(x) = u(x)$.

As $\mathrm{Maps}_{H,G}(P, G) = \mathrm{Inv}(B, G)$ we have that $\mathrm{Gauge}^H(P)$ is homeomorphic to $\mathrm{Inv}(B, G)$.

Let us now use this to calculate the Homotopy type of the Gauge groups in the fibrations in Chapter 2.

Consider a rank 2 symplectic normal bundle of $\overline{D}$. Let us fix an equivariant arbitrary compatible fibre wise almost complex structure J on $N(\overline{D})$. As this is a rank two bundle, the structure group is $Sp(2)$, and this now can be reduced to $U(1) = S^1$, and the two bundles are isomorphic. Thus the space of symplectic automorphisms of the original bundle is homeomorphic to the space of symplectic automorphisms of the reduced bundle.

In our case we have a right group action of S^1 on this bundle and we are interested in the equivariant symplectic automorphisms of this bundle.This is homotopic to the space of Equivariant symplectic automorphisms of the $U(1)$ bundle.(As the reduction of the structure group can be done equivariantly) And as $U(1) = S^1$ the space of Equivariant symplectic automorphisms of the $U(1)$ bundle is the same as $\mathrm{Gauge}^{S^1}(P)$ where P is the associated principal bundle. This is homeomorphic to $\mathrm{Inv}(S^2, S^1)$ from the above discussion.

Finally note that for any non-trivial S^1 action on S^2 (possibly non-effective) the quotient space under this action is just the interval. Hence the space $\mathrm{Inv}(S^2, S^1)$ is just the space of smooth maps from the interval to S^1.

Before we embark on trying to calculate the homotopy type of $\mathrm{Gauge}^{S^1}(N(\overline{D}))$, we would need the following technical lemma. Note that in all our calculations above we have used the C^∞-topology on $\mathrm{Gauge}^{S^1}(N(\overline{D}))$ and $\mathrm{Inv}_{S^1}(S^2, S^1)$. Let $\mathrm{Gauge}_c^{S^1}(N(\overline{D}))$ denote the space of continuous S^1-equivariant gauge transformations of the bundle $N(\overline{D})$ equipped the C^0-topology. Using the same argument at the beginning of the section we can show that $\mathrm{Gauge}_c^{S^1}(N(\overline{D}))$ is homotopic to the space $\mathrm{Inv}_{S^1,c}(S^2, S^1)$ of continuous S^1-invariant maps from S^2 to S^1. Then we have that,

Lemma A.0.1. *The space* $\mathrm{Gauge}^{S^1}(N(\overline{D}))$ *with the* C^∞*-topology is homotopic to the space* $\mathrm{Gauge}_c^{S^1}(N(\overline{D}))$ *equipped with the* C^0*-topology.*

The proof of this lemma follows from an equivariant version of the arguments used to prove Theorem 3.2.13 in [41].

Lemma A.0.2. $\mathrm{Gauge}^{S^1}(N(\overline{D})) \simeq \mathrm{Gauge}_c^{S^1}(N(\overline{D})) \simeq \mathrm{Inv}_{S^1,c}(S^2, S^1) \simeq S^1$

Proof. $\mathrm{Gauge}^{S^1}(N(\overline{D})) \simeq \mathrm{Gauge}_c^{S^1}(N(\overline{D})) \simeq \mathrm{Inv}_{S^1,c}(S^2, S^1)$ follows from Lemma A.0.1 and the discussion above.

Let $\mathrm{Maps}(S^2/S^1, S^1)$ denote the space of continuous maps from S^2/S^1 to S^1.Then the space $\mathrm{Inv}_{S^1,c}(S^2, S^1)$ is homeomorphic to the space $\mathrm{Maps}(S^2/S^1, S^1)$. Further we note that as S^2/S^1 is homeomorphic to an interval $[0, 1]$, the space $\mathrm{Maps}(S^2/S^1, S^1)$ can be identified the space of continuous maps from the interval $[0, 1]$ to S^1 which we denote by $\mathrm{Maps}([0, 1], S^1)$. Let p_0 be a fixed point for the S^1 action on S^2. Then we consider the following fibration,

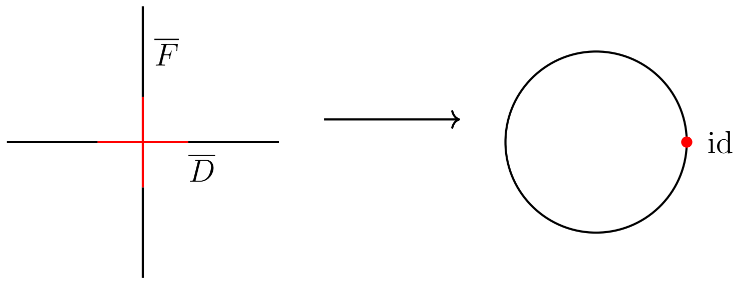
$$\{*\} \simeq \mathrm{Inv}_{S^1,c}\left((S^2, p_0), (S^1, id)\right) \hookrightarrow \mathrm{Inv}(S^2, S^1) \xrightarrow{ev} S^1$$

Where the map $ev : \mathrm{Inv}(S^2, S^1) \to S^1$ is just the evaluation map at the fixed point (for the S^1 action on S^2) p_0 and the space $\mathrm{Inv}_{S^1,c}\left((S^2, p_0), (S^1, id)\right)$ is the space of all continuous maps from S^2 to S^1, invariant under the S^1 action and send the point p_0 to the identity on S^1. As above, the space $\mathrm{Inv}\left((S^2, p_0), (S^1, id)\right)$ can be identified with the space of continuous maps from the the interval $[0, 1]$ to S^1 that send 0 to the the identity on S^1. This space of pointed maps from $[0, 1]$ to S^1 is contractible, thus completing the proof. □

Remark A.0.3. *We need the point* p_0 *to be a fixed point as the evaluation map has to be surjective.*

Lemma A.0.4. $\mathrm{Gauge}^{S^1}(N(\overline{D} \vee \overline{F})) \simeq \{*\}$

Proof. Analogous to our method above, we get that $\mathrm{Gauge}^{S^1}(N(\overline{D} \vee \overline{F}))$ is just the space of continuous maps from the following configuration to S^1 that send some neighbourhood of the crossing to the identity in S^1



And the space of such maps is indeed contractible thus completing the proof. □

We now want to carry out similar computation but for action of finite abelian groups $\mathbb{Z}_n$ on the bundle. As discussed before, we have that $\mathrm{Gauge}^{Z_n}(N(\overline{D})) \simeq \mathrm{Inv}_{Z_n}(S^2, S^1)$ where $\mathrm{Inv}_{Z_n}(S^2, S^1)$ denotes the space of Z_n invariant maps from S^2 to S^1. Further let $\mathrm{Gauge}_c^{Z_n}(N(\overline{D}))$ denote the space of continuous $\mathbb{Z}_n$ equivariant gauge transformations. As in Lemma A.0.1 we have that $\mathrm{Gauge}^{Z_n}(N(\overline{D})) \simeq \mathrm{Gauge}_c^{Z_n}(N(\overline{D}))$. Further, just as in the S^1 case we may identify $\mathrm{Gauge}_c^{Z_n}(N(\overline{D}))$ with the space $\mathrm{Inv}_{Z_n,c}(S^2, S^1)$ of continuous $\mathbb{Z}_n$ invariant maps from S^2 to S^1.

Putting all this together we have,

Lemma A.0.5. $\mathrm{Gauge}^{Z_n}(N(\overline{D})) \simeq \mathrm{Gauge}_c^{Z_n}(N(\overline{D})) \simeq \mathrm{Inv}_{Z_n,c}(S^2, S^1) \simeq S^1$

Proof. The homotopy equivalences $\mathrm{Gauge}^{Z_n}(N(\overline{D})) \simeq \mathrm{Gauge}_c^{Z_n}(N(\overline{D})) \simeq \mathrm{Inv}_{Z_n,c}(S^2, S^1)$ are all explained above. Thus we only need to show that $\mathrm{Inv}_{Z_n,c}(S^2, S^1) \simeq S^1$. In our case we know that the Z_n action on S^2 are in fact restrictions of S^1 actions on S^2, hence they are rotations about fixed points of S^2. Note that $\mathrm{Inv}_{Z_n,c}(S^2, S^1)$ is homeomorphic to the space $\mathrm{Maps}(S^2/Z_n, S^1)$ of continuous maps from $S^2/\mathbb{Z}_n$ to S^1. Further, S^2/Z_n is homeomorphic to S^2 and hence $\mathrm{Maps}(S^2/Z_n, S^1) \cong \mathrm{Maps}(S^2, S^1)$. Finally, we note that $\mathrm{Maps}(S^2, S^1) \simeq S^1$ thus completing the result. □

Lemma A.0.6. $\mathrm{Gauge}^{Z_n}(N(\overline{D} \vee \overline{F})) \simeq \{*\}$

Proof. Analogous to the proof of Lemmas A.0.4 and A.0.5, we can identify the group $\mathrm{Gauge}_{Z_n}(N(\overline{D} \vee \overline{F}))$ with maps from $S^2 \vee S^2$ that send a neighbourhood of the wedge point to the identity in S^1. The space of such maps is contractible. $\square$

Finally, we need to understand the homotopy type of $\mathbb{Z}_n$ equivariant symplectomorphisms $\mathrm{Symp}^{\mathbb{Z}_n}(S^2)$, in our analysis in Chapter 6.

Lemma A.0.7. *Consider a symplectic action of $\mathbb{Z}_n$ on S^2, then the space* $\mathrm{Symp}^{\mathbb{Z}_n}(S^2)$ *is homotopic to* $\mathrm{SO}(3)$*, if $\mathbb{Z}_n$ fixes S^2 pointwise, and is homotopic to S^1 otherwise.*

Proof. Let $\psi \in \mathrm{Symp}^{\mathbb{Z}_n}(S^2)$, consider the graph $\tilde{\psi}$ of ψ i.e

$$\begin{aligned} \tilde{\psi} : S^2 &\to S^2 \times S^2 \\ z &\mapsto (z, \psi(z)) \end{aligned}$$

Let $\mathrm{SO}(3)^{\mathbb{Z}_n}$ denote the centraliser of Z_n inside $\mathrm{SO}(3)$. Choose a $\mathbb{Z}_n$ equivariant metric for the product $\mathbb{Z}_n$ action on $S^2 \times S^2$ coming from the $\mathbb{Z}_n$ action on S^2. Then by Theorem C, Corollary C and Corollary 4.1 in [39], the mean curvature flow with respect to this equivariant metric gives us a canonical homotopy of ψ to an element inside $\mathrm{SO}(3)^{\mathbb{Z}_n}$. Further this homotopy is identity on all the elements of $\mathrm{SO}(3)^{\mathbb{Z}_n}$. Thus the map we get is in fact a deformation retract of $\mathrm{Symp}^{\mathbb{Z}_n}(S^2)$ and $\mathrm{SO}(3)^{\mathbb{Z}_n}$. Note $\mathrm{SO}(3)^{\mathbb{Z}_n} = \mathrm{SO}(3)$ *or* S^1 depending on whether $\mathbb{Z}_n$ is in the centre of $SO(3)$ or not respectively, thus proving the claim. $\square$

Appendix B

Holomorphic configurations and equivariant Gompf argument

Theorem B.0.1. *Let G be a compact group. Let A and B be two G-invariant symplectic spheres in a 4-dimensional symplectic manifold (M, ω) intersecting ω-orthogonally at a unique fixed point p for the G action. Then there exists an invariant $J \in \mathcal{J}_\omega^G$ such that both A and B are J-holomorphic. Here $\mathcal{J}_\omega^G$ denotes the space of G invariant compatible almost complex structures on M.*

Proof. The proof follows from mimicking the proof of Lemma A.1 in [18] under the presence of a group action.

□

Theorem B.0.2. *(Equivariant Gompf Argument) Let G be a compact group. Let A and B be two G-invariant symplectic spheres in a 4-dimensional symplectic manifold (M, ω) such that $A \cap B = \{p\}$ where p is a fixed point for the action and the intersection at p is transverse. Then there exists an S^1-equivariant isotopy A_t of A such that A_t intersects B transversally at p for all t, A_1 intersects B ω-orthogonally at p and the curve A_1 agrees with A outside some neighbourhood of p, .*

Proof. Since this is a local problem, we can work in a trivialising chart in $\mathbb{R}^4$ in which the action is linear. Let $B^{\perp_\omega}$ be the symplectic orthogonal to B. We can assume the image of B to be the two plane in $\mathbb{R}^4$ given by $(0,0,x,y)$, and $B^{\perp_\omega}$ to the given by the plane $(x,y,0,0)$. As A is transverse to B at p, we can assume its image is given by the graph of function (which we also call A) $A:(f,g):\mathbb{R}^2\to\mathbb{R}^2$.

Next we observe that given a function $A:=(f,g):\mathbb{R}^2\to\mathbb{R}^2$, the graph of A is a symplectic (for the standard form) submanifold of $\mathbb{R}^4$ iff $\{f,g\}>-1$. This can be proven from a direct computation. We will construct an isotopy of graphs of function of the form $A_t:=\alpha_t(r^2)A$ where α_t is a bump function depending only on the radius squared (for a fixed G invariant metric) in $\mathbb{R}^2$, and such that

- $A_0=A$,
- $A_1=0$ near (0,0),
- $A_t=A$ outside of some neighbourhood of the origin,
- A_t is symplectic for all t.

Note that as α_t is depends on the radius for a fixed G invariant metric, A_t is also G invariant.

Define $E=g\{f,r^2\}+f\{r^2,g\}$. Using the fact that $r^2(0,0)=0$ and $(r^2)'(0,0)=0$ we see that $E(0,0)=0$ and $\frac{\partial}{\partial r}E(0,0)=0$. By the intermediate value theorem, there exists $c>0$, $\epsilon>0$ and $u>0$ such that on the ball of radius $u+\epsilon$ around the origin $B(0,u+\epsilon)$ we have $E(x)\geqslant -cr^2(x)$. Choose δ such that on $B(0,u+\epsilon)$, $1+\{f,g\}>\delta>0$

Pick $\alpha:\mathbb{R}\to\mathbb{R}$ satisfying the following properties

- $\alpha(r^2)=1$ for $r^2\geqslant u$.
- $\alpha(r)=0$ for r near 0.
- $\alpha'(r^2)\leqslant\frac{\delta}{2cr^2}<\frac{1+\{f,g\}}{2cr^2}$

Define $\alpha_t := (1-t) + t\alpha(r^2)$ and $A_t := \alpha_t A$. To show that A_t is symplectic for all $0 \leqslant t \leqslant 1$ we need to check that $\{\alpha_t f, \alpha_t g\} > -1$ for all $0 \leqslant t \leqslant 1$. In the neighbourhood $B(u)$ we have

$$1 + \{\alpha_t f, \alpha_t g\} = \underbrace{1 + \alpha_t^2\{f,g\}}_{\geqslant \delta} + \underbrace{\alpha_t \alpha_t' E}_{\geqslant \frac{-\delta}{2}} \geqslant 0$$

The inequality $1 + \alpha_t^2\{f,g\} \geqslant \delta$ follows from the definition of δ and from noting that $0 \geqslant \alpha_t \geqslant 1$. $\alpha_t\alpha_t' E \geqslant \frac{-\delta}{2}$ follows from the inequality

$$\begin{aligned}
\alpha_t\alpha_t' E &\geqslant \alpha_t\alpha_t'(-cr^2) \\
&\geqslant -\alpha_t \frac{\delta}{2cr^2}(cr^2) \\
&\geqslant -\alpha_t \frac{\delta}{2} \\
&\geqslant \frac{-\delta}{2}
\end{aligned}$$

Thus in the neighbourhood $B(u)$ we have the inequality $1 + \{\alpha_t f, \alpha_t g\} > 0$ for all t. Outside of $B(u)$, the derivative α_t' is identically 0 and $\alpha_t \equiv 1$. Hence $\alpha_t\alpha_t' E \equiv 0$ outside $B(u)$ and $1 + \{\alpha_t f, \alpha_t g\} = 1 + \alpha_t^2\{f,g\} + \cancelto{0}{\alpha_t\alpha_t' E} = 1 + \{f,g\} > 0$ outside of $B(u)$.

Finally we note that $A_1 = 0$ in a neighbourhood of $(0,0)$ and it equals A outside the ball of radius u around the origin, thus proving the claim. $\square$

Appendix C

Equivariant versions of classical results from Differential Topology

Lemma C.0.1 (Relative Poincare Lemma)**.** *(see [28], Lemma 43.10) Let M be a smooth finite dimensional manifold and let $S \subset M$ be a closed submanifold. Let ω be a closed $(k+1)$-form on M which vanishes on S. Then there exists a k-form σ on an open neighborhood U of S in M such that $d\sigma = \omega$ on U and $\sigma = 0$ along S. If moreover $\omega = 0$ along S, then we may choose σ such that the first derivatives of σ vanish on S.*

Proof. By restricting to a tubular neighborhood of S in M, we may assume that M is a smooth vector bundle $p : E \to S$ and that $i : S \to E$ is the zero section. We consider $\mu : \mathbb{R} \times E \to E$, given by $\mu(t,x) = \mu_t(x) = tx$, then $\mu_1 = \mathrm{id}_E$ and $\mu_0 = i \circ p : E \to S \to E$. Let $V \in \mathfrak{X}(E)$ be the vertical vector field $V(x) = vl(x,x) = \frac{d}{dt}(x + tx)$ whose flow is $\mathrm{Fl}_t^V = \mu_{e^t}$. Locally, for t in $(0,1]$ we have

$$\frac{d}{dt}\mu_t^*\omega = \frac{d}{dt}(\mathrm{Fl}_{\log t}^V)^*\omega = \frac{1}{t}(\mathrm{Fl}_{\log t}^V)^*\mathcal{L}_V\omega = \frac{1}{t}\mu_t^*(i_V d\omega + d i_V\omega) = \frac{1}{t}d\mu_t^* i_V\omega$$

For $x \in E$ and $X_1, \dots, X_k \in T_x E$ we have

$$\begin{aligned}(\frac{1}{t}\mu_t^* i_V \omega)_x(X_1, \dots, X_k) &= \frac{1}{t}(i_V \omega_{tx}(T_x\mu_t \cdot X_1, \dots, T_x\mu_t \cdot X_k) \\ &= \frac{1}{t}\omega_{tx}(V(tx), T_x\mu_t \cdot X_1, \dots, T_x\mu_t \cdot X_k) \\ &= \omega_{tx}(vl(tx, tx), T_x\mu_t \cdot X_1, \dots, T_x\mu_t \cdot X_k)\end{aligned}$$

So the k-form $\frac{1}{t}\mu_t^* i_V \omega$ is defined and smooth in (t, x) for all $t \in [0, 1]$ and describes a smooth curve in $\Omega^k(E)$. Note that for $x \in S = 0_E$ we have $\frac{1}{t}\mu_t^* i_V \omega = 0$, and if $\omega = 0$ on $T_S M$, we also have $0 = \frac{d}{dt}\mu_t^*\omega = \frac{1}{t} d\mu_t^* i_V \omega$, so that all first derivatives of $\frac{1}{t}\mu_t^* i_V \omega$ vanish along S. Since $\mu_0^*\omega = p^* i^* \omega = 0$ and $\mu_1^*\omega = \omega$, we have

$$\begin{aligned}\omega &= \mu_1^*\omega - \mu_0^*\omega \\ &= \int_0^1 \frac{d}{dt}\mu_t^*\omega \, dt \\ &= \int_0^1 d(\frac{1}{t}\mu_t^* i_V \omega) \, dt \\ &= d\left(\int_0^1 (\frac{1}{t}\mu_t^* i_V \omega) \, dt\right) \\ &= d\sigma\end{aligned}$$

If $x \in S$, we have $\sigma = 0$, and all first derivatives of σ vanish along S whenever $\omega = 0$ on $T_S M$. □

Remark C.0.2. *If there is a symplectic action of a compact group G acting on M such that ω is G invariant and S is G-invariant, then we can construct σ as above such that in addition to the above conditions σ also is G-invariant. This is gotten by noting that*

$$\omega = \int_G \omega = \int_G d\sigma = d\int_G \sigma$$

Let $\tilde{\sigma} := \int_G \sigma$ and hence $d\tilde{\sigma} = \omega$ and $\tilde{\sigma}$ satisfies all the conditions.

Lemma C.0.3 (Moser isotopy)**.** *Let (M, ω) be a symplectic manifold and let $S \subset M$ be a submanifold. Suppose that ω_i, $i = 0, 1$, are closed 2-forms such that at each point $x \in S$, the forms ω_0 and ω_1 are equal and non-degenerate on $T_x S$. Then there exist open* *neighborhoods N_0 and N_1 of S and a diffeomorphism $\phi : N_0 \to N_1$ such that $\phi^*\omega_1 = \omega_0$, $\phi|_S = \mathrm{id}$, and $d\phi|_S = \mathrm{id}$.*

Proof. Consider the convex linear combination $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$. Since ω_0 and ω_1 are equal along S, there exists a neighborhood U_1 of S on which ω_t is non-degenerate for all $t \in [0,1]$. By restricting U_1 to a possibly smaller neighborhood U_2, the Relative PoincarÃ© Lemma C.0.1 implies that there exists a 1-form σ such that $d\sigma = (\omega_1 - \omega_0)$, $\sigma = 0$ on S, and all first derivatives of σ vanish along S. Define the time-dependent vector field X_t on U_2 by setting

$$\sigma = -i_{X_t}\omega_t$$

Since $X_t = 0$ on S, by restricting U_2 to a smaller neighborhood U_3, we can ensure that the flow ψ_t of X_t exists for $t \in [0,1]$. We then have

$$\frac{d}{dt}\psi_t^*\omega_t = \psi_t^*\left(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t\right) = \psi_t^*\left(\frac{d}{dt}\omega_t + di_{X_t}\omega_t\right) = \psi^*(\omega_1 - \omega_0 - d\sigma) = 0$$

so that $\psi^*\omega_t = \omega_0$. Finally, since $\sigma = 0$ on S, $\psi = \mathrm{id}$ on S, and since all first derivatives of σ vanish on T_SM, $d\psi = \mathrm{id}$ on T_SM. □

Remark C.0.4. *As the remark above, when both ω_1 and ω_2 are both invariant under a compact group action G, and S is an G-invariant submanifold, then there is a G-equivariant diffeomorphism ϕ that satisfies the conditions as above.*

Lemma C.0.5. *[Equivariant Symplectic neighborhoods theorem] Let (M_i, ω_i), $i = 0, 1$, be two symplectic G-manifolds(The compact group G, acting symplectically on the 2 manifolds) with two invariant symplectic submanifolds $S_i \subset M_i$ with invariant symplectic normal bundles N_i. Suppose that there is an equivariant isomorphism $A : N_0 \to N_1$ covering an equivariant symplectomorphism $\phi : S_0 \to S_1$. Then ϕ extends to a equivariant symplectomorphism of neighborhoods $\Phi : U_0 \to U_1$ whose derivative along S_0 is equal to A.*

Proof. We can extend the automorphism A to a diffeomorphism of neighborhoods $\psi : U_0 \to U_1$ by setting

$$\psi = \exp \circ A \circ \exp^{-1}$$

By construction, $d\psi = A$ along S_0, so that ω_0 and $\psi^*\omega_1$ coincides along S_0. Applying the G- equivariant Moser isotopy Lemma gives the result. $\square$

Let (M, ω) be a symplectic manifold. Let G be a compact lie group acting symplectically on M. Let S be an invariant submanifold under the G action. Let $\mathrm{Op}(S)$ be an invariant open neighbourhood of S, Further define

$$\mathrm{Symp}^G_{\mathrm{id},N}(M,S) = \{\phi \in \mathrm{Symp}^G_0(M) \mid \phi|_S = \mathrm{id},\ d\phi|_{T_SM} = \mathrm{id}\}$$

$$\mathrm{Symp}^G_{\mathrm{id},\mathrm{Op}(S)}(M,S) = \{\phi \in \mathrm{Symp}^G_0(M) \mid \phi = \mathrm{id}\ \text{ near } S\}$$

then we would like to show that $\mathrm{Symp}^G_{\mathrm{id},N}(M,S) = \mathrm{Symp}^G_{\mathrm{id},\mathrm{Op}(S)}(M,S)$. But before we do that we would need the following lemmas.

Following [23], we define a invariant tubular neighborhood of a invariant submanifold $\iota : S \hookrightarrow M$ as a smooth equivariant embeddings $f : E \hookrightarrow M$ of a vector bundle $\pi : E \to S$ such that

1. $f|_S = \iota$ after identifying S with the zero section of $\pi : E \to S$.
2. $f(E)$ is an open neighborhood of S.

In practice, it is often enough to work with the normal bundle $N \subset T_SM$ defined as the orthogonal of T_SM relative to a equivariant riemannian structure. (See Bradon pg 303 for existence of such invariant tubular neighbourhood.)

Lemma C.0.6 (Unicity of tubular neighborhoods). *(See [23], Theorem 4.5.3) Let M be a G-manifold, let $\iota : S \hookrightarrow M$ be a invariant submanifold with normal bundle N. Then,*

1. *given any two invariant tubular neighborhoods $f_i : N \hookrightarrow M$, $i = 0, 1$, there is a equivariant gauge transformation $A \in \mathcal{G}(N)$ such that f_0 and $f_1 \circ A$ are equivariant isotopic rel. to S.*
2. *The space $\mathcal{T}_S$ of all invariant tubular neighborhoods $f : N \hookrightarrow M$ is homotopy equivalent to the group of equivariant gauge transformations $\mathcal{G}(N)$.*

3. *The space $\mathcal{T}_{S,d\iota}$ of invariant tubular neighborhoods $f : N \hookrightarrow M$ such that $df|_S = d\iota$ is contractible.*

Proof. (1) We construct an equivariant isotopy F_t in two steps. Firstly, given an G-invariant smooth function $\delta : S \to (0, 1]$, let $U_\delta \subset N$ be the invariant disc bundle

$$U_\delta = \{x \in N \mid |x| < \delta(\pi(x))\}$$

Note that N smoothly retracts onto U_δ through embeddings $G_t : N \to N$ of the form

$$G_t = (1 - t)\,\mathrm{id} + t h_{\delta(x)}$$

where h_r is a equivariant one-parameter family of contracting, $C(G) \subset \mathrm{SO}(k)$-invariant diffeomorphism (where $C(g)$ is the centraliser of G in $\mathrm{SO}(k)$ and k is the rank of the bundle) $h_r : \mathbb{R}^k \to D^k(r)$, restricting to the identity on $D^k(r/2)$ and varying smoothly with r. Then, choosing an appropriate invariant function δ, and composing f_1 with G_t, we can isotope f_1 to an embedding $f_\delta = f_1 G_1$ satisfying

$$f_\delta(N) \subset f_0(N) \text{ and } f_\delta = f_1 \text{ on } U_{\delta/2} \tag{C.1}$$

so that the map $g = f_0^{-1} f_\delta : N \to N$ is well-defined. Secondly, observe that the map g is equivariantly isotopic to its vertical derivative $A_{f_\delta} = dg^{\mathrm{vert}} \in \mathcal{G}(N)$ along S via the canonical smooth isotopy

$$H_0(x) = \phi(x), \quad H_t(x) = g(tx)/t,\ 0 < t \leqslant 1$$

Note that H is indeed equivariant. An isotopy from f_δ to $f_0 \circ A_{f_\delta}$ is then given by $f_0 H_{1-t}$. The sought-for equivariant isotopy F_t is the concatenation of $f_1 G_t$ with $f_0 H_{1-t}$.

(2) Fix a invariant tubular neighborhood $f_0 : N \hookrightarrow M$ and choose once for all a smooth family of equivariant diffeomorphisms $h_r : \mathbb{R}^k \to D^k(r)$ as in the proof of (1). Given any other invariant tubular neighborhood $f : N \hookrightarrow M$, the isotopy constructed in (1) only depends on an auxiliary invariant function $\delta : S \to \mathbb{R}_+$. Although the choice of

this function is not canonical, the set $\Delta^G(f)$ of all invariant δ for which $h_\delta(N) \subset f^{-1}f_0(N)$ is a convex subset of $C^\infty(S, \mathbb{R}_+)$. Consequently, the projection of the fibre product

$$\mathcal{T}_\Delta = \{(f, \delta) \mid f \in \mathcal{T}_S,\ \delta \in \Delta^G(f)\} \to \mathcal{T}_S$$

is a homotopy equivalence. Consider the embedding

$$\phi : \mathcal{G}(N) \times C^{\infty,G}(S, \mathbb{R}^+) \hookrightarrow \mathcal{T}_\Delta \tag{C.2}$$

$$(A, \delta) \mapsto (f_0 \circ A, \delta) \tag{C.3}$$

and the continuous map

$$\psi : \mathcal{T}_\Delta \to \mathcal{G}(N) \times C^{\infty,G}(S, \mathbb{R}_+) \tag{C.4}$$

$$(f, \delta) \mapsto (A_{f_\delta}, \delta) \tag{C.5}$$

where $C^{\infty,G}(S, \mathbb{R}_+)$ denotes the space of G invariant smooth functions. Then we have $\psi\phi = \mathrm{id}_{\mathcal{G}(N)\times C^{\infty,G}(S,\mathbb{R}_+)}$, while $\phi\psi(f, \delta) = (f_0 \circ A_{f_\delta}, \delta)$, the map $f_0 \circ A_{f_\delta}$ being the terminal point of the isotopy defined in (1). This shows that ϕ and ψ are homotopy inverses.

(3) Choosing f_0 such that $df_0 = d\iota$ along S, this immediately follows from the fact that the space $\mathcal{T}_{S,d\iota}$ is homotopy equivalent to the subspace of $\mathcal{T}_\Delta$ that retracts to the contractible subspace $\{f_0\} \times C^\infty(S, \mathbb{R}_+)$ under the isotopy defined in (1). □

Lemma C.0.7. *The inclusion* $\mathrm{Symp}^G_{\mathrm{id},\mathrm{Op}(S)}(M, S) \hookrightarrow \mathrm{Symp}^G_{\mathrm{id},N}(M, S)$ *is a homotopy equivalence.*

Proof. We follow the same ideas as in the non-equivariant case. Consider the short exact sequence

$$\mathrm{Symp}^G_{\mathrm{id},\mathrm{Op}(S)}(M, S) \hookrightarrow \mathrm{Symp}^G_{\mathrm{id},N}(M, S) \to \mathcal{G}_{S,\omega}$$

where the group $\mathcal{G}_{S,\omega} := \mathrm{Symp}_{\mathrm{id},N}(M, S)/\,\mathrm{Symp}_{\mathrm{id},\mathrm{Op}(S)}(M, S)$ is the group of germs along S of equivariant symplectomorphisms $\phi \in \mathrm{Symp}_{\mathrm{id},N}(M, S)$. We wish to show that $\mathcal{G}_{S,\omega}$ is contractible.

Choose a compatible equivariant almost-complex structure J and let g be the associated equivariant metric. Let N be the symplectic orthogonal complement of TS in TM. N is invariant under the G action. Equip N with the minimal coupling form Ω and choose $\epsilon > 0$ so that the ϵ-disk subbundle $V_\epsilon \subset N$ is symplectomorphic to a tubular neighborhood U of S. Let ι denote both inclusions $U \hookrightarrow M$ and $V \hookrightarrow N$.

Let $\Omega_S^{\mathrm{loc},G}$ be the space of germs of G invariant symplectic forms defined near S and agreeing with ω along $T_S M$. Given any two germs $[\omega_0]$ and $[\omega_1]$, their linear convex combination $\omega_t = (1-t)\omega_0 + t\omega_1$ is non-degenerate in some neighborhood of S. Consequently, $\Omega_S^{\mathrm{loc},G}$ is convex, hence contractible. By the Symplectic neighborhood theorem, the group $\mathcal{G}_{S,\omega}$ acts transitively on $\Omega_S^{\mathrm{loc},G}$, giving rise to a fibration

$$\mathcal{G}_{S,\omega}^{\mathrm{loc},G} \xrightarrow{\simeq} \mathcal{G}_{S,\omega} \to \Omega_S^{\mathrm{loc},G}$$

whose fiber $\mathcal{G}_{S,\omega}^{\mathrm{loc},G}$ is the group of germs of equivariant diffeomorphisms that are symplectic near S. This space is homeomorphic to the space $\mathcal{E}_{S,\omega}$ of germs along S of equivariant symplectic embeddings $f : \mathrm{Op}(S) \to M$ such that $f|_S = \iota$ and $df|_S = d\iota$. By Lemma C.0.6 (3), we know that $\mathcal{E}_{S,\omega}$ is contractible, so that $\mathcal{G}_{S,\omega}$ and $\mathcal{G}_{S,\omega}^{\mathrm{loc},G}$ are also contractible, thus completing the proof. □

Lemma C.0.8. *Let G be a compact group acting symplectically on a compact manifold (M, ω). Let W_t be a smooth k-parameter family of symplectic submanifolds ($t \in [0,1]^k$), which are invariant under the G action. Then there exists a k-parameter family of equivariant Hamiltonian symplectomorphisms $\phi_t : M \to M$ such that $\phi_t(W_0) = W_t$*

Proof. The proof follows by mimicking the proof of Proposition 4 in [7] under the presence of a group action. □

Let X be a topological space with an action of a topological group G. We say X admits local cross sections at $x_0 \in X$ if for there is a neighbourhood U containing x_0 and a map $\chi : X \to G$ such that $\chi(u) \cdot x_0 = u$ for all $u \in U$. We say X admits local cross sections if this is true for all $x_0 \in X$.

Theorem C.0.9. *(Palais) Let X, Y be a topological spaces with a action of a topological group G. Let the G action on X admit local cross sections. Then any equivariant map f from another space Y to X is locally trivial.*

Proof. Suppose for every point $x_0 \in X$ there is a local section $\chi : U \to G$ where U is an open neighbourhood of x_0. Then we define a local trivialisation of f as follows.

$$\begin{aligned} \rho : U \times f^{-1}(x_0) &\to f^{-1}(U) \\ (u, \gamma) &\mapsto \chi(u) \cdot \gamma \end{aligned}$$

As f is equivariant we indeed have $f(\rho(u,\gamma)) = f(\chi(u)\cdot\gamma) = \chi(u)\cdot f(\gamma) = \chi(u)\cdot x_0 = u$, where the last equality follows from the definition of being a local section. Thus ρ maps $U \times f^{-1}(x_0)$ into $f^{-1}(U)$.

Conversely there is map

$$\begin{aligned} \beta : f^{-1}(U) &\to U \times f^{-1}(x_0) \\ y &\mapsto \big(f(y), \chi(f(y))^{-1} \cdot y\big) \end{aligned}$$

We can indeed check that the two maps are inverses of each other.

$\beta \circ \rho(u, \gamma) = \beta(\chi(u)\cdot\gamma) = \big(\chi(u) \cdot \gamma, \chi(\chi(u) \cdot f(\gamma))^{-1} \cdot \chi(u) \cdot \gamma\big) = (u, \chi(u)^{-1}\chi(u)\cdot\gamma) = (u, \gamma)$.

Similarly we can check that $\rho \circ \beta = id$, thus completing the proof. ☐

Appendix D

Alexander-Eells isomorphism

In this appendix, we first recall an isomorphism between the homology of a submanifold $Y \subset X$ and the homology of its complement $X - Y$ that is reminiscent of the Alexander-Pontryagin duality in the category of oriented, finite dimensional manifolds. This isomorphism, due to J. Eells, exists whenever the submanifold Y is co-oriented and holds, in particular, for infinite dimensional Fréchet manifolds. We then give a geometric realization of this isomorphism in the special case Y and $X - Y$ are orbits of a continuous action $G \times X \to X$ satisfying some mild assumptions. We closely follow Eells [17] p. 125–126.

Let X be a manifold, possibly infinite dimensional, and let Y be a co-oriented submanifold of positive codimension p. As explained in [17], there exists an isomorphism of singular cohomology groups

$$\phi : H^i(Y) \to H^{i+p}(X, X - Y)$$

called the Alexander-Eells isomorphism. We define the fundamental class (Thom class) of the pair (X, Y) as $u = \phi(1) \in H^p(X, X - Y)$.

Proposition D.0.1 (Eells, p. 113). *The pairing*

$$H^*(Y) \otimes H^*(X, X - Y) \to H^*(X, X - Y)$$

$$y \otimes x \mapsto y \cup x$$

makes $H^*(X, X - Y)$ into a free $H^*(Y)$-module of rank one, generated by u.

Let $\phi_* : H_{i+p}(X, X - Y) \to H_i(Y)$ be the dual of the Alexander-Eells isomorphism $\phi^* = \phi$. By definition, we have

$$\phi_*(a) = u \cap a$$

Suppose a topological group G acts continuously on X (on the left), leaving Y invariant, and in such a way that both $X - Y$ and Y are homotopy equivalent to orbits. We have continuous maps $\mu : G \times (X, X - Y) \to (X, X - Y)$, $\mu : G \times (X - Y) \to (X - Y)$, and $\mu : G \times Y \to Y$ inducing $H_*(G)$-module structures on $H_*(X, X - Y)$, $H_*(X - Y)$ and $H_*(Y)$. We write $\mu_*(c \otimes a) = c * a$ for the action of $c \in H_i(G)$.

Lemma D.0.2. *In this situation, the Alexander-Eells isomorphism preserves the $H_*(G)$-module structure, that is, the following diagram is commutative:*

$$\begin{array}{ccc} H_*(G) \otimes H_*(X, X - Y) & \xrightarrow{\mu_*} & H_*(X, X - Y) \\ \downarrow{\scriptstyle 1 \otimes \phi_*} & & \downarrow{\scriptstyle \phi_*} \\ H_*(G) \otimes H_*(Y) & \xrightarrow{\mu_*} & H_*(Y) \end{array}$$

Thus for any $a \in H_{i+p}(X, X - Y)$, $c \in H_i(G)$, we have $\phi_*(c * a) = c * \phi_*(a)$.

Proof. We first note that if u is the fundamental class of the pair (X, Y), then $\mu^*(u) = 1 \otimes u \in H^0(G) \otimes H^p(X, X - Y)$, because $H^i(X, X - Y) = 0$ for all $i < p$. The cap product is a bilinear pairing

$$\cap : H^*\big(G \times (X, X - Y)\big) \otimes H_*\big(G \times (X, X - Y)\big) \to H_*\big(G \times X\big)$$

that is adjoint to the cup product. Using the relative form of Künneth theorem, this bilinear map defines a pairing

$$\cap : \big(H^*(G) \otimes H^*(X, X - Y)\big) \otimes \big(H_*(G) \otimes H_*(X, X - Y)\big) \to H_*(G) \otimes H_*(X)$$

Let's write $j^* : H^*(X, X - Y) \to H^*(X)$. Then, for any $c \otimes x \in H^*(G) \otimes H^*(X, X - Y)$, and any $b \otimes a \in H_*(G) \otimes H_*(X, X - Y)$ we have

$$\begin{aligned}
\big\langle c \otimes j^* x, \mu^*(u) \cap (b \otimes a)\big\rangle &= \big\langle (c \otimes x) \cup \mu^*(u), b \otimes a\big\rangle \\
&= \big\langle (c \otimes x) \cup (1 \otimes u), b \otimes a\big\rangle \\
&= \big\langle (c \cup 1) \otimes (x \cup u), b \otimes a\big\rangle \\
&= \big\langle c, b\big\rangle\big\langle (x \cup u), a\big\rangle \\
&= \big\langle c, b\big\rangle\big\langle j^* x, u \cap a\big\rangle \\
&= \big\langle c \otimes j^* x, b \otimes (u \cap a)\big\rangle
\end{aligned}$$

It follows that $\mu^*(u) \cap (b \otimes a) = b \otimes (u \cap a) = b \otimes \phi_*(a)$. We then compute

$$\phi_*(\mu_*(c \otimes x)) = u \cap \mu_*(c \otimes x) = \mu_*\big(\mu^*(u) \cap (c \otimes x)\big) = \mu_*\big(c \otimes \phi_*(x)\big) = \mu_*\big((1 \otimes \phi_*)(c \otimes x)\big)$$

which is the desired relation. $\square$

Since $H_i(X, X - Y) = 0$ for $i < p$, the Universal Coefficient Theorem yields a canonical isomorphism $\beta : H^p(X, X - Y) \to \mathrm{Hom}(H_p(X, X - Y); \mathbb{Z})$. If Y is connected, $H_p(X, X - Y)$ is of rank one. In this case, define $a_u \in H_p(X, X - Y)$ as the unique class such that $\beta(u)a_u = 1$. Suppose the Leray-Hirsch theorem applies to the evaluation fibration $G \to Y$. Then, $H_*(X, X - Y)$ becomes a $H_*(Y)$-module by identifying $H_*(Y)$ with $1 \otimes H_*(Y) \subset H_*(G)$ and by setting

$$b * x := [1 \otimes b] * x$$

Theorem D.0.3. *Under the above assumptions, the isomorphism $\psi_* = \phi_*^{-1} : H_i(Y) \to H_{i+p}(X, X - Y)$ is given by $\psi_*(y) = y * a_u$. Thus $H_*(X, X - Y)$ is generated by a_u as a $H_*(Y)$-module.*

Proof. Since

$$\langle 1, \phi_*(a_u)\rangle = \langle 1, u \cap a_u\rangle = \langle 1 \cup u, a_u\rangle = \langle u, a_u\rangle = 1$$

it follows that $\phi_*(a_u) = 1$, which is equivalent to $\psi_*(1) = a_u$. Since ϕ_* is an isomorphism of $H_*(G)$-modules, its inverse is also an isomorphism of $H_*(G)$-modules. We can then write

$$\psi_*(y) = \psi_*(y * 1) = y * \psi_*(1) = y * a_u$$

$\square$

From the naturality of the connecting homomorphism ∂ in the long exact sequence of the pair $(X, X - Y)$, we get

Corollary D.0.4. *Suppose in addition to the above hypotheses that X is contractible. Then the isomorphism*

$$\lambda_* = \partial \circ \psi_* : H_i(Y) \to H_{i+p-1}(X - Y)$$

is given by $\lambda_*(y) = y * x_u$, where $x_u = \partial a_u$.

We now apply the Alexander-Eells isomorphism to the situation of Section 4.2. Let G be the centralizer $\mathrm{Symp}_h^{S^1}(S^2 \times S^2, \omega_\lambda)$, let $X = \mathcal{J}_{\omega_\lambda}^{S^1}$ be the contractible space of invariant, compatible, almost-complex structures, and let Y be the codimension 2 stratum $\mathcal{J}_{\omega_\lambda}^{S^1} \cap U_{m'}$. The connecting isomorphism $\partial : H_2(\mathcal{J}_{\omega_\lambda}^{S^1}, \mathcal{J}_{\omega_\lambda}^{S^1} \cap U_m) \to H_1(\mathcal{J}_{\omega_\lambda}^{S^1} \cap U_m)$ maps the generator a_u to the link of $\mathcal{J}_{\omega_\lambda}^{S^1} \cap U_{m'}$ in $\mathcal{J}_{\omega_\lambda}^{S^1} \cap U_m$, that is, to the loop $p_m(y_2)$ generated by the action of $y_2 \subset G$. Consequently, Corollary D.0.4 immediately implies the following geometric description of the Alexander-Eells isomorphism.

Proposition D.0.5. *The Alexander-Eells isomorphism*

$$\lambda_* : H_p(\mathcal{J}_{\omega_\lambda}^{S^1} \cap U_{m'}) \to H_{p+1}(\mathcal{J}_{\omega_\lambda}^{S^1} \cap U_m)$$

is given by

$$\lambda_*(y) = y * p_m(y_2) = y * y_2 * 1 = [y_2 \otimes y] * 1 = \mu_m\big([y_2 \otimes y] \otimes 1\big) = p_m\big([y_2 \otimes y]\big)$$

In particular, if $p_{m'}(\tilde{y}) = y \in H_*(Y)$, then $\lambda_*(y) = \tilde{y} * y_2 * 1 = p_m(\tilde{y} * y_2)$.

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