Rostocker Mathematisches Kolloquium

Heft 62

Universität Rostock

Institut für Mathematik

2007

https://doi.org/10.18453/rosdok_id00003244

Herausgeber: Universität Rostock
Mathematisch – Naturwissenschaftliche
Fakultät
Institut für Mathematik

Schriftleitung: Raimond Strauß

Herstellung der Druckvorlage: Susann Dittmer

Zitat–Kurztitel: Rostock. Math. Kolloq. **62** (2007)

ISSN 0138-3248

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Rostock. Math. Kolloq. **62**, 3–19 (2007)

Subject Classification (AMS)
60K25, 90B22, 60J10

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A New Criterion of Stability for Stochastic Networks With Two Stations and Two Heterogeneous Servers

ABSTRACT. We introduce and study a new notion of stability of a stochastic fluid model in terms of random stopping times (partially building on ideas used by Stolyar [10] in his deterministic setting). It may be viewed as an analog of the original criterion for random T's (which may differ for different φ's). In particular, it is shown that our notion of stability is equivalent to L_p-stability for some $p > 1$. We consider an example of a polling system with tow stations and two servers in which the corresponding fluid model may be unstable in the sense as it was written in ([10]) but stable from the generalised viewpoint that we adopt.

KEY WORDS. stability, queueing networks, polling systems.

1 Introduction

In a number of papers, the fluid approximation approach was used for the instability analysis of queueing models. Dai [2] and Meyn [9] proved that if all fluid limits are unstable, then the underlying Markov process is transient. Bramson [1] showed that a Markov process may be transient even if some of its fluid limits are stable. One should note that in order to establish the positive recurrence of a Markov process, it is sufficient (and in certain sense necessary) to show some weak stability of all corresponding fluid limits.

Kumar and Meyn [7] considered stochastic fluid limits and proposed the following notion of stability : a fluid model is L_p-stable, $p > 0$ if

$$sup_{\varphi}\mathbb{E}|\varphi(t)|^p \to 0 \quad \text{as} \quad t \to +\infty\,.$$

They showed the equivalence of the L_2-stability of the fluid model and several notions of stability for the underlying Markov process.

This paper is organised as follows. We introduce and study the notion of stability of a stochastic fluid model in terms of random stopping times (partially building on ideas used by Stolyar in his deterministic setting). It may be viewed as an analog of the original criterion for random T's (which may differ for different φ's). In particular, it is shown that our notion of stability is equivalent to L_p-stability for some $p > 1$.

We consider an example of a polling system with tow stations and two servers in which the corresponding fluid model may be unstable in the sense as it was written in ([10]), i.e.

$$\exists T^{'} > 0,\ \exists \varepsilon \in (0,1]\ :\ |\varphi(T^{'})| \leq 1 - \varepsilon \text{ a.s.}, \tag{1}$$

for any fluid limit φ, but stable from the generalised viewpoint that we adopt.

It shows that even simple queueing systems may exhibit a kind of fluid behavior (basically random bifurcations) that cannot be captured by deterministic fluid models, but nevertheless, is essential for stability analysis. This kind of behavior has already been described by Malyshev et al. in a number of papers about "random walks" in $\mathbb{Z}_+^N$ (see [8] and the list of references therein).

1.1 Positive recurrence of a Markov process via stability of its fluid limits

Let χ be a complete metric space with a metric ρ, and $\mathcal{B}$ the $\sigma - algebra$ generated by open sets. Let $\mathbf{0} \in \chi$ be a fixed element. For $x \in \chi$, put $|x| = \rho(x, \mathbf{0})$. In what follows, we make the following assumptions.

Assumption 1.1 (i) *for any constant $K \geq 0$, the set*

$$A(K) = \{x \in \chi : |x| \leq K\}$$

is compact;

(ii) *for any constant $c \geq 0$, a mapping $x \to c * x$ is defined such that*

(1) *$c * \mathbf{0} = \mathbf{0}$ for any $c \geq 0$;*

(2) *$\rho(c * x_1, c * x_2) = c\rho(x_1, x_2)$ for any $c \geq 0$ and $x_1, x_2 \in \chi$;*

(3) *if $c_n \to c$, then $c_n * x \to c * x$ for any x (and, therefore, the convergence is uniform on any set $A(K)$.)*

In fact, in (i) it is sufficient to assume that the set $A(1)$ is compact. For simplicity, we will write $\dfrac{x}{c}$ instead $\dfrac{1}{c} * x$.

Let $\mathcal{Z} = \{z_1, ..., z_d\}$ be a finite set with natural discrete topology. For each $M > 0$, denote by $\mathcal{D}[0, M]$ the space of $\chi \times \mathcal{Z}$-valued cadlag (right-continuous with LHS limits) functions

$$f(t) = (f^1(t), f^2(t)), \quad t \in [0, M]$$

endowed with the skorohod J_1-metric :

$$d_M(f_1, f_2) = \inf_{g\in\Delta}\{ \sup_{t\in[0,M]} [|g(t) - t| + \rho(f_1^1(g(t)), f_2^1(t)) + 1\!\!1_{(f_1^2(g(t))\neq f_2^2(t))}]\},$$

where

$$\Delta = \{g : [0, M] \to [0, M],\ g \text{ monotone continious},\ g(0) = 0,\ g(M) = M\}.$$

Let $\mathcal{D}[0, \infty)$ denote a space of $\chi \times \mathcal{Z}$-valued cadlag functions on $[0, \infty)$ with the metric

$$d(f_1, f_2) = \sum_1^\infty 2^{-M} \frac{d_M(f_{1,M}, f_{2,M})}{1 + d_M(f_{1,M}, f_{2,M})},$$

where $f_{i,M}$ is a restriction of f_i in $[0, M], i = 1, 2$.

Let $\chi^{'}$ be a closed subset of χ, and $P(t, x, z, B)$ a probabilistic transition kernel. Here $t \geq 0, x \in \chi^{'}, z \in \mathcal{Z}, B \in \mathcal{Z}^{'}$, where $\mathcal{Z}^{'}$ is a σ-algebra in $\chi^{'} \times \mathcal{Z}$ generated by open sets.

For $(x, z) \in \chi^{'} \times \mathcal{Z}$, let

$$(X, Z)^{(x,z)} = \{(X, Z)^{(x,z)}(t), t \geq 0\}$$

be a $\chi^{'} \times \mathcal{Z}$-valued time-homogeneous Markov process with transition kernel P, a.s. cadlag paths, and the initial state $(X, Z)^{(x,z)}(0) = (x, z)$. We assume further that the process satisfies the strong Markov property.

Remark 1.1 One can introduce a more general description of a Markov process with infinite (either countable or not) "index set" $\mathcal{Z}$. In this case, a lot of additional technicalities arise. Within this paper, we decided to confine ourselves only to finite set $\mathcal{Z}$.

Definition 1.1 *A Markov process $(X, Z) = \{(X, Z)^{(x,z)}\}$ is positive recurrent (with respect to the semi norm $|.|$) if there exists a finite K such that the set*

$$B = B(K) = \{(x, z) : |x| \ \leq K\} \subset \chi^{'}$$

is positive recurrent, i.e for some $\delta > 0$,

1. *for all $(x, z) \in \chi^{'} \times \mathcal{Z}$,*

$$\eta^{(x,z)}(B) = inf\{t \geq \delta : (X, Z)^{(x,z)}(t) \in B\} < \infty \quad a.s.;$$

2. $\sup_{(x,z)\in B} \mathbb{E}\eta^{(x,z)}(B) < \infty.$

For $x \in \chi^{'}, z, r \in \mathcal{Z}$, let

$$Y_r^{(x,z)}(t) = \int_0^t 1\!\!1_{(Z^{(x,z)}(u)=r)} du, \quad t \geq 0$$

be the process couting the sejourn time of the 2nd coordinate at r.

For each $(x, z) \in \chi^{'} \times \mathcal{Z}, |x| > 0$, introduce a family of scaled processes

$$\tilde{X}^{(x,z)} = \{\tilde{X}^{(x,z)}(t) = \frac{X^{(x,z)}(|x|t)}{|x|}, \quad t \geq 0\}$$

and for each $r \in \mathcal{Z}$,

$$\tilde{Y}_r^{(x,z)} = \{\tilde{Y}_r^{(x,z)}(t) = \frac{Y_r^{(x,z)}(|x|t)}{|x|}, \quad t \geq 0\}$$

Definition 1.2 *We call the family*

$$(\tilde{X}, \tilde{Y}) = \{\tilde{X}^{(x,z)}, \tilde{Y}_r^{(x,z)}, \quad r \in \mathcal{Z}\}_{x \in \chi^{'}, |x| \geq 1, z \in \mathcal{Z}}$$

relatively compact (at infinity) if, for each sequence

$$(\tilde{X}^{(x_n,z_n)}, \tilde{Y}_r^{(x_n,z_n)}, \quad r \in \mathcal{Z}), \quad |x_n| \to \infty, \quad z_n \in \mathcal{Z}$$

there exists a subsequence $(\tilde{X}^{(x_{n_k},z_{n_k})}, \tilde{Y}_r^{(x_{n_k},z_{n_k})}, \quad r \in \mathcal{Z})$ *that converges weakly (in Skorohod topology) to some limit process*

$$\varphi = \{\varphi(t), t \geq 0\},$$

which is called a fluid limit.

For any $t \geq 0$ and fluid limit φ, the values of $\varphi(t)$ lie in $\chi \times \mathbb{R}_+^d$.

Put $\varphi(t) = (x(t), y(t))$ and $y(t) = \{y_z(t)\}_{z \in \mathcal{Z}}$, where $x(t) \in \chi$ and $y(t) \in \mathbb{R}_+^d$.

Note that $\sum_z y_z(t) = t$ for any fluid limit and for any t.

Denote by $\Phi = \varphi$ the family of all fluid limits φ (or, equivalently, the family of their distributions).

Lemma 1.1 *If the family* $(\tilde{X}, \tilde{Y})$ *is relatively compact, then the family* Φ *is compact (i.e. any sequence of fluid limits contains a convergent subsequence).*

The following assumption applies for the rest of this section.

Assumption 1.2 *The family of processes* $\{X^{(x,z)}, \ (x, z) \in \chi^{'} \times \mathcal{Z}\}$ *is such that*

1. *for all* $t > 0$ *and* $(x, z) \in \chi' \times \mathcal{Z}$,

$$\mathbb{E}|X^{(x,z)}(t)| < \infty$$

and moreover, for any K *and any* z,

$$\sup_{|x| \le K} \mathbb{E}|X^{(x,z)}(t)| < \infty;$$

2. *for all* $0 \le u < t$, *the family of random variables*

$$\{\rho(\tilde{X}^{(x,z)}(u), \tilde{X}^{(x,z)}(t)); \ |x| \ge 1; \ z \in \mathcal{Z}\}$$

is uniformly integrable (U.I), and for any z,

$$\limsup_{|x| \to \infty} \mathbb{P}\{ \sup_{u',t' \in [u,t]} \rho(\tilde{X}^{(x,z)}(u'), \tilde{X}^{(x,z)}(t')) > C(t-u)\} = 0, \tag{2}$$

where C *is a finite constant that does not depend on* u, t.

Theorem 1.1 [4] *Assume that for some* $\varepsilon > 0$, *there exists a finite constant* T *such that*

$$\sup_{\varphi \in \Phi} \mathbb{E}|x(T)| < 1 - \varepsilon.$$

Then, the underlying Markov process (X, Z) *is positive recurrent.*

Remark 1.2 The L_p-stability implies conditions of theorem 1.1. Indeed, take any $T > 0$ such that $\sup_\varphi \mathbb{E}|x(T)|^p < 1$ and apply the Hölder inequality.

Fix $t \ge 0$, and on the event $\{|x(t)| > 0\}$, introduce the shift transformation $\varphi \to \varphi^t = (x^t, y^t)$ as follows :

$$x^t(u) = \frac{x(t + u|x(t)|)}{|x(t)|};$$

for $r \in \mathcal{Z}$,

$$y_r^t(u) = \frac{y_r(t + u|x(t)|) - y_r(t)}{|x(t)|}.$$

Along with Markov processes with fixed initial values, we consider processes with random initial values (x, z). For such processes, one can define fluid limits as follows. Consider a sequence

$$(\tilde{X}^{(x_n,z_n)}, \tilde{Y}_r^{(x_n,z_n)}, \quad r \in \mathcal{Z}), \text{ where } |x_n| \to \infty,$$

in probability, $z_n \in \mathcal{Z}$. By assumption 1.2, it contains a subsequence

$$(\tilde{X}^{(x_{n_k},z_{n_k})}, \tilde{Y}_r^{(x_{n_k},z_{n_k}}, \ r \in \mathcal{Z})$$

that converges weakly (in the Skorohod topology) to some limit process, which is also called a fluid limit. The family of all such fluid limits (or, equivalently, of their distributions) is denoted by $\tilde{\Phi}$. Introduce the following :

Assumption 1.3 *For any* $\varphi \in \tilde{\Phi}, t \geq 0, z \in \mathcal{Z}$*, the right derivative* $v_z(t) = y_z'(t+0)$ *exists a.s. on the event* $\{|x(t) > 0|\}$.

Put $v(t) = \{v_z(t)\}_{z \in \mathcal{Z}}$ and define the set

$$\mathbf{V} = \{\{v_z\}_{z \in \mathcal{Z}} : v_z \geq 0, \forall z \text{ and } \sum_{z \in \mathcal{Z}} v_z = 1\}.$$

For any stopping time τ, put

$$v^\tau(t) = v(\tau + t), t \geq 0, \text{ if } |x(\tau + t)| > 0.$$

For a set U and a fluid limit $\varphi \in \tilde{\Phi}$, put

$$\beta = \beta_\varphi = inf\{t \geq 0 : |x(t)| = 0 \vee (x^t(0), v^t(0)) \in U\}. \tag{3}$$

Denote $\chi_1 = \{x \in \chi : |x| = 1\}$.

We are ready now to formulate and prove the main result which we will make use in **section 1.2**.

Theorem 1.2 *Let Assumption 1.3 hold. Assume that there exist* $\varepsilon > 0$ *and a measurable set* $U \subseteq \chi_1 \times \boldsymbol{V}$ *such that for each* $\varphi \in \Phi$,

1. *the stopping time* β_φ *is admissible ;*
2. *if* $(x(0), v(0)) \in U$ *a.s., then* $\mathbb{E}|x(\beta)| \leq 1 - \varepsilon$*;*
3. *the family of random variables* $\{\beta_\varphi,\ \varphi \in \Phi\}$ *is uniformly integrable.*

Then, for some $\varepsilon > 0$ *and for any* $\varphi \in \Phi$*, there exists a stopping time* τ_φ *such that*

$$\mathbb{E}|x(\tau_\varphi)| \leq 1 - \varepsilon \tag{4}$$

and

$$\lim_{K \to \infty} \sup_{\varphi \in \Phi} K\mathbb{P}\{\tau_\varphi > K\} = 0.$$

In particular, the conditions of theorem 1.1 are satisfied and therefore the underlying Markov process (X, Z) *is positive recurrent.*

Proof: By the total the total probability law, conditions of the theorem imply that

- a stopping time β_φ is admissible for any $\varphi \in \tilde{\Phi}$
- the family $\{\beta_\varphi, \varphi \in \tilde{\Phi}\}$ is uniformly integrable.

Let $\Phi_0 = \{\varphi \in \tilde{\Phi} : (x(0), v(0)) \in U\}$ and $\Phi_1 = \tilde{\Phi} \setminus \Phi_0$.

If $\varphi \in \Phi_0$, put $\tau_\varphi = \beta_\varphi$.

Otherwise, put $c = \sup_{\varphi \in \Phi_1} \mathbb{E}|x(\beta_\varphi)|$ and

$$l = min\{n \geq 0 : \ (1-\varepsilon)^n c \leq 1\} + 2.$$

For $\varphi \in \Phi_1$, on define τ_φ via the following recursive procedure.

Put $T_0 = 0$, $\varphi^{(0)} = \varphi$, $T_1 = \beta_\varphi$, $\varphi^{(1)} = (x^{(1)}, y^{(1)}) = \varphi^{T_1}$

and for $i \in \{1, \ldots, l-1\}$, if $|x^{(i)}(0)| > 0$ then put

$$T_{i+1} = T_i + |x^{(i)}(0)|\beta_{\varphi^{(i)}}, \ \varphi^{(i+1)} = \varphi^{T_{i+1}}.$$

Denote $\Gamma = min(l, min\{i : |x^{(i)}(0)| = 0\})$ and $\tau_\varphi = T_\Gamma$.

Then (4) follows from inequalities

$$\begin{aligned}
\mathbb{E}|x(T_\Gamma)| &\equiv \mathbb{E}\{|x(T_\Gamma)|\mathbb{1}_{(|x(T_\Gamma)|>0)}\} \\
&= \mathbb{E}\{|x(T_l)|\mathbb{1}_{(|x(T_l)|>0)}\} \\
&\leq \mathbb{E}\{|x(T_l)|\mathbb{1}_{(|x(T_{l-1})|>0)}\} \\
&\leq (1-\varepsilon)\mathbb{E}\{|x(T_{l-1})|\mathbb{1}_{(|x(T_{l-1})|>0)}\} \\
&\vdots \\
&\leq (1-\varepsilon)^{l-1}\mathbb{E}|x(T_1)| \\
&\leq (1-\varepsilon)^{l-1} c \\
&\leq 1-\varepsilon.
\end{aligned}$$

Let us show uniform integrability of $\{\tau_\varphi, \, \varphi \in \Phi\}$. For any $u \geq 1$,

$$\mathbb{E}\{T_\Gamma \mathbb{1}_{(T_\Gamma > u)}\} \leq \sum_{i=1}^{l} \mathbb{E}\{T_i \mathbb{1}_{(\Gamma \geq i)} \mathbb{1}_{(T_i \geq u)}\}.$$

Set $g(u) = \sup_\varphi \mathbb{E}\{\beta_\varphi \mathbb{1}_{(\beta_\varphi \geq u)}\}$. Then, $g(u) \to 0$ as $u \to \infty$.

One can see that for $i = 1$,

$$\mathbb{E}\{T_1 \mathbb{1}_{(\Gamma \geq 1)} \mathbb{1}_{(T_1 \geq u)}\} \leq g(u).$$

Further, for any $v \geq 1$ and $u \geq v(2+C)$,

$$\begin{aligned}
\mathbb{E}\{T_2 1\!\!1_{(\Gamma\geq 2)} 1\!\!1_{(T_2\geq u)}\} &\leq \mathbb{E}\{[T_1 + (1+C.T_1)\beta_{\varphi^{T_1}}] 1\!\!1_{(\Gamma\geq 2)} 1\!\!1_{(T_1+(1+CT_1)\beta_{\varphi^{T_1}}\geq u)}\} \\
&\leq \mathbb{E}\{v[1+(1+C).T_1] 1\!\!1_{(T_1 \geq \frac{u-v}{v(1+C)})}\} + \\
&+ \mathbb{E}\{\mathbb{E}\{\beta_{\varphi^{T_1}} 1\!\!1_{(\beta_{\varphi^{T_1}}\geq v)}/T_1, x(T_1)\}[1+(1+C)T_1] 1\!\!1_{(\Gamma\geq 2)}\} \\
&\leq v\mathbb{P}\{T_1 \geq \frac{u-v}{v(1+C)}\} + (1+C)v\mathbb{E}\{T_1 1\!\!1_{(T_1\geq \frac{u-v}{v(1+C)})}\} + \\
&+ g(v)\mathbb{E}\{1+(1+C)T_1\} \\
&\leq (2+C)v.g(\frac{u-v}{v(1+C)}) + g(v)[1+(1+C)g(0)]
\end{aligned}$$

uniformly in $\tilde{\Phi}$. Take $v = v(u)$ such that

$$v \to \infty \text{ and } v.g(\frac{u-v}{v(1+C)}) \to 0 \text{ as } u \to \infty.$$

The proof is completed by induction. □

1.2 Application to the study of the stability of a polling system

Consider an open polling system with two stations and two "heterogenous" servers. With each station $i = 1, 2$ an input stream of customers is associated, that has i.i.d interarrival times with common distribution function $F_i^{(0)}(t)$ and finite positive mean λ_i^{-1}. The inputs to different stations $i = 1, 2$ are mutually independent. For $i, m \in \{1, 2\}$, server m has a station i i.i.d service times with common distribution function $F_i^{(m)}(t)$ and finite positive mean $(\mu_i^{(m)})^{-1}$. Both servers follow the so-called exhaustive service policy : after completing a service, a server either starts the service of a new customer (if there is any), or leaves the station ; after a finite "walking" ("switch-over") period, the server arrives to the other station. For server m, walking times from station i_1 to station i_2 form an i.i.d sequence of non-negative random variables with finite mean $W^{(m)}(i_1, i_2)$ (either $i_1 = 1$; $i_2 = 2$, or $i_1 = 2$; $i_2 = 1$). If a server arrives to a station with empty queue, it becomes "passive" and waits there for the first customer. If during this period the other server arrives to this station, it becomes passive, too, and waits for the second customer to arrive to this station.

This system can be analysed via the fluid approximation approach. In order to avoid the surplus of technical details, we make the following

Assumption 1.4 *The distribution functions* $F_i^{(m)}, i = 1, 2; m = 0, 1, 2$ *are exponential;* $\lambda_1 = \lambda_2 = 1$ *; all the walking times are equal to zero a.s.* ($W^{(m)}(i_1, i_2) = 0, m = 1, 2$).

Consider a right-continuous time-homogeneous Markov process

$$\{X(t); Z(t)\} = \{(Q_1(t), Q_2(t)), (Z^{(1)}, Z^{(2)}(t))\},\ t \geq 0,$$

where

- $Q_i(t)$ is the queue length at station $i = 1, 2$ (including the customers being served),
- for $m = 1, 2, Z^{(m)}(t) \in \{-2, -1, 1, 2\}$ is the position of server m at time instant t; $Z^{(m)}(t) = i$ means that server m is serving ("active") at station i; $Z^{(m)}(t) = -i$ means that server m is waiting ("passive") at station i.

With necessity, we have to assume that $Q_i(t) \geq 1$ if at least one of $Z^{(m)}(t)$ equals i, and $Q_i(t) \geq 2$ if $Z^{(1)}(t) = Z^{(2)}(t) = i$.

Under Assumption 1.4, a server cannot become passive if there at least two customers in the whole system.

Here $X(t)$ take its values in

$$\chi' = \{0, 1, \cdots\} \times \{0, 1, \cdots\} \subset \chi \equiv \mathbb{R}_+^2$$

and $Z(t)$ in

$$\mathcal{Z} = \{-2, -1, 1, 2\} \times \{-2, -1, 1, 2\}.$$

Put $\mathbf{0} = ((0, 0))$. For $x^{(m)} = (x_1^{(m)}, x_2^{(m)}) \in \chi$, $m = 1, 2$, introduce the metric

$$\rho(x^{(1)}, x^{(2)}) = \sum_{i=1}^{2} |x_i^{(1)} - x_i^{(2)}|.$$

Then, $|x| = x_1 + x_2$ for $x = (x_1, x_2) \in \chi$.

The process (X, Z) is piecewise deterministic (in fact, piecewise constant) and, therefore, possesses the strong Markov property.

Put

$$C = 3 + \mu_1^{(1)} + \mu_1^{(2)} + \mu_2^{(1)} + \mu_2^{(2)}.$$

Assumption 1.2 holds for the process (X, Z). Indeed, for any $t \geq 0$, $K > 0$, and $z \in \mathcal{Z}$,

$$\sup_{|x| \leq K} \mathbb{E}|X^{(x,z)}(t)| \leq K + 2t < \infty;$$

For any $0 \leq u < t$, $z \in \mathcal{Z}$, $|x| \geq 1$,

$$\rho(\tilde{X}^{(x,z)}(u), \tilde{X}^{(x,z)}(t)) \leq_{st} \frac{\sum_{i=1}^{|x|} \pi_i}{|x|},$$

where r.v.'s $\pi_1, \pi_2, \ldots$ are i.i.d and have Poisson distribution with parameter $(C-1)(t-u)$. Therefore, the family

$$\{\rho(\tilde{X}^{(x,z)}(u), \tilde{X}^{(x,z)}(t)), z \in \mathcal{Z}, |x| \geq 1\}$$

is uniformly integrable.

Since

$$\frac{\sum_{i=1}^{|x|}\pi_i}{|x|} \longrightarrow (C-1)(t-u) \text{ a.s.} \quad \text{as } |x| \to \infty,$$

then, (2) holds.

Note that if $\mu_i^{(1)} + \mu_i^{(2)} \leq 1$ for some $i = 1, 2$, then the polling system cannot be stable. Indeed, for $i = 1, 2$ and for any $\Delta > 0$ and $K > 0$, we denote by τ the first moment after Δ when the queue length at station i becomes smaller than K. If both servers start at station i with the queue length $Q \geq K$, then τ is either infinite with positive probability (if $\mu_i^{(1)} + \mu_i^{(2)} < 1$) or finite, but with infinite mean (if $\mu_i^{(1)} + \mu_i^{(2)} = 1$).

Similarly, the polling system cannot be stable if either

$$max(\mu_1^{(1)}, \mu_2^{(2)}) \leq 1 \quad \text{or} \quad max(\mu_1^{(2)}, \mu_2^{(1)}) \leq 1.$$

Let us number the stations and the servers so that

$$\mu_1^{(1)} = min\{\mu_i^{(m)};\ i, m = 1, 2\}.$$

Then, the polling model may be stable in one of the following cases :

(A1) $\mu_1^{(1)} > 1,$

(A2.1) $\mu_1^{(1)} \leq 1,\ \mu_1^{(2)} > 1,\ \mu_2^{(1)} > 1,\ \mu_2^{(2)} > 1,$

(A3.1) $\mu_1^{(1)} \leq 1,\ \mu_1^{(2)} > 1,\ \mu_2^{(1)} \leq 1,\ \mu_2^{(2)} > 1,$

(A4.1) $\mu_1^{(1)} \leq 1,\ \mu_1^{(2)} \leq 1,\ \mu_1^{(1)} + \mu_1^{(2)} > 1,\ \mu_2^{(1)} > 1,\ \mu_2^{(2)} > 1.$

We need some additional notations. First, for $m, i = 1, 2$, let

$$p_i^{(m)} = (1 - \frac{\mu_i^{(m)}}{\mu_i^{(1)} + \mu_i^{(2)}}).max(0, 1 - \mu_i^{(m)}).$$

Then put

$$c_{11} = \frac{1}{\mu_1^{(1)} + \mu_1^{(2)} - 1}, \quad c_{22} = \frac{1}{\mu_2^{(1)} + \mu_2^{(2)} - 1}, \quad c_{12} = \frac{1 - \mu_1^{(1)}}{\mu_2^{(2)} - 1},$$

$$c_{21} = \frac{1 - \mu_2^{(1)}}{\mu_1^{2} - 1} 1\!\!1_{(\mu_1^{(2)} > 1 \geq \mu_2^{1})} + \frac{1 - \mu_1^{(2)}}{\mu_2^{1} - 1} 1\!\!1_{(\mu_2^{(1)} > 1 \geq \mu_1^{2})}.$$

Introduce the families of conditions as follows :

Condition (A2). Inequalities **(A2.1)** and

$$c_{11}(c_{22}(1-p_1^{(1)})+c_{12}p_1^{(1)})<1.$$

Condition (A3). Inequalities **(A3.1)** and

$$(1-c_{11}c_{12}p_1^{(1)})^+.(1-c_{22}c_{21}p_2^{(1)})^+ > c_{11}c_{22}(1-p_1^{(1)})(1-p_2^{(1)}),$$

where $a^+ = max(a,o)$.

Condition (A4). Inequalities **(A4.1)** and

$$c_{11}(c_{22}(1-p_1^{(1)}-p_1^{(2)})+c_{12}p_1^{(1)}+c_{21}p_1^{(2)})<1. \tag{5}$$

Theorem 1.3 [4] *Under Assumption 1.4, if one of conditions* **A1-A4** *is satisfied, then the process* (X,Z) *is positive recurrent and ergodic.*

Remark 1.3 If the service times are "server-independent" (i.e. $\mu_i^{(1)}=\mu_i^{(2)}=\mu_i,\ i=1,2$), then, the stability condition

$$\frac{1}{\mu_1}+\frac{1}{\mu_2}<2 \tag{6}$$

is well known and may be easily obtained via the criterion (1). In this case, one can check that (6) holds if and only if either **(A1)**, or **(A4)** is satisfied. Similarly, if the service times are "station-independent" (i.e. $\mu_1^{(m)}=\mu_2^{(m)}=\mu^m,\ m=1,2$), then the stability condition $\mu^{(1)}+\mu^{(2)}>2$ holds if and only if **(A1)** or **(A3)** is validated.

We complete this paper with the following theorem, we assume that the set U consists only of one point $(1,0,1,1)$ and the r.v.'s β and β_0 are defined by (3).

Theorem 1.4 *Assume that any of conditions* **(A2.1)-(A4.1)** *holds. If*

$$1\le \mathbb{E}|x_0(\beta_0)|<\infty \ and\ \mathbb{E}\log|x_0(\beta_0)|<0, \tag{7}$$

then

1. *for any fluid limit* φ

$$\gamma=\gamma_\varphi=inf\{t>0:\ |x(t)|=0\}<\infty\ a.s.; \tag{8}$$

and for any non-flashing fluid limit,

$$\mathbb{E}\gamma=\infty; \tag{9}$$

2. *with any fluid limit* $\varphi = (\boldsymbol{x}, \boldsymbol{y})$ *one can associate an infinite sequence* $\{\gamma_\varphi{}^{(l)}\}$ *of r.v.'s such that* $\gamma_\varphi^{(l)} \to \infty$ *a.s. as* $l \to \infty$, *and*

$$|\boldsymbol{x}(\gamma_\varphi^{(l)})| = 0 \quad a.s. \quad for\ any\ n; \tag{10}$$

3. *the underlying Markov process is recurrent.*

Proof: 1. Put $|\mathbf{x}_0(\beta_0)| = q_1(\beta_0) = u_0 > 0$ a.s. for any $n = 0, 1, ...$, set

$$\beta_{n+1} = inf\{t > \beta_n : \ q_2(t) = 0, \ \mathbf{z}^{(1)}(t) = \mathbf{z}^{(2)}(t) = 1\}$$

and put $\gamma_0 = \lim_{n\to\infty} \beta_n \le \infty$. It follows from Theorem 1.3 that $|\mathbf{x}_0(t)| > 0$ for any $t < \gamma_0$ and $q_1(\beta_n)$ may be represented in the form

$$q_1(\beta_n) = \prod_{j=0}^{n} u_j,$$

where $\{u_j\}$ are i.i.d.

Put $\delta_0 = \beta_0$ and, for $n \ge 1$,

$$\delta_n = \frac{\beta_n - \beta_{n-1}}{q_1(\beta_{n-1})}.$$

Then, $\{\delta_n\}$ form an i.i.d sequence, δ_n does not depend on $u_0, \cdots, u_{n-1}$ for any n, and

$$\gamma_0 = \delta_0 + \sum_{i=1}^{\infty} \delta_i \prod_{j=0}^{i-1} u_j.$$

Since all r.v's are a.s. strictly positive,

$$\mathbb{E}\gamma_0 = \mathbb{E}\delta_0 (1 + \sum_{j=1}^{\infty} (\mathbb{E}u_0)^j)$$

is infinite if $\mathbb{E}u_0 \ge 1$. On the other hand, if $\mathbb{E}\log u_0 = -c < 0$, then

$$\prod_{j=0}^{i} u_j = exp\{\sum_{j=0}^{i} \log u_j\} \to 0 \quad \text{a.s.} \quad \text{as} \quad i \to \infty.$$

Set

$$\nu = min\{i : \sum_{j=0}^{k-1} \log u_j \le -ck/2, \ \forall k \ge i\} < \infty \quad \text{a.s.}$$

Then

$$\gamma_0 \le \delta_0 + \sum_{i=1}^{\nu-1} \delta_i \prod_{j=0}^{i-1} u_j + \sum_{i=1}^{\infty} \delta_i exp\{-c(i-1)/2\} < \infty \quad \text{a.s.}$$

and

$$|\mathbf{x}_0(\gamma_0)| = \lim_{n\to\infty} |\mathbf{x}_0(\beta_n)| = \lim_{n\to\infty} \prod_{j=0}^{n} u_j = 0 \quad \text{a.s.}$$

Take now any other non-flashing fluid limit φ. Note that either $\gamma_\varphi = \beta$ or φ^β has the same distribution as φ_0. Therefore, one can represent $\gamma = \gamma_\varphi$ in the form

$$\gamma = \beta + |\mathbf{x}(\beta)|.\tilde{\gamma}_0,$$

where $\tilde{\gamma}_0$ is distributed like γ_0 and does not depend on $\mathbf{x}(\beta)$.

Note that

$$\mathbb{P}(\gamma > \beta) = \mathbb{P}(|\mathbf{x}(\beta)| > 0) > 0.$$

Therefore, under conditions (7), γ is finite a.s. and $\mathbb{E}\gamma = \infty$.

2. Consider any fluid limit $\varphi = (\mathbf{x}, \mathbf{y})$. For any $t > 0$, if $|\mathbf{x}(t)| = q > 0$, then $|\mathbf{x}(t + \gamma_{\varphi^t}.q)| = 0$ a.s., that proves (10).

3. Consider the ordinary birth-death process $U(t)$ with constant birth and death intensities λ and μ respectively, and initial size $U(0) = 1$.

Set

$$\eta = inf\{t > 0 : \ U(t) = 0\} \text{ and } H(\lambda, \mu) = \mathbb{E}(\eta 1\!\!1_{(\delta<\infty)}).$$

The following is well-known :

1. If $\lambda \neq \mu$, then $H(\lambda, \mu) < \infty$.
2. If $\lambda < \mu$, then
$$\sup_{t\geq 0} \mathbb{E}\{U(t) 1\!\!1_{(\delta>t)}\} \leq H(\lambda, \mu) < \infty.$$

The state space of the Markov process (X, Z) is countable, and any two states communicate. It is sufficient to show that there exists a constant K such that the random variable

$$\beta^{(1)}(x, z, K) = inf\{t > 0 : \ Q_1^{(x,z)}(t) + Q_2^{(x,z)}(t) \leq K\}$$

is finite a.s. for any couple $x = (n_1, n_2)$ of non-negative integers and for any $z \in \mathcal{Z}$.

It is easy to see that $\beta^{(2)}(x, z) = inf\{t > 0 : \ Q_2^{(x_n,z)}(t) = 0,$

$$Z^{(1)}(t) = Z^{(2)}(t) = 1, \ Z^{(1)}(t-0) = 2 \ \vee \ Z^{(2)}(t-0) = 2\}$$

is finite a.s. (here the symbol $\vee$ stands for the union of two events).

Set

$$\beta(x, z) = min\{\beta^{(1)}(x, z, 1), \beta^{(2)}(x, z)\} \text{ and } \beta_n = \beta((n, 0), (1, 1)).$$

Set $z=(1,1),\ x_n=(n,0)$, tend n to infinity, and denote by φ the weak limit of the process $(\tilde{X}^{(x_n,z)}, \tilde{Y}^{(x_n,z)})$.

For any constant $c>0$,

$$min(\frac{\beta_n}{n}, c) \to min(\beta, c) \text{ weakly.}$$

Here, $\beta \equiv \beta_0$ is defined in the statement of the theorem).

Therefore, $\beta_n/n \to \beta$ and

$$\frac{Q_1^{(x_n,z)}(\beta_n)}{n} \longrightarrow q_1(\beta) \quad \text{weakly.}$$

Assume we have proved uniform integrability of $(\log(Q_1^{(x_n,z)}(\beta_n)/n))^+$.

Then,

$$\lim_{n\to\infty} sup \mathbb{E} \log(\frac{max(1, Q_1^{(x_n,z)}(\beta_n) + Q_2^{(x_n,z)}(\beta_n))}{n}) \leq \mathbb{E} \log q_1(\beta)$$

(since $Q_2^{(x_n,z)}(\beta_n) \leq 1$ a.s.) and there exists K such that

$$\sup_{n\geq K} \mathbb{E} \log(\frac{max(1, Q_1^{(x_n,z)}(\beta_n) + Q_2^{(x_n,z)}(\beta_n))}{n}) \leq -\varepsilon$$

for some $\varepsilon > 0$.

Start with any initial value (x,z), put $\kappa_1 = \beta^{(1)}(x,z,K)$ and for $k=1,2,\ldots,$

$\kappa_{k+1} = inf\{t > \kappa_k :\ (Q_2^{(x_n,z)}(t) = 0,\ Z^{(1)}(t) = Z^{(2)}(t) = 1,$

$(Z^{(1)}(t-0) = 2 \vee Z^{(2)}(t-0) = 2)) \vee (Q_1^{(x_n,z)}(t) + Q_2^{(x_n,z)}(t) \leq 1)\}$.

Denote $Q(k) = Q_1^{(x,z)}(\kappa_k) + Q_2^{(x,z)}(\kappa_k)$. Then

$$\mathbb{E}(\log \frac{Q(k+1)}{Q(k)} / Q(k) > K) \leq -\varepsilon$$

and, therefore,

$\gamma(x,z) = min\{k :\ Q(k) \leq K\}$ and $\beta^{(1)}(x,z,K) \leq \kappa_{\gamma(x,z)}$ are finite a.s. Thus, the Markov process is recurrent.

We give now the proof of uniform integrability for a sequence of random variables

$\{(\log(Q_1^{(x_n,z)}(\beta_n)/n))^+,\ n \geq 1\}$.

The proof is based on similar arguments in cases **A2.1-A4.1**.

Consider case (**A3.1**), as the most complicated one.

Let $\alpha_n = inf\{t > 0 :\ Z^{(1)}(t) \neq 1 \vee Z^{(2)}(t) \neq 1\}$. Then $\alpha_n \leq \beta_n$ a.s. and

$$\mathbb{E}Q_2^{(x_n,z)}(\alpha_n) = (n-1)c_{11}.$$

Put $D = \{Q_1^{(x,z)}(\beta_n) + Q_2^{(x,z)}(\beta_n) \leq 1\}$. Denote by $\overline{D}$ the complement of D, that is,

$$\overline{D} = \{\text{both servers stop not on the time interval } [0, \beta_n]\}.$$

From the total probability law,

$$\begin{aligned}\mathbb{E}Q_1^{(x_n,z)}(\beta_n) = \mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_D\} + \mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_{B_{-1}}\}+ \\ + \mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_{\tilde{B}_{-1}}\} + \sum_{i=0}^{\infty}\mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_{\overline{D}} \mid B_i\}\mathbb{P}(B_i)+ \\ + \sum_{i=0}^{\infty}\mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_{\overline{D}} \mid \tilde{B}_i\}\mathbb{P}(\tilde{B}_i),\end{aligned}$$

where the events B_i and $\tilde{B}_i$ describe the dynamics of servers within the time interval $[0, \beta_n]$. Namely,

$B_{-1} = \{$ server 2 stays at station 1 all the time ; server 1 switches to station 2 only once and returns to station 1 at time instant $\beta_n\}$;

for $i \geq 0$,

$B_i = \{$ first, both servers switch to station 2 (in any order); second, server 2 switches to station 1 and returs back i times ; finally, both servers switch to station 1 (in any order)$\}$;

$\tilde{B}_{-1} = \{$ server 1 stays at station 1 all the time; server 2 switches to station 2 only once and return back at time instant $\beta_n\}$;

for $i \geq 0$,

$\tilde{B}_i = \{$ first, both servers switch to station 2 (in any order); second, servers 1 and 2 switche to station 1 and returs back several times (alternatively) ; after the last return of server 1 to station 2, server 2 visits station 1 i times ; finally, both servers return to station 1 (in any order)$\}$.

Then

$$\mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_D\} \leq 1;$$

and routine (but space-consuming) calculations show that following inequalities are valid :

$$\begin{aligned}\mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_{\overline{D}B_{-1}}\} &\leq H(1, \mu_1^{(2)}) < \infty; \\ \mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_{\overline{D}\tilde{B}_{-1}}\} &\leq (n-1)c_{11}c_{12} + 1; \\ \mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_{\overline{D}} \mid B_0\} &\leq ((n-1)c_{11} + c_{21})max(c_{12}, c_{22}) + c_{12}; \\ \mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_{\overline{D}} \mid B_i\} &\leq ((n-1)c_{11} + c_{21})max(c_{12}, c_{22})(c_{21}c_{22})^i + c_{12}; \\ \mathbb{E}\{Q_1^{(x_n,z)}(\beta_n)\mathbb{1}_{\overline{D}} \mid \tilde{B}_i\} &\leq H(1, \mu_2^{(2)})(c_{21}c_{22})^i + c_{12}\end{aligned}$$

for all $i \geq 0$.

Thus, $\sup_{n\geq 2} \mathbb{E} \left(\frac{Q_1^{(x_n,z)}(\beta_n)}{n} \right)^{\delta}$ is finite for any $\delta \in (0,1]$ such that

$$(c_{21}c_{22})^{\delta} \frac{\mu_2^{(2)}}{\mu_2^{(1)} + \mu_2^{(2)}} < 1.$$

Therefore, the random variables $\left(\log \left(\frac{Q_1^{(x_n,z)}(\beta_n)}{n} \right) \right)^{+}$ are uniformly integrable. □

Prospects (future works) :

1. We will consider a same polling system but with $\lambda_1 \neq \lambda_2$ and a walking times $W^{(m)}(i_1, i_2) \neq 0$.
2. We will seek the conditions of transience for this new system.

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received: Juni 23, 2006

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Rostock. Math. Kolloq. **62**, 21–39 (2007)

Subject Classification (AMS)
Primary: 47H10, 47H17,
Secondary: 54H25

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Implicit fixed point iterations

ABSTRACT. Let K be a compact convex subset of a real Hilbert space H; $T : K \to K$ a continuous hemicontractive map. Let $\{\alpha_n\}$ be a real sequence in $[0, 1]$ satisfying appropriate conditions, then for arbitrary $x_0 \in K$ and $\{v_n\}$ in K, the sequence $\{x_n\}$ defined iteratively by $x_n = \alpha_n x_{n-1} + (1 - \alpha_n)Tv_n$, $n \geq 1$ converges strongly to a fixed point of T.

We also establish a strong convergence of an implicit iteration process to a common fixed point for a finite family of $\psi-$uniformly pseudocontractive and $\psi-$uniformly accretive mappings in real Banach spaces.

The results presented in this paper extend and improve the corresponding results of Refs. [4, 9, 19, 20, 22, 25, 44].

KEY WORDS. Implicit iteration process, Mann iteration, $\psi-$uniformly pseudocontractive and $\psi-$uniformly accretive mappings, Common fixed point, Banach space, Hilbert Space

1 Fundamentals

We assume that E is a real Banach space and K be a nonempty convex subset of E. Let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2 \text{ and } ||f^*|| = ||x||\},$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We shall denote the single-valued duality map by j.

Let $\Psi := \{\psi \mid \psi : [0, \infty) \to [0, \infty)$ is a strictly increasing mapping such that $\psi(0) = 0\}$.

Definition 1 *A mapping $T : K \to K$ is called $\psi-$uniformly pseudocontractive if there exist mapping $\psi \in \Psi$ and $j(x - y) \in J(x - y)$ such that*

$$\langle Tx - Ty, j(x - y) \rangle \leq ||x - y||^2 - \psi(||x - y||), \ \forall x, \ y \in K. \tag{1.1}$$

Definition 2 *A mapping $S : D(S) \subset E \to E$ is called $\psi-$uniformly accretive if there exist mapping $\psi \in \Psi$ and $j(x-y) \in J(x-y)$ such that*

$$\langle Sx - Sy, j(x-y)\rangle \geq \psi(||x-y||),\ \forall x,\ y \in E. \tag{1.2}$$

Remark 1 a) Taking $\psi(a) := \psi(a)a,\ \forall a \in [0,\infty),\ (\psi \in \Psi)$, we get the usual definitions of $\psi-$ pseudocontractive and $\psi-$ accretive mappings.

b) Taking $\psi(a) := \gamma a^2; \gamma \in (0,1),\ \forall a \in [0,\infty),\ (\psi \in \Psi)$, we get the usual definitions of strongly pseudocontractive and strongly accretive mappings.

c) T is $\psi-$uniformly pseudocontractive iff $S = I - T$ is $\psi-$uniformly accretive.

d) It is known that T is strongly pseudocontractive if and only if $(I-T)$ is strongly accretive.

Let H be a Hilbert space.

Definition 3 *A mapping $T : H \to H$ is said to be* pseudocontractive *(see e.g., [1, 2]) if*

$$||Tx - Ty||^2 \leq ||x-y||^2 + ||(I-T)x - (I-T)y||^2,\ \forall x, y \in H \tag{1.3}$$

and is said to be strongly pseudocontractive *if there exists $k \in (0,1)$ such that*

$$||Tx - Ty||^2 \leq ||x-y||^2 + k||(I-T)x - (I-T)y||^2,\ \forall x, y \in H. \tag{1.4}$$

Definition 4 *Let $F(T) := \{x \in H : Tx = x\}$ and let K be a nonempty subset of H. A map $T : K \to K$ is called* hemicontractive *if $F(T) \neq \emptyset$ and*

$$||Tx - x^*||^2 \leq ||x - x^*||^2 + ||x - Tx||^2\ \ \forall\ x \in H,\ x^* \in F(T). \tag{1.5}$$

Remark 2 It is easy to see that the class of pseudocontractive maps with fixed points is a subclass of the class of hemicontractions. The following example, due to Rhoades [35], shows that the inclusion is proper. For $x \in [0,1]$, define $T : [0,1] \to [0,1]$ by $Tx = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}$. It is shown in [35] that T is not Lipschitz and so cannot be nonexpansive. A straightforward computation (see e.g., [38]) shows that T is pseudocontractive. For the importance of fixed points of pseudocontractions the reader may consult [2].

We shall make use of the following results.

Lemma 1 [40] *Suppose that $\{\rho_n\}, \{\sigma_n\}$ are two sequences of nonnegative numbers such that for some real number $N_0 \geq 1$,*

$$\rho_{n+1} \leq \rho_n + \sigma_n\ \forall n \geq N_0.$$

(a) *If* $\sum \sigma_n < \infty$, *then,* $\lim \rho_n$ *exists.*

(b) *If* $\sum \sigma_n < \infty$ *and* $\{\rho_n\}$ *has a subsequence converging to zero, then* $\lim \rho_n = 0$.

Lemma 2 [20] *For all* $x,\, y \in H$ *and* $\lambda \in [0,1]$, *the following well-known identity holds:*

$$||(1-\lambda)x + \lambda y||^2 = (1-\lambda)||x||^2 + \lambda||y||^2 - \lambda(1-\lambda)||x-y||^2.$$

Lemma 3 [42] *Let* $J : E \to 2^E$ *be the normalized duality mapping. Then for any* $x, y \in E$, *we have*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

Lemma 4 [23] *Let* $\{\theta_n\}$ *be a sequence of nonnegative real numbers,* $\{\lambda_n\}$ *be a real sequence satisfying*

$$0 \leq \lambda_n \leq 1, \ \sum_{n=0}^{\infty} \lambda_n = \infty$$

and let $\psi \in \Psi$. *If there exists a positive integer* n_0 *such that*

$$\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \psi(\theta_{n+1}) + \sigma_n,$$

for all $n \geq n_0$, *with* $\sigma_n \geq 0$, $\forall n \in \mathbb{N}$, *and* $\sigma_n = 0(\lambda_n)$, *then* $\lim_{n\to\infty} \theta_n = 0$.

2 Implicit Mann iteration process in Hilbert spaces

In the last ten years or so, numerous papers have been published on the iterative approximation of fixed points of Lipschitz *strongly* pseudocontractive (and correspondingly Lipschitz *strongly* accretive) maps using the *Mann iteration process* (see e.g., [22]). Results which had been known only in *Hilbert spaces* and only for *Lipschitz maps* have been extended to more general Banach spaces (see e.g., [5–16, 21, 30–38, 40, 41, 43, 45] and the references cited therein) and to more general classes of maps (see e.g., [6–16, 19, 21, 27–34, 36–38, 40, 41, 43, 45] and the references cited therein). This success, however, has not carried over to arbitrary *Lipschitz pseudocontraction* T even when the domain of the operator T is a *a compact convex subset of a Hilbert space*. In fact, it is still an open question whether or not the Mann iteration process converges under this setting. In 1974, Ishikawa introduced an iteration process which, in some sense, is more general than that of Mann and which converges, under this setting, to a fixed point of T. He proved the following theorem.

Theorem 1 *If* K *is a compact convex subset of a Hilbert space* H, $T : K \mapsto K$ *is a Lipschitzian pseudocontractive map and* x_0 *is any point in* K, *then the sequence* $\{x_n\}$

converges strongly to a fixed point of T, where x_n is defined iteratively for each positive integer $n \geq 0$ by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{2.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying the conditions

$$(i)\ 0 \leq \alpha_n \leq \beta_n < 1;\ \ (ii)\ \lim_{n\to\infty} \beta_n = 0;\ \ (iii)\ \sum_{n\geq 0} \alpha_n\beta_n = \infty.$$

Since its publication in 1974, Theorem 1, as far as we know, has never been extended to more general Banach spaces. In [27], Qihou extended the theorem to the slightly more general class of Lipschitz *hemicontractions* and in [28] he proved, under the setting of Theorem 1, that the convergence of the recursion formula (2.1) to a fixed point of T when T is a continuous hemicontractive map, *under the additional hypothesis that the number of fixed points of T is finite*. The iteration process (2.1) is generally referred to as the *Ishikawa iteration process* in light of [20]. Another iteration process which has been studied extensively in connection with fixed points of pseudocontractive maps is the following:

For K a convex subset of a real normed space H, and $T : K \to K$, the sequence $\{x_n\}$ is defined iteratively by $x_1 \in K$,

$$x_{n+1} = (1-c_n)x_n + c_n T x_n,\ \ n \geq 1, \tag{2.2}$$

where $\{c_n\}$ is a real sequence satisfying the following conditions:

$$(iv)\ 0 \leq c_n < 1; (v)\ \lim_{n\to\infty} c_n = 0; (vi)\ \sum_{n=1}^{\infty} c_n = \infty.$$

The iteration process (2.2) is generally referred to as the *Mann iteration process* in light of [22].

In 1995, Liu [21] introduced what he called *Ishikawa and Mann iteration processes with errors* as follows:

(1-a) For K a nonempty subset of H and $T : K \to E$, the sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n + u_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n + v_n,\ \ n \geq 1, \end{aligned} \tag{2.3}$$

where, $\{\alpha_n\}, \{\beta_n\}$ are sequences in [0,1] satisfying appropriate conditions and $\sum ||u_n|| < \infty$, $\sum ||v_n|| < \infty$ is called the *Ishikawa Iteration process with errors.*

(1-b) With K, H and T as in part (1-a), the sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T x_n + u_n,\ n \geq 1, \end{aligned} \tag{2.4}$$

where $\{\alpha_n\}$ is a sequence in [0,1] satisfying appropriate conditions and $\sum ||u_n|| < \infty$, is called the *Mann iteration process with errors.*

While it is known that consideration of error terms in iterative processes is an important part of the theory, it is also clear that the iteration processes with errors introduced by Liu in (1-a) and (1-b) are unsatisfactory. The occurrence of errors is random so that the conditions imposed on the error terms in (1-a) and (1-b) which imply, in particular, that they tend to zero as n tends to infinity are, therefore, unreasonable. In 1997, Y. Xu [43] introduced the following more satisfactory definitions.

(1-c) Let K be a nonempty convex subset of H and $T : K \to K$ a mapping. For any given $x_1 \in K$, the sequence $\{x_n\}$ defined iteratively by

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n u_n, \\ y_n &= a_n' x_n + b_n' T x_n + c_n' v_n,\ n \geq 1, \end{aligned} \tag{2.5}$$

where $\{u_n\}, \{v_n\}$ are bounded sequences in K and $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n'\}, \{b_n'\}$ and $\{c_n'\}$ are sequences in $[0,1]$ such that $a_n + b_n + c_n = a_n' + b_n' + c_n' = 1\ \forall\ n \geq 1$ is called the *Ishikawa iteration sequence with errors in the sense of Xu.*

(1-d) If, with the same notations and definitions as in (1-c), $b_n' = c_n' = 0$, for all integers $n \geq 1$, then the sequence $\{x_n\}$ now defined by

$$\begin{aligned} x_1 &\in K \\ x_{n+1} &= a_n x_n + b_n T x_n + c_n u_n,\ n \geq 1, \end{aligned} \tag{2.6}$$

is called the *Mann iteration sequence with errors in the sense of Xu.* We remark that if K is bounded (as is generally the case), the error terms u_n, v_n are *arbitrary* in K.

In [9], Chidume and Chika Moore proved the following theorem.

Theorem 2 *Let K be a compact convex subset of a real Hilbert space H; $T : K \to K$ a continuous hemicontractive map. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n'\}, \{b_n'\}$ and $\{c_n'\}$ be real sequences in $[0,1]$ satisfying the following conditions:*

(vii) $a_n + b_n + c_n = 1 = a_n' + b_n' + c_n'\ \forall\ n \geq 1$;

(viii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_n' = 0$;

(ix) $\sum c_n < \infty; \sum c_n^{'} < \infty$;

(x) $\sum \alpha_n \beta_n = \infty; \sum \alpha_n \beta_n \delta_n < \infty, where\ \delta_n := ||Tx_n - Ty_n||^2$;

(xi) $0 \leq \alpha_n \leq \beta_n < 1 \ \forall \ n \geq 1, where\ \alpha_n := b_n + c_n; \beta_n := b_n^{'} + c_n^{'}$.

For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n u_n, \\ y_n &= a_n^{'} x_n + b_n^{'} T x_n + c_n^{'} v_n, n \geq 1, \end{aligned}$$

where $\{u_n\}, \{v_n\}$ are arbitrary sequences in K. Then, $\{x_n\}$ converges strongly to a fixed point of T.

They also gave the following remark in [9].

Remark 3 d) In connection with the iterative approximation of fixed points of pseudocontractions, the following question is still open. Does the Mann iteration process always converge for continuous pseudocontractions, or for even Lipschitz pseudocontractions?

e) Let H be a Banach space and K be a nonempty compact convex subset of H. Let $T : K \to K$ be a Lipschitz pseudocontractive map. Under this setting, even for H, as a Hilbert space, the answer to the above question is not known. There is, however, an example [19] of a *discontinuous* pseudocontractive map T with a unique fixed point for which the Mann iteration process does not always converge to the fixed point of T. Let H be the complex plane and $K := \{z \in H : |z| \leq 1\}$. Define $T : K \to K$ by

$$T(re^{i\theta}) = \begin{cases} 2re^{i(\theta + \frac{\pi}{3})}, & \text{for } 0 \leq r \leq \frac{1}{2}, \\ e^{i(\theta + \frac{2\pi}{3})}, & \text{for } \frac{1}{2} < r \leq 1. \end{cases}$$

Then, zero is the only fixed point of T. It is shown in [15] that T is pseudocontractive and that with $c_n = \frac{1}{n+1}$, the sequence $\{z_n\}$ defined by $z_{n+1} = (1 - c_n) z_n + c_n T z_n$, $z_0 \in K$, $n \geq 1$, does not converge to zero. Since the T in this example is not continuous, the above question remains open.

In [10], Chidume and Mutangadura, provide an example of a Lipschitz pseudocontractive map with a unique fixed point for which the Mann iteration sequence failed to converge and they stated there ”This resolves a long standing open problem”.

We introduce the following Mann type implicit iteration process associated with continuous hemicontractive mappings to have a strong convergence in the setting of Hilbert spaces.

Let K be a closed convex subset of a real normed space H and $T : K \to K$ be a mapping. For a sequence $\{v_n\}$ in K, define $\{x_n\}$ in the following way:

$$\begin{aligned} x_0 &\in K, \\ x_n &= \alpha_n x_{n-1} + (1-\alpha_n)Tv_n, \end{aligned} \tag{2.7}$$

where $\{\alpha_n\}$ be a real sequence in $[0,1]$ satisfying some appropriate conditions.

Now we prove our main results.

Theorem 3 *Let K be a compact convex subset of a real Hilbert space H; $T : K \to K$ a continuous hemicontractive map. Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ satisfying $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. For arbitrary $x_0 \in K$ and $\{v_n\}$ in K, define the sequence $\{x_n\}$ by (2.7) satisfying $\sum_{n\geq 1} \|v_n - x_n\| < \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof: The existence of a fixed point of T follows from Schauders fixed point theorem. Let $x^* \in K$ be a fixed point of T and $M = dim(K)$. Using the fact that T is hemicontractive we obtain

$$\|Tv_n - x^*\|^2 \leq \|v_n - x^*\|^2 + \|v_n - Tv_n\|^2. \tag{2.8}$$

With the help of (2.7), Lemma 2 and (2.8), we obtain the following estimates:

$$\begin{aligned} \|x_n - x^*\|^2 &= \|\alpha_n x_{n-1} + (1-\alpha_n)Tv_n - x^*\|^2 \\ &= \|\alpha_n(x_{n-1} - x^*) + (1-\alpha_n)(Tv_n - x^*)\|^2 \\ &= \alpha_n \|x_{n-1} - x^*\|^2 + (1-\alpha_n)\|Tv_n - x^*\|^2 \\ &\quad -\alpha_n(1-\alpha_n)\|x_{n-1} - Tv_n\|^2. \end{aligned} \tag{2.9}$$

Substituting (2.8) in (2.9), we get

$$\begin{aligned} \|x_n - x^*\|^2 \leq\ & \alpha_n \|x_{n-1} - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2 \\ & +(1-\alpha_n)\|v_n - Tv_n\|^2 - \alpha_n(1-\alpha_n)\|x_{n-1} - Tv_n\|^2. \end{aligned} \tag{2.10}$$

Also

$$\begin{aligned} \|v_n - x^*\|^2 &\leq \|v_n - x_n\|^2 + \|x_n - x^*\|^2 \\ &\quad +2M\|x_n - x^*\|\|v_n - x_n\| \\ &\leq \|v_n - x_n\|^2 + \|x_n - x^*\|^2 \\ &\quad +2M\|v_n - x_n\|, \end{aligned} \tag{2.11}$$

$$\begin{aligned} \|v_n - Tv_n\|^2 &\leq \|v_n - x_n\|^2 + \|x_n - Tv_n\|^2 \\ &\quad +2M\|x_n - Tv_n\|\|v_n - x_n\| \\ &\leq \|v_n - x_n\|^2 + \|x_n - Tv_n\|^2 \\ &\quad +2M\|v_n - x_n\|, \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \|x_n - Tv_n\|^2 &= \|\alpha_n x_{n-1} + (1-\alpha_n)Tv_n - Tv_n\|^2 \\ &= \alpha_n^2 \|x_{n-1} - Tv_n\|^2 . \end{aligned} \quad (2.13)$$

Substituting (2.11-2.13) in (2.10), we get

$$\begin{aligned} \|x_n - x^*\|^2 \leq\ & \alpha_n \|x_{n-1} - x^*\|^2 + (1-\alpha_n)(\|v_n - x_n\|^2 \\ & + \|x_n - x^*\|^2 + 2M \|v_n - x_n\|) \\ & +(1-\alpha_n)(\|v_n - x_n\|^2 + \alpha_n^2 \|x_{n-1} - Tv_n\|^2 + 2M \|v_n - x_n\|) \\ & -\alpha_n(1-\alpha_n) \|x_{n-1} - Tv_n\|^2 , \end{aligned}$$

implies

$$\begin{aligned} \|x_n - x^*\|^2 \leq\ & \|x_{n-1} - x^*\|^2 + 2\frac{1-\alpha_n}{\alpha_n} \|v_n - x_n\|^2 + 4M\frac{1-\alpha_n}{\alpha_n} \|v_n - x_n\| \\ & -(1-\alpha_n)^2 \|x_{n-1} - Tv_n\|^2 , \end{aligned}$$

and from the condition $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$, we conclude that the inequality

$$\|x_n - x^*\|^2 \leq \|x_{n-1} - x^*\|^2 - \delta^2 \|x_{n-1} - Tv_n\|^2 + \delta_n, \quad (2.14)$$

holds for all fixed points x^* of T provided

$$\delta_n = 2\frac{1-\delta}{\delta} \|v_n - x_n\|^2 + 4M\frac{1-\delta}{\delta} \|v_n - x_n\| .$$

Moreover

$$\delta^2 \|x_{n-1} - Tv_n\|^2 \leq \|x_{n-1} - x^*\|^2 - \|x_n - x^*\|^2 + \delta_n,$$

and thus

$$\begin{aligned} \delta^2 \sum_{j=1}^{\infty} \|x_{j-1} - Tv_j\|^2 &\leq \sum_{j=1}^{\infty} (\|x_{j-1} - x^*\|^2 - \|x_j - x^*\|^2) + \sum_{j=1}^{\infty} \delta_j \\ &= \|x_0 - x^*\|^2 + \sum_{j=1}^{\infty} \delta_j . \end{aligned}$$

Hence due to the condition $\sum_{n\geq 1} \|v_n - x_n\| < \infty$, we obtain

$$\sum_{j=1}^{\infty} \|x_{j-1} - Tv_j\|^2 < \infty. \quad (2.15)$$

It implies that

$$\lim_{n\to\infty} \|x_{n-1} - Tv_n\| = 0.$$

From (2.13) it further implies that

$$\lim_{n\to\infty} \|x_n - Tv_n\| = 0.$$

Also the condition $\sum_{n\geq 1} \|v_n - x_n\| < \infty$ implies $\lim\limits_{n\to\infty} \|v_n - x_n\| = 0$ and the continuity of T further implies that $\lim\limits_{n\to\infty} \|Tv_n - Tx_n\| = 0$. Now from

$$\|x_n - Tx_n\| \leq \|x_n - Tv_n\| + \|Tv_n - Tx_n\|,$$

implies that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

By compactness of K this immediately implies that there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to a fixed point of T, say y^*. Since (2.14) holds for all fixed points of T we have

$$\|x_n - y^*\|^2 \leq \|x_{n-1} - y^*\|^2 - \delta^2 \|x_{n-1} - Tv_n\|^2 + \delta_n,$$

and in view of (2.15) and Lemma 1 we conclude that $\|x_n - y^*\| \to 0$ as $n \to \infty$, i.e., $x_n \to y^*$ as $n \to \infty$. The proof is complete. □

Corollary 1 *Let H, K, T, be as in Theorem 3 and $\{\alpha_n\}$ be a real sequence in $[0,1]$ satisfying $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. Let $P_K : H \to K$ be the projection operator of H onto K. For arbitrary $x_0 \in K$ and $\{v_n\}$ in K, define the sequence $\{x_n\}$ by*

$$x_n = P_K(\alpha_n x_{n-1} + (1-\alpha_n)Tv_n),\ n \geq 1,$$

satisfying $\sum_{n\geq 1} \|v_n - x_n\| < \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof: The operator P_K is nonexpansive (see e.g., [1]). K is a Chebyshev subset of H so that, P_K is a single-valued map. Hence, we have the following estimate:

$$\begin{aligned}
\|x_n - x^*\|^2 &= \|P_K(\alpha_n x_{n-1} + (1-\alpha_n)Tv_n) - P_K x^*\|^2 \\
&\leq \|\alpha_n x_{n-1} + (1-\alpha_n)Tv_n - x^*\|^2 \\
&= \|\alpha_n(x_{n-1} - x^*) + (1-\alpha_n)(Tv_n - x^*)\|^2 \\
&\leq \alpha_n \|x_{n-1} - x^*\|^2 + (1-\alpha_n)(\|v_n - x_n\|^2 \\
&\quad + \|x_n - x^*\|^2 + 2M\|v_n - x_n\|) \\
&\quad +(1-\alpha_n)(\|v_n - x_n\|^2 + \alpha_n^2 \|x_{n-1} - Tv_n\|^2 + 2M\|v_n - x_n\|) \\
&\quad -\alpha_n(1-\alpha_n)\|x_{n-1} - Tv_n\|^2,
\end{aligned}$$

implies

$$\begin{aligned}
\|x_n - x^*\|^2 &\leq \|x_{n-1} - x^*\|^2 + 2\frac{1-\alpha_n}{\alpha_n}\|v_n - x_n\|^2 + 4M\frac{1-\alpha_n}{\alpha_n}\|v_n - x_n\| \\
&\quad -(1-\alpha_n)^2\|x_{n-1} - Tv_n\|^2.
\end{aligned}$$

The set $K \cup T(K)$ is compact and so the sequence $\{\|x_n - Tx_n\|\}$ is bounded. The rest of the argument follows exactly as in the proof of Theorem 3 and the proof is complete. □

3 Multi-step iterations in Hilbert spaces

Let K be a nonempty closed convex subset of a real normed space H and T_1, T_2, ..., T_p : $K \to K$ $(p \geq 2)$ be a family of selfmappings.

Algorithm 1 *For a given $x_0 \in K$, compute the sequence $\{x_n\}$ by the implicit iteration process of arbitrary fixed order $p \geq 2$,*

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)T_1 y_n^1, \\ y_n^i &= \beta_n^i x_{n-1} + (1-\beta_n^i)T_{i+1} y_n^{i+1};\ i = 1,\ 2,\ ...,\ p-2, \\ y_n^{p-1} &= \beta_n^{p-1} x_{n-1} + (1-\beta_n^{p-1})T_p x_n,\ n \geq 1, \end{aligned} \tag{3.1}$$

which is called the multi-step implicit iteration process, where $\{\alpha_n\}$, $\{\beta_n^i\} \subset [0,1]$, $i = 1$, 2, ..., $p-1$.

For $p = 3$, we obtain the following three-step implicit iteration process:

Algorithm 2 *For a given $x_0 \in K$, compute the sequence $\{x_n\}$ by the iteration process*

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)T_1 y_n^1,\ , \\ y_n^1 &= \beta_n^1 x_{n-1} + (1-\beta_n^1)T_2 y_n^2, \\ y_n^2 &= \beta_n^2 x_{n-1} + (1-\beta_n^2)T_3 x_n,\ n \geq 1, \end{aligned} \tag{3.2}$$

where $\{\alpha_n\}$, $\{\beta_n^1\}$ and $\{\beta_n^2\}$ are three real sequences in $[0,1]$ satisfying some certain conditions.

For $p = 2$, we obtain the following two-step implicit iteration process:

Algorithm 3 *For a given $x_0 \in K$, compute the sequence $\{x_n\}$ by the iteration process*

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)T_1 y_n^1, \\ y_n^1 &= \beta_n^1 x_{n-1} + (1-\beta_n^1)T_2 x_n, \quad n \geq 1, \end{aligned} \tag{3.3}$$

where $\{\alpha_n\}$ and $\{\beta_n^1\}$ are two real sequences in $[0,1]$ satisfying some certain conditions.

If $T_1 = T$, $T_2 = I$, $\beta_n^1 = 0$ in (3.3), we obtain the implicit Mann iteration process:

Algorithm 4 *For any given $x_0 \in K$, compute the sequence $\{x_n\}$ by the iteration process*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)Tx_n,\ n \geq 1, \tag{3.4}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ satisfying some certain conditions.

Theorem 4 *Let K be a compact convex subset of a real Hilbert space H and $T_1, T_2, \ldots, T_p$ $(p \geq 2)$ be selfmappings of K. Let T_1 be a continuous hemicontractive map. Let $\{\alpha_n\}$, $\{\beta_n^i\} \subset [0,1]$, $i = 1, 2, \ldots, p-1$ be real sequences in $[0,1]$ satisfying $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$ and $\sum_{n\geq 1}(1-\beta_n^1) < \infty$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by (3.1). Then $\{x_n\}$ converges strongly to the common fixed point of $\bigcap_{i=1}^{p} F(T_i) \neq \phi$.*

Proof: By applying Theorem 3 under assumption that T_1 is continuous hemicontractive, we obtain Theorem 4 which proves strong convergence of the iteration process defined by (3.1). Consider by taking $T_1 = T$ and $v_n = y_n^1$,

$$
\begin{aligned}
\|v_n - x_n\| &= \left\|y_n^1 - x_n\right\| \\
&= \left\|\beta_n^1 x_{n-1} + (1-\beta_n^1) T_2 y_n^2 - x_n\right\| \\
&= \left\|\beta_n^1 (x_{n-1} - x_n) + (1-\beta_n^1)\left(T_2 y_n^2 - x_n\right)\right\| \\
&\leq \beta_n^1 \|x_{n-1} - x_n\| + (1-\beta_n^1)\left\|T_2 y_n^2 - x_n\right\| \\
&\leq \beta_n^1 \|x_{n-1} - x_n\| + M(1-\beta_n^1), \qquad (3.5)
\end{aligned}
$$

and

$$
\begin{aligned}
\|x_{n-1} - x_n\| &= \|x_{n-1} - \alpha_n x_{n-1} - (1-\alpha_n) T v_n\| \\
&= (1-\alpha_n)\|x_{n-1} - T v_n\|. \qquad (3.6)
\end{aligned}
$$

From (3.5) and (3.6), we have

$$
\begin{aligned}
\|v_n - x_n\| &\leq \beta_n^1 (1-\alpha_n)\|x_{n-1} - T v_n\| + M(1-\beta_n^1) \\
&\leq \beta_n^1 (1-\delta)\|x_{n-1} - T v_n\| + M(1-\beta_n^1).
\end{aligned}
$$

Now from (2.15) and the condition $\sum_{n\geq 1}(1-\beta_n^1) < \infty$, it can be easily seen that $\sum_{n\geq 1}\|v_n - x_n\| < \infty$. □

Corollary 2 *Let K be a compact convex subset of a real Hilbert space H; $T : K \to K$ a hemicontractive map. Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ satisfying $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by (3.4). Then $\{x_n\}$ converges strongly to a fixed point of T.*

4 Implicit iteration process for a finite family of ψ-uniformly pseudocontractive mappings

Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_i : i \in I\}$ be N self-mappings of K.

In 2001, Xu and Ori [44] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in I\}$ (here $I = \{1, 2, ..., N\}$), with $\{\alpha_n\}$ a real sequence in $(0, 1)$, and an initial point $x_0 \in K$:

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1 x_1, \\ x_2 &= \alpha_2 x_1 + (1-\alpha_2)T_2 x_2, \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N, \\ x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1 x_{N+1}, \\ &\vdots \end{aligned}$$

which can written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \ \forall n \geq 1, \tag{4.1}$$

where $T_n = T_{n \pmod N}$ (here the $\pmod N$ function takes values in I). Xu and Ori proved the weak convergence of this process to a common fixed point of the finite family defined in a Hilbert space. They further remarked that it is yet unclear what assumptions on the mappings and/or the parameters $\{\alpha_n\}$ are sufficient to guarantee the strong convergence of the sequence $\{x_n\}$.

In [24], Oslilike proved the following theorem.

Theorem 5 *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_i : i \in I\}$ be N strictly pseudocontractive self-mappings of K with $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence satisfying the conditions:*

(i) $0 < \alpha_n < 1$,

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$,

(iii) $\sum_{n=1}^{\infty}(1-\alpha_n)^2 < \infty$.

From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (4.1). Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$ if and only if $\lim_{n\to\infty} d(x_n, F) = 0$.

Definition 5 [24] *A normed space E is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$.*

In [4], Chen et al proved the following theorem.

Theorem 6 *Let K be a nonempty closed convex subset of a q-uniformly smooth and p-uniformly convex Banach space E that has the Opial property. Let s be any element in $(0,1)$ and let $\{T_i\}_{i=1}^{N}$be a finite family of strictly pseudocontractive self-maps of K such that $\{T_i\}_{i=1}^{N}$, have at least one common fixed point. For any point x_0 in K and any sequence $\{\alpha_n\}_{n=1}^{\infty}$ in $(0,s)$, define the sequence $\{x_n\}$ by the implicit iteration process (4.1). Then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^{N}$.*

The purpose of this section is to study the strong convergence of the implicit iteration process (4.1) to a common fixed point for a finite family of $\psi-$uniformly pseudocontractive and $\psi-$uniformly accretive mappings in real Banach spaces.

Theorem 7 *Let $\{T_1, T_2, \dots, T_N\} : K \to K$ be N, $\psi-$uniformly pseudocontractive mappings with $F = \bigcap\limits_{i=1}^{N} F(T_i) \neq \phi$. From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (4.1) satisfying $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ and $\lim\limits_{n\to\infty}(1-\alpha_n) = 0$. If the sequence $\{T_n x_n\}$ is bounded, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, ..., T_N\}$.*

Proof: Since each T_i is $\psi-$uniformly pseudocontractive, we have from (1.1)

$$\langle T_i x - T_i y, j(x-y)\rangle \leq ||x-y||^2 - \psi(||x-y||), \quad i = 1, 2, \dots, N\,. \tag{4.2}$$

We know that the mappings $\{T_1, T_2, \dots, T_N\}$ have a common fixed point in K, say w, then the fixed point set $F = \bigcap\limits_{i=1}^{N} F(T_i) \neq \phi$ is nonempty. We will show that w is the unique fixed point of F. Suppose there exists $q \in F(T_1)$ such that $w \neq q$ i.e., $\|w-q\| > 0$. Then

$$\psi(\|w-q\|) > 0. \tag{AR}$$

Since ψ is strictly increasing with $\psi(0) = 0$. Then, from the definition of $\psi-$uniformly pseudocontractive mapping,

$$\begin{aligned} ||w-q||^2 &= \langle w-q, j(w-q)\rangle = \langle T_1 w - T_1 q, j(w-q)\rangle \\ &\leq ||w-q||^2 - \psi(||w-q||), \end{aligned}$$

implies

$$\psi(||w-q||) \leq 0,$$

contracditing (AR), which implies the uniqueness. Hence $F(T_1) = \{w\}$. Similarly we can prove that $F(T_i) = \{w\}$; $i = 2, 3, \dots, N$. Thus $F = \{w\}$.

Since the sequence $\{T_n x_n\}$ is bounded, we set

$$M_1 = ||x_0 - w|| + \sup_{n\geq 1}||T_n x_n - w||.$$

Obviously $M_1 < \infty$.

It is clear that $||x_0 - w|| \leq M_1$. Let $||x_{n-1} - w|| \leq M_1$. Next we will prove that $||x_n - w|| \leq M_1$.

Consider

$$\begin{aligned} ||x_n - w|| &= ||\alpha_n x_{n-1} + (1-\alpha_n)T_n x_n - w|| \\ &= ||\alpha_n(x_{n-1} - w) + (1-\alpha_n)(T_n x_n - w)|| \\ &\leq \alpha_n||x_{n-1} - w|| + (1-\alpha_n)||T_n x_n - w|| \\ &\leq \alpha_n M_1 + (1-\alpha_n)M_1 \\ &= M_1. \end{aligned}$$

So, from the above discussion, we can conclude that the sequence $\{x_n - w\}$ is bounded. Let $M_2 = \sup_{n\geq 1}||x_n - w||$.

Denote $M = M_1 + M_2$. Obviously $M < \infty$.

The real function $f : [0, \infty) \to [0, \infty)$, defined by $f(t) = t^2$ is increasing and convex. For all $\lambda \in [0, 1]$ and $t_1, t_2 > 0$ we have

$$((1-\lambda)t_1 + \lambda t_2)^2 \leq (1-\lambda)t_1^2 + \lambda t_2^2. \tag{4.3}$$

Consider

$$\begin{aligned} ||x_n - w||^2 &= ||\alpha_n x_{n-1} + (1-\alpha_n)T_n x_n - w||^2 \\ &= ||\alpha_n(x_{n-1} - w) + (1-\alpha_n)(T_n x_n - w)||^2 \\ &\leq [\alpha_n \|x_{n-1} - w\| + (1-\alpha_n)\|T_n x_n - w\|]^2 \\ &\leq \alpha_n \|x_{n-1} - w\|^2 + (1-\alpha_n)\|T_n x_n - w\|^2 \\ &\leq \|x_{n-1} - w\|^2 + M^2(1-\alpha_n). \end{aligned} \tag{4.4}$$

From Lemma 3 and (4.1), we have

$$\begin{aligned} \|x_n - w\|^2 &= \|\alpha_n x_{n-1} + (1-\alpha_n) T_n x_n - w\|^2 \\ &= \|\alpha_n(x_{n-1} - w) + (1-\alpha_n)(T_n x_n - w)\|^2 \\ &\leq {\alpha_n}^2||x_{n-1} - w||^2 + 2(1-\alpha_n)\langle T_n x_n - w, j(x_n - w)\rangle \\ &\leq {\alpha_n}^2||x_{n-1} - w||^2 + 2(1-\alpha_n)\|x_n - w\|^2 \\ &\quad -2(1-\alpha_n)\psi(\|x_n - w\|), \end{aligned} \tag{4.5}$$

Substituting (4.4) in (4.5), we get

$$\begin{aligned}
\|x_n - w\|^2 &\leq [\alpha_n^2 + 2(1-\alpha_n)]||x_{n-1} - w||^2 - 2(1-\alpha_n)\psi(\|x_n - w\|) \\
&\quad +2M^2(1-\alpha_n)^2 \\
&= \left[1 + (1-\alpha_n)^2\right] ||x_{n-1} - w||^2 - 2(1-\alpha_n)\psi(\|x_n - w\|) \\
&\quad +2M^2(1-\alpha_n)^2 \\
&\leq ||x_{n-1} - w||^2 - 2(1-\alpha_n)\psi(\|x_n - w\|) + 3M^2(1-\alpha_n)^2. \qquad (4.6)
\end{aligned}$$

Denote

$$\begin{aligned}
\theta_n &= ||x_{n-1} - w||, \\
\lambda_n &= 2(1-\alpha_n), \\
\sigma_n &= 3M^2(1-\alpha_n)^2.
\end{aligned}$$

Condition $\lim_{n\to\infty}(1-\alpha_n) = 0$ assures the existence of $n_0 \in \mathbb{N}$ such that $\lambda_n = 2(1-\alpha_n) \leq 1$, for all $n \geq n_0$. Now with the help of $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ and Lemma 4, we obtain from (4.6) that

$$\lim_{n\to\infty} ||x_n - w|| = 0,$$

completing the proof. □

Remark 4 Theorem 7 extend and improve the Theorems 5-6 in the following directions:

- The strictly pseudocontractive mappings are replaced by the more general $\psi-$uniformly pseudocontractive and $\psi-$uniformly accretive mappings;
- Theorem 7 holds in real Banach space whereas the results of Theorem 6 are in q-uniformly smooth and p-uniformly convex Banach space;
- We do not need the assumption $\lim_{n\to\infty} d(x_n, F)$ as in Theorem 5.
- One can easily see that if we take $\alpha_n = 1 - \frac{1}{n^\sigma}$; $0 < \sigma < 1$, then $\sum(1-\alpha_n) = \infty$, but $\sum(1-\alpha_n)^2 = \infty$. Hence the conclusion of Theorem 5 is false.

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received: October 2, 2006

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Rostock. Math. Kolloq. **62**, 41–59 (2007)

Subject Classification (AMS)
26-99, 26A27, 39B22

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On the extrema and the improper derivatives of Takagi's continuous nowhere differentiable function

ABSTRACT. In this paper we derive functional relations for Takagi's continuous nowhere differentiable function T, and we give an explicit representation of T at dyadic points. As application of these functional relations we derive a limit relation at dyadic points which implies that at these points T attains locally minima. Further, T is maximal on a perfect set of Lebesgue measure zero. Though the points, where T has a locally maximum, are dense it is remarkable that there is no point where T has a *proper* maximum. Moreover, we verify the existence of the improper derivatives $T'(x) = +\infty$ or $T'(x) = -\infty$ for rational x which have an odd length of period in the binary representation. Finally we investigate one-side upper and lower derivatives.

KEY WORDS. Takagi's continuous nowhere differentiable function, functional equations, improper derivatives, upper and lower derivatives.

1 Introduction

In 1903, T. Takagi [4] discovered an example of a continuous, nowhere differentiable function that was simpler than a well-known example of K. Weierstrass. Takagi's function T is defined by

$$T(x) = \sum_{n=0}^{\infty} \frac{\Delta(2^n x)}{2^n} \qquad (x \in \mathbb{R}) \tag{1.1}$$

where $\Delta(x) = \text{dist}(x, \mathbb{Z})$ is an periodic function with period 1. This function T satisfies for $0 \le x \le 1$ the following system of functional equations

$$T\left(\frac{x}{2}\right) = \frac{x}{2} + \frac{1}{2}T(x), \qquad T\left(\frac{1+x}{2}\right) = \frac{1-x}{2} + \frac{1}{2}T(x), \tag{1.2}$$

cf. [3], [2], [7]. The graph of the Takagi function is illustrated in Figure 1.

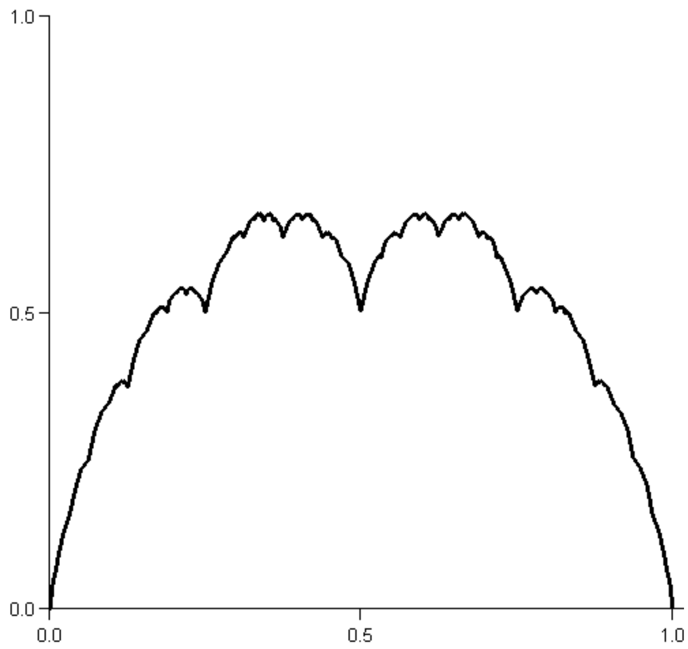

Figure 1: The graph of the Takagi function

For Takagi's function we derive functional relations and give some applications for it. First we show that at dyadic points $x = \frac{k}{2^\ell}$, $(k, \ell \in \mathbb{Z})$, there exists the limit

$$\lim_{h \to 0} \frac{T(x+h) - T(x)}{|h| \log_2 \frac{1}{|h|}} = 1. \tag{1.3}$$

Consequently, T has at all dyadic points a locally minimum, and its point out that only these points are locally minima of T (Proposition 4.1). It holds $\max T = \frac{2}{3}$ and the set M of points $x \in [0, 1]$ with $T(x) = \frac{2}{3}$ is given by

$$M = \left\{ \sum_{k=1}^{\infty} \frac{a_k}{4^k} \, : \, a_k \in \{1, 2\} \right\},$$

which is a perfect set of measure zero (Proposition 4.2). Further, the set of points where T is locally maximal is a set of first category with Lebesgue measure zero, and there is no point where T has a proper locally maximum (Proposition 4.4).

A further consequence of (1.3) is the fact that at each dyadic point x there exist the right-side improper derivative $T'_+(x) = +\infty$ and the left-side improper derivative $T'_-(x) = -\infty$. We give a simple criterion for the existence of the improper derivatives $T'(x) = +\infty$ and $T'(x) = -\infty$ (Proposition 5.3). In particular, for rational x with odd length of period in the binary representation always there exists the improper derivative (Proposition 5.4).

Moreover, we investigate the four derivatives

$$D^+(x) = \limsup_{h\to+0} \frac{T(x+h)-T(x)}{h}, \qquad D_+(x) = \liminf_{h\to+0} \frac{T(x+h)-T(x)}{h},$$
$$D^-(x) = \limsup_{h\to-0} \frac{T(x+h)-T(x)}{h}, \qquad D_-(x) = \liminf_{h\to-0} \frac{T(x+h)-T(x)}{h},$$

cf. [5], p. 354. We show that if for $x \in \mathbb{R}$ the right-side derivatives $D^+(x)$ and $D_+(x)$ are finite then

$$D^+(x) - D_+(x) \geq 2. \tag{1.4}$$

In view of the symmetry $T(1-x) = T(x)$ this is true also for the left-side derivatives. Furthermore, if all four derivatives are finite then for the upper and lower derivatives

$$\overline{D}(x) = \limsup_{h\to0} \frac{T(x+h)-T(x)}{h} = \max\{D^+(x), D^-(x)\} \tag{1.5}$$

$$\underline{D}(x) = \liminf_{h\to0} \frac{T(x+h)-T(x)}{h} = \min\{D_+(x), D_-(x)\} \tag{1.6}$$

it holds

$$\overline{D}(x) - \underline{D}(x) \geq 3. \tag{1.7}$$

We show that the estimates (1.4) and (1.7) are best possible.

In the textbook [3] you can find in detail investigations on Takagi's function. Unfortunately the representation contains errors which we correct in Section 7.3.

2 Functional relations

In order to derive functional relations for Takagi's function we use the binary sum-of-digit function $s(k)$ which for integers $k \geq 0$ with the dyadic representation $k = a_0a_1\dots a_m$, $a_j \in \{0,1\}$, is defined by

$$s(k) = \sum_{j=0}^{m} a_j \tag{2.1}$$

and which has the properties $s(2k) = s(k)$ and $s(2k+1) = s(k)+1$.

Proposition 2.1 *For $\ell \in \mathbb{N}$, $k = 0, 1, \dots, 2^\ell - 1$, $x \in [0,1]$, the Takagi function T satisfies the functional equations*

$$T\left(\frac{k+x}{2^\ell}\right) = T\left(\frac{k}{2^\ell}\right) + \frac{\ell - 2s(k)}{2^\ell}x + \frac{1}{2^\ell}T(x) \tag{2.2}$$

and

$$T\left(\frac{k-x}{2^\ell}\right) = T\left(\frac{k}{2^\ell}\right) + \frac{2s(k-1)-\ell}{2^\ell}x + \frac{1}{2^\ell}T(x). \tag{2.3}$$

Moreover, for $x = \frac{n}{2^\ell}$ with $n = 0, \ldots, 2^\ell$ the function T has the representation

$$T\left(\frac{n}{2^\ell}\right) = \frac{n\ell}{2^\ell} - \frac{1}{2^{\ell-1}} \sum_{k=0}^{n-1} s(k). \tag{2.4}$$

Proof: Equation (2.2) for $\ell = 1$ turns over into (1.2). Assume that (2.2) is true for an integer $\ell \geq 1$. Replacing x by $\frac{x}{2}$ and applying (1.2) we get

$$\begin{aligned} T\left(\frac{2k+x}{2^{\ell+1}}\right) &= T\left(\frac{k}{2^\ell}\right) + \frac{\ell - 2s(k)}{2^\ell}\frac{x}{2} + \frac{1}{2^\ell} T\left(\frac{x}{2}\right) \\ &= T\left(\frac{2k}{2^{\ell+1}}\right) + \frac{\ell - 2s(k)}{2^{\ell+1}} x + \frac{x}{2^{\ell+1}} + \frac{1}{2^{\ell+1}} T(x). \end{aligned}$$

In view of $s(2k) = s(k)$ we obtain (2.2) with $2k$ instead of k and $\ell + 1$ instead of ℓ. If we replace x by $\frac{x+1}{2}$ in (2.2) then in view of (1.2) we obtain

$$\begin{aligned} T\left(\frac{2k+1+x}{2^{\ell+1}}\right) &= T\left(\frac{k}{2^\ell}\right) + \frac{\ell - 2s(k)}{2^\ell}\frac{x+1}{2} + \frac{1}{2^\ell} T\left(\frac{x+1}{2}\right) \\ &= T\left(\frac{k}{2^\ell}\right) + \frac{\ell - 2s(2k)}{2^{\ell+1}}(x+1) + \frac{1-x}{2^{\ell+1}} + \frac{1}{2^{\ell+1}} T(x). \end{aligned}$$

For $x = 0$ we find in view of $T(0) = 0$

$$T\left(\frac{2k+1}{2^{\ell+1}}\right) = T\left(\frac{k}{2^\ell}\right) + \frac{\ell - 2s(2k)}{2^{\ell+1}} - \frac{1}{2^{\ell+1}}$$

and it follows

$$T\left(\frac{2k+1+x}{2^{\ell+1}}\right) = T\left(\frac{2k+1}{2^\ell}\right) + \frac{\ell + 1 - 2s(2k+1)}{2^{\ell+1}} x + \frac{1}{2^{\ell+1}} T(x)$$

where we have used $s(2k+1) = s(k) + 1$, so that (2.2) is proved by induction.

From (2.2) with $k - 1$ instead of k and $1 - x$ instead of x we get in view of the symmetry of T the equation

$$T\left(\frac{k-x}{2^\ell}\right) = T\left(\frac{k-1}{2^\ell}\right) + \frac{\ell - 2s(k-1)}{2^\ell}(1-x) + \frac{1}{2^\ell} T(x) \qquad (0 \leq x \leq 1). \tag{2.5}$$

It follows for $x = 0$ that

$$T\left(\frac{k}{2^\ell}\right) = T\left(\frac{k-1}{2^\ell}\right) + \frac{\ell - 2s(k-1)}{2^\ell},$$

so that (2.5) can be written as (2.3). Finally, equation (2.4) follows from (2.2) for $x = 1$ and by summation in view of $T(0) = T(1) = 0$. □

Corollary 2.2 *For $\ell \in \mathbb{N}$, $k = 1, \ldots, 2^\ell - 1$, $x \in [0, 1]$, the Takagi function T satisfies*

$$T\left(\frac{k+x}{2^\ell}\right) - T\left(\frac{k-x}{2^\ell}\right) = \frac{\ell - s(k) - s(k-1)}{2^{\ell-1}} x. \tag{2.6}$$

For $k = \ell = 1$ this means the symmetry of T with respect to $\frac{1}{2}$.

It is easy to see that the partial sum

$$S_\ell(x) = \sum_{n=0}^{\ell-1} \frac{\Delta(2^n x)}{2^n} \tag{2.7}$$

of Takagi's function T from (1.1) is linear in the intervals

$$I_{k\ell} = \left[\frac{k}{2^\ell}, \frac{k+1}{2^\ell}\right] \tag{2.8}$$

where $\ell \in \mathbb{N}$ and $k \in \{0, 1, \ldots, 2^\ell - 1\}$. Moreover, for $n \geq \ell$ and $k \in \{0, 1, \ldots, 2^\ell\}$ we have $S_n(\frac{k}{2^\ell}) = S_\ell(\frac{k}{2^\ell})$ and hence also $T(\frac{k}{2^\ell}) = S_\ell(\frac{k}{2^\ell})$, cf. Figure 2.

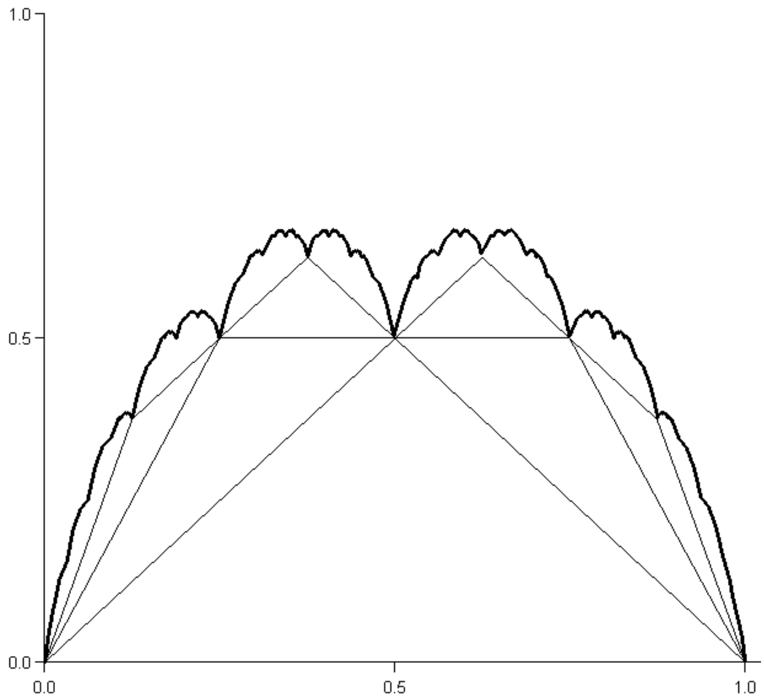

Figure 2: The partial sums S_1, S_2, S_3

Proposition 2.3 *For $\ell \in \mathbb{N}$, $k = 0, 1, \ldots, 2^\ell - 1$, the partial sum (2.7) of (1.1) is linear in the interval (2.8) and it holds*

$$S_\ell\left(\frac{k+x}{2^\ell}\right) = S_\ell\left(\frac{k}{2^\ell}\right) + \frac{\ell - 2s(k)}{2^\ell} x \qquad (x \in [0, 1]). \tag{2.9}$$

Proof: Clearly, $S_\ell(x)$ is linear in $I_{k\ell}$ so that

$$S_\ell\left(\frac{k+x}{2^\ell}\right) = S_\ell\left(\frac{k}{2^\ell}\right) + ax \qquad (x \in [0,1]).$$

In view of $T(\frac{k}{2^\ell}) = S_\ell(\frac{k}{2^\ell})$ and $T(\frac{k+1}{2^\ell}) = S_\ell(\frac{k+1}{2^\ell})$ we obtain from (2.2) with $x = 1$ that

$$S_\ell\left(\frac{k+1}{2^\ell}\right) - S_\ell\left(\frac{k}{2^\ell}\right) = \frac{\ell - 2s(k)}{2^\ell}$$

which implies the assertion. □

3 A limit relation at dyadic points

In order to derive the limit relation (1.3) first we show

Lemma 3.1 *For $0 < x \le \frac{1}{2}$ the Takagi function T satisfies the estimate*

$$x \log_2 \frac{1}{x} \le T(x) \le x \log_2 \frac{1}{x} + cx \tag{3.1}$$

with a constant $c < \frac{2}{3}$.

Proof: For $0 < x \le \frac{1}{2}$ we put

$$C(x) = \frac{T(x)}{x \log_2 \frac{1}{x}}$$

and we show that for $\frac{1}{2^{\ell+1}} < x \le \frac{1}{2^\ell}$ $(\ell \in \mathbb{N})$ it holds

$$1 \le C(x) \le 1 + \frac{c}{\ell+1}. \tag{3.2}$$

Applying (1.2) we obtain

$$C\left(\frac{x}{2}\right) \log_2 \frac{2}{x} = \frac{2}{x} T\left(\frac{x}{2}\right) = 1 + C(x) \log_2 \frac{1}{x}$$

which implies

$$\left\{C\left(\frac{x}{2}\right) - 1\right\} \log_2 \frac{2}{x} = \{C(x) - 1\} \log_2 \frac{1}{x}. \tag{3.3}$$

1. First we show that $C(x) \ge 1$ for $\frac{1}{4} \le x \le \frac{1}{2}$. We use the estimate $T(x) \ge S_3(x)$ where the partial sum $S_3(x)$ from (2.7) has for $\frac{1}{4} \le x \le \frac{1}{2}$ the form

$$S_3(x) = \begin{cases} \frac{1}{2} + x & \text{for} \quad \frac{1}{4} \le x \le \frac{3}{8} \\ 1 - x & \text{for} \quad \frac{3}{8} \le x \le \frac{1}{2}\,, \end{cases}$$

cf. Figure 2. For $\frac{1}{4} < x < \frac{3}{8}$ we have for the function $f(x) = x \log_2 \frac{1}{x}$

$$f'(x) = \frac{-1 + \log \frac{1}{x}}{\log 2} < \frac{\log \frac{4}{e}}{\log 2} < 1 = S_3'(x)$$

so that from $S_3(\frac{1}{4}) = f(\frac{1}{4}) = \frac{1}{2}$ it follows $S_3(x) \geq f(x)$. For $\frac{3}{8} < x < \frac{1}{2}$ we have

$$f'(x) = \frac{-1 - \log x}{\log 2} > \frac{-\log \frac{e}{2}}{\log 2} > -1 = S_3'(x)$$

so that from $S_3(\frac{1}{2}) = f(\frac{1}{2}) = \frac{1}{2}$ it follows $S_3(x) \geq f(x)$. So we have $T(x) \geq S_3(x) \geq f(x)$ for $\frac{1}{4} \leq x \leq \frac{1}{2}$, i.e. $C(x) \geq 1$ for these x. The relation (3.3) implies that $C(x) \geq 1$ is valid for all $x \in (0, \frac{1}{2}]$.

2. Next we show that (3.2) is valid for $\ell = 1$. Since $\frac{1}{f(x)}$ is increasing for $0 < x < \frac{1}{e}$ and decreasing for $\frac{1}{e} < x$, it follows that in interval $[\frac{1}{4}, \frac{1}{2}]$ the function $\frac{1}{f(x)}$ is maximal for $x = \frac{1}{4}$ or for $x = \frac{1}{2}$. Because of $f(\frac{1}{4}) = f(\frac{1}{2}) = \frac{1}{2}$ it follows in view of $T(\frac{1}{4}) < \frac{2}{3}$ and $T(\frac{1}{2}) < \frac{2}{3}$ that $C(x) < \frac{4}{3}$ for $\frac{1}{4} \leq x \leq \frac{1}{2}$, i.e. (3.2) is true for $\ell = 1$ with a constant $c < \frac{2}{3}$. If (3.2) is true for a certain $\ell \in \mathbb{N}$ then by (3.3) we have

$$\frac{C\left(\frac{x}{2}\right) - 1}{C(x) - 1} = \frac{\log_2 \frac{1}{x}}{1 + \log_2 \frac{1}{x}} = 1 - \frac{1}{1 + \log_2 \frac{1}{x}} \leq 1 - \frac{1}{\ell + 2}$$

for $\frac{1}{2^{\ell+1}} \leq x \leq \frac{1}{2^\ell}$. This implies

$$C\left(\frac{x}{2}\right) - 1 \leq \{C(x) - 1\}\frac{\ell + 1}{\ell + 2} \leq \frac{c}{\ell + 2}$$

for $\frac{1}{2^{\ell+2}} \leq \frac{x}{2} \leq \frac{1}{2^{\ell+1}}$, i.e. (3.2) is valid also for $\ell + 1$ and hence by induction for all $\ell \in \mathbb{N}$.

Finally, for $\frac{1}{2^{\ell+1}} < x$ we have $\ell + 1 > \log_2 \frac{1}{x}$, so that for the right hand side of (3.2) we get

$$C(x) \leq 1 + \frac{c}{\ell + 1} \leq 1 + \frac{c}{\log_2 \frac{1}{x}}$$

which yields the assertion. □

Proposition 3.2 *The Takagi function T satisfies at each dyadic point $x = \frac{k}{2^\ell}$ the limit relation*

$$\lim_{h \to 0} \frac{T(x + h) - T(x)}{|h| \log_2 \frac{1}{|h|}} = 1.$$

Proof: For $x = 0$ the limit relation is a consequence of Lemma 3.1. Let $x = \frac{k}{2^\ell}$ ($\ell \in \mathbb{N}, 0 \leq k \leq 2^\ell - 1$), and $0 < h < \frac{1}{2^\ell}$. According to (2.2) we have

$$T(x + h) - T(x) = \{\ell - 2s(k)\}h + \frac{1}{2^\ell}T(2^\ell h)$$

and

$$\frac{T(x+h)-T(x)}{h\log_2\frac{1}{h}} = \frac{\ell-2s(k)}{\log_2\frac{1}{h}} + \frac{1}{2^\ell h}\,\frac{T(2^\ell h)}{\log_2\frac{1}{h}}.$$

With $t = 2^\ell h$ the last term can be written as

$$\frac{T(t)}{t\log_2\frac{2^\ell}{t}} = \frac{T(t)}{t\log_2\frac{1}{t}(1-\frac{\ell}{\log_2 t})} \to 1 \qquad (t\to+0).$$

We obtain the same limit for

$$\frac{T(x-h)-T(x)}{-h\log_2\frac{1}{h}}$$

by means of (2.3) which yields the assertion. □

4 The extreme values of Takagi's function

Clearly, since Takagi's function T is continuous and nowhere differentiable there is no interval where T is monotone. The function T has at the point x_0 a locally maximum if $T(x_0) \geq T(x)$ for all x of a certain neighbourhood U of x_0. If even $T(x_0) > T(x)$ for $x \in U$ with $x \neq x_0$ then T has at x_0 a proper locally maximum. Analogous notations are used for a proper locally minimum, cf. e.g. [3].

Proposition 4.1 *The Takagi function T attains its locally minima exactly at the dyadic points $x = \frac{k}{2^\ell}$ where all these $T(x)$ are proper minima.*

Proof: The limit relation (1.3) implies

$$\lim_{h\to 0}\frac{T(x+h)-T(x)}{|h|} = +\infty$$

so that T has at each dyadic point a proper locally minimum. Now let $x \in [0,1]$ be a nondyadic point then for arbitrary $\ell \in \mathbb{N}$ there is $k \in \{0,1,\dots,2^\ell-1\}$ such that $\frac{k}{2^\ell} < x < \frac{k+1}{2^\ell}$, i.e. $x = t\frac{k}{2^\ell} + (1-t)\frac{k+1}{2^\ell}$ with a certain $t \in (0,1)$. For the partial sum $S_\ell(x)$ from (2.7) it holds $T(\frac{k}{2^\ell}) = S_\ell(\frac{k}{2^\ell})$ and $T(\frac{k+1}{2^\ell}) = S_\ell(\frac{k+1}{2^\ell})$, and $T(x) > S_\ell(x) = tS_\ell(\frac{k}{2^\ell}) + (1-t)S_\ell(\frac{k+1}{2^\ell})$. This implies $T(x) > \min\{T(\frac{k}{2^\ell}), T(\frac{k+1}{2^\ell})\}$ so that T cannot have a proper minimum at x. □

Next we investigate the global maxima of Takagi's function.

Proposition 4.2 *We have $\max T = \frac{2}{3}$ and the set M of points $x \in [0,1]$ with $T(x) = \frac{2}{3}$ is given by*

$$M = \left\{ x = \sum_{k=1}^{\infty}\frac{a_k}{4^k} \,:\, a_k \in \{1,2\} \right\}. \tag{4.1}$$

M is a perfect set of measure zero with $\min M = \frac{1}{3}$ and $\max M = \frac{2}{3}$.

Proof: By means of the partial sum $S_2(x)$ from (2.7) the series (1.1) can be written as

$$T(x) = \sum_{n=0}^{\infty} \frac{S_2(4^n x)}{4^n} \qquad (x \in \mathbb{R}). \tag{4.2}$$

Since $S_2(x)$ is 1-periodic and has in $[0,1]$ the form

$$S_2(x) = \begin{cases} 2x & \text{for} \quad 0 \le x < \frac{1}{4} \\ \frac{1}{2} & \text{for} \quad \frac{1}{4} \le x < \frac{3}{4} \\ 2 - 2x & \text{for} \quad \frac{3}{4} \le x \le 1, \end{cases} \tag{4.3}$$

cf. Figure 2, it follows that

$$T(x) \le \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{2}{3}$$

and that $T(x) = \frac{2}{3}$ if and only if $S_2(4^n x) = \frac{1}{2}$ for all $n \in \mathbb{N}$. According to (4.3) this is valid for $x \in [0,1]$ exactly for $x \in M$ from (4.1). M is a perfect set since for $x \in M$ with given a_k in (4.1) also $x_n = x + \frac{3-2a_n}{4^n} \in M$ (exchange of the digits 1 and 2) and $x_n \to x$ as $n \to \infty$. Moreover, M has the measure zero, since the representations of $x \in M$ do not contain at least one digit, here 0 and 3, cf. [5], p. 329-330. From (4.1) we get $\min M = \frac{1}{3}$ and $\max M = \frac{2}{3}$. □

In order to investigate the locally maxima of Takagi's function we determine the maxima of it in the closed intervals (2.8).

Proposition 4.3 *For $\ell \in \mathbb{N}$, $k = 0, 1, \ldots, 2^\ell - 1$ the set $A_{k\ell}$ of points in $I_{k\ell}$ from (2.8), where $T(x)$ is maximal, is a perfect set of measure zero so that it is nowhere dense in $[0,1]$. For the maximum it holds*

$$\max_{x \in I_{k\ell}} T(x) = \begin{cases} T\left(\frac{k}{2^\ell}\right) + \frac{2}{3 \cdot 4^{s(k)}} & 2s(k) \ge \ell \\ T\left(\frac{k+1}{2^\ell}\right) + \frac{2}{3 \cdot 4^{\ell - s(k)}} & 2s(k) < \ell \end{cases}$$

with $s(k)$ from (2.1).

Proof: According to Proposition 2.3 in the interval $I_{k\ell}$ it holds relation (2.9), so that S_ℓ is linear in $I_{k\ell}$ with the slope $p = \ell - 2s(k)$. In case $p = 0$ we get from (2.2) that

$$T\left(\frac{k+x}{2^\ell}\right) = T\left(\frac{k}{2^\ell}\right) + \frac{1}{2^\ell} T(x) \qquad (x \in [0,1]).$$

According to Proposition 4.2 it follows that T attains its maximum in $I_{k\ell}$ on a nowhere dense perfect set $A_{k\ell}$ with measure $|A_{k\ell}| = 0$, and for the maximal value we have

$$\max_{x \in I_{k\ell}} T(x) = T\left(\frac{k}{2^\ell}\right) + \frac{2}{3}\frac{1}{2^\ell}.$$

This is the assertion in case $\ell = 2s(k)$.

In case $p < 0$ the partial sum S_ℓ is strictly decreasing in $I_{k\ell}$. The partial sum $S_{\ell+|p|}$ is decreasing in $I_{k\ell}$, where more precisely we have $S_{\ell+|p|}(x) = S_\ell(\frac{k}{2^\ell})$ for $x \in I_{2^{|p|}k,\ell+|p|} \subset I_{k\ell}$ and $S_{\ell+|p|}(x) < S_\ell(\frac{k}{2^\ell})$ for all another x in $I_{k\ell}$. Therefore, the maximum of T in $I_{k\ell}$ we find in $I_{2^{|p|}k,\ell+|p|}$ where in view of Proposition 2.1 we have

$$T\left(\frac{2^{|p|}k+x}{2^{\ell+|p|}}\right) = T\left(\frac{k}{2^\ell}\right) + \frac{1}{2^{\ell+|p|}}T(x) \qquad (x \in [0,1]).$$

Thus, for the maximum of T in $I_{k\ell}$ we have in view of $\ell + |p| = 2s(k)$ and Proposition 4.2 that

$$\max_{x\in I_{k\ell}} T(x) = T\left(\frac{k}{2^\ell}\right) + \frac{2}{3}\frac{1}{4^{s(k)}}.$$

Finally, if $p > 0$ then the partial sum S_ℓ is strictly increasing in $I_{k\ell}$, and $S_{\ell+p}$ is increasing. Now, in this case we have $S_{\ell+p}(x) = S_\ell(\frac{k+1}{2^\ell})$ for $x \in I_{2^p(k+1)-1,\ell+p}$ and $S_{\ell+p}(x) < S_\ell(\frac{k+1}{2^\ell})$ for all another x in $I_{k\ell}$. Therefore, the maximum of T in $I_{k\ell}$ we find in $I_{2^p(k+1)-1,\ell+p}$ where in view of Proposition 2.1 we have

$$T\left(\frac{2^p(k+1)-x}{2^{\ell+p}}\right) = T\left(\frac{k+1}{2^\ell}\right) + \frac{1}{2^{\ell+p}}T(x) \qquad (x \in [0,1]).$$

As before it follows in view of $\ell + p = 2\ell - 2s(k)$ that

$$\max_{x\in I_{k\ell}} T(x) = T\left(\frac{k+1}{2^\ell}\right) + \frac{2}{3}\frac{1}{4^{\ell-s(k)}}.$$

According to Proposition 4.2 the set $A_{k\ell}$ where T is maximal in $I_{k\ell}$ is a nowhere dense set of measure zero. □

It follows from Proposition 4.3 and (2.4) that the maximum of T in $I_{k\ell}$ has the form $\frac{1}{3}\frac{m}{2^n}$ with certain integers m, n. As consequence we get

Proposition 4.4 *The set $A \subseteq [0,1]$, where T attains its locally maxima, is a set of first category, i.e. it is representable as union of at most countable many perfect nowhere dense sets. This set A has the power* **c** *and the measure zero. For $x \in A$ the values are $T(x) = \frac{1}{3}\frac{m}{2^n}$ with certain $n \in \mathbb{N}_0$ and $m \in \{1, 2, \dots, 2^{n+1}\}$. There is no point where T has a proper maximum.*

5 Improper derivatives

As already mentioned in the introduction formula (1.3) implies that for dyadic points $x = \frac{k}{2^\ell}$ it holds

$$\lim_{h\to+0} \frac{T(x+h)-T(x)}{h} = +\infty$$

and
$$\lim_{h\to -0}\frac{T(x+h)-T(x)}{h}=-\infty.$$
Hence, T has at all dyadic points x the one-side improper derivatives $T'_+(x)=+\infty$ and $T'_-(x)=-\infty$.

For arbitrary numbers $x,y\in[0,1]$ we consider the dyadic representations

$$x=\xi_0,\xi_1\xi_2\dots,\qquad y=\eta_0,\eta_1\eta_2\dots \tag{5.1}$$

with $\xi_0=\eta_0=0$ and $\xi_n,\eta_n\in\{0,1\}$, and we put

$$x_n=0,\xi_n\xi_{n+1}\dots,\qquad y_n=0,\eta_n\eta_{n+1}\dots \tag{5.2}$$

for $n\geq 0$.

Proposition 5.1 *Let x and y are different points in $[0,1]$ with $\xi_\nu=\eta_\nu$ for $\nu<n\in\mathbb{N}$. Then also x_n and y_n are different, and we have*

$$\frac{T(x)-T(y)}{x-y}=\sum_{\nu=0}^{n-1}(-1)^{\xi_\nu}+\frac{T(x_n)-T(y_n)}{x_n-y_n}. \tag{5.3}$$

In particular, if $\eta_\nu=1-\xi_\nu$ for $\nu\geq n$, i.e. $x_n+y_n=1$, then we have $|x-y|\leq\frac{1}{2^n}$ and

$$\frac{T(x)-T(y)}{x-y}=\sum_{\nu=0}^{n-1}(-1)^{\xi_\nu}. \tag{5.4}$$

Proof: We put $k_n=[2^{n-1}x]$, i.e. $k_n=\sum_{\nu=0}^{n-1}2^{n-\nu}\xi_\nu$ then we have
$$x=\frac{2k_n+x_n}{2^n},\qquad y=\frac{2k_n+y_n}{2^n}$$
and
$$x-y=\frac{x_n-y_n}{2^n}. \tag{5.5}$$
From equation (2.2) we get
$$T(x)=T\left(\frac{2k_n+x_n}{2^n}\right)=T\left(\frac{2k_n}{2^n}\right)+\frac{n-2s(2k_n)}{2^n}x_n+\frac{1}{2^n}T(x_n)$$
and
$$T(y)=T\left(\frac{2k_n+y_n}{2^n}\right)=T\left(\frac{2k_n}{2^n}\right)+\frac{n-2s(2k_n)}{2^n}y_n+\frac{1}{2^n}T(y_n).$$
It follows
$$\frac{T(x)-T(y)}{x-y}=n-2s(2k_n)+\frac{T(x_n)-T(y_n)}{x_n-y_n}$$
and the relations (5.5) and $1-2\xi_\nu=(-1)^{\xi_\nu}$ for $\nu=0,\dots n-1$ yield the assertion (5.3). In case $y_n=1-x_n$ we have $T(x_n)=T(1-x_n)=T(y_n)$, and hence (5.4). □

Corollary 5.2 *Formula (5.4) implies:*

1. *There is no point where T has a finite derivative since as $n \to \infty$ the right-hand side is not convergent to a finite value.*

2. *If there exists the improper derivative $T'(x) = +\infty$ then*

$$\sum_{k=0}^{\infty} (-1)^{\xi_k} = +\infty \tag{5.6}$$

and if $T'(x) = -\infty$ then

$$\sum_{k=0}^{\infty} (-1)^{\xi_k} = -\infty. \tag{5.7}$$

Proposition 5.3 *If in the dyadic representation of the number x the number of both zeros and ones which occur one after the other is bounded then (5.6) implies the existence of the improper derivative $T'(x) = +\infty$, and (5.7) implies $T'(x) = -\infty$.*

Proof: For $y \neq x$ let n be the smallest integer such that $\eta_n = \xi_n$, cf. (5.1). Then by Proposition 5.1 it holds

$$\left| \frac{T(x) - T(y)}{x - y} - \sum_{\nu=0}^{n-1} (-1)^{\xi_\nu} \right| = \left| \frac{T(x_n) - T(y_n)}{x_n - y_n} \right|$$

with x_n, y_n from (5.2). If d denotes the maximal number of equals digits ξ_ν which occur one after the other then in case $\xi_n = 1$, $\eta_n = 0$ we have $x_n > \frac{1}{2} + \frac{1}{2^{d+3}}$ and $y_n < \frac{1}{4} + \frac{1}{8} + \ldots = \frac{1}{2}$ so that $|x_n - y_n| > \frac{1}{2^{d+3}}$. In case $\xi_n = 0$, $\eta_n = 1$ we have $x_n < \frac{1}{4} + \ldots + \frac{1}{2^{d+2}} = \frac{1}{2} - \frac{1}{2^{d+3}}$ and $y_n \geq \frac{1}{2}$ so that $|x_n - y_n| > \frac{1}{2^{d+3}}$, too. Hence

$$\left| \frac{T(x_n) - T(y_n)}{x_n - y_n} \right| < \frac{2}{3} 2^{d+3}.$$

This implies the assertion. □

So for rational x we summarize

Proposition 5.4 *For the Takagi function T we have the following statements at rational points x:*

1. *If $x = \frac{k}{2^\ell}$ is a dyadic point then $T'_+(x) = +\infty$ and $T'_-(x) = -\infty$.*

2. *If $x \neq \frac{k}{2^\ell}$ has a dyadic representation with the period $\xi_{k+1} \ldots \xi_{k+p}$ then it holds:*

$$\xi_{k+1} + \xi_{k+2} + \ldots + \xi_{k+p} \begin{cases} < \frac{p}{2} & \Longrightarrow \quad T'(x) = +\infty \\ > \frac{p}{2} & \Longrightarrow \quad T'(x) = -\infty \\ = \frac{p}{2} & \Longrightarrow \quad T'(x) \text{ does not exists.} \end{cases}$$

In the last case, where p must be even, $\overline{D}(x)$ from (1.5) and $\underline{D}(x)$ from (1.6) are finite.

Remark 5.5 It follows from Proposition 5.4 that for rational x with an odd length of period in the dyadic representation always there exists the improper derivative $T'(x)$. For instance $x_1 = \frac{1}{7} = 0,001001\ldots$ has the period 001 and hence there exists the improper derivative $T'(x_1) = +\infty$, and $x_2 = \frac{6}{7} = 0,110110\ldots$ has the period 110 and hence there exists the improper derivative $T'(x_2) = -\infty$.

Remark 5.6 We know that for dyadic points $x = \frac{k}{2^\ell}$ there exists the limit (1.3). Let us mention that a similar argument as in the proof of Proposition 5.3 also yields (1.3) and moreover, that for rational $x \neq \frac{k}{2^\ell}$ it holds

$$\lim_{h\to 0} \frac{T(x+h) - T(x)}{h \log_2 \frac{1}{|h|}} = 1 - \frac{2(\xi_{k+1} + \ldots + \xi_{k+p})}{p}$$

where $\xi_{k+1} \ldots \xi_{k+p}$ is a period in the dyadic representation of x.

6 Upper and lower derivatives

Finally, we investigate the four derivatives $D^+(x)$, $D_+(x)$, $D^-(x)$, $D_-(x)$ of Takagi's function T, which are defined in the introduction. We begin with

Lemma 6.1 *For $0 < x < \frac{1}{3}$ the T satisfies the inequality $T(x) \geq 2x$ where we have equality if and only if $x = x_m$ with*

$$x_m = \sum_{\mu=1}^{m} \frac{1}{4^\mu} = \frac{4^m - 1}{3 \cdot 4^m} \qquad (m \in \mathbb{N}). \tag{6.1}$$

Proof: First we show by induction on m that $T(x_m) = 2x_m$. For $m = 1$ we have $x_1 = \frac{1}{4}$, and according to (2.4) it holds $T(\frac{1}{4}) = \frac{1}{2}$. Formula (6.1) implies $x_m = \frac{k_m}{4^m}$ with $k_m = 1 + 4 + \ldots + 4^{m-1}$ so that $s(k_m) = m$, cf. (2.1). Moreover, we have

$$x_{m+1} = x_m + \frac{1}{4^{m+1}} = \frac{4k_m + 1}{4^{m+1}}. \tag{6.2}$$

Assume that for a fixed m it holds

$$T(x_m) = 2x_m = \frac{2(4^m - 1)}{3 \cdot 4^m}$$

then by (2.2) with $k = 4k_m$, $\ell = 2m + 2$ and $x = 1$ we get in view of $s(4k_m) = m$ and $T(1) = 0$ that

$$\begin{aligned} T\left(\frac{4k_m + 1}{2^{2m+2}}\right) &= T(x_m) + \frac{2m + 2 - 2s(4k_m)}{2^{2m+2}} \\ &= \frac{2(4^m - 1)}{3 \cdot 4^m} + \frac{2}{4^{m+1}} = \frac{2(4^{m+1} - 1)}{3 \cdot 4^{m+1}}, \end{aligned}$$

i.e. $T(x_{m+1}) = 2x_{m+1}$. It follows that $T(x_{m+1}) = T(x_m) + 2(x_{m+1} - x_m)$. Since $x_m = \frac{4k_m}{2^{2m+2}}$ and $x_{m+1} = \frac{4k_m+1}{2^{2m+2}}$ the equation (2.2) implies in view of $T(t) > 0$ for $0 < t < \frac{1}{4^{m+1}}$ that $T(x_m + t) > T(x_m) + 2t$. □

Lemma 6.2 *The Takagi function has at the point $x = \frac{1}{3}$ the derivatives*

$$D^+\left(\frac{1}{3}\right) = 0, \quad D_+\left(\frac{1}{3}\right) = -1, \quad D^-\left(\frac{1}{3}\right) = 2, \quad D_-\left(\frac{1}{3}\right) = 1.$$

Proof: We know that $T(x) \le T(\frac{1}{3}) = \frac{2}{3}$ and that the set M of points x in $[0,1]$ with $T(x) = \frac{2}{3}$ is a perfect set. Since $\frac{1}{3} = \min M$ it follows $D^+(\frac{1}{3}) = 0$. The symmetry $T(1-x) = T(x)$ implies $D^-(\frac{2}{3}) = 0$, too.

Let x_ν be a sequence with $x_\nu \to x$ as $\nu \to \infty$. From the first equation in (1.2) with $2x$ instead of x we get for $x_\nu \ne x$ and $x_\nu, x < \frac{1}{2}$ that

$$\frac{T(x_\nu) - T(x)}{x_\nu - x} = 1 + \frac{T(2x_\nu) - T(2x)}{2(x_\nu - x)}. \tag{6.3}$$

It follows $D^-(\frac{1}{3}) = 1$ since $D^-(\frac{2}{3}) = 0$. Moreover, Lemma 6.1 implies $D_-(\frac{1}{3}) = 2$ so that $D_+(\frac{2}{3}) = -2$ since the symmetry of T. Now, (6.3) implies $D_+(\frac{1}{3}) = -1$. □

Proposition 6.3 *If for $x \in \mathbb{R}$ the right-side derivatives $D^+(x)$ and $D_+(x)$ of the Takagi function T are finite then we have*

$$D^+(x) - D_+(x) \ge 2$$

where we have equality if x has the form

$$x = \frac{k}{2^n} + \frac{1}{3 \cdot 2^n}$$

with $k, n \in \mathbb{N}_0$. Moreover, the upper and lower derivatives $\overline{D}(x)$ and $\underline{D}(x)$ of T satisfy the inequality

$$\overline{D}(x) - \underline{D}(x) \ge 3$$

where we have equality if x has above form.

Proof: For dyadic $x = \frac{k}{2^\ell}$ we know from Proposition 3.2 that $D^+T(x) = +\infty$. Let x be a nondyadic point with the representation $x = 0, \xi_1\xi_2\ldots$ and for $n \in \mathbb{N}$ let be $y = 0, \eta_1\eta_2\ldots$ with $\eta_\nu = \xi_\nu$ for $\nu \le n$ and $\eta_\nu = 1 - \xi_\nu$ for $\nu > n$. In case $\xi_{n+1} = 0$ we have $y > x$ since $y \ge 0, \xi_1\ldots\xi_n 1 > 0, \xi_1\ldots\xi_n 0\ldots = x$ and x is notdyadic. Equation (5.4) implies that $D_r(x) := D^+(x) - D_+(x) \ge 1$ where $D_r(x) \ge 2$ and the case $D_r(x) = 1$ may be only possible if there is an integer n such that for $\nu \geq 0$ it holds $\xi_{n+2\nu} = 1$ and $\xi_{n+2\nu+1} = 0$. This means that $x = x_0$ necessarily must be of the form $x_0 = 0, \xi_1 \ldots \xi_n 0101 \ldots$, i.e.

$$x_0 = \sum_{\nu=1}^{n} \frac{\xi_\nu}{2^\nu} + \frac{1}{2^{n+2}} \sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{k}{2^n} + \frac{1}{3 \cdot 2^n}$$

and according to (2.2) we get for $0 < |h| < \frac{1}{3 \cdot 2^n}$ that

$$\frac{T(x_0 + h) - T(x_0)}{h} = n - 2s(k) + \frac{T(\frac{1}{3} + 2^n h) - T(\frac{1}{3})}{2^n h}.$$

It follows $D_r(x_0) = D_r(\frac{1}{3}) = 2$ by Lemma 6.2. Consequently, for an arbitrary nondyadic point x we have $D_r(x) \geq 2$.

As before formula (5.4) implies that $S = \overline{D}(x) - \underline{D}(x) \geq 2$ where the case $S = 2$ may be only possible if there is an integer n such that for $\nu \geq 0$ it holds $\xi_{n+2\nu} = 1$ and $\xi_{n+2\nu+1} = 0$, i.e. if $x = x_0$. But for x_0 we get from Lemma 6.2 as before that $\overline{D}(x_0) - \underline{D}(x_0) = 3$. □

7 Supplements

Finally we give three supplements.

7.1. Improper derivatives at irrational points. There exists irrational points such that there exists the improper derivative. In order to give an example first we put $x = \xi_0, \xi_1\xi_2 \ldots$ where ξ_k is $s(k)$ mod 2 with values from $\{0, 1\}$ which is the Morse sequence, cf. [1]. Relations $s(2k) = s(k)$ and $s(2k + 1) = s(k) + 1$ imply that $d = 2$ is the maximal number of the same digit which occur one after the other. For $k = 2^\ell + 1$ $(\ell = 1, 2, \ldots)$ we have $s(k) = 2$ and hence $\xi_k = 0$. We put $y = \eta_0, \eta_1\eta_2 \ldots$, where $\eta_k = 1$ for $k = 2^\ell + 1$ and $\eta_k = 0$ elsewhere. We show that $z = x + y$ is irrational and that Takagi's function has at this point the improper derivative $T'(z) = -\infty$. First we show that x is irrational. Assume the representation $x = \xi_0, \xi_1\xi_2 \ldots$ contains a period, i.e. there is an integer $p > 1$ such that $\xi_{k+p} = \xi_k$ for $k \geq k_0$. If $s(p) \equiv 0 \mod 2$ then for $k = 2^n \geq k_0$ we have $s(k) = 1$ but $s(kp) = s(p) \not\equiv s(k) \mod 2$ which is impossible. In case $s(p) \equiv 1 \mod 2$ we note that $\xi_{k+p'} = \xi_k$ for each multiply p' of p. In particular for $p' = (2^n + 1)p$ with $2^n > p$ we get $s(p') = 2s(p)$, and as before we get an contradiction so that x cannot be rational. Now it follows easy that also $z = \zeta_0, \zeta_1 \ldots$ with $\zeta_k = \xi_k + \eta_k$ does not have a period in view of $2^{\ell+1} + 1 - (2^\ell + 1) \to \infty$ as $\ell \to \infty$. In order to apply Proposition 5.3 we have to show that

$$\sum_{k=0}^{\infty} (-1)^{\xi_k + \eta_k} = -\infty. \tag{7.1}$$

But the sequence $\sum_{k=0}^{n}(-1)^{\xi_k} \in \{0, +1, -1\}$ is bounded and $\eta_k = 1$ only for $k = 2^\ell + 1$ where $\xi_k = 0$. This implies (7.1), and by Proposition 5.3 it holds $T'(z) = -\infty$.

7.2. An example for the case $\overline{D}(x) = +\infty$ and $\underline{D}(x) = -\infty$. In order to show that the condition (5.6) is not sufficient for the existence of the improper derivative $T'(x) = +\infty$ we use the following

Lemma 7.1 *Assume that $x = \frac{k+r}{2^n}$ and $y = \frac{k-2r}{2^n}$ where k is an odd integer and $0 < r < \frac{1}{4}$. Then we have*

$$\frac{T(x) - T(y)}{x - y} = n + 2 - 2s(k) - \frac{T(r)}{3r} \tag{7.2}$$

with $s(k)$ from (2.1).

Proof: According to equation (2.2) we have

$$T(x) = T\left(\frac{k+r}{2^n}\right) = T\left(\frac{k}{2^n}\right) + \frac{n - 2s(k)}{2^n} r + \frac{1}{2^n} T(r)$$

and by equation (2.3) we get

$$T(y) = T\left(\frac{k-2r}{2^n}\right) = T\left(\frac{k}{2^n}\right) + \frac{2s(k) - 2 - n}{2^n} 2r + \frac{1}{2^n} T(2r)$$

where we have used that $s(k-1) = s(k) - 1$ since k is an odd integer. It follows

$$T(x) - T(y) = \frac{n - 2s(k)}{2^n} 3r + \frac{4r}{2^n} + \frac{T(r) - T(2r)}{2^n}$$

and in view of $x - y = \frac{3r}{2^n}$ we find

$$\frac{T(x) - T(y)}{x - y} = n - 2s(k) + \frac{4}{3} + \frac{T(r) - T(2r)}{3r}$$

From the first equation in (1.2) we get for $0 < r < \frac{1}{4}$ that

$$\frac{T(r) - T(2r)}{3r} = \frac{T(r) - \{2T(r) - 2r\}}{3r} = -\frac{T(r)}{3r} + \frac{2}{3}$$

and hence it follows the assertion. □

Example 7.2 For

$$x = \sum_{n=1}^{\infty} \frac{1}{2^{a_n}}$$

with $a_n \in \mathbb{N}$ such that $a_{n+1} \geq 4a_n$. Then $\sum(-1)^{\xi_\nu} = +\infty$ and hence $\overline{D}(x) = +\infty$. We show that $\underline{D}(x) = -\infty$. For this we put

$$x = \frac{k_n + r_n}{2^{a_n}}, \qquad y = \frac{k_n - 2r_n}{2^{a_n}}$$

with

$$k_n = 2^{a_n} \sum_{k=1}^{n} \frac{1}{2^{a_k}}, \qquad r_n = 2^{a_n} \sum_{k=n+1}^{\infty} \frac{1}{2^{a_k}},$$

i.e. $y = x + h_n$ with $h_n = -\frac{3}{2^{\alpha_n}} r_n$. By Lemma 7.1 we have in view of $s(k_n) = n$ that

$$\frac{T(x) - T(x + h_n)}{-h_n} = a_n - 2n - \frac{T(r_n)}{3r_n}$$

and by Proposition 3.1 it holds

$$\frac{T(r_n)}{r_n} \geq \log_2 \frac{1}{r_n} \geq a_{n+1} - a_n$$

since $\frac{1}{r_n} \geq 2^{a_{n+1} - a_n}$. In view of $a_{n+1} \geq 4a_n$ we get

$$\frac{T(x) - T(x + h_n)}{-h_n} \leq a_n - 2n - \frac{a_{n+1} - a_n}{3} \leq -2n$$

i.e. $\underline{D}(x) = -\infty$.

7.3. Some remarks to the representations in textbook [3]. The textbook [3] of K. Strubecker: "EINFÜHRUNG IN DIE HÖHERE MATHEMATIK", vol. II, contains a beautiful introduction in the foundations of the analysis. So you can find in detail a treatise on the function $f = T$ of T. Kakagi, among other things very interested investigations due to W. Wunderlich [6]. Unfortunately, in the passage on Takagi's function are misrepresentations and since it is not planed a new edition of [3], we want to make here two remarks.

1. The first remark concern the formula (56.45) in [3]:

$$D_\nu = \frac{f(x_\nu) - f(x)}{x_\nu - x} = \sum_{n=1}^{\nu} (-1)^{\tau_n} = (-1)^{\tau_1} + (-1)^{\tau_2} + \ldots + (-1)^{\tau_\nu} \tag{7.3}$$

where

$$x = 0, \tau_1 \tau_2 \ldots \tau_\nu \ldots$$

and

$$x_\nu = 0, \tau_1 \tau_2 \ldots \tau_{\nu-1} \tau'_\nu \tau_{\nu+1} \ldots$$

with $\tau'_\nu = 1 - \tau_\nu$ are the dyadic representations of x and x_ν, respectively. This formula cannot be correct as the following example shows. In case $x = \frac{2}{3} = 0,10101\ldots$ we have $\tau_{2\nu} = 0$ and $\tau_{2\nu-1} = 1$ for $\nu \geq 1$ and $x_1 = 0,0010101\ldots = \frac{1}{6}$. Now, $f(x) = \frac{2}{3}$ and $f(x_1) = \frac{1}{6} + \frac{1}{2} f(\frac{1}{3}) = \frac{1}{2}$, cf. (1.2), so that

$$\frac{f(x_1) - f(x)}{x_1 - x} = \frac{1}{3}$$

but formula (7.3) yields integer values. Let us mention that instead of (7.3) it holds

$$\frac{f(x_\nu)-f(x)}{x_\nu-x}=1+\sum_{n=1}^{\nu-1}(-1)^{\tau_n}-\sum_{k=0}^{\infty}\frac{\tau_{\nu+k+1}}{2^k} \tag{7.4}$$

which follows from Proposition 5.1.

2. The second remark concern Satz 3 and Satz 4 on p. 255. Both theorems base on formula (7.3) which we have recognize as not correct. Moreover, x_ν is only a special sequence which converges to x so that the fact $\lim\limits_{\nu\to\infty} D_\nu=+\infty$ does not imply the existence of (one-side) improper derivatives. Therefore the statements in Satz 3 and Satz 4 concerning the existence of (one-side) improper derivatives are not proved.

On p. 255 it says literal: "Zum Beispiel hat $f(x)$ an der Stelle

$$x=\frac{1}{7}=0,001001001\dots \qquad \text{(periodisch)}$$

nach (56.45) (i.e. (7.3)) die uneigentliche Ableitung $f'(x)=\lim\limits_{\nu\to\infty} D_\nu=+\infty$ und ...". By Proposition 5.4 indeed $f'(x)=+\infty$, cf. also Remark 5.5. But this a not a consequence of $\lim\limits_{\nu\to\infty} D_\nu=+\infty$ as Example 7.2 shows.

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received: December 11, 2006

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Rostock. Math. Kolloq. **62**, 61–70 (2007)

Subject Classification (AMS)
06A, 06B

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Bell Numbers and Engel's Conjecture

ABSTRACT. In this paper, we prove some new properties of the sequence of the Bell numbers and present some results in connection with Engel's conjecture. In addition, using a new approach we state a stronger conjecture.

KEY WORDS. Partition lattice; Bell number; variance; mean; convexity; concavity.

1 Introduction

A partition of the set $[n] = \{1, 2, \ldots, n\}$ is a collection of nonempty, pairwise disjoint subsets of $[n]$ called *blocks* whose union is $[n]$. A partition π_1 is said to refine another partition π_2, denoted by $\pi_1 \leq \pi_2$, if every block of π_1 is contained in some block of π_2. Hence, the refinement relation is a partial ordering of the set $\prod_n$ of all partitions of $[n]$. The number of partitions of $[n]$ having exactly k blocks is the Stirling number of the second kind $S(n, k)$. The total number of partitions of $[n]$ is the n^{th} Bell number B_n. Therefore,

$$B_n = \sum_{k=1}^{n} S(n, k), \ n \geq 1 .$$

Also, recall Dobinski's formula [1]

$$B_n = \frac{1}{e} \sum_{i=0}^{\infty} \frac{i^n}{i!} .$$

Now, for all $x \in \mathbb{R}$, we set

$$B(x) = \frac{1}{e} \sum_{i=0}^{\infty} \frac{i^x}{i!}, \ x \in \mathbb{R} .$$

Note that the series $\sum_{i \geq 0} \frac{i^x}{i!}$ converges for all $x \in \mathbb{R}$ and $B(n) = B_n$ for all $n \in \mathbb{N}$.

2 New properties of the Bell numbers

Theorem 1 *Let $p \in]1, +\infty[$ and let q be the conjugate exponent of p. Then, for all $x_1, x_2 \in \mathbb{R}$, we have*

$$B(x_1 + x_2) \leq B^{1/p}(px_1) B^{1/q}(qx_2) .$$

Proof: Let Z be the discrete random variable with distribution function

$$P(Z = i) = \frac{1}{e} \cdot \frac{1}{i!}, \ i \in \mathbb{N} .$$

Then

$$E(Z^x) = B(x), \quad \text{for all} \ \ x \in \mathbb{R} . \tag{1}$$

From Hölder's inequality we obtain

$$E\left(Z^{x_1+x_2}\right) \leq E^{1/p}\left(Z^{px_1}\right) \cdot E^{1/q}\left(Z^{qx_2}\right), \ \ \text{for all} \ \ x_{1,}x_2 \in \mathbb{R} .$$

Hence, if we use (1), the result follows immediately. □

Theorem 2 *For all $x_1, \ x_2 \in \mathbb{R}$, we have*

$$2B(x_1 + x_2) \leq B(2x_1) + B(2x_2) .$$

Proof: Using the same discrete random variable in the above proof we have

$$E((Z^{x_1} - Z^{x_2})^2) \geq 0 .$$

Thus

$$2E(Z^{x_1+x_2}) \leq E(Z^{2x_1}) + E(X^{2x_2}) .$$

Consequently, using (1), we get

$$2B(x_1 + x_2) \leq B(2x_1) + B(2x_2) .$$

□

As a consequence of Theorem 1, with $x_1, \ x_2 \in \mathbb{N}$ and $p = q = 2$ we obtain the first following new property of the sequence of the Bell numbers.

Corollary 3 *The inequality*

$$B_{n+m}^2 \leq B_{2n} B_{2m}$$

holds for every $n, \ m \in \mathbb{N}$.

Corollary 4 *The sequence* $(B_{n+1}/B_n)_n$ *is increasing, and, equivalently, the sequence* $(B_n)_n$ *is logarithmically convex, i.e.*

$$B_{n+1}^2 \le B_n B_{n+2}, \ \textit{for all } n \ge 0\,.$$

Proof: The assertion follows from Theorem 1 with $x_1 = \frac{n}{2}$ $x_2 = \frac{n+2}{2}$, and $p = q = 2$. □

Corollary 5 *The sequence* $(B_n)_n$ *is convex, i.e.*

$$2B_n \le B_{n-1} + B_{n+1}, \ \textit{for all } n \ge 1\,.$$

Proof: This inequality follows easily from Theorem 2 with $x_1 = \frac{n+1}{2}$ and $x_2 = \frac{n-1}{2}$. □

Henceforth, let τ_n (resp. σ_n^2) denote the average (resp. the variance) of the number of blocks in a partition of the generic n-set $[n]$, i.e.

$$\tau_n = \frac{1}{B_n}\sum_{k=1}^{n} kS\left(n,k\right)$$

and

$$\sigma_n^2 = \frac{1}{B_n}\sum_{k=1}^{n} k^2 S\left(n,k\right) - \quad \frac{1}{B_n}\sum_{k=1}^{n} kS\left(n,k\right)\Bigg)^2 .$$

Using the recurrence relation

$$S\left(n+1,k\right) = S\left(n,k-1\right) + kS\left(n,k\right) \tag{2}$$

we obtain

$$\tau_n = \frac{B_{n+1}}{B_n} - 1$$

and

$$\sigma_n^2 = \frac{B_{n+2}}{B_n} - \left(\frac{B_{n+1}}{B_n}\right)^2 - 1\,.$$

In studying Alekseev's inequality [4] on the principal ideal of the partition lattice, it was shown in [3] that the inequality is equivalent to

$$\tau_{n_1} + \tau_{n_2} \ge \tau_{n_1+n_2}, \quad \text{for all } n_1, n_2 \in \mathbb{N}\,.$$

Furthermore, K. Engel [5] showed that the inequality above is true if the sequence $(\tau_n)_n$ is concave, and he was led to the conjecture that the sequence $(\tau_n)_n$ is concave, i.e.

$$\tau_n \ge \frac{1}{2}\left(\tau_{n-1} + \tau_{n+1}\right), \text{for all } n \ge 1\,.$$

We verified the last inequality for $n \leq 1500$ using a computer [3], but no general proof has been found yet. The second purpose of this paper is to contribute to the study of this conjecture by using a new approach.

Let, for $x \in \mathbb{R}$,

$$B_n(x) = \sum_{k=0}^{n} S(n,k)\, x^k . \tag{3}$$

It is clear that $(B_n(x))_n$ is a sequence of polynomials, with $B_0(x) \equiv 1$ and $B_n(1) = B_n$ (Bell number). Recall that the polynomial $B_n(x)$ admits n distinct roots, where only one of them is equal to zero and all others are strictly negative. This result is due to L. N. Harper [8] and the detailed proof can be found in [3]. From now on, let $-\alpha_1(n), -\alpha_2(n), \cdots, -\alpha_{n-1}(n)$ denote the $(n-1)$ negative roots of $B_n(x)$, and let $I_n = \{0\} \cup \{-\alpha_i(n), i = 1, ..., n-1\}$. This allows us to write

$$B_n(x) = x \prod_{i=1}^{n-1} (x + \alpha_i(n)) .$$

In this section we assume that $x \notin I_n$. Setting

$$\tau_n(x) = \frac{B_{n+1}(x)}{B_n(x)} - x$$

and

$$\sigma_n^2(x) = \frac{B_{n+2}(x)}{B_n(x)} - \left(\frac{B_{n+1}(x)}{B_n(x)}\right)^2 - x, \tag{4}$$

we have $\tau_n(1) = \tau_n$ and $\sigma_n^2(1) = \sigma_n^2$.

Theorem 6 *For every $n \in \mathbb{N}^*$,*

i) $\tau_n(x) = 1 + \sum_{j=1}^{n-1} \frac{x}{x+\alpha_j(n)} = n - \sum_{k=1}^{n-1} \frac{\alpha_j(n)}{x+\alpha_j(n)}$,

ii) $\sigma_n^2(x) = x\, d(\tau_n(x))$,

where d is the differential operator $\frac{d}{dx}$.

Proof: Without restriction, we only consider here the case when $x > 0$.

i) It is easy to verify from (2) and (3) that

$$B_{n+1}(x) = x\,(d(B_n(x)) + B_n(x)) . \tag{5}$$

It follows that

$$\begin{aligned}\tau_n(x) &= x\frac{d(B_n(x))}{B_n(x)} \\ &= x\ d(\log(B_n(x))) \\ &= x\ d\left(\log x + \sum_{j=1}^{n-1}\log(x+\alpha_j(n))\right) \\ &= 1+\sum_{j=1}^{n-1}\frac{x}{x+\alpha_j(n)} \\ &= n-\sum_{j=1}^{n-1}\frac{\alpha_j(n)}{x+\alpha_j(n)}\ .\end{aligned}$$

To prove ii), we have

$$\begin{aligned}x\ d(\tau_n(x)) &= x\left(\frac{d(B_{n+1}(x))}{B_n(x)}-\frac{B_{n+1}(x)\,d(B_n(x))}{B_n^2(x)}-1\right) \\ &= \frac{B_{n+2}(x)-xB_{n+1}(x)}{B_n(x)}-\frac{B_{n+1}(x)(B_{n+1}(x)-xB_n(x))}{B_n^2(x)}-x \\ &= \frac{B_{n+2}(x)}{B_n(x)}-\left(\frac{B_{n+1}(x)}{B_n(x)}\right)^2-x \\ &= \sigma_n^2(x)\ .\end{aligned}$$

Thus, the theorem is proved. □

Corollary 7 *For every $n \geq 2$,*

a) $1 < \tau_n(x) < n$, *for each* $x > 0$,

b) $0 < \sigma_n^2(x) < \frac{n-1}{4}$, *for each* $x > 0$,

c) *The sequence* $(B_n(x))_n$ *is logarithmically convex for* $x > 0$ *and logarithmically concave for* $x < 0$,

d) *For every* $n \geq 1$, *the polynomials* $\frac{B_{n+1}(x)}{x}$ *and* $\frac{B_n(x)}{x}$ *are coprime,*

e) $\sigma_{n+1}^2(x)+\sigma_{n-1}^2(x)-2\sigma_n^2(x) = x\ d(\tau_{n+1}(x)+\tau_{n-1}(x)-2\tau_n(x))$.

Proof: a) By the fact that $\alpha_j(n)$ is positive for every j, the inequality a) follows immediately from i) of Theorem 6.

b) using i) and ii) of Theorem 6, we obtain

$$\sigma_n^2(x) = x\sum_{j=1}^{n-1}\frac{\alpha_j(n)}{(x+\alpha_j(n))^2}\ . \tag{6}$$

Thus, it is sufficient to notice that the maximum value of the function $x \longrightarrow \frac{\alpha_j(n)x}{(x+\alpha_j(n))^2}$ is $\frac{1}{4}$, for $x > 0$.

c) This result is an immediate consequence of (6). Indeed, if $x > 0$ (*resp.* $x < 0$), then $\sigma_n^2(x) > 0$ (*resp.* $\sigma_n^2(x) < 0$), i.e. using (4)

$$B_{n+2}(x) B_n(x) - B_{n+1}^2(x) - xB_n^2(x) > 0$$

$$(\text{resp.} \quad B_{n+2}(x) B_n(x) - B_{n+1}^2(x) - xB_n^2(x) < 0) .$$

Hence

$$B_{n+2}(x) B_n(x) > B_{n+1}^2(x) \ ,$$

$$(\text{resp.} \quad B_{n+2}(x) B_n(x) < B_{n+1}^2(x)) .$$

From (5), we get

$$B_{n+1}(-\alpha_j(n)) = \alpha_j^2(n) \prod_{\substack{i=1 \\ i \neq j}}^{n-1} (-\alpha_j(n) + \alpha_i(n)) \neq 0 . \tag{7}$$

Thus d) is proved.

To prove e), it is sufficient to use ii) of Theorem 6. □

Corollary 8 *We have*

$$2B_n < B_{n+1} < (n+1)B_n .$$

Proof: Use a) of Corollary 7 and choose $x = 1$. □

Remark 1 Note that the inequality $2B_n < B_{n+1}$ is stronger than the convexity of the sequence $(B_n)_n$.

Let $u_n(x) = \tau_n(x) + x$. Then we have the following result.

Lemma 9 *For every* $n \geq 3$,

$$\frac{1}{u_{n-1}(x)} = \sum_{j=1}^{n-1} \frac{\beta_j(n)}{x + \alpha_j(n)} ,$$

where $\beta_j(n) \in]0, 1[$.

Proof: We have

$$\frac{1}{u_{n-1}(x)} = \frac{B_{n-1}(x)}{B_n(x)} = \frac{\prod_{i=1}^{n-2}(x+\alpha_i(n-1))}{\prod_{i=1}^{n-1}(x+\alpha_i(n))}.$$

By a decomposition into partial fractions, we get

$$\frac{1}{u_{n-1}(x)} = \sum_{j=1}^{n-1} \frac{\beta_j(n)}{x+\alpha_j(n)},$$

where

$$\beta_j(n) = \frac{B_{n-1}(-\alpha_j(n))}{-\alpha_j(n)\prod_{\substack{i=1\\ i\neq j}}^{n-1}(-\alpha_j(n)+\alpha_i(n))}.$$

Moreover, from (7) we obtain

$$\beta_j(n) = \frac{-\alpha_j(n)\, B_{n-1}(-\alpha_j(n))}{B_{n+1}(-\alpha_j(n))}.$$

On the other hand in view of (6) we have

$$\sigma_{n-1}^2(x) = \frac{B_{n+1}(x)}{B_{n-1}(x)} - \left(\frac{B_n(x)}{B_{n-1}(x)}\right)^2 - x < 0, \text{ for all } x<0 \text{ and } x \notin I_{n-1}.$$

Therefore

$$\frac{B_{n-1}(x)}{B_{n+1}(x)} < \left(\frac{B_n(x)}{B_{n+1}(x)}\right)^2 + x\left(\frac{B_{n-1}(x)}{B_{n+1}(x)}\right)^2, \text{ for all } x<0 \text{ and } x\notin I_{n+1}.$$

If we replace x in the above inequality by $-\alpha_j(n)$, then we obtain

$$\beta_j(n)\,(1-\beta_j(n)) > 0,$$

thus $\beta_j(n) \in]0,1[\cdot$ □

Theorem 10 *For every $n \geq 2$,*

$$\tau_{n+1}(x) + \tau_{n-1}(x) - 2\tau_n(x) = x\left(\frac{u_{n-1}(x)}{u_n(x)}\right) d\left(\frac{u_n(x)}{u_{n-1}(x)}\right),$$

with $\frac{u_n(x)}{u_{n-1}(x)} = 1 + x\sum_{j=1}^{n-1}\frac{\beta_j(n)}{(x+\alpha_j(n))^2}$ *and* $\beta_j(n) \in]0,1[$.

Proof: From (5) we obtain

$$\tau_n(x) = x\frac{d(B_n(x))}{B_n(x)}.$$

Hence

$$\begin{aligned}
\tau_{n+1}(x) + \tau_{n-1}(x) - 2\tau_n(x) &= xd(\log(B_{n+1}(x)) + \log(B_{n-1}(x)) - 2\log(B_n(x))) \\
&= xd\left(\log\left(\frac{B_{n+1}(x)B_{n-1}(x)}{B_n^2(x)}\right)\right) \\
&= xd\left(\log\left(\frac{u_n(x)}{u_{n-1}(x)}\right)\right).
\end{aligned}$$

We also have

$$\begin{aligned}
d\left(\frac{1}{u_{n-1}(x)}\right) &= d\left(\frac{B_{n-1}(x)}{B_n(x)}\right) \\
&= \frac{d(B_{n-1}(x))}{B_n(x)} - \frac{d(B_n(x))}{B_n(x)} \cdot \frac{B_{n-1}(x)}{B_n(x)} \\
&= \frac{d(B_{n-1}(x))}{B_{n-1}(x)} \cdot \frac{B_{n-1}(x)}{B_n(x)} - \frac{d(B_n(x))}{B_n(x)} \cdot \frac{B_{n-1}(x)}{B_n(x)} \\
&= \frac{1}{x}\frac{1}{u_{n-1}(x)}(\tau_{n-1}(x) - \tau_n(x)) \\
&= \frac{1}{x}\left(1 - \frac{u_n(x)}{u_{n-1}(x)}\right).
\end{aligned}$$

Hence

$$\frac{u_n(x)}{u_{n-1}(x)} = 1 - x\, d\left(\frac{1}{u_{n-1}(x)}\right).$$

Therefore, from Lemma 9 we obtain

$$\frac{u_n(x)}{u_{n-1}(x)} = 1 + x\sum_{j=1}^{n-1}\frac{\beta_j(n)}{(x+\alpha_j(n))^2}.$$

This completes the proof. □

3 Strong Conjecture

Recall that K. Engel conjectured that the sequence $(\tau_n)_n$ is concave in n, i.e.

$$2\tau_n - \tau_{n-1} - \tau_{n+1} \geq 0, \quad \text{for every } n \geq 2.$$

In this section we show that this conjecture is in fact a consequence of the following stronger conjecture.

Theorem 11 *If the positive roots of the equation* $d\left(\frac{u_n(x)}{u_{n-1}(x)}\right)=0$ *are less than 1, then the sequence* $(\tau_n)_n$ *is concave.*

Proof: Using Theorem 10, we have

$$\frac{u_n(x)}{u_{n-1}(x)} \geq 1 \text{ when } x \geq 0 \quad \text{and} \quad \lim_{x\to+\infty} \frac{u_n(x)}{u_{n-1}(x)} = 1\,.$$

Then, assuming that the positive roots of the equation $d\left(\frac{u_n(x)}{u_{n-1}(x)}\right)=0$ are less than 1, the function $x \mapsto \frac{u_n(x)}{u_{n-1}(x)}$ would be necessarily decreasing in the neighborhood of 1, which means that

$$\tau_{n+1}+\tau_{n-1}-2\tau_n = \left(\frac{u_{n-1}(1)}{u_n(1)}\right) d\left(\frac{u_n(1)}{u_{n-1}(1)}\right) < 0\,.$$

□

Conjecture 1 *For every* $n \geq 1$*, the positive roots of the equation* $d\left(\frac{u_n(x)}{u_{n-1}(x)}\right)=0$ *are all less than 1.*

Using Maple9, we checked this new conjecture for $2 \leq n \leq 400$.

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received: September 21, 2006

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Rostock. Math. Kolloq. **62**, 71–85 (2007)

Subject Classification (AMS)
47H09,47H10

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Generalized Noor iterations with errors for asymptotically nonexpansive mappings

ABSTRACT. In the present paper, we define and study a new three-step iterative schemes with errors. Several strong convergence theorems of this scheme are established for asymptotically nonexpansive mappings. Our results extend and improve the recent ones announced by Osilike and Aniagbosor, Cho et.al, Liu and Kang, Nammanee et al., and many others.

KEY WORDS. asymptotically nonexpansive mapping, uniformly convex Banach space, Mann-type iteration, Ishikawa-type iteration, Noor-type iteration

1 Introduction

Let X be a real Banach space and C be a nonempty subset of X. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|,$$

for all $x, y \in C$ and all $n \geq 1$. The mapping T is called *uniformly L-Lipschitzian* if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leq L\|x - y\|,$$

for all $x, y \in C$ and all $n \geq 1$. It is easy to see that if T is asymptotically nonexpansive, then it is uniformly L-Lipschitzian with the uniform Lipschitz constant $L = \sup\{k_n : n \geq 1\}$.

In 2002, Xu and Noor [10] introduced and studied a three-step scheme to approximate fixed points of asymptotically nonexpansive mappings in a Banach space. Glowinski and Le Tallec [2] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown in [2] that the three-step iterative scheme gives better numerical results than the two-step and one-step approximate iterations. Haubruge, Nguyen and Strodiot [3] studied the convergence analysis of three-step schemes of Glowinski and Le Tallec [2] and applied these schemes to obtain new splitting-type algorithms for solving variation inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions. Thus we conclude that three-step scheme plays an important and significant part in solving various problems, which arise in pure and applied sciences. In 2004, Cho, Zhou, and Guo [1], and Liu and Kang [4] extended the preceding scheme to the three-step iterative scheme with errors and gave weak and strong convergence theorems for asymptotically nonexpansive mappings in a Banach space. Recently, Nammanee, Noor and Suantai [5] defined a three-step iterative scheme with errors which is an extension of schemes in [1] and [4] iterations and gave some weak and strong convergence theorems for asymptotically nonexpansive mappings in a uniformly convex Banach space. The authors of the present paper [6] defined a new three-step iterative schemes and gave some strong convergence theorems for asymptotically nonexpansive mappings. Inspired by the preceding iteration scheme, we define a new iteration scheme with errors as follows.

Let C be a nonempty convex subset of a real Banach space X and $T : C \to C$ be a mapping.
Algorithm 1 For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes, for all $n \geq 1$,

$$\begin{aligned} z_n &= a'_n x_n + b'_n T^n x_n + r_n u_n, \\ y_n &= a_n x_n + b_n T^n x_n + c_n T^n z_n + s_n v_n, \\ x_{n+1} &= \alpha_n x_n + \beta_n T^n x_n + \gamma_n T^n y_n + \delta_n T^n z_n + t_n w_n, \end{aligned} \tag{1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ are appropriate sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n + \delta_n + t_n = a_n + b_n + c_n + s_n = a'_n + b'_n + r_n = 1$, and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C. The iterative schemes (1) is called the *three-step mean value iterative scheme with errors.*

If $\beta_n \equiv 0$, then Algorithm 1 reduces to
Algorithm 2 [5] For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes, for all $n \geq 1$,

$$\begin{aligned} z_n &= a'_n x_n + b'_n T^n x_n + r_n u_n, \\ y_n &= a_n x_n + b_n T^n x_n + c_n T^n z_n + s_n v_n, \\ x_{n+1} &= \alpha_n x_n + \gamma_n T^n y_n + \delta_n T^n z_n + t_n w_n, \end{aligned} \tag{2}$$

where $\{\alpha_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ are appropriate sequences in $[0, 1]$ with $\alpha_n + \gamma_n + \delta_n + t_n = a_n + b_n + c_n + s_n = a'_n + b'_n + r_n = 1$, and $\{u_n\}$,

$\{v_n\}$ and $\{w_n\}$ are bounded sequences in C. The iterative schemes (2) is called the *modified Noor iterative scheme with errors.*

If $\beta_n = \delta_n = b_n \equiv 0$, then Algorithm 1 reduces to
Algorithm 3 [1, 4] For a given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative schemes, for all $n \geq 1$,

$$\begin{aligned} z_n &= a'_n x_n + b'_n T^n x_n + r_n u_n, \\ y_n &= a_n x_n + c_n T^n z_n + s_n v_n, \\ x_{n+1} &= \alpha_n x_n + \gamma_n T^n y_n + t_n w_n, \end{aligned} \tag{3}$$

where $\{\alpha_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ are appropriate sequences in $[0, 1]$ with $\alpha_n + \gamma_n + t_n = a_n + c_n + s_n = a'_n + b'_n + r_n = 1$, and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C. The iterative schemes (3) is called the *Noor iterative scheme with errors.*

2 Auxiliary Lemmas

For convenience, we use the notations $\lim_n \equiv \lim_{n\to\infty}$, $\liminf_n \equiv \liminf_{n\to\infty}$, and $\limsup_n \equiv \limsup_{n\to\infty}$. In the sequel, we shall need the following lemmas.

Lemma 1 ([7], Lemma 1) *Let $\{a_n\}, \{b_n\}$ and $\{\lambda_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \lambda_n)a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_n a_n$ exists.

Lemma 2 *Let X be a real Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ (i.e., $F(T) := \{x \in C : x = Tx\} \neq \varnothing$) and a sequence $\{k_n\}$ of real numbers such that $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence defined by Algorithm 1 with the restrictions that $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n s_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n c_n r_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n r_n < \infty$. Then we have the following conclusions.*

(i) $\lim_n \|x_n - p\|$ *exists for any $p \in F(T)$.*

(ii) $\lim_n d(x_n, F(T))$ *exists, where $d(x, F(T))$ denotes the distance from x to the fixed-point set $F(T)$.*

Proof: Let $p \in F(T)$. We note that $\{u_n - p\}$, $\{v_n - p\}$, and $\{w_n - p\}$ are bounded sequences in C. Let

$$L = \sup\{k_n : n \geq 1\} \text{ and } M = \sup\{\|u_n - p\|, \|v_n - p\|, \|w_n - p\| : n \geq 1\}.$$

By using (1), we have

$$\begin{aligned} \|z_n - p\| &\leq a'_n\|x_n - p\| + b'_n\|T^n x_n - p\| + r_n\|u_n - p\| \\ &\leq (1 - b'_n)\|x_n - p\| + b'_n k_n\|x_n - p\| + Mr_n \\ &\leq (1 + b'_n(k_n - 1))\|x_n - p\| + Mr_n \\ &\leq k_n\|x_n - p\| + Mr_n, \end{aligned} \tag{1}$$

$$\begin{aligned} \|y_n - p\| &\leq a_n\|x_n - p\| + b_n\|T^n x_n - p\| + c_n\|T^n z_n - p\| + s_n\|v_n - p\| \\ &\leq (1 - b_n - c_n)\|x_n - p\| + b_n k_n\|x_n - p\| + c_n k_n\|z_n - p\| + Ms_n \\ &\leq (1 + (b_n + c_n + c_n k_n)(k_n - 1))\|x_n - p\| + M(s_n + c_n r_n k_n) \\ &\leq (1 + (L + 2)(k_n - 1))\|x_n - p\| + M(s_n + Lc_n r_n), \end{aligned} \tag{2}$$

and so

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n\|x_n - p\| + \beta_n\|T^n x_n - p\| + \gamma_n\|T^n y_n - p\| \\ &\quad + \delta_n\|T^n z_n - p\| + t_n\|w_n - p\| \\ &\leq (1 - \beta_n - \gamma_n - \delta_n)\|x_n - p\| + \beta_n k_n\|x_n - p\| \\ &\quad + \gamma_n k_n\|y_n - p\| + \delta_n k_n\|z_n - p\| + Mt_n \\ &\leq (1 + (\beta_n + \gamma_n + \gamma_n k_n(L + 2) + \delta_n(k_n + 1))(k_n - 1))\|x_n - p\| \\ &\quad + M(t_n + \gamma_n k_n s_n + L\gamma_n k_n c_n r_n + \delta_n k_n r_n) \\ &\leq (1 + (L^2 + 3L + 3)(k_n - 1))\|x_n - p\| \\ &\quad + M(t_n + L\gamma_n s_n + L^2\gamma_n c_n r_n + L\delta_n r_n). \end{aligned}$$

By assumption, the conclusions of the lemma follow from Lemma 1. This completes the proof. □

We also need the following lemma proved by Schu [8].

Lemma 3 *Let X be a uniformly convex Banach space, let $\{\lambda_n\}$ be a sequence of real numbers such that $0 < b \leq \lambda_n \leq c < 1$ for all $n \geq 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences of X such that* $\limsup_n \|x_n\| \leq a$, $\limsup_n \|y_n\| \leq a$ *and* $\lim_n \|\lambda_n x_n + (1 - \lambda_n)y_n\| = a$ *for some $a \geq 0$. Then* $\lim_n \|x_n - y_n\| = 0$.

By Schu's Lemma, we have the following lemma.

Lemma 4 *Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences in a uniformly convex Banach space X. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\limsup_n \|x_n\| \leq a$, $\limsup_n \|y_n\| \leq a$, $\limsup_n \|z_n\| \leq a$, and $\lim_n \|\alpha_n x_n + \beta_n y_n + \gamma_n z_n\| = a$, where $a \geq 0$. If $\liminf_n \alpha_n > 0$ and $\liminf_n \beta_n > 0$, then $\lim_n \|x_n - y_n\| = 0$.*

Proof: We may assume without loss of generality that $\alpha_n > 0$ and $\beta_n > 0$ for all $n \in \mathbb{N}$. Let $\{n_k\}$ be a subsequence of $\{n\}$ such that

$$\lim_k \left\| \frac{\alpha_{n_k}}{\alpha_{n_k} + \beta_{n_k}} x_{n_k} + \frac{\beta_{n_k}}{\alpha_{n_k} + \beta_{n_k}} y_{n_k} \right\| = \liminf_n \left\| \frac{\alpha_n}{\alpha_n + \beta_n} x_n + \frac{\beta_n}{\alpha_n + \beta_n} y_n \right\|.$$

Then

$$\begin{aligned} a &= \liminf_k \|\alpha_{n_k} x_{n_k} + \beta_{n_k} y_{n_k} + \gamma_{n_k} z_{n_k}\| \\ &\leq \liminf_k \left((\alpha_{n_k} + \beta_{n_k}) \left\| \frac{\alpha_{n_k}}{\alpha_{n_k} + \beta_{n_k}} x_{n_k} + \frac{\beta_{n_k}}{\alpha_{n_k} + \beta_{n_k}} y_{n_k} \right\| + \gamma_{n_k} \|z_{n_k}\| \right) \\ &\leq \liminf_k (\alpha_{n_k} + \beta_{n_k}) \left\| \frac{\alpha_{n_k}}{\alpha_{n_k} + \beta_{n_k}} x_{n_k} + \frac{\beta_{n_k}}{\alpha_{n_k} + \beta_{n_k}} y_{n_k} \right\| + \limsup_k \gamma_{n_k} \|z_{n_k}\| \\ &\leq \liminf_k (\alpha_{n_k} + \beta_{n_k}) \liminf_n \left\| \frac{\alpha_n}{\alpha_n + \beta_n} x_n + \frac{\beta_n}{\alpha_n + \beta_n} y_n \right\| + a \limsup_k \gamma_{n_k}. \end{aligned}$$

This implies that

$$\begin{aligned} &\liminf_k (\alpha_{n_k} + \beta_{n_k}) a \\ &= (1 - \limsup_k \gamma_{n_k}) a \\ &\leq \liminf_k (\alpha_{n_k} + \beta_{n_k}) \liminf_n \left\| \frac{\alpha_n}{\alpha_n + \beta_n} x_n + \frac{\beta_n}{\alpha_n + \beta_n} y_n \right\|. \end{aligned}$$

Since $\liminf_n (\alpha_n + \beta_n) \geq \liminf_n \alpha_n + \liminf_n \beta_n > 0$, it follows that

$$a \leq \liminf_n \left\| \frac{\alpha_n}{\alpha_n + \beta_n} x_n + \frac{\beta_n}{\alpha_n + \beta_n} y_n \right\| \leq \limsup_n \left\| \frac{\alpha_n}{\alpha_n + \beta_n} x_n + \frac{\beta_n}{\alpha_n + \beta_n} y_n \right\| \leq a.$$

We now observe that

$$\liminf_n \frac{\alpha_n}{\alpha_n + \beta_n} \geq \liminf_n \alpha_n > 0 \quad \text{and} \quad \liminf_n \frac{\beta_n}{\alpha_n + \beta_n} \geq \liminf_n \beta_n > 0.$$

By Lemma 3, we have $\lim_n \|x_n - y_n\| = 0$. This completes the proof. □

The following lemmas are the important ingredients for proving our main results in the next section.

Lemma 5 *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with the nonempty* *fixed-point set $F(T)$ and a sequence $\{k_n\}$ of real numbers such that $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence defined by Algorithm 1 with the restrictions that $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n s_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n c_n r_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n r_n < \infty$. Then we have the following assertions.*

(i) *If $0 < \liminf_n \gamma_n \leq \limsup_n(\beta_n + \gamma_n + \delta_n) < 1$ and $\limsup_n(b_n + c_n) < 1$, then* $\lim_n \|T^n x_n - x_n\| = 0$.

(ii) *If $0 < \liminf_n \delta_n \leq \limsup_n(\beta_n + \gamma_n + \delta_n) < 1$ and $\limsup_n b'_n < 1$, then $\lim_n \|T^n x_n - x_n\| = 0$.*

(iii) *If $0 < \liminf_n \beta_n \leq \limsup_n(\beta_n + \gamma_n + \delta_n) < 1$, then $\lim_n \|T^n x_n - x_n\| = 0$.*

Proof: Let $p \in F(T)$. By Lemma 2, we have $\lim_n \|x_n - p\| = a$ for some $a \geq 0$. Since $\lim_n t_n = 0$,

$$\begin{aligned} a &= \lim_n \|x_{n+1} - p\| \\ &= \lim_n \|(1 - \beta_n - \gamma_n - \delta_n)(x_n - p) + \beta_n(T^n x_n - p) + \gamma_n(T^n y_n - p) \\ &\quad + \delta_n(T^n z_n - p) + t_n(w_n - x_n)\| \\ &= \lim_n \|(1 - \beta_n - \gamma_n - \delta_n)(x_n - p) + \beta_n(T^n x_n - p) \\ &\quad + \gamma_n(T^n y_n - p) + \delta_n(T^n z_n - p)\|. \end{aligned} \tag{3}$$

We first observe that

$$\limsup_n \|T^n x_n - p\| \leq \limsup_n k_n \|x_n - p\| = a. \tag{4}$$

To prove (i), let $\{m_j\}$ be a subsequence of $\{n\}$. We show that there is a subsequence $\{n_k\}$ of $\{m_j\}$ such that $\lim_k \|T^{n_k} y_{n_k} - x_{n_k}\| = 0$.
As $\liminf_n \gamma_n > 0$, $\sum_{n=1}^{\infty} \gamma_n s_n < \infty$, and $\sum_{n=1}^{\infty} \gamma_n c_n r_n < \infty$, $\lim_n s_n = c_n r_n = 0$. By using (2), we have

$$\limsup_j \|T^{m_j} y_{m_j} - p\| \leq \limsup_j k_{m_j} \|y_{m_j} - p\| \leq a. \tag{5}$$

If $\liminf_j \delta_{m_j} > 0$, then $\lim_j r_{m_j} = 0$. By (1), we gives

$$\limsup_j \|T^{m_j} z_{m_j} - p\| \leq \limsup_j k_{m_j} \|z_{m_j} - p\| \leq a. \tag{6}$$

It follows from (3)-(6) and Lemma 4 that

$$\lim_j \|T^{m_j} y_{m_j} - x_{m_j}\| = 0.$$

On the other hand, if $\liminf_j \delta_{m_j} = 0$, then we may extract a subsequence $\{\delta_{n_k}\}$ of $\{\delta_{m_j}\}$ so that $\lim_k \delta_{n_k} = 0$, it follows that

$$\lim_k \delta_{n_k} \|x_{n_k} - p\| = 0 = \lim_k \delta_{n_k} \|T^{n_k} z_{n_k} - p\|.$$

This together with (3) gives

$$\begin{aligned} a = \lim_k \|&(1 - \beta_{n_k} - \gamma_{n_k})(x_{n_k} - p) \\ &+ \beta_{n_k}(T^{n_k} x_{n_k} - p) + \gamma_{n_k}(T^{n_k} y_{n_k} - p)\|. \end{aligned} \tag{7}$$

It follows from (4), (5), (7), and Lemma 4 that

$$\lim_k \|T^{n_k} y_{n_k} - x_{n_k}\| = 0.$$

By double extract subsequence principle,

$$\lim_n \|(x_n - p) - (T^n y_n - p)\| = \lim_n \|T^n y_n - x_n\| = 0. \tag{8}$$

It follows that $\lim_n \|T^n y_n - p\| = a$. Also

$$a = \liminf_n \|T^n y_n - p\| \leq \liminf_n k_n \|y_n - p\| = \liminf_n \|y_n - p\|.$$

From (2), we gives $\limsup_n \|y_n - p\| \leq a$, so that $\lim_n \|y_n - p\| = a$.
Next we prove that

$$\lim_n \|T^n x_n - x_n\| = 0, \tag{9}$$

let $\{\ell_j\}$ be a subsequence of $\{n\}$. It suffices to show that there is a subsequence $\{n_k\}$ of $\{\ell_j\}$ such that $\lim_k \|T^{n_k} x_{n_k} - x_{n_k}\| = 0$. Since $\lim_n s_n = 0$,

$$\begin{aligned} a &= \lim_j \|y_{\ell_j} - p\| \\ &= \lim_j \|(1 - b_{\ell_j} - c_{\ell_j})(x_{\ell_j} - p) + b_{\ell_j}(T^{\ell_j} x_{\ell_j} - p) \\ &\quad + c_{\ell_j}(T^{\ell_j} z_{\ell_j} - p) + s_{\ell_j}(v_{\ell_j} - x_{\ell_j})\| \\ &= \lim_j \|(1 - b_{\ell_j} - c_{\ell_j})(x_{\ell_j} - p) + b_{\ell_j}(T^{\ell_j} x_{\ell_j} - p) + c_{\ell_j}(T^{\ell_j} z_{\ell_j} - p)\|. \end{aligned}$$

If $\liminf_j c_{\ell_j} > 0$, by Lemma 4 and $\limsup_n (b_n + c_n) < 1$, then

$$\lim_j \|T^{\ell_j} z_{\ell_j} - x_{\ell_j}\| = 0. \tag{10}$$

On the other hand, if $\liminf_j c_{\ell_j} = 0$, then we may extract a subsequence $\{c_{n_k}\}$ of $\{c_{\ell_j}\}$ so that $\lim_k c_{n_k} = 0$, it follows that

$$\lim_k c_{n_k} \|T^{n_k} z_{n_k} - x_{n_k}\| = 0. \tag{11}$$

By using (1), we have

$$\begin{aligned}\|T^{n_k}x_{n_k} - x_{n_k}\| &\le \|T^{n_k}x_{n_k} - T^{n_k}y_{n_k}\| + \|T^{n_k}y_{n_k} - x_{n_k}\| \\ &\le k_{n_k}\|x_{n_k} - y_{n_k}\| + \|T^{n_k}y_{n_k} - x_{n_k}\| \\ &\le k_{n_k}b_{n_k}\|T^{n_k}x_{n_k} - x_{n_k}\| + k_{n_k}c_{n_k}\|T^{n_k}z_{n_k} - x_{n_k}\| \\ &\quad + k_{n_k}s_{n_k}\|v_{n_k} - x_{n_k}\| + \|T^{n_k}y_{n_k} - x_{n_k}\|.\end{aligned}$$

This together with (8), (10), and (11) gives

$$\lim_k (1 - k_{n_k}b_{n_k})\|T^{n_k}x_{n_k} - x_{n_k}\| = 0.$$

As $\liminf_n(1 - k_n b_n) = 1 - \limsup_n b_n \ge 1 - \limsup_n(b_n + c_n) > 0$, we have

$$\lim_k \|T^{n_k}x_{n_k} - x_{n_k}\| = 0.$$

By double extract subsequence principle, we obtain (9) and the proof of (i) is finished.

By using a similar method, it can be shown that (ii) is satisfied.

(iii) To show that

$$\lim_n \|T^n x_n - x_n\| = 0, \tag{12}$$

let $\{m_j\}$ be a subsequence of $\{n\}$. It suffices to show that there is a subsequence $\{n_k\}$ of $\{m_j\}$ such that $\lim_k \|T^{n_k}x_{n_k} - x_{n_k}\| = 0$. We consider the following cases.

Case 1: $\liminf_j \gamma_{m_j} > 0$.

Subcase 1.1: $\liminf_j \delta_{m_j} > 0$. Then we obtain (3)-(6). It follows from Lemma 4 that $\lim_j \|T^{m_j}x_{m_j} - x_{m_j}\| = 0$.

Subcase 1.2: $\liminf_j \delta_{m_j} = 0 = \lim_k \delta_{n_k}$, where $\{\delta_{n_k}\} \subset \{\delta_{m_j}\}$. Then we obtain (7), and so

$$\lim_k \|T^{n_k}x_{n_k} - x_{n_k}\| = 0.$$

Case 2: $\liminf_j \gamma_{m_j} = 0$. Choose $\{\gamma_{\ell_k}\} \subset \{\gamma_{m_j}\}$ such that $\lim_k \gamma_{\ell_k} = 0$, it follows that

$$\lim_k \gamma_{\ell_k}\|x_{\ell_k} - p\| = 0 = \lim_k \gamma_{\ell_k}\|T^{\ell_k}y_{\ell_k} - p\|.$$

This together with (3) gives

$$a = \lim_k \|(1 - \beta_{\ell_k} - \delta_{\ell_k})(x_{\ell_k} - p) + \beta_{\ell_k}(T^{\ell_k}x_{\ell_k} - p) + \delta_{\ell_k}(T^{\ell_k}z_{\ell_k} - p)\|. \tag{13}$$

Subcase 2.1: $\liminf_k \delta_{\ell_k} > 0$. By (1), we have $\limsup_k \|T^{\ell_k}z_{\ell_k} - p\| \le a$. It follows from (4), (13) and Lemma 4,

$$\lim_k \|T^{\ell_k}x_{\ell_k} - x_{\ell_k}\| = 0.$$

Subcase 2.2: $\liminf_k \delta_{\ell_k} = 0 = \lim_i \delta_{n_i}$, where $\{\delta_{n_i}\} \subset \{\delta_{\ell_k}\}$. It follows that

$$\lim_i \delta_{n_i} \|T^{n_i} z_{n_i} - p\| = 0.$$

This together with (13) gives

$$a = \lim_i \|(1 - \beta_{n_i})(x_{n_i} - p) + \beta_{n_i}(T^{n_i} x_{n_i} - p)\|.$$

It follows from Lemma 3, $\lim_i \|T^{n_i} x_{n_i} - x_{n_i}\| = 0$. By double extract subsequence principle, we obtain (12). This completes the proof. □

Lemma 6 *Let X be a real Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ of real numbers such that $k_n \geq 1$ and $\lim_n k_n = 1$ and, $\{x_n\}$ be a sequence defined in C by Algorithm 1 with the restrictions that $\lim_n t_n = \lim_n \gamma_n s_n = \lim_n \gamma_n c_n r_n = \lim_n \delta_n r_n = 0$. If $\lim_n \|T^n x_n - x_n\| = 0$, then $\lim_n \|T x_n - x_n\| = 0$.*

Proof: Using (1), we have

$$\begin{aligned}
\|T^n z_n - x_n\| &\leq \|T^n z_n - T^n x_n\| + \|T^n x_n - x_n\| \\
&\leq k_n \|z_n - x_n\| + \|T^n x_n - x_n\|, \\
&\leq (b'_n k_n + 1)\|T^n x_n - x_n\| + r_n k_n \|u_n - x_n\|,
\end{aligned}$$

$$\begin{aligned}
\|T^n y_n - x_n\| &\leq \|T^n y_n - T^n x_n\| + \|T^n x_n - x_n\| \\
&\leq k_n \|y_n - x_n\| + \|T^n x_n - x_n\|, \\
&\leq b_n k_n \|T^n x_n - x_n\| + c_n k_n \|T^n z_n - x_n\| \\
&\quad + s_n k_n \|v_n - x_n\| + \|T^n x_n - x_n\| \\
&\leq (b_n k_n + c_n b'_n k_n^2 + c_n k_n + 1)\|T^n x_n - x_n\| \\
&\quad + s_n k_n \|v_n - x_n\| + c_n r_n k_n^2 \|u_n - x_n\|,
\end{aligned}$$

and so

$$\begin{aligned}
&\|x_{n+1} - T^n x_{n+1}\| \\
&\le \|x_{n+1} - x_n\| + \|T^n x_{n+1} - T^n x_n\| + \|T^n x_n - x_n\| \\
&\le (1 + k_n)\|x_{n+1} - x_n\| + \|T^n x_n - x_n\| \\
&\le \beta_n(1 + k_n)\|T^n x_n - x_n\| + \gamma_n(1 + k_n)\|T^n y_n - x_n\| \\
&\quad + \delta_n(1 + k_n)\|T^n z_n - x_n\| + t_n(1 + k_n)\|w_n - x_n\| + \|T^n x_n - x_n\| \\
&\le \beta_n(1 + k_n)\|T^n x_n - x_n\| \\
&\quad + \gamma_n(1 + k_n)(b_n k_n + c_n b'_n k_n^2 + c_n k_n + 1)\|T^n x_n - x_n\| \\
&\quad + \gamma_n s_n k_n(1 + k_n)\|v_n - x_n\| + \gamma_n c_n r_n(1 + k_n)k_n^2\|u_n - x_n\| \\
&\quad + \delta_n(1 + k_n)(b'_n k_n + 1)\|T^n x_n - x_n\| + \delta_n r_n(1 + k_n)k_n\|u_n - x_n\| \\
&\quad + t_n(1 + k_n)\|w_n - x_n\| + \|T^n x_n - x_n\| \to 0.
\end{aligned}$$

Thus

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &\le \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - Tx_{n+1}\| \\
&\le \|x_{n+1} - T^{n+1}x_{n+1}\| + k_1\|T^n x_{n+1} - x_{n+1}\| \to 0,
\end{aligned}$$

which implies $\lim_n \|Tx_n - x_n\| = 0$. This completes the proof. □

3 Main results

In this section, we establish several strong convergence theorems of the three-step mean value iterative scheme with errors for asymptotically nonexpansive mappings.

Theorem 7 *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ and a sequence $\{k_n\}$ of real numbers such that $k_n \ge 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

(i) $0 < \liminf_n \gamma_n \le \limsup_n(\beta_n + \gamma_n + \delta_n) < 1$,

(ii) $\limsup_n(b_n + c_n) < 1$, *and*

(iii) $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$, $\sum_{n=1}^{\infty} c_n r_n < \infty$, $\sum_{n=1}^{\infty} \delta_n r_n < \infty$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Let $\{x_n\}$ be a given sequence in C. Recall that a mapping $T : C \to C$ with the nonempty fixed-point set $F(T)$ in C satisfies *Condition (A) with respect to the sequence* $\{x_n\}$ ([9]) if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$f(d(x_n, F(T))) \leq \|x_n - Tx_n\|, \text{ for all } n \geq 1.$$

Proof. By Lemma 5(i) and Lemma 6, we have

$$\lim_n \|Tx_n - x_n\| = 0.$$

Let f be a nondecreasing function corresponding to Condition (A) with respect to $\{x_n\}$. Then

$$f(d(x_n, F(T))) \leq \|Tx_n - x_n\| \to 0,$$

and so

$$d(x_n, F(T)) \to 0.$$

Therefore, the conclusion of the theorem follows exactly from [6]. This completes the proof. □

Remark 8 Suppose we rewrite our scheme by treating the additional terms as error terms in the sense of Xu [11] in this way: $x_1 \in C$,

$$\begin{aligned} z_n &= a'_n x_n + b'_n T^n x_n + r_n u_n, \\ y_n &= a_n x_n + c_n T^n z_n + (b_n + s_n)\left(\frac{b_n}{b_n + s_n} T^n x_n + \frac{s_n}{b_n + s_n} v_n\right), \\ x_{n+1} &= \alpha_n x_n + \gamma_n T^n y_n + (\beta_n + \delta_n + t_n) \\ &\quad \times \left(\frac{\beta_n}{\beta_n + \delta_n + t_n} T^n x_n + \frac{\delta_n}{\beta_n + \delta_n + t_n} T^n z_n + \frac{t_n}{\beta_n + \delta_n + t_n} w_n\right), \end{aligned}$$

for all $n \geq 1$. To obtain a strong convergence theorem by Theorem 2.4 of [1], we are restricted to the following

$$\sum_{n=1}^{\infty} (\beta_n + \delta_n + t_n) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} (b_n + s_n) < \infty,$$

from which $\lim_n \beta_n = \lim_n \delta_n = \lim_n b_n = 0$, $\sum_{n=1}^{\infty} s_n < \infty$, and $\sum_{n=1}^{\infty} t_n < \infty$. But our Theorem 7 still gives the result for more general restriction. For example, our result is applicable to the case of $\beta_n = \delta_n = b_n = 1/4$ and $s_n = t_n = 1/2^n$.

Consequently, we obtain the following corollaries. When $\beta_n \equiv 0$, we have

Corollary 9 *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with* *the nonempty fixed-point set and a sequence* $\{k_n\}$ *of real numbers such that* $k_n \geq 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{x_n\}$ *be a sequence in* C *defined by Algorithm 2 with the following restrictions:*

(i) $0 < \liminf_n \gamma_n \leq \limsup_n(\gamma_n + \delta_n) < 1$,

(ii) $\limsup_n(b_n + c_n) < 1$, *and*

(iii) $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$, $\sum_{n=1}^{\infty} c_n r_n < \infty$, $\sum_{n=1}^{\infty} \delta_n r_n < \infty$.

If T *satisfies Condition (A) with respect to the sequence* $\{x_n\}$, *then* $\{x_n\}$ *converges strongly to a fixed point of* T.

When $\beta_n = \delta_n = b_n \equiv 0$ in Theorem 7, we also have

Corollary 10 *Let* X *be a uniformly convex Banach space and* C *be a nonempty closed convex subset of* X. *Let* $T : C \to C$ *be an asymptotically nonexpansive mapping with the nonempty fixed-point set and a sequence* $\{k_n\}$ *of real numbers such that* $k_n \geq 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{x_n\}$ *be a sequence in* C *defined by Algorithm 3 with the following restrictions:*

(i) $0 < \liminf_n \gamma_n \leq \limsup_n \gamma_n < 1$,

(ii) $\limsup_n c_n < 1$, *and*

(iii) $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$, $\sum_{n=1}^{\infty} c_n r_n < \infty$.

If T *satisfies Condition (A) with respect to the sequence* $\{x_n\}$, *then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Remark 11 1. Corollary 9 extends and improves Theorem 2.3 of [5] in the following ways:

   (i) The condition $\liminf_n c_n > 0$ is removed.

   (ii) The restriction $\sum_{n=1}^{\infty} r_n < \infty$ is weakened and replaced by $\sum_{n=1}^{\infty} c_n r_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n r_n < \infty$.

   (iii) The complete continuity imposed on T is replaced by the more general Condition (A) with respect to $\{x_n\}$ (see also [1, Corollary 2.5]).

2. Corollary 10 extends and improves Theorem 2.4 of [1]. The restriction $\sum_{n=1}^{\infty} r_n < \infty$ is weakened and replaced by $\sum_{n=1}^{\infty} c_n r_n < \infty$.

3. Corollary 10 also extends and improves Theorem 3.2 of [4] in the following ways:

   (i) The semi-compactness imposed on T is weakened by assuming that T satisfies Condition (A) with respect to $\{x_n\}$ [1, Corollary 2.5].

   (ii) The condition $\lim_n c_n = 0$ is weakened and replaced by $\limsup_n c_n < 1$.

Next, as consequences of Lemma 5(ii), (iii) and Lemma 6, we have the following theorems.

Theorem 12 *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set and a sequence $\{k_n\}$ of real numbers such that $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

(i) $0 < \liminf_n \delta_n \leq \limsup_n(\beta_n + \gamma_n + \delta_n) < 1$,

(ii) $\limsup_n b'_n < 1$, *and*

(iii) $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} \alpha_n s_n < \infty$, $\sum_{n=1}^{\infty} r_n < \infty$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 13 *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping with the nonempty fixed-point set and a sequence $\{k_n\}$ of real numbers such that $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be a sequence in C defined by Algorithm 1 with the following restrictions:*

(i) $0 < \liminf_n \beta_n \leq \limsup_n(\beta_n + \gamma_n + \delta_n) < 1$ *and*

(ii) $\sum_{n=1}^{\infty} t_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n s_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n c_n r_n < \infty$, $\sum_{n=1}^{\infty} \delta_n r_n < \infty$.

If T satisfies Condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Remark 14 By using the same ideas and techniques, we can also discuss the weak convergence for asymptotically nonexpansive mappings with errors and thereby improve the corresponding results obtained by Cho, Zhou and Guo [1], Liu and Kang [4], and Nammanee, Noor and Suantai [5].

4 Acknowledgement

The second author is supported by the Thailand Research Fund and the Commission on Higher Education under grant MRG4980022.

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received: February 2, 2007

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Rostock. Math. Kolloq. **62**, 87–106 (2007) Subject Classification (AMS)
35Q51, 35Q58, 37K50

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Integral Bifurcation Method and Its Application for Solving the Modified Equal Width Wave Equation and Its Variants

ABSTRACT. In this paper, a improved method named the integral bifurcation is introduced. In order to demonstrate its effectiveness for obtaining travelling waves of the nonlinear wave equations, we studied the modified equal width wave equation and its variants by this new method. Under the different parameter conditions, many integral bifurcations are obtained. According to these integral bifurcations, different kinds of travelling wave solutions are figured out. Compared with [1], many new travelling wave solutions are obtained.

KEY WORDS. integral bifurcation method, the modified equal width equation, integral bifurcations, travelling wave solutions

1 Introduction

In recent years, the sine-cosine method (see Refs. [1–4] and cited therein), the tanh-function method (see Refs. [5–8] and cited therein) and the bifurcation theory of the planar dynamical system (see Refs. [11–21] and cited therein) have been often used to study the problem of all kinds of travelling wave solutions in the nonlinear wave equation domain. These mathematical methods have been, and continue to be, popular tools for nonlinear analysis. However, by using the sine-cosine and tanh-function methods to solve nonlinear wave equations, we cannot obtain the solutions of the type of elliptic function. Among these three mathematical methods, the bifurcation theory of the planar dynamical system is acceptable on discussion of the existence of travelling wave solutions, using this method, we can obtain all kinds of travelling wave solutions, but its analysis of phase portraits and discussion of bifurcation are more complicated. Therefore, in this paper, we shall introduce a improved method (are also called simplified method) named the integral bifurcation based on the bifurcation theory of the planar dynamical system. This improved method needn't make complicated analysis of phase portraits like the bifurcation theory and is easy enough in practice. In order to demonstrate its effectiveness for obtaining travelling waves of the nonlinear wave equations, we shall consider the following nonlinear modified equal width (MEW) wave equation

$$u_t + a(u^3)_x + bu_{xxt} = 0\,, \tag{1.1}$$

and its variants

$$u_t + a(u^n)_x - b(u^n)_{xxt} = 0\,, \tag{1.2}$$

and

$$u_t + a(u^{-n})_x - b(u^{-n})_{xxt} = 0\,, \tag{1.3}$$

where a, b nonzero real parameters, n is positive integer and $n > 1$.

The modified equal width (MEW) wave equation has been discussed in Refs. [1, 9, 10]. Just as A. M. Wazwaz said, the MEW equation, which is related to the regularized long wave (RLW) equation [22], has solitary waves with both positive and negative amplitudes, all of which have the same width. The MEW equation is a nonlinear wave equation with cubic nonlinearity with a pulse-like solitary wave solution. The MEW equation's variants, (1.2) and (1.3) can be reduced to $K(m, n)$ type equations which is well known, so these two equations are also very good application.

By using tanh and sine-cosine methods, A. M. Wazwaz studied the equations (1.1), (1.2) and (1.3), many solutions including compactons and periodic solutions are given (see [1]). In fact, by using the integral bifurcation method, we shall obtain more exact travelling wave solutions.

Making the transformation $u(x, t) = \phi(x - ct) = \phi(\xi)$, then substituting $\phi(x - ct)$ into (1.1), (1.2) and (1.3) respectively, we obtain the following three nonlinear ODE equations

$$-c\phi' + a(\phi^3)' - bc\phi''' = 0\,, \tag{1.4}$$

and

$$-c\phi' + a(\phi^n)' + bc(\phi^n)''' = 0\,, \tag{1.5}$$

and

$$-c\phi' + a(\phi^{-n})' + bc(\phi^{-n})''' = 0\,, \tag{1.6}$$

where "$'$" is the derivative with respect to ξ (i.e. $\phi' = \phi_\xi$) and c is wave speed.

Integrating (1.4), (1.5) and (1.6) once and setting the integral constant as zero, we obtain the following three wave equations, respectively

$$-c\phi + a\phi^3 - bc\phi'' = 0\,, \tag{1.7}$$

and

$$-c\phi + a\phi^n + bcn(n-1)\phi^{n-2}(\phi')^2 + bcn\phi^{n-1}\phi'' = 0\,, \tag{1.8}$$

and

$$-c\phi + a\phi^{-n} + bcn(n+1)\phi^{-(n+2)}(\phi')^2 - bcn\phi^{-(n+1)}\phi'' = 0\,. \tag{1.9}$$

Letting $\phi' = y$, the equations (1.7), (1.8) and (1.9) become the following three two-dimensional systems, respectively

$$\frac{d\phi}{d\xi} = y, \qquad \frac{dy}{d\xi} = -\frac{1}{b}\phi + \frac{a}{bc}\phi^3\,, \tag{1.10}$$

and

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{c\phi - a\phi^n - bcn(n-1)\phi^{n-2}y^2}{bcn\phi^{n-1}}\,, \tag{1.11}$$

and

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{-c\phi^{n+3} + a\phi^2 + bcn(n+1)y^2}{bcn\phi}\,. \tag{1.12}$$

Systems (1.10), (1.11) and (1.12) are all integral systems. Clearly, system (1.10) has the following first integral

$$y^2 = -\frac{1}{b}\phi^2 + \frac{a}{2bc}\phi^4 + C\,, \tag{1.13}$$

where C is integral constant. We define

$$F_1(\phi, y^2) = y^2 + \frac{1}{b}\phi^2 - \frac{a}{2bc}\phi^4\,. \tag{1.14}$$

System (1.11) has the following first integral

$$y^2 = \frac{2}{bn(n+1)}\phi^{3-n} - \frac{a}{bcn^2}\phi^2 + C\phi^{2-2n}\,. \tag{1.15}$$

Similarly we define

$$F_2(\phi, y^2) = \phi^{2n-2}y^2 + \frac{a}{bcn^2}\phi^{2n} - \frac{2}{bn(n+1)}\phi^{n+1}\,. \tag{1.16}$$

System (1.12) has the following first integral

$$y^2 = \frac{2}{bn(n-1)}\phi^{n+3} - \frac{a}{bcn^2}\phi^2 + C\phi^{2n+2}\,. \tag{1.17}$$

We also define

$$F_3(\phi, y^2) = \frac{y^2}{\phi^{2n+2}} + \frac{a}{bcn^2}\phi^{-2n} - \frac{2}{bn(n-1)}\phi^{-(n-1)}\,. \tag{1.18}$$

In the next,we shall introduce integral bifurcation method. By using this method, under the different parameter conditions and choosing the proper integral constant C and using (1.13), (1.15) and (1.17), we shall derive all kinds of integral bifurcations. Utilizing these integral bifurcations, we can obtain all kinds of travelling wave solutions of (1.1), (1.2) and (1.3).

The rest of this paper is organized as follows: In Section 2, we shall introduce the integral bifurcation method. In section 3, by using the integral bifurcation method, we shall derive the travelling wave solutions of equation (1.1). In section 4, by using the integral bifurcation method, we will derive the travelling wave solutions of equation (1.2). In section 5, by using the integral bifurcation method, we shall derive the travelling wave solutions of equation (1.3).

2 Integral bifurcation method

For a given $(n+1)-$dimensional nonlinear partial differential equation

$$E[t, x_i, u_{x_i}, u_{x_i x_i}, u_{x_i x_j}, u_{tt}, \cdots] = 0, \quad (i, j = 1, 2, \cdots, n)\,. \tag{2.1}$$

The integral bifurcation method simply proceeds as follows:

Step1. Making a transformation $u(t, x_1, x_2, \cdots, x_n) = \phi(\xi)$, $\xi = \sum_{i=1}^{n} \mu_i x_i - ct$, (2.1) can be reduced to a nonlinear ODE

$$P(\xi, \phi, \phi_\xi, \phi_{\xi\xi}, \phi_{\xi\xi\xi} \cdots) = 0\,, \tag{2.2}$$

where μ_i, $(i = 1, 2, \cdots, n)$ are arbitrary nonzero constants. After integrating Eq. (2.2) several times, if it can be reduced to the following second-order nonlinear ODE

$$G(\phi, \phi_\xi, \phi_{\xi\xi}) = 0\,, \tag{2.3}$$

then we go on the next process.

Step2. Let $\phi_\xi = \frac{d\phi}{d\xi} = y$. Eq. (2.3) can be reduced to a two-dimensional planar systems

$$\frac{d\phi}{d\xi} = y, \qquad \frac{dy}{d\xi} = f(\phi, y)\,, \tag{2.4}$$

where $f(\phi, y)$ is an integral expression or a fraction. If $f(\phi, y)$ is a fraction such as $f(\phi, y) = \frac{f^*(\phi,y)}{g(\phi)}$ and $g(\phi_s) = 0$, then $\phi_{\xi\xi}$ (i.e $\frac{dy}{d\xi}$) does not exist when $\phi = \phi_s$. In this case, we make a transformation $d\xi = g(\phi)d\tau$, Eq. (2.4) can be rewritten as

$$\frac{d\phi}{d\tau} = g(\phi)y, \qquad \frac{dy}{d\tau} = f^*(\phi, y)\,, \tag{2.5}$$

where τ is a parameter. If the system (2.4) is an integral system, then Eqs. (2.4) and (2.5) have the same first integral as the follows

$$H(\phi, y) = h\,, \tag{2.6}$$

where h is integral constant. Commonly, the function y of (2.6) is satisfied the following relationship:

$$y = y(\phi, h)\,. \tag{2.7}$$

Substituting (2.7) into the first equation of (2.4) and integrating it, we obtain

$$\int_{\phi(0)}^{\phi} \frac{d\varphi}{y(\varphi, h)} = \int_0^{\xi} d\nu\,, \tag{2.8}$$

where $\phi(0)$ and 0 are initial constants. Taking proper initial constants and integrating equation (2.8), we can obtain exact travelling wave solution of Eq. (2.1). In fact, the initial constants can be taken by some extreme points or inflection points of the travelling waves. In other words, $\phi(0)$ is root of the equation (2.7) when $y = 0$ or the equation $\frac{dy}{d\xi} = 0$. Particularly, the initial constants can be also taken by $(\phi_s, 0)$ and a beforehand given $(\phi(0), \xi_0)$. As the value of parameters of Eq. (2.1) and constant h of Eqs. (2.6), (2.7) are varied, so are the integral expression (2.8). Therefore, we call these integral expressions integral bifurcations. The different integral bifurcations correspond to different travelling wave solutions. This is the whole process of the integral bifurcation method. Using this method, we shall investigate travelling wave solutions of the equations (1.1), (1.2) and (1.3). See the below computations.

3 Travelling wave solutions of the equation (1.1)

It is easy to see that the system (1.10) has three equilibrium points $(0,0)$ and $(\pm\sqrt{\frac{c}{a}}, 0)$ as $ac > 0$. From (1.14), we have

$$F_1(0,0) = 0, \qquad F_1\left(\sqrt{\frac{c}{a}}, 0\right) = F_1\left(-\sqrt{\frac{c}{a}}, 0\right) = \frac{c}{2ab}\,. \tag{3.1}$$

According to the analysis of the section 2, we shall calculate the explicit expressions of all kinds of travelling wave solutions of (1.1).

3.1 Under the conditions of $ac > 0,\ b > 0$,

(1) taking $C = F_1(\pm\sqrt{\frac{c}{a}}, 0) = \frac{c}{2ab}$ and substituting it into (1.13), it yields

$$y = \pm\sqrt{\frac{a}{2bc}}\left(\phi^2 - \frac{c}{a}\right)\,. \tag{3.2}$$

Letting $\frac{dy}{d\xi} = 0$ in (1.10), we obtain $\phi(0) = 0$. Under the initial condition $\phi(0) = 0$, substituting (3.2) into (2.8), we obtain the following integral bifurcations

$$\int_0^{\phi} \frac{d\phi}{\frac{c}{a} - \phi^2} = \sqrt{\frac{a}{2bc}} \int_0^{\xi} d\xi, \quad \text{for} \quad \xi \geq 0\,, \tag{3.3}$$

$$\int_{\phi}^{0} \frac{d\phi}{\frac{c}{a} - \phi^2} = -\sqrt{\frac{a}{2bc}} \int_{\xi}^{0} d\xi, \quad \text{for} \quad \xi < 0\,. \tag{3.4}$$

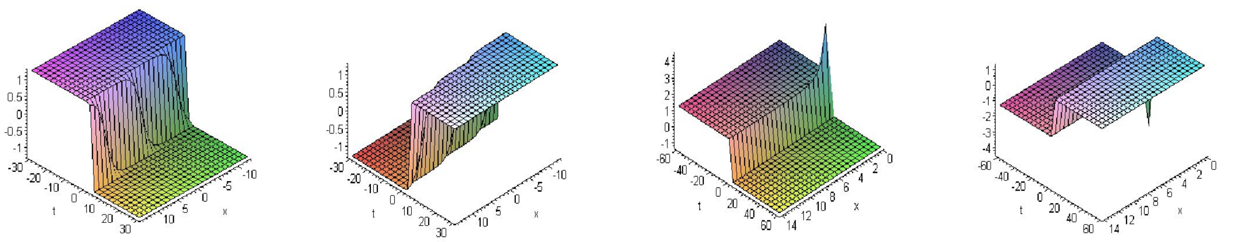
Integrating (3.3) and (3.4), we obtain two couple of kink and anti-kink wave solutions,

$$u_1(x,t) = \phi(x - ct) = \pm\sqrt{\frac{c}{a}}\ \tanh \frac{1}{\sqrt{2b}}(x - ct)\,, \tag{3.5}$$

and

$$u_2(x,t) = \phi(x - ct) = \pm\sqrt{\frac{c}{a}}\ \coth \frac{1}{\sqrt{2b}}(x - ct)\,, \tag{3.6}$$

the 3D graphs of kink and anti-kink wave solutions are shown in Fig. 1. In the graphs, the abscissa axis is t, the ordinate axis is x and the vertical axis is u.



(a) *kink wave 1* (b) *anti-kink wave 1* (c) *kink wave 2* (d) *anti-kink wave 2*

Fig. 1: The 3D graphs of (3.5) and (3.6) as $a = 2.5,\ b = 2,\ c = 4,\ C = 0$.

(2) When $0 < C < \frac{c}{2ab}$, from (1.13), it yields

$$y = \pm\sqrt{\frac{a}{2bc}}\sqrt{\frac{2bcC}{a} - \frac{2c}{a}\phi^2 + \phi^4} = \pm\sqrt{\frac{a}{2bc}}\sqrt{(\alpha^2 - \phi^2)(\beta^2 - \phi^2)}\,, \tag{3.7}$$

where $\alpha^2 = \frac{c}{a}(1+\sqrt{1 - \frac{2abC}{c}})$, $\beta^2 = \frac{c}{a}(1-\sqrt{1 - \frac{2abC}{c}})$ and $\alpha > \beta > \phi > 0$. Taking the initial conditions $\phi(0) = 0$ and substituting (3.7) into the (2.8), we obtain the following integral bifurcations,

$$\int_0^{\phi} \frac{d\phi}{\sqrt{(\alpha^2 - \phi^2)(\beta^2 - \phi^2)}} = \sqrt{\frac{a}{2bc}} \int_0^{\xi} d\xi \quad \text{for } \xi \geq 0\,. \tag{3.8}$$

$$\int_{\phi}^{0} \frac{d\phi}{\sqrt{(\alpha^2 - \phi^2)(\beta^2 - \phi^2)}} = -\sqrt{\frac{a}{2bc}} \int_{\xi}^{0} d\xi, \quad \text{for } \xi < 0\,. \tag{3.9}$$

By using the elliptic integral formulas, we obtain the following a family of periodic wave solutions,

$$u(x,t)=\phi(x-ct)=\beta\ sn(\omega_1(x-ct),k_1)\,, \tag{3.10}$$

where $\omega_1=\pm\alpha\sqrt{\frac{a}{2bc}}$, $k_1=\frac{\beta}{\alpha}$.

(3) Taking $C=0$, from (1.13) we have

$$y=\pm\sqrt{\frac{a}{2bc}}\phi\sqrt{\phi^2-\frac{2c}{a}}\,, \tag{3.11}$$

and $\phi_{1,2}(0)=\pm\sqrt{\frac{2c}{a}}$. Similarly, Under the initial condition $(\phi(0),\xi_0)=(\pm\sqrt{\frac{2c}{a}},\xi_0)$, substituting (3.11) into (2.8) and integrating it, we obtain a periodic wave solution:

$$u(x,t)=\phi(\xi)=\sqrt{\frac{2c}{a}}\sec\frac{1}{\sqrt{b}}(x-ct-\xi_0)\,, \tag{3.12}$$

where ξ_0 is an arbitrary constant. Especially, when $\xi_0=0$ or $\xi_0=\frac{\pi}{2}$, we obtain the following two results which are the same as in Ref. [1],

$$u(x,t)=\phi(\xi)=\sqrt{\frac{2c}{a}}\sec\frac{1}{\sqrt{b}}(x-ct)\,, \tag{3.13}$$

or

$$u(x,t)=\phi(\xi)=\sqrt{\frac{2c}{a}}\csc\frac{1}{\sqrt{b}}(x-ct)\,. \tag{3.14}$$

3.2 Under the conditions of $ac>0,\ b<0$,

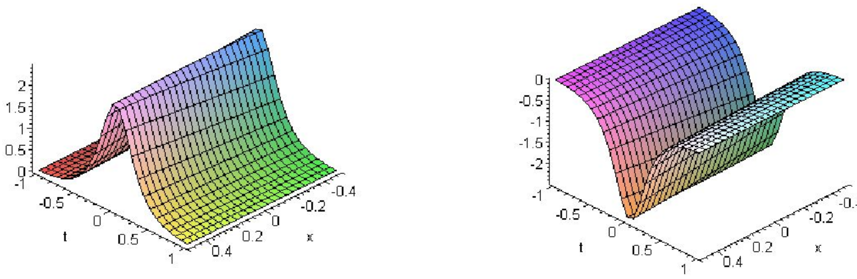
(1) taking $C=0$, from (1.13), it yields

$$y=\pm\sqrt{-\frac{a}{2bc}}\phi\sqrt{\frac{2c}{a}-\phi^2}\,, \tag{3.15}$$

and $\phi(0)=\pm\sqrt{\frac{2c}{a}}$. Under the initial condition $\phi(0)=\pm\sqrt{\frac{2c}{a}}$, substituting (3.15) into (2.8) and integrating it, we obtain two smooth solitary wave solutions,

$$u(x,t)=\phi(\xi)=\pm\sqrt{\frac{2c}{a}}\operatorname{sech}\frac{1}{\sqrt{-b}}(x-ct)\,, \tag{3.16}$$

the 3D graphs of solitary wave solutions of (3.16) are shown in Fig. 2. In the graphs, the abscissa axis is t, the ordinate axis is x and the vertical axis is u.



(a) *solitary wave of peak form* (b) *solitary wave of valley form*

Fig. 2: The 3D graphs of (3.16) as $a = 5,\ b = -8,\ c = 15,\ C = 0,\ x \in (-0.5, 0.5)$.

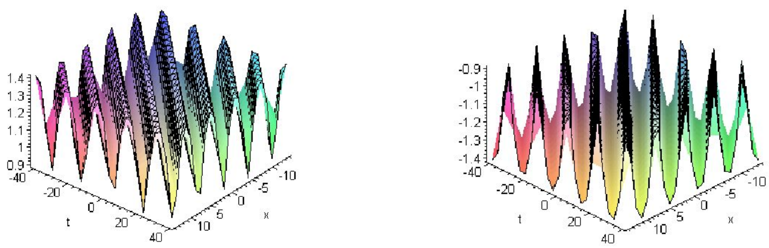
(2) When $\frac{c}{2ab} < C < 0$, from (3.13), it yields

$$y = \pm\sqrt{-\frac{a}{2bc}}\sqrt{(\alpha^2 - \phi^2)(\phi^2 - \beta^2)}\,, \tag{3.17}$$

and $\phi(0) = \alpha$, where $\alpha^2,\ \beta^2$ are given above. Corresponding to the Eq. (3.17), we obtain two families of periodic wave solutions of the type of elliptic function,

$$u(x,t) = \phi(x - ct) = \pm\sqrt{\alpha^2 - (\alpha^2 - \beta^2)sn^2[\omega_2(x - ct), k_2]}\,, \tag{3.18}$$

where $\omega_2 = \alpha\sqrt{-\frac{a}{2bc}},\ k_2 = \sqrt{\frac{\alpha^2-\beta^2}{\alpha^2}}$ and the 3D graphs of (3.18) are shown in Fig. 3. In the graphs, the abscissa axis is t, the ordinate axis is x and the vertical axis is u.



(a) + (b) −

Fig. 3: The 3D graphs of (3.18) as $a = 2.5,\ b = -2,\ c = 4,\ C = -0.3,\ x \in (-14, 14)$.

(3) When $C > 0$, from (1.13), we obtain

$$y = \pm\sqrt{-\frac{a}{2bc}}\sqrt{(\beta^2 + \phi^2)(\alpha^2 - \phi^2)}\,, \tag{3.19}$$

and $\phi(0) = \alpha$, where $\alpha^2 > 0,\ \beta^2 < 0$ are given above. Corresponding to the Eq. (3.19), we obtain a family of periodic wave solutions,

$$u(x,t) = \phi(x-ct) = \alpha\ cn[\omega_3(x-ct), k_3]\,, \tag{3.20}$$

where $\omega_3 = \sqrt{-\frac{a(\alpha^2+\beta^2)}{2bc}},\ k_3 = \sqrt{\frac{\alpha^2}{\alpha^2+\beta^2}}$.

Under the other parameter conditions, according to the above results, we can obtain the other travelling wave solutions without difficulty:

(i) When $ac > 0,\ b < 0,\ C = \frac{c}{2ab}$, from (3.5) and (3.6), we obtain

$$u(x,t) = \phi(x-ct) = \pm i\sqrt{\frac{c}{a}}\ \tan\frac{1}{\sqrt{-2b}}(x-ct)\,, \tag{3.21}$$

or

$$u(x,t) = \phi(x-ct) = \mp i\sqrt{\frac{c}{a}}\ \cot\frac{1}{\sqrt{-2b}}(x-ct)\,, \tag{3.22}$$

where $i = \sqrt{-1}$.

(ii) When $ac < 0,\ b > 0,\ C = 0$, from (3.13), we obtain

$$u(x,t) = \phi(\xi) = \pm i\sqrt{-\frac{2c}{a}}\ \text{sech}\,\frac{1}{\sqrt{b}}(x-ct)\,. \tag{3.23}$$

(iii) When $ac < 0,\ b > 0,\ C = \frac{a}{2bc}$, from (3.5) and (3.6), we obtain

$$u(x,t) = \phi(x-ct) = \pm i\sqrt{-\frac{c}{a}}\ \tanh\frac{1}{\sqrt{2b}}(x-ct)\,, \tag{3.24}$$

or

$$u(x,t) = \phi(x-ct) = \mp i\sqrt{-\frac{c}{a}}\ \coth\frac{1}{\sqrt{2b}}(x-ct)\,. \tag{3.25}$$

(iv) When $ac < 0,\ b < 0,\ C = 0$, from (3.16), we obtain

$$u(x,t) = \phi(\xi) = \pm i\sqrt{-\frac{2c}{a}}\ \text{sech}\,\frac{1}{\sqrt{-b}}(x-ct)\,. \tag{3.26}$$

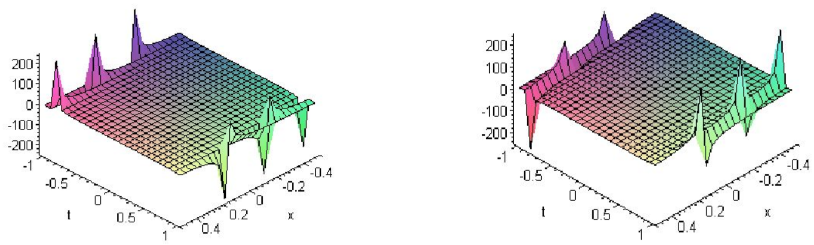
(v) When $ac < 0,\ b < 0,\ C = \frac{a}{2bc}$, from (3.5) and (3.6), we obtain

$$u(x,t) = \phi(x-ct) = \pm\sqrt{-\frac{c}{a}}\ \tan\frac{1}{\sqrt{-2b}}(x-ct)\,, \tag{3.27}$$

or

$$u(x,t) = \phi(x-ct) = \mp\sqrt{-\frac{c}{a}}\ \cot\frac{1}{\sqrt{-2b}}(x-ct)\,. \tag{3.28}$$

The 3D graphs of travelling wave solutions of the type of tangent function are shown in Fig. 4. In the graphs, the abscissa axis is t, the ordinate axis is x and the vertical axis is u.



(a) + (b) −

Fig. 4: The 3D graphs of (3.27) as $a=-2,\ b=-3,\ c=4,\ C=0,\ x\in(-0.5,0.5)$.

(vi) When $ac>0,\ b<0,\ C=0$, from (3.14) yields

$$u(x,t) = \phi(x-ct) = i\sqrt{\frac{2c}{a}}\ \operatorname{csch}\frac{1}{\sqrt{-b}}(x-ct)\,. \tag{3.29}$$

4 Travelling wave solutions of the equation (1.2)

Because the equation (1.15) has a term of $C\phi^{2-2n}$, we only consider those travelling wave solutions in which the integral constant C is zero.

4.1 Under the conditions of $ac>0,\ b>0,\ C=0$, from (1.15), we obtain

$$y^2 = \frac{\frac{2}{bn(n+1)}\phi^{n-1} - \frac{a}{bcn^2}\phi^{2n-2}}{\phi^{2n-4}}\,, \tag{4.1}$$

i.e.

$$y = \pm\frac{\frac{1}{n}\sqrt{\frac{a}{bc}}\sqrt{\frac{2cn}{a(n+1)}\phi^{n-1} - (\phi^{n-1})^2}}{\phi^{n-2}}\,, \tag{4.2}$$

and $\phi_1(0)=0,\ \phi_2(0)=[\frac{2cn}{a(n+1)}]^{\frac{1}{n-1}}$. Where $\phi_1(0)=0\neq\phi_s$ as $n=2$, and $\phi_1(0)=0=\phi_s$ as

$n > 2$. Substituting (4.2) into (2.8), we obtain the following integral bifurcations

$$\int_0^{\phi} \frac{d\phi^{n-1}}{\sqrt{\frac{2cn}{a(n+1)}\phi^{n-1} - (\phi^{n-1})^2}} = \frac{n-1}{n}\sqrt{\frac{a}{bc}}\int_0^{\xi} d\xi \quad \text{for } \xi \geq 0\,, \tag{4.3}$$

$$-\int_{\phi}^{0} \frac{d\phi^{n-1}}{\sqrt{\frac{2cn}{a(n+1)}\phi^{n-1} - (\phi^{n-1})^2}} = \frac{n-1}{n}\sqrt{\frac{a}{bc}}\int_{\xi}^{0} d\xi \quad \text{for } \xi < 0\,, \tag{4.4}$$

and

$$\int_{[\frac{2cn}{a(n+1)}]^{\frac{1}{n-1}}}^{\phi} \frac{d\phi^{n-1}}{\sqrt{\frac{2cn}{a(n+1)}\phi^{n-1} - (\phi^{n-1})^2}} = \frac{n-1}{n}\sqrt{\frac{a}{bc}}\int_0^{\xi} d\xi \quad \text{for } \xi \geq 0\,, \tag{4.5}$$

$$-\int_{\phi}^{[\frac{2cn}{a(n+1)}]^{\frac{1}{n-1}}} \frac{d\phi^{n-1}}{\sqrt{\frac{2cn}{a(n+1)}\phi^{n-1} - (\phi^{n-1})^2}} = \frac{n-1}{n}\sqrt{\frac{a}{bc}}\int_{\xi}^{0} d\xi \quad \text{for } \xi < 0\,. \tag{4.6}$$

Integrating (4.3) and (4.4), we obtain

$$\phi^{n-1} = \frac{cn}{a(n+1)}\left[1 - \cos\frac{n-1}{n}\sqrt{\frac{a}{bc}}(x-ct)\right]\,, \tag{4.7}$$

or

$$\phi^{n-1} = \frac{2cn}{a(n+1)}\sin^2\frac{n-1}{2n}\sqrt{\frac{a}{bc}}(x-ct)\,. \tag{4.8}$$

Integrating (4.5) and (4.6), we obtain

$$\phi^{n-1} = \frac{cn}{a(n+1)}\left[1 + \cos\frac{n-1}{n}\sqrt{\frac{a}{bc}}(x-ct)\right]\,, \tag{4.9}$$

or

$$\phi^{n-1} = \frac{2cn}{a(n+1)}\cos^2\frac{n-1}{2n}\sqrt{\frac{a}{bc}}(x-ct)\,. \tag{4.10}$$

Thus, when $n = 2$, we obtain two smooth periodic wave solutions

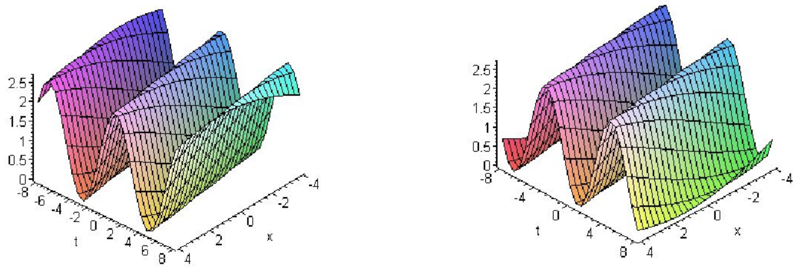
$$u(x,t) = \phi(x-ct) = \frac{cn}{a(n+1)}\left[1 \pm \cos\frac{n-1}{n}\sqrt{\frac{a}{bc}}(x-ct)\right]\,, \tag{4.11}$$

or

$$u(x,t) = \phi(x-ct) = \frac{2cn}{a(n+1)}\sin^2\frac{n-1}{2n}\sqrt{\frac{a}{bc}}(x-ct)\,, \tag{4.12}$$

$$u(x,t) = \phi(x-ct) = \frac{2cn}{a(n+1)}\cos^2\frac{n-1}{2n}\sqrt{\frac{a}{bc}}(x-ct)\,, \tag{4.13}$$

for $x - ct \in (-\infty, +\infty)$. The 3D graphs of periodic wave solutions of (4.11) are shown in Fig. 5. In the graphs, the abscissa axis is t, the ordinate axis is x and the vertical axis is u.



(a) + (b) −

Fig. 5: The 3D graphs of (4.11) as $n = 2,\ a = -2,\ b = -3,\ c = 4,\ C = 0,\ x \in (-0.5, 0.5)$.

When n is an even number and $n > 2$, we obtain two periodic cusp wave solutions

$$u(x,t) = \phi(x - ct) = \left\{ \frac{cn}{a(n+1)} \left[1 \pm \cos \frac{n-1}{n} \sqrt{\frac{a}{bc}} (x - ct) \right] \right\}^{\frac{1}{n-1}}, \tag{4.14}$$

or

$$u(x,t) = \phi(x - ct) = \left[\frac{2cn}{a(n+1)} \sin^2 \frac{n-1}{2n} \sqrt{\frac{a}{bc}} (x - ct) \right]^{\frac{1}{n-1}}, \tag{4.15}$$

$$u(x,t) = \phi(x - ct) = \left[\frac{2cn}{a(n+1)} \cos^2 \frac{n-1}{2n} \sqrt{\frac{a}{bc}} (x - ct) \right]^{\frac{1}{n-1}}. \tag{4.16}$$

When n is an odd number and $n > 1$, we obtain four periodic cusp wave solutions

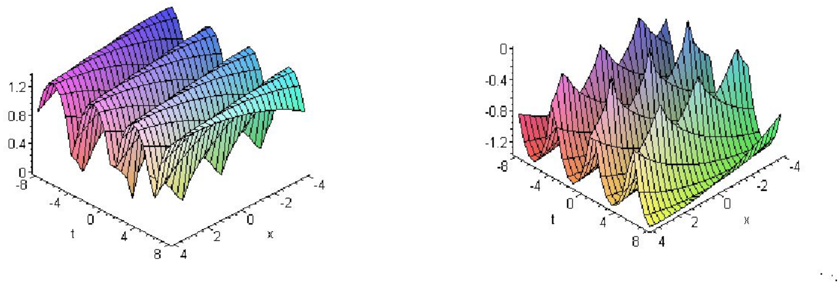
$$u(x,t) = \phi(x - ct) = \pm \left\{ \frac{cn}{a(n+1)} \left[1 \pm \cos \frac{n-1}{n} \sqrt{\frac{a}{bc}} (x - ct) \right] \right\}^{\frac{1}{n-1}}, \tag{4.17}$$

or

$$u(x,t) = \phi(x - ct) = \pm \left[\frac{2cn}{a(n+1)} \sin^2 \frac{n-1}{2n} \sqrt{\frac{a}{bc}} (x - ct) \right]^{\frac{1}{n-1}}, \tag{4.18}$$

$$u(x,t) = \phi(x - ct) = \pm \left[\frac{2cn}{a(n+1)} \cos^2 \frac{n-1}{2n} \sqrt{\frac{a}{bc}} (x - ct) \right]^{\frac{1}{n-1}}. \tag{4.19}$$

The 3D graphs of periodic cusp wave solutions of (4.18) are shown in Fig. 6. In the graphs, the abscissa axis is t, the ordinate axis is x and the vertical axis is u.



(a) + (b) −

Fig. 6: The 3D graphs of (4.18) as $n = 5,\ a = 2,\ b = 3,\ c = 4,\ C = 0,\ x \in (-4, 4)$.

When n is an arbitrary positive integer, from (4.15), (4.16), (4.18) and (4.19) we can obtain two compacton solutions of peak type which have been given in reference [1]:

$$\begin{cases} u(x,t) = \left[\frac{2cn}{a(n+1)} \sin^2 \frac{n-1}{2n}\sqrt{\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n-1}}, & 0 \le (x-ct) \le \frac{2n\pi}{n-1}\sqrt{\frac{bc}{a}}, \\ 0, & \textit{otherwise}. \end{cases} \tag{4.20}$$

and

$$\begin{cases} u(x,t) = \left[\frac{2cn}{a(n+1)} \cos^2 \frac{n-1}{2n}\sqrt{\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n-1}}, & |(x-ct)| \le \frac{n\pi}{n-1}\sqrt{\frac{bc}{a}}, \\ 0, & \textit{otherwise}. \end{cases} \tag{4.21}$$

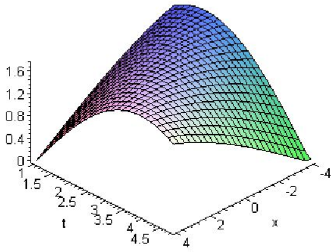
In fact, when n is an odd number, we also obtain two compacton solutions of valley type

$$\begin{cases} u(x,t) = -\left[\frac{2cn}{a(n+1)} \sin^2 \frac{n-1}{2n}\sqrt{\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n-1}}, & 0 \le (x-ct) \le \frac{2n\pi}{n-1}\sqrt{\frac{bc}{a}}, \\ 0, & \textit{otherwise}. \end{cases} \tag{4.22}$$

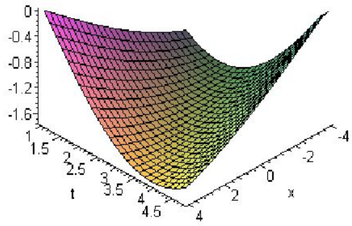
and

$$\begin{cases} u(x,t) = -\left[\frac{2cn}{a(n+1)} \cos^2 \frac{n-1}{2n}\sqrt{\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n-1}}, & |(x-ct)| \le \frac{n\pi}{n-1}\sqrt{\frac{bc}{a}}, \\ 0, & \textit{otherwise}. \end{cases} \tag{4.23}$$

The 3D graphs of compacton solutions of (4.20) and (4.22) are shown in Fig. 7. In the graphs, the abscissa axis is t, the ordinate axis is x and the vertical axis is u.



(a) compacton of (4.20)



(b) compacton of (4.22)

Fig. 7: The 3D graphs of (4.18) and (4.22) as $n = 3,\ a = 2,\ b = 3,\ c = 4, C = 0, x \in (-4, 4)$.

4.2 Suppose that $ac < 0,\ b > 0,\ C = 0$, from (1.15), it yields

$$y = \pm \frac{\frac{1}{n}\sqrt{-\frac{a}{bc}}\sqrt{-\frac{2cn}{a(n+1)}\phi^{n-1} + (\phi^{n-1})^2}}{\phi^{n-2}}, \tag{4.24}$$

$$\tag{4.25}$$

and $\phi(0) = 0$. Thus, corresponding the Eq. (4.24), we obtain

$$\phi^{n-1} = -\frac{cn}{a(n+1)}\left[\cosh \frac{n-1}{n}\sqrt{-\frac{a}{bc}}(x - ct) - 1\right]. \tag{4.26}$$

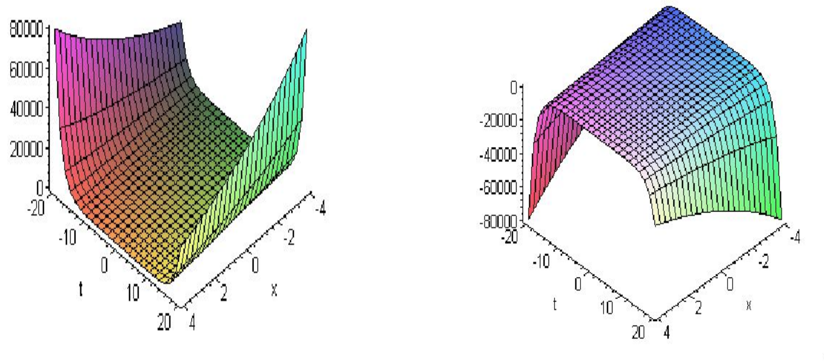
When n is an even number, we obtain a unbounded travelling wave solution of hyperbolic cosine type

$$u(x,t) = \phi(x - ct) = \left\{-\frac{cn}{a(n+1)}\left[\cosh \frac{n-1}{n}\sqrt{-\frac{a}{bc}}(x - ct) - 1\right]\right\}^{\frac{1}{n-1}}. \tag{4.27}$$

When n is an odd number, we obtain two unbounded travelling wave solutions of hyperbolic cosine type

$$u(x,t) = \phi(x - ct) = \pm\left\{-\frac{cn}{a(n+1)}\left[\cosh \frac{n-1}{n}\sqrt{-\frac{a}{bc}}(x - ct) - 1\right]\right\}^{\frac{1}{n-1}}. \tag{4.28}$$

The 3D graphs of unbounded travelling wave solutions of (4.28) are shown in Fig. 8. In the graphs, the abscissa axis is t, the ordinate axis is x and the vertical axis is u.



(a) $+$ (b) $-$

Fig. 8: The 3D graphs of (4.28) as $n = 5,\ a = -2,\ b = 3,\ c = 4,\ C = 0,\ x \in (-4, 4)$.

Similarly, under the other parameter conditions, according to the above results, we can obtain the other travelling wave solutions without difficulty:

(1) When n is an even number and $ac > 0,\ b < 0,\ C = 0$, from (4.14) and (4.15), we obtain three unbounded travelling wave solutions:

$$u(x,t) = \phi(x - ct) = \left\{ \frac{cn}{a(n+1)} \left[1 \pm \cosh \frac{n-1}{n} \sqrt{-\frac{a}{bc}}(x - ct) \right] \right\}^{\frac{1}{n-1}}, \tag{4.29}$$

and

$$u(x,t) = \phi(x - ct) = \left[-\frac{2cn}{a(n+1)} \sinh^2 \frac{n-1}{2n} \sqrt{-\frac{a}{bc}}(x - ct) \right]^{\frac{1}{n-1}}. \tag{4.30}$$

(2) When n is an odd number and $ac > 0,\ b < 0,\ C = 0$, from (4.17), (4.18) and (4.19), we obtain six unbounded travelling wave solutions:

$$u(x,t) = \phi(x - ct) = \pm \left\{ \frac{cn}{a(n+1)} \left[1 + \cosh \frac{n-1}{n} \sqrt{-\frac{a}{bc}}(x - ct) \right] \right\}^{\frac{1}{n-1}}, \tag{4.31}$$

and

$$u(x,t) = \phi(x - ct) = \pm \left[-\frac{2cn}{a(n+1)} \sinh^2 \frac{n-1}{2n} \sqrt{-\frac{a}{bc}}(x - ct) \right]^{\frac{1}{n-1}}, \tag{4.32}$$

$$u(x,t) = \phi(x - ct) = \pm \left[\frac{2cn}{a(n+1)} \cosh^2 \frac{n-1}{2n} \sqrt{-\frac{a}{bc}}(x - ct) \right]^{\frac{1}{n-1}}. \tag{4.33}$$

5 Travelling wave solutions of the equation (1.3)

Since (1.16) is a high order equation, we only consider the case of integral constant $C = 0$ in this section.

5.1 Suppose that $ac > 0,\ b > 0,\ C = 0$ or $ac < 0,\ b < 0,\ C = 0$, from (1.17), we obtain

$$y = \pm\sqrt{\frac{2}{bn(n-1)}}\phi\sqrt{\phi^{n+1} - A^2}\,, \tag{5.1}$$

and $\phi_1(0) = A^{\frac{2}{n+1}}$ or $\phi_{1,2}(0) = \pm A^{\frac{2}{n+1}}$, where $A = \sqrt{\frac{a(n-1)}{2cn}}$. Substituting (5.1) into (2.8), we obtain the following integral bifurcations

$$\int_{\pm[A^{\frac{2}{n+1}}]}^{\phi} \frac{d\phi}{\phi\sqrt{\phi^{n+1} - A^2}} = \frac{n-1}{n}\sqrt{\frac{a}{bc}}\int_{\xi_k}^{\xi} d\xi, \quad \text{for } \xi \geq 0\,, \tag{5.2}$$

$$-\int_{\phi}^{\pm[A^{\frac{2}{n+1}}]} \frac{d\phi}{\phi\sqrt{\phi^{n+1} - A^2}} = \frac{n-1}{n}\sqrt{\frac{a}{bc}}\int_{\xi}^{\xi_k} d\xi, \quad \text{for } \xi < 0\,, \tag{5.3}$$

where ξ_k is an arbitrary constant. Integrating (5.2) and (5.3), we obtain

$$\phi^{n+1} = \frac{a(n-1)}{2cn}\sec^2\frac{n+1}{2n}\sqrt{\frac{a}{bc}}(\xi - \xi_k)\,. \tag{5.4}$$

When n is an even number, we obtain a family of periodic wave solutions

$$u(x,t) = \phi(x - ct) = \left[\frac{a(n-1)}{2cn}\sec^2\frac{n+1}{2n}\sqrt{\frac{a}{bc}}(\xi - \xi_k)\right]^{\frac{1}{n+1}}. \tag{5.5}$$

Taking $\xi_k = 0$ and $\xi_k = \frac{\pi}{2}$ respectively, we obtain the following two periodic wave solutions which are the same as Ref. [1],

$$u(x,t) = \phi(x - ct) = \left[\frac{a(n-1)}{2cn}\sec^2\frac{n+1}{2n}\sqrt{\frac{a}{bc}}(x - ct)\right]^{\frac{1}{n+1}}, \tag{5.6}$$

and

$$u(x,t) = \phi(x - ct) = \left[\frac{a(n-1)}{2cn}\csc^2\frac{n+1}{2n}\sqrt{\frac{a}{bc}}(x - ct)\right]^{\frac{1}{n+1}}. \tag{5.7}$$

When n is an odd number, we obtain two families of periodic wave solutions:

$$u(x,t) = \phi(x - ct) = \pm\left[\frac{a(n-1)}{2cn}\sec^2\frac{n+1}{2n}\sqrt{\frac{a}{bc}}(\xi - \xi_k)\right]^{\frac{1}{n+1}}. \tag{5.8}$$

Taking $\xi_k = 0$ and $\xi_k = \frac{\pi}{2}$ respectively, we obtain the following four periodic wave solutions:

$$u(x,t) = \phi(x-ct) = \pm\left[\frac{a(n-1)}{2cn}\sec^2\frac{n+1}{2n}\sqrt{\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n+1}}. \tag{5.9}$$

and

$$u(x,t) = \phi(x-ct) = \pm\left[\frac{a(n-1)}{2cn}\csc^2\frac{n+1}{2n}\sqrt{\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n+1}}. \tag{5.10}$$

5.2 Suppose that $ac > 0,\ b < 0,\ C = 0$ or $ac < 0,\ b > 0,\ C = 0$. According to the results (5.7), (5.8), (5.9), (5.10), we can obtain the other traveling wave solutions without difficulty:

(i) When n is an even number, we obtain two solitary wave solutions:

$$u(x,t) = \phi(x-ct) = \left[\frac{a(n-1)}{2cn}\ \operatorname{sech}^2\frac{n+1}{2n}\sqrt{-\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n+1}}, \tag{5.11}$$

and

$$u(x,t) = \phi(x-ct) = \left[-\frac{a(n-1)}{2cn}\ \operatorname{csch}^2\frac{n+1}{2n}\sqrt{-\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n+1}}. \tag{5.12}$$

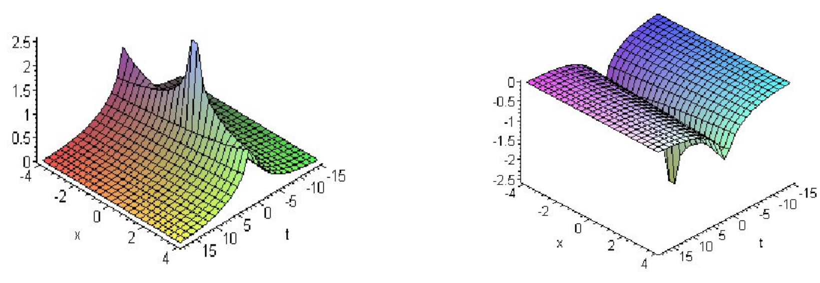
(ii) When n is an odd number, we obtain four solitary wave solutions:

$$u(x,t) = \phi(x-ct) = \pm\left[\frac{a(n-1)}{2cn}\ \operatorname{sech}^2\frac{n+1}{2n}\sqrt{-\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n+1}}, \quad \text{for } ac > 0, \tag{5.13}$$

and

$$u(x,t) = \phi(x-ct) = \pm\left[-\frac{a(n-1)}{2cn}\ \operatorname{csch}^2\frac{n+1}{2n}\sqrt{-\frac{a}{bc}}(x-ct)\right]^{\frac{1}{n+1}}, \quad \text{for } ac < 0. \tag{5.14}$$

The 3D graphs of solitary wave solutions of (5.14) are shown in Fig. 9. In the graphs, the abscissa axis is t, the ordinate axis is x and the vertical axis is u.



(a) solitary of peak form (b) solitary of valley form

Fig. 9: The 3D graphs of (5.14) as $n = 5,\ a = -2,\ b = 3,\ c = 4,\ C = 0,\ x \in (-4, 4)$.

From the above process of deriving, it is easy to see that this method is also available to many nonlinear integral systems.

6 Conclusion

In this paper, we introduced a new method named integral bifurcation. By using this method, we studied the modified equal width wave equation and its variants and obtained many new traveling wave solutions in addition to the results in reference [1]. Clearly, this method is available to many nonlinear partial equations. However, when we solve the universal nonlinear partial equations by this method, are all the effects good? We will continue to thoroughly pay attention to this question.

Acknowledgements

This research was supported by the Science Research Foundation of Yunnan Provincial Educational Department (06Y147A) and the Natural Science Foundation of China (10571062).

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received: May 15, 2007

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