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Approximation of the Bochner integral by means of Riemann sums

Dedicated to professor Harry Poppe on his 65th birthday

The natural way to approximate the integral (of some kind) of a function f on a measure space T into a Banach space E via the values of f is - in the author's opinion - that by Riemann sums. The Bochner integral does not offer this possible way of approximation in an immediate way. The goal of this paper is to investigate in which sense the desired approximation is nevertheless possible for the Bochner integral of f.

More precisely, we consider a Banach space E endowed with its norm topology and a measure space (T, μ) with a finite measure μ. There are essentially two ways to introduce the Bochner integral: one extends the natural integral of step functions having *finite* range (i.e. of simple functions), the other extends the natural integral of step functions having *countable* range. We denote the first type of Bochner integral by ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu$, the second type by ${}^{\mathrm{B}}\!\!\int \cdot\, d\mu$. Each integral defined on a subset of E^T and ranging in E is interpreted as T-ary partial operation in E, thus as a subset of $E^T \times E$. (Every function is considered as a relation.) Obviously, one has ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu \subseteq {}^{\mathrm{B}}\!\!\int \cdot\, d\mu$ (see Theorem 2.1) and, if μ is complete, one obtains ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu = {}^{\mathrm{B}}\!\!\int \cdot\, d\mu$ (see Theorem 3.2). (Although these facts must be well-known, the author could not find them explicitly in the literature.)

In the papers [6] and [7], Riemann sums were used to introduce, for each "μ-partition system" $\mathfrak{Y}$ (consisting of certain subdivisions of T), the $(\mu, \mathfrak{Y})$-integral ${}^{\mathfrak{Y}}\!\!\int \cdot\, d\mu$, called now "$(\mu, \mathfrak{Y})$-subdivision integral", which maps $\mathfrak{E}^T$ into a set $\mathfrak{E}$, having just one element more than E. (The role of this additional element is played by the empty set $\emptyset$.) ${}^{\mathfrak{Y}}\!\!\int \cdot\, d\mu$ is a T-ary operation in $\mathfrak{E}$. Certain T-ary operations in $\mathfrak{E}$, for instance ${}^{\mathfrak{Y}}\!\!\int \cdot\, d\mu$, are algebraic completions of T-ary partial operations in E, and every T-ary partial operation in E can be extended to a T-ary operation in $\mathfrak{E}$. There is a one-to-one correspondance $(\cdot)^\wedge$ between all T-ary partial operations in E and certain T-ary operations in $\mathfrak{E}$ its inverse being denoted by $(\cdot)^\vee$. For instance, $({}^{\mathfrak{Y}}\!\!\int \cdot\, d\mu)^\vee$ is a T-ary partial operation in E such that $(({}^{\mathfrak{Y}}\!\!\int \cdot\, d\mu)^\vee)^\wedge = {}^{\mathfrak{Y}}\!\!\int \cdot\, d\mu$.

(Actually, we shall consider even an extension of the mapping $(\cdot)^\wedge$ (denoted the same way) which assigns, especially, to the generalized partial operation lim defined on *certain* filtered families in E a generalized (algebraic) operation $\lim^\wedge$ in $\mathfrak{E}$ defined on *all* filtered families in $\mathfrak{E}$. For details, see Section 1 and [11].)

The most important μ-partition systems are: (a) that consisting of all *finite* "μ-partitions" (intuitively: subdivisions) of T, which we denote (later, beginning with Lemma 1.4) by $\mathfrak{X}$, and (b) that consisting of all *countable* μ-partitions of T, denoted by Ω. (Because of the implicit occurence of ${}^{\Omega}\!\!\int \cdot\, d\mu$ in [2], we call this integral *Birkhoff integral.*) In every case, one has

$$ {}^{\mathrm{B}}\!\!\int \cdot\, d\mu \subseteq \left({}^{\Omega}\!\!\int \cdot\, d\mu\right)^\vee \tag{1}$$

(see Theorem 2.1); for bounded $f : T \longrightarrow E$, one has

$$ {}^{\mathrm{D}}\!\!\int^\wedge f^\sim d\mu \subseteq {}^{\mathfrak{Y}}\!\!\int f^\sim d\mu \text{ if } \mathfrak{X} \subseteq \mathfrak{Y} \tag{2}$$

(see Theorem 2.2); for bounded measurable f and complete μ, one obtains

$$ \emptyset \neq {}^{\mathrm{D}}\!\!\int^\wedge f^\sim d\mu = {}^{\mathrm{B}}\!\!\int^\wedge f^\sim d\mu = {}^{\mathfrak{Y}}\!\!\int f^\sim d\mu \text{ if } \mathfrak{X} \subseteq \mathfrak{Y} \tag{3}$$

(see Theorem 3.4), where $^\sim$ denotes the natural injective mapping on E^T into $\mathfrak{E}^T$. The interpretation for instance of the latter assertion (3) in the traditional language is the following one: f is D-Bochner integrable, B-Bochner integrable, and $\mathfrak{Y}$-integrable, and the values of the corresponding integrals of f coincide (if $\mathfrak{X} \subseteq \mathfrak{Y}$). Each of the three assertions contains the statement that the Bochner integral of Bochner integrable functions can be approximated by means of Riemann sums either such having countably many summands (assertion (1)) or such having finitely many summands (assertions (2) and (3)). In the assertion (1), generally, the inclusion is a proper inclusion; the validity of the sign "=" implies that E is finite-dimensional provided there exists an infinite μ-partition of T into sets having positive measure (see Theorem 3.3/b).

As a byproduct of this comparison of several approaches to integration in Banach spaces we obtain, for complete μ, a characterization of the "μ-summation" in the case $E = \mathbb{R}$, i.e. of the restriction of classical μ-integration to μ-summable functions $f : T \longrightarrow \mathbb{R}$ (see Theorem 3.1). This allows us to interpret, for complete μ, the $(\mathfrak{Y}, \mu)$-subdivision integration as a strict extension of μ-summation, no matter which approach to the Lebesgue integration one might choose (while originally, in [6], we (the authors of [6]) have restricted ourselves to the approach to integration presented in [13]).

Finally, we discuss the relationship of the Bochner integral and the $(\mu, \mathfrak{Y})$-subdivision integral with the Riemann integral redefined in the expected way in the Banach space E with a compact metric space as domain T being endowed with a measure μ compatible in a precise sense with the metric of T. As an example for the possible transfer of classical statements for the Riemann integral into the language of the subdivision integral in Banach spaces we discuss the rearrangement of the order of integration and uniform convergence of generalized sequences in E^T (compare Proposition 4.7 with Theorem 2.3).

In addition to the references given in [6], the author should mention HENSTOCK's books [15] and [16] (both containing numerous references to the Riemann-type integration), his paper [14], furthermore, for $T = [a, b] \subseteq \mathbb{R}$ and a suitable μ-partition system $\mathfrak{Y} \subseteq \mathfrak{X}$, GORDON's article [8] on Riemann-type integration in Banach spaces (see Example 4.2 below), where especially GRAVES' article [9] occurs under the references.

The author is grateful to Professor W. STRAUSS for discussion and critical remarks as well as for valuable hints to the literature.

This paper is a continuation of the papers [6] and [7]. In order to make this paper selfcontained as much as possible, we collect in Section 1 definitions and some results (used here) introduced or discussed there already.

1 Terminology and some lemmas

a) *Set theory.* Sets are special classes. Functions are special (binary) relations. If A and B are classes, a function (= mapping) $h \subseteq A \times B$ is called a function *from* A into B , a function *on* A into B, if $\mathrm{Dmn}\, h = A$ (symbol $h : A \longrightarrow B$). If h is a function, then $\underset{i \in \mathrm{Dmn}\, h}{\mathrm{P}} h(i)$ or $\mathrm{P}h$ denotes the cartesian product of h. If M is a set, $\mathcal{P}M$ denotes $\mathrm{P}h$ with $h = \{(m, m) \mid m \in M\}$, while $\mathfrak{P}M$ denotes the powerset of M. For the whole paper, we fix a set E (denoting a Banach space later) and define $\mathfrak{E}$ to be the set $\{X \mid X \subseteq E \textit{ and } \mathrm{card}\, X \le 1\}$. For each set I and each $h : I \longrightarrow E$, we define by $h^\sim$ the mapping defined by $h^\sim(i) = \{h(i)\}$ for all $i \in I$. Then $h^\sim : I \longrightarrow \mathfrak{E}$. For all $X \in \mathfrak{E}\backslash\{\emptyset\}$, $\mathrm{e}X$ denotes the element of X. Then, e^{-1} maps E one-to-one onto $\mathfrak{E}\backslash\{\emptyset\}$, and $h^\sim = \mathrm{e}^{-1} \circ h$ holds for all $h : I \longrightarrow E$.

b) *Ordered sets, directed sets.* If $(E, \le)$ is an ordered set, then the relation $\le^\star$ is defined by letting, for all $X, Y \in \mathfrak{E}$, $X \le^\star Y$, if $x \le y$ holds for all $(x, y) \in X \times Y$. In the case $Y = \{y\}$, we write also $X \le^\star y$ instead of $X \le^\star Y$. Finally, we write $X \le Y$ and $X \le y$ instead of $X \le^\star Y$ and $X \le^\star y$, respectively. – Similar agreements hold for $<$ instead of $\le$. – If $(M, \preceq)$ is a directed set, then $\mathcal{F}(M, \preceq)$ or $\mathcal{F}M$ denotes the filter on M generated by the sections $\{y \in M \mid x \preceq y\}$ with $x \in M$ (*filter of perfinality of* M). The filter $\mathcal{F}\mathbb{N}$ on $\mathbb{N}(= \{1, 2, \cdots\})$ is called the *Fréchet filter.*

c) *Universal algebra.* Let I be a non-empty set. Each mapping Θ from E^I into E is called an *I-ary partial operation in E* (resp. an *I-ary operation in E* if $\mathrm{Dmn}\, \Theta = E^I$). Let Θ be an

I-ary partial operation in E (resp. an I-ary operation in E). Then (E, Θ) is called an *I-ary partial algebra* (resp. *I-ary algebra*). We denote by $\Theta^\wedge$ the I-ary operation in $\mathfrak{E}$ defined by

$$\Theta^\wedge f = \{x \in E \mid \textit{ for some } \phi \in \mathrm{P}\ f,\ x = \Theta\phi\} \textit{ for all } f \in \mathfrak{E}^I .$$

The I-ary algebra $(\mathfrak{E}, \Theta^\wedge)$ is a natural extension of (E, Θ), called *1-point-completion of* (E, Θ).The mapping e^{-1} is an isomorphism on (E, Θ) into $(\mathfrak{E}, \Theta^\wedge)$, namely onto the *trace* $(F,\, \Theta^\wedge \cap (F^I \times F))$ of $(\mathfrak{E}, \Theta^\wedge)$ in the set $F = \mathfrak{E}\backslash\{\emptyset\}$. For each $g \in E^I$,

$$g \in \operatorname{Dmn}\Theta \text{ if and only if } \Theta^\wedge g^\sim \neq \emptyset ,$$

furthermore

$$\text{if } g \in \operatorname{Dmn}\Theta ,\ \text{ then } \mathrm{e}^{-1}(\Theta g) = \Theta^\wedge g^\sim .$$

Denote, for the moment, by $\mathcal{A}(E, I)$ the set of all I-ary partial operations in E and by $\mathcal{B}(\mathfrak{E}, I)$ the set of all I-ary operations Λ in $\mathfrak{E}$ such that for all $\phi \in \mathfrak{E}^I$

$$\text{if } \Lambda\phi \neq \emptyset \text{ then } \phi(i) \neq \emptyset \text{ for all } i \in I .$$

Then, the mapping $\Theta \longmapsto \Theta^\wedge(\Theta \in \mathcal{A}(E, I))$ is one-to-one onto $\mathcal{B}(\mathfrak{E}, I)$; its inverse mapping $(\cdot)^\vee$ is defined by letting for each $\Lambda \in \mathcal{B}(\mathfrak{E}, I)$

$$(\psi, x) \in \Lambda^\vee \text{ if and only if } \{x\} = \Lambda\psi$$

for all $\psi \in E^I$ and all $x \in E$. For each $f \in \mathfrak{E}^I$, we write Θf or $\underset{i \in I}{\Theta} f(i)$ instead of $\Theta^\wedge f$ without any danger of confusion. *For the remainder of this paper, let* $(E, \|\cdot\|)$ *be a Banach space.* If I is countable, the unconditional summation Σ_I in E (see [11],p.119 and p.128) is an I-ary partial operation in E.

d) *Convergence.* Let τ be the norm topology of $(E, \|\cdot\|)$. Let ΦM denote, for each set M, the class of all filtered families $(g, K, \mathfrak{b})$ in M, i.e. with $g(K) \subseteq M$. Denote by $\lim_\tau$ or just by lim the limit operation induced by τ; lim is a mapping from ΦE into E. Let $\mathcal{M}$ be the class of all filtered sets $(K, \mathfrak{b})$. Then lim is an $\mathcal{M}$-ary partial operation in E (see [11], p.120), and for the $\mathcal{M}$-ary operation $\lim^\wedge$ in $\mathfrak{E}$ the following holds for all $(g, K, \mathfrak{b}) \in \Phi\mathfrak{E}$:

$$\lim{}^\wedge(g, K, \mathfrak{b}) = \{x \in E \mid \exists B \in \mathfrak{b}\ (\exists \phi \in \underset{k \in B}{\mathrm{P}}\ g(k) :\ x = \lim(\phi, B, \mathfrak{b}_B))\} ,$$

where $\mathfrak{b}_B$ denotes the *trace* $\{B \cap C \mid C \in \mathfrak{b}\}$ *of the filter* $\mathfrak{b}$ *in* the set B.– The mapping e^{-1} is an isomorphism on the $\mathcal{M}$-ary partial algebra $(E, \lim)$ into the $\mathcal{M}$-ary algebra $(\mathfrak{E}, \lim^\wedge)$, namely onto the $\mathcal{M}$-ary partial algebra

$$(F,\, \lim{}^\wedge \cap (\Phi(F) \times F)) \text{ with } F = \mathfrak{E}\backslash\{\emptyset\} ,$$

being called the *trace of* $(\mathfrak{E}, \lim^{\wedge})$ *in* the set F. This means that, for each $x \in E$ and all $(g, K, \mathfrak{b}) \in \Phi(E)$, (1) the statement $(g, K, \mathfrak{b}) \in \mathrm{Dmn}(\lim)$ holds (i.e., $(g, K, \mathfrak{b})$ converges in E w. r. to the norm topology τ) if and only if $\lim^{\wedge}(g^{\sim}, K, \mathfrak{b}) \neq \emptyset$, and (2) $x = \lim(g, K, \mathfrak{b})$ holds if and only if $\{x\} = \lim^{\wedge}(g^{\sim}, K, \mathfrak{b})$.– In the sequel, we write, for each $(g, K, \mathfrak{b}) \in \Phi\mathfrak{E}$, also $\lim(g, K, \mathfrak{b})$ or ${}^{\mathfrak{b}}\lim\limits_{k \in K} g(k)$ instead of $\lim^{\wedge}(g, K, \mathfrak{b})$. In the special case that $K = \mathbb{N}$ and $\mathfrak{b} = \mathcal{F}\mathbb{N}$ (see b)), we replace ${}^{\mathfrak{b}}\lim\limits_{k \in K} g(k)$ by $\lim\limits_{k \in \mathbb{N}} g(k)$. (Of course, these considerations under d) hold in every Hausdorff space E.)

e) *Banach spaces.* The scalar domain of $(E, \|\cdot\|)$ is denoted by $\mathbb{K}$. The scalar multiplication in E, $(x, \alpha) \longmapsto x\alpha((x, \alpha) \in E \times \mathbb{K})$, is written in a right-hand notation. If $X, Y \in \mathfrak{E}$ and $\alpha \in \mathbb{K}$, then we define $X\alpha = \{x\alpha \mid x \in X\}$, $\|X\| = \{\|x\| \mid x \in X\}$, and (in accordance with the agreements in a) above for $\Theta = +$ or $\Theta = -$) $X \pm Y = \{x \pm y \mid x \in X \text{ and } y \in Y\}$. For each mapping h on a set T into E, we define $\|h\|$ to be the mapping $t \longmapsto \|h(t)\|\ (t \in T)$; in the special case $E = \mathbb{R}$, $h^{\sim}$ is defined as under a) with T instead of I.

f) *Integration.* In this paper, we assume that a non-empty measure space (T, μ) with a *finite measure* μ is given. Each countable partition $\mathfrak{x}$ of T such that $\mathfrak{x} \subseteq \mathrm{Dmn}\,\mu$ is called a *μ-partition.* We recall that $\Omega(\mu)$ or Ω denotes the class of all μ-partitions of T. Let $\mathfrak{X} \subseteq \Omega$. Given $\mathfrak{x}, y \in \mathfrak{X}$, we agree that $\mathfrak{x} \preceq y$ holds, if, for each $Y \in \mathfrak{y}$, there exists an $X \in \mathfrak{x}$ such that $Y \subseteq X$. $\mathfrak{X}$ is called a *μ-partition system*, if $(\mathfrak{X}, \preceq)$ is a directed set. Define the binary operation $\vee$ in Ω by letting, for each $\mathfrak{x}, y \in \Omega$, $\mathfrak{x} \vee y = \{X \cap Y \mid X \in \mathfrak{x} \text{ and } Y \in \mathfrak{y}\} \backslash \{\emptyset\}$. We say that $\mathfrak{X}$ is $\vee$*-closed*, if $\vee(\mathfrak{X} \times X) \subseteq X$. If $\mathfrak{X}$ is $\vee$-closed, then $\mathfrak{X}$ is a μ-partition system; e.g., Ω is a μ-partition system, and the set $\mathfrak{X}_{fi}$ of all finite μ-partitions is a μ-partition system. Let $\mathfrak{X}$ be a μ-partition system of T. Then, we denote by $\mathfrak{X}^{\#}$ the set $\{(\mathfrak{x}, \phi) \mid x \in \mathfrak{X} \text{ and } \phi \in \mathcal{P}\mathfrak{x}\}$. We define $\preceq^{\#}$ to be the relation such that, for all $(\mathfrak{x}, \phi), (\mathfrak{y}, \psi) \in \mathfrak{X}^{\#}$, $(\mathfrak{x}, \phi) \preceq^{\#} (\mathfrak{y}, \psi)$, if $\mathfrak{x} \preceq \mathfrak{y}$. $(\mathfrak{X}^{\#}, \preceq^{\#})$ is a directed set. By the agreement in b), $\mathcal{F}\mathfrak{X}$ and $\mathcal{F}\mathfrak{X}^{\#}$ denote the filters of perfinality on $\mathfrak{X}$ and $\mathfrak{X}^{\#}$. Given $f : T \longrightarrow \mathfrak{E}$, the *Riemann sum* $\mathrm{R}(f, \mathfrak{x}, \phi)$ of f belonging to $(\mathfrak{x}, \phi) \in \mathfrak{X}^{\#}$ is defined to be the set $\sum\limits_{X \in \mathfrak{x}} f(\phi(X))\mu X$. The $(\mu, \mathfrak{X})$*-integral* ${}^{\mathfrak{X}}\!\!\int f\, d\mu$ of f is defined to be the set

$$
{}^{\mathcal{F}\mathfrak{X}^{\#}}\lim_{(\mathfrak{x}, \phi) \in \mathfrak{X}^{\#}} \mathrm{R}(f, \mathfrak{x}, \phi)\,.
$$

If $h : T \longrightarrow \mathbb{R}$ (where $\mathbb{R}$ denotes the real line), then, tacitly, ${}^{\mathfrak{X}}\!\!\int h^{\sim} d\mu$ denotes the $(\mu, \mathfrak{X})$-integral of $h^{\sim}$ for the Banach space $(\mathbb{R}, |\cdot|)$ instead of $(E, \|\cdot\|)$. – We call the $(\mu, \mathfrak{X})$-integral also $(\mu, \mathfrak{X})$-*subdivision integral.*

We add some results being proved in [6] and [7], in which f is a mapping on T into $\mathfrak{E}$ and $\mathfrak{X}, \mathfrak{Y}$ are μ-partition systems of T :

Lemma 1.1 ([6],Theorem 4). *$\mathfrak{X} \subseteq \mathfrak{Y}$ implies ${}^{\mathfrak{X}}\!\!\int f\,d\mu \subseteq {}^{\mathfrak{Y}}\!\!\int f\,d\mu$.*

We note a simple consequence of the preceding lemma applied in the Banach space $(\mathbb{R}, |\cdot|)$:

If $g \in E^T$ and ${}^{\mathfrak{X}}\!\!\int \|g\|^{\sim} d\mu = \{0\}$ and the function $\|g\|$ is μ-measurable, then $g(t) = 0$ holds for μ-a.a. $t \in T$.

[**Proof:** By Lemma 1, one has $0 \in {}^{\Omega}\!\!\int \|g\|^{\sim} d\mu$, thus (by Theorem 12 in [7]) $0 = {}^{\mathrm{H}}\!\!\int \|g\|\,d\mu$. Now apply Satz 8 on p.100 in [13]. $\square$]

f is called an $\mathfrak{X}-$*step function*, if there is an $\mathfrak{x} \in \mathfrak{X}$ and a mapping $\chi : \mathfrak{x} \longrightarrow \mathfrak{E}$ such that $f(x) = \chi(X)$ for all (X, x) with $X \in \mathfrak{x}$ and $x \in X$. If f is in such a relationship with $\mathfrak{x}$ and χ, we say that f is the $\mathfrak{X}$-step function determined by $(\mathfrak{x}, \chi)$. The class of all $\mathfrak{X}$-step functions on T into $\mathfrak{E}$ is denoted by $\mathcal{ST}(\mathfrak{X}, \mathfrak{E})$. Similarly, one defines the class $\mathcal{ST}(\mathfrak{X}, E)$ of all $\mathfrak{X}$-step funtions on T into E. Partially, the next lemma follows from Lemma 1.1:

Lemma 1.2 ([6], Theorem 5). *Let $\mathfrak{x} \in \mathfrak{X}$, $\chi : \mathfrak{x} \longrightarrow \mathfrak{E}$, and f be the $\mathfrak{X}$-step function determined by $(\mathfrak{x}, \chi)$. Then, one has*

$$ {}^{\mathfrak{X}}\!\!\int f\,d\mu = \sum_{X \in \mathfrak{x}} \chi(X)\mu X. $$

Given a μ-partition system $\mathfrak{X}$ of T and $\emptyset \neq R \in \operatorname{Dmn}\mu$, let $\mathfrak{X}_R = \{\mathfrak{x}_R \mid \mathfrak{x} \in \mathfrak{X}\}$, where $\mathfrak{x}_R = \{R \cap X \mid X \in \mathfrak{x}\} \backslash \{\emptyset\}$. Then, $\mathfrak{X}_R$ is a μ_R-partition system of R, where $\mu_R = \mu \mid (\operatorname{Dmn}\mu \cap \mathfrak{P}R)$. Then, one has the

Lemma 1.3 ([7], Theorem 9). *Assume $\mathfrak{X}$ to be $\vee$-closed and $\mathfrak{y} \in \mathfrak{X}$. If $\mathfrak{y}$ is finite, then, one has ${}^{\mathfrak{X}}\!\!\int f\,d\mu = \sum_{Y \in \mathfrak{y}} {}^{\mathfrak{X}_Y}\!\!\int (f \mid Y) d\mu_Y$.*

Corollary 1.1 *Let $g : T \longrightarrow E$ and $h : T \longrightarrow E$. Assume that $\mathfrak{X}$ is $\vee$-closed and that there is a set $M \in \operatorname{Dmn}\mu$ with $\mu M = 0$ such that $\{M, T\backslash M\} \in \mathfrak{X}$ and $g(t) = h(t)$ holds for all $t \in T\backslash M$. Then ${}^{\mathfrak{X}}\!\!\int g^{\sim} d\mu = {}^{\mathfrak{X}}\!\!\int h^{\sim} d\mu$.*

In particular, if $\mathfrak{X} = \{\mathfrak{x} \in \Omega \mid \mathfrak{x}$ *is finite* $\}$ or $\mathfrak{X} = \Omega$ and $g(t) = h(t)$ holds for μ-almost all $t \in T$, then ${}^{\mathfrak{X}}\!\!\int g^{\sim} d\mu = {}^{\mathfrak{X}}\!\!\int h^{\sim} d\mu$ (cf. Proposition 30 in [7], p.18).

For the remainder of this paper, *we reserve the symbol $\mathfrak{X}$ for the system of all finite μ-partitions of T* and the symbol $\mathfrak{Y}$ *for some (given) μ-partition system of T.*

We shall make use of the following link to the integration of bounded and measurable real-valued functions:

Lemma 1.4 *Let $(E, \|\cdot\|)$ be the Banach space $(\mathbb{R}, |\cdot|)$, and assume $g : T \longrightarrow E$ to be μ-measurable. Then,* (a)$\Longrightarrow$(c) *and* (b)$\Longrightarrow$(c) *hold true:*

(a) *g is bounded;*

(b) *μ is complete and g is bounded μ-a.e.;*

(c) ${}^{\mathfrak{Y}}\!\!\int g^{\sim} d\mu \neq \emptyset$*, namely* $\mathrm{e}({}^{\mathfrak{Y}}\!\!\int g^{\sim} d\mu) = {}^{\mathrm{H}}\!\!\int g\, d\mu$*, if* $\mathfrak{X} \subseteq \mathfrak{Y}$. (For the terminology, see [7],p.15.)

Proof: (a)$\Longrightarrow$(c). For abbreviation, put $s(\mathfrak{x}) = \sum\limits_{X \in \mathfrak{x}} \inf\limits_{t \in X} g(t)\mu X$ and $S(\mathfrak{x}) = \sum\limits_{X \in \mathfrak{x}} \sup\limits_{t \in X} g(t)\mu X$ for all $\mathfrak{x} \in \mathfrak{X}$, furthermore $j(\mathfrak{x}, \phi) = s(\mathfrak{x})$ and $J(\mathfrak{x}, \phi) = S(\mathfrak{x})$ for all $(\mathfrak{x}, \phi) \in \mathfrak{X}^{\#}$. Then,

$$s(\mathfrak{x}) \leq \mathrm{e}(\mathrm{R}(g^{\sim}, \mathfrak{x}, \phi) \leq S(\mathfrak{x}) \text{ holds for all } (\mathfrak{x}, \phi) \in \mathfrak{X}^{\#}\,. \tag{1}$$

Since g is bounded and μ-measurable and $\mu T < +\infty$, there is (cf. the proof of Satz 1a, p.95 in [13]) an $r \in \mathbb{R}$ such that

$r = \sup\limits_{\mathfrak{x} \in \mathfrak{X}} s(\mathfrak{x}) = {}^{\mathcal{F}\mathfrak{X}^{\#}} \lim\limits_{(x, \phi) \in \mathfrak{X}^{\#}} j(\mathfrak{x}, \phi)$ and

$r = \inf\limits_{\mathfrak{x} \in \mathfrak{X}} S(\mathfrak{x}) = {}^{\mathcal{F}\mathfrak{X}^{\#}} \lim\limits_{(\mathfrak{x}, \phi) \in \mathfrak{X}^{\#})} J(\mathfrak{x}, \phi)$

and therefore, by (1), $r \in {}^{\mathfrak{X}}\!\!\int g^{\sim} d\mu$. Furthermore, use Lemma 1.1 here and Theorem 12 in [7].

(b)$\Longrightarrow$(c). There is a set $M \in \mathrm{Dmn}\,\mu$ such that $\mu M = \emptyset$ and g bounded on $T \backslash M$. Consider the non-trivial case $M \neq \emptyset$. Define the function $f : T \longrightarrow E$ by letting $f(t) = g(t)$ on $T \backslash M$ and $f(t) = 0$ on M. Then f is bounded and (μ being complete) μ-measurable. Now, one applies (a)$\Longrightarrow$(c) to f instead of g occuring there. In view of Corollary 1.1, one has ${}^{\mathfrak{X}}\!\!\int f^{\sim} d\mu = {}^{\mathfrak{X}}\!\!\int g^{\sim} d\mu$. Thus (c) holds true by Lemma 1.1. □

In this paper, a function $g : T \longrightarrow \overline{\mathbb{R}}$ will be called *μ-summable* if $g \in \mathrm{Dmn}({}^{\mathrm{H}}\!\!\int \cdot\, d\mu)$ and $| {}^{\mathrm{H}}\!\!\int g\, d\mu |< +\infty$.

2 More on the relationship between Bochner integral and subdivision integral

For each set M and each mapping $g : M \longrightarrow E^T$, we write in this paper, for each $m \in M$, also g_m instead of $g(m)$. We use this agreement especially, if g is a sequence in E^T, i.e. if $M = \mathbb{N}$.

In the next considerations, E^T is considered as a linear space w.r. to the pointwise summation $+ : E^T \times E^T \longrightarrow E^T$ and the pointwise scalar multiplication $\cdot : E^T \times \mathbb{K} \longrightarrow E^T$.

Definition 2.1 (a) *A filtered family $(g, K, \mathfrak{b})$ in E^T is called a $(\mu, \mathfrak{Y})$-*Cauchy family*, if $0 \in {}^{\mathfrak{b}\times^\star\mathfrak{b}}\lim_{(k,l)\in K\times K} {}^{\mathfrak{Y}}\!\!\int \|g_k - g_l\|^\sim d\mu$, where $\mathfrak{b}\times^\star\mathfrak{b}$ denotes the cardinal product of the filter $\mathfrak{b}$ with itself. (Of course, we will speak of a $(\mu, \mathfrak{Y})$-*Cauchy sequence *in the case $K = \mathbb{N}$ and $\mathfrak{b} = \mathcal{F}\mathbb{N}$.)*

(b) *A linear subspace L of E^T is called to be $(\mu, \mathfrak{Y})$-*absolutely integrable*, if $f \in L$ implies ${}^{\mathfrak{Y}}\!\!\int f^\sim d\mu \neq \emptyset$ and ${}^{\mathfrak{Y}}\!\!\int \|f\|^\sim d\mu \neq \emptyset$.*

Obviously, a filtered family $(g, K, \mathfrak{b})$ in E^T is a $(\mu, \mathfrak{Y})$-Cauchy family if and only if, for all $\epsilon > 0$, there is a $B \in \mathfrak{b}$ such that $\emptyset \neq {}^{\mathfrak{Y}}\!\!\int \|g_k - g_l\|^\sim d\mu < \epsilon$ holds for all k, $l \in B$.

Definition 2.2 *Let $S \subseteq T$, $f : T \longrightarrow E$, and $(h, K, \mathfrak{b})$ be a filtered family in E^T.*

(a) $(h, K, \mathfrak{b})$ converges uniformly on S *to f, if for each $\epsilon > 0$ there is a $B \in \mathfrak{b}$ such that $\|f(t) - h_k(t)\| < \epsilon$ holds for all $k \in B$ and all $t \in S$. If $S = T$, we omit mostly the reference "on S" to S.*

(b) $(h, K, \mathfrak{b})$ converges uniformly μ-a.e. *(*on T*) to f, if there is a set $M \in \operatorname{Dmn}\mu$ with $\mu M = 0$ such that $(h, K, \mathfrak{b})$ converges uniformly on $T\backslash M$ to f.*

(c) $(h, K, \mathfrak{b})$ $(\mu, \mathfrak{Y})$-converges in mean to f, *if $0 \in {}^{\mathfrak{b}}\lim_{k\in K} {}^{\mathfrak{Y}}\!\!\int \|f - h_k\|^\sim d\mu$.*

Note a possible terminological conflict between the relation "converges uniformly μ-a.e." occuring here and the relation "converges μ-almost uniformly" appearing in connection with Egoroff's theorem (see, e.g., [12], p.88) but not occuring in this paper.

Definition 2.1/b is illustrated and the relationship of the notions introduced in the Definitions 2.1/a and 2.2/c to the corresponding classical notions is discussed in the following remark, in which we refer to results and a definition formulated later:

Remark 2.1 A) The linear subspaces L_1, L_2, $\cdots$, L_5 of E^T defined in (a)–(d) are $(\mu, \mathfrak{Y})$-absolutely integrable (for a linear subspace of E^T which is not $(\mu, \mathfrak{Y})$-absolutely integrable see the remarks after Theorem 3.3):

(a) Let $\mathfrak{Y} = \mathfrak{X}$ and $L_1 = \mathcal{ST}(\mathfrak{X}, E)$. (See Lemma 1.2.)

(b) Let $\mathfrak{Y} = \mathfrak{X}$ and $L_2 = \{f \in \operatorname{Dmn}\,({}^{\mathrm{D}}\!\!\int \cdot\, d\mu) \mid f \text{ is bounded } \mu\text{-a.e.}\}$. (See Definition 2.3, Theorem 2.2/a, and Proposition 2.10.)

(c) Let $\mathfrak{Y} = \Omega$ and let L_3 be the set of all $f \in \mathcal{ST}(\mathfrak{Y}, E)$ such that ${}^{\Omega}\!\!\int \|f\|^{\sim} d\mu \neq \emptyset$. (In view of Lemma 1.2, the last condition for f is just a short form for a condition about the absolute convergence of a certain sum. – See Theorem 14 in [11], p.128.)

(d) Let $\mathfrak{Y} = \Omega$, let $L_4 = \mathrm{Dmn}(\ {}^{\mathrm{D}}\!\!\int \cdot\, d\mu)$ and $L_5 = \mathrm{Dmn}(\ {}^{\mathrm{B}}\!\!\int \cdot\, d\mu)$. (See Definition 2.3, Proposition 2.8, Definition 8 in [7], p.20, Theorem 2.1, and Remark 2.3.)

B) Let $\mathfrak{Z}$ be a μ-partition system of T with $\mathfrak{Y} \subseteq \mathfrak{Z}$. Then, one has by Lemma 1.1: If a linear subspace L of E^T is $(\mu, \mathfrak{Y})$-absolutely integrable, it is also $(\mu, \mathfrak{Z})$-absolutely integrable.

C) If L is $(\mu, \mathfrak{Y})$-absolutely integrable, then the mapping $f \longmapsto \mathrm{e}({}^{\mathfrak{Y}}\!\!\int \|f\|^{\sim} d\mu)$ $(f \in L)$ is a semi-norm (denoted by $\mathrm{sn}(\mu, \mathfrak{Y})$) on L. (For the terminology, see Section 1/a.)

D) Let (L, sn) be a semi-normed linear space. Denote by σ the topology on L induced in the usual way by sn. Let $(h, K, \mathfrak{b}) \in \Phi(L)$ and $x \in L$. $(h, K, \mathfrak{b})$ is called a *Cauchy family in* the topological vector space (L, σ), if $h(\mathfrak{b})$ is a Cauchy filter (see, e.g., Definition 1 in [18], p.128) in (L, σ). Then, $(h, K, \mathfrak{b})$ is a Cauchy family in (L, σ) if and only if, for each $\epsilon > 0$, there is a $B \in \mathfrak{b}$ such that $\mathrm{sn}(h(k) - h(l)) < \epsilon$ for all $k, l \in B$. Denote by $\mathrm{Lim}_\sigma : \Phi(L) \longrightarrow \mathfrak{P}L$ the limit operator belonging to σ (see [10], p.183). Then $x \in \mathrm{Lim}_\sigma(h, K, \mathfrak{b})$ holds if and only if, for all $\epsilon > 0$, there is a $B \in \mathfrak{b}$ such that $\mathrm{sn}(h(k) - x) < \epsilon$ for all $k \in B$. If $\emptyset \neq \mathrm{Lim}_\sigma(h, K, \mathfrak{b})$, then $(h, K, \mathfrak{b})$ is a Cauchy family in (L, σ). – Now, let especially L be a linear subspace of E^T being $(\mu, \mathfrak{Y})$-absolutely integrable and let sn be the semi-norm $\mathrm{sn}(\mu, \mathfrak{Y})$ on L defined under C. Then $(h, K, \mathfrak{b})$ is a $(\mu, \mathfrak{Y})$-Cauchy family if and only if $(h, K, \mathfrak{b})$ is a Cauchy family in (L, σ). Furthermore, $(h, K, \mathfrak{b})$ $(\mu, \mathfrak{Y})$-converges in mean to x if and only if $x \in \mathrm{Lim}_\sigma(h, K, \mathfrak{b})$.

Proposition 2.1 *Let L be a linear subspace of E^T being $(\mu, \mathfrak{Y})$-absolutely integrable. Let $(g, K, \mathfrak{b})$ be a filtered family in L. Then, one has* (a), (b), (c), *and* (d)*:*

(a) *If $(g, K, \mathfrak{b})$ is a $(\mu, \mathfrak{Y})$-Cauchy family, then* ${}^{\mathfrak{b}}\!\lim\limits_{k \in K} \ {}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu \neq \emptyset$.

(b) *If there is an $f \in E^T$ such that $(g, K, \mathfrak{b})$ converges uniformly to f, then $(g, K, \mathfrak{b})$ is a $(\mu, \mathfrak{Y})$-Cauchy family.*

(c) *If $\mathfrak{Y}$ is $\vee$-closed and, for each set $M \in \mathrm{Dmn}\,\mu$ having the properties $\mu M = 0$, $\emptyset \neq M \neq E$, the set $\{M, T\backslash M\}$ is an element of $\mathfrak{Y}$, then* (b) *holds with "uniformly μ-a.e." instead of "uniformly".*

(d) *If there is an $f \in E^T$ such that $(g, K, \mathfrak{b})$ $(\mu, \mathfrak{Y})$-converges in mean to f, then $(g, K, \mathfrak{b})$ is a $(\mu, \mathfrak{Y})$-Cauchy family.*

(Remark: Of course, the premise within the statement (c) is satisfied, e.g., if $\mathfrak{Y} = \mathfrak{X}$.)

Proof: Ad (a) Let $h = (\mathrm{e}({}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu))_{k\in K}$. Then, by the premise within (a), $(h, K, \mathfrak{b})$ is a Cauchy family in the topological vector space (E, τ) (for the terminology, see Remark 2.1/D), therefore (since (E, τ) is complete) the conclusion within (a) is true.

Ad (b) and (c). We skip the trivial case $\mu T = 0$. Assume that $(g, K, \mathfrak{b})$ converges uniformly μ-a.e. to f $(\in E^T)$. Then, there a set $M \in \mathrm{Dmn}\,\mu$ with $\mu M = 0$ such that $(g, K, \mathfrak{b})$ converges uniformly on $T\backslash M$ to f. Let $\epsilon > 0$. Then, there is a $B \in \mathfrak{b}$ such that $\|f(t) - g_k(t)\| < \frac{\epsilon}{3(\mu T)}$ for all $k \in B$ and all $t \in T\backslash M$. Therefore, with the abbreviation $N = T\backslash M$, the chain

$$\begin{aligned} \emptyset \neq {}^{\mathfrak{Y}}\!\!\int \|g_k - g_l\|^{\sim} d\mu &\leq \sup_{t\,\in\, N} \,(\|g_k(t) - g_l(t)\|)\mu_N(N) \leq \\ &\leq \sup_{t\,\in\, N}\,(\|f(t) - g_k(t)\| + \|f(t) - g_l(t)\|)\mu T < \epsilon \end{aligned}$$

holds for all $k, l \in B$ in the case $M = \emptyset$ without any further assumptions (thus (b) is true), while, in the case $M \neq \emptyset$, it holds for all $k, l \in B$ under the assumption made in (c) (use of Lemma 1.3).

Ad (d). Use Proposition 26 in [7], p.17, the elementary fact that, for each pair of mappings $f, j : T \longrightarrow \mathbb{R}$, $f(t) \leq j(t)$ for all $t \in T$ implies ${}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu \leq {}^{\mathfrak{Y}}\!\!\int j^{\sim} d\mu$, furthermore the linearity of L and the supposition for L. □

For the remainder of this paper, let $f : T \longrightarrow E$.

By Theorem 12 in [7], p.16, for a sequence g in E^T the following is evident: g (μ, Ω)-converges in mean to f if and only if, for some $p \in \mathbb{N}$, the function $\|f - g_n\|$ is μ-summable for all $n \geq p$, and $\lim\limits_{n\,\geq\, p} {}^{\mathrm{H}}\!\!\int \|f - g_n\|\, d\mu = 0$ holds.

In the general case of $(\mu, \mathfrak{Y})$-convergence in mean of filtered families one has the following: If $(g, K, \mathfrak{b})(\in \Phi(E^T))$ $(\mu, \mathfrak{Y})$-converges in mean to f, then (by Lemma 1.1) there is a $B \in \mathfrak{b}$ such that $\emptyset \neq {}^{\mathfrak{Y}}\!\!\int \|f - g_k\|^{\sim} d\mu = {}^{\Omega}\!\!\int \|f - g_k\|^{\sim} d\mu$, thus (by Theorem 12 in [7]) ${}^{\mathrm{H}}\!\!\int \|f - g_k\|\, d\mu = \mathrm{e}({}^{\Omega}\!\!\int \|f - g_k\|^{\sim} d\mu)$ holds for all $k \in B$, and $0 = {}^{\mathfrak{c}}\!\lim\limits_{k\,\in\, B} {}^{\mathrm{H}}\!\!\int \|f - g_k\|\, d\mu$ with $\mathfrak{c} = \mathfrak{b}_B$ (= trace of the filter $\mathfrak{b}$ in the set B). – Let L be a linear subspace of E^T such that $j \in L$ implies that $\|j\|$ is μ-measurable. (If the measure μ is complete, the linear space $L = \mathrm{Dmn}\,({}^{\mathrm{D}}\!\!\int \cdot\, d\mu)$ (see Definition 2.3) has this property (see Proposition 2.8/b).) Then, the intersection of the relation "$(\mu, \mathfrak{Y})$-converges in mean to" (between the class $\Phi(E^T)$ and the set E^T) with the class $\Phi(L) \times L$ is "μ-almost" a function, i.e., if $(g, K, \mathfrak{b}) \in \Phi(L)$, $f, j \in L$, and $(g, K, \mathfrak{b})$ $(\mu, \mathfrak{Y})$-converges in mean to f and j, then $f(t) = j(t)$ holds for μ-a.a. $t \in T$. (Use of the preceding lines, of [13], p.102, 4.3.4, Satz 1, and of the note following to Lemma 1.1.)

The assertion (a)$\Longrightarrow$(b) in the following proposition (cf. [5], p.130, Proposition 14) generalizes a well-known relationship between the classical mean convergence and pointwise convergence (see, e.g., [21], p.219, within a proof):

Proposition 2.2 *Assume the measure μ to be complete. Let g be a sequence in E^T. Then* (a) *implies* (b)*:*

(a) *g $(\mu, \mathfrak{Y})$-converges in mean to f.*

(b) *There is a subsequence of g converging pointwise μ-a.e. to f.*

If, additionally, for almost all (i.e. eventually all) $n \in \mathbb{N}$, $f - g_n$ is bounded μ-a.e., then (a) *implies* (c)*:*

(c) *g $(\mu, \mathfrak{X})$-converges in mean to f.*

Proof: Assume (a). Then, by Lemma 1.1, $g\,(\mu, \Omega)$-converges in mean to f, thus (by Theorem 12 in 3.Satz in [13], p.96), for some $p \in \mathbb{N}$, the function $\|f - g_n\|$ is μ-summable, therefore (by 3.Satz in [13], p.96) μ-measurable (since μ is complete) for all $n \geq p$, and one has $\lim\limits_{n \geq p} {}^{\mathrm{H}}\!\!\int \|f - g_n\|\, d\mu = 0$. Now, *formally*, we have the same situation as in the proof of Proposition 14 in [5], p.130, and we can continue *formally* almost word by word like there: For each $k \in \mathbb{N}$ there exists an $n_k \in \mathbb{N}$ with $n_k \geq p$ such that

$$ {}^{\mathrm{H}}\!\!\int \|f - g_{n_k}\|\, d\mu < (1/(2^{2k})\,, \tag{1} $$

and, of course, one can reach that $n_k < n_{k+1}$ holds for each $k \in \mathbb{N}$. The sequence (g_{n_k}) converges pointwise μ-a.e. to f.

In order to show this, one observes that, for each $k \in \mathbb{N}$, the set E_k defined by

$$ E_k = \{t \in T \mid 1/(2^k) \leq \|f(t) - g_{n_k}(t)\|\} \tag{2} $$

is a member of $\mathrm{Dmn}\,\mu$ for all $k \in \mathbb{N}$, because $\|f - g_{n_k}\|$ is μ-measurable. Using 4.Satz in [13], p.100, one obtains (by means of (1) and (2)) that

$$ \mu(E_k) \leq 1/(2^k) \text{ holds for all } k \in \mathbb{N}\,. \tag{3} $$

Let, now, $F = \bigcap_{k \in \mathbb{N}} \bigcup_{i=k}^{\infty} E_i$. Then, one has (in view of (3) and of the completeness of μ) $\mu F = 0$. Using (2), one obtains that on $T \backslash F$ the sequence (g_{n_k}) converges pointwise to f. We have showed (b).

Now assume $f - g_n$ to be bounded μ-a.e. for almost all $n \in \mathbb{N}$, say for all $n \geq q$ for some $q \geq p$. Let $n \geq q$. The function $\|f - g_n\|$ being bounded μ-a.e. and μ-measurable, one obtains (by Lemma 1.4) ${}^{\mathrm{H}}\!\!\int \|f - g_n\|\, d\mu \in {}^{\mathfrak{X}}\!\!\int \|f - g_n\|^{\sim} d\mu = {}^{\Omega}\!\!\int \|f - g_n\|^{\sim} d\mu$. Therefore, since g (μ, Ω)-converges in mean to f, one has (c). □

(Note: For the relationship between the relation "(μ, Ω)-converges in mean to" and the strict specialization of the relation "converges in mean to" in the space $\mathcal{L}^1_E(m)$ defined in [5],p.127, to the present situation of a positive scalar measure, see Remark 2.4 below.)

Lemma 2.1 (Proposition 32 in [7]). *Assume that* (a) *E is finite-dimensional or* (b) $\mathfrak{Y} \subseteq \mathfrak{X}$. *Let $(h, K, \mathfrak{b})$ be a filtered family in E^T. Then, one has: If $(h, K, \mathfrak{b})$ $(\mu, \mathfrak{Y})$-converges in mean to f, then* $^{\mathfrak{b}}\lim\limits_{k \in K} {}^{\mathfrak{Y}}\!\!\int h_k{}^{\sim} d\mu \subseteq {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$.

Proposition 2.3 *Let μ be complete and f be bounded μ-a.e. Let g be a sequence in $\mathcal{ST}(\mathfrak{X}, E)$.*

Then (a) *implies* (b)*:*

(a) *g (μ, Ω)-converges in mean to f.*

(b) $\lim\limits_{n \in \mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu \subseteq {}^{\mathfrak{X}}\!\!\int f^{\sim} d\mu$.

Proof: By Proposition 2.2, (a) implies that g $(\mu, \mathfrak{X})$-converges in mean to f, thus, by Lemma 2.1, one has (b). $\square$

Remark 2.2 As an immediate consequence of Proposition 31 in [7], one obtains the following statement: If ${}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu \neq \emptyset$ and the filtered family $(h, K, \mathfrak{b})$ in $E^T (\mu, \mathfrak{Y})$-converges in mean to f, then $^{\mathfrak{b}}\lim\limits_{k \in K} {}^{\mathfrak{Y}}\!\!\int h_k{}^{\sim} d\mu \subseteq {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$.

The next considerations assure that our setting of convergence in mean (using the subdivision integral, in particular the Birkhoff integral ${}^{\Omega}\!\!\int \cdot \, d\mu$, in its definition) fits in the usual approach to convergence in mean occuring in a natural way in the theory of completion of certain seminormed spaces (see, e.g., [13], p.137, [21], p.218–220, [5], p.127–133).

Lemma 2.2 *Let g be a $(\mu, \mathfrak{Y})$-Cauchy sequence in $\mathcal{ST}(\mathfrak{Y}, E)$. Then there are a $(\mu, \mathfrak{Y})$-Cauchy sequence h in $\mathcal{ST}(\mathfrak{Y}, E)$, a strictly increasing mapping $\phi : \mathbb{N} \longrightarrow \mathbb{N}$, and a mapping $j : T \longrightarrow E$ such that the statements* (1)–(7) *defined next hold true:*

(1) *$h_n(t) = g_n(t)$ for all $n \in \mathbb{N}$ and for μ-a.a. $t \in T$.*

(2) *$j(t) = \lim\limits_{k \in \mathbb{N}} h_{\phi(k)}(t)$ for all $t \in T$.*

(3) *$(j - g_n)(t) = (j - h_n)(t)$ for all $n \in \mathbb{N}$ and for μ-a.a. $t \in T$.*

(4) *$\|j\|$ and all functions $\|j - h_n\|$ (with $n \in \mathbb{N}$) are μ-measurable.*

(5) *g and h (μ, Ω)-converge in mean to j.*

(6) *If μ is complete, then $\|j - g_n\|$ is μ-measurable for all $n \in \mathbb{N}$.*

(7) *If $j - g_n$ is bounded μ-a.e. for almost all (i.e. for eventually all) $n \in \mathbb{N}$, then g and h $(\mu, \mathfrak{X})$-converge in mean to j.*

(Remark: Within the scope of the lemma, the four implications

$$(1)\Longrightarrow(3);\ (2)\Longrightarrow(4);\ (1)\wedge(2)\Longrightarrow(6);\ (3)\wedge(4)\wedge(5)\Longrightarrow(7)$$

hold true.)

Proof: (We adapt the main part of the proof of Theorem 1 in [21], p.218–219, where the Daniell integration for measurable real-valued functions instead of ${}^{\mathrm{H}}\!\!\int \cdot\, d\mu$ is used and, up to the different terminology, our case $\mathfrak{Y} = \mathfrak{X}$ for complete measure μ is covered; cf. also the proof of Satz 1 in [13], p.137.) Without any danger of confusion, the sign ${}^{c}\Sigma$ denotes the conditional summation either in $\overline{\mathbb{R}}$ or in E, depending on its application (for the terminology, see [10], p.187, or [11], p.119, respectively). Tacitly, we will use of Theorem 12 in [7].

Since g ist a $(\mu, \mathfrak{Y})$-Cauchy sequence (thus, by Lemma 1.1, a (μ, Ω)-Cauchy sequence), there is a stricly increasing mapping $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$\underset{k\,\in\,\mathbb{N}}{{}^{\mathrm{c}}\Sigma}\ {}^{\mathrm{H}}\!\!\int \|g_{\phi(k+1)} - g_{\phi(k)}\|\, d\mu < +\infty\,. \tag{a}$$

For remainder of the proof, we put $\phi(k) = n_k$ for all $k \in \mathbb{N}$. Since, for each $k \in \mathbb{N}$, the function $\|g_{n_{k+1}} - g_{n_k}\|$ is μ-measurable, one has by (a) and Levi's theorem on monotone convergence (see 3.Satz, p.105 in [13])

$${}^{\mathrm{H}}\!\!\int \Big(\underset{k\,\in\,\mathbb{N}}{{}^{\mathrm{c}}\Sigma}\ \|g_{n_{k+1}}(t) - g_{n_k}(t)\|\Big)_{t\in T} d\mu < +\infty\,. \tag{b}$$

Define the function $G : T \longrightarrow \overline{\mathbb{R}}$ by

$$G(t) = \|g_{n_1}(t)\| + \underset{k\,\in\,\mathbb{N}}{{}^{\mathrm{c}}\Sigma}\ \|g_{n_{k+1}}(t) - g_{n_k}(t)\|$$

for all $t \in T$. By (b) (cf. [13], p.96, 2.Satz), there is a set $M \in \mathrm{Dmn}\,\mu$ such that $\mu M = 0$ and $G(t) < +\infty$ for all $t \in T\backslash M$. Define the sequence h in $\mathcal{ST}(\mathfrak{Y}, E)$ by letting, for each $n \in \mathbb{N}$, $h_n(t) = g_n(t)$ for all $t \in T\backslash M$ and $h_n(t) = 0$ for all $t \in M$ (where 0 denotes the zero element of E). (Thus, one has (1). Since g is a $(\mu, \mathfrak{Y})$-Cauchy sequence in $\mathcal{ST}(\mathfrak{Y}, E)$, so also h.) Then, for each $t \in T$, the set $L(t)$, defined by

$$L(t) = \{h_{n_1}(t)\} + \underset{k\,\in\,\mathbb{N}}{{}^{\mathrm{c}}\Sigma}\ \{h_{n_{k+1}}(t) - h_{n_k}(t)\}\,,$$

is non-empty, since absolute convergence of a series in E implies its conditional convergence (for the notation, see Theorem 14 in [11], p.128). Define the mapping $j : T \longrightarrow E$ by letting $j(t) = \mathrm{e}L(t)$ for all $t \in T$. Then, (2) and, therefore, also (4) holds, since one has, e.g., $\|j(t) - h_n(t)\| = \lim\limits_{k \in \mathbb{N}} \|h_{n_k}(t) - h_n(t)\|$ for all $t \in T$ and all $n \in \mathbb{N}$. Furthermore, one has, for each $q \in \mathbb{N}$ and each $t \in T$, $j(t) - h_{n_q}(t) = {}^{\mathrm{c}}\!\!\sum\limits_{k \geq q} (h_{n_{k+1}}(t) - h_{n_k}(t))$, thus

$$\lim_{q \in \mathbb{N}} {}^{\mathrm{H}}\!\!\int \|j - h_{n_q}\| \, d\mu = 0 \,. \tag{c}$$

[We have used the following: By the definition of h, the statement (a) and therefore (b) holds for h instead of g. Consequently, for each $q \in \mathbb{N}$, the function H_q, defined by $H_q(t) = {}^{\mathrm{c}}\!\!\sum\limits_{k \geq q} \|h_{n_{k+1}}(t) - h_{n_k(t)}\|$ for all $t \in T$, is μ-summable. Now apply Fatou's lemma (see 2.Satz, p.105 in [13]) to the sequence (H_q) and use (4).] – Since

$${}^{\mathrm{H}}\!\!\int \|j - h_n\| \, d\mu \leq {}^{\mathrm{H}}\!\!\int \|h_n - h_{n_q}\| \, d\mu + {}^{\mathrm{H}}\!\!\int \|h_{n_q} - j\| \, d\mu$$

holds for all n, $q \in \mathbb{N}$, one obtains (using (c) and the fact that h is a $(\mu, \mathfrak{Y})$-Cauchy sequence) that h (μ, Ω)-converges in mean to j, therefore, using (3) and Corollary 1.1, the statement (5). Assume the premise within the statement (7) and choose a $p \in \mathbb{N}$ such that the function $j - g_n$ is bounded μ-a.e. for all $n \geq p$. Let $n \geq p$. Then, in view of (3), (4), Corollary 1.1, and Lemma 1.4, the statement (d) formulated next holds true:

$$\emptyset \neq {}^{\mathfrak{X}}\!\!\int \|j - h_n\|^{\sim} d\mu = {}^{\mathfrak{X}}\!\!\int \|j - g_n\|^{\sim} d\mu = {}^{\Omega}\!\!\int \|j - g_n\|^{\sim} d\mu \,. \tag{d}$$

Combining (d) with (5), one obtains the conclusion within (7). □

In the following remark, where we slightly generalize Proposition 33 and Theorem 13 in [7], p.21, the preceding lemma is applied for $\mathfrak{Y} = \Omega$:

Remark 2.3 In order to have the μ-partition system $\mathfrak{X}$ available, we have the measure μ assumed to be *finite* in this paper. For this remark, we replace this assumption by requiring μ to be (only) σ-finite. Then, e.g., Definitions 2.1 and 2.2 need not be changed and the parts (a) and (d) of Proposition 2.1 keep to be valid. Also, Lemma 2.2, reformulated with the words "and if μ is finite" added to the premise within (7), remains true. We assume these modifications, now, tacitly: In Definition 8 in [7], p.20, the statements (a) and (b) imply (by Proposition 2.1/d) that the sequence (g_n), occuring there, is a (μ, Ω)-Cauchy sequence. (Observe that, by (a), $g_n \in L_3$ (see Remark 2.1) holds for all $n \in \mathbb{N}$.) Therefore (by Lemma 2.2 and Corollary 1.1) the T-ary partial operation ${}^{\mathrm{B}}\!\!\int \cdot \, d\mu$ in E is the set of all $(j, x) \in E^T \times E$ such there exists a (μ, Ω)-Cauchy sequence h in $\mathcal{ST}(\Omega, E)$ having the properties (α), (β), and (γ) defined next:

(α) ${}^{\Omega}\!\!\int \|h_n\|^{\sim} d\mu \neq \emptyset$ for all $n \in \mathbb{N}$.

(β) h converges pointwise μ-a.e. to j.

(γ) $x \in \lim\limits_{n \in \mathbb{N}} {}^{\Omega}\!\!\int h_n{}^{\sim} d\mu$.

The elements of $\mathrm{Dmn}({}^{\mathrm{B}}\!\!\int \cdot\, d\mu)$ are now called to be B-*Bochner integrable*. We assert: *If f is* B-*Bochner integrable, then $\|f\|$ is* B-*Bochner integrable in the Banach space* $(\mathbb{R}, |\cdot|)$, *and one has* ${}^{\mathrm{B}}\!\!\int \|f\|\, d\mu \in {}^{\Omega}\!\!\int \|f\|^{\sim} d\mu$, *thus $\|f\|$ is μ-summable.* [*Proof.* Choose $x \in E$ and a (μ, Ω)-Cauchy sequence h in $\mathcal{ST}(\Omega, E)$ having the properties (α), (β) with f instead of j, and (γ). Then, the sequence $(\|h_n\|)$ is a (μ, Ω)-Cauchy sequence in $\mathcal{ST}(\Omega, \mathbb{R})$ (use of Lemma 1.2 for $E = \mathbb{R}$) with the property (α), thus (by Proposition 2.1/a, applied to $L = L_3$ (see Remark 2.1) with $E = \mathbb{R}$) $\emptyset \neq \lim\limits_{n \in \mathbb{N}} {}^{\Omega}\!\!\int \|h_n\|^{\sim} d\mu$. If one combines this with (α), the modified (β), and the continuity of $\|\cdot\|$, one obtains that $\|f\|$ is B-Bochner integrable in $\mathbb{R}$. Since $\mathbb{R}$ is finite-dimensional and $(\|h_n\|)\,(\mu, \Omega)$-converges in mean to $\|f\|$ (use of Lemma 2.2 and Corollary 1.1, one has (by Lemma 2.1) $\lim\limits_{n \in \mathbb{N}} {}^{\Omega}\!\!\int \|h_n\|^{\sim} d\mu \subseteq {}^{\Omega}\!\!\int \|f\|^{\sim} d\mu$, thus the remainder of the assertion is proven (use of Theorem 12 in [7],p.16).$\square$] Therefore, in Proposition 33 in [7], p.21, consequently in Theorem 13 in [7], p.21,(which uses this proposition) the supposition of the completeness of μ is allowed to be deleted. (In the proof of the latter theorem, the second occurence of "$h^\star$" should be replaced by "h" (printing error).)

We apply Lemma 2.2 for $\mathfrak{Y} = \mathfrak{X}$:

Proposition 2.4 *Let g be a $(\mu, \mathfrak{X})$-Cauchy sequence in $\mathcal{ST}(\mathfrak{X}, E)$ converging pointwise μ-a.e. to f. Then, one has* (a) *and* (b)*:*

(a) *g (μ, Ω)-converges in mean to f and* $\emptyset \neq \lim\limits_{n \in \mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu$. *If the linear space E is finite-dimensional, then, additionally,* $\lim\limits_{n \in \mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu = {}^{\Omega}\!\!\int f^{\sim} d\mu$.

(b) *If $f - g_n$ is bounded μ-a.e. for almost all (i.e. for eventually all) $n \in \mathbb{N}$, then g $(\mu, \mathfrak{X})$-converges in mean to f, and* $\emptyset \neq \lim\limits_{n \in \mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu = {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$ *holds, provided that* $\mathfrak{X} \subseteq \mathfrak{Y}$.

Proof: Choose a $(\mu, \mathfrak{X})$-Cauchy sequence h in $\mathcal{ST}(\mathfrak{X}, E)$, a strictly increasing mapping $\phi : \mathbb{N} \longrightarrow \mathbb{N}$, and a mapping $j : T \longrightarrow E$ such that the statements (1)–(7) in Lemma 2.2 hold true. Then, by (1) and (2), $f(t) = j(t)$ holds for μ-a.a. $t \in T$, therefore, by the

Corollary 1.1 and by (5), the sequence g (μ,Ω)-converges in mean to f. By Proposition 2.1/a, one has $\emptyset \neq \lim\limits_{n\in\mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu$. Therefore, if the linear space E is finite-dimensional, then, by Lemma 2.1, $\lim\limits_{n\in\mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu = {}^{\Omega}\!\!\int f^{\sim} d\mu$. (Observe that in any case ${}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu = {}^{\Omega}\!\!\int g_n{}^{\sim} d\mu$ by Lemma 1.2.) This settles the proof of (a). Combining (a) with (7) and Lemma 1.1, one obtains (b). □

We adapt now the following definition of the Bochner integral using only finite μ-partitions of T:

Definition 2.3 (cf. Definition 1 in [5], p.120, there for vector measures). *We define the relation* ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu$ *between* E^T *and* E *to be the set of all* $(j,x) \in E^T \times E$ *such that there is a* $(\mu,\mathfrak{X})$*-Cauchy sequence* g *in* $\mathcal{ST}(\mathfrak{X},E)$ *converging pointwise* μ*-a.e. to* j *and having the property* $x \in \lim\limits_{n\in\mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu$. *We call* ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu$ *the* D-Bochner integral. *(Remark: The relation* ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu$ *is a* T*-ary partial operation in* E *(use, e.g., Proposition 2.7). The elements of* Dmn $({}^{\mathrm{D}}\!\!\int \cdot\, d\mu)$ *are called to be* D-Bochner integrable. *Instead of* $({}^{\mathrm{D}}\!\!\int \cdot\, d\mu)^{\wedge}$ *we write also* ${}^{\mathrm{D}}\!\!\int^{\wedge} \cdot\, d\mu$ *(see Section 1/c).*

Proposition 2.5 *If* f *is an* $\mathfrak{X}$*-step function determined by* $(\mathfrak{x},\chi)$ *with* $\mathfrak{x} \in \mathfrak{X}$ *and* $\chi : \mathfrak{x} \longrightarrow E$*, then* f *is D-Bochner integrable, and one has* ${}^{\mathrm{D}}\!\!\int f\, d\mu = \sum\limits_{X\in\mathfrak{x}} f(\chi(X))\mu X =$ e$({}^{\mathfrak{X}}\!\!\int f^{\sim} d\mu)$.

Proof: Ad (a). The sequence $(f, f, \cdots)$ is a $(\mu,\mathfrak{X})$-Cauchy sequence. Use, furthermore, Lemma 1.2 and the finiteness of $\mathfrak{x}$. □

We denote by ${}^{\mathrm{el}}\!\!\int \cdot\, d\mu$ the set of all $(f,x) \in \mathcal{ST}(\mathfrak{X},E) \times E$ such that $x = \sum\limits_{X\in\mathfrak{x}} \chi(X)\mu X$ holds true for some pair $(\mathfrak{x},\lambda)$ with $\mathfrak{x} \in \mathfrak{X}$ and $\lambda : \mathfrak{x} \longrightarrow E$ which determines f. The relation ${}^{\mathrm{el}}\!\!\int \cdot\, d\mu$ is a function on $\mathcal{ST}(\mathfrak{X},E)$ into E being called the *elementary integral* for (μ,E).

By Proposition 2.5, one has $({}^{\mathfrak{X}}\!\!\int \cdot\, d\mu)^{\vee} | \mathcal{ST}(\mathfrak{X},E) = {}^{\mathrm{el}}\!\!\int \cdot\, d\mu$.

Avoiding the use of ${}^{\mathfrak{X}}\!\!\int \cdot\, d\mu$, but using the elementary integral ${}^{\mathrm{el}}\!\!\int \cdot\, d\mu$ twice (first in the Banach space $\mathbb{R}$, secondly in E)), one can describe the mapping ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu$ in the following way:

Proposition 2.6 ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu$ *is the set of all* $(f,x) \in E^T \times E$ *such there exists a sequence* g *in* $\mathcal{ST}(\mathfrak{X},E)$ *such that* (a), (b), *and* (c) *hold true:*

(a) $0 = \lim\limits_{\substack{n\in\mathbb{N}\\ m\in\mathbb{N}}} {}^{\mathrm{el}}\!\!\int \|g_n - g_m\|\, d\mu$;

(b) *g converges pointwise μ-a.e. to f;*

(c) $x = \lim_{n \in \mathbb{N}} {}^{\mathrm{el}}\!\!\int g_n \, d\mu$.

Proof: Use of Lemma 1.2. □.

The following characterization of the D-Bochner integral provides its description via mean convergence, which becomes more familiar for measurable integrands j (see Proposition 3.2 below and [4]).

Proposition 2.7 *In every case (resp. in the case of complete μ) ${}^{\mathrm{D}}\!\!\int \cdot \, d\mu$ is the set of all $(j, x) \in E^T \times T$ such there exists a sequence g in $\mathcal{ST}(\mathfrak{X}, E)$ having the properties* (a), (b), *and* (c) *(resp. the properties* (b) *and* (c)*) defined next:*

(a) *g converges to j pointwise μ-a.e..*

(b) *g (μ, Ω)-converges in mean to j.*

(c) $x = \lim_{n \in \mathbb{N}} {}^{\mathrm{el}}\!\!\int g_n \, d\mu$.

Proof: Observe first that, for each sequence g in $\mathcal{ST}(\mathfrak{X}, E)$, g is a $(\mu, \mathfrak{X})$-Cauchy sequence if and only if g is a (μ, Ω)-Cauchy sequence. Now, apply the Propositions 2.4 and 2.1/d in the general case, and, additionally, Proposition 2.2 in the special case of a complete μ. □

Recall that the T-ary partial operation $({}^{\Omega}\!\!\int \cdot \, d\mu)^{\vee}$ in E (being also denoted by ${}^{\Omega}\!\!\int^{\vee} \cdot \, d\mu$) is defined to be the function $\{(g, x) \in E^T \times E \mid x \in {}^{\Omega}\!\!\int g^{\sim} d\mu\}$ (see Corollary to Proposition 18 and Section 0/a in [6]).

The Bochner integral ${}^{\mathrm{B}}\!\!\int \cdot \, d\mu$, as it has been described in [7], Definition 8 there, is related to ${}^{\mathrm{D}}\!\!\int \cdot \, d\mu$ and to ${}^{\Omega}\!\!\int \cdot \, d\mu$ in the following way:

Theorem 2.1 *The following chain holds true:*

$$ {}^{\mathrm{D}}\!\!\int \cdot \, d\mu \subseteq {}^{\mathrm{B}}\!\!\int \cdot \, d\mu \subseteq ({}^{\Omega}\!\!\int \cdot \, d\mu)^{\vee} . $$

Proof: For the first inclusion, use Proposition 2.7, for the second one use Theorem 13 in [7], where the completeness of μ need not be supposed (see Remark 2.3). □

As a consequence, one has: If f is D-Bochner integrable, then ${}^{\mathrm{D}}\!\!\int f \, d\mu$ is the limit of a filtered family of Riemann sums which are constructed by means of f. In Theorem 2.2, it will be assured that, if f is bounded μ- a.e., those Riemann sums can be chosen as *finite* sums. (This remark justifies the title of the paper.)

Proposition 2.8 (For the first the assertion in (a), see Proposition 4 in [5], p.122.) *Let f be* D*-Bochner integrable. Then, one has* (a) *and* (b)*:*

(a) $\|f\|$ *is* D*-Bochner integrable in the Banach space* $(\mathbb{R}, |\cdot|)$ *and μ-summable; more precisely, one has the chain*

$$^{\mathrm{D}}\!\!\int \|f\| \, d\mu = {}^{\Omega}\!\!\int^{\vee} \|f\| \, d\mu = {}^{\mathrm{H}}\!\!\int \|f\| \, d\mu < +\infty.$$

(b) *If μ is complete, then $\|f\|$ is μ-measurable.*

Proof: Proving the first assertion in (a), use Proposition 2.5, the continuity of $\|\cdot\|$, and the fact that, for each $(\mu, \mathfrak{X})$-Cauchy sequence g in $\mathcal{ST}(\mathfrak{X}, E)$, the sequence $\|g_n\|_{n \in \mathbb{N}}$ is a $(\mu, \mathfrak{X})$-Cauchy sequence in $\mathcal{ST}(\mathfrak{X}, \mathbb{R})$. Proving the μ-summability of f, combine Theorem 12 in [7] with Theorem 2.1. In order to prove (b), apply either additionally 3.Satz, p.96 in [13] or combine the continuity of $\|\cdot\|$ with the μ-measurability of each function $\|g_n\|$. □

Remark 2.4 Denote, for this remark, the linear space Dmn$({}^{\mathrm{D}}\!\!\int \cdot \, d\mu)$ by L and define the relation "D*-converges in mean to*" $(= R$ for abbreviation) to be the set of all $(h, g) \in L^{\mathbb{N}} \times L$ such that $0 = \lim\limits_{n \in \mathbb{N}} {}^{\mathrm{D}}\!\!\int \|g - h_n\| \, d\mu$ holds (use of Proposition 2.8). For abbreviation, let S denote the relation "(μ, Ω)-converges in mean to". Then, by Proposition 2.8, one has the equation $R = S \cap (L^{\mathbb{N}} \times L)$ (where, of course, each sequence j in E^T is identified with the filtered family $(j, \mathbb{N}, \mathcal{F}\mathbb{N})$). The notion of the "D-convergence in mean" is a strict specialization of the notion of "mean convergence" in $\mathcal{L}^1_E(m)$ defined in [5], p.127. Now, reconsider the proof of Proposition 2.2: If we would have supposed f and the functions g_n to be D-Bochner integrable, the greatest part of the proof would have become a strict special case of the cited proof in [5].

Lemma 2.3 (cf. [5], p.128, Proposition 12). *If g is a $(\mu, \mathfrak{X})$-Cauchy sequence in $\mathcal{ST}(\mathfrak{X}, E)$ converging pointwise μ-a.e to f, then*

$$0 = \lim_{n \in \mathbb{N}} {}^{\mathrm{D}}\!\!\int \|f - g_n\| \, d\mu$$

holds in the Banach space $(\mathbb{R}, |\cdot|)$*, i.e.* " g D*-converges in mean to* f " (see Remark 2.4).

Proof: Formulating the lemma, we have tacitly used Propositions 2.4/a and 2.8. In order to prove the assertion, just reformulate the proof of the cited Proposition in [5] in our terminology. □

Within the preceding lemma, ${}^{\mathrm{D}}\!\!\int \|f - g_n\| \, d\mu \in {}^{\Omega}\!\!\int \|f - g_n\|^{\sim} d\mu$ holds for all $n \in \mathbb{N}$ by Proposition 2.8. Thus, the lemma implies the first assertion of Proposition 2.4/a. [But, in view of the implicit dependence of Proposition 2.8 on Proposition 2.4/a, this implication is not another proof of that assertion.] On the other hand, one can use it in the proof of the following description of the D-Bochner integral (for complete μ) related to $(E, \|\cdot\|)$ in terms of the D-Bochner integral related to the Banach space $(\mathbb{R}, |\cdot|)$ and the $(\mu, \mathfrak{X})$-subdivision integral of $\mathfrak{X}$-step functions:

Proposition 2.9 *Let M be the set of all $(j, x) \in E^T \times E$ such that there exists a sequence g in $\mathcal{ST}(\mathfrak{X}, E)$ having the properties*

$$0 \in \lim_{n \in \mathbb{N}} {}^{\mathrm{D}}\!\!\int^{\wedge} \|j - g_n\|^{\sim}\, d\mu \text{ and } x \in \lim_{n \in \mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu\,.$$

Then ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu \subseteq M$ holds in any case. If μ is complete, then $M = {}^{\mathrm{D}}\!\!\int \cdot\, d\mu$.

Proof: Proving ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu \subseteq M$, one uses Definition 2.3, Lemma 2.3, and Proposition 2.7. Proving $M \subseteq {}^{\mathrm{D}}\!\!\int \cdot\, d\mu$, one uses Proposition 2.8/b (in the Banach space $(\mathbb{R}, |\cdot|)$), the completeness of μ, and Proposition 2.7. $\square$

Now, we raise the question, whether or under which conditions in the statement ${}^{\mathrm{D}}\!\!\int^{\wedge} f^{\sim} d\mu \subseteq {}^{\Omega}\!\!\int f^{\sim} d\mu$ (in Theorem 2.1) the μ-partition system Ω is allowed to be replaced by $\mathfrak{X}$. Here is a partial answer:

Theorem 2.2 (a) *If f is bounded μ-a.e., then ${}^{\mathrm{D}}\!\!\int^{\wedge} f^{\sim} d\mu \subseteq {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$ holds, provided that $\mathfrak{X} \subseteq \mathfrak{Y}$.*

(b) *If the sequence g in $\mathcal{ST}(\mathfrak{X}, E)$ converges uniformly μ-a.e. to f, then $\emptyset \neq \lim\limits_{n \in \mathbb{N}} {}^{\mathfrak{Y}}\!\!\int g_n{}^{\sim} d\mu =$*
$= {}^{\mathrm{D}}\!\!\int^{\wedge} f^{\sim} d\mu = {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$ holds, provided that $\mathfrak{X} \subseteq \mathfrak{Y}$.

Proof: Ad (a). Note: If μ is complete, then, one has by Proposition 2.7, Proposition 2.3, and Lemma 1.1 the validity of (a). In the general case we conclude the following way:

Assume the premise of (a) and the inclusion $\mathfrak{X} \subseteq \mathfrak{Y}$. Let $x \in {}^{\mathrm{D}}\!\!\int^{\wedge} f^{\sim} d\mu$. Then, there exists a $(\mu, \mathfrak{X})$-Cauchy sequence g in $\mathcal{ST}(\mathfrak{X}, E)$ converging pointwise μ-a.e. to f such that $x \in \lim\limits_{n \in \mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu$. Choose such a g. Then, by the premise, there is an $M \in \operatorname{Dmn} \mu$ such that $\mu M = 0$ and, for each $n \in \mathbb{N}$, the function $(f - g_n)|T \backslash M$ is bounded. By Proposition 2.4/b, one obtains $x \in {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$.

Ad (b). Assume the premise of (b) and the inclusion $\mathfrak{X} \subseteq \mathfrak{Y}$. Then g is a $(\mu, \mathfrak{X})$-Cauchy sequence (by Proposition 2.1/b) converging pointwise μ-a.e. to f, thus (by Proposition 2.1/a and Definition 2.3) $\emptyset \neq \lim\limits_{n \in \mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu = {}^{\mathrm{D}}\!\!\int^{\wedge} f^{\sim} d\mu$. Because g converges even uniformly μ-a.e. to f, there is a $p \in \mathbb{N}$ and an $M \in \operatorname{Dmn} \mu$ such that $\mu M = 0$ and for each $n \geq p$ the function $(f - g_n)|T \backslash M$ is bounded. By Proposition 2.4/b, one obtains $\emptyset \neq \lim\limits_{n \in \mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu = {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$. Observe, furthermore, that ${}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu = {}^{\mathfrak{Y}}\!\!\int g_n{}^{\sim} d\mu$ holds for all $n \in \mathbb{N}$ by Lemma 1.2, since $\mathfrak{X} \subseteq \mathfrak{Y}$. □

Next, we note a simple consequence of the preceding theorem applied in the Banach space $(\mathbb{R}, |\cdot|)$:

Proposition 2.10 *If f is* D*-Bochner integrable and bounded μ-a.e., then* ${}^{\mathrm{D}}\!\!\int \|f\|\, d\mu \in {}^{\mathfrak{Y}}\!\!\int \|f\|^{\sim} d\mu$*, provided that $\mathfrak{X} \subseteq \mathfrak{Y}$.*

Proof: By Proposition 2.8/b, $\|f\|$ is D-Bochner integrable in the Banach space $(\mathbb{R}, |\cdot|)$; thus, the assertion holds by Theorem 2.2/a and Lemma 1.1. □

Next, we give another proof of Theorem 2.2/b not using implicitly Lemma 2.2 (via Proposition 2.4) but based on the following elementary lemma the validity of which is not restricted to classical sequences as it is the case for Lemma 2.2.

Lemma 2.4 *Let $(h, K, \mathfrak{b})$ be a filtered family in E^T. If $(h, K, \mathfrak{b})$ converges uniformly μ-a.e. to f and ${}^{\mathfrak{X}}\!\!\int \|f - h_k\|^{\sim} d\mu \neq \emptyset$ holds for $\mathfrak{b}$-almost all $k \in K$* (for the terminology, see [11], p.116)*, then $(h, K, \mathfrak{b})$ $(\mu, \mathfrak{X})$-converges in mean to f and one has*

$$^{\mathfrak{b}}\lim_{k \in K} {}^{\mathfrak{X}}\!\!\int h_k{}^{\sim} d\mu \subseteq {}^{\mathfrak{X}}\!\!\int f^{\sim} d\mu.$$

(Remark. By the following proof, the lemma is logically equivalent with the statement one gets from it by canceling the quantifier "μ-a.e.")

Proof: We skip the trivial case $\mu T = 0$.

1. By the supposition, there is a set $M \in \operatorname{Dmn} \mu$ such that $\mu M = 0$ and $(h, K, \mathfrak{b})$ converges uniformly on $T \backslash M$ to f. Define a mapping $g : K \longrightarrow E^T$ by letting, for each $k \in K$, $g_k(t) = h_k(t)$ for all $t \in T \backslash M$ and $g_k(t) = f(t)$ for all $t \in M$. Then, the filtered family $(g, K, \mathfrak{b})$ converges uniformly to f, and (by Corollary 1.1) the equations ${}^{\mathfrak{X}}\!\!\int \|f - h_k\|^{\sim} d\mu = {}^{\mathfrak{X}}\!\!\int \|f - g_k\|^{\sim} d\mu$, ${}^{\mathfrak{X}}\!\!\int h_k{}^{\sim} d\mu = {}^{\mathfrak{X}}\!\!\int g_k{}^{\sim} d\mu$ hold for all $k \in K$. Therefore, it suffices to prove the lemma under the supposition that $(\star)$ $(h, K, \mathfrak{b})$ converges uniformly to f.

2. Assume $(\star)$. By the premise, there is a $B \in \mathfrak{b}$ such that

$$\emptyset \neq {}^{\mathfrak{X}}\!\!\int \|f - h_k\|^{\sim} d\mu \text{ for all } k \in B\,. \tag{1}$$

Let $\epsilon > 0$. By the assumption $(\star)$, there is a $C \in \mathfrak{b}$ such that

$$\|f(t) - h_k(t)\| < \frac{\epsilon}{\mu T} \text{ for all } t \in T \text{ and all } k \in C\,.$$

Therefore, we have for each $(\mathfrak{x}, \phi) \in \mathfrak{X}^{\#}$

$$\mathrm{R}(\|f - h_k\|, \mathfrak{x}, \phi) < \epsilon \text{ for all } k \in C\,,$$

thus, by (1), $\emptyset \neq {}^{\mathfrak{X}}\!\!\int \|f - h_k\|^{\sim} d\mu \leq \epsilon$ for all $k \in B \cap C$, and therefore, by the choice of ϵ, $(h, K, \mathfrak{b})$ $(\mu, \mathfrak{X})$-converges in mean to f. In view of Lemma 2.1, one obtains ${}^{\mathfrak{b}}\!\lim\limits_{k \in K} {}^{\mathfrak{X}}\!\!\int h_k{}^{\sim} d\mu \subseteq {}^{\mathfrak{X}}\!\!\int f^{\sim} d\mu$. $\square$

Second proof of Theorem 2.2/b: We skip the trivial case $\mu T = 0$. By the premise within Theorem 2.2/b, there is a set $M \in \mathrm{Dmn}\,\mu$ such that $\mu M = 0$ and

($\star$) the sequence $(g_n|(T\backslash M))_{n \in \mathbb{N}}$ converges uniformly to $f|(T\backslash M)$.

For abbreviation, we put $Q = T\backslash M$, $\mathfrak{Z} = \mathfrak{X}_Q$, $\nu = \mu_Q$, $j = f|Q$, and $h_n = g_n|Q$ for all $n \in \mathbb{N}$. Observe that $Q \neq \emptyset$, since $\mu T > 0$. Let $m \in \mathbb{N}$. Since $\|h_n - h_m\|$ is ν-measurable for all $n \in \mathbb{N}$ and $\|j - h_m\|(t) = \|\lim\limits_{n \in \mathbb{N}} h_n(t) - h_m(t)\| = \lim\limits_{n \in \mathbb{N}} \|h_n(t) - h_m(t)\|$ holds for all $t \in Q$, the function $\|j - h_m\|$ is ν-measurable. Because of ($\star$), there is an $n_0 \in \mathbb{N}$ such that $\|j - h_n\|$ is bounded for all $n \geq n_0$. Therefore, by Lemma 1.4, ${}^{\mathfrak{Z}}\!\!\int \|j - h_n\|^{\sim} d\nu$ is non-empty for all those n. Let $n \geq n_0$. If $M = \emptyset$, the inequality ${}^{\mathfrak{X}}\!\!\int \|f - g_n\|^{\sim} d\mu \neq \emptyset$ is proved; otherwise, one has $\{M, Q\} \in \mathfrak{X}$, and, therefore, this inequality holds by Lemma 1.3. Now use Lemma 2.4, Proposition 2.1, Definition 2.3, and Lemma 1.1. $\square$

Theorem 2.2/b is generalized by the next theorem clearing up the (expected) behavior of the subdivision integral w.r. to the uniform convergence of filtered families in E^T. Observe that its proof does not take the detour via mean convergence we have taken implicitly in the special case (by using Lemma 2.2 or Lemma 2.4).

Theorem 2.3 *Let $(g, K, \mathfrak{b})$ be a filtered family in E^T converging uniformly μ-a.e. to f. If (α) f is $\mathfrak{Y}$-integrable or (β) $\mathfrak{Y} = \mathfrak{X}$, then* (a) *and* (b) *hold true:*

(a) ${}^{\mathfrak{b}}\!\lim\limits_{k \in K} {}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu \subseteq {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu.$

(b) *If there exists a $\mathfrak{Y}$-absolutely integrable linear subspace L of E^T such that $g(K) \subseteq L$, then $\emptyset \neq {}^{\mathfrak{b}}\!\lim\limits_{k \in K} {}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu = {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$.*

Note: If the quantifier "μ-a.e." is canceled, then "$\mathfrak{Y} = \mathfrak{X}$" is allowed to be replaced by "$\mathfrak{Y} \subseteq \mathfrak{X}$".

Proof: (b) follows from (a) by means of Proposition 2.1.

Ad (a). We consider the non-trivial case $\mu T > 0$.

1. Assume first that $(g, K, \mathfrak{b})$ converges uniformly to f. We assume (α) or (γ) $\mathfrak{Y} \subseteq \mathfrak{X}$. ($(\gamma)$ replaces (β) above.) Let $x \in {}^{\mathfrak{b}}\lim\limits_{k\,\in\,K}\ {}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu$ and $\epsilon > 0$. Then there is a $B \in \mathfrak{b}$ such that

(1) $\emptyset \neq \| {}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu - \{x\}\| < \frac{\epsilon}{2}$ for all $k \in B$

and

(2) $\|f(t) - g_k(t)\| < \frac{\epsilon}{4\mu T}$ for all $k \in B$ and all $t \in T$.

Since $\mathfrak{b}$ is a filter, B is non-empty; choose a $k \in B$. In view of (1), one has $\emptyset \neq {}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu$. Thus, there is an $\mathfrak{x}_0 \in \mathfrak{Y}$ such that

(3) $\emptyset \neq \|\mathrm{R}(g_k^{\sim}, \mathfrak{x}, \phi) - {}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu\| < \frac{\epsilon}{4}$ for all $\mathfrak{x} \in \mathfrak{Y}$ with $\mathfrak{x}_0 \preceq \mathfrak{x}$ and all $\phi \in \mathcal{P}\mathfrak{x}$.

Now the proof branches: Case 1. If (α) is valid, then there is an $\mathfrak{x}_1 \in \mathfrak{Y}$ such that $\mathrm{R}(f^{\sim}, \mathfrak{x}, \phi) \neq \emptyset$ for all $\mathfrak{x} \in \mathfrak{Y}$ with $\mathfrak{x}_1 \preceq \mathfrak{x}$ and all $\phi \in \mathcal{P}\mathfrak{x}$; in this case $\mathfrak{x}_2 = \mathfrak{x}_1 \vee\ \mathfrak{x}_0$. Case 2. If (γ) but not (α) is valid then $\mathrm{R}(f^{\sim}, \mathfrak{x}, \phi) \neq \emptyset$ holds true for all $\mathfrak{x} \in \mathfrak{Y}$; in this case put $\mathfrak{x}_2 =\ \mathfrak{x}_0$.

In both branches we continue in the following way: Let $\mathfrak{x} \in \mathfrak{Y}$ with $\mathfrak{x}_2 \preceq \mathfrak{x}$ and $\phi \in \mathcal{P}\mathfrak{x}$. Then, in view of $\mathrm{R}(g_k^{\sim}, \mathfrak{x}, \phi) \neq \emptyset$ (use of "$\neq$" in (3)) and of (2), one obtains the chain

(4) $\emptyset \neq \|\mathrm{R}(f^{\sim}, \mathfrak{x}, \phi) - \mathrm{R}(g_k^{\sim}, \mathfrak{x}, \phi)\| \leq \mathrm{R}(\|f - g_k\|^{\sim}, \mathfrak{x}, \phi) < \frac{\epsilon}{4}$,

where the third Riemann sum (which is non-empty) is related to the Banach space $(\mathbb{R}, |\cdot|)$. (Cf. Proposition 10 and (0.12) in [6], p.490 and p.487.) The combination of (3) and (4) provides

(5) $\emptyset \neq \|\mathrm{R}(f^{\sim}, \mathfrak{x}, \phi) - {}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu\| < \frac{\epsilon}{2}$.

Combining (5) with (1), one gets $\emptyset \neq \|\mathrm{R}(f^{\sim}, \mathfrak{x}, \phi) - \{x\}\| < \epsilon$. In view of the choice of $(\mathfrak{x}, \phi)$ we have proved $x \in {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$.

2. Since, by the supposition, $(g, K, \mathfrak{b})$ converges uniformly μ-a.e. to f, there is a set $M \in \operatorname{Dmn}\mu$ such $(g, K, \mathfrak{b})$ converges uniformly on $T \backslash M$ to f. Define a mapping $h : K \longrightarrow E^T$ by letting, for each $k \in K$, $h_k(t) = g_k(t)$ for all $t \in T \backslash M$ and $h_k(t) = f(t)$ for all $t \in M$. Then, the filtered family $(h, K, \mathfrak{b})$ converges uniformly to f. Under the supposition (α), one obtains by Lemma 1.1, Corollary 1.1, and Part **1** of this proof

$$ {}^{\mathfrak{b}}\lim_{k\,\in\,K}\ {}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu \subseteq {}^{\mathfrak{b}}\lim_{k\,\in\,K}\ {}^{\Omega}\!\!\int g_k{}^{\sim} d\mu = {}^{\mathfrak{b}}\lim_{k\,\in\,K}\ {}^{\Omega}\!\!\int h_k{}^{\sim} d\mu \subseteq {}^{\Omega}\!\!\int f^{\sim} d\mu = {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu. $$

Under supposition (β), one has by Corollary 1.1 and Part **1** of this proof

$$ {}^{\mathfrak{b}}\lim_{k\,\in\,K}\ {}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu = {}^{\mathfrak{b}}\lim_{k\,\in\,K}\ {}^{\mathfrak{Y}}\!\!\int h_k{}^{\sim} d\mu \subseteq {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu\,. \qquad \square $$

Note: By Definition 2.3, the statement one gets from Theorem 2.3/b by putting there $K = \mathbb{N}$, $\mathfrak{b} = \mathcal{F}\mathbb{N}$, $\mathfrak{Y} = \mathfrak{X}$, and $L = \mathcal{ST}(\mathfrak{X}, E)$ coincides logically with Theorem 2.2/b.

Remark 2.5 The translation of G. Birkhoff´s Theorem 22 in [2], p.372, into our language is following nice assertion: "If g is a sequence in E^T converging uniformly to f and having the property that ${}^{\Omega}\!\!\int g_n{}^{\sim} d\mu \neq \emptyset$ for all $n \in \mathbb{N}$, then $\emptyset \neq \lim\limits_{n \in \mathbb{N}} {}^{\Omega}\!\!\int g_n{}^{\sim} d\mu = {}^{\Omega}\!\!\int f^{\sim} d\mu$." Unfortunately, we cannot follow the arguments of Birkhoff´s proof. On the other hand, we do not have an example contradicting the assertion. For $\mathfrak{Y} = \Omega$ and $(K, \mathfrak{b}) = (\mathbb{N}, \mathcal{F}\mathbb{N})$ the preceding theorem is contained in Birkhoff´s theorem. For arbitrary $\mathfrak{Y}$ it contains a (possibly well-known) analogue for the D- and B-Bochner integral formulated next.

Corollary 2.1 *Read the superscript "C" as "D" or as "B". Let $f \in \mathrm{Dmn}\ {}^{\mathrm{C}}\!\!\int \cdot\, d\mu$. Let $(g, K, \mathfrak{b})$ be a filtered family in $\mathrm{Dmn}\ {}^{\mathrm{C}}\!\!\int \cdot\, d\mu$ converging uniformly to f. Then, one has*

$$ {}^{\mathfrak{b}}\!\lim_{k \in K} {}^{\mathrm{C}}\!\!\int g_k^{\sim}\, d\mu = {}^{\mathrm{C}}\!\!\int f^{\sim}\, d\mu. $$

Proof: Since $f \in \mathrm{Dmn}\ {}^{\mathrm{C}}\!\!\int \cdot\, d\mu$ and $g_k \in \mathrm{Dmn}\ {}^{\mathrm{C}}\!\!\int \cdot\, d\mu$ for all $k \in K$, one has $\emptyset \neq {}^{\mathrm{C}}\!\!\int f^{\sim}\, d\mu = {}^{\Omega}\!\!\int f^{\sim} d\mu$ and $\emptyset \neq {}^{\mathrm{C}}\!\!\int g_k^{\sim}\, d\mu = {}^{\mathrm{B}}\!\!\int g_k^{\sim}\, d\mu = {}^{\Omega}\!\!\int g_k{}^{\sim} d\mu$ for all $k \in K$ by Theorem 2.1. Now apply Theorem 2.3/b for $\mathfrak{Y} = \Omega$ and the linear subspace $L = \mathrm{Dmn}\ {}^{\mathrm{B}}\!\!\int \cdot\, d\mu$ of E^T (which is Ω-absolutely integrable (see Remark 2.3). □

We conclude this section by reconsidering Proposition 2.4/a. For this purpose, we define the statement (A) by the following lines:

(A) *If $h : \mathbb{N} \longrightarrow \mathcal{ST}(\mathfrak{X}, \mathbb{R})$ is a $(\mu, \mathfrak{X})$-Cauchy sequence and $j : T \longrightarrow \mathbb{R}$ a function such that h converges pointwise μ-a.e. to j, then j is μ-summable, and one has*

$$ \lim_{n \in \mathbb{N}} {}^{\mathrm{H}}\!\!\int \mid j - h_n \mid\, d\mu = 0 \ \textit{and}\ {}^{\mathrm{H}}\!\!\int j\, d\mu = \lim_{n \in \mathbb{N}} {}^{\mathrm{H}}\!\!\int h_n\, d\mu. $$

We define the statement **(B)** to be the statement one obtains from (A) by canceling there " $\lim\limits_{n \in \mathbb{N}} {}^{\mathrm{H}}\!\!\int \mid j - h_n \mid\, d\mu = 0$ and ".

Then, we obtain

Proposition 2.11 *Each of the two statements* (A), (B) *is logically equivalent to* Proposition 2.4/a.

Proof: **1.** By Theorem 12 in [7], p.16, Proposition 2.4/a implies (A).

2. Trivially, (A) implies (B).

3. Assume (B). Assume the premise within Proposition 2.4/a. Define $k : \mathbb{N} \longrightarrow (\mathcal{ST}(\mathfrak{X}, \mathbb{R}))^{\mathbb{N}}$ by $k(n) = (\|g_n - g_m\|)_{m\in\mathbb{N}}$ for all $n \in \mathbb{N}$. By the choice of g and f in the premise of Proposition 2.4/a, for each $n \in \mathbb{N}$, $k(n)$ turns out to be a $(\mu, \mathfrak{X})$-Cauchy sequence in $\mathcal{ST}(\mathfrak{X}, \mathbb{R})$ converging pointwise μ-a.e. to the function $\|f - g_n\|$. By (B), one obtains therefore, for all $n \in \mathbb{N}$,

$$ {}^{\mathrm{H}}\!\!\int \|f - g_n\| \, d\mu = \lim_{m \in \mathbb{N}} {}^{\mathrm{H}}\!\!\int (k(n))(m) \, d\mu < +\infty. \tag{1}$$

Let $\epsilon > 0$. By the choice of g, there is a $p \in \mathbb{N}$ such that ${}^{\mathrm{H}}\!\!\int \|g_n - g_m\| \, d\mu \in {}^{\mathfrak{X}}\!\!\int \|g_n - g_m\|^{\sim} d\mu < \epsilon$ holds for all $n, m \geq p$, thus, by (1), ${}^{\mathrm{H}}\!\!\int \|f - g_n\| \, d\mu \leq \epsilon$ holds for all $n \geq p$. (So far, we have just translated the proof of Proposition 12 in Dinculeanu [5], p.128, into our language.) Thus, by Theorem 12 in [7], p.16, g (μ, Ω)-converges in mean to f. The remainder of the assertion of Proposition 2.4/a follows by means of Proposition 2.1/a and Lemma 2.1. □

Remark 2.6 Let $(E, \|\cdot\|) = (\mathbb{R}, |\cdot|)$. Then, one has by Theorem 2.1 here and Theorem 12 in [7], p.16, the chain ${}^{\mathrm{D}}\!\!\int \cdot \, d\mu \subseteq {}^{\mathrm{B}}\!\!\int \cdot \, d\mu \subseteq ({}^{\mathrm{H}}\!\!\int \cdot \, d\mu)|M$, where M denotes the set of all μ-summable functions $j : T \longrightarrow \mathbb{R}$. Of course, the inclusion ${}^{\mathrm{D}}\!\!\int \cdot \, d\mu \subseteq ({}^{\mathrm{H}}\!\!\int \cdot \, d\mu)|M$ follows also from the statement (A) which is true by Proposition 2.11.

3 Summability of real-valued functions and integrability of vector-valued functions

It is usual (see, e.g., [4], p.41) to call f *μ-measurable* – for clarity we say also *$(\mu, \|\cdot\|)$-measurable* – if there exists a sequence g in $\mathcal{ST}(\mathfrak{X}, E)$ converging pointwise μ-a.e. to f. [For a generalization of this notion, see Definition 4 in [5], p.89.] If μ is complete, f is μ-measurable in the sense of Dinculeanu, loc.cit., if and only if f is $(\mu, \|\cdot\|)$-measurable (use of Theorem 1 in [5], p.94, and Theorem 2 in [5], p.99).] For later use we note that:

(3.1) *Every element of $\mathcal{ST}(\Omega, E)$ is $(\mu, \|\cdot\|)$-measurable.* (Use of $\mu T < +\infty$.)

For the remainder of this section, we assume μ to be complete.

If $(E, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, then f is $(\mu, \|\cdot\|)$-measurable if and only if f is μ-measurable. Therefore, there cannot arise any confusion if we just say "μ-measurable function" instead of "$(\mu, \|\cdot\|)$-measurable function", as we do in the following. We denote by $\mathcal{M}(\mu, T, E)$ the set of all μ-measurable functions on T into E.

In the remainder of this section, we rediscuss the relationships between the integrals occuring in Theorems 2.1 and 2.2. In the sequel, we say that f be B-*Bochner integrable* resp. Ω-*integrable* if $f \in \mathrm{Dmn}\ {}^{\mathrm{B}}\!\!\int \cdot \, d\mu$ resp. ${}^{\Omega}\!\!\int f^{\sim} d\mu \neq 0$.

For later use, we note a well-known fact:

(3.2) *If g is a sequence of μ-measurable functions $g_n : T \longrightarrow E$ converging pointwise μ-a.e. to f, then f is μ-measurable.*

Preparing the general case of integration of μ-measurable functions on T into E, we consider the integration of functions on T into $\mathbb{R}$:

For the next considerations, the elementary integral ${}^{\mathrm{el}}\!\!\int \cdot\, d\mu$ (defined in Section 2) is related to the Banach space $\mathbb{R}$.

Definition 3.1 *A linear mapping J* from *$\mathbb{R}^T$ (i.e. defined on a subset of $\mathbb{R}^T$) into $\mathbb{R}$ is called a μ-*summation *if* (J 1) *through* (J 4) *hold true:*

(J 1) $\mathcal{ST}(\mathfrak{X}, \mathbb{R}) \subseteq \mathrm{Dmn}\, J \subseteq \mathcal{M}(\mu, T, \mathbb{R})$ *and* $J|\mathcal{ST}(\mathfrak{X}, \mathbb{R}) = {}^{\mathrm{el}}\!\!\int \cdot\, d\mu$.

(J 2) $f \in \mathrm{Dmn}\, J$ *and* $A \in \mathrm{Dmn}\, \mu$ *implies* $f \cdot 1_A \in \mathrm{Dmn}\, J$.

(J 3) *If* $f, g \in \mathrm{Dmn}\, J$ *then* $f \leq g$ *implies* $J(f) \leq J(g)$.

(J 4) *If* (g_n) *is an increasing sequence in* $\mathrm{Dmn}\, J$ *converging pointwise to a function* $f (\in \mathbb{R}^T)$ *and having the property that the sequence* $(J(g_n))$ *is bounded, then* $f \in \mathrm{Dmn}\, J$ *and* $J(f) = \lim\limits_{n \in \mathbb{N}} J(g_n)$.

We recall that in the Banach space $E = \mathbb{R}$ the relationship between ${}^{\Omega}\!\!\int^{\vee} \cdot\, d\mu$ and the integral ${}^{\mathrm{H}}\!\!\int \cdot\, d\mu$ is described by

(1) ${}^{\Omega}\!\!\int^{\vee} \cdot\, d\mu = {}^{\mathrm{H}}\!\!\int \cdot\, d\mu \cap (\mathbb{R}^T \times \mathbb{R})$

(see Theorem 12 in [7], p.16). Let ${}^{\mathrm{Ba}}\!\!\int \cdot\, d\mu$ denote the "(μ)-Integral" defined in [1], p.64, Definition 12.1 (being considered here as a mapping *from* $\overline{\mathbb{R}}^T$ into $\overline{\mathbb{R}}$, defined on the set of all "(μ)-integrierbaren Funktionen" $g : T \longrightarrow \overline{\mathbb{R}}$). In analogy to (1) we define now the mapping ${}^{\mathrm{su}}\!\!\int \cdot\, d\mu$ by

${}^{\mathrm{su}}\!\!\int \cdot\, d\mu = {}^{\mathrm{Ba}}\!\!\int \cdot\, d\mu \cap (\mathbb{R}^T \times \mathbb{R})$. (We agree in the definitions $f^+ = \max\{f, 0\}$ and $f^- = \min\{f, 0\}$ for $f : T \longrightarrow \mathbb{R}$.)

In this terminology, we obtain

Theorem 3.1 *Let $E = \mathbb{R}$. There is exactly one μ-summation J, which we denote by $J(\mu)$, and one has $J(\mu) = {}^{\mathrm{su}}\!\!\int \cdot\, d\mu = {}^{\mathrm{D}}\!\!\int \cdot\, d\mu = {}^{\mathrm{B}}\!\!\int \cdot\, d\mu = {}^{\Omega}\!\!\int^{\vee} \cdot\, d\mu$. The function f is μ-summable if and only if $f \in J(\mu)$.*

Proof: 1. Clearly, there is a μ-summation, for instance the mapping ${}^{\Omega}\!\!\int^{\vee} \cdot\, d\mu$ (cf. Theorem 12 in [7], p.16, Lemma 1.2, [13], p.93, Satz 3, p.105, Satz 3, Theorems 7 and 8 in [7], p.12, and [13], p.105, Satz 3).

2. There is at most one μ-summation. Let J be a μ-summation. We show that $J = {}^{\mathrm{su}}\!\!\int \cdot\, d\mu$. By the way, we show $J \subseteq {}^{\mathrm{D}}\!\!\int \cdot\, d\mu$.

2a. Let $(f, x) \in J$. By (J 1), f is μ-measurable, thus (by (J 2)) $f^+, f^- \in \operatorname{Dmn} J$, thus $x = J(f) = J(f^+) + J(f^-)$, since $f = f^+ + f^-$ and J is linear.

2aα**.** Since f^+ is μ-measurable and non-negative, there exists an increasing sequence (g_n) in $\mathcal{ST}(\mathfrak{X}, \mathbb{R})$ with $g_n \geq 0$ for all $n \in \mathbb{N}$ converging pointwise to f^+. By (J 1) through (J 4) one has $f^+ \in \operatorname{Dmn}({}^{\mathrm{su}}\!\!\int \cdot\, d\mu)$ and

(2) $J(f^+) = \lim\limits_{n \in \mathbb{N}} {}^{\mathrm{el}}\!\!\int g_n \, d\mu = {}^{\mathrm{su}}\!\!\int f^+ \, d\mu.$

Furthermore, (g_n) being increasing, one has

$$ {}^{\mathrm{el}}\!\!\int |g_n - g_m| \, d\mu = |\, {}^{\mathrm{el}}\!\!\int g_n \, d\mu - {}^{\mathrm{el}}\!\!\int g_m \, d\mu| \longrightarrow 0 \text{ as } n, m \longrightarrow \infty. $$

Using this and the first sign $=$ in (2), one obtains (by Definition 2.3 and Proposition 2.8/a) $J(f^+) = {}^{\mathrm{D}}\!\!\int f^+ \, d\mu$.

2aβ**.** Since mapping J is linear, one has $-f^- \in \operatorname{Dmn} J$. Paraphrasing the preceding paragraph **2a**α with $-f^-$ replacing f^+, one gets $J(-f^-) = {}^{\mathrm{su}}\!\!\int -f^- \, d\mu = {}^{\mathrm{D}}\!\!\int -f^- \, d\mu$.

Combining paragraphs **2a**α and **2a**β and the linearity of the mappings J, ${}^{\mathrm{su}}\!\!\int \cdot\, d\mu$, and ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu$, one obtains $(f, x) \in {}^{\mathrm{su}}\!\!\int \cdot\, d\mu \cap {}^{\mathrm{D}}\!\!\int \cdot\, d\mu$.

2b. We show ${}^{\mathrm{su}}\!\!\int \cdot\, d\mu \subseteq J$. Let $(f, x) \in {}^{\mathrm{su}}\!\!\int \cdot\, d\mu$. Then, there exists an increasing sequence (g_n) in $\mathcal{ST}(\mathfrak{X}, \mathbb{R})$ converging pointwise to f^+ and having the property ${}^{\mathrm{su}}\!\!\int f^+ \, d\mu = \lim\limits_{n \in \mathbb{N}} {}^{\mathrm{el}}\!\!\int g_n \, d\mu$. Using (J 1), the boundedness of the sequence $({}^{\mathrm{el}}\!\!\int g_n \, d\mu)$, and (J 4), one gets $f^+ \in \operatorname{Dmn} J$ and ${}^{\mathrm{su}}\!\!\int f^+ \, d\mu = J(f^+)$. Analogously, one has ${}^{\mathrm{su}}\!\!\int (-f^-) \, d\mu = J(-f^-)$. Since ${}^{\mathrm{su}}\!\!\int \cdot\, d\mu$ and J are linear mappings, we have proved $(f, x) \in J$.

3. Combining Part 1 with Part 2 of this proof, we have proved the existence and the uniqueness of a μ-summation and the chain $J(\mu) = {}^{\mathrm{su}}\!\!\int \cdot\, d\mu = {}^{\Omega}\!\!\int^{\vee} \cdot\, d\mu \subseteq {}^{\mathrm{D}}\!\!\int \cdot\, d\mu$. The remainder of the assertion follows by means of Theorem 2.1 and the definition of " μ-summable". □

The conditions (J 2) and (J 4) in Definition 3.1 are allowed to be replaced by weaker ones:

Proposition 3.1 *A linear mapping J from $\mathbb{R}^T$ into $\mathbb{R}$ is a μ-summation in $\mathbb{R}$ if and only if the above statements* (J 1) *and* (J 3) *and the next statements* (J 2′) *and* (J 4′) *hold true:*

(J 2′) *If $f \in \operatorname{Dmn} J$ then $f^+ \in \operatorname{Dmn} J$.*

(J 4′) *If (g_n) is an increasing sequence in $\mathcal{ST}(\mathfrak{X}, \mathbb{R})$ converging pointwise to an element f of E^T and having the property that the sequence $(J(g_n))$ is bounded, then $f \in \operatorname{Dmn} J$ and $\lim\limits_{n \in \mathbb{N}} J(g_n) = J(f)$.*

Proof: If a linear mapping J has the properties (J 1), (J 2′), (J 3), and (J 4′), then (paraphrasing Part 2 of the proof of Theorem 3.1, one gets) $J = {}^{\mathrm{su}}\!\!\int \cdot \, d\mu$, thus, by Theorem 3.1, $J = J(\mu)$. □

We return to the general case of integration in the Banach space E.

Theorem 2.1 and Proposition 2.8/b are supplemented by the next two theorems. [Note: For the statements (a) $\Longleftrightarrow$ (b) and (a) $\Longleftrightarrow$ (c), occuring in the following theorem, cf. Proposition 20 in [5], p.136, *or* Theorem 2 in [4], p.45, and Theorem 3.74 in [17], p.80, respectively.]

Theorem 3.2 *The statements* (a), (b), (c), (d) *defined next are logically equivalent:*

(a) *f is μ-measurable and $\|f\|$ is μ-summable;*

(b) *f is* D*-Bochner integrable.*

(c) *f is* B*-Bochner integrable;*

(d) $\emptyset \neq {}^{\mathrm{D}}\!\!\int^{\wedge} f^{\sim} d\mu = {}^{\mathrm{B}}\!\!\int^{\wedge} f^{\sim} d\mu = {}^{\Omega}\!\!\int f^{\sim} d\mu.$

In particular, (a) *implies* (e)*, defined next:*

(e) *f is Ω-integrable.*

In the case $(E, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, each of the statements (a) *through* (d) *is equivalent to the statement* (e).

Proof: Trivially, (d) implies (b). By Theorem 2.2, (b) implies (d). For the conclusion from (a) to (b) we refer to [4], loc. cit. (which we are allowed to in view of Proposition 3.2 below). (b) implies (c) by Theorem 2.1, while (c) implies (a) by Proposition 33 in [7], p.21, (or by Remark 2.3) and (2.2). The assertion for $E = \mathbb{R}$ follows from Theorem 3.1 by means of (a) $\Longleftrightarrow$ (b). □

Theorem 3.3 *One has* (a) *and, if there is an infinite* $\mathfrak{x} \in \Omega$ *such that* $\mu X > 0$ *for all* $X \in \mathfrak{x}$, (b)*:*

(a) ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu = {}^{\mathrm{B}}\!\!\int \cdot\, d\mu$;

(b) *if* ${}^{\mathrm{B}}\!\!\int \cdot\, d\mu = ({}^{\Omega}\!\!\int \cdot\, d\mu)^{\vee}$ *then the Banach space* E *is finite-dimensional.*

Proof: Ad (a). Use Theorem 3.2.

Ad (b). Choose an $\mathfrak{x} \in \Omega$ as required to exist above; let $\mathfrak{x} = \{X_1, X_2, \cdots\}$. Assume the Banach space E to be infinite-dimensional. Then, by the Dvoretzky-Rogers theorem (see, e.g., [19], p.27,Theorem 6), there is a sequence $(a_n)_{n\in\mathbb{N}}$ in E such that $\underset{n\,\in\,\mathbb{N}}{\Sigma}\ \{a_n\} \neq \emptyset$ and $\underset{n\,\in\,\mathbb{N}}{\Sigma}\ \{\|a_n\|\} = \emptyset$. Define $\chi : \mathfrak{x} \longrightarrow E$ by $\chi(X_n) = a_n \cdot \frac{1}{\mu X_n}$ for all $n \in \mathbb{N}$. Let f be the Ω-step function determined by $(\mathfrak{x}, \chi)$. Then one obtains, by Lemma 1.2, ${}^{\Omega}\!\!\int f^{\sim} d\mu = \underset{n\,\in\,\mathbb{N}}{\Sigma}\ \{a_n\} \neq \emptyset$ and ${}^{\Omega}\!\!\int \|f\|^{\sim} d\mu = \underset{n\,\in\,\mathbb{N}}{\Sigma}\ \{\|a_n\|\} = \emptyset$, therefore, by Theorem 3.2 or Proposition 33 in [7],p.21 (use of the completeness of μ), ${}^{\mathrm{B}}\!\!\int^{\wedge} f^{\sim} d\mu = \emptyset$. Thus ${}^{\mathrm{B}}\!\!\int \cdot\, d\mu \neq ({}^{\Omega}\!\!\int \cdot\, d\mu)^{\vee}$. □

As a consequence of Theorem 3.3 one has: If the Banach space E is infinite-dimensional and if there is an μ-partition as described in Theorem 3.3, then (by Theorem 3.2) the linear space $L = \mathrm{Dmn}(({}^{\Omega}\!\!\int \cdot\, d\mu)^{\vee})$ (see Corollary to Proposition 18 in [6], p.495) is not a (μ, Ω)-absolutely integrable subspace of E^T (cf. Definition 2.1 /b).

Theorem 2.2 is supplemented by the following theorem, where the superscript "D" is allowed to be replaced by "B" (in view of Theorem 3.3).

Theorem 3.4 *Assume* f *to be bounded* μ*-a.e. Then the following statements* (a), (b), *and* (c) *are equivalent:*

(a) f *is* μ*-measurable;*

(b) f *is* D*-Bochner integrable;*

(c) $\emptyset \neq {}^{\mathrm{D}}\!\!\int^{\wedge} f^{\sim} d\mu = {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$, *provided that* $\mathfrak{X} \subseteq \mathfrak{Y}$.

Proof: By Theorem 3.2, (b) implies (a). Assume (a). Then, since μ is complete, the function $\|f\|$ is μ-measurable and (by supposition) bounded μ-a.e., therefore (by Lemma 1.4) it is μ-summable. Thus, the combination of Theorems 3.2 and 2.2 provides the statements (b) and (b) $\Longleftrightarrow$ (c). □

Next, we make sure that in the present situation the D-Bochner integral coincides with the Bochner integral used in [4].

Proposition 3.2 *${}^{\mathrm{D}}\!\!\int \cdot\, d\mu$ is the set of all $(f,x) \in \mathcal{M}(\mu,T,E) \times E$ such that, for some sequence g in $\mathcal{ST}(\mathfrak{X},E)$,* (a) *and* (b) *defined next hold true:*

(a) $0 = \lim\limits_{n \in \mathbb{N}} {}^{\mathrm{H}}\!\!\int \|f - g_n\|\, d\mu;$

(b) $x = \lim\limits_{n \in \mathbb{N}} {}^{\mathrm{el}}\!\!\int g_n\, d\mu.$

Proof: 1. By Theorem 3.2, one has $\mathrm{Dmn}\ {}^{\mathrm{D}}\!\!\int \cdot\, d\mu \subseteq \mathcal{M}(\mu,T,E)$.

2. If g is a sequence in $\mathcal{ST}(\mathfrak{X},E)$ Ω-converging in mean to an element f of $\mathcal{M}(\mu,T,E)$, then ${}^{\Omega}\!\!\int \|f - g_n\|^{\sim} d\mu \neq \emptyset$ for eventually all $n \in \mathbb{N}$, say for all $n \geq n_0$, and for those n ${}^{\mathrm{H}}\!\!\int \|f - g_n\|\, d\mu \in {}^{\Omega}\!\!\int \|f - g_n\|^{\sim} d\mu$. Thus, then (a) holds true.

3. If (a) holds true for some $f \in \mathcal{M}(\mu,T,E)$ and some sequence g in $\mathcal{ST}(\mathfrak{X},E)$, then eventually all functions $\|f - g_n\|$ are μ-summable, thus g Ω-converges in mean to f.

4. Furthermore, use Proposition 2.7. $\square$

4 Integration w. r. to an arbitrarily fine μ-partition system

Up to now, on the set T no topological structure was given. For the remainder of this paper, we assume that (T,d) be a metric space. For each $X \subseteq T$, we denote by $\mathrm{diam}(X)$ the diameter of X w. r. to d; we define the mapping $\mathrm{Diam}\colon \mathfrak{P}\mathfrak{P}T \longrightarrow \overline{\mathbb{R}}$ by letting $\mathrm{Diam}(\mathfrak{x}) = \sup\limits_{X \in \mathfrak{x}} \mathrm{diam}(X)$ for all $\mathfrak{x} \in \mathfrak{P}\mathfrak{P}T$.

Definition 4.1 *We call a set $\mathfrak{N}$ of partitions of T to be* arbitrarily fine w. r. to *d* (*or d*-arbitrarily fine)*, if, for each $\epsilon > 0$, there is an $\mathfrak{x} \in \mathfrak{N}$ such that Diam$(\mathfrak{x}) < \epsilon$.*

Obviously, $\mathfrak{N}$ is d-arbitrarily fine if and only if there exists a sequence $(\mathfrak{x}_n)_{n\in\mathbb{N}}$ in $\mathfrak{N}$ such that $\lim\limits_{n \in \mathbb{N}}{}_{\sigma} \mathrm{Diam}(\mathfrak{x}_n) = 0$, where $\lim_{\sigma}$ denotes the limit operation in the extended real line w.r. to its usual topology σ. If $\mathfrak{R}$ and $\mathfrak{S}$ are sets of partitions of T with $\mathfrak{R} \subseteq \mathfrak{S}$ then the statement "$\mathfrak{R}$ is d-arbitrarily fine" implies "$\mathfrak{S}$ is d-arbitrarily fine".

Example 4.1 Let $n \in \mathbb{N}$ and ν be the Lebesgue-Borel measure (for the terminology, see [1], p.37) or the Lebesgue measure on $\mathbb{R}^n$; let T be a non-empty compact subset of $\mathbb{R}^n$. Define μ by $\mu X = \nu X$ for all $X \in \mathrm{Dmn}\,\nu$ with $X \subseteq T$. Then, $\mathfrak{X}$ is arbitrarily fine w. r. to the Euclidian metric of $\mathbb{R}^n$ restricted to $T \times T$.

Remark 4.1 The comparison of the notion of an "arbitrarily fine μ-partition system" with proposition 8.3.3.1 in [13], p.184, on a measure being "eng adaptiert" to a topology suggests to generalize the preceding considerations by replacing there the metric space (T, d) by a topological space (T, ρ) or a uniform space $(T, \mathfrak{U})$ and modifying the notion "d-arbitrarily fine" in a suitable way. Given $(T, \mathfrak{U})$, a set $\mathfrak{N}$ of partitions of T should be called $\mathfrak{U}$-*arbitrarily fine*, if, for each $U \in \mathfrak{U}$, there is an $\mathfrak{x} \in \mathfrak{N}$ such that $X \times X \subseteq U$ holds for all $X \in \mathfrak{x}$. Of a special interest, of course, would be, in this context, the topology $\rho(\mathfrak{U})$ induced by $\mathfrak{U}$. Is there a natural relationship between the property of a μ-partition system to be "$\mathfrak{U}$-arbitrarily fine" and and the property of a measure to be "eng adaptiert" to $\rho(\mathfrak{U})$?

For the remainder of this paper, let $\mathfrak{Z}$ *be a* third μ*-partition system* (beside $\mathfrak{X}$ and $\mathfrak{Y}$).

Proposition 4.1 *Let* (T, d) *be compact and assume that* $\mathfrak{Z}$ *is* d*-arbitrarily fine. If* f *is continuous* ($w.r.$ *to* d *and the norm of* E)*, then one has* (a) *and* (b)*:*

(a) *There exists a sequence* $g : \mathbb{N} \longrightarrow \mathcal{ST}(\mathfrak{Z}, E)$ *converging uniformly to* f.

(b) *If* $\mathfrak{Z} \subseteq \mathfrak{X} \subseteq \mathfrak{Y}$*, then* ${}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$ *is non-empty.*

Proof: Since $\mathfrak{Z}$ is d-arbitrarily fine, there exists a sequence $(\mathfrak{z}_n)_{n\in\mathbb{N}}$ in $\mathfrak{Z}$ with the property $\lim\limits_{n \in \mathbb{N}}{}^{\sigma} \mathrm{Diam}(\mathfrak{z}_n) = 0$. Choose $\phi \in \underset{n \in \mathbb{N}}{\mathrm{P}} \mathcal{P}_{\mathfrak{Z}_n}$ and put $\phi(n) = \phi_n$ for each $n \in \mathbb{N}$. Define, for each $n \in \mathbb{N}$, a $\mathfrak{Z}$-step function $g_n : \mathfrak{z}_n \longrightarrow E$ by letting, for each $t \in T$,

$$g_n(t) = f(\phi_n(Z)), \text{ if } t \in Z \in \mathfrak{z}_n .$$

Let $\epsilon > 0$. Since f is continuous and T is compact, there is a $\delta > 0$ such that for all $t_1, t_2 \in T$

$$\|f(t_1) - f(t_2)\| < \epsilon, \text{ if } \mathrm{d}(t_1, t_2) < \delta . \tag{1}$$

There is an $n_0 \in \mathbb{N}$ such that

$$\mathrm{Diam}(\mathfrak{z}_n) < \delta \text{ for all } n \geq n_0 . \tag{2}$$

Let $n \geq n_0$ and $t \in T$. Then, there is a Z with $t \in Z \in \mathfrak{z}_n$. One has $\mathrm{diam}(Z) < \delta$ by (2), therefore, by (1), $\|g_n(t) - f(t)\| < \epsilon$, since $g_n(t) = f(\phi_n(Z))$. Thus (a) is proven. (b) follows from (a) by means of Theorem 2.2/b. □

We conclude this paper with a short view to the Riemann integration:

Definition 4.2 (a) *For each* $\mathfrak{x}, y \in \mathfrak{Y}$*, let* $\mathfrak{x} \preceq_{\mathrm{R}} \mathfrak{y}$*, if Diam*$(\mathfrak{y}) \leq$ *Diam*$(\mathfrak{x})$. (*Remark: Since* $(\mathfrak{Y}, \preceq)$ *is a directed set, also* $(\mathfrak{Y}, \preceq_{\mathrm{R}})$ *is a directed set.*)
(b) *For each* $(\mathfrak{x}, \phi), (\mathfrak{y}, \psi) \in \mathfrak{Y}^{\#}$*, let* $(\mathfrak{x}, \phi)(\preceq_{\mathrm{R}})^{\#}(\mathfrak{y}, \psi)$*, if* $\mathfrak{x} \preceq_{\mathrm{R}} \mathfrak{y}$. (*Remark:* $(\mathfrak{Y}^{\#}, (\preceq_{\mathrm{R}})^{\#})$ *is a directed set. We denote by* $\mathcal{F}(\mathfrak{X}^{\#}, (\preceq_{\mathrm{R}})^{\#})$ *the filter of perfinality on this directed set* (*see Section 1/b*).)
(c) *The* $(\mu, \mathfrak{Y})$-Riemann integral ${}^{\mathrm{R}\mathfrak{Y}}\!\!\int \cdot \, d\mu$ *is the mapping on* $\mathfrak{E}^T$ *into* $\mathfrak{E}$ *being defined by letting, for each* $g \in \mathfrak{E}^T$,

$$ {}^{\mathrm{R}\mathfrak{Y}}\!\!\int g\, d\mu \;=\; {}^{\mathcal{F}(\mathcal{Y}^{\#},\,(\preceq_{\mathrm{R}})^{\#})}\lim_{(\mathfrak{x},\,\phi)\,\in\,\mathfrak{Y}^{\#}} \mathrm{R}(f,\mathfrak{x},\phi)\,. $$

(*For the terminology, see Section* 1/d *and* f.)

The following example opens a wide range of investigation in the survey paper [8]:

Example 4.2 Let $T = [a,b] \subseteq \mathbb{R}$ with $a < b$. Take μ as in Example 4.1. Define $\mathfrak{Y}_1$ to be set of all finite partitions of $[a,b[$ into half-open intervals of the form $[c,d[$ with $a \leq c < d \leq b$ and let $\mathfrak{Y} = \{\mathfrak{y} \cup \{\{b\}\} \mid \mathfrak{y} \in \mathfrak{Y}_1\}$. Alternatively, define $\mathfrak{Y}$ to be set of all finite partitions of T into intervals, where also half-open intervals and singletons $[c,c]$ with $c \in T$ are admitted. In both cases, $({}^{\mathfrak{Y}}\!\!\int \cdot\, d\mu)^{\vee}$ (see Propositon 18 in [6]) coincides with the integral defined in [8], p.924, Definition 2(b) there, while $({}^{\mathrm{R}\mathfrak{Y}}\!\!\int \cdot\, d\mu)^{\vee}$ coincides with the integral defined in [8], p.924, Definition 2(a) there. (For the latter use of ${}^{\vee}$, see the terminology introduced in Section 0 in [6], which is applicable here by Proposition 18 in [6] and Proposition 4.2 below.) By Theorem 3 in [8], p.924, one has ${}^{\mathrm{R}\mathfrak{Y}}\!\!\int \cdot\, d\mu = {}^{\mathfrak{Y}}\!\!\int \cdot\, d\mu$.

Since $(\preceq^{\#}) \subseteq ((\preceq_{\mathrm{R}})^{\#})$, one obtains

Proposition 4.2 *For each $g \in \mathfrak{E}^T$, one has the inclusion ${}^{\mathrm{R}\mathfrak{Y}}\!\!\int g\, d\mu \subseteq {}^{\mathfrak{Y}}\!\!\int g\, d\mu$.*

Proposition 4.3 *If ${}^{\mathrm{R}\mathfrak{Y}}\!\!\int f^{\sim} d\mu$ and ${}^{\mathrm{R}\mathfrak{Z}}\!\!\int f^{\sim} d\mu$ are non-empty, then*

$$ {}^{\mathrm{R}\mathfrak{Y}}\!\!\int f^{\sim} d\mu = {}^{\mathrm{R}\mathfrak{Z}}\!\!\int f^{\sim} d\mu\,. $$

Proof: By Lemma 1.1 and Proposition 4.2, one has ${}^{\mathrm{R}\mathfrak{Y}}\!\!\int f^{\sim} d\mu \subseteq {}^{\Omega}\!\!\int f^{\sim} d\mu$ and ${}^{\mathrm{R}\mathfrak{Z}}\!\!\int f^{\sim} d\mu \subseteq {}^{\Omega}\!\!\int f^{\sim} d\mu$. Now, use the premise. □

Proposition 4.4 *Let (T, d) be compact; assume $\mathfrak{Y} \subseteq \mathfrak{X}$ and $\mathfrak{Y}$ to be* d*-arbitrarily fine. Then, one has: If f is continuous, then ${}^{\mathrm{R}\mathfrak{Y}}\!\!\int f^{\sim} d\mu$ is non-empty.*

Proof: We skip the trivial case $\mu T = 0$ and assume $\mu T > 0$. Let $\epsilon > 0$. Since T is compact and f is continuous, there is a $\delta > 0$ such that

$$ \text{for all } s,t \in T,\ \mathrm{d}(s,t) < \delta \text{ implies } \|f(s) - f(t)\| < \frac{\epsilon}{\mu T}\,. \tag{1} $$

Because $\mathfrak{Y}$ is d-arbitrarily fine, there is an $\mathfrak{x}_0 \in \mathfrak{Y}$ such that

$$ \mathrm{Diam}(\mathfrak{x}_0) < \frac{\delta}{2}\,. \tag{2} $$

Let $\mathfrak{x}, \mathfrak{y} \in \mathfrak{Y}$ such that $\mathfrak{x}_0 \preceq_{\mathrm{R}} \mathfrak{x}, \mathfrak{y}$. Furthermore, let $\phi \in \mathcal{P}\mathfrak{x}$ and $\psi \in \mathcal{P}\mathfrak{y}$. Since $(\mathfrak{Y}, \preceq)$ is a directed set, there is a $\mathfrak{z} \in \mathfrak{Y}$ with $\mathfrak{x}, \mathfrak{y} \preceq \mathfrak{z}$. Choose such a $\mathfrak{z}$. For all $Z \in \mathfrak{z}$, there are exactly one $\alpha(Z) \in \mathfrak{x}$ such that $Z \subseteq \alpha(Z)$ and exactly one $\beta(Z) \in \mathfrak{y}$ such that $Z \subseteq \beta(Z)$. Define the mappings $g, h : \mathfrak{z} \longrightarrow E$ by letting $g(Z) = f(\phi(\alpha(Z)))$ and $h(Z) = f(\psi(\beta(Z)))$ for all $Z \in \mathfrak{z}$. Using (2) and the relationship between $\mathfrak{x}_0$, $\mathfrak{x}$, $\mathfrak{y}$, and $\mathfrak{z}$, one obtains $\mathrm{d}(\phi(\alpha(Z)), \psi(\beta(Z))) < \delta$ for all $Z \in \mathfrak{z}$, thus, using (1) and the relationship between $\mathfrak{x}$, $\mathfrak{y}$, and $\mathfrak{z}$, the chain

$$\|\mathrm{R}(f^{\sim},\mathfrak{x},\phi)-\mathrm{R}(f^{\sim},\mathfrak{y},\psi)\| \le \sum_{Z\,\in\,\mathfrak{z}} \|g^{\sim}(Z)-h^{\sim}(Z)\|\mu Z < \epsilon\,,$$

where the three occuring sums are singletons, since $\mathfrak{x}$, $\mathfrak{y}$, and $\mathfrak{z}$ are finite sets. (Observe that, e.g., $\sum_{Z\,\in\,\mathfrak{z}} g(Z)\mu Z \in \mathrm{R}(f^{\sim},\mathfrak{x},\phi)$ holds, since $\mathfrak{x}\preceq\mathfrak{z}$ (use of Proposition 20 and Lemma 1 in [6]).) We have shown that the net

$$(j,\mathfrak{Y}^{\#},(\preceq_{\mathrm{R}})^{\#}) \text{ with } j(\mathfrak{u},\kappa)=\mathrm{e}(\mathrm{R}(f^{\sim},\mathfrak{u},\kappa)) \text{ for all } (\mathfrak{u},\kappa)\in\mathfrak{Y}^{\#}$$

is a Cauchy net in the Banach space $(E,\|\cdot\|)$, therefore (by Definition 4.2) the set ${}^{\mathrm{R}\mathfrak{Y}}\!\!\int f^{\sim}d\mu$ is non-empty. (For the definition of the mapping e, see Section 1/a.) □

Of course, Proposition 4.1/b (even if one supposes there $\mathfrak{Z}\subseteq\mathfrak{X}\cap\mathfrak{Y}$ instead of $\mathfrak{Z}\subseteq\mathfrak{X}\subseteq\mathfrak{Y}$) follows also from the Propositions 4.2 and 4.4 (use of Lemma 1.1); see also Proposition 4.6/b.

For the remainder of this paper, we assume $\mathfrak{Z}$ to be d-arbitrarily fine and T to be compact. We add now some remarks on the integration of continuous functions. – As in Remark 2.1, we denote by L_2 the linear space of all those D-Bochner integrable functions $g:T\longrightarrow E$, which are bounded μ-a.e. In the following, $\mathcal{C}(T,E)$ denotes the set of all $(\mathrm{d},\|\cdot\|)$-continuous functions on T into E.

Proposition 4.5 *If $f\in\mathcal{C}(T,E)$ and $\mathfrak{Z}\subseteq\mathfrak{X}$, then ${}^{\mathrm{R}\mathfrak{Z}}\!\!\int f^{\sim}d\mu = {}^{\mathrm{R}\mathfrak{X}}\!\!\int f^{\sim}d\mu$.*

Proof: Combine Propositions 4.4 and 4.3. □

In the sense of Proposition 4.5, for continuous f, the integral ${}^{\mathrm{R}\mathfrak{X}}\!\!\int f\,d\mu$ represents all integrals ${}^{\mathrm{R}\mathfrak{W}}\!\!\int f\,d\mu$, $\mathfrak{W}$ being a d-arbitrarily fine μ-partition system of T with $\mathfrak{W}\subseteq\mathfrak{X}$. So, it seems to be justified to reserve a special symbol, namely ${}^{\mathrm{R}}\!\!\int\cdot\,d\mu$, for ${}^{\mathrm{R}\mathfrak{X}}\!\!\int\cdot\,d\mu$, and to call ${}^{\mathrm{R}}\!\!\int\cdot\,d\mu$ the *μ-Riemann integral*. (This terminology deviates from that used in [8], p.925, definition of "Riemann integrable". See our Example 4.2: There, for continuous f (by Proposition 4.4 and Lemma 1.1) ${}^{\mathrm{R}\mathfrak{Y}}\!\!\int f^{\sim}d\mu = {}^{\mathrm{R}}\!\!\int f^{\sim}d\mu = {}^{\mathfrak{X}}\!\!\int f^{\sim}d\mu$.)

Proposition 4.6 *One has* (a) *and* (b):

(a) *If $\mathfrak{X}$ is d-arbitrarily fine, then $\mathcal{C}(T,E)\subseteq L_2$.*

(b) *If $f\in\mathcal{C}(T,E)$, then ${}^{\mathrm{R}\mathfrak{Z}}\!\!\int f^{\sim}d\mu = {}^{\mathrm{D}}\!\!\int^{\wedge} f^{\sim}d\mu = {}^{\mathfrak{Y}}\!\!\int f^{\sim}d\mu \neq\emptyset$, provided that $\mathfrak{Z}\subseteq\mathfrak{X}\cap\mathfrak{Y}$.*

Proof: Ad (a). Assume $\mathfrak{X}$ to be d-arbitrarily fine. Let $f\in\mathcal{C}(T,E)$. Then, by Proposition 4.1, there is a sequence $h:\mathbb{N}\longrightarrow\mathcal{ST}(\mathfrak{X},E)$ converging uniformly to f. Thus, by

Theorem 2.2/b, one has ($\star$) $\emptyset \neq {}^{\mathrm{D}}\!\!\int^{\wedge} f^{\sim} d\mu = {}^{\mathfrak{X}}\!\!\int f^{\sim} d\mu$; therefore, the function f is D-Bochner integrable, while, being continuous on a compact set, it is bounded. Thus, one has $f \in L_2$.

Ad (b). Let $\mathfrak{Z} \subseteq \mathfrak{X} \cap \mathfrak{Y}$. Using the Propositions 4.2, 4.4, and Lemma 1.1, one gets ${}^{\mathrm{R}\mathfrak{Z}}\!\!\int f^{\sim} d\mu = {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu = {}^{\mathfrak{X}}\!\!\int f^{\sim} d\mu$, therefore (using the statement ($\star$) which is applicable, since $\mathfrak{X}$ containing $\mathfrak{Z}$ as a subset is d-arbitrarily fine) the assertion (b). $\square$

From Theorem 2.3, we regain a classical result on the Riemann integration being contained in

Proposition 4.7 *If $(g, K, \mathfrak{b})$ is a filtered family in $\mathcal{C}(T, E)$ converging uniformly to f, then one has:*

$$\emptyset \neq {}^{\mathfrak{b}}\lim_{k \in K} {}^{\mathrm{R}\mathfrak{Z}}\!\!\int g_k{}^{\sim} d\mu = {}^{\mathrm{R}\mathfrak{Z}}\!\!\int f^{\sim} d\mu, \textit{ provided that } \mathfrak{Z} \subseteq \mathfrak{X}\,.$$

Proof: By the premise, f is continuous. Now, apply Proposition 4.6 (to f and all g_k with $k \in K$) and then Corollary 2.1. $\square$

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[1]Printing errors: On p.489, line 23 from above, replace "0" by "$\emptyset$". On p.490, line 23 from above, move the subscript "$i \in I$" of the sign "$\sum$" as a subscript to the sign "P".

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[2]Printing errors: On p.20, lines 17–18 from below, move the subscript "$n \in \mathbf{N}$" of the integral sign as a subscript to the sign "lim". On p.21, lines 13 from below, replace "$h^\star$" by "h". On p.21, line 10 from below, replace the superscript "$^\wedge$" by the superscript "$^\vee$".

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A note on antitonic convergence

This paper is dedicated in honour to Professor Harry Poppe with congratulations to his 65th birthday and also to Professors Gerhard Maeß and Günther Wildenhain on the occasions of their 60th birthdays

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0 Introduction

In the joint paper [2], Bentley, Herrlich and Lowen-Colebunders noted that $Conv_S$, the category of symmetric convergence spaces, and Chy, the category of Cauchy spaces, can be fully embedded into the Katětov's category Fil of filter-merotopic spaces [9]. Fil is a bicoreflective subcategory of Mer, the category of merotopic spaces, which is closely related to the concept of nearness introduced by Herrlich [8] who basically uses notions of set systems which are near. Katětov proved that Fil is cartesian closed and that the corresponding function space structure is the one of continuous convergence. As pointed out above, not all convergence spaces can be described by Fil, only the symmetric ones can be. Supertopological spaces in the sense of Doitchinov [5] or, more generally, the so called neighborhood spaces of Tozzi and Wyler [17], are a common generalization of all topological spaces and proximity spaces but fail to form a cartesian closed category.

Preuß [15] introduced semiuniform convergence spaces as a common generalization of (symmetric) limit spaces (and thus of symmetric topological spaces) as well as of uniform limit spaces (and thus of uniform spaces) with many convenient properties like cartesian closedness, hereditariness, and the fact that products of quotients are quotients again, which together build a strong topological universe. The latter mentioned two concepts lead us to define a notion of convergence in a more general setting. Moreover, it looks like as a first step in finding a more general concept which contains all the above mentioned spaces and, additionally, is being more set-like enough to allow constructions such as formation of functions spaces or universal one-point extensions. As a basic concept we use the convergence of filters to bounded subsets defined similarly as in the Wyler's paper [18]. But as for construction of function spaces with respect to evaluation maps our definition seems to be more natural because in some special cases (such as limit spaces or merotopic spaces) we obtain the usual function space structures.

1 Superfilterformities and related structures

Definition 1.1 *For a set X, a subset $\mathcal{B}^X \subseteq \mathcal{P}X$ (where $\mathcal{P}X$ denotes the set of all subsets of X) is called a* prebornology *or shortly a* B-structure *on X and the elements of $\mathcal{B}^X$ are called* bounded sets *if the following axioms are satisfied:*

(B1) $B' \subseteq B \in \mathcal{B}^X$ *implies* $B' \in \mathcal{B}^X$,

(B2) $\emptyset \in \mathcal{B}^X$,

(B3) $x \in X$ *implies* $\{x\} \in \mathcal{B}^X$.

Given a pair of B-structures $\mathcal{B}^X$ and $\mathcal{B}^Y$ on sets X and Y respectively, a map $f : X \to Y$ is called bounded *iff* $\{f[B]; B \in \mathcal{B}^X\} \in \mathcal{B}^Y$.

Remark 1.2 The category $BOUND$ whose objects are pairs $(X, \mathcal{B}^X)$ where X is a set and $\mathcal{B}^X$ is a B-structure on X and whose morphisms are bounded maps is topological and cartesian closed and has universal one-point extensions, which means that $BOUND$ is a *topological universe.*

Definition 1.3 *Given a B-structure $\mathcal{B}^X$ on a set X, a map $C : \mathcal{B}^X \to \mathcal{P}(\mathcal{P}(\mathcal{P}X))$ is called an* antitonic convergence operator *provided that the following axioms are satisfied:*

(ac1) $\mathcal{H}_1 \in C(B) \wedge \mathcal{H}_1 \ll \mathcal{H}_2$ *imply* $\mathcal{H}_2 \in C(B)$, *where* $\mathcal{H}_1 \ll \mathcal{H}_2$ *iff for each* $F_1 \in \mathcal{H}_1$ *there exists* $F_2 \in \mathcal{H}_2$ *such that* $F_2 \subseteq F_1$,

(ac2) $\emptyset \notin C(\emptyset)$ *and* $\{\emptyset\} \in C(X)$,

(ac3) $B_1 \subseteq B_2 \in \mathcal{B}^X$ *implies* $C(B_2) \subseteq C(B_1)$.

Examples 1.4 (i) For a preuniform convergence space $(X, \mathcal{J}_X)$ and $A \in \mathcal{P}X$ we put $C_{\mathcal{J}_X}(A) := \{\mathcal{F}; \mathcal{F} \times \dot{A} \in \mathcal{J}_X\}$ where $\mathcal{F} \times \dot{A}$ denotes the set generated by $\{F \times B; F \in \mathcal{F}, B \supseteq A\}$.

(ii) For a syntopogenous structure $\mathcal{S}$ on a set X in the sense of Császár [3] we put $C_{\mathcal{S}}(A) := \{\mathcal{F}; \exists <\in \mathcal{S}\ \forall x \in A :< (x) \subseteq \mathcal{F}\}$ where $< (x) := \{T \subseteq X; \{x\} < T\}$.

(iii) For a symmetrical semi-topogenous order $<$ on a set X in the sense of Császár [3] we put $C_{<}(A) := \{\mathcal{H}; < (B) \ll \mathcal{H}\}$ where $< (B) := \{F \subseteq X; X \setminus B < X \setminus F\}$.

(iv) For a convergence structure q on a set X we put $C_q(A) := \{\mathcal{F}^*; \forall x \in A\ \exists \mathcal{F} \in q(x) : \mathcal{F} \ll \mathcal{F}^*\}$.

(v) For a merotopy Γ on a set X we put $C_\Gamma(A) := \Gamma$.

At last we will mention an example with respect to generalized supertopological spaces in the sense of Doitchinov [5], [16]:

(vi) For a neighborhood space $(\mathcal{B}^X, \Theta)$ where Θ is a function from $\mathcal{B}^X$ to $\mathcal{P}(\mathcal{P}(\mathcal{P}X)))$ satisfying certain axioms we put $C_\Theta(B) := \{\mathcal{F}; \forall x \in B : \Theta(\{x\}) \ll \mathcal{F}\}$.

Remark 1.5 In all the above examples, if we reduce set systems to filters, we obtain the notion of *filter convergence*. Moreover, it is interesting to note that in the case $\mathcal{B}^X = \mathcal{P}X$ we can easily construct a bijection between the set of all *antitonic convergence operators* and the set of all *isotonic operators* as follows: For an antitonic convergence operator C and $A \in \mathcal{P}X$ we put $N_C(A) := \{\mathcal{H}; sec\mathcal{H} \in C(X \setminus A)\}$ where $sec\mathcal{H} := \{T \subseteq X; \forall F \in \mathcal{H} : F \cap T \neq \emptyset\}$. Then N_C fulfils the following axioms:

(in1) $\mathcal{H}_1 \in N_C(A) \wedge \mathcal{H}_2 \ll \mathcal{H}_1$ imply $\mathcal{H}_2 \in N_C(A)$,

(in2) $\emptyset \in N_C(\emptyset)$ and $\{\emptyset\} \notin N_C(X)$,

(in3) $B_1 \subseteq B_2$ implies $N_C(B_1) \subseteq N_C(B_2)$.

By imposing some next axioms we obtain the notion of a *near operator* and, more generally, if $\mathcal{B}^X$ is not specified, we get a *supernearness*, both introduced by myself [12], [13].

Now, after giving the previous background, we will define *superfilterformities* and consider some their relationships to other structures and also construct for them natural function space structures with respect to evaluation maps.

Definition 1.6 *For a set X, $FIL(X)$ denotes the set of all filters on X. A function $F : \mathcal{B}^X \to \mathcal{P}(FIL(X))$, where $\mathcal{B}^X$ is a B-structure on X, is called a* superfilterformity *and the pair $(\mathcal{B}^X, F)$ is called a* superfilterformic space *iff the following axioms are satisfied:*

(SF1) *If $B \in \mathcal{B}^X$, $\mathcal{F}_1 \in F(B)$ and $\mathcal{F}_2 \in FIL(X)$, then $\mathcal{F}_1 \subseteq \mathcal{F}_2$ implies $\mathcal{F}_2 \in F(B)$,*

(SF2) *$x \in X$ implies $\dot{x} \in F(x)$ where $\dot{x} := \{T \subseteq X; x \in T\}$,* (SF3) $B_1 \subseteq B_2 \in \mathcal{B}^X$ *implies* $F(B_2) \subseteq F(B_1)$, *which means that* F *is an* antitonic *function.*

Elements of $F(B)$ *are called* B-convergent *filters.*

Given a pair of superfilterformic spaces $(\mathcal{B}^X, F_X)$ *and* $(\mathcal{B}^Y, F_Y)$, *a bounded map* $f : X \to Y$ *is called an* sf-map *iff*

(sf) $B \in \mathcal{B}^X$ *and* $\mathcal{F}_1(B)$ *imply* $f(\mathcal{F}) \in F_2(B)$, *where* $f(\mathcal{F}) := \{G \subseteq Y; f^{-1}[G] \in \mathcal{F}\}$.

We also refer as to an sf-map f *by saying it preserves convergent filters in the above mentioned sense.*

We denote by $SUPFILFORM$ the category whose objects are the superfilterformic spaces and whose morphisms are the bounded maps which preserve convergent filters.

Examples 1.7 (i) Let $(\mathcal{B}^X, \Theta)$ be a subadditive neighborhood space, which means Θ is a function from $\mathcal{B}^X$ to $FIL(X)$ satisfying certain axioms including the following one:

(SA) $B \in \mathcal{B}^X$ implies $\bigcap_{x \in B} \Theta(\{x\}) = \Theta(B)$.

Note that each subadditive neighborhood space $(\mathcal{B}^X, \Theta)$ is additive, which means Θ has the property:

(A) $\Theta(B' \cup B) = \Theta(B') \cap \Theta(B)$ whenever $B' \cup B \in \mathcal{B}^X$.

Further note that every neighborhood structure Θ on $\mathcal{D}^X := \{\emptyset\} \cup \{\{x\}; x \in X\}$, i.e. each pretopology is subadditive. Define $F_\Theta(B) := \{\mathcal{F} \in FIL(X); \forall x \in B : \Theta(\{x\}) \subseteq \mathcal{F}\}$.

(ii) For a limit space (X, q), we consider $\mathcal{P}X$ equipped with the superfilterformity F_q defined as follows:

$F_q(B) := \{\mathcal{F}' \in FIL(X); \forall x \in B\ \exists \mathcal{F} \in q(x) : \mathcal{F} \subseteq \mathcal{F}'\}$.

(iii) For a filtermerotopic space (X, Γ), we consider $\mathcal{P}X$ equipped with the superfilterformity F_Γ defined as follows:

$F_\Gamma(B) := \Gamma$ for each set $B \in \mathcal{P}X$.

(iv) Lastly, let $(X, \mathcal{J}_X)$ be a preuniform convergence space and let $\mathcal{B}^X$ be a prebornology. We consider $\mathcal{B}^X$ equipped with the superfilterformity $F_{\mathcal{J}_X}$ defined as follows:

$F_{\mathcal{J}_X}(B) := \{\mathcal{F} \in FIL(X); \mathcal{F} \times \dot{B} \in \mathcal{J}_X\}$ where $\mathcal{F} \times \dot{B}$ denotes the filter generated by $\{F \times B'; F \in \mathcal{F}, B' \supseteq B\}$.

In this context it seems to be of interest to search the question whether each superfilterformity is induced by a preuniform convergence space or, more precisely, whether we can construct to each superfilterformity F' a preuniform convergence structure $\mathcal{J}$ such that $F_{\mathcal{J}} = F'$. But this is not the aim of this paper. Nevertheless, an affirmative answer to this question is given at the end of the paper.

Theorem 1.8 *The category $SASNBD$ of subadditive neighborhood spaces and usual morphisms is isomorphic to a full subcatefory of $SUPFILFORM$.*

Proof: With respect to 1.7(i), let $(\mathcal{B}^X, F^*)$ be a superfilterformic space. Define an underlying neighborhood structure Θ_{F^*} by setting $\Theta_{F^*}(B) := \{U \subseteq X; \forall x \in B \forall \mathcal{F} \in F^*(\{x\}) : U \in \mathcal{F}\}$ for each $B \in \mathcal{B}^X$. Then we get an isomorphism between the category $SASNBD$ and a corresponding full subcategory of $SUPFILFORM$.

Theorem 1.9 *The category LIM of limit spaces and convergence preserving maps is isomorphic to a full subcatergory of $SUPFILFORM$.*

Proof: With respect to 1.7(ii), let $(\mathcal{P}X, F)$ be a superfilterformic space. Define an underlying convergence space [6] (X, q_F) by setting $q_F(X) := F(\{x\})$ for each $x \in X$. Imposing some aditional axiom on F, q_F is a limit structure. Therefore we obtain an isomorphism between the category LIM and a corresponding full subcategory of $SUPFILFORM$.

Theorem 1.10 *The category $FMER$ of filter-merotopic spaces and convergence preserving maps is isomorphic to a full subcategory of $SUPFILFORM$.*

Proof: With respect to 1.7(iii), let $(\mathcal{P}X, F)$ be a superfilterformic space. Define an underlying filter merotopy Φ_F by setting $\Phi_F := F(\emptyset)$. We obtain an isomorphism between the catergory $FMER$ and a corresponding full subcategory of $SUPFILFORM$.

2 Categorical properties of $SUPFILFORM$

First we note that superfilterformities on a B-structure $\mathcal{B}^X$ can be naturally ordered by putting $F_1 \leq F_2 :\Leftrightarrow \forall B \in \mathcal{B}^X : F_1(B) \subseteq F_2(B)$. The $SUPFILFORM$ fiber of $\mathcal{B}^X$ is a set and it will be denoted by $SUPFILFORM(\mathcal{B}^X)$ (i.e. $SUPFILFORM(\mathcal{B}^X)$ is the set of all superfilterformities on $\mathcal{B}^X$). In addition we observe that from the categorical point of view $SUPFILFORM(\mathcal{B}^X)$ has the terminal separator property, which means that $SUPFILFORM(\mathcal{B}^X)$ is a singleton whenever X is a singleton. We state that $\mathcal{B}^X$ equals to $\{\emptyset, X\}$ in this case. Moreover, the null filter generated by the empty set is allowed to be an element of $FIL(X)$. Thus, to prove that $SUPFILFORM$ is a topological category it is sufficient to show that it has initial structures.

Theorem 2.1 *For any set X, any family $(\mathcal{B}^{X_i}, F_i)_{i\in I}$ of superfilterformic spaces and any family $(f_i : X \to X_i)_{i\in I}$ of functions there exists a unique prebornology $\mathcal{B}^X_{\{f_i^{-1}\}}$ and a unique superfilterformity $F_{\{f_i^{-1}\}}$ on $\mathcal{B}^X_{\{f_i^{-1}\}}$ which is initial with respect to the given data $(\mathcal{B}^X_{\{f_i^{-1}\}}, f_i, (\mathcal{B}^{X_i}, F_i), I)$, i.e. such that for any superfilterformic space $(\mathcal{B}^Y, F)$ a map $g : (\mathcal{B}^Y, F) \to (\mathcal{B}^X_{\{f_i^{-1}\}}, F_{\{f_i^{-1}\}})$ is an sf-map iff for every $i \in I$ the composite map $f_i \circ g : (\mathcal{B}^Y, F) \to (\mathcal{B}^{X_i}, F_i)$ is an sf-map.*

Proof: We define the prebornology $\mathcal{B}^X_{\{f_i^{-1}\}}$ as follows:

$$\mathcal{B}^X_{\{f_i^{-1}\}} := \{B \subseteq X; \forall i \in I : f_i[B] \in \mathcal{B}^{X_i}\}\,.$$

Note that $\mathcal{B}^X_{\{f_i^{-1}\}}$ is the initial B-structure on X with respect to the above mentioned data. Moreover, we define $F_{\{f_i^{-1}\}}$ analogously in such a natural manner, namely:

$$F_{\{f_i^{-1}\}}(B) := \{\mathcal{F} \in FIL(X); \forall i \in I : f_i(\mathcal{F}) \in F_i(f_i[B])\} \text{ for each } B \in \mathcal{B}^X_{\{f_i^{-1}\}})\,.$$

Then $(\mathcal{B}^X_{\{f_i^{-1}\}}, F_{\{f_i^{-1}\}}$ is a superfilterformic space, and if $(\mathcal{B}^Y, F)$ is a superfilterformic space such that $g : (\mathcal{B}^Y, F) \to (\mathcal{B}^X_{\{f_i^{-1}\}}, F_{\{f_i^{-1}\}})$ is an sf-map, for every $i \in I$ the composite map $f_i \circ g : (\mathcal{B}^Y, F) \to (\mathcal{B}^{X_i}, F_i)$ is also an sf-map. Conversely, let latter situation be given. We are to show that $g : (\mathcal{B}^Y, F) \to (\mathcal{B}^X_{\{f_i^{-1}\}}, F_{\{f_i^{-1}\}})$ is an sf-map. So, let $\mathcal{F}$ be a B-convergent filter in Y. Our goal is to verify that $g(\mathcal{F})$ is a $g[B]$-convergent filter in X. To this end, let $i \in I$. We have $(f_i \circ g)(\mathcal{F}) \in F_i((f_i \circ g)[B]) = F_i(f_i[g[B]])$. Hence $g(\mathcal{F}) \in F_{\{f_i^{-1}\}}(g[B])$. Note also that $F_{\{f_i^{-1}\}}$ is uniquely determined.

Corollary 2.2 *$SUPFILFORM$ is a topological category.*

Remark 2.3 The existence of initial structures in $SUPFILFORM$ implies, by using purely categorical arguments, the existence of final structures in $SUPFILFORM$. So $SUPFILFORM$ is complete and its limits (subspaces or products) are formed by supplying the corresponding limits in SET (the category of sets and functions) with the initial structures. Moreover, it is cocomplete and colimits (quotients or sums) are formed by supplying the corresponding colimits in SET with the final structures. Finally, we mention the fact that $SUPFILFORM$ is also wellpowered and cowellpowered in consequence of the above details.

3 Function space structures with respect to evaluation maps

Theorem 3.1 *For any pair $(\mathcal{B}^X, F_1)$, $(\mathcal{B}^Y, F_2)$ of superfilterformic spaces, the set $Y^X := \{f; f : (\mathcal{B}^X, F_1) \to (\mathcal{B}^Y, F_2)$ is an sf-map$\}$ can be supplied in a natural way with a prebornology $\mathcal{B}^{Y^X}$ and this prebornology (which is a B-structure) can be then supplied with a* *superfilterformity F_{Y^X} such that the evaluation map $e : (\mathcal{B}^X, F_1) \times (\mathcal{B}^{Y^X}, F_{Y^X}) \to (\mathcal{B}^Y, F_2)$, $(x, f) \mapsto f(x)$, preserves convergent filters.*

Proof: We define $\mathcal{B}^{Y^X}$ by setting $\mathcal{B}^{Y^X} := \{B^* \subseteq Y^X; \forall B \in \mathcal{B}^X : B^*(B) \in \mathcal{B}^Y\}$, where $B^*(B) := \{f(b); f \in B^*, b \in B\}$. Now we define a superfilterformity F_{Y^X} on $\mathcal{B}^{Y^X}$ by setting for $B^* \in \mathcal{B}^{Y^X}$: $F_{Y^X}(B^*) := \{\mathcal{F}^* \in FIL(Y^X); \forall B \in \mathcal{B}^X \forall \mathcal{F} \in F_1(B) : e(\mathcal{F} \times \mathcal{F}^*) \in F_2(B^*(B))\}$, where $e(\mathcal{F} \times \mathcal{F}^*)$ denotes the filter generated by $\{e[G \times G^*]; G \in \mathcal{F}, G^* \in \mathcal{F}^*\}$. It is easy to verify that F_{Y^X} fulfils the axioms (SF1) and (SF3) in the definition of a superfilterformity (Definition 1.6). We will prove that F_{Y^X} fulfils also the axiom (SF2). To this end, let $f : (\mathcal{B}^X, F_1) \to (\mathcal{B}^Y, F_2)$ be an sf-map. We will show that $\dot{f} \in F_{Y^X}(\{f\})$. To this account, let B be an element of $\mathcal{B}^X$ and let $\mathcal{F}$ be a B-convergent filter. Our goal is to verify that $e(\mathcal{F} \times f) \in F_2(f[B])$. By the supposition it remains to prove the inclusion $f(\mathcal{F}) \subseteq e(\mathcal{F} \times f)$. As $C \in f(\mathcal{F})$ implies $f^{-1}[C] \in \mathcal{F}$ (see Definition 1.6), we have $f^{-1}[C] \times \{f\} \in \mathcal{F} \times \dot{f}$. Further, $C \supseteq e(f^{-1}[C] \times \{f\})$ because $z \in e(f^{-1}[C] \times \{f\})$ implies $z = e(x, f)$ for some $x \in f^{-1}[C]$ and hence $z \in C$. So the above mentioned inclusion is proved.

In the following step we show that the evaluation map is an sf-map. Since $BOUND$ is a topological universe (see Remark 1.2), the evaluation map $e : (\mathcal{B}^X, F_1) \times (\mathcal{B}^{Y^X}, F_{Y^X}) \to (\mathcal{B}^Y, F_2)$ is bounded with respect to the product prebornology $\mathcal{B}^X \times \mathcal{B}^{Y^X}$ on $X \times Y^X$. So, let R be an element of $\mathcal{B}^X \times \mathcal{B}^{Y^X}$ and let $\mathcal{F}$ be an R-convergent filter. We denote by $F_1 \times F^{Y^X}$ the product superfilterformity on $\mathcal{B}^X \times \mathcal{B}^{Y^X}$, i.e. the superfilterformity which is initial with respect to the data $(\mathcal{B}^X \times B^{Y^X}, P_X, P_{Y^X}, ((X, F_1), (Y^X, F_{Y^X}))$ (see Theorem 2.1), where P_X denotes the projection from $X \times Y^X$ to X, and P_{Y^X} denotes the projection from $X \times Y^X$ to Y^X. We are to show that $e(\mathcal{F}) \in F_2(e[R])$. By the supposition we have $P_X(\mathcal{F}) \in F_1(P_X[R])$ and $P_{Y^X}(\mathcal{F}) \in F^{Y^X}(P_{Y^X}[R])$. By definition of F^{Y^X} we get $e(P_X(\mathcal{F}) \times P_{Y^X}(\mathcal{F})) \in F_2(P_{Y^X}[R](P_X[R]))$. Now, according to Definition 1.6, it is sufficient to show the validity of the following two conditions:

(i) $P_{Y^X}[R](P_X[R]) \supseteq e[R]$,

(ii) $e(P_X(\mathcal{F}) \times P_{Y^X}(\mathcal{F})$.

Proof of (i): From $y \in e[R]$ it follows that $y = e(r)$ for some $r \in R$ which means $r = (x, f)$ for some $x \in X$ and $f \in Y^X$, hence $y = f(x)$. We have $f = P_{Y^X}(x, f) = P_{Y^X}(r)$ and $x = P_X(x, f) = P_X(r)$, thus $y \in P_{Y^X}[R](P_X[R])$.

Proof of (ii): From $S \in e(P_X(\mathcal{F}) \times P_{Y^X}(\mathcal{F}))$ it follows that $S \supseteq e[F \times F^*]$ for some $F \in P_X(\mathcal{F})$ and $F^* \in P_{Y^X}(\mathcal{F})$. We have $F \supseteq P_X[F_1]$ for some $F_1 \in \mathcal{F}$ and $F^* \supseteq P_{Y^X}[F_2]$ for some $F_2 \in \mathcal{F}$. Hence $F_1 \cap F_2 \in \mathcal{F}$ because $\mathcal{F}$ is a filter. Now, our goal is to show that $e[F_1 \cap F_2] \subseteq S$. We have $e[F_1 \cap F_2] \subseteq e[P_X[F_1 \cap F_2] \times P_{Y^X}[F_1 \cap F_2]]$, and the following two inclusions hold:

(1) $P_X[F_1] \supseteq P_X[F_1 \cap F_2]$,

(2) $P_{Y^X}[F_2] \supseteq P_{Y^X}[F_1 \cap F_2]$.

Consequently, $P_X[F_1] \times P_{Y^X}[F_2] \supseteq P_X[F_1 \cap F_2] \times P_{Y^X}[F_1 \cap F_2]$ and therefore $e[P_X[F_1 \cap F_2] \times P_{Y^X}[F_1 \cap F_2]]$ is a subset of the set $e[F \times F^*]$. But now we have $e[F_1 \cap F_2] \subseteq S$, which concludes the proof.

Now, on the other hand, we shall prove that for given superfilterformic spaces $(\mathcal{B}^X, F_1)$,$(\mathcal{B}^Y, F_2)$ ans $(\mathcal{B}^Z, F_3)$ the superfilterformity F^{Y^X} is weak enough to ensure that for any sf-map $f : (\mathcal{B}^X \times \mathcal{B}^Z, F_1 \times F_2) \to (\mathcal{B}^Y, F_2)$ the associated function $\bar{f} : (\mathcal{B}^Z, F_3) \to (\mathcal{B}^{Y^X}, F^{Y^X})$ defined by $\bar{f}(z)(x) := f(x, z)$ for each $z \in Z$ and each $x \in X$ is an sf-map.

Theorem 3.2 *For a triple of superfilterformic spaces $(\mathcal{B}^X, F_1), (\mathcal{B}^Y, F_2), (\mathcal{B}^Z, F_3)$ let $f : (\mathcal{B}^X \times \mathcal{B}^Z, F_1 \times F_3) \to (\mathcal{B}^Y, F_2)$ be an sf-map. Then the function $\bar{f} : (\mathcal{B}^Z, F_3) \to (\mathcal{B}^{Y^X}, F^{Y^X})$ defined by $\bar{f}(z)(x) := f(x, z)$ for each $z \in Z$ and each $x \in X$ is an sf-map.*

Proof: With respect to Remark 1.2 it suffices to show that for a given bounded set $\bar{B} \in \mathcal{B}^Z$ and a $\bar{B}$-convergent filter $\bar{\mathcal{F}}$ the image $\bar{f}(\bar{\mathcal{F}})$ is an $\bar{f}[\bar{B}]$-convergent filter, i.e. $\bar{f}(\bar{\mathcal{F}}) \in F^{Y^X}(\bar{f}[\bar{B}])$. We apply the definition of F^{Y^X}. Let $B \in \mathcal{B}^X$ and $\mathcal{F}$ be a B-convergent filter. We have to show that $e(\mathcal{F} \times \bar{f}(\bar{\mathcal{F}})) \in F_2(\bar{f}[\bar{B}](B))$. To show this we will verify that $\mathcal{F} \times \bar{\mathcal{F}} \in (F_1 \times F_3)(B \times \bar{B})$.

Since F is an sf-map, by supposition we have $f(\mathcal{F} \times \bar{\mathcal{F}}) \in F_2(f[B \times \bar{B}])$. Moreover, the inclusion $F_2(f[B \times \bar{B}]) \subseteq F_2(\bar{f}[\bar{B}](B))$ - note that $\bar{f}[\bar{B}](B) \subseteq f[B \times \bar{B}]$ is valid - results in $f(\mathcal{F} \times \bar{\mathcal{F}}] \in F_2(\bar{f}[\bar{B}])$. The desired result will be obtained as a consequence of the fact that $f(\mathcal{F} \times \bar{\mathcal{F}})$ is coarser then the filter $e(\mathcal{F} \times \bar{f}(\bar{\mathcal{F}}))$. To prove this fact, let $\bar{E}$ be an element of $f(\mathcal{F} \times \bar{\mathcal{F}}]$, i.e. $\bar{E} \supseteq f[F \times \bar{F}]$ for some $F \in \mathcal{F}$ and some $\bar{F} \in \bar{\mathcal{F}}$. We have $F \times \bar{f}[\bar{F}] \in \mathcal{F} \times \bar{f}(\bar{\mathcal{F}})$. It remains to show that $f[F \times \bar{F}] \supseteq e[F \times \bar{f}[\bar{F}]]$. From $y \in e[F \times \bar{f}[\bar{F}]]$ it follows that $y = e(x, f)$ for some $x \in F$ and some $f \in \bar{f}[\bar{F}]$. Now $f = \bar{f}(z)$ for some $z \in \bar{F}$, and we have $y = e(x, \bar{f}(z)) = \bar{f}(z)(x) = f(x, y)$ where $(x, y) \in F \times \bar{F}$.

It remains to prove the still open statement that $\mathcal{F} \times \bar{\mathcal{F}} \in (F_1 \times F_3)(B \times \bar{B})$. In other words, we have to check that

(i) $P_X(\mathcal{F} \times \bar{\mathcal{F}}) \in F_1(P_X[B \times \bar{B}])$,

(ii) $P_Z(\mathcal{F} \times \bar{\mathcal{F}}) \in F_3(P_Z[B \times \bar{B}])$

are valid. But the above two conditions are fulfilled whenever the following four inclusions hold:

(1) $P_X[B \times \bar{B}] \subseteq B$,

(2) $P_X(\mathcal{F} \times \bar{\mathcal{F}}) \supseteq \mathcal{F}$,

(3) $P_Z[B \times \bar{B}] \subseteq \bar{B}$,

(4) $P_Z(\mathcal{F} \times \bar{\mathcal{F}}) \supseteq \bar{\mathcal{F}}$.

Indeed, by supposition we firstly get $\mathcal{F} \in F_1(B) \subseteq F_1(P_X[B \times \bar{B}])$ and $\bar{\mathcal{F}} \in F_3(\bar{B}] \subseteq F_3(P_Z[B \times \bar{B}])$, and secondly we have $P_X(\mathcal{F} \times \bar{\mathcal{F}}) \in F_1(P_X[B \times \bar{B}])$ and $P_Z(\mathcal{F} \times \bar{\mathcal{F}}) \in F_3(P_Z[B \times \bar{B}])$. Thus, it remains to prove (1)-(4). But it is trivial to verify (1) and (3). To prove (2), let F be an element of $\mathcal{F}$. Since $\bar{\mathcal{F}} \neq \emptyset$, we can choose $\bar{F} \in \bar{\mathcal{F}}$ such that $F \times \bar{F} \in \mathcal{F} \times \bar{\mathcal{F}}$. Now, with respect to (1), we get $P_X[F \times \bar{F}] \subseteq F$, hence $F \in P_X(\mathcal{F} \times \bar{\mathcal{F}})$. The inclusion (4) can be shown in an analogous way.

4 $SUPFILFORM$ and exponential laws

To give a short summary we note that $SUPFILFORM$ is a topological category with well-behaved function space structures, i.e. it is cartesian closed. With respect to Theorem 3.1, Example 1.7(ii) and Theorem 1.9 or, respectively, Example 1.7(iii) and Theorem 1.10, the corresponding „reduced“ function space structures are the ones of continuous convergence. Moreover, $FMER$, $FCONV$ and some interesting neighborhood spaces can be fully embedded up to isomorphism into $SUPFILFORM$.

By purely categorical arguments (see [7]) the following three exponential laws hold in $SUPFILFORM$:

(1) First exponential law: $X^{Y \times Z}$ is isomorphic to $(X^Y)^Z$,

(2) Second exponential law: $(\prod\limits_{i \in I} X_i)^Y$ is isomorphic to $\prod\limits_{i \in I}(X_i^Y)$,

(3) Third exponential law: $X^{\coprod\limits_{i \in I} Y_i}$ is isomorphic to $\prod\limits_{i \in I}(X^{Y_i})$.

At least we mention that in $SUPFILFORM$ there also holds the interesting distributive law:

$X \times \coprod\limits_{i \in I} Y_i$ is isomorphic to $\coprod\limits_{i \in I}(X \times Y_i)$.

5 Filterunitopic spaces

In Example 1.7(iv) the question raises whether each superfilterformity is induced by a pre-uniform convergence. We will give an affirmative answer by constructing a more general category which includes both the superfilterformic spaces and the preuniform convergence spaces. The construction is very natural and seems to have some nice properties as discussed in the precedent paragraphs.

Definition 5.1 *For a set X, $FIL(X \times X)$ denotes the set of all filters on $X \times X$, i.e. on the cartesian product of X with itself. A function $\mu : \mathcal{B}^X \to \mathcal{P}(FIL(X \times X))$ from a B-structure $\mathcal{B}^X$ on X to subsets of $FIL(X \times X)$ is called a* filterunitopy *(or a* filterunitopic operator*), and the pair $(\mathcal{B}^X, \mu)$ is called a* filterunitopic space *iff the following axioms are satisfied:*

(fut1) $B \in \mathcal{B}^X$ *and* $\mathcal{W} \in \mu(B)$ *with* $\mathcal{W} \subseteq \mathcal{V} \in FIL(X \times X)$ *imply* $\mathcal{V} \in \mu(B)$,

(fut2) $x \in X$ *implies* $\dot{x} \times \dot{x} \in \mu(\{x\})$,

(fut3) $B_1 \subseteq B_2 \in \mathcal{B}^X$ *implies* $\mu(B_2) \subseteq \mu(B_1)$.

$\mathcal{W}$ is called a B-uniform filter iff $\mathcal{W} \in \mu(B)$.

Now, let $(\mathcal{B}^X, \mu_X)$ and $(\mathcal{B}^Y, \mu_Y)$ be filterunitopic spaces. A bounded map f is called an fut-map iff

(fut) $B \in \mathcal{B}^X$ *and* $\mathcal{W} \in \mu_X(B)$ *imply* $(f \times f)(\mathcal{W}) \in \mu_Y(f[B])$, *where* $(f \times f)(\mathcal{W}) := \{R \subseteq Y \times Y; (f \times f)^{-1}[R] \in \mathcal{W}\}$.

We also refer as to an fut-map by saying that it preserves uniform filters in the above mentioned sense.

We denote by F-$UNITOP$ the category whose objects are the filterunitopic spaces and whose morphisms are the bounded maps which preserve uniform filters.

Examples 5.2 (i) For a preuniform convergence space $(X, \mathcal{J}_X)$, we consider $\mathcal{P}X$ equipped with the filterunitopy $\mu_{\mathcal{J}_X}$ defined as follows:

$\mu_{\mathcal{J}_X} := \mathcal{J}_X$ for each $B \in \mathcal{P}X$.

(ii) For a superfilterformic space $(\mathcal{B}^X, F)$ we define a filterunitopic operator μ_F by setting

$\mu_F(B) := \{\mathcal{W} \in FIL(X \times X); \exists \mathcal{F} \in F(B) : \mathcal{F} \times \mathcal{F} \subseteq \mathcal{W}\}$.

Theorem 5.3 *The category $PUConv$ of preuniform convergence spaces and uniformly continuous maps is isomorphic to a full subcategory of F-$UNITOP$.*

Proof: With respect to Example 5.2(i), let $(\mathcal{P}X, \mu)$ be a filterunitopic space. Define an underlying preuniform convergence structure $\mathcal{J}_\mu$ by setting $\mathcal{J}_\mu := \mu(\emptyset)$. By this way we obtain an isomorphism between the category $PUConv$ and a corresponding full subcategory of F-$UNITOP$.

Theorem 5.4 *The category SUPFILFORM is isomorphic to a full subcategory of F-UNITOP.*

Proof: With respect to Example 5.2(ii), let $(\mathcal{B}^X, \mu)$ be a filterunitopic space. Define an underlying superfilterformity F_μ by setting $F_\mu(B) := \{\mathcal{F} \in FIL(X); \mathcal{F} \times \mathcal{F} \in \mu(B)\}$ for each $B \in \mathcal{B}^X$. By this way we obtain an isomorphism between the category *SUPFILFORM* and a corresponding full subcategory of *F-UNITOP*.

Remark 5.5 Now it seems to be of interest to study categorical properties of *F-UNITOP* and, moreover, to deal with the possibility of constructing function space structures with respect to evaluation maps. But it is not the aim of this paper.

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Active boundaries of upper semicontinuous and compactoid relations; closed and inductively perfect maps

Dedicated to professor Harry Poppe on his 65th birthday

ABSTRACT. It is shown that upper semicontinuity (of a relation) is a weak variant of compactoidness; in particular, for the relations that are the inverses of maps, the upper semicontinuity amounts to closedness of the map, and the compactoidness to the perfectness of the map. Therefore the Choquet theorem on the compactness of active boundaries, and the Vainštein lemma on the compactness of boundaries of fibers are instances of the same quest on conditions that make coincide upper semicontinuity and compactoidness. The role of Fréchetness and its variants in this quest is discussed.

1 Introduction

A filter $\mathcal{F}$ is *compactoid* (resp., *finitely compactoid*) *in* A whenever every open cover (resp., finite open cover) of A admits a finite subfamily the union of which belongs to $\mathcal{F}$. The upper semicontinuity of a relation Ω (at a point y) amounts to the finite compactoidness of the filter $\mathcal{F} = \Omega\mathcal{N}(y) \vee (\Omega y)^c$ in the set $A = \Omega y$. The latter property can be rephrased as

$$A^c \in \mathcal{F} \geq \mathcal{N}(A)\,. \tag{1.1}$$

In this paper I show how some compactness properties of A (like paracompactness), and some set-theoretic compactness properties of $\mathcal{F}$ (like strong Fréchetness) can improve original finite compactoidness to compactoidness. We have to do with the composition of a chain of three compactness-like properties. A general theory of such properties in the framework of convergence spaces is developed in [4]. Applied to relations of the type f^-, where f is a continuous closed map, the mentioned results entail the compactness of the boundaries of fibers.

Let X, Y be topological spaces. A relation $\Omega : Y \Longrightarrow X$ is *upper semicontinuous at* y if for every open set P such that $\Omega y \subset P$, there is a neighborhood W of y such that $\Omega W \subset P$. A relation Ω is *compactoid at* y if for every family $\mathcal{P}$ of open sets fulfilling $\bigcup \mathcal{P} \supset \Omega y$, there exists a finite $\mathcal{P}_0 \subset \mathcal{P}$ and a neighborhood W of y such that $\Omega W \subset \bigcup \mathcal{P}_0$.

It is known (and follows immediately from the definitions above) that a relation which is compactoid at y, is upper semicontinuous at y ([1]). It is clear that the converse is true if Ωy is compact, and that this is in some sense also a necessary condition. However under some additional conditions on the spaces (that need not look like variants of compactness), a partial converse is true. G. Choquet announced without proof [2] the following

Theorem 1.1 (Choquet) *If X and Y are metrizable* ([2])*, and if $\Omega : Y \Longrightarrow X$ is upper semicontinuous at y, then there exists a compact set $K \subset \Omega y$ such that for each $U \in \mathcal{N}(K)$, there exists $W \in \mathcal{N}(y)$ for which $\Omega W \subset \Omega y \cup U$.*

Later it was observed [3] that the least set K fulfilling the conditions of Theorem 1.1, is equal to $\partial_{\#}\Omega y$, the *active boundary of* Ω *at* y:

$$\partial_{\#}\Omega y = \operatorname{adh}(\Omega\mathcal{N}(y) \vee (\Omega y)^c), \tag{1.2}$$

where the *adherence* $\operatorname{adh} \mathcal{F}$ of a filter $\mathcal{F}$ is the union of the limits of all the filters that are finer than $\mathcal{F}$; if $\mathcal{F}$ is a filter in a topological space (which is the framework of this paper), then

$$\operatorname{adh} \mathcal{F} = \bigcap_{F \in \mathcal{F}} \operatorname{cl} F.$$

In [9] S. Rolewicz and the present author proved, in the language of measures of non compactness, that if Ω is valued in a completely metrizable space, and upper semicontinuous at a point of countable character ([3]) y, then $\Omega\mathcal{N}(y) \vee (\Omega y)^c$ is compactoid. A. Lechicki and the present author proved in [7] the same result, but under merely the Dieudonné completeness of X.

At about the same time as the appearance of Theorem 1.1 of G. Choquet, I. A. Vainštein proved in [20] that

Theorem 1.2 (Vainštein) *If f is a (continuous) closed map from a metrizable space onto a topological space Y, then for every y of countable character, $\partial f^{-}(y)$ is compact,*

[1] For example, R. E. Smithson [18] calls Ω *subcontinuous* at y if for each cover $\mathcal{P}$ of X, there exist a neighborhood W of y and a finite subfamily that covers ΩW, and shows that if Ω is subcontinuous and graph-closed at y (hence, compactoid at y), then it is upper semicontinuous at y.

[2] It was noticed [3] that it is enough that Y be first-countable.

[3] A point is of *countable character* if its neighborhood filter is countably based.

where ∂A denotes the boundary of A.

It is well-known (and immediate) that the relation $f^- : Y \rightrightarrows X$ ([4]) is upper semicontinuous if and only if the map f is closed. On the other hand (if f is continuous), f^- is compactoid if and only if f is perfect, that is, closed with compact fibers; hence if the relation ∂f^- is compactoid (and f is continuous), then f is inductively perfect ([5]). Therefore Y is metrizable by the theorem of Hanai-Morita-Stone on preservation of metrizability by perfect maps [19, 17]. Now,

Proposition 1.3 *If f is continuous, then the boundary of $f^-(y)$ is included in the active boundary of f^- at y ([6]).*

Proof: If x is a boundary point of $f^-(y)$, then $Q \setminus f^-(y) \neq \emptyset$ for every $Q \in \mathcal{N}(x)$. If f is continuous, then for every $W \in \mathcal{N}(y)$, there exists $Q \in \mathcal{N}(x)$ such that $Q \subset f^-(W)$. Thus $Q \cap f^-(W) \setminus f^-(y) \neq \emptyset$. $\square$

Therefore Theorem 1.2 is a consequence of Theorem 1.1 (strengthened as in the footnote).

A space X is called a *q-space* if every $x \in X$ is a *q-point*, i.e., such that there exists a sequence $(Q_n)_n$ of neighborhoods of x with the property that if $x_n \in Q_n$, then $\mathrm{adh}(x_n) \neq \emptyset$ [16]. E. Michael proved [16, Theorem 2.1] that can be stated as follows:

Theorem 1.4 (Michael) *Let X be T_1, and let $f : X \to Y$ be continuous, closed and onto. If y is a q-point, then $\partial f^-(y)$ is pseudocompactoid in X ([7]).*

In [11] R. Hansell, J. Jayne, I. Labuda and C. A. Rogers extended Theorem 1.4 from the relations f^- to arbitrary relations Ω (of course, the active boundary of Ω at y generalizes the boundary of $f^-(y)$), but under the provision that Y is regular.

As every point of countable character is a q-point, and since in paracompact spaces pseudocompactness implies compactness, Theorem 1.4 generalizes Theorem 1.2.

However, the argument of Vainštein uses only the fact that f is closed at a given point y (that is, that f^- is upper semicontinuous at y), while the argument of Michael requires the upper semicontinuity of f^- on the whole Y. Therefore, the quest of Choquet and of Vainštein (but not that of Michael) can be reduced to that on compactness of the adherence of a filter $\mathcal{F}$ that is finer than the neighborhood filter of a set A as formulated in (1.1), so

[4]For a relation Γ, the inverse relation is denoted by Γ^-; in particular, f^- stands for the inverse relation of a map f.

[5]A continuous map is *inductively perfect* if there exists its restriction (preserving the range) which is perfect.

[6]On the other hand, if the graph of f is closed, then the active boundary is included in $\partial f^-(y)$.

[7]A subset A of X is *pseudocompactoid* if for each $f \in C(X)$, $\sup |f(A)| < \infty$.

that I can reformulate the (strengthened version of) Theorem 1.1, in terms of filters on a single space ([8]).

Theorem 1.5 *If X is metrizable, $\mathcal{F}$ is a countably based filter on X, $A \subset X$, and (1.1) holds, then*

$$\operatorname{adh} \mathcal{F} \subset A\,; \tag{1.3}$$

$$\operatorname{adh} \mathcal{F} \text{ is compact}\,; \tag{1.4}$$

$$\mathcal{F} \geq \mathcal{N}(\operatorname{adh} \mathcal{F})\,. \tag{1.5}$$

Moreover (1.4), (1.5) imply immediately that $\mathcal{F}$ is compactoid in $\operatorname{adh} \mathcal{F}$.

In this paper I investigate the conditions on X, A, and $\mathcal{F}$ that guarantee, separately, (1.3), (1.4), the compactoidness of $\mathcal{F}$, or (1.5).

2 $\mathfrak{J}$-compactoid filters

Recall that two families $\mathcal{A}, \mathcal{B}$ of subsets (of a set) *mesh* (in symbols, $\mathcal{A} \# \mathcal{B}$) if $A \cap B \neq \emptyset$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$; the *grill* $\mathcal{A}^{\#}$ of $\mathcal{A}$ is the family of all subsets that intersect every element of $\mathcal{A}$. Recall also that $\mathcal{A}$ is *finer* than $\mathcal{B}$ (in symbols, $\mathcal{A} \geq \mathcal{B}$) if for every $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$ such that $A \subset B$.

Let $\mathfrak{J}$ be a class of filters, and let X be a topological space. Specializing the definitions of [6, 4], I say that a filter $\mathcal{F}$ on X is $\mathfrak{J}$-*compactoid in* a subset A of X if

$$\underset{\mathcal{F} \# \mathcal{H} \in \mathfrak{J}}{\forall} \quad \operatorname{adh} \mathcal{H} \cap A \neq \emptyset\,. \tag{2.1}$$

If $\mathfrak{J}$ is the class of, respectively, all filters, countably based filters, functional filters ([9]), sequences ([10]), and principal filters, then $\mathfrak{J}$-*compactoid* becomes *compactoid, countably compactoid, pseudocompactoid, sequence-compactoid, and finitely compactoid.*

One should not confuse sequence-compactoidness and sequential compactoidness, the latter being the compactoidness with respect to the sequential modification of the underlying topology.

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[8]If the topology is T_1, then (1.1) implies that $\mathcal{F}$ is free.

[9]By a *functional filter*, I understand a filter generated by $(\{|f| \geq n\})$, where f is a continuous real-valued function.

[10]I use the term *sequence* both in the usual sense and for each filter generated by a sequence; (x_n) is used rather than $(x_n)_n$ if no ambiguity is feared. By a *subsequence* of a sequence (x_n), I understand (x_{n_k}) with (n_k) tending to ∞ (not necessarily increasing), so that the filter generated by (y_m) is finer than the filter generated by (x_n) if and only if (y_{m_0+m}) for some m_0, is a subsequence of (x_n) in our sense.

$\mathfrak{J}$ (class of filters)	$\mathfrak{J}$-compactoid
all	compactoid
countably based	countably compactoid
functional	pseudocompactoid
sequences	sequence-compactoid
principal	finitely compactoid

Tabelle 2.1: Some notions of compactoidness of filters

In particular, the principal filter of a set F is compactoid (resp., countably compactoid, pseudocompactoid) in A if and only if F is compact (resp., countably compact, pseudocompact) relative to A in the usual sense (but without separation hypotheses).

The notion of cover in the theory of convergence spaces specializes to topological spaces as follows. A family $\mathcal{P}$ is a *cover* of a subset A of a topological space whenever

$$\bigcup_{P\in\mathcal{P}} \operatorname{int} P \supset A.$$

It coincides with the classical concept of cover under the provision that its elements are open. The cover definitions of compactness, countable compactness, and so on, with respect to the two concepts of cover, are tantamount. I use here the convergence-theoretic cover, because this makes several subsequent arguments and formulations simpler.

Given a family $\mathcal{P}$ of subsets of a set, let us denote by $\mathcal{P}_c = \{P^c : P \in \mathcal{P}\}$. If $\mathfrak{P}$ is a class of families of sets, then we set

$$\mathfrak{P}_c = \{\mathcal{P}_c : \mathcal{P} \in \mathfrak{P}\},$$

and denote by $\mathfrak{P}_\star$ the class of all (possibly) degenerate filters generated by the elements of $\mathfrak{P}_c$. It is straightforward that

Theorem 2.1 [4] *A family $\mathcal{F}$ is $\mathfrak{P}_\star$-compactoid in A if and only if for every cover $\mathcal{P} \in \mathfrak{P}$ of A, there exists a finite subfamily $\mathcal{P}_0$ of $\mathcal{P}$ such that $\bigcup \mathcal{P}_0 \in \mathcal{F}$.*

In particular, a filter $\mathcal{F}$ is finitely compactoid in a subset A of a topological space if and only if every open set Q that includes A belongs to $\mathcal{F}$; in other words, whenever $\mathcal{F} \geq \mathcal{N}(A)$.

A *functional cover* of A is a cover of the form $(\{|f| < n\})$, where f is a real-valued map continuous at every point of A. Hence, a filter $\mathcal{F}$ on X is *pseudocompactoid* in A [4] if for every real-valued function f on X that is continuous on A (that is, $f \in C_A(X)$), there exists $F \in \mathcal{F}$ such that $\sup |f(F)| < \infty$ ([11]).

[11] This notion is different from the one introduced in [6] under the same name.

Of course, compactoidness implies countable compactoidness, and the latter implies pseudocompactoidness, finite compactoidness and sequence-compactoidness, and the implications are in general strict. Indeed, the principal filter of the whole space X is compactoid (resp., countably compactoid, pseudocompactoid, finitely compactoid, sequence-compactoid) whenever the topological space X is compact (resp., countably compact, pseudocompact, finitely compact, sequence-compact). Notice that every space is finitely compact, and X is sequence-compact if and only if it is countably compact.

Example 3.4 shows that there exists a filter that is simultaneously finitely compactoid and sequence-compactoid, but not countably compactoid (actually not even pseudocompactoid). In the next section I will consider conditions on filters that enable one to reverse these implications.

I shall say that $\mathcal{F}$ is *nearly $\mathfrak{J}$-compactoid in A* if the following variant of (2.1) holds ([12])

$$\forall_{\mathcal{F} \leq \mathcal{H} \in \mathfrak{J}} \quad \operatorname{adh} \mathcal{H} \cap A \neq \emptyset. \tag{2.2}$$

It is clear that (2.1) implies (2.2). Of course, near compactoidness coincides with compactoidness. It was observed in [6, Proposition 6.4] that countable near compactoidness coincides with near sequence-compactoidness, and is strictly weaker than countable compactoidness [6, Proposition 6.5]; in fact, it is strictly weaker than sequence-compactoidness (a free ultrafilter on a countable discrete space X is nearly sequence-compactoid in X, but not sequence-compactoid).

The following proposition slightly extends [13, Lemma 1].

Proposition 2.2 *If X is T_1 and if $A^c \in \mathcal{F} \geq \mathcal{N}(A)$, then $\mathcal{F}$ is nearly sequence-compactoid in A.*

Proof: Suppose that, on the contrary, there exists a sequence $(x_n) \geq \mathcal{F}$ such that $\operatorname{adh}(x_n) \cap A = \emptyset$. As $A^c \in \mathcal{F}$, we can suppose without loss of generality that $\{x_n : n \in \mathbb{N}\} \cap A = \emptyset$. Consequently, $\operatorname{cl}\{x_n : n \in \mathbb{N}\} = \{x_n : n \in \mathbb{N}\} \cup \operatorname{adh}(x_n)$ is disjoint from A, thus $\left(\operatorname{cl}\{x_n : n \in \mathbb{N}\}\right)^c$ is a one-element cover of A, hence by finite compactoidness, $\operatorname{cl}\{x_n : n \in \mathbb{N}\} \notin \mathcal{F}^\#$, thus in particular $\{x_n : n \in \mathbb{N}\} \notin \mathcal{F}^\#$ in contradiction with $(x_n) \geq \mathcal{F}$. $\square$

As for classical variants of compactness, certain compactoidness-type properties of a space combined (we can even say, composed) with compactoidness properties of filters improve the latter [4]. For example, a countably compactoid filter on a Lindelöf space is compactoid, or

Proposition 2.3 *A pseudocompactoid filter on a normal topological space is sequence-compactoid.*

[12] In [6] *quasi* was used for what I call now *nearly*; because *quasi compact* has had frequently a different meaning in the literature, I decided to change the terminology of [6].

Proof: If $\mathcal{F}$ is not sequence-compactoid, then there exists (x_n) such that $(x_n)\#\mathcal{F}$ and $\mathrm{adh}(x_n) = \emptyset$. Consequently, the set $\{x_n : n \in \mathbb{N}\}$ is closed and discrete. Therefore the function $f(x_n) = n$, that is continuous on $\{x_n : n \in \mathbb{N}\}$, can be continuously extended to the whole space. Since $\{x_k : k \geq n\} \in \mathcal{F}^{\#}$ for every n, one has $\sup |f(F)| = \infty$ for every $F \in \mathcal{F}$. $\square$

However, many other classical results in this direction do not extend (from principal) to larger classes of filters. Example 3.4 shows that a sequence-compactoid filter in a completely metrizable space need not be even pseudocompactoid. In the case of principal filters, sequence-compactoid is countably compactoid, and thus compactoid in completely metrizable spaces.

3 Set-theoretic properties of filters improving compactoidness

If a filter enjoys both one of the compactoidness properties, and a set theoretic property, like Fréchetness or strong Fréchetness, then the type of its compactoidness can be improved.

Let $\mathfrak{D}, \mathfrak{J}$ be classes of filters. A filter $\mathcal{F}$ is said to be a $\mathfrak{J}/\mathfrak{D}$-*filter* if for every $\mathcal{H} \in \mathfrak{J}$ such that $\mathcal{H}\#\mathcal{F}$, there exists a filter $\mathcal{D} \in \mathfrak{D}$ such that $\mathcal{D} \geq \mathcal{H}$ and $\mathcal{D}\#\mathcal{F}$. Notice that a filter $\mathcal{F}$ is a $\mathfrak{J}/\mathfrak{D}$-filter if and only if for every set-theoretic cover of an element of $\mathcal{F}$ by a family $\mathcal{P} \in \mathfrak{D}_c$, there exists a refinement $\mathcal{R}$ of $\mathcal{P}$ which is a set-theoretic cover of an element of $\mathcal{F}$, and such that $\mathcal{R} \in \mathfrak{J}_c$.

Properties of the type $\mathfrak{J}/\mathfrak{D}$ can be considered as variants of compactness of filters $\mathcal{F}$ with respect to the discrete topology ([13]). Conjugate with compactoidness properties of $\mathcal{F}$ in A, and with compactness-like properties of A, they imply stronger compactoidness properties of $\mathcal{F}$ in A.

A filter $\mathcal{F}$ is said to be a *super*-$\mathfrak{J}/\mathfrak{D}$-*filter* if for every $\mathcal{H} \in \mathfrak{J}$ such that $\mathcal{H}\#\mathcal{F}$, there exists a filter $\mathcal{D} \in \mathfrak{D}$ such that $\mathcal{D} \geq \mathcal{F} \vee \mathcal{H}$. It is obvious that

Theorem 3.1 *If a $\mathfrak{J}/\mathfrak{D}$-filter is $\mathfrak{D}$-compactoid in A, then it is $\mathfrak{J}$-compactoid in A. If a super-$\mathfrak{J}/\mathfrak{D}$-filter is nearly $\mathfrak{D}$-compactoid in A, then it is $\mathfrak{J}$-compactoid in A.*

A filter $\mathcal{F}$ is called *Fréchet* [5] if for every $A \in \mathcal{F}^{\#}$, there is a sequence (x_n) such that $(x_n) \geq \mathcal{F} \vee A$, that is, whenever

$$\mathcal{F}^{\#} = \bigcup_{(x_n) \geq \mathcal{F}} (x_n),$$

[13] More general schemes are studied in [4].

where (x_n) is a shorthand for the filter generated by (x_n). In other words, a filter is Fréchet if and only if it is a super-principal/sequence filter (super-$\mathfrak{J}/\mathfrak{D}$-filter, where $\mathfrak{J}$ is the class of principal filters, and $\mathfrak{D}$ is the class of sequences) ([14]).

On the other hand, Fréchet filters belong to the class of super-sequence/sequence filters (that is, super-$\mathfrak{D}/\mathfrak{D}$-filters for the class $\mathfrak{D}$ of sequences). Indeed,

Proposition 3.2 *If $\mathcal{F}$ is a Fréchet filter and a sequence (x_n) meshes $\mathcal{F}$, then there exists a subsequence (x_{n_k}) that is finer than $\mathcal{F}$.*

Proof: We decompose $\mathcal{F} = \mathcal{F}_\bullet \wedge \mathcal{F}_\circ$, where $\mathcal{F}_\bullet$ is the principal filter of the kernel $\bigcap \mathcal{F}$ of $\mathcal{F}$ and $\mathcal{F}_\circ$ is the trace of $\mathcal{F}$ on the complement of the kernel. Of course, $\mathcal{F}_\bullet$ and $\mathcal{F}_\circ$ are Fréchet filters. Let (x_n) be a sequence that meshes $\mathcal{F}$; hence (x_n) meshes either with $\mathcal{F}_\bullet$ or with $\mathcal{F}_\circ$. In the first case the trace of (x_n) on $\mathcal{F}_\bullet$ is a subsequence of (x_n) that is finer than $\mathcal{F}$. In the second case, (x_n) must be free, and $\mathcal{F}_\circ \vee \{x_n : n \in \mathbb{N}\}$ is a Fréchet filter, hence there is a sequence $(y_k) \geq \mathcal{F}_\circ$, and since $\mathcal{F}_\circ$ is free, (y_k) is finer than the cofinite filter of the set $\{x_n : n \in \mathbb{N}\}$, that is, (x_n). $\square$

Remark that no ultrafilter is a super-sequence/sequence filter.

A filter $\mathcal{F}$ is called *strongly Fréchet* if for every countably based filter $\mathcal{H}$ with $\mathcal{H}\#\mathcal{F}$, there is a sequence $(x_n) \geq \mathcal{H} \vee \mathcal{F}$; in other words, strongly Fréchet filters coincide with super-countably-based/sequence filters (super-$\mathfrak{J}/\mathfrak{D}$-filters, where $\mathfrak{J}$ is the class of countably based filters, and $\mathfrak{D}$ is the class of sequences) ([15]). It immediately follows from Theorem 3.1 that

Proposition 3.3 *A Fréchet filter that is nearly sequence-compactoid in A is sequence-compactoid in A and finitely compactoid in A.*

A strongly Fréchet filter that is nearly sequence-compactoid in A is countably compactoid in A.

Example 3.4 (*A finitely compactoid, sequence-compactoid non pseudocompactoid filter*) Consider $\mathcal{F} = \mathcal{N}(\mathbb{Z}) \vee \mathbb{Z}^c$ on the real line equipped with the natural topology. No sequence is finer than $\mathcal{F}$, hence $\mathcal{F}$ is nearly countably compactoid. Since $\mathcal{F}$ is a Fréchet filter, by Proposition 3.3, $\mathcal{F}$ is sequence-compactoid. $\mathcal{F}$ is not pseudocompactoid, because the function $f(r) = r$ is continuous and unbounded on every element of $\mathcal{F}$.

By Propositions 2.2 and 3.3,

Corollary 3.5 *If $\mathcal{F}$ is a strongly Fréchet filter on a T_1 space, such that $A^c \in \mathcal{F} \geq \mathcal{N}(A)$, then $\mathcal{F}$ is countably compactoid in A.*

[14] It is known that a filter is Fréchet [5] if and only if it is an intersection of sequential filters.

[15] A topological space is *Fréchet* if and only if its every neighborhood filter is Fréchet; a topological space is *strongly Fréchet* if and only if its every neighborhood filter is strongly Fréchet.

In a regular topological space every compactoid filter has compact adherence, but from the countable compactoidness of a filter we cannot conclude that the adherence is countably compact (think of a relatively countably compact set with non countably compact closure) ([16]). The latter is true in normal spaces, because pseudocompactoid closed sets are countably compact therein.

As countable compactoidness implies pseudocompactoidness of a filter, and the latter implies the pseudocompactoidness of the adherence of the filter, we have

Corollary 3.6 *If $\mathcal{F}$ is a strongly Fréchet filter on a T_1 space, such that $A^c \in \mathcal{F} \geq \mathcal{N}(A)$, then* $\operatorname{adh}\mathcal{F}$ *is pseudocompactoid in A.*

Recall [10, page 568] that a topology is *Dieudonné complete* if it is homeomorphic to a closed subset of a product of metrizable spaces; equivalently, if its topology admits a complete uniformity $\mathfrak{U}$. Every paracompact space is Dieudonné complete. A filter $\mathcal{F}$ in a uniform space is *totally bounded* if for every $U \in \mathfrak{U}$, there exists a finite set A such that $UA \in \mathcal{F}$. It is known [6, Proposition 7.1] that every compactoid filter is totally bounded, and that the converse statement is true for complete uniformities. Here is a refinement of [6, Theorem 7.2] that required $\mathcal{F}$ to be countably based.

Proposition 3.7 *A strongly Fréchet, countably compactoid filter in a Dieudonné complete space, is compactoid.*

Proof: It is enough to show that a strongly Fréchet filter $\mathcal{F}$ which is countably compactoid on a Dieudonné complete space, is totally bounded. Let $\mathfrak{U}$ be a complete uniformity of X, and suppose that $\mathcal{F}$ is not totally bounded: there exists $U \in \mathfrak{U}$ such that $UA \notin \mathcal{F}$ for every finite subset A of X. Consider $W \in \mathfrak{U}$ for which $W^5 \subset U$.

I claim that there exists a sequence $(x_n)_n$ and sequences $(x_{(n,k)})_k \geq \mathcal{F}$ such that $x_n \in \operatorname{adh}(x_{(n,k)})_k$, and $\{x_{(n,k)} : k \in \mathbb{N}\} \subset Wx_n$, and moreover $W^2x_n \cap W^2x_m = \emptyset$ for $n \neq m$. Indeed, as $\mathcal{F}$ is Fréchet and countably compactoid, there exists a sequence $(x_{0,k})_k \geq \mathcal{F}$ and x_0 such that $x_0 \in \operatorname{adh}(x_{0,k})_k$, and $(x_{0,k})_k$ is included in Wy_0, so that the induction hypothesis of order 0 is fulfilled. Suppose that the induction hypothesis holds up to the order n. Then, by assumption, $U\{x_0, x_1, \ldots, x_n\} \notin \mathcal{F}$, so that by Fréchetness and countable compactoidness, there exist x_{n+1} and in $Wx_{n+1} \setminus U\{x_0, x_1, \ldots, x_n\}$ a sequence $(x_{n+1,k}) \geq \mathcal{F}$ such that $x_{n+1} \in \operatorname{adh}(x_{n+1,k})_k$. Clearly, $W^2x_m \cap W^2x_{n+1} = \emptyset$ for $0 \leq m \leq n$.

Now, by strong Fréchetness, there exists a sequence $(x_{(n_p,k_p)})_p \geq \mathcal{F}$, for which $(n_p)_p$ tends to ∞, and $\operatorname{adh}(x_{(n_p,k_p)})_p \neq \emptyset$ by countable compactoidness. But it follows from the construction

[16] In order to avoid this inconvenience I. Labuda assumes in several arguments of [13] that each relatively countably compact set is relatively compact. Dieudonné complete spaces have this property.

that $\{x_{(n_p,k_p)} : p \in \mathbb{N}\}$ is discrete, hence its every subset is closed, and thus $\mathrm{adh}(x_{(n_p,k_p)})_p = \emptyset$: a contradiction. □

Corollary 3.8 *If $\mathcal{F}$ is a strongly Fréchet filter on a Dieudonné complete space, and $A^c \in \mathcal{F} \geq \mathcal{N}(A)$, then $\mathcal{F}$ is compactoid in A.*

Since each a Dieudonné complete space is regular, it follows that $\mathrm{adh}\,\mathcal{F}$ is compact.

A. V. Arhangel'skii and A. Bella proved [1, Theorem 9] that (for T_1 topologies) the boundary of each fiber of a continuous closed map to a countably fan-tight space is pseudocompactoid in the domain space. I say that a filter $\mathcal{F}$ on a set X is *countably fan-tight* if for every (decreasing) sequence (A_n) of sets such that $A_n \in \mathcal{F}^{\#}$, there exists a sequence (K_n) of finite sets such that $K_n \subset A_n$ and $\bigcup_{n\in\mathbb{N}} K_n \in \mathcal{F}^{\#}$. A topological space is countably fan-tight if and only if its every neighborhood filter is countably fan-tight. A Fréchet filter is countably fan-tight if and only if it is strongly Fréchet [1]. Therefore the following theorem generalizes Corollary 3.5 and, because of Corollary 3.6, it generalizes also [1, Theorem 9].

Theorem 3.9 *If $\mathcal{F}$ is a countably fan-tight filter on a T_1 space, and if $A^c \in \mathcal{F} \geq \mathcal{N}(A)$, then $\mathcal{F}$ is countably compactoid in A.*

Proof: Let $\mathcal{H}$ be a countably based filter that meshes $\mathcal{F}$. As $A^c \in \mathcal{F}$, the filter $\mathcal{H} \vee A^c$ also meshes $\mathcal{F}$. Consider (H_n) be a (decreasing) base of $\mathcal{H} \vee A^c$ such that $H_n \cap A = \emptyset$. By countable fan-tightness, there exists a sequence (K_n) of finite sets such that $K_n \subset H_n$ such $B = \bigcup_{n\in\mathbb{N}} K_n \in \mathcal{F}^{\#}$. By T_1, the filter $\mathcal{F}$ is free, and hence $B_n = \bigcup_{m\geq n} K_m \in \mathcal{F}^{\#}$ for every n. Therefore $\mathrm{cl}\, B_n \cap A \neq \emptyset$, because $\mathcal{F}$ is finitely compactoid in A.

I claim that $\mathrm{adh}(B_n) \cap A \neq \emptyset$. If this were not the case, then on one hand $\mathrm{adh}(B_n) \cap A = \emptyset$, and by construction $B \cap A = \emptyset$. As the filter generated by (B_n) is a sequence in a T_1-space, $\mathrm{cl}\, B = B \cup \mathrm{adh}(B_n)$, that is a contradiction. As $(B_n) \geq \mathcal{H}$, we conclude that $\mathcal{F}$ is countably compactoid in A. □

In order to come back to the starting point, that of upper semicontinuous relations, let me notice that it is straightforward that if $\mathcal{N}(y)$ is, respectively, countably based, Fréchet, strongly Fréchet, countably fan-tight, then $\Omega\mathcal{N}(y) \vee (\Omega y)^c$ is, respectively, countably based, Fréchet, strongly Fréchet, countably fan-tight. As a result, the following corollary generalizes [1, Theorem 9] in several respects.

Corollary 3.10 *Let X be T_1, and let y be a countably fan-tight point of Y. If $\Omega : Y \Longrightarrow X$ is upper semicontinuous at y, then $\Omega\mathcal{N}(y) \vee (\Omega y)^c$ is countably compactoid, and thus its active boundary $\partial_{\#}\Omega y$ is pseudocompactoid. If moreover X is Dieudonné complete, then $\partial_{\#}\Omega y$ is compact.*

4 Active boundaries

The *active boundary* $\partial_{\mathcal{F}}A$ of a filter $\mathcal{F}$ with respect to a set A is defined by $\partial_{\mathcal{F}}A = \operatorname{adh}\left(\mathcal{F}\vee A^c\right)$ [6]. It follows right away that $\partial_{\mathcal{F}}A$ is disjoint from $\operatorname{int}A$. Of course, if $A^c \in \mathcal{F}$, then the active boundary of $\mathcal{F}$ with respect to A is equal to $\operatorname{adh}\mathcal{F}$. By Theorem 1.5, if $\mathcal{F}$ is a countably based filter on a metrizable space, and if A is a subspace such that

$$A^c \in \mathcal{F} \geq \mathcal{N}(A), \tag{4.1}$$

then the active boundary of $\mathcal{F}$ with respect to A is a subset of A, hence of ∂A. Observe that A is not assumed to be closed. In general, (4.1) does not imply that $\operatorname{adh}\mathcal{F} \subset A$, even if A is closed.

Example 4.1 Let X be a non regular Hausdorff topological space. Then there exist a closed set A and $x \notin A$ such that the filter $\mathcal{F} = \mathcal{N}(x) \vee \mathcal{N}(A)$ is non degenerate. This is a free filter finer than $\mathcal{N}(A)$ and than $\mathcal{N}(x)$, hence convergent to x. Therefore $\{x\} = \operatorname{adh}\mathcal{F}$, the active boundary of $\mathcal{F}$ with respect to A, is disjoint from A.

Example 4.1 is a sort of characterization, because if A is a closed subset of a regular space, then $\operatorname{adh}\mathcal{N}(A) \subset A$. Therefore in this case (4.1) implies $\operatorname{adh}\mathcal{F} \subset A$.

A subset of a topological space is said to be G_δ*-closed* if it is closed for the topology obtained from the original topology by taking G_δ sets as a base of open sets. I. Labuda proved in [13, Lemma 6] a result to the effect that if a countably based filter $\mathcal{F}$ on a regular (Hausdorff) space fulfills (4.1) and if A is G_δ-closed, then $\operatorname{adh}\mathcal{F} \subset A$.

Theorem 4.2 *Let A be a G_δ-closed subset of a regular T_1 space and let $\mathcal{F}$ be strongly Fréchet. If* (4.1) *holds, then* $\operatorname{adh}\mathcal{F} \subset A$.

Proof: Let $x \notin A$ and let (P_n) be a (decreasing) sequence of closed neighborhoods of x such that $\bigcap_n P_n \cap A = \emptyset$. If $x \in \operatorname{adh}\mathcal{F}$, then $P_n \in \mathcal{F}^\#$ for each n. Hence there exists a sequence $(x_n) \geq \mathcal{F}$ such that $x_n \in P_n$, because $\mathcal{F}$ is strongly Fréchet. It follows that $\operatorname{adh}(x_n) \subset \bigcap_n P_n$. On the other hand by Proposition 2.2, $\mathcal{F}$ is nearly sequence-compactoid in A, thus $\operatorname{adh}(x_n) \cap A \neq \emptyset$. A contradiction. □

In the case of Fréchet filters $\mathcal{F}$ in Hausdorff spaces, (4.1) implies that the sequential adherence of $\mathcal{F}$ is included in A. Therefore in this case we prove that $\operatorname{adh}\mathcal{F} \subset A$ if we can prove that the sequential adherence of $\mathcal{F}$ coincides with the adherence of $\mathcal{F}$.

Recall that $\operatorname{adh}_{\mathrm{Seq}}\mathcal{F}$ stands the *sequential adherence* of $\mathcal{F}$, that is,

$$\operatorname{adh}_{\mathrm{Seq}}\mathcal{F} = \bigcup_{(x_n)\#\mathcal{F}} \lim(x_n)\,.$$

By Proposition 2.2, each sequence (x_n) on A^c finer than $\mathcal{N}(A)$ is nearly sequence-compactoid in A. This fact slightly refines [13, Lemma 1]. It follows that

Proposition 4.3 *Suppose that each sequence has at most one limit point. If $\mathcal{F}$ is Fréchet and if $A^c \in \mathcal{F} \geq \mathcal{N}(A)$, then*

$$\operatorname{adh}_{\mathrm{Seq}} \mathcal{F} \subset A\,.$$

5 Minimal kernels

I say that a set A is a *kernel* of a filter $\mathcal{F}$ if (4.1) holds ([17]). If $\mathcal{F}$ is a countably based filter on a metrizable space, then by Theorem 1.5, (4.1) implies that $\operatorname{adh} \mathcal{F} \subset A$ and that $(\operatorname{adh} \mathcal{F})^c \in \mathcal{F} \geq \mathcal{N}(\operatorname{adh} \mathcal{F})$, so that $\operatorname{adh} \mathcal{F}$ is a the least kernel of $\mathcal{F}$. In general, (4.1) does not imply that $\operatorname{adh} \mathcal{F}$ is a kernel of $\mathcal{F}$.

Example 5.1 (A filter with empty adherence that admits kernels). Let $X = \mathbb{R}$ and $\mathcal{F} = \mathcal{N}(\mathbb{N}) \vee \mathbb{N}^c \vee \mathcal{B}$, where the filter $\mathcal{B}$ is generated by $\{\{x : |x| \geq n\} : n \in \mathbb{N}\}$. The adherence of $\mathcal{F}$ is empty, thus $\mathcal{N}(\emptyset)$ is the degenerate filter, so that $\mathcal{F}$ is not finer than $\mathcal{N}(\emptyset)$.

On the other hand, it follows from Example 4.1 that if $\operatorname{adh} \mathcal{F}$ is a kernel of $\mathcal{F}$, it need not be the least one. Indeed in that example $\{x\} = \operatorname{adh} \mathcal{F}$ is a kernel of $\mathcal{F}$ disjoint from another kernel: A.

It is known [6] that if $\mathcal{F}$ is compactoid and $(\operatorname{adh} \mathcal{F})^c \in \mathcal{F}$, then $\operatorname{adh} \mathcal{F}$ is a kernel of $\mathcal{F}$. This follows from the fact that if $\mathcal{F}$ is compactoid (in A), then $\operatorname{adh} \mathcal{H} \cap \operatorname{adh} \mathcal{F} \neq \emptyset$ for every $\mathcal{H} \# \mathcal{F}$, because there exists a (convergent) ultrafilter $\mathcal{U} \geq \mathcal{H} \vee \mathcal{F}$, so that $\mathcal{F}$ is compactoid in $\operatorname{adh} \mathcal{F}$. If $\mathcal{F}$ is merely countably compactoid (in A) and $\mathcal{H}$ is a countably based filter such that $\mathcal{H} \# \mathcal{F}$, then $\operatorname{adh} \mathcal{H}$ need not intersect $\operatorname{adh} \mathcal{F}$, because there need not be a convergent filter $\mathcal{G}$ finer than $\mathcal{H} \vee \mathcal{F}$.

For a given filter $\mathcal{F}$, consider the set $\mathfrak{U}(\mathcal{F}) = \{\mathcal{U} \in \beta(\mathcal{F}) : \lim \mathcal{U} \neq \emptyset\}$ of convergent ultrafilters. If $\mathfrak{U}(\mathcal{F})$ is Stone closed, that is, if the filter $\mathcal{F}^+ = \bigwedge \mathfrak{U}(\mathcal{F})$ is compactoid, then $\mathcal{F}$ is called *solid*. The infimum of a compactoid filter and of a non adherent filter is an example of a solid non compactoid filter.

Proposition 5.2 *If $\mathcal{F}$ is a free solid non compactoid filter in a regular topological space, then $\operatorname{adh} \mathcal{F}$ is not a kernel of $\mathcal{F}$.*

Proof: Take any $\mathcal{U} \in \beta(\mathcal{F}) \setminus \beta(\mathcal{F}^+)$. Since by definition $\operatorname{adh} \mathcal{U} = \emptyset$, for every $x \in \operatorname{adh} \mathcal{F} = \operatorname{adh} \mathcal{F}^+$, there exists an open set O_x that contains x and such that $\mathcal{O}_x \notin \mathcal{U}$. Of course, $O_x \in \mathcal{W}$ for every $\mathcal{W} \in \beta(\mathcal{F})$ with $x \in \lim \mathcal{W}$. Since $\mathcal{F}^+$ is compactoid and X is regular,

[17] A similar notion of an $\mathcal{F}$-kernel of a set B was introduced in [6] as an abstraction of a concept from [7]; It is a kernel of $\mathcal{F}$ in the present sense that is moreover a subset of B.

$\operatorname{adh} \mathcal{F}^+ = \operatorname{adh} \mathcal{F}$ is compact, and thus there exist $x_1, \ldots x_n$ such that $O_{x_1} \cup \ldots \cup O_{x_n} \in \mathcal{N}(\operatorname{adh} \mathcal{F})$. On the other hand, $O_{x_1} \cup \ldots \cup O_{x_n} \notin \mathcal{U}$, thus $O_{x_1} \cup \ldots \cup O_{x_n} \notin \mathcal{F}$. $\square$

The *Fréchet adherence* of a filter $\mathcal{F}$:

$$\operatorname{adh}^{\varepsilon} \mathcal{F} = \bigcup_{(x_n) \geq \mathcal{F}} \operatorname{adh}(x_n) \tag{5.1}$$

has been introduced by A. Lechicki [14] ([18]) as an abstraction of an object considered by I. Labuda [13]. This is not a notion of adherence with respect to some convergence modification of the underlying topology; $\operatorname{adh}^{\varepsilon} \mathcal{F}$ depends only on the *Fréchet core* of $\mathcal{F}$, that is, on

$$\mathcal{F}^{\varepsilon} = \bigcap_{(x_n) \geq \mathcal{F}} (x_n),$$

where as usual (x_n) is a shorthand for the filter generated by (x_n). A filter is Fréchet if and only if it is equal to its Fréchet core. If $\mathcal{F}$ is Fréchet, then $\operatorname{adh}^{\varepsilon} \mathcal{F} = \bigcup_{(x_n) \# \mathcal{F}} \operatorname{adh}(x_n)$, hence $\operatorname{adh}_{\mathrm{Seq}} \mathcal{F} \subset \operatorname{adh}^{\varepsilon} \mathcal{F}$. The following theorem has been proved for countably based filters by A. Lechicki in [14, 3.1].

Theorem 5.3 *If $\mathcal{F}$ is a Fréchet filter and if $A^c \in \mathcal{F} \geq \mathcal{N}(A)$, then*

$$\mathcal{F} \geq \mathcal{N}\big(A \cap \operatorname{adh}^{\varepsilon} \mathcal{F}\big). \tag{5.2}$$

Proof: Because $\mathcal{F}$ is Fréchet, it is enough to show that $(x_n) \geq \mathcal{N}\big(A \cap \operatorname{adh}^{\varepsilon} \mathcal{F}\big)$ for every $(x_n) \geq \mathcal{F}$. If this were not the case, then there would exist an open set $O \supset A \cap \operatorname{adh}^{\varepsilon} \mathcal{F}$ and a subsequence (x_{n_k}) such that $x_{n_k} \notin O$ for each k. By Proposition 2.2, $\operatorname{adh}(x_{n_k}) \cap A \neq \emptyset$, but on the other hand, $\operatorname{adh}(x_{n_k}) \cap O = \emptyset$, which is a contradiction. $\square$

It follows from Theorem 5.3 and from Proposition 4.3 that in Hausdorff spaces, $\operatorname{adh}_{\mathrm{Seq}} \mathcal{F}$ is included in every kernel of $\mathcal{F}$. If $\mathcal{F}$ is Fréchet, then $\operatorname{adh}_{\mathrm{Seq}} \mathcal{F} \subset \operatorname{adh}^{\varepsilon} \mathcal{F}$, thus it follows [13, Lemma 2] that if $\operatorname{adh}_{\mathrm{Seq}} \mathcal{F}$ is a kernel of $\mathcal{F}$, then it is the smallest one.

A topological space is said to have the *Fréchet property with respect to a class* $\mathcal{B}$ of subsets if for every $B \in \mathcal{B}$ if $x \in \operatorname{cl} B$, then there exists a sequence (x_n) in B such that $x \in \lim(x_n)$. Of course, if a topological space has the Fréchet property with respect to all its subsets, then the topology is Fréchet by definition.

I. Labuda proved (for Hausdorff spaces, and in different terms) [13] the following

Proposition 5.4 *If $\mathcal{F}$ is a filter on a T_1 space with the Fréchet property with respect to countably compactoid sets, then*

$$\operatorname{adh}^{\varepsilon} \mathcal{F} \subset \operatorname{adh}_{\mathrm{Seq}} \mathcal{F}.$$

[18] Without any name.

Proof: If $x \in \operatorname{adh}^{\varepsilon} \mathcal{F}$, then by definition there exists a sequence $(x_n) \geq \mathcal{F}$ such that $x \in \operatorname{adh}(x_n) \subset \operatorname{cl}\{x_n : n \in \mathbb{N}\}$. By the Fréchet property with respect to countably compactoid sets, there exists a sequence on $\{x_n : n \in \mathbb{N}\}$ that converges to x, and since the topology is T_1, this is a subsequence of (x_n). Therefore $x \in \operatorname{adh}_{\mathrm{Seq}} \mathcal{F}$. $\square$

Corollary 5.5 *Let $\mathcal{F}$ be a Fréchet filter on a Hausdorff space with the Fréchet property with respect to countably compactoid sets. If (4.1) holds, then $\operatorname{adh}_{\mathrm{Seq}} \mathcal{F} = \operatorname{adh}^{\varepsilon} \mathcal{F}$ is the least kernel of $\mathcal{F}$.*

We conclude that $\operatorname{adh} \mathcal{F}$ is the least kernel of a Fréchet filter $\mathcal{F}$ in a Hausdorff topological space if and only if $\mathcal{F}$ admits a kernel, and if

$$\operatorname{adh} \mathcal{F} = \operatorname{adh}_{\mathrm{Seq}} \mathcal{F} . \tag{5.3}$$

It turns out that the equality (5.3) can be characterized in terms of the Fréchetness of the product of certain spaces. It is known [15] that if the product of X with a non discrete space is Fréchet, then X must be strongly Fréchet. Consider $\mathfrak{Fr}$ the class of Fréchet filters, and let $\mathfrak{E}, \mathfrak{F}$ classes of filters included in the class of strongly Fréchet filters; I assume that moreover $\mathfrak{F}$ has the property

$$\mathcal{F} \in \mathfrak{F}(V), A \subset V \times X \Longrightarrow\ A\mathcal{F} \in \mathfrak{F}(X) .$$

Theorem 5.6 *The following assertions are equivalent:*

$$\mathcal{N}(x) \in \mathfrak{E}, \mathcal{F} \in \mathfrak{F} \Longrightarrow\ \big(x \in \operatorname{adh}_X \mathcal{F} \Longrightarrow\ x \in \operatorname{adh}_{\mathrm{Seq}\,X} \mathcal{F}\big) ; \tag{5.4}$$

$$\mathcal{N}(x) \in \mathfrak{E}, \mathcal{N}(y) \in \mathfrak{F} \Longrightarrow\ \mathcal{N}(x,y) \in \mathfrak{Fr} . \tag{5.5}$$

Proof: Assume (5.4) and consider $A \subset X \times Y$ and $(x,y) \in \operatorname{cl} A$. This amounts to $\mathcal{N}(x) \# A^{-}\mathcal{N}(y)$, that is, to $x \in \operatorname{adh} A^{-}\mathcal{N}(y)$. By the assumption, there exists a sequence $(x_n) \# A^{-}\mathcal{N}(y)$ that converges to x. It follows that $A(x_n)_n \# \mathcal{N}(y)$ and since $\mathcal{N}(y)$ is strongly Fréchet, there exist sequences $(y_k) \geq \mathcal{N}(y)$ and x_{n_k} such that $(x_{n_k}, y_k) \in A$. We conclude that $(x,y) \in \operatorname{cl}_{\mathrm{Seq}} A$.

Assume (5.5) and let $x \in \operatorname{adh} \mathcal{G}$, that is $\mathcal{N}(x) \# \mathcal{G}$. Define on $Y = \{\infty\} \cup X$ the finest topology in which $\mathcal{G}$ converges to ∞. The set $\Delta = \{(x,x) : x \in X\}$ belongs to the grill of $\mathcal{N}(x) \times \mathcal{G}$ (hence to the grill of $\mathcal{N}(x) \times \mathcal{N}(y)$), thus by the assumption there exists a sequence (x_n, y_n) on Δ that is finer than $\mathcal{N}(x) \times \mathcal{G}$. Consequently, $x_n = y_n$ for each n, and $x \in \operatorname{adh}_{\mathrm{Seq}} \mathcal{G}$. $\square$

Theorem 5.6 constitutes a return to the original setting, that of a relation between two spaces.

Proposition 5.7 *If X is Hausdorff, if $X \times Y$ is Fréchet, and if $\Omega : \Longrightarrow X$ is upper semicontinuous at y, then the active boundary of Ω at y is a kernel of Ω at y.*

Proof: Indeed, under the assumptions, the adherence of $\Omega\mathcal{N}(y) \vee (\Omega y)^c$ is equal to the sequential adherence: $x \in \operatorname{adh}\big(\Omega\mathcal{N}(y) \vee (\Omega y)^c\big)$ if and only if $(x, y) \in \operatorname{cl}\Omega \setminus \Omega$. By Fréchetness, there exists a free sequence (x_n, y_n) in Ω that converges to (x, y). Hence $(x_n) \geq \Omega\mathcal{N}(y) \vee (\Omega y)^c$ and converges to x. As the active boundary of Ω at y is equal to the adherence of $\Omega\mathcal{N}(y) \vee (\Omega y)^c$, by Proposition 4.3, the proof is complete. $\square$

The two-fold theorem of [8, Theorem 4] (by T. Nogura and the present author) is a common generalization and refinement of the known theorems on the Fréchetness of products. A convergent bisequence

$$x_{(n,k)} \xrightarrow[k]{} x_n \xrightarrow[n]{} x \tag{5.6}$$

is *free* if $x_n \neq x$, and *stationary* if $x_n = x$ for every n. A space X is an α_3-space (respectively, a *β_3-space*) [8] if for every stationary (respectively, free) convergent bisequence, there exists a compact metrizable subset C of $\{x\} \cup \{x_n : n \in \omega\} \cup \{x_{n,k} : n, k \in \omega\}$ such that

$$|\{n : |C \cap \{x_{n,k} : k \in \omega\}| = \omega\}| = \omega. \tag{5.7}$$

The property α_3 (but not β_3) is a property of single neighborhood filters. Therefore, it can be formulated for filters without relation to any topological structure. A filter $\mathcal{F}$ is α_3 if for every countable collection of sequences $(x_{n,k})_k$ finer than $\mathcal{F}$, there exists a set C such that (5.7) holds, and the cofinite filter of C is finer than $\mathcal{F}$. This property is preserved by relations.

As a consequence of Proposition 4.3, of the two-fold theorem and of Theorem 5.6, we have

Theorem 5.8 *If X is Hausdorff and $\mathcal{F}$ admits kernels, then* $\operatorname{adh}\mathcal{F}$ *is the least kernel, provided that one of the following holds:*

1. *X is first-countable and $\mathcal{F}$ is strongly Fréchet;*
2. *X is strongly Fréchet and $\mathcal{F}$ is first-countable;*
3. *X is a regular β_3 Fréchet q-space, and $\mathcal{F}$ is strongly Fréchet;*
4. *X is a regular Fréchet q-space, and $\mathcal{F}$ is α_3 Fréchet.*

Each first-countable space is a β_3 Fréchet q-space, but need not be regular; a regular Fréchet q-space is strongly Fréchet; a first-countable filter is α_3 Fréchet, and the latter is strongly Fréchet. However, because no regularity is required in the first two conditions, they are not a consequence of the two latter.

By virtue of Theorem 5.7, one can formulate other conditions for the adherence to be the least kernel. For example,

Corollary 5.9 *If Y is a regular Fréchet q-space, X is a Hausdorff α_3 Fréchet space, and $\Omega : Y \Longrightarrow X$ is upper semicontinuous, then (at each point) the active boundary of Ω is its least kernel. Moreover, if X is Dieudonné complete, then the active boundary at every point is compact.*

The last conclusion of the the corollary follows from the results of [11] by R. Hansell, J. Jayne, I. Labuda and C. A. Rogers, that I have mentioned in the introduction.

6 Disjoint kernels and non normality number

A filter can have many disjoint kernels. By definition a topological space X is not normal if for there exist closed disjoint sets F_0, F_1 such that $\mathcal{N}(F_0)\#\mathcal{N}(F_1)$. The supremum of cardinal numbers κ for which there exists a family $\mathcal{A}$ of disjoint closed subsets of X such that $|\mathcal{A}| = \kappa$ and the filter

$$\mathcal{F} = \bigvee_{A \in \mathcal{A}} \mathcal{N}(A) \tag{6.1}$$

is non degenerate is called the *non normality number* of X. It is clear that in each space of non normality κ, there is a filter that admits κ disjoint closed kernels ([19]).

Example 6.1 (Non normality of the Niemytzki plane is $\mathfrak{c}$). Recall [10, Example 1.2.4] that the Niemytzki plane can be defined as the upper half of the Euclidean plane, where the neighborhood filters of the elements of the strict upper half of the plane are induced from the natural topology, while for each $(x, 0)$, the neighborhood filter admits a base consisting of the Euclidean-closed discs tangent to $\mathbb{R} \times \{0\}$ at $(x, 0)$. The non normality of the Niemytzki plane is $\mathfrak{c}$.

The argument below slightly refines that employed in [12, p. 514] to prove the Bernstein theorem. Let $\{C_\beta : \beta < \mathfrak{c}\}$ be the set of the Cantor subsets of the real line. Set $C_{(\alpha,\beta)} = C_\beta$ for each $\beta \le \alpha < \mathfrak{c}$, and order $\{(\alpha, \beta) : \beta \le \alpha < \mathfrak{c}\}$ lexicographically (denote $l(\alpha, \beta)$ the ordinal that corresponds to (α, β) in the lexicographic order). Arrange $\mathbb{R}$ in the set $\{x_\gamma : \gamma < \mathfrak{c}\}$ of distinct terms.

Let $x_{\gamma_{(0,0)}}$ be the first element that belongs to C_0. If $\delta < \mathfrak{c}$ and $\{x_{\gamma_{(\alpha,\beta)}} : l(\alpha, \beta) < \delta\}$ is such that $x_{\gamma_{(\alpha,\beta)}} \in C_\beta$, let $\gamma_{(\alpha_\delta,\beta_\delta)}$ be the first γ such that x_γ has not been already selected and belongs to C_{β_δ}. Such an element exists, because C_{β_δ} is of cardinality $\mathfrak{c}$.

Let $B_\alpha = \{x_{\gamma_{(\alpha+\beta,\beta)}} : \beta < \mathfrak{c}\}$. The family $\{B_\alpha : \alpha < \mathfrak{c}\}$ is of cardinality $\mathfrak{c}$, all its elements are disjoint, dense, of cardinality $\mathfrak{c}$. For each α_0, the union $\bigcup_{\alpha_0 \neq \alpha < \mathfrak{c}} B_\alpha$ is a totally imperfect

[19] Dr Roberto Peirone (University of Rome, Tor Vergata) proved that for every regular cardinal κ, there exists a completely regular space which the non normality number is equal κ.

set (includes no Cantor set), hence each B_α is of second category in every open interval [12, Section 40, Theorem 3].

The sets $B_\alpha \times \{0\}$ are closed disjoint subsets of the Niemytzki space. Let $O_1, \dots, O_n$ be Niemytzki open sets such that $B_{\alpha_k} \times \{0\} \subset O_k$ for $1 \leq k \leq n$; for every $x \in B_{\alpha_k}$, let $r(x)$ stand for the radius of a disc tangent to $(x, 0)$ and included in O_k. There are open subsets $W_1 \supset W_k \supset \ldots \supset W_n$ of $\mathbb{R}$ and $r_k > 0$ such that $\{x \in B_{\alpha_k} : r(x) > r_k\}$ is dense in W_k. Therefore $\bigcap_{1 \leq k \leq n} O_k \neq \emptyset$.

On the other hand, the cardinality of the Niemytzki plane is $\mathfrak{c}$, so that there are no more than $\mathfrak{c}$ mutually disjoint closed sets.

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Cauchy filters and strong completeness of quasi-uniform spaces

ABSTRACT. We introduce and study the notions of a strongly completable and of a strongly complete quasi-uniform space. A quasi-uniform space (X, U) is said to be strongly complete if every Cauchy filter (in the sense of Sieber and Pervin) clusters in the uniform space $(X,\mathcal{U} \vee \mathcal{U}^{-1})$. An interesting motivation for the study of this notion of completeness is the fact, proved here, that the quasi-uniformity induced by the complexity space is strongly complete but not Corson complete. We recall that the (quasi-metric) complexity space was introduced by Schellekens to study complexity analysis of programs. We characterize those quasi-uniform space that are strongly completable and show that a quasi-uniform space is strongly complete if and only if it is bicomplete and strongly completable. We observe that every T_0 strongly complete quasi-uniform space is Smyth complete. We also show that every T_1 strongly complete quasi-uniform space is small-set symmetric, so every T_1 strongly complete quasi-metric space is (completely) metrizable.

KEY WORDS. Cauchy filter, strongly complete, Corson complete, Smyth complete, bicomplete, small set-symmetric, complexity space.

1 Introduction

Throughout this paper the letters $\mathbb{R}$, $\mathbb{N}$ and ω will denote the sets of reals, positive integers and nonnegative integers, respectively.

Terms and undefined concepts may be found in [4] and [7].

Given a quasi-uniform space $(X,\mathcal{U})$ (we shall denote by $\mathcal{U}^s$ the coarsest uniformity finer than $\mathcal{U}$ and its conjugate $\mathcal{U}^{-1}$ (i.e. $\mathcal{U}^s = \mathcal{U} \vee \mathcal{U}^{-1}$). If $U \in \mathcal{U}$ we denote by U^s the entourage $U \cap U^{-1}$ of $\mathcal{U}^s$.

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Let us recall that every quasi-uniformity $\mathcal{U}$ on a set X induces a topology $T(\mathcal{U}) = \{A \subseteq X|$ for each $x \in A$ there is $U \in \mathcal{U}$ such that $U(x) \subseteq A\}$, where $U(x) = \{y \in X|(x, y) \in U\}$.

According to [4], a quasi-uniform space $(X,\mathcal{U})$ is called bicomplete if $(X,\mathcal{U}^s)$ is a complete uniform space. A bicompletion of (X, U) is a bicomplete quasi-uniform space $(Y, \mathcal{V})$ which has a $T(\mathcal{V}^s)$-dense subspace quasi-unimorphic to $(X,\mathcal{U})$. It was shown in [15] and in [4] that every quasi-uniform space $(X,\mathcal{U})$ admits a bicompletion $(\widetilde{X},\widetilde{\mathcal{U}})$ such that if $(X,\mathcal{U})$ is a T_0 quasi-unifom space then $(\widetilde{X},\widetilde{\mathcal{U}})$ is T_0 and it is the unique (up to quasi-unimorphism) bicompletion of $(X,\mathcal{U})$.

In the context of this paper, a quasi-metric on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$ and (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

If d is a quasi-metric on a set X and $x \in X$, the set $\{y \in X|d(x, y) < r\}$ is called the open r-sphere around x and is denoted by $S_d(x, r)$. The conjugate d^{-1} of the quasi-metric d is given by $d^{-1}(x, y) = d(y, x)$. Then we shall denote by d^s the metric defined on X by $d^s = d \vee d^{-1}$.

Every quasi-metric d on a set X induces a quasi-uniformity U_d on X which has as a base the family of sets of the form $\{(x, y) \in X \times X|d(x, y) < 2^{-n}\}$, for $n \in \mathbb{N}$ (see [4] p. 3]). The topology $T(U_d)$ will be denoted simply by $T(d)$.

In [16] M. Schellekens introduced the quasi-metric complexity space as a part of the development of a topological foundation for the complexity analysis of programs. Via the analysis of its dual it was proved in [13] that the complexity space is Smyth complete. In Section 3 of this paper we shall show that actually the (dual) complexity space admits a stronger form of completeness based on the use of Cauchy filters (in the sense of Sieber and Pervin) having a sup-cluster point. This kind of completeness will be called "strong completeness". Thus, in Section 2 we define the notions of a strongly completable and of a strongly complete quasi-uniform space and obtain some properties of such spaces. In particular, we characterize both strongly completable and strongly complete quasi-uniform spaces and deduce that every strongly completable quasi-uniform space is Smyth completable and that every T_0 strongly complete quasi-uniform space is Smtyh complete. We give examples which show that the converse implications do not hold. We also observe that a quasi-uniform space is totally bounded if and only if it is precompact and strongly completable. We show that every T_1 strongly complete quasi-uniform space is small-set symmetric, so every T_1 strongly complete quasi-metric space is completely metrizable. Finally, in Section 3 we show, in addition to the result cited above, that the (dual) complexity space is not Corson complete (in the sense of [11]) and give an example of a weightable Smyth complete quasi-metric space having a maximum which is not strongly complete.

2 Strongly complete quasi-uniform spaces

Let us recall that a filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is Cauchy [17] provided that for each $U \in \mathcal{U}$ there is $x \in X$ such that $U(x) \in \mathcal{F}$. $\mathcal{F}$ is left K-Cauchy [12] provided that for each $U \in \mathcal{U}$ there is an $F \in \mathcal{F}$ such that $U(x) \in \mathcal{F}$ for all $x \in F$.

$(X,\mathcal{U})$ is said to be (left K-) complete if every (left K-) Cauchy filter has a $T(\mathcal{U})$-cluster point. lt is clear that every left K-complete quasi-uniform space is complete and it is well-known that the converse is not true. On the other hand, although every left K-Cauchy filter converges to its cluster points, there exist complete quasi-uniform spaces having non $T(\mathcal{U})$-convergent Cauchy filters (see [4], [6]).

In [8] H.P.A. Künzi proved that a quasi-uniform space $(X,\mathcal{U})$ is Smyth complete if and only if each left K-Cauchy filter is $T(\mathcal{U}^s)$-convergent to a unique point of X and that $(X,\mathcal{U})$ is Smyth completable if and only if every left K-Cauchy filter is a Cauchy filter on the uniform space $(X,\mathcal{U}^s)$.

Definition 1 *A quasi-uniform space $(X,\mathcal{U})$ is called* strongly complete *if each Cauchy filter on $(X,\mathcal{U})$ has a $T(\mathcal{U}^s)$-cluster point.*

Definition 2 *Let $(X,\mathcal{U})$ be a quasi-uniform space. A* strong completion *of $(X,\mathcal{U})$ is a strongly complete quasi-unifom space $(Y,\mathcal{V})$ in which $(X,\mathcal{U})$ can be quasi-uniformly embedded as a $T(\mathcal{V}^s)$-dense subspace. In this case we say that $(X,\mathcal{U})$ is* strongly completable.

We will say that a quasi-metric space (X,d) is strongly completable *(resp.* strongly complete*) if the quasi-uniform space (X,U_d) is strongly completable (resp. strongly complete). Similarly, (X,d) is called Smyth complete if (X,U_d) is Smyth complete.*

Lemma 1 *Let $(Y,\mathcal{V})$ be a quasi-uniform space and let X be a $T(\mathcal{V}^s)$-dense subset of Y. If $\mathcal{F}$ is a Cauchy filter on $(Y,\mathcal{V})$, then*

$$\mathcal{G} = \{U^s(F) \cap X | F \in \mathcal{F}, U \in \mathcal{V}\}$$

is a Cauchy filter base on $(X,\mathcal{V}|X \times X)$.

Proof: Let $\mathcal{F}$ be a Cauchy filter on $(Y,\mathcal{V})$. Let $U \in \mathcal{V}$. We shall show that there is $x \in X$ such that $U(x) \in \mathcal{G}$. Choose a $V \in \mathcal{V}$ such that $V^3 \subseteq U$. Then there exists a $y \in Y$ such that $V(y) \in \mathcal{F}$. Since X is $T(\mathcal{V}^s)$-dense in Y, there exists an $x \in V^s(y) \cap X$. Since $V^s(V(y)) \cap X \in \mathcal{G}$, it will be sufficient to see that $(V^s(V(y)) \cap X) \subseteq U(x)$. Indeed, given $z \in V^s(V(y)) \cap X$, there is $a \in V(y)$ such that $z \in V^s(a)$. Thus $z \in V^2(y)$. Since $y \in V(x)$, we conclude that $z \in V^3(x) \subseteq U(x)$. The proof is complete. ■

Theorem 1 *A quasi-uniform space $(X,\mathcal{U})$ is strongly completable if and only if every Cauchy filter on $(X,\mathcal{U})$ is contained in a Cauchy filter on the uniform space $(X,\mathcal{U}^s)$.*

Proof: Suppose that $(X,\mathcal{U})$ is strongly completable. Then there is a quasi-unimorphism f from $(X,\mathcal{U})$ to a $T(\mathcal{V}^s)$-dense subspace of a strongly complete quasi-uniform space $(Y,\mathcal{V})$. Let $\mathcal{F}$ be a Cauchy filter on $(X,\mathcal{U})$. Clearly $\{f(F) : F \in \mathcal{F}\}$ is a Cauchy filter base on $(Y,\mathcal{V})$, so it has a $T(\mathcal{V}^s)$-cluster point $y \in Y$. Then, the filter generated by $\{f^{-1}(V^s(y)) \cap F | F \in \mathcal{F}, V \in \mathcal{V}\}$ is a Cauchy filter on $(X,\mathcal{U}^s)$ which contains $\mathcal{F}$.

Conversely, let $(\widetilde{X},\widetilde{\mathcal{U}})$ be a bicompletion of $(X,\mathcal{U})$. If $\mathcal{F}$ is a Cauchy filter on $(\widetilde{X},\widetilde{\mathcal{U}})$, then it follows from Lemma 1 that the filter base $\mathcal{G} = \{U^s(F) \cap X | F \in \mathcal{F}, U \in \widetilde{\mathcal{U}}\}$ is Cauchy on $(X,\mathcal{U})$. So $\mathcal{G} \subseteq \mathcal{H}$ for some Cauchy filter $\mathcal{H}$ on $(X,\mathcal{U}^s)$. Denote by $\widetilde{\mathcal{H}}$ the filter generated on $\widetilde{X}$ by $\mathcal{H}$. Then $\widetilde{\mathcal{H}}$ is $T(\widetilde{\mathcal{U}}^s)$-convergent to some point $y \in \widetilde{X}$. Hence y is a $T(\widetilde{\mathcal{U}}^s)$-cluster point of $\mathcal{F}$. We conclude that $(X,\mathcal{U})$ is strongly completable. ■

Remark 1 It follows from the preceding result that if $(X,\mathcal{U})$ is a T_o strongly completable quasi-uniform space, then its bicompletion is the unique strong completion of $(X,\mathcal{U})$.

Recall that a quasi-uniform space $(X,\mathcal{U})$ is precompact provided that for each $U \in \mathcal{U}$ there is a finite subset A of X such that $U(A) = X$. $(X,\mathcal{U})$ is said to be totally bounded if the uniform space $(X,\mathcal{U}^s)$ is precompact (see [4], [7]). Every totally bounded quasi-uniform space is precompact but the converse does not hold. On the other hand, a quasi-uniform space is precompact if and only if every ultrafilter is a Cauchy filter [4].

Corollary 1 *A quasi-uniform space is totally bounded if and only if its precompact and strongly completable.*

Proof: Let $(X,\mathcal{U})$ be a precompact strongly completable quasi-uniform space. Let $\mathcal{F}$ be an ultrafilter on X. By the precompactness of $(X,\mathcal{U})$, $\mathcal{F}$ is a Cauchy (ultra) filter. So, by Theorem 1, $\mathcal{F}$ is a Cauchy filter on the uniform space $(X,\mathcal{U}^s)$. Therefore $(X,\mathcal{U})$ is totally bounded. The converse follows from Theorem 1 and the preceding observations. ■

It is essentially known (see, for instance, the proof of [11] Corollary 3) that a left K-Cauchy filter on a quasi-uniform space $(X,\mathcal{U})$ is Cauchy on $(X,\mathcal{U}^s)$ if and only if it is contained in a Cauchy filter on $(X,\mathcal{U}^s)$. From this fact and Theorem 1 we deduce the following result.

Corollary 2 *Every strongly completable quasi-uniform space is Smyth completable.*

Theorem 2 *A quasi-uniform space $(X,\mathcal{U})$ is strongly complete if and only if it is bicomplete and strongly completable.*

Proof: Suppose that $(X,\mathcal{U})$ is bicomplete and strongly completable. Let $\mathcal{F}$ be a Cauchy filter on $(X,\mathcal{U})$. By Theorem 1, $F \in \mathcal{G}$ for some Cauchy filter $\mathcal{G}$ on $(X,\mathcal{U}^s)$. Hence $\mathcal{G}$ is $T(\mathcal{U}^s)$-convergent to a point $x \in X$. So x is a $T(\mathcal{U}^s)$-cluster point of $\mathcal{F}$. We conclude that $(X,\mathcal{U})$ is strongly complete. The converse is obvious. ■

Corollary 3 *A T_0 quasi-uniform space $(X,\mathcal{U})$ is strongly complete if and only if it is Smyth complete and strongly completable.*

Proof: Suppose that $(X,\mathcal{U})$ is a T_0 strongly complete quasi-uniform space. Let $\mathcal{F}$ be a left K-Cauchy filter on $(X,\mathcal{U})$. By Corollary 2, $\mathcal{F}$ is a Cauchy filter on $(X,\mathcal{U}^s)$. So it is $T(\mathcal{U}^s)$-convergent to a point of X, by Theorem 2. Hence $(X,\mathcal{U})$ is Smyth complete. The converse follows from Theorem 2. ■

The following is a simple example of a compact Hausdorff Smyth complete quasi-metric space which is not strongly complete.

Example 1 Let d be the quasi-metric defined on ω by $d(0,n) = 1/n$ for all $n \in \mathbb{N}$, $d(n,m) = 1$ for all $n \in \mathbb{N}$ and $m \in \omega$ with $n \neq m$, and $d(n,n) = 0$ for all $n \in \omega$. Clearly (ω, d) is a compact Hausdorff Smyth complete quasi-metric space. However, it is not strongly complete because the filter generated by $\{\{m \in \mathbb{N} : m \geq n\} : n \in \mathbb{N}\}$ is a Cauchy filter on (ω, U_d) without $T((U_d)^s)$-cluster points.

Proposition 1 *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then the uniform space $(X,\mathcal{U}^s)$ is compact if and only if $(X,\mathcal{U})$ is precompaet and strongly complete.*

Proof: Suppose that $(X,\mathcal{U})$ is precompact and strongly complete. Then $(X,\mathcal{U})$ is bicomplete. Moreover, it is totally bounded by Corollary 1. We conclude that $(X,\mathcal{U}^s)$ is a compact uniform space. The converse is obvious. ■

In [1] P. Fletcher and W. Hunsaker introduced the notion of a small-set symmetric quasi-uniform space. It was shown in [9] that a quasi-uniform space $(X,\mathcal{U})$ is small-set symmetric if and only if $T(U^{-1}) \subseteq T(\mathcal{U})$.

Proposition 2 *A T_1 quasi-uniform space is strongly complete if and only if it is complete and small-set symmetric.*

Proof: Let $(X,\mathcal{U})$ be a strongly complete T_1 quasi-uniform space. Obviously, it is complete. In order to prove that $(X,\mathcal{U})$ is also small-set symmetric suppose that there exists $x \in X$ and $U \in \mathcal{U}$ such that $V(x)\backslash U^{-1}(x) \neq \emptyset$ for all $V \in \mathcal{U}$. Thus, the filter generated by $\{V(x)\backslash U^{-1}(x) | V \in \mathcal{U}\}$ is a Cauchy filter on $(X,\mathcal{U})$. Let $y \in X$ be a $T(\mathcal{U}^s)$-cluster point of such a filter. Then $x = y$. Indeed, given $V \in \mathcal{U}$ there is $W \in \mathcal{U}$ such that $W^2 \subseteq V$. Since $W^s(y) \cap W(x) \neq \emptyset$, it follows that $y \in V(x)$. Consequently $y \in \cap_{V\in\mathcal{U}} V(x)$, so $y = x$. Therefore $U^s(x) \cap (U(x)\backslash U^{-1}(x)) \neq \emptyset$, a contradiction. We conclude that $T(\mathcal{U}^{-1}) \subseteq T(\mathcal{U})$. Hence $(X,\mathcal{U})$ is small-set symmetric. The converse is almost obvious, so its simple proof is omitted. ■

Corollary 4 *Every strongly complete T_1 quasi-metric space is completely metrizable.*

Proof: Let (X, d) be a strongly complete T_1 quasi-metric space. By Proposition 2 we obtain that $T(d^{-1}) \subseteq T(d)$, so $(X, T(d))$ is a metrizable space. Now the conclusion follows from the fast, proved in [2], that every metrizable space which admits a complete quasi-metric is completely metrizable. ∎

3 Strong completeness of the complexity space

Let us recall [16] that the complexity space is the pair $(\mathcal{C}, d_{\mathcal{C}})$, where

$$\mathcal{C} = \left\{ f : \omega \to (0, +\infty] | \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < +\infty \right\}$$

and $d_{\mathcal{C}}$ is the quasi-metric on $\mathcal{C}$ defined by

$$d_{\mathcal{C}}(f, g) = \sum_{n=0}^{\infty} 2^{-n} \left[\left(\frac{1}{g(n)} - \frac{1}{f(n)} \right) \vee 0 \right]$$

whenever $f, g \in \mathcal{C}$.

The dual complexity space $(\mathcal{C}^*, d_{\mathcal{C}^*})$ is defined in [13] as follows:

$$\mathcal{C}^* = \left\{ f : \omega \to [0, +\infty) | \sum_{n=0}^{\infty} 2^{-n} f(n) < +\infty \right\}$$

and $d_{\mathcal{C}^*}$ is the quasi-metric on $\mathcal{C}^*$ given by

$$d_{\mathcal{C}^*}(f, g) = \sum_{n=0}^{\infty} 2^{-n} [(g(n) - f(n)) \vee 0]$$

whenever $f, g \in \mathcal{C}^*$.

It is observed in [13] that the inversion function $\Psi : \mathcal{C}^* \to \mathcal{C}$ is a quasi-isometry from $(\mathcal{C}^*, d_{\mathcal{C}^*})$ to $(\mathcal{C}, d_{\mathcal{C}})$, because $d_{\mathcal{C}}(\Psi(f), \Psi(g)) = d_{\mathcal{C}}(1/f, 1/g) = d_{\mathcal{C}^*}(f, g)$, whenever $f, g \in \mathcal{C}^*$.

The fact that the dual complexity space admits a structure of quasi-normed semilinear space (see [14]) provided a first motivation to the authors for the use of the dual complexity space rather than the original one in the study of the properties of completeness, compactness and total boundedness of the complexity space (see [13]). A second motivation for the use of the dual space is the fact that the definition of the dual is mathematically somewhat more appealing, since $d_{\mathcal{C}^*}$ is "derived" from the restriction to $[0, +\infty)$ of the standard quasi-metric u defined on $\mathbb{R} \times \mathbb{R}$, by $u(x, y) = (y - x) \vee 0$. Consequently, the presentation of the proofs becomes somewhat more elegant.

The quasi-metric of pointwise convergence of u is the quasi-metric u_P defined on $\mathbb{R}^\omega \times \mathbb{R}^\omega$ by $u_P(f,g) = \sum_{n=0}^{\infty} 2^{-n} \min\{u(f(n),g(n)),1\}$. Thus the metric $(u_P)^s$ induces the usual topology of pointwise convergence on $\mathbb{R}^\omega$.

A quasi-metric space (X,d) is called weightable [10] if there is a function $w : X \to [0,+\infty)$ such that for all $x,y \in X$:

$$d(x,y) + w(x) = d(y,x) + w(y).$$

In this case, we say that w is a weighting function for (X,d).

It was proved in [16] that the complexity space $(C, d_{\mathcal{C}})$ is weigthable with weighting function $w_{\mathcal{C}}$ defined by $w_{\mathcal{C}}(f) = \sum_{n=0}^{\infty} 2^{-n}(i/f(n))$, for all $f \in \mathcal{C}$. Similary, the dual complexity space $(\mathcal{C}^*, d_{\mathcal{C}^*})$ is weightable with weighting function $w_{\mathcal{C}^*}$ defined by $w_{\mathcal{C}^*}(f) = \sum_{n=0}^{\infty} 2^{-n} f(n)$, for all $f \in \mathcal{C}^*$.

A quasi-metric space (X,d) *has a maximum* provided there is $x_0 \in X$ such that $d(x,x_0) = 0$ for all $x \in X$. It is obvious that the (dual) complexity space has a maximum. Before we prove that the (dual) complexity space is strongly complete, it seems interesting to note that there exists a non strongly complete weightable Smyth complete quasi-metric space which has a maximum.

Example 2 Let $X = \omega \cup \{\infty\}$. Define a quasi-metric d on X by $d(0,x) = 0$ for all $x \in X$, $d(\infty,n) = 1$ for all $n \in \mathbb{N}$, $d(\infty,0) = 2$, $d(n,m) = 1$ for all $n,m \in \mathbb{N}$ with $n \neq m$, $d(n,\infty) = 0$ for all $n \in \mathbb{N}$, $d(n,0) = 1$ for all $n \in \mathbb{N}$, and $d(x,x) = 0$ for all $x \in X$. It is immediate to check that (X,d) is Smyth complete and that ∞ is a maximum for (X,d). Furthermore (X,d) is weightable with weighting function w given by $w(0) = 2$, $w(\infty) = 0$ and $w(n) = 1$ for all $n \in \mathbb{N}$.

Proposition 3 *The dual complexity space $(\mathcal{C}^*, d_{\mathcal{C}^*})$ is strongly complete.*

Proof: Let $\mathcal{F}$ be a Cauchy filter on $(\mathcal{C}^*, \mathcal{U}_{d_{\mathcal{C}^*}}$. Then, for each $k \in \mathbb{N}$ there is an $f_k \in \mathcal{C}^*$ such that $S_{d_{\mathcal{C}^*}}(f_k, 2^{-3k}) \in \mathcal{F}$. Put $F_k = S_{d_{\mathcal{C}^*}}(f_k, 2^{-3k})$ for all $k \in \mathbb{N}$.

Furthermore, for each $f \in F_1$, $w_{\mathcal{C}^*}(f) \leq w_{\mathcal{C}^*}(f_1) + d_{\mathcal{C}^*}(f_1, f)$. Hence $\sum_{n=0}^{\infty} 2^{-n} f(n) < w_{\mathcal{C}^*}(f_1) + 2^{-3}$ and thus $f(n) < 2^n(w_{\mathcal{C}^*}(f_1) + 1)$ for all $f \in F_1$ and $n \in \omega$.

Denote by K the compact space $\prod_{n=0}^{\infty}[0, 2^n(w_{\mathcal{C}^*}(f_1)+1)]$, and by $\overline{F \cap K}$ the closure of $F \cap K$ in K for all $F \in \mathcal{F}$. (Note that for each $F \in \mathcal{F}$, $F \cap K \neq \emptyset$ because $F_1 \subseteq K$.)

Next we show that for each $F \in \mathcal{F}$, $(\overline{F \cap K}) \cap (\cap_{k=1}^{\infty} \overline{F_k \cap K}) \neq \emptyset$.

Indeed, fix $F \in \mathcal{F}$. For each $k \in \mathbb{N}$ there is $g_k \in F \cap F_k$, so $(g_k)_{k \in \mathbb{N}}$ is a sequence in K and, thus, it clusters to a function $g \in K$ with respect to $T((u_P)^s)$. Therefore $g \in (\overline{F \cap K}) \cap (\cap_{k=1}^{\infty} \overline{F_k \cap K})$.

In particular, it follows from the above observation that $\cap_{k=1}^{\infty}\overline{F_k \cap K}$ is a nonempty compact subset of K, so the filter base $\{(\overline{F \cap K}) \cap (\cap_{k=1}^{\infty}\overline{F_k \cap K})|F \in \mathcal{F}\}$ clusters to some function $h \in \cap_{k=1}^{\infty}\overline{F_k \cap K}$ with respect to $T((u_P)^s)$.

Now we want to show that $h \in \mathcal{C}^*$ and $\mathcal{F}$ clusters to h with respect to $T((d_{\mathcal{C}^*})^s)$. Thus $(\mathcal{C}^*, d_{\mathcal{C}^*})$ will be strongly complete.

Suppose that $h \notin \mathcal{C}^*$. Then, for each $j \in \mathbb{N}$ there is an $m_j \in w$ such that $j < \sum_{n=0}^{m_j} 2^{-n}h(n)$. Since $h \in \overline{F_1 \cap K}$, there exists $f \in F_1$ such that $|h(n) - f(n)| < 2^{-j}$ for $n = 0, 1, \ldots, m_j$. So

$$\sum_{n=0}^{m_j} 2^{-n}|h(n) - f(n)| < 2^{-j}\sum_{n=0}^{m_j} 2^{-n} < 2^{-(j-1)}.$$

Since $d_{\mathcal{C}^*}(f_1, f) < 2^{-3}$, it follows that

$$\begin{aligned} j < \sum_{n=0}^{m_j} 2^{-n}h(n) &\le 2^{-(j-1)} + \sum_{n=0}^{m_j} 2^{-n}f(n) \\ &< 2^{(j-1)} + 2^{-3} + \sum_{n=0}^{m_j} 2^{-n}f_1(n), \end{aligned}$$

which contradicts that $f_1 \in \mathcal{C}^*$. Therefore $h \in \mathcal{C}^*$.

Finally, we shall prove that $\mathcal{F}$ clusters to h with respect to $T((d_{\mathcal{C}^*})^s)$.

Fix $k \in \mathbb{N}$ and $F \in \mathcal{F}$. Since f_k and h are in $\mathcal{C}^*$, there is $n_0 \in \mathbb{N}$ with $n_0 > 3k$ such that

$$\sum_{n=n_0}^{\infty} 2^{-n}f_k(n) < 2^{-3k} \quad \text{and} \quad \sum_{n=n_0}^{\infty} 2^{-n}h(n) < 2^{-3k}.$$

On the other hand, since $h \in \overline{F \cap F_k \cap K}$, there is $f \in F \cap F_k$ such that $(u_P)^s(h, f) < 2^{-n_0}$, which implies that $\sum_{n=0}^{n_0-1} 2^{-n}\min\{1, |h(n) - f(n)|\} < 2^{n_0}$, i.e. $\sum_{n=0}^{n_0-1} 2^{-n}|h(n) - f(n)| < 2^{-n_0}$. Therefore,

$$\begin{aligned} (d_{\mathcal{C}^*})^s(h, f) &\le \sum_{n=0}^{\infty} 2^{-n}|h(n) - f(n)| \\ &< 2^{-n_0} + \sum_{n=n_0}^{\infty} 2^{-n}h(n) + \sum_{n=n_0}^{\infty} 2^{-n}f(n). \end{aligned}$$

From $\sum_{n=n_0}^{\infty} 2^{-n}u(f_k(n), f(n)) \le d_{\mathcal{C}^*}(f_k, f) < 2^{-3k}$ we deduce that $\sum_{n=n_0}^{\infty} 2^{-n}f(n) < 2^{-3k} + \sum_{n=n_0}^{\infty} 2^{-n}f_k(n) < 2^{-3k} + 2^{-3k}$, so

$$(d_{\mathcal{C}^*})^s(h, f) < 2^{-n_0} + 2^{-3k} + 2^{-3k} + 2^{-3k} < 4 \cdot 2^{-3k} \le 2^{-k}.$$

We have shown that $\mathcal{F}$ clusters to h with respect to $T((d_{\mathcal{C}^*})^s)$. Consequently, the dual complexity space is strongly complete. ∎

By using the quasi-isometry Ψ from $(\mathcal{C}^*, d_{\mathcal{C}^*})$ to $(\mathcal{C}, d_{\mathcal{C}})$ constructed above and the preceding theorem, we immediately deduce the following result.

Corollary 5 *The complexity space $(\mathcal{C}, d_{\mathcal{C}})$ is strongly complete.*

The notion of a Corson complete quasi-uniform space was introduced in [11]. A quasi-uniform space $(X,\mathcal{U})$ is said to be Corson complete if every weakly Cauchy filter on $(X,\mathcal{U})$ has a $T(\mathcal{U}^s)$-cluster point, where a filter $\mathcal{F}$ on $(X,\mathcal{U})$ is weakly Cauchy provided that for each $U \in \mathcal{U}$, $\cap_{F\in\mathcal{F}} U^{-1}(F) \neq \emptyset$ (see, for instance, [3]). Clearly, every Corson complete quasi-uniform space is strongly complete. The converse does not hold even for uniform spaces as a well-known example of J.R. Isbell shows (see [5]).

We conclude the paper by showing that the quasi-uniform space $(\mathcal{C}^*, U_{d_{\mathcal{C}^*}})$ is not Corson complete.

Example 3 For each $j, k \in \mathbb{N}$ define a function $f_k^j : \omega \to [0, +\infty)$ by $f_k^j(n) = j$ if $n < k$ and $f_k^j(n) = j + 2^{k-j}$ if $n \geq k$.

An easy computation shows that $\sum_{n=0}^{\infty} 2^{-n} f_k^j(n) = 2(j + 2^{-j})$, so $f_k^j \in \mathcal{C}^*$ for all $j, k \in \mathbb{N}$.

Now, for each $m \in \mathbb{N}$, define $F_m = \{f_k^j : j \geq 1 \text{ and } k \geq m\}$. Then $\{F_m : m \in \mathbb{N}\}$ is a base for a filter $\mathcal{F}$ on $\mathcal{C}^*$.

For each $j \in \mathbb{N}$ consider the constant function $g_j : \omega \to [0, +\infty)$ defined by $g_j(n) = j$ for all $n \in \omega$. Clearly each g_j is in $\mathcal{C}^*$ and, for each $j \in \mathbb{N}$, the sequence $(f_k^j)_{k\in\mathbb{N}}$ converges to g_j with respect to $T((u_P)^s)$.

Furthermore, for each $j, k \in \mathbb{N}$, we have

$$d_{\mathcal{C}^*}(g_j, f_k^j) = \sum_{n=k}^{\infty} 2^{-n}(j + 2^{k-j} - j) = 2^{-(j-1)},$$

which implies that $\mathcal{F}$ is a Corson filter on $(\mathcal{C}^*, \mathcal{U}_{d_{\mathcal{C}^*}})$.

Finally, suppose that $\mathcal{F}$ clusters to a function g with respect to $T((d_{\mathcal{C}^*})^s)$. Then there is a sequence $(f_{k_m}^{j_m})_{m\in\mathbb{N}}$ of distinct elements of $\mathcal{F}$ such that $f_{k_m}^{j_m} \in F_m$ for all $m \in \mathbb{N}$ and $(d_{\mathcal{C}^*})^s(g, f_{k_m}^{j_m}) \to 0$.

We have two cases:

Case 1. The sequence $(j_m)_{m\in\mathbb{N}}$ is bounded. Then there is an $i \in \mathbb{N}$ and a subsequence $(h_m)_{m\in\mathbb{N}}$ of $(f_{k_m}^{j_m})_{m\in\mathbb{N}}$ which is also a subsequence $(f_k^i)_{k\in\mathbb{N}}$. Hence, $(d_{\mathcal{C}^*})^s(g, h_m) \to 0$, so $(u_P)^s(g, h_m) \to 0$, and, by the triangle inequality, $(u_P)^s(g, g_i) = 0$, i.e. $g = g_i$, which is a contradiction because $d_{\mathcal{C}^*}(g_i, h_m) = 2^{-(i-1)}$ for all $m \in \mathbb{N}$.

Case 2. The sequence $(j_m)_{m\in\mathbb{N}}$ is not bounded. Let $m_0 \in \mathbb{N}$ such that $(d_{\mathcal{C}^*})^s(g, f_{k_m}^{j_m}) < 1$ for all $m \geq m_0$. Then, in particular, $|f_{k_m}^{j_m}(0) - g(0)| < 1$ for all $m \geq m_0$, which provides a new contradiction because $f_{k_m}^{j_m}(0) = j_m$, for all $m \in \mathbb{N}$.

We conclude that the Corson filter $\mathcal{F}$ has no cluster point in $(\mathcal{C}^*, (d_{\mathcal{C}^*})^s)$, and, thus, the quasi-uniform space $(\mathcal{C}^*, \mathcal{U}_{d_{\mathcal{C}^*}})$ is not Corson complete.

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On the chromatic number of order-interval hypergraphs

ABSTRACT. Let P a finite poset. We consider the hypergraph $\mathfrak{H}(P)$ whose vertices are the points of P and whose edges are the maximal intervals in P. We verify in the present paper the NP-completeness of the determination of the (strong) chromatic number $\gamma(\mathfrak{H}(P))$. We give some exact or asymptotic values of $\gamma(\mathfrak{H}(P))$ if P is the subposet of the Boolean lattice induced by consecutive levels $P_l \cup \cdots \cup P_u$. Moreover, we determine $\gamma(\mathfrak{H}(P))$ for interval orders P.

KEY WORDS. Partial order, Boolean lattice, interval, chromatic number, packing system, Steiner system, large set, NP-completeness

1 Preliminaries

Let P be a finite poset. A subset I of P of the form $I = \{v \in P : p \le v \le q\}$ (denoted $[p, q]$) is called an *interval*. If p resp. q is a minimal resp. maximal element of P, then $[p, q]$ is called *maximal interval*. Let $\mathfrak{I}(P)$ be the family of maximal intervals of P. The hypergraph $\mathfrak{H}(P) = (P, \mathfrak{I}(P))$, briefly denoted $\mathfrak{H} = (P, \mathfrak{I})$, whose vertices are the elements of P and whose edges are the maximal intervals of P is said to be the *order-interval hypergraph of P*.

Several interesting results exist about the matching, covering and independence numbers of $\mathfrak{H}$ such as the algorithmic complexity and min-max relations (see [2, 3, 6]).

In this paper, we investigate another parameter, the chromatic number, and study it in the case of the order-interval hypergraph $\mathfrak{H}_{n;l,u}$ of the subposet $P_{n;l,u}$ induced by consecutive levels of the Boolean lattice B_n, where more precisely $P_{n;l,u} = \{X \subseteq [n] : \ l \le |X| \le u\}$ with $0 \le l < u \le n$ and $[n] = \{1, \ldots, n\}$.

Let $k \ge 2$ be an integer. Recall that a *(strong vertex) k-colouring* of a hypergraph $\mathfrak{H} = (V, \mathcal{E})$ is a partition $\{C_1, \ldots, C_k\}$ of V into k (colour) classes (or independent sets) such that no colour appears twice in the same edge. The (strong) chromatic number of a hypergraph $\mathfrak{H}$, denoted by $\gamma(\mathfrak{H})$, is the smallest integer k for which $\mathfrak{H}$ admits a k-colouring. For the special order-interval hypergraph $\mathfrak{H}_{n;l,u}$ of the poset $P_{n;l,u}$, we denote the chromatic number by $\gamma_{n;l,u}$.

The design theory enables us in some cases to get results for the colouring problem. Let us recall some notions. A *packing system* $P(t,k,n)$ of order n (briefly *packing*) is a pair (Q,q), where Q is an n-set and q is a collection of k-subsets of Q called blocks, having the property that every t-subset of Q is a subset of at most one block of q. In particular, a $P(2,3,n)$ is called a *packing triple system* PT of order n (briefly $PT(n)$). A PT is *optimal* if there is no PT of the same order with more blocks. Two PTs, (Q,q_1) and (Q,q_2), are said to be disjoint if $q_1 \cap q_2 = \emptyset$. If we replace "at most" with "exactly" in the definition of the packing system, we have a *Steiner system* $S(t,k,n)$. In particular an $S(2,3,n)$ is called a Steiner triple system STS of order n (briefly $STS(n)$). Such a system exists iff $n \equiv 1$ or 3 (mod 6). It is easy to show that the maximum number of pairwise disjoint $STSs$ of order n is at most $n-2$. If this number is exactly $n-2$, i.e., the set of all triples can be partitioned into $n-2$ $STS(n)$, this partition will be called a *large set of disjoint Steiner triple system* (briefly $LTS(n)$). An $LTS(n)$ exists iff $n \equiv 1$ or $3 \pmod 6$, $n \neq 7$ (see [13, 14, 18]).

The objective of the paper is to begin the study of the chromatic number of the order-interval hypergraph of P where P is either the subposet induced by consecutive levels of the Boolean lattice B_n or an interval order. We also show that the problem of the determination of the chromatic number $\gamma(\mathfrak{H}(P))$ is NP-complete if there is no restriction on P.

2 NP-completeness of the colouring problem

Let us mention that the order-interval hypergraphs of chromatic number 2 can be recognized in polynomial time since $\gamma(\mathfrak{H}(P)) \leq 2$ iff P is a bipartite poset. Let us denote the problem of deciding $\gamma(\mathfrak{H}(P)) \leq k$ (with input P and k, with $k > 2$) the *colouring problem for the order-interval hypergraph.*

Theorem 1 *The colouring problem for the order-interval hypergraph is NP-complete.*

Proof: It is clear that this problem belongs to the class NP. We prove the completeness by a polynomial reduction of the colouring problem in a graph to our problem. For a graph $G=(V,E)$, the chromatic number $\gamma(G)$ is given by

$$\gamma(G) = \min\{k,\ \text{there exists a partition of } G \text{ into}\\ k \text{ sets of mutually nonadjacent vertices}\}.$$

It is known (see [10]) that the decision problem $\gamma(G) \leq k$ is NP-complete for $k > 2$. Now we present a construction which leads to the relation $2\gamma(G) = \gamma(\mathfrak{H})$. Label the vertices of G

by $v_1, \ldots, v_n$ and the edges by $e_1, \ldots, e_m$. We associate with G a poset $P = Q^* \oplus Q$ where Q is the incidence poset of G, i.e., $P = E \cup V \cup V' \cup E'$, where $V' = \{v' : v \in V\}$ (copy of V) and $E' = \{e' : e \in E\}$ (copy of E) and the ordering is given in the following way:

$$\begin{array}{ll} \text{for } e \in E, v \in V & \text{we have } e < v \text{ iff } v \text{ is an endpoint of } e \text{ in } G, \\ \text{for } v \in V, v' \in V' & \text{we have } v < v', \\ \text{for } v' \in V', e' \in E' & \text{we have } v' < e' \text{ iff } v \text{ is an endpoint of } e \text{ in } G. \end{array}$$

We show that, G is t-colourable iff $\mathfrak{H}(P)$ is $2t$-colourable whenever $t > 2$. Indeed, for a colouring $\{C_1, \ldots, C_t\}$ of G whose colours are chosen among the set $\mathcal{C} = \{\alpha_1, \ldots, \alpha_t\}$, we shall assign to the vertices of $\mathfrak{H}$ the following colours: For $e \in E$, e will get any colour $\alpha \in \mathcal{C}$ provided that it is different to the colours of its adjacent vertices in G (note $t > 2$). For $v \in V$, v will get the same colour as in G. For $v' \in V'$, v' will get the colour $\alpha' \in \mathcal{C}' = \{\alpha'_1, \ldots, \alpha'_t\}$ (with $\alpha'_i \neq \alpha'_j$ for all $i \neq j$) if α is the colour of v. For $e' \in E'$, e' will get the colour $\alpha' \in \mathcal{C}'$ if e has colour α. In this way, we obtain a $2t$-colouring of $\mathfrak{H}$, that is $\gamma(\mathfrak{H}) \leq 2\gamma(G)$.

Conversely, if $\mathfrak{H}$ admits a $2t$-colouring, then there are in the first or in the second level of P at most t colours. Thus there is a k-colouring of the vertices of G with $k \leq t$. Consequently, $2\gamma(G) \leq \gamma(\mathfrak{H})$. □

3 Some results for the level induced subposet of the Boolean lattice

Proposition 1 *We have $\gamma_{n;1,n} = 2^{n-1}$.*

Proof: The inequality "$\geq$" follows by observing that each edge of $\mathfrak{H}_{n;1,n}$ has 2^{n-1} elements. Now, if we assign the same colour i to both elements, p_i and its complement $[n] \setminus p_i$ ($p_i \neq [n]$), for all $i = 1, \ldots, 2^{n-1} - 1$ and the colour 2^{n-1} to the element $[n]$, we obtain the inequality $\gamma_{n;1,n} \leq 2^{n-1}$. □

We need two preliminary lemmas for the next results.

Lemma 1 *We have*

$$\max\{n - l + 1, \frac{\binom{n}{l+1}}{\lfloor \frac{n}{l+1} \lfloor \frac{n-1}{l} \lfloor \ldots \lfloor \frac{n-l+1}{2} \rfloor \ldots \rfloor}\} \leq \gamma_{n;l,l+2},$$

where, as usual, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.

Proof: Proceeding by induction on l, Schönheim [17] proved that a packing system $P(l, l+1, n)$ does not contain more than $\Psi(n, l) = \lfloor \frac{n}{l+1} \lfloor \frac{n-1}{l} \lfloor \ldots \lfloor \frac{n-l+1}{2} \rfloor \ldots \rfloor$ $(l+1)$-subsets. Since each colour-class of the level $N_{l+1} = \{X \subseteq [n] : |X| = l + 1\}$ is obviously a packing system,

$$\frac{\binom{n}{l+1}}{\Psi(n, l)} \leq \gamma_{n;l,l+2}.$$

Besides this inequality, we have also $\gamma_{n;l,l+2} \geq n-l+1$ since the elements $\{1,\ldots,l\}$ and $\{1,\ldots,l,i\}$, $i=l+1,\ldots,n$, must have pairwise different colours. $\square$

Lemma 2 *Suppose that there is a partition of the $(l+1)$-subsets of an n-set into $\varphi(n)$ disjoint packings $P(l,l+1,n)$.*

If $\varphi(n) > \max\{n-l, \frac{(l+1)(l+2)}{2}+l+2$, then $\gamma_{n;l,l+2} \leq \varphi(n)$.
If $\varphi(n) = n-l$ and $n > 2l+2$, then $\gamma_{n;l,l+2} \leq \varphi(n)+1$.

Proof: Let $\{C_1,\ldots,C_{\varphi(n)}\}$ be the partition of $\binom{[n]}{l+1}$ into disjoint packings $P(l,l+1,n)$. It is easy to see that $\varphi(n) \geq n-l$. Assign to each element of C_i the colour i. If $\varphi(n) > n-l$, then each l-subset has at least $\varphi(n)-(n-l)$ choices for its colour among the elements of the set $[\varphi(n)]$. After that, we observe that each $(l+2)$-subset can get at least $\varphi(n)-\binom{l+2}{l}-\binom{l+2}{l+1}$ colours from $[\varphi(n)]$. Hence, we obtain a $\varphi(n)$-colouring of $\mathfrak{H}_{n;l,l+2}$ for $\varphi(n) > \frac{(l+2)(l+1)}{2}+l+2$. If $\varphi(n)=n-l$, then all l-subsets get a new colour $\varphi(n)+1$ and as above, since each $(l+2)$-subset has at least $\varphi(n)-\binom{l+2}{l+1}$ choices for its colour in $[\varphi(n)]$, we obtain a $(\varphi(n)+1)$-colouring of $\mathfrak{H}_{n;l,l+2}$ for $\varphi(n) > l+2$. $\square$

Now we study the values of $\gamma_{n;l,l+2}$ for some $l \leq n$. The following theorem is proved for $l=1$ and $n>5$, nevertheless, it is easy to verify that $\gamma_{3;1,3}=4$, $\gamma_{4;1,3}=5$ and $\gamma_{5;1,3}=6$.

Theorem 2 *For $n>5$, we have $\gamma_{n;1,3}=n$.*

Proof: The inequality "$\geq$" follows by Lemma 1 ($l=1$). Let n be even. The well known one-factorization $F=\{F_1,\ldots,F_{n-1}\}$ of the complete graph K_n is a partition of the 2-subsets of $[n]$ into $n-1$ disjoint packings $P(1,2,n)$.

Let n be odd. It suffices to consider a one-factorization $F'=\{F'_1,\ldots,F'_n\}$ of K_{n+1} and to delete all 2-subsets which contain the element $n+1$. Hence, we obtain from Lemma 2 the inequality "$\leq$". $\square$

It is not difficult to verify that $\gamma_{5;2,4}=5$, $\gamma_{6;2,4}=6$ and $\gamma_{7;2,4}=7$.

Theorem 3 *We have*

$$\gamma_{n;2,4} = \begin{cases} n-1 & \text{if } n \equiv 0,1,2 \text{ or } 3 \pmod 6 \quad n \neq 6,7,8 \\ n & \text{if } n \equiv 4 \pmod 6 \quad n \neq 10 \end{cases}$$

and

$$\gamma_{n;2,4} \in \{n-1,n\} \text{ if } n \equiv 5 \pmod 6,\ n \neq 5.$$

Proof: By Lemma 1 with $l = 2$ we have

$$\gamma_{n;2,4} \geq \begin{cases} n-1 & \text{if } n \equiv 0,1,2,3,5 \pmod 6 \\ n & \text{if } n \equiv 4 \pmod 6). \end{cases}$$

Let us determine upper bounds for $\gamma_{n;2,4}$. To this end, we need, using Lemma 2, only a partition of the 3-subsets into disjoint packings $P(2, 3, n)$.

Case 1. $n \equiv 1$ or $3 \pmod 6$ and $n \neq 7$.
From the partition which is defined by the large set of disjoint Steiner triple system $LTS(n)$, we deduce an $(n-1)$-colouring of $\mathfrak{H}_{n;2,4}$ and the inequality $\gamma_{n;2,4} \leq n-1$ follows.

Case 2. $n \equiv 0$ or $2 \pmod 6$ and $n \neq 6, 8$.
In this case $n+1 \equiv 1$ or $3 \pmod 6$. Therefore all triples of order $n+1$ can be partitioned into $(n+1)-2 = n-1$ pairwise disjoint STSs whenever $n \neq 6$. By deleting all 3-subsets containing the element $n+1$, we obtain a partition into $n-1$ packings $P(2, 3, n)$ and then an $(n-1)$-colouring of $\mathfrak{H}_{n;2,4}$ for $n > 11$.

Case 3. $n \equiv 5 \pmod 6$ and $n \neq 5$.
The condition $n \equiv 5 \pmod 6$ is equivalent to $n+1 \equiv 0 \pmod 6$. Hence, $\gamma_{n+1;2,4} = (n+1)-1 = n$ for $n \neq 5$. If we delete all subsets containing $n+1$, we obtain an n-colouring of $\mathfrak{H}_{n;2,4}$, i.e, $\gamma_{n;2,4} \leq n$.

Let us mention that the above inequality can be improved if the weak conjecture of Etzion [8] is proved. It states that for every $n \equiv 5 \pmod 6$, $n > 5$, there exists a partition of triples of order n into $n-1$ PTs.
More precisely, we shall have, $\gamma_{n;2,4} = n-1$ if $n \equiv 5 \pmod 6$, $n > 11$.

Case 4. $n \equiv 4 \pmod 6$ and $n \neq 10$.
In Etzion's papers [7, 8] it is proved for $n \equiv 4 \pmod 6$ that there is a partition of the triples into $n-1$ optimal PTs and one PT of size $(n-1)/3$. Hence, we can infer an n-colouring of $\mathfrak{H}_{n;2,4}$ for $n > 10$. □

In the case $l = 3$, it is well known [4] that an optimal packing $P(3, 4, n)$ has size $\frac{n(n-1)(n-3)}{24}$ if $n \equiv 1$ or $3 \pmod 6$ and $\frac{n(n^2-3n-6)}{24}$ if $n \equiv 0 \pmod 6$. For $n \equiv 5 \pmod 6$, it has, by the Johnson bound [12], at most size $\frac{(n-1)(n^2-3n-4)}{24} + \lfloor \frac{n-5}{12} \rfloor$. Consequently, the minimal number $\theta(n)$ of pairwise disjoint packings $P(3, 4, n)$ which partition all quadruples of an n-set satisfies $\theta(n) \geq n-1$ if $n \equiv 5 \pmod 6$ and $\theta(n) \geq n-2$ otherwise.

In [9], Etzion proved that $\theta(n) \leq n-2$ if $n = 2^i$ $(i \geq 3)$. With Lemma 2, we infer the equality $\gamma_{n;3,5} = n-2$ for this value of n. Graham and Sloane [11] proved that $\theta(n) \leq n$ for all n. Hence, we obtain $\gamma_{n;3,5} \in \{n-1, n\}$ if $n \equiv 5 \pmod 6$ and $\gamma_{n;3,5} \in \{n-2, n-1, n\}$ otherwise. In particular, it was proved by Van Pul and Etzion [19] that $\theta(n) \leq n-1$ if $n = 3 \cdot 2^i$ and by Brouwer et al [5] that $\theta(n) \le n-1$ if $n = 5 \cdot 2^i$ or $n = 7 \cdot 2^i$, therefore, we have again $\gamma_{n;3,5} \in \{n-2, n-1\}$ if $n = 3 \cdot 2^i$, $\gamma_{n;3,5} = n-1$ if $n = 5 \cdot 2^i$ and $\gamma_{n;3,5} \in \{n-2, n-1\}$ if $n = 7 \cdot 2^i$.

Theorem 4 *For every fixed l, we have $\gamma_{n;l,l+2} = n(1+o(1))$ as $n \to \infty$.*

In order to prove the theorem, we need the following result due to Spencer and Pippenger [16]. Let $\mathfrak{H} = (V, \mathcal{E})$ be a hypergraph. The *degree* of x in $\mathfrak{H}$ (denoted *deg(x)*) is the number of edges of $\mathfrak{H}$ containing x and the *codegree* of x and y in $\mathfrak{H}$ (denoted *codeg(x,y)*) is the number of edges of $\mathfrak{H}$ containing both x and y. A packing in $\mathfrak{H}$ is a set P of edges of $\mathfrak{H}$ such that each vertex of $\mathfrak{H}$ is in at most one edge of P. The chromatic index of $\mathfrak{H}$, denoted by $\chi(\mathfrak{H})$, is the smallest number of packings into which the edges of $\mathfrak{H}$ may be partitioned.

Theorem 5 *[16]. Let k, $k \ge 2$, be a fixed natural number. Let $\mathfrak{H}_n = ([n], E_n)$ be a sequence of k-uniform hypergraphs such that for $n \to \infty$*

$$\min_{x\in[n]} deg(x) = (\max_{x\in[n]} deg(x))(1+o(1)),$$

$$\max_{x,y\in[n],x\neq y} codeg(x,y) = (\max_{x\in[n]} deg(x))o(1).$$

Then

$$\chi(\mathfrak{H}_n) = (\max_{x\in[n]} deg(x))(1+o(1)).$$

Proof of Theorem 4: Let us consider the uniform hypergraph $\mathfrak{H} = (V, \mathcal{E})$ whose vertices are all l-subsets of $[n]$ and whose edges are the $(l+1)$ l-subsets of an $(l+1)$-set. $\mathfrak{H}$ satisfies both hypotheses of the above theorem since we have for all $x, y \in V$, $x \neq y$, $deg(x) = n-l$ and $codeg(x,y) = 1$ if $|x \cap y| = l-1$ and 0 otherwise. It follows that $\chi(\mathfrak{H})$, which is nothing else than the minimum number of disjoint packings $P(l, l+1, n)$ which partition the set of all $(l+1)$-subsets, is asymptotically equal to n. Hence, from Lemma 1, we have for $n \to \infty$

$$n - l \le \gamma_{n;l,l+2} \le n(1+o(1))$$

i.e., $\gamma_{n;l,l+2} = n(1+o(1))$. $\square$

Theorem 6 *We have for $n > 7$*

$$\gamma_{n;1,4} = \begin{cases} 2n-1 & \textit{if } n \equiv 0 \textit{ or } 2 \pmod 6 \\ 2n-2 & \textit{if } n \equiv 1 \textit{ or } 3 \pmod 6 \end{cases}$$

and

$$\gamma_{n;1,4} \in \begin{cases} \{2n-2, 2n-1, 2n\} & \textit{if } n \equiv 5 \pmod 6 \\ \{2n-1, 2n\} & \textit{if } n \equiv 4 \pmod 6. \end{cases}$$

Proof: Let A be a subset of the vertex set of $\mathfrak{H}_{n;1,4}$ whose elements contain 1. We claim that A has not less than $2n-2$ colours if n is odd and $2n-1$ if n is even. Indeed, to show this, it suffices to observe that the set $A_1 = \{\{1\}, \{1,i\}, i = 2,\ldots,n\}$ needs exactly n colours and the set $A_2 = \{\{1,i,j\}, i,j \in \{2,\ldots,n\}, i \neq j\}$, which is isomorphic to $\binom{[n-1]}{2}$, needs at least $n-2$ colours if $n-1$ is even (from the factorization of the complete graph on $\{2,\ldots,n\}$) and $n-1$ colours if $n-1$ is odd. Moreover, each colour in A_1 cannot appear in A_2 because otherwise, there would exist an edge containing both vertices having these colours. Thus, $\gamma_{n;1,4} \geq 2n-2$ if n is odd and $\gamma_{n;1,4} \geq 2n-1$ if n is even.

Now, let us determine upper bounds for $\gamma_{n;1,4}$. We only must obtain a colouring of $\binom{[n]}{1} \cup \binom{[n]}{2} \cup \binom{[n]}{3}$ since in all cases the colour of each quadruple may be chosen among colours of triples.

Case 1. $n \equiv 1$ or $3 \pmod 6$, $n \neq 7$.
Consider a near-one-factorization $\{F_1,\ldots,F_n\}$ of K_n (i.e., a one-factorization of K_{n+1} where the pairs $\{i, n+1\}$ are substituted by isolated vertices $\{i\}$) and a large set of disjoint Steiner triple systems of order n $\{S_1,\ldots,S_{n-2}\}$ (which exists since $n \equiv 1$ or $3 \pmod 6$, $(n \neq 7)$. If we choose the colour i (resp. $n+j$) for the elements of F_i (resp. S_j), we have immediately $\gamma_{n;1,4} \leq 2n-2$.

Case 2. $n \equiv 0$ or $2 \pmod 6$, $n \neq 6$.
Using the parity of n and the fact that $n+1 \equiv 1$ or $3 \pmod 6$, $n+1 \neq 7$, we get on the one hand a factorization of $\binom{[n]}{2}$ into $n-1$ factors and on the other hand the existence of a large set $LTS(n+1)$. The deletion of the $n/2$ triples containing $n+1$ in each $STS(n+1)$, and one additional colour for all singletons yield $\gamma_{n;1,4} \leq (n-1)+(n-1)+1 = 2n-1$.

Case 3. $n \equiv 5 \pmod 6$, $n \neq 5$.
From Etzion's papers [7, 8], we know that $\binom{[n]}{3}$ may be partitioned into n PTs, and with a near-one-factorization of K_n we infer that $\mathfrak{H}_{n;1,4}$ is $2n$-colourable for $n > 5$.

Case 4. $n \equiv 4 \pmod 6$, $n \neq 4$.
The factorization of $\binom{[n]}{2}$ and the partition of $\binom{[n]}{3}$ into n PTs [7, 8] produce $2n-1$ colours. If we add one new colour for the singletons, we have $\gamma_{n;1,4} \leq 2n-1$ whenever $n > 4$. $\square$

4 The chromatic number for interval orders

P is called an *interval order* if there is a mapping f from P into the set of closed intervals on the real line, such that for all $p, q \in P$:

$$p < q \text{ in } P \Longleftrightarrow \sup f(p) < \inf f(q).$$

Let us denote the right (resp. left) endpoint of the interval $f(p)$ by $r(p)$ (resp. $l(p)$). Let $\min(P) = \{p_1, \dots, p_s\}$ and $\max(P) = \{q_1, \dots, q_t\}$ be the minimal and maximal elements of P, respectively. Assume without loss of generality that $r(p_1) \le \cdots \le r(p_s)$ and $l(q_1) \le \cdots \le l(q_t)$. In [3], we studied the order-interval hypergraph $\mathfrak{H}$ of the interval order and in particular, we presented polynomial-time algorithms which lead min-max relations $\alpha = \rho$ and $\nu = \tau$ where α, τ, ν and ρ are the independence number, the point covering number, the matching number, and the edge covering number of $\mathfrak{H}$, respectively.

Proposition 2 *We have* $\gamma(\mathfrak{H}(P)) = n - s - t + 2$.

Proof: Since the interval $[p_1, q_t]$ in P contains $n - s - t + 2$ different colours, we obtain $n - s - t + 2 \le \gamma(\mathfrak{H})$. If we assign to the remaining elements $p_2, \dots, p_s, q_1, \dots, q_{t-1}$ the colours $c(p_i) = c(p_1)$ for all $i \ge 2$ and $c(q_j) = c(q_t)$ for all $j \le t - 1$, we obtain the inequality $\gamma(\mathfrak{H}) \le n - s - t + 2$. Hence, $\gamma(\mathfrak{H}) = n - s - t + 2$.

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Taylor Series of the Pick Function and the Loewner Variation within the Class *S* and Applications[0]

ABSTRACT. We find the unknown Taylor series of the Pick function (4) (or (8)) and the Loewner variation (5). As an application we prove the local Bieberbach conjecture for the Loewner variation (5) in a simple way (see Corollaries 3.2 and 3.3 and their consequences).

KEY WORDS. Pick function, Loewner variation, Gauss hypergeometric function, Taylor series, local Bieberbach conjecture, Bombieri proof in the class S, our proof for the Loewner variation within the class S.

1 Introduction

Let S denote the class of all functions

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 = 1, \tag{1}$$

analytic and univalent in the unit disk $D(z) = \{z : |z| < 1\}$. The family $f_\lambda(z)$ $(0 \leqq \lambda < \lambda_0$ for some $\lambda_0 > 0)$ of functions is called a variation of $f(z)$ within S if $f_0(z) = f(z)$ and $f_\lambda(z) \in S$ and if

$$\dot{f}_0 = \frac{\partial}{\partial \lambda} f_\lambda(z)\Big|_{\lambda=0} = \lim_{\lambda \to 0} \frac{f_\lambda(z) - f(z)}{\lambda} \tag{2}$$

exists locally uniformly in $D(z)$ (Pommerenke [1, p. 185]). Let the function (1) be the rotated Koebe function

$$g(z) = \frac{z}{(1+\bar{\zeta}z)^2} = \sum_{n=1}^{\infty} n(-\bar{\zeta})^{n-1} z^n \in S, \quad |\zeta| = 1, \tag{3}$$

and G be its inverse function, i.e. $G(g(z)) = z$. If $\lambda > 0$, then the function

$$w = G(e^{-\lambda} g(z)) = e^{-\lambda} z + \dots \tag{4}$$

[0] Änderung zur Druckvorlage! Korrigiert am 19.4.2002 Formel (4) und Formel (21) letzte Zeile

maps $D(z)$ onto $D(w)$ minus some slit. The function (4) and its normalization are known as the Pick function [2] or as the bounded Koebe function (Goodman [3, pp. 36-38]). It follows from (1) and (4) that the function

$$f_\lambda(z) = e^\lambda f(G(e^{-\lambda} g(z))) = z + \cdots \in S. \tag{5}$$

Further it follows from (2), (5) and (3) that

$$\dot{f}_0(z) = f(z) - \frac{g(z)}{g'(z)} f'(z) = f(z) - z f'(z) \frac{\zeta + z}{\zeta - z}. \tag{6}$$

The familiy (5) with $0 \leqq \lambda < \lambda_0$ is called the Loewner variation within S (Pommerenke [1, p. 185, Example 7.3]). Having in mind (1), the Taylor series of (6) is easily obtained (Pommerenke [1, p. 191, Formula (10)]):

$$\dot{f}_0(z) = -\sum_{n=2}^{\infty} \left[(n-1) a_n + 2 \sum_{j=1}^{n-1} (n-j) a_{n-j} \overline{\zeta}^j \right] z^n, \quad a_1 = 1, \tag{7}$$

for $z \in D(z)$ and $|\zeta| = 1$. But the explicit form of the Taylor series of (4) and (5) are unknown. Now we will give a method with the help of which we discover the Taylor series of (4) and (5) even in a wider range $0 \leqq \lambda < +\infty$.

2 Taylor series of the natural powers of the Pick function (4)

It is clear from (3) and (4) that the Pick function (4) can also be determined by the equation

$$\frac{z}{(1 + \overline{\zeta} z)^2} = \frac{e^\lambda w}{(1 + \overline{\zeta} w)^2}, \quad z \in D(z), \quad 0 \leqq \lambda < +\infty, \quad |\zeta| = 1. \tag{8}$$

The function w, determined by (8), maps the disk $D(z)$ onto the disk $D(w)$ except for a slit along the rectilinear segment from the point

$$G\left(\frac{e^{-\lambda}\zeta}{4}\right) = \zeta \left[2e^\lambda - 1 - 2\sqrt{e^\lambda (e^\lambda - 1)} \right]$$

to the point

$$G\left(\zeta \left[1 - 2e^{-\lambda} \pm 2i \sqrt{e^{-\lambda}(1 - e^{-\lambda})} \right]\right) = \zeta,$$

where

$$|w| \leqq |z|, \quad z \in D(z), \quad 0 \leqq \lambda < +\infty, \tag{9}$$

where, for $0 < \lambda < +\infty$, the equality holds only for $z = 0$. Further we will use the Pochhammer symbol

$$(a)_\nu = a(a+1)\dots(a+\nu-1), \quad \nu = 1,2,\dots; \quad (a)_0 = 1, \tag{10}$$

for an arbitrary number a, the Gauss hypergeometric series (function, respectively)

$$F(\alpha,\beta;\gamma;x) = \sum_{\nu=0}^{\infty} \frac{(\alpha)_\nu(\beta)_\nu}{(\gamma)_\nu \nu!} x^\nu \tag{11}$$

for arbitrary parameters (or elements) α, β and γ with $\gamma \neq 0, -1, -2, \dots$, and a variable x with $|x| < 1$, the summation formula

$$F(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \quad Re(\gamma-\alpha-\beta) > 0, \tag{12}$$

where Γ is the Gamma function, and the transformation formula

$$F(\alpha,\beta;\gamma;x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta;\gamma;x) \tag{13}$$

(see, for example, in [4, pp. 1065-1071]). For $\alpha = -n$, $n = 0,1,2,\dots$, the series (11) is reduced to the hypergeometric polynomial

$$F\left(-n,\beta;\gamma;x\right) = \sum_{\nu=0}^{n} \frac{(-n)_\nu(\beta)_\nu}{(\gamma)_\nu \nu!} x^\nu \tag{14}$$

for all x. If $Re(\gamma+n-\beta) \leqq 0$, then the formula (12) is not always applicable to (14) when $x = 1$. But there is another general summation formula which is given without a proof in the Gould tables for binomial coefficent summations [5, p. 58, Identity (7.1) and p. 61, Identity (7.20)] and in the Bailey monograph [6, p. 3], namely:

$$F(-n,\beta;\gamma;1) = \frac{(\gamma-\beta)_n}{(\gamma)_n}, \quad n = 1,2,\dots, \quad \gamma \neq 0,-1,-2,\dots,-n+1 \tag{15}$$

having in mind the notation (10). A simple proof of the identity (15) can be obtained with the help of the following procedure. We have the identity

$$(-n)_\nu = (-n+1)_\nu - \nu(-n+1)_{\nu-1}, \quad n = 1,2,\dots, \quad \nu = 1,2,\dots,n. \tag{16}$$

From (14) for $x = 1$ and (16) we obtain the identity

$$F(-n,\beta;\gamma;1) = \frac{\gamma-\beta}{\gamma} F(-n+1,\beta;\gamma+1;1), \quad n = 1,2,\dots. \tag{17}$$

By induction on the first and third elements in (17) we attain to (15).

Theorem 1 *The natural powers w^p, $p=1,2,\dots,$ of the Pick function w, determined by (4) (or (8)), have the following Taylor series*

$$w^p = e^{-p\lambda}\sum_{n=p}^{\infty}(-\bar\zeta)^{n-p}\binom{n+p-1}{2p-1}F(p-n,p+n;2p+1;e^{-\lambda})z^n \tag{18}$$

for $|z|<1,\ |\zeta|=1$ and $0\leqq\lambda<+\infty$, where

$$F(p-n,p+n;2p+1;e^{-\lambda}) = \sum_{\nu=0}^{n-p}\frac{(p-n)_\nu(p+n)_\nu}{(2p+1)_\nu\nu!}e^{-\nu\lambda} \tag{19}$$

are the hypergeometric polynomials in $e^{-\lambda}$, determined according to (10)-(11) and (14).

Proof: Let the complex number z and the real number r be fixed with $0\leqq|z|<r<1$. Then, on the basis of (9) for the Pick function w, determined by (4) (or(8)), and the Cauchy theorem on residues, we have the integral representation

$$w^p = \frac{1}{2\pi i}\int\limits_{|u|=r} u^p\frac{e^\lambda g'(u)}{e^\lambda g(u)-g(z)}du \tag{20}$$

for $0\leqq\lambda<+\infty$, and $p=1,2,\dots,$ where the integration along the circle $|u|=r$ is performed in positive direction. From (20) and (3) we obtain the Taylor series as follows

$$\begin{aligned} w^p &= \frac{1}{2\pi i}\int\limits_{|u|=r}\frac{u^{p-1}(1-\bar\zeta u)}{1+\bar\zeta u}\frac{du}{1-\frac{(1+\bar\zeta u)^2}{e^\lambda u}\frac{z}{(1+\bar\zeta z)^2}} = \\ &= \sum_{\nu=p}^{\infty}\frac{z^\nu}{(1+\bar\zeta z)^{2\nu}}\frac{e^{-\nu\lambda}}{2\pi i}\int\limits_{|u|=r}\frac{(1-\bar\zeta u)(1+\bar\zeta u)^{2\nu-1}}{u^{\nu-p+1}}du = \\ &= \sum_{\nu=p}^{\infty}(-1)^\nu\bar\zeta^{-p}\frac{p}{\nu}\binom{2\nu}{\nu-p}e^{-\nu\lambda}\sum_{n=\nu}^{\infty}\binom{n+\nu-1}{n-\nu}(-\bar\zeta)^n z^n = \\ &= e^{-p\lambda}\sum_{n=p}^{\infty}(-\bar\zeta)^{n-p}z^n\sum_{\nu=0}^{n-p}c_\nu(n,p)e^{-\nu\lambda} \end{aligned} \tag{21}$$

for $|z|<1$, $0\leqq\lambda<+\infty$, $p=1,2,\dots,$ where

$$c_\nu(n,p) = (-1)^\nu\frac{p}{\nu+p}\binom{2\nu+2p}{\nu}\binom{n+\nu+p-1}{n-p-\nu} \tag{22}$$

for $0\leqq\nu\leqq n-p$, $n\geqq p$, $p=1,2,\dots.$ It follows from (22) that

$$\frac{c_\nu(n,p)}{c_{\nu-1}(n,p)} = \frac{(p-n+\nu-1)(p+n+\nu-1)}{(2p+\nu)\nu} \tag{23}$$

for $1 \leqq \nu \leqq n-p$, $n \geqq p+1$, $p = 1, 2, \ldots,$ with

$$c_0(n,p) = \binom{n+p-1}{2p-1}, \quad n \geqq p, \quad p = 1, 2, \ldots. \tag{24}$$

From (23) and (24) we deduce that

$$c_\nu(n,p) = \binom{n+p-1}{2p-1} \frac{(p-n)_\nu (p+n)_\nu}{(2p+1)_\nu \nu!} \tag{25}$$

for $0 \leqq \nu \leqq n-p$, $n \geqq p$, $p = 1, 2, \ldots,$ having in mind the notation (10). With the help of (25), having in mind (11) and (14), the expansion (21) takes the form (18)-(19).

This completes the proof of Theorem 1.

Corollary 2.1 *The natural powers w^p, $p = 1, 2, \ldots,$ of the Pick function w, determined by (4) (or (8)), have the following Taylor series*

$$\begin{aligned} w^p = {} & e^{-p\lambda} z^p + e^{-p\lambda}(1-e^{-\lambda}) \cdot \\ & \cdot \sum_{n=p+1}^{\infty} (-\bar{\zeta})^{n-p} \binom{n+p-1}{2p-1} F(p+1-n, p+1+n; 2p+1; e^{-\lambda}) z^n \end{aligned} \tag{26}$$

for $|z| < 1$, $|\zeta| = 1$ and $0 \leqq \lambda < +\infty$, where

$$F(p+1-n, p+1+n; 2p+1; e^{-\lambda}) = \sum_{\nu=0}^{n-p-1} \frac{(p+1-n)_\nu (p+1+n)_\nu}{(2p+1)_\nu \nu!} e^{-\nu\lambda} \tag{27}$$

are the hypergeometric polynomials in $e^{-\lambda}$, determined according to (10)-(11) and (14).

Proof: By (13) we have the factorization

$$F(p-n, p+n; 2p+1; e^{-\lambda}) = (1-e^{-\lambda}) F(p-n+1, p+n+1; 2p+1; e^{-\lambda}) \tag{28}$$

for $n \geqq p+1$. Thus from (28) and (18) we obtain (26)-(27).

3 Taylor series of the Loewner variation (5)

These series originate from (18)-(19), namely:

Theorem 2 *The Loewner variation $f_\lambda(z)$, determined by (5), has the following Taylor series*

$$f_\lambda(z) = \sum_{n=1}^{\infty} a_n(\lambda) z^n \in S, \quad |z| < 1, \quad 0 \leqq \lambda < +\infty, \quad a_1(\lambda) = 1, \tag{29}$$

where

$$a_n(\lambda)=\sum_{p=1}^{n}a_p(-\overline{\zeta})^{n-p}e^{-(p-1)\lambda}\binom{n+p-1}{2p-1}F(p-n,p+n;2p+1;e^{-\lambda}), \tag{30}$$

where a_p, $1\leqq p\leqq n$, are the coefficients of the function $f(z)$ in (1), $|\zeta|=1$, and F are the hypergeometric polynomials in $e^{-\lambda}$, determined by (19) for $p=1,2,\ldots,n$, $n\geqq 1$.

Proof: From (5), (4) and (1) we have

$$f_\lambda(z)=e^\lambda f(w)=e^\lambda\sum_{p=1}^{n}a_pw^p. \tag{31}$$

Now we use (18)-(19) to (31) to obtain (29)-(30).

Corollary 3.1 *The Loewner variation $f_\lambda(z)$, determined by (5) has the following representation*

$$f_\lambda(z)=e^\lambda f(ze^{-\lambda})+(1-e^{-\lambda})\sum_{n=2}^{\infty}b_n(\lambda)z^n\in S,\quad |z|<1,\quad 0\leqq\lambda<+\infty, \tag{32}$$

where

$$b_n(\lambda)=\sum_{p=1}^{n-1}a_p(-\overline{\zeta})^{n-p}e^{-(p-1)\lambda}\binom{n+p-1}{2p-1}F(p+1-n,p+1+n;2p+1;e^{-\lambda}) \tag{33}$$

where a_p, $1\leqq p\leqq n-1$, $n\geqq 2$, are the coefficients of the function $f(z)$ in (1), $|\zeta|=1$, and F are the hypergeometric polynomials in $e^{-\lambda}$, determined by (27) for $p=1,2,\ldots,n-1,n\geqq 2$.

Proof: The transformation of (30) for $1\leqq p\leqq n-1$, $n\geqq 2$, by means of (28) leads us to (32)-(33). The representation (32)-(33) can also be obtained from (31) and (26).

Corollary 3.2 *The Loewner variation (5) tends to the rotated Koebe function (3) as $\lambda\to+\infty$, i.e.*

$$\lim_{\lambda\to+\infty}f_\lambda(z)=g(z),\quad |z|<1. \tag{34}$$

Proof: As $\lambda\to+\infty$ it follows that the series (29), determined by (30) and (19), is reduced to the series (3) and hence the relation (34) holds. Also the relation (34) follows from the representation (32), determined by (33) and (27).

Corollary 3.3 *Let $n\geqq 2$ be an arbitrary fixed integer. Then the Taylor coefficient (30) of the Loewner variation (5) has the following asymptotic representation*

$$a_n(\lambda)=n(-\overline{\zeta})^{n-1}+e^{-\lambda}(-\overline{\zeta})^{n-2}\binom{n+1}{3}(a_2+2\overline{\zeta})+O(e^{-2\lambda}),\quad 0\leqq\lambda<+\infty, \tag{35}$$

where $|\zeta|=1$, a_2 is determined by (1) and $O(e^{-2\lambda})$ denotes a magnitude, the ratio of which to $e^{-2\lambda}$ for $0\leqq\lambda<+\infty$ is bounded by some positive constant depending only on n.

Proof: This follows from (30) and the Bieberbach result [7] that the coefficient body $(a_2, a_3, \ldots, a_n)$ is bounded in the class S (see also Schaeffer and Spencer [8, p. 12, Lemma II]).

In particular, for $\zeta = -1$ it follows from (35) that

$$a_n(\lambda) = n + e^{-\lambda}\binom{n+1}{3}(a_2 - 2) + O(e^{-2\lambda}), \quad 0 \leqq \lambda < +\infty. \tag{36}$$

By the Bieberbach inequality $|a_2| \leqq 2$ [7] with equality only for the Koebe function (3) (see also Pommerenke [1, p. 24]) it follows from (36) that

$$Re\, a_n(\lambda) \leqq n \tag{37}$$

for sufficiently large positive λ where the equality holds if and only if the function f in (5) is the Koebe function g in (3). Thus it follows from (37) that the Bieberbach conjecture $|a_n(\lambda)| \leqq n$ for the Loewner variation (5) within the class S is true for sufficiently large positive λ. Hence in accordance with Corollary 3.2 we proved the local Bieberbach conjecture for the Loewner variation (5) in a new, simpler and direct method in comparison with the Bombieri proof [9] of this conjecture in the class S (see also Pommerenke [1, p. 26] and Hummel [10]). We must note that the global Bieberbach conjecture $|a_n| \leqq n$ in the full class S with equality only for the Koebe function (3) was proved by Louis de Branges and others in different methods (see the Grinshpan reviews [11]).

Remark 1 In particular, for $\lambda = 0$, the series (29)-(30) is reduced to the series (1). In fact, for $\lambda = 0$ and $1 \leqq p \leqq n-1$, $n \geqq 2$, we have the values

$$F(p-n, p+n; 2p+1; 1) = 0 \tag{38}$$

according to the formula (12) since $\Gamma(2p+1) = (2p)!$, $\Gamma(1) = 1$, $\Gamma(p+n+1) = (p+n)!$ and $\Gamma(p-n+1) = \infty$. Of course, the values (38) follow from the formula (15) as well. Also it is evident that the equation $f_0(z) = f(z)$ is immediately obtained from (32).

Remark 2 In particular, the derivative (2) applied to the expansions (29)-(30) for λ =0 yields the equation (7). In fact, for $1 \leqq p \leqq n-1$, $n \geqq 2$, from (19) we have the relations

$$\frac{d}{d\lambda}F(p-n, p+n; 2p+1; e^{-\lambda}) = \frac{(n-p)(n+p)}{(2p+1)e^{\lambda}}F(p-n+1, p+n+1; 2p+2; e^{-\lambda}) \tag{39}$$

and for $\lambda = 0$ we have the values

$$F(p-n+1, p+n+1; 2p+2; 1) = (-1)^{n-p-1}\frac{(n-p-1)!}{(2p+2)_{n-p-1}} \tag{40}$$

according to the formula (15) since the formula (12) is not applicable in this case. Now it follows from (30) and (38)-(40) that

$$\frac{d}{d\lambda}a_n(\lambda)\Big|_{\lambda=0} = -(n-1)a_n - 2\sum_{p=1}^{n-1}(n-p)a_{n-p}\overline{\zeta}^p, \quad n \geqq 2, \quad |\zeta| = 1. \tag{41}$$

The formula (41) in accordance with (29) leads us to (7). The equation (7) can also be obtained from the derivative of (32) with respect to λ and (33) for $\lambda = 0$, having in mind the values

$$F(p+1-n, p+1+n; 2p+1; 1) = (-1)^{n-p-1}\frac{(n-p)!}{(2p+1)_{n-p-1}}$$

for $1 \leqq p \leqq n-1$, $n \geqq 2$, according to (15).

Remark 3 The Theorems 1 and 2 and their Corollaries appear to be basic results in the theory of the Pick function (4) (or (8)) and the Loewner variation (5). Using the Gauss hypergeometric function theory (see, for example, [4], [6], [12]-[13]) for these results, we can obtain other formulas that suit our purposes better.

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