ON THE NUMBER OF REPRESENTATIONS BY QUADRATIC FORMS AND TRIANGULAR NUMBERS

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Abstract

In this thesis, we study the problem of representing integers by quadratic forms. The formulas for the number of representations are obtained as a sum of an Eisenstein part and a cusp part. We begin by solving the representation problem for binary quadratic forms of discriminant $-D < 0$ where the number field $\mathbb{Q}(\sqrt{-D})$ has class number 3. We obtain formulas for the number of representations of an integer as a sum of k triangular numbers, denoted by $\delta_k(n)$, for even values of k. As special cases, for $k = 14, 16$ and 18, new formulas are provided in which the cusp part is given as a linear combination of certain eta products. At the end, for even values of k, we study the first and the second moments of $\delta_k(n)$ and prove an analogue of the Wagon's conjecture for the second moment of $\delta_k(n)$.

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Chapter 1

Introduction

1.1 The number of representations by quadratic forms

The study of binary quadratic forms dates back to the 7th century C.E. when Brahmagupta, an Indian mathematician, found integer solutions to the Pell equation $x^2 - 92y^2 = 1$. The solution to the general Pell equation was given by Bhaskara, in the 11th century C.E. using the *chakravala* method he developed upon the works of Brahmagupta. The name of the method refers to the circular nature of the technique (*chakra* in Sanskrit means circle).

The problem of representing integers by some specific quadratic forms $Q(x_1, x_2, \cdots, x_n)$, i.e., solving the equation

$$n = Q(x_1, x_2, \cdots, x_n),$$

was considered by Fermat in the 17th century. He proposed several observations including when a prime could be represented as a sum of two squares, proofs of which were later provided by Euler. It marked the beginning and a general theory was not so far away.

Lagrange was the first to recognize the importance of the discriminant of a quadratic form and defined the notion of equivalence. He found out that there were only finitely many equivalence classes of binary quadratic forms of a given discriminant, i.e., the class number was finite and hence the study of quadratic forms was reduced to the study of equivalent forms.

Historically, one of the most extensively studied problem in the theory of quadratic forms is finding the number of representations of an integer by a quadratic form. A well known result in this area is Jacobi's four square theorem which says that the number of representations of an integer as a sum of four squares is given by

$$r_4(n) = 8\sigma(n) - 32\sigma(n/4),$$

where $\sigma(n)$ is the sum of divisors of n. This thesis deals with a few problems in this area which we will describe in Section 1.3. The next section builds up the necessary notations and definitions needed to describe our work.

1.2 Modular forms

In this section, we give a brief description of the theory of modular forms. For more information, see [5], [7] and [10]. There are different approaches to study the number of representations of an integer by quadratic forms including combinatorial methods, elementary methods and the circle method of Hardy and Littlewood. In this thesis, we use the theory of modular forms to investigate such problems.

The full modular group $SL_2(\mathbb{Z})$ acts on the complex upper half plane via fractional linear transformations defined by

$$\gamma(z) = \frac{az+b}{cz+d},$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and z belongs to the upper half plane $\mathbb{H}$. We define $\gamma(\infty) = a/c$. This action partitions the upper half plane into equivalence classes. The action can be restricted to certain subgroups. For a positive integer N, define

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

to be the *principal congruence subgroup* of level N. A subgroup of the full modular group is called a *congruence subgroup* if it contains $\Gamma(N)$ for some $N > 0$. For a congruence subgroup Γ, we define a *cusp* to be an equivalence class of $\mathbb{Q} \cup \{i\infty\}$ under the action of Γ. A cusp can also refer to a representative of an equivalence class. For example, the full modular group has only 1 inequivalent cusp $i\infty$. Now we define what a modular form is.

Definition 1.1. *A function f on the upper half plane $\mathbb{H}$ is said to be a modular form of weight k, level N, and character χ if*

(i) f is holomorphic on $\mathbb{H}$.

(ii) $f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$ for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and for any $z \in \mathbb{H}$.

(iii) f is holomorphic at each cusp of $\Gamma_0(N)$, where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\}.$$

For the notion of holomorphicity at a cusp, see [5, Section 5.5]. We denote by $M_k(\Gamma_0(N),\chi)$, the vector space of weight k modular forms on $\Gamma_0(N)$ with character χ. When the character is trivial, we denote the space by $M_k(\Gamma_0(N))$. A modular form $f \in M_k(\Gamma_0(N),\chi)$ is said to be a *cusp form* if f vanishes at every cusp of $\Gamma_0(N)$. The space of cusp forms in $M_k(\Gamma_0(N),\chi)$ is a subspace of $M_k(\Gamma_0(N),\chi)$ and is denoted by $S_k(\Gamma_0(N),\chi)$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a matrix in $SL_2(\mathbb{Z})$. Then, for an integer k, the weight k operator $[\gamma]_k$ on functions $f : \mathbb{H} \to \mathbb{C}$ is defined as

$$(f[\gamma]_k)(z) = (cz+d)^{-k} f(\gamma(z)).$$

The space $M_k(\Gamma_0(N),\chi)$ is a finite dimensional vector space over $\mathbb{C}$ and the dimension can be computed explicitly. We can write $M_k(\Gamma_0(N),\chi)$ as a direct sum

$$M_k(\Gamma_0(N),\chi) = E_k(\Gamma_0(N),\chi) \bigoplus S_k(\Gamma_0(N),\chi),$$

where $S_k(\Gamma_0(N),\chi)$ is the *space of cusp forms* and $E_k(\Gamma_0(N),\chi)$ is called the *space of Eisenstein forms*. The Eisenstein space is the space generated by the Eisenstein series. The Eisenstein series will be described in Chapter 3, Sections 3.1 and 3.2. As a consequence of the above direct sum decomposition, every modular form $f \in M_k(\Gamma_0(N), \chi)$ can be written as a sum of an Eisenstein form and a cusp forms for $\Gamma_0(N)$ and character χ. For more details, see [5, Chapter 8]. An important function in the study of Eisenstein series is the *generalised divisor function* given by

$$\sigma_{t,\chi_1,\chi_2}(n) = \sum_{d|n} \chi_1(d)\chi_2(n/d)d^t,$$

where χ_1 and χ_2 are two Dirichlet characters and t is a non-negative integer. When χ_1 and χ_2 are both *trivial*, i.e., $\chi_1(n) = \chi_2(n) = 1$ for all integers n, this reduces to the normal divisor function $\sigma_t(n) = \sum_{d|n} d^t$. When studying the space of cusp forms, it is important to study Dedekind's eta function. The *Dedekind eta function* is a holomorphic function on the upper half plane defined by

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n), \;\; q = e^{2\pi i z}.$$

An *eta product* is defined to be a finite product of eta functions given by

$$f(z) = \prod_{m} \eta(mz)^{a_m}$$

for positive integers m and integers a_m. The least common multiple N of all the m's is called the *level* of f. The product being finite ensures that the least common multiple N exists. Since every such m divides N, we may also write,

$$f(z) = \prod_{m|N} \eta(mz)^{a_m}.$$

We next state a result due to Gordon, Hughes and Newman which describes the modular transformation properties of the eta products.

Theorem 1.2. ([14, Theorem 1.64]) *Let $f(z) = \prod_{m|N} \eta(mz)^{a_m}$ be an eta quotient with $k =$*

$\frac{1}{2}\sum_{m|N} a_m \in \mathbb{Z}$, *with the additional properties that*

$$\sum_{m|N} m a_m \equiv 0 \pmod{24}$$

and

$$\sum_{m|N} \frac{N}{m} a_m \equiv 0 \pmod{24},$$

Then

$$f(z) \in M_k(\Gamma_0(N), \chi),$$

where, the character χ *is defined by* $\chi(d) = \left(\frac{(-1)^k \prod_{m|N} m^{a_m}}{d}\right)$.

1.3 This thesis

In this section, we briefly describe the contents of this thesis.

Let $a(n,Q)$ be the number of representations of n by a quadratic form Q. Fred van der Blij, in a 1952 paper [23], gives exact formulas for $a(n,Q)$ for all three equivalence classes of quadratic forms Q of discriminant -23. He proves that

$$a(n,Q_1) = \frac{2}{3}\sum_{d|n}\left(\frac{d}{23}\right) + \frac{4}{3}t(n)$$

and

$$a(n,Q_2) = a(n,Q_3) = \frac{2}{3}\sum_{d|n}\left(\frac{d}{23}\right) - \frac{2}{3}t(n),$$

where $\left(\frac{\cdot}{23}\right)$ is the Legendre symbol mod 23, the quadratic forms $Q_i's$ are

$$Q_1 = x^2 + xy + 6y^2,$$

$$Q_2 = 2x^2 + xy + 3y^2,$$

$$Q_3 = 2x^2 + xy - 3y^2,$$

and the coefficients $t(n)$ arise out of the cusp form given by

$$\sum_{n=1}^{\infty} t(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)(1-q^{23n}). \tag{1.1}$$

The cusp form on the right hand side of (1.1) is an eta product of weight 1, level 23 and character $\chi = \left(\frac{-23}{\cdot}\right)$. The proof in [23] is based on combinatorial arguments. Observe that $\mathbb{Q}(\sqrt{-23})$ is one of the 16 imaginary quadratic fields of class number 3, where the class number of $\mathbb{Q}(\sqrt{-D})$ is the order of the class group of $\mathbb{Q}(\sqrt{-D})$. We generalise the above result to the case of binary quadratic forms with class number 3 and obtain the following.

Theorem 2.6. *For $D > 0$, let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field with class number 3. Let Q_1, Q_2, Q_3 be the reduced binary quadratic forms, where Q_1 represents the principle form. Let $a(n,Q)$ be the number of representations of n by the quadratic form Q. For $q = e^{2\pi iz}$ with $z \in \mathbb{H}$, let*

$$F_D(z) = \frac{1}{2}\left(\sum_{a,b\in\mathbb{Z}} q^{Q_1(a,b)} - \sum_{a,b\in\mathbb{Z}} q^{Q_2(a,b)}\right) = \sum_{n=0}^{\infty} t(n)q^n.$$

Then F_D is a cusp form of weight 1, level D, character $\chi = \left(\frac{-D}{\cdot}\right)$ and $t(1) = 1$. Moreover,

$$a(n,Q_1) = \frac{2}{3}\sum_{d|n}\left(\frac{d}{D}\right) + \frac{4}{3}t(n)$$

and

$$a(n,Q_2) = a(n,Q_3) = \frac{2}{3}\sum_{d|n}\left(\frac{d}{D}\right) - \frac{2}{3}t(n).$$

In the above theorem, a different feature compared to the result of Fred van der Blij is that for values of D other than 23, we do not have an eta product representation for the cusp form $F_D(z)$. A result of Eholzer and Skoruppa [8, Section 2] (See also Lemma 2.8) guarantees that $F_D(z)$ has a representation as an infinite product given by

$$F_D(z) = q\prod_{n=1}^{\infty}(1-q^n)^{c(n)}, \;\; q = e^{2\pi i z},$$

for some integers $c(n)$ and for sufficiently small q. Observe that for an eta product, the exponents $c(n)$'s are bounded. We prove in Section 2.4 the following result, which, for $D \neq 23$, implies that $F_D(z)$ does not have an eta product representation.

Theorem 2.15. *Let $D \neq 23$ be such that $\mathbb{Q}\left(\sqrt{-D}\right)$ has class number 3. Let Q_1, Q_2, Q_3 be the three reduced forms of discriminant $-D$, where Q_1 is the principal form. Let*

$$F_D(z) = \frac{1}{2}\left(\sum_{a,b\in\mathbb{Z}} q^{Q_1(a,b)} - \sum_{a,b\in\mathbb{Z}} q^{Q_2(a,b)}\right).$$

Then the integers $c(n)$ in the expansion

$$F_D(z) = q\prod_{n=1}^{\infty}(1-q^n)^{c(n)}$$

are unbounded.

The proof of the above theorem relies on an analytic statement that we prove in Section 2.4 (See Lemma 2.9). It should be mentioned that the proof of Lemma 2.9 is inspired by the exercise in Section 2.1.3 of Serre's monograph [18].

In Chapter 3, we study the number of representations of an integer as a sum of triangular numbers. This problem is intimately connected to the number of representations of integers by quadratic forms. The number of representations of an integer as a sum of squares has been extensively studied. The q-series of interest in the representation problem for squares is the *Jacobi's theta function* defined as

$$\theta(q) = \sum_{n\in\mathbb{Z}} q^{n^2} = \sum_{n\geq 0} q^{n^2} + \sum_{n<0} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots.$$

Let $r_k(n)$ be the number of representations of an integer n as a sum of k squares. Then $r_k(n)$ is intrinsically connected to the Jacobi's theta function by the identity

$$\theta^k(q) = \sum_{n\geq 0} r_k(n)q^n.$$

The theta function is a modular form of integral weight for even values of k. The k^{th} *triangular number* T_k is defined as the number of dots in a triangular arrangement of dots with k dots on each side. Explicitly, the k^{th} triangular number is given by $T_k = k(k+1)/2$. Let $\delta_k(n)$ be the number of representations of n as a sum of k triangular numbers. To study $\delta_k(n)$, we define the *Psi function* as

$$\Psi(q) = \sum_{n=0}^{\infty} q^{T_n} = 1 + q + q^3 + q^6 + \cdots.$$

The role of this function in studying $\delta_k(n)$ is analogous to the role of Jacobi's theta function $\theta(q)$ in studying $r_k(n)$. The Psi function is connected to $\delta_k(n)$ by

$$\Psi^k(q) = \sum_{n=0}^{\infty} \delta_k(n)q^n.$$

In [15], Ono, Robins, and Wahl employ the Psi function to describe formulas for $\delta_k(n)$ for $k = 4, 6, 8, 10, 12, 24$ (Also, formulas for $k = 2, 3$ are given by elementary means). As an example, for $k = 4$, they note that

$$q\Psi^4(q^2) = \sum_{n=0}^{\infty} \delta_4(n)q^{2n+1} \in M_2(\Gamma_0(4))$$

and proceed by observing that it is the Eisenstein series given by

$$\sum_{n=0}^{\infty} \sigma_1(2n+1)q^{2n+1}. \tag{1.2}$$

Motivated by the work of Rankin [16] for the sums of squares, Atanasov et al. [1] obtain formulas for $\delta_{4k}(n)$ in terms of a divisor function and the coefficients of a cusp form. They consider the modular form $q^k\Psi^{4k}(q^2) \in M_{2k}(\Gamma_0(4))$ and decompose it into an Eisenstein series and a cusp form to get their result. Moreover, they also give a basis for the cusp space $S_{2k}(\Gamma_0(4))$ in terms of Jacobi's theta function and (1.2). In Chapter 3, we give a new proof of the main result of [1] using a method due to Aygin employed in [2]. Let χ_0 and χ_1 be the principal Dirichlet characters mod 4 and mod 1 respectively.

Theorem 3.6. *For any positive k, we have*

$$\delta_{4k}(n) = \frac{1}{d_k}\sigma_{2k-1,\chi_1,\chi_0}(2n+k) + c(2n+k),$$

where

$$\sum_{n=1}^{\infty} c(n)q^n \in \mathrm{S}_{2k}(\Gamma_0(4))$$

is a cusp form and

$$d_k = -\frac{(-16)^k\left(4^k-1\right)B_{2k}}{8k}$$

in which B_{2k} is the $2k^{th}$ Bernoulli number.

Next, we extend Theorem 3.6 to obtain formulas for $\delta_{4k+2}(n)$. Our proof of Theorem 3.6 relies on the study of Eisenstein series of even integral weight for $\Gamma_0(4)$. For $\delta_{4k+2}(n)$, we observe that

$$q^{2k+1}\Psi^{4k+2}(q^4) = \sum_{n=0}^{\infty} \delta_{4k+2}(n)q^{4n+2k+1} \in M_{2k+1}(\Gamma_0(8),\chi_{-4}),$$

where χ_{-4} is the Dirichlet character mod 4 taking the values $\chi_{-4}(1) = 1$ and $\chi_{-4}(3) = -1$. A study of the Eisenstein series of odd integral weight for $\Gamma_0(8)$ and character χ_{-4} enables us to obtain our next result.

Theorem 3.12. *For any positive k, we have*

$$\delta_{4k+2}(n) = \frac{-(2k+1)}{2^{4k}B_{2k+1,\chi_{-4}}}\sigma_{2k,\chi_{-4},\chi_1}(4n+2k+1) + t(4n+2k+1),$$

where

$$\sum_{n=1}^{\infty} t(n)q^n \in \mathrm{S}_{2k+1}(\Gamma_0(8),\chi_{-4})$$

is a cusp form and $B_{n,\chi}$ is the n^{th} generalised Bernoulli number associated to χ.

It is worth noting that the terms involving the generalised divisor functions in Theorem 3.6 and Theorem 3.12 are asymptotically dominating terms in the formulas for $\delta_{4k}(n)$ and $\delta_{4k+2}(n)$ respectively. Next, we shift our attention to the cusp part in the formulas for $\delta_{4k}(n)$ and $\delta_{4k+2}(n)$. Towards the end of Chapter 3, we obtain bases for $S_{2k}(\Gamma_0(4))$ and $S_{2k+1}(\Gamma_0(8),\chi_{-4})$ in terms of eta products employing the ideas of Aygin used in [3, Chapter 5].

Theorem 3.16. *The collection*

$$\left\{C(2k,v,4,z) \quad ; \quad 1 \le v \le k-2\right\}$$

forms a basis of $S_{2k}(\Gamma_0(4))$, where

$$C(2k,v,4,z) = \left(\frac{\eta^{10}(2z)}{\eta^4(z)\eta^4(4z)}\right)^{2k}\left(\frac{\eta^8(z)\eta^{16}(4z)}{\eta^{24}(2z)}\right)^{v}\left(\frac{\eta^{16}(z)\eta^8(4z)}{\eta^{24}(2z)}\right).$$

Theorem 3.17. *The collection*

$$\left\{C(2k+1,v,8,z) \quad ; \quad 1 \le v \le 2k-2\right\}$$

forms a basis of $S_{2k+1}(\Gamma_0(8),\chi_{-4})$, where

$$C(2k+1,v,8,z) = \left(\frac{\eta^4(z)}{\eta^2(2z)}\right)^{2k+1}\left(\frac{\eta^2(2z)\eta^4(8z)}{\eta^4(z)\eta^2(4z)}\right)^{v}\left(\frac{\eta^{10}(2z)\eta^6(4z)}{\eta^{12}(z)\eta^4(8z)}\right).$$

By employing the above results, at the end of Chapter 3, we provide new explicit formulas for $\delta_k(n)$ for a few values of k.

Proposition 3.18. *We have*

$$\begin{aligned}
\delta_{14}(n) =& -\frac{1}{124928}\left(\sigma_{6,\chi_{-4},\chi_1}(4n+7) - c(4n+7)\right),\\
\delta_{16}(n) =& \frac{1}{17408}\left(\sigma_{7,\chi_1,\chi_0}(2n+4) - d(2n+4)\right),\\
\delta_{18}(n) =& \frac{1}{45383680}\left(\sigma_{8,\chi_{-4},\chi_1}(4n+9) - e(4n+9)\right),
\end{aligned}$$

where

$$\begin{aligned}
\sum_{n=1}^{\infty} c(n)q^n =& 728\left(\eta^4(z)\eta^2(2z)\eta^8(8z) + 4\frac{\eta^4(2z)\eta^{12}(8z)}{\eta^2(4z)}\right),\\
\sum_{n=1}^{\infty} d(n)q^n =& 128\eta^8(2z)\eta^8(4z),\\
\sum_{n=1}^{\infty} e(n)q^n =& \frac{\eta^{20}(z)\eta^4(4z)}{\eta^6(2z)} + 20\frac{\eta^{16}(z)\eta^2(4z)\eta^4(8z)}{\eta^4(2z)} + 144\frac{\eta^{12}(z)\eta^8(8z)}{\eta^2(2z)}\\
&+448\frac{\eta^8(z)\eta^{12}(8z)}{\eta^2(4z)} + 391168\frac{\eta^4(z)\eta^2(2z)\eta^{16}(8z)}{\eta^4(4z)}\\
&+1562624\frac{\eta^4(2z)\eta^{12}(20z)}{\eta^6(4z)}.
\end{aligned}$$

In Chapter 4, we obtain asymptotic formulas for the first and the second moments of $\delta_k(n)$ for even values of k using the explicit formulas obtained in Chapter 3. Let $f(n)$ be an *arithmetic function* and let

$$L_f(s) = \sum_{n=0}^{\infty} \frac{f(n)}{n^s}$$

be the *formal Dirichlet series* associated to $f(n)$. In [4], Borwein and Choi study formulas for the Dirichlet series associated to $r_k(n)$ and $r_k^2(n)$. Their motivation for considering these explicit representations was to settle the Wagon's conjecture (See [4, Page 97]), which says that for $N \geq 3$,

$$\sum_{n\leq x} r_N^2(n) \sim W_N x^{N-1}, \text{ as } x \to \infty,$$

where

$$W_N = \frac{\pi^N}{(1-2^{-N})(N-1)\Gamma^2\left(\frac{N}{2}\right)} \frac{\zeta(N-1)}{\zeta(N)}.$$

Here $\zeta(.)$ is the Riemann zeta function and $\Gamma(.)$ is the gamma function. By using the explicit formulas for $r_N(n)$, the Wagon's conjecture for $N = 4, 6, 8$ is settled in [4]. Inspired by the above, we phrase an analogue of the Wagon's conjecture for $\delta_N(n)$. For $N \geq 3$, we conjecture that

$$\sum_{n \leq x} \delta_N^2(n) \sim Y_N x^{N-1}, \text{ as } x \to \infty,$$

where

$$Y_N = \frac{\pi^N}{2^N(N-1)\Gamma^2\left(\frac{N}{2}\right)} \frac{L(N-1,\chi_0)}{L(N,\chi_0)},$$

where χ_0 is the principal Dirichlet character mod 4 and $L(s,\chi_0)$ is the Dirichlet series associated to χ_0. Aiming to settle the conjecture for even values of N, we use the explicit formulas obtained in Chapter 3 for $\delta_{4k}(n)$ and $\delta_{4k+2}(n)$ to study the twisted Dirichlet series associated to $\delta_{2k}^i(n)$ for $i = 1, 2$ and obtain the following results for the first and the second moment.

Theorem 4.9. *For an even value of $N > 2$ and any $\varepsilon > 0$, we have*

$$\sum_{n \leq x} \delta_N(n) = \frac{\pi^{N/2}}{2^{N/2}\Gamma(N/2+1)} x^{N/2} + O\left(x^{N/2-1/2+\varepsilon}\right).$$

Theorem 4.10. *For an even value of $N > 2$ and any $\varepsilon > 0$, we have*

$$\sum_{n \leq x} \delta_N^2(n) = Y_N x^{N-1} + O\left(x^{N-1/2+\varepsilon}\right),$$

where

$$Y_N = \frac{\pi^N}{2^N(N-1)\Gamma^2\left(\frac{N}{2}\right)} \frac{L(N-1,\chi_0)}{L(N,\chi_0)}.$$

Hence, the analogue of the Wagon's conjecture $\delta_N(n)$ is true for even values of N. For odd values of N, we cannot use the same method due to the lack of explicit formulas for

$\delta_N(n)$ for odd values of N.

At the end, we will devote Chapter 5 to list a few problems relevant to the works in this thesis, which we plan to investigate in the near future.

Chapter 2

Explicit Formulas for Representation by Binary Quadratic Forms

In this chapter, we generalise a result of Fred van der Blij [23] to the case of imaginary quadratic fields with class number three using the theory of theta functions. Before we proceed to our main results, we give a brief description of the theory of binary quadratic forms and their associated theta functions.

2.1 Binary quadratic forms

An *integral binary quadratic form* is a homogenous quadratic polynomial in two variables given by

$$Q(x,y) = ax^2 + bxy + cy^2,$$

where a, b and c are integers. Moreover, the form is said to be *primitive* if a, b and c are relatively prime. From here onwards we only deal with primitive forms. The discriminant of Q is defined to be $D = b^2 - 4ac$. It can be shown that when the discriminant is negative, Q represents either positive or negative integers depending on the sign of the leading coefficient. So when $D < 0$ and $a > 0$, Q only represents positive integers and the form is said to be a *positive definite* form. The following definition regarding the equivalence of forms is due to Lagrange.

Definition 2.1. *Two forms $Q_1(x,y)$ and $Q_2(x,y)$ are said to be equivalent if there exists* *integers p,q,r and s such that $ps-qr=\pm 1$ and*

$$Q_1(x,y)=Q_2(px+qy,rx+sy).$$

Moreover, this equivalence is called proper if $ps-qr=1$.

In the context of positive definite forms, we only talk about the proper equivalence and going forward, we drop the word proper. From the definition, observe that two equivalent forms are going to represent the same set of integers in $\mathbb{Z}$. We want to know if there are finitely many equivalent classes of positive definite quadratic forms and whether there is a simple description for the representative of each class. To answer these questions, we first define what a reduced form is.

Definition 2.2. *A primitive positive definite form $Q(x,y)=ax^2+bxy+cy^2$ is said to be a reduced form if*

$$|b|\leq a\leq c, \text{ and } b\geq 0 \text{ if either } |b|=a \text{ or } a=c.$$

It can be shown that every equivalence class of positive definite binary quadratic forms contains a unique reduced form (see [6, Theorem 2.8]). For $D<0$, let $h(D)$ be the *class number* associated to D which is defined to be the number of equivalence classes of primitive positive definite forms of discriminant D. Then $h(D)$ is equal to the number of solutions of $b^2-4ac=D$ satisfying the condition in the above definition. Hence, the class number is finite.

Proposition 2.3. *Let $Q=ax^2+bxy+cy^2$ be a reduced form. Then, the least positive integer represented by the reduced form is equal to a.*

Proof. Observe that

$$Q(x,y)\geq(a-|b|+c)\min(x^2,y^2).$$

Since Q is a reduced form, $c-|b|>0$, hence

$$Q(x,y) \geq a\min(x^2,y^2).$$

Thus, $Q(x,y) \geq a$ when $(x,y) \neq (0,0)$. The value a is achieved by Q at the point $(1,0)$ and hence we are done. □

For a detailed description of the theory of binary quadratic forms, see [6, Chapter 1, Section 2].

Let $a(n,Q)$ be the number of representations of n by the quadratic form Q. The theta series which is informally the generating function for these coefficients has a special behaviour on the complex upper half plane. We discuss the theta functions in detail in the next section.

2.2 Theta functions

To every positive definite binary quadratic form $Q(x,y) = ax^2+bxy+cy^2$, we associate a *theta function* $\theta_Q : \mathbb{H} \to \mathbb{C}$, given by

$$\theta_Q(z) = \sum_{m,n\in\mathbb{Z}} q^{Q(m,n)},$$

where

$$q = e^{2\pi i z}.$$

As it turns out, these theta functions are actually modular forms. To describe the level and the character associated to a theta function, let

$$A = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}.$$

Set N to be the smallest positive integer such that the matrix NA^{-1} is integral and has even diagonal entries, and let χ be the Kronecker symbol

$$\left(\frac{-\det A}{\cdot}\right).$$

Theorem 2.4. ([24, Theorem 10.1]) *For a positive definite binary quadratic form Q, the theta function given by $\theta_Q(z)$ is a modular form of weight 1, level N and character χ described above.*

In the next section, we obtain a generalisation of the main result of [23].

2.3 A generalisation of a theorem of Fred van der Blij

Let $a(n,Q)$ be the number of representations of an integer n by a quadratic form $Q = ax^2+bxy+cy^2$ and $\left(\frac{\cdot}{D}\right)$ be the Legendre symbol.

Proposition 2.5. *The total number of representations of a positive integer n by the three reduced forms Q_i's of discriminant $-D$ such that $\mathbb{Q}(\sqrt{-D})$ has class number 3 is given by*

$$a(n,Q_1)+a(n,Q_2)+a(n,Q_3) = 2\sum_{d|n}\left(\frac{d}{D}\right).$$

Proof. Observe that there is a one to one correspondence between the ideal classes of the number field $K = \mathbb{Q}(\sqrt{-D})$ and the classes of positive definite quadratic forms of discriminant $-D$ (See [6, Theorem 5.30]). Let C_1, C_2, C_3 be the three ideal classes of K and let Q_1, Q_2, Q_3 be the corresponding quadratic forms. The Dedekind zeta function of K is defined by

$$\zeta(K,s) = \sum_{\mathfrak{a}\subset O_K} \frac{1}{\mathrm{N}(\mathfrak{a})^s},$$

where $\mathfrak{a}$ denotes an ideal in O_K and $\mathrm{N}(\mathfrak{a})$ denotes the norm of $\mathfrak{a}$. We can split this sum into three different sums as we have three ideal classes in K. We get,

$$\zeta(K,s) = \sum_{\mathfrak{a}\subset C_1} \frac{1}{\mathrm{N}(\mathfrak{a})^s} + \sum_{\mathfrak{a}\subset C_2} \frac{1}{\mathrm{N}(\mathfrak{a})^s} + \sum_{\mathfrak{a}\subset C_3} \frac{1}{\mathrm{N}(\mathfrak{a})^s} \tag{2.1}$$

By using [19, Lemma 27], we get

$$\zeta(K,s) = \frac{1}{2}\sum_{n}\left(\frac{a(n,Q_1)}{n^s} + \frac{a(n,Q_2)}{n^s} + \frac{a(n,Q_3)}{n^s}\right). \tag{2.2}$$

On the other hand, for $Re(s) > 1$, we have

$$\begin{aligned}\zeta(K,s) &= \prod_{\mathfrak{p}}\left(1 - \frac{1}{\mathrm{N}(\mathfrak{p})^s}\right)^{-1} \\ &= \left(1 - \frac{1}{D^s}\right)^{-1} \prod_{\left(\frac{-D}{p}\right)=-1}\left(1 - \frac{1}{p^{2s}}\right)^{-1} \prod_{\left(\frac{-D}{p}\right)=1}\left(1 - \frac{1}{p^s}\right)^{-2} \\ &= \prod_{p}\left(1 - \frac{1}{p^s}\right)^{-1} \prod_{p}\left(1 - \frac{\left(\frac{-D}{p}\right)}{p^s}\right)^{-1}.\end{aligned} \tag{2.3}$$

Here we used the fact that when $\left(\frac{-D}{p}\right) = -1$, pO_K stays a prime and has norm p^2 and when $\left(\frac{-D}{p}\right) = 1$, pO_K splits into two prime ideals of norm p (See [6, Proposition 5.16]). Using the law of quadratic reciprocity,

$$\left(\frac{D}{p}\right) = (-1)^{\frac{p-1}{2}}\left(\frac{p}{D}\right)$$

since $D \equiv 3 \bmod 4$ (See the list of imaginary quadratic fields of class number 3 in Table 2.1). So we get

$$\left(\frac{-D}{p}\right) = \left(\frac{p}{D}\right).$$

Substituting this back in (2.3) gives

$$\begin{aligned}\zeta(K,s) &= \prod_{p}\left(1 - \frac{1}{p^s}\right)^{-1} \cdot \prod_{p}\left(1 - \frac{\left(\frac{p}{D}\right)}{p^s}\right)^{-1} \\ &= \sum_{k\in\mathbb{Z}}\frac{1}{k^s} \cdot \sum_{l\in\mathbb{Z}}\frac{\left(\frac{l}{D}\right)}{l^s} = \sum_{n}\frac{\sum_{d|n}\left(\frac{d}{D}\right)}{n^S}.\end{aligned} \tag{2.4}$$

Combining (2.2) and (2.4) and using the fact that a function can have at most one representation as a Dirichlet series (See [21, Section 9.6]), we have our result. □

Now, we state the main result of this section.

Theorem 2.6. *For $D > 0$, let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field with class number 3. Let Q_1, Q_2, Q_3 be the reduced binary quadratic forms, where Q_1 represents the principle form. Let $a(n,Q)$ be the number of representations of n by the quadratic form Q. For $q = e^{2\pi iz}$ with $z \in \mathbb{H}$, let*

$$F_D(z) = \frac{1}{2}\left(\sum_{a,b\in\mathbb{Z}} q^{Q_1(a,b)} - \sum_{a,b\in\mathbb{Z}} q^{Q_2(a,b)}\right) = \sum_{n=0}^{\infty} t(n)q^n.$$

Then F_D is a cusp form of weight 1, level D, character $\chi = \left(\frac{-D}{\cdot}\right)$ and $t(1) = 1$. Moreover,

$$a(n,Q_1) = \frac{2}{3}\sum_{d|n}\left(\frac{d}{D}\right) + \frac{4}{3}t(n) \tag{2.5}$$

and

$$a(n,Q_2) = a(n,Q_3) = \frac{2}{3}\sum_{d|n}\left(\frac{d}{D}\right) - \frac{2}{3}t(n). \tag{2.6}$$

Proof. First, we will give a description of the three binary quadratic forms of discriminant $-D$. Recall from Section 2.1 that each class of positive definite binary quadratic forms contains a unique form, called the reduced form, such that

$$|b| \le a \le c, \;\; b \ge 0 \;\text{ if }\; |b| = a \;\text{ or }\; a = c. \tag{2.7}$$

Observe that there are only 16 values of squarefree D for which $\mathbb{Q}(\sqrt{-D})$ has class number 3 (see Table 2.1) and all these values are primes congruent to -1 mod 4. The computations are done using SAGE [20]. Since the class number is 3, the equation $b^2 - 4ac = -D$ is going to have 3 solutions that satisfy (2.7). As the values of D are congruent to -1 modulo 4, one of the solutions of the ordered pair (a,b,c) is $(1,1,(1+D)/4)$, which gives rise to the principal form

$$Q_1 = x^2 + xy + \frac{(1+D)}{4}y^2.$$

Table 2.1: Imaginary quadratic fields $\mathbb{Q}\left(\sqrt{-D}\right)$ with class number three and reduced forms Q_1, Q_2, Q_3.

D	Q_1	Q_2	Q_3	$\dim(M_1(\Gamma_0(D),\chi))$	$\dim(S_1(\Gamma_0(D),\chi))$
23	$x^2+xy+6y^2$	$2x^2+xy+3y^2$	$2x^2-xy+3y^2$	2	1
31	$x^2+xy+8y^2$	$2x^2+xy+4y^2$	$2x^2-xy+4y^2$	2	1
59	$x^2+xy+15y^2$	$3x^2+xy+5y^2$	$3x^2-xy+5y^2$	2	1
83	$x^2+xy+21y^2$	$3x^2+xy+7y^2$	$3x^2-xy+7y^2$	2	1
107	$x^2+xy+27y^2$	$3x^2+xy+9y^2$	$3x^2-xy+9y^2$	2	1
139	$x^2+xy+35y^2$	$5x^2+xy+7y^2$	$5x^2-xy+7y^2$	2	1
211	$x^2+xy+53y^2$	$5x^2+3xy+11y^2$	$5x^2-3xy+11y^2$	2	1
283	$x^2+xy+71y^2$	$7x^2+5xy+11y^2$	$7x^2-5xy+11y^2$	4	3
307	$x^2+xy+77y^2$	$7x^2+xy+11y^2$	$7x^2-xy+11y^2$	2	1
331	$x^2+xy+83y^2$	$5x^2+3xy+17y^2$	$5x^2-3xy+17y^2$	4	3
379	$x^2+xy+95y^2$	$5x^2+xy+19y^2$	$5x^2-xy+19y^2$	2	1
499	$x^2+xy+125y^2$	$5x^2+xy+25y^2$	$5x^2-xy+25y^2$	2	1
547	$x^2+xy+137y^2$	$11x^2+5xy+13y^2$	$11x^2-5xy+13y^2$	2	1
643	$x^2+xy+161y^2$	$7x^2+xy+23y^2$	$7x^2-xy+23y^2$	4	3
883	$x^2+xy+221y^2$	$13x^2+xy+17y^2$	$13x^2-xy+17y^2$	2	1
907	$x^2+xy+227y^2$	$13x^2+9xy+19y^2$	$13x^2-9xy+19y^2$	2	1

Now suppose (a_0,b_0,c_0) is a solution of $b^2-4ac=-D$ that satisfies (2.7) with $a_0>1$, giving rise to $Q_2 = a_0x^2+b_0xy+c_0y^2$. Then by (2.7), $(a_0,-b_0,c_0)$ is going to be another solution given that $a_0 \neq |b_0|$ and $a_0 \neq c_0$. Hence, the last form will be given by $Q_3 = a_0x^2-b_0xy+c_0y^2$. We will prove $a_0 \neq |b_0|$ and $a_0 \neq c_0$ by contradiction. First let us prove $a_0 \neq |b_0|$. Suppose $a_0 = |b_0|$, then

$$a_0^2 - 4a_0c_0 = -D.$$

Since a_0 divides the left hand side, a_0 divides D. Therefore $a_0 = D$ since $a_0 > 1$ and D is prime, which contradicts the upper bound $\left(\sqrt{\frac{D}{3}}\right)$ of a_0 for a reduced form (See [6, Page

29]). Now let us prove $a_0 \neq c_0$. Suppose $a_0 = c_0$. Then

$$b_0^2 - 4a_0^2 = -D,$$

i.e.,

$$D = (2a_0 - b_0)(2a_0 + b_0),$$

contradicting the fact that D is prime. Here we also used the fact that $|b_0| \leq a_0$ and $a_0 > 1$ to guarantee that the factors $(2a_0 \pm b_0)$ are non trivial. So we have

$$Q_1 = x^2 + xy + \frac{(1+D)}{4}y^2,$$

$$Q_2 = a_0x^2 + b_0xy + c_0y^2,$$

and

$$Q_3 = a_0x^2 - b_0xy + c_0y^2.$$

Now that we have a description of the quadratic forms, we prove that $F_D(z)$ is a cusp form of weight 1, level D and character $\chi = \left(\frac{-D}{\cdot}\right)$. Recall that $M_1(\Gamma_0(D), \chi)$ is the vector space of modular forms of weight 1, level D and character $\chi = \left(\frac{-D}{\cdot}\right)$. Then by Theorem 2.4,

$$\theta_{Q_1}(z), \theta_{Q_2}(z), \theta_{Q_3}(z) \in M_1(\Gamma_0(D), \chi)$$

and

$$F_D(z) = \frac{1}{2}(\theta_{Q_1}(z) - \theta_{Q_2}(z)) \in M_1(\Gamma_0(D), \chi)$$

is a modular form. To prove that it is in fact a cusp form, we have to prove that $F_D(z)$ vanishes at all the inequivalent cusps of $\Gamma_0(D)$. Since D is prime, $\Gamma_0(D)$ will have 2 inequivalent cusps namely 1 and $i\infty$ (See [7, Page 103]). Hence we first find out the Fourier expansions of the theta functions $\theta_{Q_1(z)}$ and $\theta_{Q_2(z)}$ at both cusps. At the cusp $i\infty$, the Fourier expansions of the theta functions are given by

$$\theta_{Q_1}(z) = \sum_{n=0}^{\infty} a(n,Q_1)q^n = 1+2q+O\left(q^2\right),$$

and

$$\theta_{Q_2}(z) = \sum_{n=0}^{\infty} a(n,Q_2)q^n = 1+O\left(q^2\right).$$

Here we have used the fact that the least positive integer represented by a reduced form is equal to the coefficient of x^2 in the reduced form (See Proposition 2.3). Hence, $F_D(z)$ has the expansion given by

$$F_D(z) = q+O\left(q^2\right). \tag{2.8}$$

Therefore $F_D(z)$ vanishes at the cusp $i\infty$. Now we find out the Fourier expansions of the theta functions at the cusp 1. For Q_1, we set

$$A = \begin{pmatrix} 2 & 1 \\ 1 & (1+D)/2 \end{pmatrix}$$

and

$$\rho = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Note that $\rho(\infty) = 1/1$. Using the formula for the Fourier expansion of theta functions in the proof of [24, Theorem 10.1], we get

$$(z+1)^{-1}\theta_{Q_1}(\rho z) = \frac{-i}{\sqrt{D}} \sum_{(x,y)\in\mathbb{Z}^2} q^{\frac{1}{D}\left(\frac{1+D}{4}x^2-xy+y^2\right)} e^{\frac{2\pi i}{D}\left(\frac{1+D}{4}x^2-xy+y^2\right)}.$$

Changing x to y and y to $-x$ yields

$$\theta_{Q_1}[\rho]_1(z) = \frac{-i}{\sqrt{D}} \sum_{(x,y)\in\mathbb{Z}^2} q^{\frac{Q_1(x,y)}{D}} e^{\frac{2\pi i Q_1(x,y)}{D}}.$$

Hence,

$$\theta_{Q_1}[\rho]_1(z) = \frac{-i}{\sqrt{D}}\left(1 + c_1 q^{\frac{1}{D}} + \cdots\right). \tag{2.9}$$

Similarly for the quadratic form $Q_2 = a_0x^2 + b_0xy + c_0y^2$,

$$\theta_{Q_2}[\rho]_1(z) = \frac{-i}{\sqrt{D}}\left(1 + c_2 q^{\frac{a_0}{D}} + \cdots\right). \tag{2.10}$$

Thus, from (2.9) and (2.10),

$$F_D[\rho]_1(z) = \frac{1}{2}\left(\theta_{Q_1}[\rho]_1(z) - \theta_{Q_2}[\rho]_1(z)\right) = \frac{-ic_1}{2\sqrt{D}} q_D + O\left(q_D^2\right), \tag{2.11}$$

where $q_D = q^{\frac{1}{D}}$. Therefore $F_D(z)$ vanishes at the cusp 1.

At last, we prove the identities (2.5) and (2.6). Since

$$F_D(z) = \frac{1}{2}\left(\sum_{p,q\in\mathbb{Z}} q^{Q_1(p,q)} - \sum_{p,q\in\mathbb{Z}} q^{Q_2(p,q)}\right) = \sum_{n=1}^{\infty} t(n)q^n.$$

We have

$$a(n, Q_1) - a(n, Q_2) = 2t(n). \tag{2.12}$$

Also, observe that

$$Q_2(x_0, y_0) = Q_2(-x_0, -y_0) = Q_3(x_0, -y_0) = Q_3(-x_0, y_0).$$

Hence,

$$a(n, Q_2) = a(n, Q_3). \tag{2.13}$$

Thus, Proposition 2.5 together with (2.12) and (2.13) yield the result. □

Now, we also give a new modular proof of the second part of [23, Theorem 1] which describes the cusp part $F_{23}(z)$.

Proposition 2.7. *The cusp form $F_{23}(z)$ is given by*

$$F_{23}(z) = \eta(q)\eta(q^{23}).$$

Proof. The 3 reduced binary quadratic forms of discriminant -23 are given by

$$F_1 = x^2 + xy + 6y^2,$$

$$F_2 = 2x^2 + xy + 3y^2,$$

$$F_3 = 2x^2 - xy + 3y^2.$$

Let

$$F_{23}(z) = \frac{1}{2}\left(\sum_{p,q\in\mathbb{Z}} q^{F_1(p,q)} - \sum_{p,q\in\mathbb{Z}} q^{F_2(p,q)}\right) = \sum_{n=0}^{\infty} t(n)q^n.$$

Then, by Theorem 2.6,

$$F_{23}(z) = \sum_{n=0}^{\infty} t(n)q^n \in S_1\left(\Gamma_0(23), \left(\frac{-23}{.}\right)\right)$$

and the number of representations by the reduced forms F_1, F_2 and F_3 are given by

$$a(n, F_1) = \frac{2}{3}\sum_{d|n}\left(\frac{d}{23}\right) + \frac{4}{3}t(n) \tag{2.14}$$

and

$$a(n, F_2) = a(n, F_3) = \frac{2}{3}\sum_{d|n}\left(\frac{d}{23}\right) - \frac{2}{3}t(n). \tag{2.15}$$

Note that by Theorem 1.2,

$$\eta(q)\eta(q^{23}) \in S_1\left(\Gamma_0(23), \left(\frac{-23}{d}\right)\right).$$

The cusp space $S_1\left(\Gamma_0(23),\left(\frac{-23}{d}\right)\right)$ is one dimensional (see Table 2.1). Hence,

$$\sum_{n=0}^{\infty} t(n)q^n = k\eta(q)\eta(q^{23})$$

for some $k \in \mathbb{C}$. Comparing the first coefficients, we see that $k = 1$ and hence we are done. □

2.4 Infinite product representation of $F_D(z)$

We saw in the last section that the cusp form $F_{23}(z)$ has a closed form representation as an eta product. In this section, we investigate whether $F_D(z)$ enjoys similar representations for other values of D in Table 2.1. We first describe a proposition due to Eholzer and Skoruppa on the infinite product expansions of periodic holomorphic functions on the upper half plane.

Lemma 2.8. ([8, Section 2]) *Let $f(q) = \sum_{n=0}^{\infty} a_f(n)q^n$ be a holomorphic function on the upper half plane and $a_f(0) = 1$. Then there exists a unique sequence of complex numbers $c(n)$ such that*

$$f(q) = \prod_{n=1}^{\infty} (1-q^n)^{c(n)} \tag{2.16}$$

for sufficiently small q. The $c(n)$'s are integral if f has integral Fourier coefficients.

We will use Lemma 2.8 to investigate the possibility of representing $F_D(z)$ as an eta product. We will prove that for none of the values of D listed in Table 2.1 other than 23, $F_D(z)$ can be written as an eta product. The important thing to observe here is that the exponents $c(n)$ in (2.16) are not necessarily bounded. We next prove an analytic result which reveals the nature of these exponents depending on whether or not the modular form has a zero on the upper half plane.

Lemma 2.9. *Let $f(q) = \sum_{n=0}^{\infty} a_f(n)q^n$ be a holomorphic function on the upper half plane* *and* $a_f(0) = 1$*. Then the complex numbers* $c(n)$ *in the expansion*

$$f(q) = \prod_{n=1}^{\infty} (1 - q^n)^{c(n)}$$

are unbounded if f *has a zero on the upper half plane.*

Proof. We know by Lemma 2.8 that $f(q)$ has an expansion given by

$$f(q) = \prod_{n \geq 1} (1 - q^n)^{c(n)} \tag{2.17}$$

for some complex numbers $c(n)$. We are given that f has a zero in $\mathbb{H}$. Since f is holomorphic, the zeroes of f are discrete and hence there exists a neighborhood $N(i\infty)$ of $i\infty$ such that f has no zeroes in $N(i\infty) \cap \mathbb{H}$. Therefore, there exists a $T > 0$ such that for every z with $\mathrm{Im}(z) > T$, $f(z) \neq 0$ on $\mathbb{H}$. Let z_0 be the zero with the largest imaginary part. We claim that the radius of convergence of the power series

$$L(q) = \log(f(q))$$

around $q = 0$ is $e^{-2\pi \mathrm{Im}(z_0)}$. To prove this claim, we set $q_0 = e^{2\pi i z_0}$. Then the point q_0 is a point of singularity for $L(q)$. For the convergence of the power series $L(q)$, the function $f(q)$ should stay away from zero. We observe that whenever $\mathrm{Im}(z) > \mathrm{Im}(z_0)$ for $z \in \mathbb{H}$, $f(q)$ stays away from 0. Lastly, we see that

$$\mathrm{Im}(z) > \mathrm{Im}(z_0) \text{ iff } |q| < |q_0|$$

for $q = e^{2\pi i z}$. Hence, whenever $|q| < |q_0|$, then the power series $L(q)$ has a convergent expansion and our claim holds. Thus, $e^{-2\pi \mathrm{Im}(z_0)}$ is the radius of convergence of $L(q)$.

Next, by (2.17), we have

$$\begin{aligned} L(q) &= \sum_{n=1}^{\infty} c(n)\log(1-q^n) \\ &= \sum_{n=1}^{\infty} -c(n)\left(q^n + \frac{q^{2n}}{2} + \frac{q^{3n}}{3}\cdots\right) \\ &= -\sum_{n=1}^{\infty} \frac{1}{n}\sum_{d|n} dc(d)q^n. \end{aligned}$$

Using the limsup formula for the radius of convergence R of the power series $L(q)$, we have

$$\mathrm{R} = \left(\limsup\left(\sqrt[n]{\left|\frac{1}{n}\sum_{d|n} dc(d)\right|}\right)\right)^{-1}.$$

We have already proved that the radius of convergence of $L(q)$ is $e^{-2\pi \mathrm{Im}(z_0)}$. Hence,

$$\limsup\left(\sqrt[n]{\left|\frac{1}{n}\sum_{d|n} dc(d)\right|}\right) = e^{2\pi \mathrm{Im}(z_0)}.$$

If $c(n)$'s are bounded, $\sum_{d|n} dc(d) = O(\sigma(D)) = O(n^{\varepsilon})$ for some $\varepsilon > 0$. Hence,

$$\limsup\left(\sqrt[n]{\left|\frac{1}{n}\sum_{d|n} dc(d)\right|}\right) \leq 1,$$

which is a contradiction since $\mathrm{Im}(z_0) > 0$. Therefore $c(n)$'s must be unbounded. Thus, we have proved that the existence of a zero in $\mathbb{H}$ implies that the exponents $c(n)$'s are unbounded. □

Before we consider Lemma 2.9 for our case regarding $F_D(z)$, let us describe the notion of the width of a cusp and the order of vanishing $\nu_z(f)$ of a modular form f at a point $z \in \overline{\mathbb{H}}$.

Definition 2.10. *Let f be a modular form of weight k for a subgroup Γ of $SL_2(\mathbb{Z})$. The width of the cusp $i\infty$ is defined to be the least positive integer such that $\pm\left(\begin{smallmatrix} 1 & w \\ 1 & 0 \end{smallmatrix}\right) \in \Gamma$ for a suitable sign. In the first case, when $\left(\begin{smallmatrix} 1 & w \\ 1 & 0 \end{smallmatrix}\right) \in \Gamma$, it follows that f is periodic with period w.*

In the second case, when $-\left(\begin{smallmatrix}1 & w\\ 1 & 0\end{smallmatrix}\right) \in \Gamma$,

$$f(z) = (-1)^k f(z+w)$$

and f is periodic with period $2w$ when k is odd and periodic with period w when k is even. If a/c is a cusp of Γ such that $\gamma(i\infty) = a/c$ for some $\gamma \in SL_2(\mathbb{Z})$, then the width of the cusp a/c for Γ is defined to be the width of the cusp $i\infty$ for $\gamma^{-1}\Gamma\gamma$.

Definition 2.11. *Suppose f is a modular form for a subgroup Γ of $SL_2(\mathbb{Z})$ on the upper half plane. For $z_0 \in \mathbb{H}$, $\nu_{z_0}(f)$ is defined to be the order of f at z_0. If f has width w at $i\infty$, write $f(z) = g(e^{2\pi iz/w})$ and define $\nu_{i\infty}(f) = \nu_0(g)$, where $\nu_0(g)$ is the least positive exponent of $e^{2\pi iz/w}$ in the expansion. For a cusp $z_0 \in \mathbb{Q}$, there exists $\gamma \in SL_2(\mathbb{Z})$, such that $z_0 = \gamma(i\infty)$ and we set $\nu_{z_0}(f) = \nu_{i\infty}(f[\gamma]_k)$.*

For more information on Definition 2.11, see [5, Page184]. Using the above definition, we compute the order of vanishing of $F_D(z)$ at the inequivalent cusps of $\Gamma_0(D)$. Since D is prime, $\Gamma_0(D)$ has 2 inequivalent cusps namely $i\infty$ and $1/1$. We have the following proposition about the order of vanishing at these two cusps.

Proposition 2.12. *The order of vanishing $\nu_z(F_d)$ in $\Gamma_1(D)$ at the cusps $i\infty$ and $1/1$ are*

$$\nu_{i\infty}(F_D) = 1$$

and

$$\nu_{1/1}(F_D) = 1.$$

Proof. For a prime D in Table 2.1, we know from (2.8) and (2.11) that

$$F_D(z) = q + O\left(q^2\right)$$

and

$$F_D[\rho z]_1(z) = \frac{-ic_1}{2\sqrt{D}} q_D + O\left(q_D^2\right),$$

where ρ is a matrix in $SL_2(\mathbb{Z})$ such that $\rho\infty = 1/1$ and $q_D = e^{2\pi iz/D}$. Using [5, Proposition 6.3.20], the width of the cusps $i\infty$ and $1/1$ in $\Gamma_1(D)$ is equal to 1 and D respectively. Hence by Definition 2.11, the order of vanishing in $\Gamma_1(D)$ at each cusp is 1. $\square$

We next state the *valence formula* for modular forms for a finite index subgroup of $SL_2(\mathbb{Z})$.

Theorem 2.13. ([5, Theorem 5.6.11]) Let Γ be a subgroup of $SL_2(\mathbb{Z})$ of finite index and let $f \neq 0$ be a modular form of weight k for Γ. We have

$$\sum_{z \in \Gamma \backslash \overline{\mathbb{H}}} \frac{\nu_z(f)}{e_z} = \frac{k}{12}[\overline{SL_2(\mathbb{Z})} : \overline{\Gamma}],$$

where $e_z = 2$ or 3 if z is $SL_2(\mathbb{Z})$ equivalent to i or $e^{2\pi i/3}$ respectively, and $e_z = 1$ otherwise. Here $\overline{SL_2(\mathbb{Z})} = SL_2(\mathbb{Z})/\{\pm 1\}$ and $\overline{\Gamma} = \Gamma/\Gamma \cap \{\pm 1\}$.

The theorem mentioned above works for modular forms on a finite index subgroup with the trivial character. Although we have that $F_D(z) \in M_1\left(\Gamma_0(D), \left(\frac{-D}{d}\right)\right)$, we can still use the valence formula on $F_D(z)$ by considering it as a modular form on $\Gamma_1(N)$. To do so, we will have to look at the order of vanishing of $F_D(z)$ at all the cusps of $\Gamma_1(D)$. For D prime, $\Gamma_1(D)$ has $D-1$ inequivalent cusps (See [7, Page 102]). The next lemma describes the order of vanishing of $F_D(z)$ at these cusps.

Lemma 2.14. *The order of vanishing of $F_D(z)$ at every cusps of $\Gamma_1(D)$ is 1.*

Proof. The Group $\Gamma_1(D)$ has $D-1$ inequivalent cusps. Using the computation as in [7, Page 102]), a set of representatives for the inequivalent cusps can be given by $S_1 \cup S_2$, where

$$S_1 = \left\{ i\infty, \frac{2}{D}, \frac{3}{D}, \cdots, \frac{(D-1)/2}{D} \right\}$$

and

$$S_2 = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{(D-1)/2} \right\}.$$

Observe that any two cusps in S_1 or S_2 are $\Gamma_0(D)$ equivalent. This is true as you can find a matrix in $\Gamma_0(D)$ which takes one cusp to the other. Next, using [5, Proposition 6.3.20] to compute the widths of the cusps in $\Gamma_1(D)$, we see that any cusp in S_1 has width 1 and any cusp in S_2 has width D. Hence, two cusps equivalent over $\Gamma_0(D)$ have the same width in $\Gamma_1(D)$. Now suppose x_1 and x_2 are two inequivaent cusps of $\Gamma_1(D)$ that are $\Gamma_0(D)$ equivalent. Let w be their width. Then, from the proof of [10, Proposition 16], the smallest exponents in the q_D expansion of $F_D(z)$ at the cusps x_1 and x_2 are the same. Since the widths of x_1 and x_2 are the same and equal to w, the smallest exponent with non-zero coefficients in q_w expansion of $F_D(z)$ at the cusps x_1 and x_2 are the same. Hence $F_D(z)$ has the same order of vanishing in $\Gamma_1(D)$ at the cusps x_1 and x_2. Now from Proposition 2.12, the order of vanishing of $i\infty$ and $1/1$ in $\Gamma_1(D)$ is equal to 1. Since every cusp of $\Gamma_1(D)$ is $\Gamma_0(D)$ equivalent to either $i\infty$ or $1/1$, we are done. □

Now we are ready to state the main result of the section.

Theorem 2.15. *Let $D \neq 23$ be such that $\mathbb{Q}\left(\sqrt{-D}\right)$ has class number 3. Let Q_1, Q_2, Q_3 be the three reduced forms of discriminant $-D$, where Q_1 is the principal form. Let*

$$F_D(z) = \frac{1}{2}\left(\sum_{a,b\in\mathbb{Z}} q^{Q_1(a,b)} - \sum_{a,b\in\mathbb{Z}} q^{Q_2(a,b)} \right).$$

Then the integers $c(n)$'s in the expansion

$$F_D(z) = q\prod_{n=1}^{\infty}(1-q^n)^{c(n)}$$

are unbounded.

Proof. Considering $F_D(z)$ as a modular form on $\Gamma_1(N)$, by Theorem 2.13, we have

$$\sum_{z\in\Gamma\backslash\overline{\mathbb{H}}} \frac{\nu_z(F_D)}{e_z} = \frac{1}{12}[\overline{SL_2(\mathbb{Z})} : \overline{\Gamma_1(D)}].$$

By [5, Corollary 6.2.13], we have

$$[\overline{SL_2(\mathbb{Z})} : \overline{\Gamma_1(D)}] = \frac{D^2-1}{2}.$$

Hence, we have

$$\sum_{z\in\Gamma\backslash\overline{\mathbb{H}}} \frac{\nu_z(F_D)}{e_z} = \frac{D^2-1}{24}.$$

From Lemma 2.14, $\nu_z(F_D) = 1$ for each of the $D-1$ cusps of $\Gamma_1(D)$. Hence,

$$\frac{1}{D-1}\sum_{z\in\Gamma\backslash\mathbb{H}} \frac{\nu_z(F_D)}{e_z} = \frac{D-23}{24}.$$

Therefore for all values of D in Table 2.1 except 23, the right hand side will be positive and the formula above will guarantee the existence of a zero in $\mathbb{H}$. Now applying Lemma 2.9 on $F_D(z)/q$ finishes the proof. □

Chapter 3

Formulas for Sums of Triangular Numbers

In this chapter, we study the number of representations of an integer as a sum of triangular numbers. First, we provide another proof of the main result in [1] to get formulas for $\delta_{4k}(n)$. Then, we extend the result to obtain formulas for $\delta_{4k+2}(n)$. We end the chapter by obtaining a basis for $S_{2k}(\Gamma_0(4))$ and $S_{2k+1}(\Gamma_0(8),\chi_{-4})$ and provide new explicit formulas for $\delta_k(n)$ for $k = 14, 16$ and 18.

3.1 An explicit formula for $\delta_{4k}(n)$

We start this section by some definitions. The k^{th} Bernoulli number B_k is defined to be $k!$ times the k^{th} coefficient in the Laurent expansion of $t/(e^t-1)$. That is,

$$\sum_{k=0}^{\infty} B_k \cdot \frac{t^k}{k!} = \frac{t}{e^t-1} = 1 - \frac{1}{2}t + \frac{1}{12}t^2 - \cdots .$$

For $k > 1$, the Eisenstein series of weight $2k$ for $\Gamma_0(1)$ is given by

$$E_{2k}(z) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n. \tag{3.1}$$

Let $\varepsilon_\infty(\Gamma_0(N))$ be the number of inequivalent cusps of $\Gamma_0(N)$. We denote the weight $2k$ Eisenstein space for $\Gamma_0(N)$ with the trivial character by $E_{2k}(\Gamma_0(N))$. Then, for $k \geq 2$, the dimension of the Eisenstein space $E_{2k}(\Gamma_0(N))$ is given by

$$\dim(E_{2k}(\Gamma_0(N))) = \varepsilon_\infty(\Gamma_0(N)) = \sum_{d|N} \phi(\gcd(d, N/d)) \tag{3.2}$$

(See [7, Page 103 and Page 111, equation (4.3)] for a proof). As a consequence, $\Gamma_0(4)$ has 3 inequivalent cusps. The complete set of inequivalent cusps is given by $\{1, \frac{1}{2}, i\infty\}$. By looking at the expansions of $E_{2k}(z), E_{2k}(2z)$ and $E_{2k}(4z)$ using (3.1), we see that $E_{2k}(z), E_{2k}(2z)$ and $E_{2k}(4z)$ are linearly independent vectors in $E_{2k}(\Gamma_0(4))$. Since the dimension of $E_{2k}(\Gamma_0(4))$ is 3, the Eisenstein space $E_{2k}(\Gamma_0(4))$ has the basis

$$\mathbb{B} = \{E_{2k}(z), E_{2k}(2z), E_{2k}(4z)\}. \tag{3.3}$$

Next we are interested in finding the constants associated to the Fourier expansions of the Eisenstein series at each cusp. Recall that for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and an integer k, the weight k operator $[\gamma]_k$ on functions $f : \mathbb{H} \to \mathbb{C}$ is defined as

$$(f[\gamma]_k)(z) = (cz+d)^{-k} f(\gamma(z)).$$

Suppose that f is a modular form of weight k. Pick a cusp α and a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ such that $\gamma\alpha = \infty$. Then, the Fourier expansion of f at the cusp α is given by the Fourier expansion of

$$f[\gamma^{-1}]_k(z)$$

at $i\infty$. Let $[0]_c(f)$ be the constant in the Fourier expansion of $f(z)$ at the cusp $1/c$. We will denote the constant in the Fourier expansion of f at $i\infty$ by $[0]_0(f)$. Then, we have the following proposition.

Proposition 3.1. *For $k > 1$, the values in the following table are established to be true.*

cusp	$[0]_c\,(E_{2k}(z))$	$[0]_c\,(E_{2k}(2z))$	$[0]_c\,(E_{2k}(4z))$
1	1	$\left(\frac{1}{2}\right)^{2k}$	$\left(\frac{1}{4}\right)^{2k}$
$\frac{1}{2}$	1	1	$\left(\frac{1}{2}\right)^{2k}$
$i\infty$	1	1	1

Proof. We use [3, Theorem 6.2.1] to get

$$[0]_c\,(E_{2k}(dz)) = \left[\frac{\gcd(d,c)}{d}\right]^{2k},$$

where $\gcd(d,c)$ is the greatest common divisor of d and c. Plugging in different values of c and d, we get the table (Note that for the cusp $i\infty$, $c = 0$). □

Now we describe first the *multiplier system* for the eta function and then a proposition [11, Proposition 2.1] that gives the Fourier expansion of $\eta(mz)$ at different cusps. The eta product $f(z) = \prod_{m|N}\eta(mz)^{a_m}$ behaves like a modular form of weight $k = (\sum a_m)/2$ with some multiplier system on $\Gamma_0(N)$. That is, for every $L \in \Gamma_0(N)$,we have

$$f(Lz) = f\left(\frac{az+b}{cz+d}\right) = v_f(L)(cz+d)^k f(z),$$

where $v_f(L)$ is a 24^{th} root of unity. See [11, Chapter 1] for more details. The next proposition gives the values of the multiplier system explicitly.

Proposition 3.2. ([11, Theorem 1.7]) *For $L = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, the multiplier system for the eta function is given by*

$$v_\eta(L) = \left(\frac{d}{c}\right)^* e\left(\frac{1}{24}\left((a+d)c - bd\left(c^2-1\right) - 3c\right)\right) \qquad \textit{if } c \textit{ is odd},$$

$$v_\eta(L) = \left(\frac{c}{d}\right)_* e\left(\frac{1}{24}\left((a+d)c - bd\left(c^2-1\right) + 3d - 3 - 3cd\right)\right) \qquad \textit{if } c \textit{ is even},$$

where

$$\left(\frac{d}{c}\right)^* = \left(\frac{d}{|c|}\right)$$

and

$$\left(\frac{c}{d}\right)_* = \left(\frac{c}{|d|}\right) \cdot (-1)^{\frac{1}{4}(\text{sgn}(c)-1)(\text{sgn}(d)-1)}$$

for $\text{sgn}(x) = \frac{x}{|x|}$, $\left(\frac{a}{b}\right)$ *being the Kronecker symbol and* $e(z) = e^{2\pi iz}$.

Proposition 3.3. ([11, Proposition 2.1]) *Let* $f_m(z) = \eta(mz)$ *with* $m \in \mathbb{N}$, *and let* $r = \frac{-d}{c} \in \mathbb{Q}$ *be a reduced fraction with* $c \neq 0$. *Let* a, b *be chosen such that* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. *Then*

$$\begin{aligned} f_m\left(A^{-1}z\right) =& v_\eta(L)\left(\frac{\gcd(c,m)}{m}(-cz+a)\right)^{1/2} \\ &\times \sum_{n=1}^{\infty}\left(\frac{12}{n}\right) e\left(\frac{n^2}{24m}\left((\gcd(c,m))^2 z + \nu \gcd(c,m)\right)\right), \end{aligned}$$

where $L = \left(\begin{smallmatrix} x & y \\ u & v \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, $x = \frac{md}{\gcd(c,m)}$, $u = \frac{-c}{\gcd(c,m)}$, *and* $\nu = -mbv - ya$.

We use the transformation formula for the eta function to find the constants in the Fourier expansion of

$$F = q\Psi^4(q^2) = \frac{\eta^8(4\tau)}{\eta^4(2\tau)} \tag{3.4}$$

at different cusps.

Proposition 3.4. *We have*

$$[0]_1(F) = -\tfrac{1}{64},$$

$$[0]_2(F) = \tfrac{1}{16},$$

$$[0]_0(F) = 0.$$

Proof. The idea of the proof comes from [3, Section 6.1]. From (3.4), we see that the constant coefficient of F at the cusp $i\infty$ is 0. For the other two constants, we use Proposition 3.3 to find the Fourier expansions of the eta functions $\eta_t(z) = \eta(tz)$ for $t = 2, 4$ at the cusps 1 and $1/2$.

For $c \in \mathbb{Z}$, define

$$A_c = \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix}.$$

Then the matrix A_c^{-1} takes the cusp $i\infty$ to the cusp $1/c$.

Consider the cusp $1/1$. Taking

$$L_2 = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } L_4 = \begin{pmatrix} -4 & 1 \\ -1 & 0 \end{pmatrix},$$

Proposition 3.2 yields $v_\eta(L_2) = e^{5\pi i/12}$ and $v_\eta(L_4) = e^{7\pi i/12}$. Observe that for L_2 and L_4, $\nu = 1$. Using Proposition 3.3, we get

$$\eta_2\left(A_1^{-1}z\right) = \frac{e^{5\pi i/12}}{2^{1/2}}(-z-1)^{1/2}\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{48}(z+1)\right)$$

and

$$\eta_4\left(A_1^{-1}z\right) = \frac{e^{7\pi i/12}}{2}(-z-1)^{1/2}\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{96}(z+1)\right).$$

Hence, we have

$$\eta_2^4\left(A_1^{-1}z\right) = \frac{e^{5\pi i/3}}{2^2}(z+1)^2\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{48}(z+1)\right)\right)^4$$

and

$$\eta_4^8\left(A_1^{-1}z\right) = \frac{e^{2\pi i/3}}{2^8}(z+1)^4\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{96}(z+1)\right)\right)^8.$$

Thus,

$$F\left(A_1^{-1}z\right) = \frac{\eta_4^8\left(A_1^{-1}z\right)}{\eta_2^4\left(A_1^{-1}z\right)} = \frac{-1}{64}(z+1)^2\frac{\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{96}(z+1)\right)\right)^8}{\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{48}(z+1)\right)\right)^4},$$

i.e.,

$$(F[A_1^{-1}]_2)(z) = (z+1)^{-2}\frac{\eta_4^8\left(A_1^{-1}z\right)}{\eta_2^4\left(A_1^{-1}z\right)} = \frac{-1}{64}\frac{\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{96}(z+1)\right)\right)^8}{\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{48}(z+1)\right)\right)^4}. \tag{3.5}$$

Next consider the cusp $1/2$. Taking

$$L_2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } L_4 = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix},$$

Proposition 3.2 yields $v_\eta(L_2) = e^{\pi i/3}$ and $v_\eta(L_4) = e^{5\pi i/12}$. Also $\nu = 1$ for L_2 and L_4. Using Proposition 3.3, we get

$$\eta_2\left(A_2^{-1}z\right) = e^{\pi i/3}(-2z-1)^{1/2}\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{48}(4z+2)\right)$$

and

$$\eta_4\left(A_2^{-1}z\right) = \frac{e^{5\pi i/12}}{2^{1/2}}(-2z-1)^{1/2}\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{96}(4z+2)\right).$$

Hence, we have

$$\eta_2^4\left(A_2^{-1}z\right) = e^{4\pi i/3}(2z+1)^2\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{48}(4z+2)\right)\right)^4$$

and

$$\eta_4^8\left(A_2^{-1}z\right) = \frac{e^{4\pi i/3}}{2^4}(2z+1)^4\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{96}(4z+2)\right)\right)^8.$$

Thus,

$$F\left(A_2^{-1}z\right) = \frac{\eta_4^8\left(A_2^{-1}z\right)}{\eta_2^4\left(A_2^{-1}z\right)} = \frac{1}{16}(2z+1)^2\frac{\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{96}(4z+2)\right)\right)^8}{\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{48}(4z+2)\right)\right)^4},$$

i.e.,

$$(F[A_2^{-1}]_2)(z) = (2z+1)^{-2}\frac{\eta_4^8\left(A_2^{-1}z\right)}{\eta_2^4\left(A_2^{-1}z\right)} = \frac{1}{16}\frac{\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{96}(4z+2)\right)\right)^8}{\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{48}(4z+2)\right)\right)^4}. \tag{3.6}$$

Looking at the constant coefficients in the Fourier expansions (3.5) and (3.6), we are done. □

The following is an immediate corollary of Proposition 3.4.

Corollary 3.5. *The constants in the Fourier expansion of $F^k = q^k\Psi^{4k}(q^2)$ at the 3 inequivalent cusps of $\Gamma_0(4)$ are*

$$[0]_1\left(F^k\right) = \left(-\tfrac{1}{64}\right)^k,$$
$$[0]_2\left(F^k\right) = \left(\tfrac{1}{16}\right)^k,$$
$$[0]_0\left(F^k\right) = 0.$$

Before we give our alternate proof of [1, Theorem 2.5], recall that χ_0 and χ_1 are principal Dirichlet characters mod 4 and mod 1 respectively and the generalised divisor function for Dirichlet characters ψ_1 and ψ_2 is given by

$$\sigma_{k,\psi_1,\psi_2}(n) = \sum_{d|n,d>0}\psi_1(d)\psi_2(n/d)d^k.$$

Theorem 3.6 and Theorem 3.12 can be considered as special cases of [2, Theorem 1.1]. Here we give detailed proofs using ideas from [2].

Theorem 3.6. *For any integer $k\geq 2$, we have*

$$\delta_{4k}(n) = \frac{1}{d_k}\sigma_{2k-1,\chi_1,\chi_0}(2n+k) + c(2n+k), \tag{3.7}$$

where

$$\sum_{n=1}^{\infty} c(n)q^n \in \mathrm{S}_{2k}(\Gamma_0(4))$$

is a cusp form and

$$d_k = -\frac{(-16)^k\left(4^k-1\right)B_{2k}}{8k} \tag{3.8}$$

in which B_{2k} is the $2k^{th}$ Bernoulli number.

Proof. Using Theorem 1.2, we have that

$$F^k(z) = q^k\Psi^{4k}(q^2) \in M_{2k}(\Gamma_0(4)).$$

So we can write $F^k(z)$ as

$$F^k(z) = E(z) + S(z),$$

where $E(z) \in E_{2k}(\Gamma_0(4))$ and $S(z) \in S_{2k}(\Gamma_0(4))$. The Eisenstein form $E(z)$ can be written as a linear combination of the Eisenstein basis (3.3). Hence,

$$F^k(z) = aE_{2k}(z) + bE_{2k}(2z) + cE_{2k}(4z) + S(z) \tag{3.9}$$

for some complex numbers a, b, c. Observing that the cusp form $S(z)$ vanishes at every cusp of $\Gamma_0(4)$, we compare the constants in the Fourier expansions of the LHS and the RHS of (3.9) at different cusps. Using Proposition 3.1 and Corollary 3.5, we get the following system of equations.

$$\begin{cases} a + \left(\frac{1}{2}\right)^{2k} b + \left(\frac{1}{4}\right)^{2k} c = \left(-\frac{1}{64}\right)^k, \\ a + b + \left(\frac{1}{2}\right)^{2k} c = \left(\frac{1}{16}\right)^k, \\ a + b + c = 0. \end{cases}$$

For k even, this system solves to

$$a = 0,\ b = \frac{1}{(2^{2k}-1)2^{2k}},\ \text{and } c = \frac{-1}{(2^{2k}-1)2^{2k}}.$$

For k odd, this system solves to

$$a = \frac{-2}{(4^k-1)2^{4k}},\ b = \frac{2^{2k}+1}{(2^{2k}-1)2^{4k}},\ \text{and } c = \frac{-1}{(2^{2k}-1)2^{2k}}.$$

For even k, we get

$$F^k(z) = \frac{1}{(2^{2k}-1)2^{2k}}E_{2k}(2z) + \frac{-1}{(2^{2k}-1)2^{2k}}E_{2k}(4z) + S(z).$$

Substituting the Fourier expansion of the Eisenstein series $E_{2k}(z)$ from (3.1), we get

$$q^k\Psi^{4k}(q^2) = \frac{-4k}{B_k\left(2^{2k}-1\right)2^{2k}}\left(\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^{2n} - \sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^{4n}\right) + S(z).$$

Next, we compare the coefficients of q^{2n+k}. If $4|2n+k$,

$$\delta_{4k}(n) = \frac{-4k}{B_k\left(2^{2k}-1\right)2^{2k}}\left(\sigma_{2k-1}\left(\frac{2n+k}{2}\right) - \sigma_{2k-1}\left(\frac{2n+k}{4}\right)\right) + c(2n+k).$$

Now, when $4 \mid 2n+k$,

$$\sigma_{2k-1}\left(\frac{2n+k}{2}\right) - \sigma_{2k-1}\left(\frac{2n+k}{4}\right) = \sigma_{2k-1,\chi_1,\chi_0}\left(\frac{2n+k}{2}\right) = \frac{1}{2^{2k-1}}\sigma_{2k-1,\chi_1,\chi_0}(2n+k).$$

Hence substituting this back gives

$$\delta_{4k}(n) = \frac{-4k}{B_k\left(2^{2k}-1\right)2^{4k-1}}\sigma_{2k-1,\chi_1,\chi_0}(2n+k) + c(2n+k).$$

If $4 \nmid 2n+k$, but $2 \mid 2n+k$,

$$\begin{aligned}\delta_{4k}(n) &= \frac{-4k}{B_k\left(2^{2k}-1\right)2^{2k}}\sigma_{2k-1}\left(\frac{2n+k}{2}\right) + c(2n+k)\\ &= \frac{-4k}{B_k\left(2^{2k}-1\right)2^{4k-1}}\sigma_{2k-1,\chi_1,\chi_0}(2n+k) + c(2n+k).\end{aligned}$$

For odd k, we get

$$F^k(z) = \frac{-2}{(4^k-1)2^{4k}}E_{2k}(z) + \frac{2^{2k}+1}{(2^{2k}-1)2^{4k}}E_{2k}(2z) + \frac{-1}{(2^{2k}-1)2^{2k}}E_{2k}(4z) + S(z).$$

Substituting the Fourier expansion of the Eisenstein series $E_{2k}(z)$ from (3.1), we get

$$\begin{aligned} q^k\Psi^{4k}(q^2) = &\frac{4k}{B_k(4^k-1)2^{4k}} + \frac{8k}{B_k(4^k-1)2^{4k}}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n \\ &- \frac{4k(2^{2k}+1)}{B_k(4^k-1)2^{4k}}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^{2n} + \frac{4k}{B_k(4^k-1)2^{2k}}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^{4n} + S(z). \end{aligned}$$

Comparing the Fourier coefficients of q^{2n+k}, we are done. □

3.2 An explicit formula for $\delta_{4k+2}(n)$

We next use the approach employed in the previous section to derive a formula for $\delta_{4k+2}(n)$. The main difference is that our generating function is a modular form of odd weight with a quadratic character. We start with a description of the generating function of $\delta_{4k+2}(n)$.

We saw that the Psi function has the product expression given by

$$\Psi(q) = q^{\frac{-1}{8}}\frac{\eta^2(q^2)}{\eta(q)}.$$

Hence,

$$q^{2k+1}\Psi^{4k+2}(q^4) = \frac{\eta^{8k+4}(q^8)}{\eta^{4k+2}(q^4)}. \tag{3.10}$$

Let $J(q) = q\psi^2(q^4)$. Then

$$J^{2k+1}(q) = q^{2k+1}\Psi^{4k+2}(q^4) = \sum\delta_{4k+2}(n)q^{4n+2k+1}. \tag{3.11}$$

Using Theorem 1.2 we see that

$$J^{2k+1}(q) \in M_{2k+1}(\Gamma_0(8), \chi_{-4}),$$

where χ_{-4} is the character mod 4 taking the values $\chi_{-4}(1) = 1$, and $\chi_{-4}(3) = -1$.

Next, we find out the constants in the Fourier expansion of $J^{2k+1}(q)$ at different cusps. We obtain these values by using Proposition 3.3. From (3.2), we see that $\Gamma_0(8)$ has 4 inequivalent cusps. A complete set is given by $C_{\Gamma_0(8)} = \{i\infty, 1, \frac{1}{2}, \frac{1}{4}\}$.

Proposition 3.7. *The constants in the Fourier expansion of $J^{2k+1}(q)$ are*

$$\begin{aligned}
[0]_1 \left(J^{2k+1}\right) &= \tfrac{i(-1)^{k+1}}{2^{8k+4}}, \\
[0]_2 \left(J^{2k+1}\right) &= \tfrac{i(-1)^{k}}{2^{6k+3}}, \\
[0]_4 \left(J^{2k+1}\right) &= \tfrac{1}{2^{4k+2}}, \\
[0]_0 \left(J^{2k+1}\right) &= 0.
\end{aligned}$$

Proof. From (3.11), we see that the constant coefficient at the cusp $i\infty$ is 0. For the other three constants, we find the Fourier expansions of $\eta_t(z) = \eta(tz)$ for $t = 4$ and 8 at the cusps 1, $1/2$ and $1/4$ using the same methodology as in Proposition 3.4. For $c \in \mathbb{Z}$, define

$$A_c = \begin{pmatrix} -1 & 0 \\ c & -1 \end{pmatrix}.$$

At the cusp $1/1$, we get

$$\eta_8\left(A_1^{-1}z\right) = \frac{e^{11\pi i/12}}{8^{1/2}}(-z-1)^{1/2}\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{192}(z+1)\right)\right)$$

and

$$\eta_4\left(A_1^{-1}z\right) = \frac{e^{7\pi i/12}}{2}(-z-1)^{1/2}\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{96}(z+1)\right)\right).$$

Hence, we have

$$\eta_8^{8k+4}\left(A_1^{-1}z\right) = \frac{e^{11\pi i(8k+4)/12}}{8^{4k+2}}(z+1)^{4k+2}\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{192}(z+1)\right)\right)^{8k+4},$$

$$\eta_4^{4k+2}\left(A_1^{-1}z\right) = \frac{e^{7\pi i(4k+2)/12}}{2^{4k+2}}(-1)(z+1)^{2k+1}\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{96}(z+1)\right)\right)^{4k+2},$$

and

$$J^{2k+1}\left(A_1^{-1}z\right) = \frac{\eta_8^{8k+4}\left(A_1^{-1}z\right)}{\eta_4^{4k+2}\left(A_1^{-1}z\right)} = \frac{i(-1)^{k+1}}{2^{8k+4}}(z+1)^{2k+1}\frac{\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{192}(z+1)\right)\right)^{8k+4}}{\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{96}(z+1)\right)\right)^{4k+2}},$$

i.e.,

$$(J^{2k+1}[A_1^{-1}]_{2k+1})(z) = (z+1)^{-2k-1}\frac{\eta_8^{8k+4}\left(A_1^{-1}z\right)}{\eta_4^{4k+2}\left(A_1^{-1}z\right)} = \frac{i(-1)^{k+1}}{2^{8k+4}}\frac{\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{192}(z+1)\right)\right)^{8k+4}}{\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{96}(z+1)\right)\right)^{4k+2}}. \tag{3.12}$$

At the cusp $1/2$, we get

$$\eta_8\left(A_2^{-1}z\right) = \frac{e^{7\pi i/12}}{4^{1/2}}(-2z-1)^{1/2}\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{192}(4z+2)\right)\right)$$

and

$$\eta_4\left(A_2^{-1}z\right) = \frac{e^{5\pi i/12}}{2^{1/2}}(-2z-1)^{1/2}\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{96}(4z+2)\right)\right).$$

Hence, we have

$$\eta_8^{8k+4}\left(A_2^{-1}z\right) = \frac{e^{7\pi i(8k+4)/12}}{2^{8k+4}}(2z+1)^{4k+2}\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{192}(4z+2)\right)\right)^{8k+4},$$

$$\eta_4^{4k+2}\left(A_2^{-1}z\right) = \frac{e^{5\pi i(4k+2)/12}}{2^{2k+1}}(-1)(2z+1)^{2k+1}\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{96}(4z+2)\right)\right)^{4k+2},$$

and

$$J^{2k+1}\left(A_2^{-1}z\right)=\frac{\eta_8^{8k+4}\left(A_2^{-1}z\right)}{\eta_4^{4k+2}\left(A_2^{-1}z\right)}=\frac{i(-1)^k}{2^{6k+3}}(2z+1)^{2k+1}\frac{\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{192}(4z+2)\right)\right)^{8k+4}}{\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{96}(4z+2)\right)\right)^{4k+2}},$$

i.e.,

$$(J^{2k+1}[A_2^{-1}]_{2k+1})(z)=(2z+1)^{-2k-1}\frac{\eta_8^{8k+4}\left(A_2^{-1}z\right)}{\eta_4^{4k+2}\left(A_2^{-1}z\right)}=\frac{i(-1)^k}{2^{6k+3}}\frac{\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{192}(4z+2)\right)\right)^{8k+4}}{\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{96}(4z+2)\right)\right)^{4k+2}}. \tag{3.13}$$

At the cusp $1/4$, we get

$$\eta_8\left(A_4^{-1}z\right)=\frac{e^{5\pi i/12}}{2^{1/2}}(-4z-1)^{1/2}\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{192}(16z+4)\right)\right)$$

and

$$\eta_4\left(A_4^{-1}z\right)=e^{4\pi i/12}(-4z-1)^{1/2}\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{96}(16z+4)\right)\right).$$

Hence, we have

$$\eta_8^{8k+4}\left(A_4^{-1}z\right)=\frac{e^{5\pi i(8k+4)/12}}{2^{4k+2}}(4z+1)^{4k+2}\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{192}(16z+4)\right)\right)^{8k+4},$$

$$\eta_4^{4k+2}\left(A_4^{-1}z\right)=-e^{4\pi i(4k+2)/12}(4z+1)^{2k+1}\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{96}(16z+4)\right)\right)^{4k+2},$$

and

$$J^{2k+1}\left(A_4^{-1}z\right)=\frac{\eta_8^{8k+4}\left(A_4^{-1}z\right)}{\eta_4^{4k+2}\left(A_4^{-1}z\right)}=\frac{1}{2^{4k+2}}(4z+1)^{2k+1}\frac{\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{192}(16z+4)\right)\right)^{8k+4}}{\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{96}(16z+4)\right)\right)^{4k+2}},$$

i.e.,

$$(J^{2k+1}[A_4^{-1}]_{2k+1})(z) = (4z+1)^{-2k-1}\frac{\eta_8^{8k+4}\left(A_4^{-1}z\right)}{\eta_4^{4k+2}\left(A_4^{-1}z\right)} = \frac{1}{2^{4k+2}}\frac{\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{192}(16z+4)\right)\right)^{8k+4}}{\left(\sum_{n\geq 1}\left(\frac{12}{n}\right)e\left(\frac{n^2}{96}(16z+4)\right)\right)^{4k+2}}. \tag{3.14}$$

Looking at the constant coefficients in the Fourier expansions (3.12), (3.13) and (3.14), we get the result. □

We next need a basis for the Eisenstein space $E_{2k+1}(\Gamma_0(8),\chi_{-4})$. We follow the construction given in [5, Section 8.5]. Let ψ_1 and ψ_2 be Dirichlet characters mod N_1 and N_2 respectively. set $\psi = \psi_1\psi_2$ viewed as a character mod N, where $N = N_1N_2$. For an integer $k > 2$, define

$$G_{k,\psi_1,\psi_2}(z) = \frac{1}{2}\sum_{\substack{N_1|c \\ (c,d)\neq(0,0)}}\frac{\overline{\psi_1}(d)\psi_2(c/N_1)}{(cz+d)^k}.$$

From [5, Corollary 8.5.5], we see that when ψ_1 is primitive, $G_{k,\psi_1,\psi_2}(z) \in M_k(\Gamma_0(N),\psi)$ and has the Fourier expansion given by

$$G_{k,\psi_1,\psi_2}(z) = \delta_{N_2,1}L(k,\overline{\psi_1}) + \left(\frac{-2\pi i}{N_1}\right)^k\frac{\mathfrak{g}(\overline{\psi_1})}{(k-1)!}\sum_{n\geq 1}\sigma_{k-1,\psi_1,\psi_2}(n)q^n, \tag{3.15}$$

where $\mathfrak{g}$ is the Gauss sum $\mathfrak{g}(\overline{\psi_1}) = \sum_{r \bmod N_1}\overline{\psi_1(r)}e^{2\pi ir/N_1}$, $\sigma_{t,\psi_1,\psi_2}(n)$ is the generalised divisor function, $\delta_{a,b} = 1$ if $a = b$ and 0 otherwise, and $L(k,\overline{\psi_1})$ is the value of the Dirichlet L-function associated with $\overline{\psi_1}$ at integer k. A basis for $E_k(\Gamma_0(N),\psi)$ can be constructed with the help of the following Theorem.

Theorem 3.8. ([5, Theorem 8.5.17]) *Let $k > 2$ and ψ be a character modulo N such that $\psi(-1) = (-1)^k$. A basis for the subspace $E_k(\Gamma_0(N),\psi)$ is given by the Eisenstein Series $G_{k,\psi_1,(\psi\psi_1^{-1})_f}(dz)$, where ψ_1 ranges through all primitive characters and d through positive integers such that $d\mathfrak{f}(\psi_1)\mathfrak{f}((\psi\psi_1^{-1})_f) \mid N$. Here, $\mathfrak{f}(\psi)$ is the conductor of ψ and $(\psi)_f$ is the primitive character equivalent to ψ.*

Using Theorem 3.8, we see that the set

$$\mathbb{B} = \{G_{2k+1,\chi_1,\chi_{-4}}(z), G_{2k+1,\chi_{-4},\chi_1}(z), G_{2k+1,\chi_1,\chi_{-4}}(2z), G_{2k+1,\chi_{-4},\chi_1}(2z)\}$$

forms a basis for $E_{2k+1}(\Gamma_0(8),\chi_{-4})$, where χ_1 is the trivial Dirichlet character of conductor 1. Also, given a Dirichlet character χ of modulus M, the generalised Bernoulli number $B_{n,\chi}$ is defined by the relation

$$\sum_{a=1}^{M} \frac{\chi(a)te^{at}}{e^{Mt}-1} = \sum_{n=0}^{\infty} \frac{B_{n,\chi}}{n!} t^n. \tag{3.16}$$

Now, computing the Fourier expansions of the functions in $\mathbb{B}$ at $i\infty$ using (3.15) yields,

$$G_{2k+1,\chi_1,\chi_{-4}}(z) = \frac{(-2\pi i)^{2k+1}}{(2k)!} \sum_{n\geq 1} \sigma_{2k,\chi_1,\chi_{-4}}(n) q^n,$$

$$G_{2k+1,\chi_1,\chi_{-4}}(z) = \frac{(-2\pi i)^{2k+1}}{(2k)!} \sum_{n\geq 1} \sigma_{2k,\chi_1,\chi_{-4}}(n) q^{2n},$$

$$G_{2k+1,\chi_{-4},\chi_1}(z) = L(2k+1,\chi_{-4}) + \left(\frac{-2\pi i}{4}\right)^{2k+1} \frac{2i}{(2k)!} \sum_{n\geq 1} \sigma_{2k,\chi_{-4},\chi_1}(n) q^n,$$

and

$$G_{2k+1,\chi_{-4},\chi_1}(z) = L(2k+1,\chi_{-4}) + \left(\frac{-2\pi i}{4}\right)^{2k+1} \frac{2i}{(2k)!} \sum_{n\geq 1} \sigma_{2k,\chi_{-4},\chi_1}(n) q^n,$$

where the value of the Dirichlet L-function $L(2k+1,\chi_{-4})$ can be computed, using [5, Theorem 3.4.14] as

$$L(2k+1,\chi_{-4}) = \frac{(-1)^{k-1}(2\pi)^{2k+1} B_{2k+1,\chi_{-4}}}{2^{4k+2}(2k+1)!}. \tag{3.17}$$

Next, define the normalised Eisenstein series $E_{2k+1,\psi_1,\psi_2}(z)$ by

$$E_{2k+1,\psi_1,\psi_2}(z) = \frac{1}{L(2k+1,\chi_{-4})} G_{2k+1,\psi_1,\psi_2}(z).$$

Then, as a corollary of Theorem 3.8, we have a basis for $E_{2k+1}(\Gamma_0(8),\chi_{-4})$ given by

$$\mathbb{B}_1 = \{E_{2k+1,\chi_1,\chi_{-4}}(z), E_{2k+1,\chi_{-4},\chi_1}(z), E_{2k+1,\chi_1,\chi_{-4}}(2z), E_{2k+1,\chi_{-4},\chi_1}(2z)\}. \tag{3.18}$$

We now describe a result given in [5] that will be used to compute the constants associated to the Fourier expansions of the functions in $\mathbb{B}_1$ at different cusps of $\Gamma_0(8)$.

Theorem 3.9. ([5, Proposition 8.5.6]) *Let $k \geq 3$, and let ψ_1, ψ_2 be two primitive characters such that $\psi = \psi_1\psi_2$ satisfies $\psi(-1) = (-1)^k$. The value $V(A/C)$ of $G_{k,\psi_1,\psi_2}(ez)$ at a cusp A/C with $\gcd(A,C) = 1$ is given by*

$$V(A/C) = \begin{cases} 0 & \text{if } N_1 \nmid C, \\ (e/h)^{-k}\overline{\psi_1}((e/h)A)\psi_2(-(C/N_1)/h)L(k,\overline{\psi_1}\psi_2) & \text{if } N_1 \mid C, \end{cases}$$

where $h = \gcd(e, C/N_1)$.

As a direct corollary of this theorem, we have

Corollary 3.10. *The value $V(A/C)$ of $E_{k,\psi_1,\psi_2}(ez)$ at a cusp A/C with $\gcd(A,C) = 1$ is given by*

$$V(A/C) = \frac{1}{L(2k+1,\chi_{-4})}\begin{cases} 0 & \text{if } N_1 \nmid C, \\ (e/h)^{-k}\overline{\psi_1}((e/h)A)\psi_2(-(C/N_1)/h)L(k,\overline{\psi_1}\psi_2) & \text{if } N_1 \mid C. \end{cases}$$

Using the corollary stated above, we have the following proposition.

Proposition 3.11. *The values in the following table are valid.*

cusp $\frac{1}{c}$	$[0]_c\left(E_{2k+1,\chi_1,\chi_{-4}}(z)\right)$	$[0]_c\left(E_{2k+1,\chi_1,\chi_{-4}}(2z)\right)$	$[0]_c\left(E_{2k+1,\chi_{-4},\chi_1}(z)\right)$	$[0]_c\left(E_{2k+1,\chi_{-4},\chi_1}(2z)\right)$
1	-1	$-\left(\frac{1}{2}\right)^{2k+1}$	0	0
$\frac{1}{2}$	0	-1	0	0
$\frac{1}{4}$	0	0	1	0
$i\infty$	0	0	1	1

We are now ready to state the main result of this section.

Theorem 3.12. *For any positive k, we have*

$$\delta_{4k+2}(n) = \frac{-(2k+1)}{2^{4k}B_{2k+1,\chi_{-4}}}\sigma_{2k,\chi_{-4},\chi_1}(4n+2k+1) + t(4n+2k+1),$$

where

$$\sum_{n=1}^{\infty} t(n)q^n \in S_{2k+1}(\Gamma_0(8), \chi_{-4})$$

is a cusp form and $B_{n,\chi}$ is the n^{th} generalised Bernoulli number associated to χ.

Proof. Using Theorem 1.2, we have that

$$J^{2k+1}(z) = q^{2k+1}\Psi^{4k+2}(q^4) \in M_{2k+1}(\Gamma_0(8), \chi_{-4}).$$

So we can write $J^{2k+1}(z)$ as

$$J^{2k+1}(z) = E(z) + S(z),$$

where $E(z) \in E_{2k+1}(\Gamma_0(8), \chi_{-4})$ and $S(z) \in S_{2k+1}(\Gamma_0(8), \chi_{-4})$. The Eisenstein form $E(z)$ can be written as a linear combination of the Eisenstein basis $\mathbb{B}_1$ given in (3.18). Thus, we have

$$J^{2k+1}(z) = aE_{2k+1,\chi_1,\chi_{-4}}(z) + bE_{2k+1,\chi_1,\chi_{-4}}(2z) + \\ cE_{2k+1,\chi_{-4},\chi_1}(z) + dE_{2k+1\chi_{-4},\chi_1}(2z) + S(z) \quad (3.19)$$

for some complex numbers a, b, c, and d.

Next we observe that $S(z)$ vanishes at each cusp of $\Gamma_0(8)$. Using Proposition 3.7 and Proposition 3.11, we compare the constants in the Fourier expansions of the LHS and the

RHS of (3.19) at different cusps to get the following system of equations:

$$\begin{cases} c+d=0, \\ -a-\left(\frac{1}{2}\right)^{2k+1}b=\frac{i(-1)^{k+1}}{2^{8k+4}}, \\ -b=\frac{i(-1)^k}{2^{6k+3}}, \\ c=\frac{1}{2^{4k+2}}. \end{cases}$$

The system has the solution $a=2i(-1)^k/2^{8k+4}, b=i(-1)^{k+1}/2^{6k+3}, c=1/2^{4k+2}$, and $d=-1/2^{4k+2}$. Comparing the coefficient of $q^{4n+2k+1}$ on both sides of (3.19), we have

$$\begin{aligned} \delta_{4k+2}(n) = a\frac{i2^{4k+2}(2k+1)\chi_{-4}(4n+2k+1)}{B_{2k+1,\chi_{-4}}}\sigma_{2k,\chi_{-4},\chi_1}(4n+2k+1) \\ +c\frac{(-1)2.(2k+1)}{B_{2k+1,\chi_{-4}}}\sigma_{2k,\chi_{-4},\chi_1}(4n+2k+1)+t(4n+2k+1), \end{aligned} \quad (3.20)$$

where we also used the fact that

$$\sigma_{2k,\chi_1,\chi_{-4}}(4n+2k+1)=\chi_{-4}(4n+2k+1)\sigma_{2k,\chi_{-4},\chi_1}(4n+2k+1).$$

Substituting the values of a and c in the above equation and employing the fact that $\chi_{-4}(4n+2k+1)=(-1)^k$, we deduce

$$\delta_{4k+2}(n)=\frac{-(2k+1)}{2^{4k}B_{2k+1,\chi_{-4}}}\sigma_{2k,\chi_{-4},\chi_1}(4n+2k+1)+t(4n+2k+1).$$

□

3.3 Bases for $S_{2k+1}(\Gamma_0(8),\chi_{-4})$ and $S_{2k}(\Gamma_0(4))$

In this section, inspired by the work of Aygin [3, Chapter 5], we obtain bases consisting of eta products for $S_{2k+1}(\Gamma_0(8),\chi_{-4})$ and $S_{2k}(\Gamma_0(4))$. We start by obtaining a basis for $S_{2k+1}(\Gamma_0(8),\chi_{-4})$. To do this, we first compute the dimension of the above mentioned space using the following result.

Proposition 3.13. ([5, Theorem 7.4.1]) *Let N and k be positive integers and let χ be a Dirichlet character modulo N with conductor $\mathfrak{f}(\chi)$ such that $\chi(-1)=(-1)^k$. For $k\geq 2$ set*

$$A_1 = \frac{k-1}{12}N\prod_{p|N}\left(1+\frac{1}{p}\right),$$
$$A_2 = \left(\frac{k-1}{3}-\left\lfloor\frac{k}{3}\right\rfloor\right)\sum_{\substack{x \bmod N\\ x^2+x+1\equiv 0 \bmod N}}\chi(x)+\left(\frac{k-1}{4}-\left\lfloor\frac{k}{4}\right\rfloor\right)\sum_{\substack{x \bmod N\\ x^2+1\equiv 0 \bmod N}}\chi(x),$$
$$A_3 = \frac{1}{2}\sum_{\substack{0<d|N\\ \gcd(d,N/d)|N/\mathfrak{f}(\chi)}}\phi(\gcd(d,N/d)),$$

and $A_4=1$ if $k=2$ and χ is trivial, otherwise $A_4=0$. Then

$$\dim(S_k(\Gamma_0(N))) = A_1-A_2-A_3+A_4.$$

As a consequence of the above theorem, we have the following assertion.

Corollary 3.14. *The cusp space $S_{2k+1}(\Gamma_0(8),\chi_{-4})$ has dimension $2k-2$.*

Since $S_{2k+1}(\Gamma_0(8),\chi_{-4})$ is $2k-2$ dimensional, we obtain a basis by constructing $2k-2$ eta quotients with distinct orders of vanishing at the cusp $i\infty$. The distinct orders of vanishing at $i\infty$ will force the linear independence of the eta products. If for each v, we can find an eta product with the order of vanishing equal to v for all $1\leq v\leq 2k-2$, we would be done.

To define an eta quotient of weight $2k+1$ and order of vanishing v at $i\infty$, we define it as a product of three factors. The first factor ensures its weight, the second factor ensures its order of vanishing at $i\infty$ and the third factor ensures that it is a cusp form. Let $C(2k+1,v,8,z)$ be defined as

$$C(2k+1,v,8,z) = (C_{(1,0)}(z))^{2k+1}(C_{(0,1)}(z))^{v}C_{(0,0)}(z),$$

where $C_{(a,b)}(z)$ is an eta quotient of weight a and order of vanishing b at $i\infty$. Let us look at the available options for $C_{(a,b)}(z)$. Suppose

$$C_{(a,b)}(z) = \eta^j(z)\eta^k(2z)\eta^l(4z)\eta^m(8z).$$

Then, by [14, Theorem 1.64], for $C_{(a,b)}$ to be in $S_a(\Gamma_0(8),\chi_{-4})$ and have order of vanishing b, we must have

$$j+k+l+m = 2a, \tag{3.21a}$$

$$j+2k+4l+8m = 24b, \tag{3.21b}$$

$$24|(8j+4k+2l+m), \tag{3.21c}$$

where the first condition is for the weight, the second for the order of vanishing, and the third condition is a necessary condition to make it an eta quotient. Some of the available options for $C_{(1,0)}(z)$ and $C_{(0,1)}(z)$ are listed in the first columns of Table 3.1 and Table 3.2 below. We now investigate when

$$C(2k+1,v,8,z) = (C_{(1,0)}(z))^{2k+1}(C_{(0,1)}(z))^v C_{(0,0)}(z),$$

where

$$C_{(1,0)}(z) = \eta^{j_1}(z)\eta^{k_1}(2z)\eta^{l_1}(4z)\eta^{m_1}(8z), \tag{3.22}$$

$$C_{(0,1)}(z) = \eta^{j_2}(z)\eta^{k_2}(2z)\eta^{l_2}(4z)\eta^{m_2}(8z), \tag{3.23}$$

$$C_{(0,0)}(z) = \eta^{j_3}(z)\eta^{k_3}(2z)\eta^{l_3}(4z)\eta^{m_3}(8z), \tag{3.24}$$

is a holomorphic cusp form. We state a result due to Kohler to aid us in the investigation.

Theorem 3.15 ([11], Corollary 2.3)**.** *An eta product $f(z) = \prod_{m|N}\eta(mz)^{a_m}$ is holomorphic* *if and only if the inequalities*

$$\sum_{m|N} \frac{(\gcd(c,m))^2}{m} a_m \geq 0$$

holds for all positive divisors c of N. It is a cusp form if and only if all the inequalities hold strictly.

Using the theorem stated above, we have the following four conditions in order to make $C(2k+1,v,8,z)$ a cusp form.

$$(2k+1)(8j_1+4k_1+2l_1+m_1)+v(8j_2+4k_2+2l_2+m_2)+(8j_3+4k_3+2l_3+m_3)>0, \tag{3.25a}$$

$$(2k+1)(2j_1+4k_1+2l_1+m_1)+v(2j_2+4k_2+2l_2+m_2)+(2j_3+4k_3+2l_3+m_3)>0, \tag{3.25b}$$

$$(2k+1)(j_1+2k_1+4l_1+2m_1)+v(j_2+2k_2+4l_2+2m_2)+(j_3+2k_3+4l_3+2m_3)>0, \tag{3.25c}$$

$$(2k+1)(j_1+2k_1+4l_1+8m_1)+v(j_2+2k_2+4l_2+8m_2)+(j_3+2k_3+4l_3+8m_3)>0. \tag{3.25d}$$

These conditions should hold irrespective of the values of k and v. The fact that $1 \leq v \leq 2k-2$ helps us in deciding what the eta products $C_{(1,0)}(z)$ and $C_{(0,1)}(z)$ should be. Using (3.21), we record a few options available to us for $C_{(1,0)}(z)$ and $C_{(0,1)}(z)$ in Table 3.1 and Table 3.2 respectively.

For the last product $C_{(0,0)}(z)$, (3.21) gives the system of linear equations

$$j_3+k_3+l_3+m_3=0,$$

$$j_3+2k_3+4l_3+8m_3=0,$$

and

$$8j_3+4k_3+2l_3+m_3=24t,$$

Table 3.1: Options for $C_{(1,0)}(z)$

(j_1,k_1,l_1,m_1)	$8j_1+4k_1+2l_1+m_1$	$2j_1+4k_1+2l_1+m_1$	$j_1+2k_1+4l_1+2m_1$	$j_1+2k_1+4l_1+8m_1$
(2,-11,17,-6)	0	-12	36	0
(0,-4,10,-4)	0	0	24	0
(-2,3,3,-2)	0	12	12	0
(-4,10,-4,0)	0	24	0	0
(-6,17,-11,2)	0	36	-12	0
(6,-9,7,-2)	24	-12	12	0
(4,-2,0,0)	24	0	0	0
(2,5,-7,2)	24	12	-12	0
(0,12,-14,4)	24	24	-24	0

Table 3.2: Options for $C_{(0,1)}(z)$

(j_2,k_2,l_2,m_2)	$8j_2+4k_2+2l_2+m_2$	$2j_2+4k_2+2l_2+m_2$	$j_2+2k_2+4l_2+2m_2$	$j_2+2k_2+4l_2+8m_2$
(0,4,-12,8)	0	0	-24	24
(4,2,-10,4)	24	0	-24	0
(-4,2,-2,4)	-24	0	0	24
(-2,-5,5,2)	-24	-12	12	24

for some integer t, which solves to $j_3 = (32t-2k_3)/7$, $k_3 = k_3$, $l_3 = -8t-k_3$, and $m_3 = (24t+2k_3)/7$. Using the tuples $(j_1,k_1,l_1,m_1) = (4,-2,0,0)$ and $(j_2,k_2,l_2,m_2) = (-4,2,-2,4)$, the conditions (3.25) for making $C(2k+1,v,z)$ a cusp form imply

$$24(2k+1)-24v+168t/7 > 0, \;\; k_3 > 2t, \;\; \text{and} \;\; k_3+12t < 0.$$

We can verify that the above inequalities hold for $t=-2$ and $k_3=10$ for all values $1 \le \nu \le 2k-2$. Hence, we get the eta product

$$C(2k+1,v,8,z) = \left(\frac{\eta^4(z)}{\eta^2(2z)}\right)^{2k+1}\left(\frac{\eta^2(2z)\eta^4(8z)}{\eta^4(z)\eta^2(4z)}\right)^{v}\left(\frac{\eta^{10}(2z)\eta^6(4z)}{\eta^{12}(z)\eta^4(8z)}\right)$$

which is a holomorphic cusp form of weight $2k+1$ and order of vanishing v at the cusp $i\infty$ for all values of $1 \le v \le 2k-2$. Therefore, we have the following theorem.

Theorem 3.16. *The collection*

$$\left\{C(2k+1,v,8,z) \quad ; \quad 1 \leq v \leq 2k-2\right\}$$

forms a basis of $S_{2k+1}(\Gamma_0(8),\chi_{-4})$*, where*

$$C(2k+1,v,8,z) = \left(\frac{\eta^4(z)}{\eta^2(2z)}\right)^{2k+1} \left(\frac{\eta^2(2z)\eta^4(8z)}{\eta^4(z)\eta^2(4z)}\right)^v \left(\frac{\eta^{10}(2z)\eta^6(4z)}{\eta^{12}(z)\eta^4(8z)}\right).$$

We next obtain a basis for $S_{2k}(\Gamma_0(4))$ using the same ideas. By Theorem 3.13 we see that the cusp space $S_{2k}(\Gamma_0(4))$ is $k-2$ dimensional. Let $C(2k,v,4,z)$ be defined as

$$C(2k,v,4,z) = (D_{(1,0)}(z))^{2k}(D_{(0,1)}(z))^v D_{(0,0)}(z),$$

where $D_{(a,b)}(z)$ is an eta product of weight a and order of vanishing b at $i\infty$ on $\Gamma_0(4)$. Then, employing [14, Theorem 1.64] and Theorem 3.15, and considering the available options for $D_{(a,b)}(z)$ similar to the case of $S_{2k+1}(\Gamma_0(8),\chi_{-4})$, we get the following result.

Theorem 3.17. *The collection*

$$\left\{C(2k,v,4,z) \quad ; \quad 1 \leq v \leq k-2\right\}$$

forms a basis of $S_{2k}(\Gamma_0(4))$*, where*

$$C(2k,v,4,z) = \left(\frac{\eta^{10}(2z)}{\eta^4(z)\eta^4(4z)}\right)^{2k} \left(\frac{\eta^8(z)\eta^{16}(4z)}{\eta^{24}(2z)}\right)^v \left(\frac{\eta^{16}(z)\eta^8(4z)}{\eta^{24}(2z)}\right).$$

3.4 Special cases

In [15], Ono, Robins and Wahl find formulas for $\delta_k(n)$ for some values of k including $k = 6,8,10,12$. The first three sections in this chapter offers another way of computing these formulas. As a consequence of Theorem 3.6, Theorem 3.12, Proposition 3.16 and

Proposition 3.17, We have

$$\begin{aligned}
\delta_6(n) =& -\frac{1}{8}\sigma_{2,\chi_{-4},\chi_1}(4n+3), \\
\delta_8(n) =& \frac{1}{8}\sigma_{3,\chi_1,\chi_0}(2n+2), \\
\delta_{10}(n) =& \frac{1}{640}\left(\sigma_{4,\chi_{-4},\chi_1}(4n+5) - a(4n+5)\right), \\
\delta_{12}(n) =& \frac{1}{256}\left(\sigma_{5,\chi_1,\chi_0}(2n+3) - b(2n+3)\right),
\end{aligned}$$

where

$$\begin{aligned}
\sum_{n=1}^{\infty} a(n)q^n =& \eta^4(z)\eta^2(2z)\eta^4(4z) + 4\eta^4(2z)\eta^2(4z)\eta^4(8z), \\
\sum_{n=1}^{\infty} b(n)q^n =& \eta^{12}(2z).
\end{aligned}$$

In addition to the above, we also obtain formulas for $\delta_k(n)$ for $k = 14, 16$ and 18. To our knowledge, these cases are new and have not been studied before.

Proposition 3.18. *We have*

$$\begin{aligned}
\delta_{14}(n) =& -\frac{1}{124928}\left(\sigma_{6,\chi_{-4},\chi_1}(4n+7) - c(4n+7)\right), \\
\delta_{16}(n) =& \frac{1}{17408}\left(\sigma_{7,\chi_1,\chi_0}(2n+4) - d(2n+4)\right), \\
\delta_{18}(n) =& \frac{1}{45383680}\left(\sigma_{8,\chi_{-4},\chi_1}(4n+9) - e(4n+9)\right),
\end{aligned}$$

where

$$\begin{aligned}
\sum_{n=1}^{\infty} c(n)q^n =& 728\left(\eta^4(z)\eta^2(2z)\eta^8(8z) + 4\frac{\eta^4(2z)\eta^{12}(8z)}{\eta^2(4z)}\right), \\
\sum_{n=1}^{\infty} d(n)q^n =& 128\eta^8(2z)\eta^8(4z),
\end{aligned}$$

$$\sum_{n=1}^{\infty} e(n)q^n = \frac{\eta^{20}(z)\eta^4(4z)}{\eta^6(2z)} + 20\frac{\eta^{16}(z)\eta^2(4z)\eta^4(8z)}{\eta^4(2z)} + 144\frac{\eta^{12}(z)\eta^8(8z)}{\eta^2(2z)}$$
$$+448\frac{\eta^8(z)\eta^{12}(8z)}{\eta^2(4z)} + 391168\frac{\eta^4(z)\eta^2(2z)\eta^{16}(8z)}{\eta^4(4z)}$$
$$+1562624\frac{\eta^4(2z)\eta^{12}(20z)}{\eta^6(4z)}.$$

Proof. We explain the proof for $\delta_{18}(n)$. By the proof of Theorem 3.12, we have

$$\sum_{n=0}^{\infty} \delta_{18}(n)q^{4n+9} = \frac{1}{45383680}\left(\sum_{n=0}^{\infty} \sigma_{8,\chi_{-4},\chi_1}(4n+1)q^{4n+1} - \sum_{n=1}^{\infty} e(n)q^n\right), \tag{3.27}$$

where

$$\sum_{n=1}^{\infty} e(n)q^n \in S_9(\Gamma_0(8),\chi_{-4}).$$

Employing Proposition 3.16, a basis for $S_9(\Gamma_0(8),\chi_{-4})$ is given by the set

$$\mathbb{B}_2 = \{b_1, b_2, b_3, b_4, b_5, b_6\},$$

where

$$b_1 = \eta^{20}(z)\eta^4(4z)/\eta^6(2z), \qquad b_2 = \eta^{16}(z)\eta^2(4z)\eta^4(8z)/\eta^4(2z),$$
$$b_3 = \eta^{12}(z)\eta^8(8z)/\eta^2(2z), \qquad b_4 = \eta^8(z)\eta^{12}(8z)/\eta^2(4z),$$
$$b_5 = \eta^4(z)\eta^2(2z)\eta^{16}(8z)/\eta^4(4z) \qquad b_6 = \eta^4(2z)\eta^{12}(20z)/\eta^6(4z).$$

By (3.27), we see that $e(2) = e(3) = e(4) = e(6) = 0$, $e(1) = 1$ and $e(5) = 5^8 + 1$. Suppose

$$\sum_{n=1}^{\infty} e(n)q^n = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 + a_5b_5 + a_6b_6.$$

Then considering the q-expansions of the basis elements in Appendix A, we get the following system of equations:

$$\begin{aligned}
a_1 &= 1,\\
-20a_1 + a_2 &= 0,\\
176a_1 - 16a_2 + a_3 &= 0,\\
-880a_1 + 108a_2 - 12a_3 + a_4 &= 0,\\
2658a_1 - 384a_2 + 56a_3 - 8a_4 + a_5 &= 5^8 + 1,\\
-4544a_1 + 688a_2 - 112a_3 + 20a_4 - 4a_5 + a_6 &= 0.
\end{aligned}$$

Solving the system of equations leads to the result. Formulas for the other cases can be obtained similarly by obtaining suitable bases and consulting their q-expansions in Appendix A. □

Theoretically there is no obstacle in extending the results of Proposition 3.18 for higher values of k. The difficulty lies in computing the Fourier expansions of eta quotients and solving a system of equations to write the cusp form as a linear combination of the basis elements. Note that our results do not cover the cases $\delta_2(n)$ and $\delta_4(n)$ as our method leads to working with Eisenstein series of weight 1 and weight 2 in these cases. The formulas for $k = 2$ and 4 can be computed using other means (see [15] for details).

Chapter 4

Moments of Sums of Triangular Numbers

Let $f(n)$ be an arithmetic function. The Dirichlet series associated with f is defined by

$$L_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

The Dirichlet series for the divisor function $\sigma_k(n)$ has the formula

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s} = \zeta(s)\zeta(s-k) \text{ for } \Re(s) > \max(1, k+1),$$

in terms of the Riemann zeta function, which one can verify by looking at the coefficients of n^s on both sides. We can also prove the following identity due to Ramanujan

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}, \tag{4.1}$$

for $\Re(s) > \max(1, a+1, b+1, a+b+1)$ (See [9, Chapter 17]). In [4], Borwein and Choi investigated other arithmetic functions whose Dirichlet series have explicit representations in terms of the Riemann zeta function and Dirichlet L-functions. They generalised (4.1) and proved the following result.

Theorem 4.1. ([4, Theorem 2.1]) *Suppose f_1, f_2, g_1, and g_2 are completely multiplicative arithmetic functions. Then for $\Re(s) \geq \max(\sigma(f_i), \sigma(g_i))$, we have*

$$\sum_{n=1}^{\infty} \frac{(f_1 * g_1)(n).(f_2 * g_2)(n)}{n^s} = \frac{L_{f_1 f_2}(s) L_{g_1 g_2}(s) L_{f_1 g_2}(s) L_{g_1 f_2}(s)}{L_{f_1 f_2 g_1 g_2}(2s)},$$

where $(f * g)(n) = \sum_{d|n} f(d) g(\frac{n}{d})$ is the convolution of f and g and $\sigma(f)$ is the abscissa of absolute convergence of f.

Let $r_k(n)$ be the number of representations of n as a sum of k squares and let

$$L_k(s) = \sum_{n=1}^{\infty} \frac{r_k(n)}{n^s} \text{ and } R_k(s) = \sum_{n=1}^{\infty} \frac{r_k^2(n)}{n^s}$$

be the Dirichlet series associated to $r_k(n)$ and $r_k^2(n)$. Formulas for $L_k(s)$ in terms of better understood L-functions can be obtained for specific even values of k by the explicit formulas known for $r_k(n)$. For example, we have

$$L_2(s) = 4\zeta(s) L(s, \chi_{-4}), \tag{4.2}$$

$$L_4(s) = 8(1 - 4^{1-s}) \zeta(s) \zeta(s-1), \tag{4.3}$$

$$L_6(s) = 16\zeta(s-2) L(s, \chi_{-4}) - 4\zeta(s) L(s-2, \chi_{-4}), \tag{4.4}$$

$$L_8(s) = 16(1 - 2^{1-s} + 4^{2-s}) \zeta(s) \zeta(s-3). \tag{4.5}$$

Using the results for $L_k(s)$ and Theorem 4.1, in [4], Formulas are obtained for $R_k(n)$ for $N = 2, 4, 6, 8$. They found

$$R_2(s) = \frac{(4\zeta(s) L(s, \chi_{-4}))^2}{(1 + 2^{-s}) \zeta(2s)}, \tag{4.6}$$

$$R_4(s) = 64 \frac{(8.2^{3-3s} - 10.2^{2-2s} + 2^{1-s} + 1) \zeta(s-2) \zeta^2(s-1) \zeta(s)}{(1 + 2^{1-s}) \zeta(2s-2)}, \tag{4.7}$$

$$R_6(s) = 16\frac{(17-32.2^{-s})\zeta(s-4)L^2(s-2,\chi_{-4})\zeta(s)}{(1-16.2^{-2s})\zeta(2s-4)} - \frac{128L(s-4,\chi_{-4})\zeta^2(s-2)L(s,\chi_{-4})}{(1+4.2^{-s})\zeta(2s-4)}, \tag{4.8}$$

$$R_8(s) = 256\frac{\left(32\cdot 2^{6-2s}-3\cdot 2^{3-s}+1\right)\zeta(s-6)\zeta^2(s-3)\zeta(s)}{\left(1+2^{3-s}\right)\zeta(2s-6)}. \tag{4.9}$$

See [4, Section 3] for proofs of (4.2)-(4.9). The motivation for considering these explicit representations was to settle the Wagon's conjecture (see [4, Page 97]), which says that for $N \geq 3$,

$$\sum_{n\leq x} r_N^2(n) \sim W_N x^{N-1}, \text{ as } x\to\infty,$$

where

$$W_N = \frac{1}{(N-1)\left(1-2^{-N}\right)}\frac{\pi^N}{\Gamma^2\left(\frac{1}{2}N\right)}\frac{\zeta(N-1)}{\zeta(N)}.$$

In the next sections, we will obtain closed forms similar to (4.2)-(4.9) for the twisted Dirichlet series related to $\delta_k(n)$ and $\delta_k^2(n)$ for even values of k and prove an analogue of the Wagon's conjecture for the sums of triangular numbers for even values of N. We will start by studying the twisted Dirichlet series for the sums of triangular numbers. The Ramanujan-Peterson conjecture gives an estimate $O(n^{(k-1)/2+\varepsilon})$, for any $\varepsilon > 0$, for the Fourier coefficients $a(n)$ of a holomorphic cusp form on a congruence subgroup. Using the Ramanujan-Peterson bound [5, Section 9.2.3] for the cusp parts in Theorem 3.6 and Theorem 3.12, we get

$$\delta_{4k}(n) = \frac{-8k}{(-16)^k(4^k-1)B_{2k}}\sigma_{2k-1,\chi_1,\chi_0}(2n+k) + O\left(n^{\frac{2k-1}{2}+\varepsilon}\right) \tag{4.10}$$

and

$$\delta_{4k+2}(n) = \frac{-(2k+1)}{2^{4k}B_{2k+1,\chi_{-4}}}\sigma_{2k,\chi_{-4},\chi_1}(4n+2k+1) + O\left(n^{k+\varepsilon}\right) \tag{4.11}$$

for any $\varepsilon > 0$.

Let $\chi \in \{\chi_0, \chi_{-4}\}$ and $\psi \in \{\chi_0, \chi_1\}$ be Dirichlet characters. Then, for $i = 1$ and 2, set

$$F_{i,a}(x) = \sum_{\substack{n \leq x \\ n \equiv a(4)}} \sigma^i_{2k,\chi_{-4},\chi_1}(n),$$

$$F_i(x;\chi) = \sum_{n \leq x} \chi(n)\sigma^i_{2k,\chi_{-4},\chi_1}(n),$$

$$G_{i,a}(x) = \sum_{\substack{n \leq x \\ n \equiv a(2)}} \sigma^i_{2k-1,\chi_1,\chi_0}(n),$$

and

$$G_i(x;\psi) = \sum_{n \leq x} \psi(n)\sigma^i_{2k-1,\chi_1,\chi_0}(n).$$

Let χ_0 be the principal character mod 4. By employing the orthogonality of characters (See [13, Page 122]), we deduce, for $i = 1, 2$ and odd a,

$$F_{i,a}(x) = \frac{1}{2}\left(\chi_0(a)F_i(x;\chi_0) + \chi_{-4}(a)F_i(x;\chi_{-4})\right). \tag{4.12}$$

Also, by examining the cases for even and odd values of a, we see that

$$G_{i,a}(x) = \begin{cases} G_i(x;\chi_o) & \text{if } a \text{ is odd,} \\ G_i(x;\chi_1) - G_i(x,\chi_0) & \text{if } a \text{ is even.} \end{cases} \tag{4.13}$$

In the next section, we will determine closed expressions for the Dirichlet series associated to the twisted sums $F_i(x;\chi)$ and $G_i(x;\psi)$ which will later be used to obtain asymptotic formulas for the twisted sums.

4.1 Twisted Dirichlet Series

Let the Dirichlet series associated to the twisted sums $F_i(x;\chi)$ and $G_i(x;\psi)$ for Dirichlet characters $\chi \in \{\chi_0, \chi_{-4}\}$ and $\psi \in \{\chi_0, \chi_1\}$, for $i = 1,2$ be defined as

$$L_{4k+2,\chi}(s) = \sum_{n=1}^{\infty} \frac{\chi(n)\sigma_{2k,\chi_{-4},\chi_1}(n)}{n^s},$$

$$R_{4k+2,\chi}(s) = \sum_{n=1}^{\infty} \frac{\chi(n)\sigma^2_{2k,\chi_{-4},\chi_1}(n)}{n^s},$$

$$S_{4k,\psi}(s) = \sum_{n=1}^{\infty} \frac{\psi(n)\sigma_{2k-1,\chi_1,\chi_0}(n)}{n^s},$$

and

$$T_{4k,\psi}(s) = \sum_{n=1}^{\infty} \frac{\psi(n)\sigma^2_{2k-1,\chi_1,\chi_0}(n)}{n^s}.$$

Then the following assertions holds.

Theorem 4.2. *The formal Dirichlet series associated to the twisted sums $F_i(x;\chi)$, for $i = 1,2$ and $\chi \in \{\chi_{-4}, \chi_0\}$, are given by*

$$\begin{aligned}
L_{4k+2,\chi_0}(s) &= L(s-2k,\chi_{-4})L(s,\chi_0),\\
L_{4k+2,\chi_{-4}}(s) &= L(s-2k,\chi_0)L(s,\chi_{-4}),\\
R_{4k+2,\chi_0}(s) &= \frac{L(s-4k,\chi_0)L(s,\chi_0)L^2(s-2k,\chi_{-4})}{L(2s-4k,\chi_0)},\\
R_{4k+2,\chi_{-4}}(s) &= \frac{L(s-4k,\chi_{-4})L(s,\chi_{-4})L^2(s-2k,\chi_0)}{L(2s-4k,\chi_0)}.
\end{aligned}$$

Proof. We use Theorem 4.1 to prove the above expressions.

- For $L_{4k+2,\chi_0}(s)$, consider

$$f_1(n) = \chi_{-4}(n)n^{2k},\ g_1(n) = 1,\ f_2(n) = \chi_0(n),\ g_2(n) = \delta(n) \tag{4.14}$$

to get $L_{f_1 f_2}(s) = L(s-2k,\chi_{-4}),\ L_{g_1 g_2}(s) = 1,\ L_{f_1 g_2}(s) = 1,\ L_{g_1 f_2}(s) = L(s,\chi_0)$, and

$L_{f_1 f_2 g_1 g_2}(2s) = 1$, where $\delta(n)$ is given by

$$\delta(n) = \begin{cases} 0, & n \neq 1, \\ 1, & n = 1. \end{cases}$$

Hence,

$$L_{4k+2,\chi_0}(s) = L(s-2k, \chi_{-4}) L(s, \chi_0).$$

- For $L_{4k+2,\chi_{-4}}(s)$, consider

$$f_1(n) = \chi_{-4}(n) n^{2k},\ g_1(n) = 1,\ f_2(n) = \chi_{-4}(n),\ g_2(n) = \delta(n)$$

to get $L_{f_1 f_2}(s) = L(s-2k, \chi_0)$, $L_{g_1 g_2}(s) = 1$, $L_{f_1 g_2}(s) = 1$, $L_{g_1 f_2}(s) = L(s, \chi_{-4})$, and $L_{f_1 f_2 g_1 g_2}(2s) = 1$. Hence,

$$L_{4k+2,\chi_{-4}}(s) = L(s-2k, \chi_0) L(s, \chi_{-4}).$$

- For $R_{4k+2,\chi_0}(s)$, consider

$$f_1(n) = n^{2k} \chi_{-4}(n) \chi_0(n),\ g_1(n) = \chi_0(n),\ f_2(n) = n^{2k} \chi_{-4}(n),\ g_2(n) = 1$$

to get $L_{f_1 f_2}(s) = L(s-4k, \chi_0)$, $L_{g_1 g_2}(s) = L(s, \chi_0)$, $L_{f_1 g_2}(s) = L(s-2k, \chi_{-4})$, $L_{g_1 f_2}(s) = L(s-2k, \chi_{-4})$, and $L_{f_1 f_2 g_1 g_2}(2s) = L(2s-4k, \chi_0)$. Hence,

$$R_{4k+2,\chi_0}(s) = \frac{L(s-4k, \chi_2) L(s, \chi_0) L^2(s-2k, \chi_{-4})}{L(2s-4k, \chi_0)}.$$

- For $R_{4k+2,\chi_{-4}}(s)$, consider

$$f_1(n) = n^{2k} \chi_{-4}(n) \chi_{-4}(n),\ g_1(n) = \chi_{-4}(n),\ f_2(n) = n^{2k} \chi_{-4}(n),\ g_2(n) = 1$$

to get $L_{f_1f_2}(s) = L(s-4k,\chi_{-4})$, $L_{g_1g_2}(s) = L(s,\chi_{-4})$, $L_{f_1g_2}(s) = L(s-2k,\chi_0)$, $L_{g_1f_2}(s) = L(s-2k,\chi_0)$, and $L_{f_1f_2g_1g_2}(2s) = L(2s-4k,\chi_0)$. Hence,

$$R_{4k+2,\chi_{-4}}(s) = \frac{L(s-4k,\chi_{-4})L(s,\chi_{-4})L^2(s-2k,\chi_0)}{L(2s-4k,\chi_0)}.$$

□

Similarly, the following identities hold.

Theorem 4.3. *The formal Dirichlet series associated to the twisted sums $G_i(x;\psi)$, for $i = 1,2$ and $\psi \in \{\chi_0,\chi_1\}$, are given by*

$$\begin{aligned}
S_{4k,\chi_0}(s) &= L(s+1-2k,\chi_0)L(s,\chi_0),\\
S_{4k,\chi_1}(s) &= L(s+1-2k,\chi_1)L(s,\chi_0),\\
T_{4k,\chi_0}(s) &= \frac{L(s+2-4k,\chi_0)L^2(s+1-2k,\chi_0)L(s,\chi_0)}{L(2s+2-4k,\chi_0)},\\
T_{4k,\chi_1}(s) &= \frac{L(s+2-4k,\chi_1)L^2(s+1-2k,\chi_0)L(s,\chi_0)}{L(2s+2-4k,\chi_0)}.
\end{aligned}$$

4.2 Asymptotic formulas for the twisted sums

The formulas for $L_{4k+2,\chi}(s)$ and $R_{4k+2,\chi}(s)$ obtained in Theorem 4.2 allow us to obtain asymptotic formulas for the first and the second moments of $\delta_{4k+2}(n)$. In this Section, we first use Perron's formula to write each of the twisted sums $F_i(x;4k+2,\chi)$ in terms of a contour integral and then estimate the integral using Cauchy's residue theorem. For this purpose, we will need the following results, which we state with a reference to their proofs.

Theorem 4.4. ([22, Lemma 3.19]) *Let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad (s > 1),$$

where $a_n = O\{\rho(n)\}$, $\rho(n)$ *being non-decreasing and*

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma}} = O\left(\frac{1}{(\sigma-1)^{\alpha}}\right),$$

as $\sigma \to 1$. *Then if* $c > 0$, $c + \sigma > 1$, x *is not an integer and* N *is the integer nearest to* x,

$$\begin{aligned}\sum_{n \leqslant x} \frac{a_n}{n^w} =& \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(w+s) \frac{x^s}{s} ds + O\left\{\frac{x^c}{T(\sigma+c-1)^{\alpha}}\right\} \\ &+ O\left\{\frac{\rho(2x)x^{1-\sigma}\log x}{T}\right\} + O\left\{\rho(N)x^{-\sigma} \min\left(\frac{x}{T|x-N|}, 1\right)\right\}.\end{aligned}$$

Next, we state Rademacher's version of the Phragmén-Lindelöf theorem which provides estimates using the convexity argument for L-functions satisfying certain properties. It is worth stating that better estimates breaking this convexity bounds have been obtained for the Riemann zeta function and Dirichlet L-functions, but for the purpose of our study Rademacher's results are more than enough.

Let

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be a Dirichlet series absolutely convergent for $\Re(s) > 1$ such that it admits an Euler product

$$L(s) = \prod_{p} L_p(s),$$

where, $L_p(s)$ is the inverse of a polynomial of degree d_p in p^{-s} given by

$$L_p(s) = \prod_{i=1}^{d_p} \left(1 - \alpha_i p^{-s}\right)^{-1},$$

where $|\alpha_i| = 1$. Suppose there exists a positive integer d such that $d_p \leq d$ for all p and the equality holds for all but finitely many p. We will call such a d the degree of the L-function.

Set

$$L^*(s) = \prod_p L_p^*(s)$$

for

$$L_p^*(s) = \prod_{i=1}^{d_p} \left(1 - \overline{\alpha_i} p^{-s}\right)^{-1}.$$

Assume that the function $L(s)$ can be analytically continued to an entire function and has a functional equation of the form

$$\Lambda(s) = w\Lambda^*(1-s),$$

where $w \in \mathbb{C}$, $|w| = 1$, and

$$\Lambda(s) = A^{s/2} \prod_{i=1}^{m} \Gamma(\alpha_i s + r_i) L(s)$$

and

$$\Lambda^*(s) = A^{s/2} \prod_{i=1}^{m} \Gamma(\alpha_i s + r_i) L^*(s)$$

be such that $A > 0$, $\alpha_i, r_i \in \mathbb{R}$, m is a positive integer and Γ is the Gamma function given by

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx.$$

Then for such an L-function, the following assertion holds.

Proposition 4.5. ([12, Page 336]) *For* $0 \le \sigma \le 1$,

$$|L(\sigma + it)| \ll \left(A(|t|+2)^d\right)^{(1-\sigma)/2} \left(\log\left(A(|t|+2)^d\right)\right)^d.$$

As an immediate corollary for the Dirichlet character χ_0, we have

Corollary 4.6. *Let χ_0 be the principal Dirichlet character mod 4. Then, for $0 \le \sigma \le 1$,*

$$L(\sigma + it, \chi_0) = O(|t|^{1/2} \log(|t| + 2)).$$

Proof. The Dirichlet L-function $L(s, \chi_0)$ has the Euler product

$$L(s, \chi_0) = \prod_p \frac{1}{1 - \frac{\chi_0(p)}{p^s}}.$$

We can see that the degree d_p of all the local factors $L_p(s) = \left(1 - \frac{\chi_0(p)}{p^s}\right)^{-1}$ is 1 except L_2(s) which has degree 0. Hence, the degree d of $L(s, \chi_0)$ is 1. We also know from [13, Corollary 10.8] that the Dirichlet L-function $L(s, \chi_0)$ satisfies the functional equation

$$\xi(s, \chi_0) = \varepsilon(\chi_0)\xi(1 - s, \overline{\chi_0}),$$

where

$$\xi(s, \chi_0) = (2/\pi)^{s/2} L(s, \chi_0)\Gamma(s/2)$$

and $\varepsilon(\chi_0)$ is a root of unity. Hence, by taking $A = 2/\pi$ in Theorem 4.5, we have our result. □

Now, we use Perron's formula (Theorem 4.4) and the estimates given by the Phragmén-Lindelöf theorem (Corollary 4.6) to obtain asymptotic formulas for the twisted sums $F_i(x; \chi)$.

Proposition 4.7. *For the twisted sums $F_i(x; \chi)$ for Dirichlet characters $\chi \in \{\chi_0, \chi_{-4}\}$ and $i = 1, 2$. We have, for any $\varepsilon > 0$, the following estimates:*

(*i*) $F_1(4x + 2k + 1; \chi_0) = O\left(x^{2k+\varepsilon}\right)$.

(*ii*) $F_1(4x + 2k + 1; \chi_{-4}) = \dfrac{L(2k + 1, \chi_{-4})(4x + 2k + 1)^{2k+1}}{2(2k + 1)} + O\left(x^{2k+1/2+\varepsilon}\right)$.

(iii) $F_2(4x+2k+1;\chi_0) = \dfrac{L^2(2k+1,\chi_{-4})L(4k+1,\chi_0)(4x+2k+1)^{4k+1}}{2(4k+1)L(4k+2,\chi_0)} + O\left(x^{4k+1/2+\varepsilon}\right).$

(iv) $F_2(4x+2k+1;\chi_{-4}) = O\left(x^{4k+\varepsilon}\right).$

Proof. We work out the proof for $F_2(4x+2k+1;4k+2,\chi_0)$. The other statements follow similarly. From Theorem 4.2 and Theorem 4.4 for $c > 4k+1$, we have

$$F_2(4x+2k+1;\chi_0) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{L(s-4k,\chi_0)L(s,\chi_0)L^2(s-2k,\chi_{-4})(4x+2k+1)^s}{sL(2s-4k,\chi_0)}ds + O\left(\frac{x^c}{T}\right). \quad (4.15)$$

Now, the integrand in the above integral has a pole of order 1 at $s = 4k+1$, so by Cauchy's residue theorem

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{L(s-4k,\chi_0)L(s,\chi_0)L^2(s-2k,\chi_{-4})(4x+2k+1)^s}{sL(2s-4k,\chi_0)}ds$$

is equal to

$$\frac{L^2(2k+1,\chi_{-4})L(4k+1,\chi_0)(4x+2k+1)^{4k+1}}{2(4k+1)L(4k+2,\chi_0)} - \frac{1}{2\pi i}\left(\int_{c+iT}^{b+iT} + \int_{b+iT}^{b-iT} + \int_{b-iT}^{c-iT}\right) \frac{L(s-4k,\chi_0)L(s,\chi_0)L^2(s-2k,\chi_{-4})(4x+2k+1)^s}{sL(2s-4k,\chi_0)}ds, \quad (4.16)$$

where b is chosen to be equal to $4k$.

Let us estimate the integrals in (4.16). For

$$\mathrm{H}_+ := \frac{1}{2\pi i}\int_{c+iT}^{b+iT} \frac{L(s-4k,\chi_0)L(s,\chi_0)L^2(s-2k,\chi_{-4})(4x+2k+1)^s}{sL(2s-4k,\chi_0)}ds,$$

by substituting $s = \sigma + iT$, we get

$$\mathrm{H}_+ = \frac{1}{2\pi i}\int_{c}^{b} \frac{L(\sigma+iT-4k,\chi_0)L(\sigma+iT,\chi_0)L^2(\sigma+iT-2k,\chi_{-4})(4x+2k+1)^{\sigma+iT}}{(\sigma+iT)L(2\sigma+2iT-4k,\chi_0)}d\sigma.$$

Using the fact that $|\sigma+iT| \geq T$ and Corollary 4.6, we have

$$H_{+} = O\left(\int_b^c \frac{T^{\frac{1}{2}}\log T}{T} x^{\sigma} d\sigma\right) = O\left(\frac{x^c \log T}{T^{1/2}\log x}\right). \tag{4.17}$$

The integral

$$\mathrm{H}_{-} := \frac{1}{2\pi i}\int_{b-iT}^{c-iT} \frac{L(s-4k,\chi_0)L(s,\chi_0)L^2(s-2k,\chi_{-4})(4x+2k+1)^s}{sL(2s-4k,\chi_0)} ds$$

has a similar estimate.

For the vertical integral,

$$\mathrm{V} := \frac{1}{2\pi i}\int_{b+iT}^{b-iT} \frac{L(s-4k,\chi_0)L(s,\chi_0)L^2(s-2k,\chi_{-4})(4x+2k+1)^s}{sL(2s-4k,\chi_0)} ds,$$

by substituting $s = b+it$, we get

$$\mathrm{V} = -\frac{1}{2\pi i}\int_{-T}^{T} \frac{L(b+it-4k,\chi_0)L(b+it,\chi_0)L^2(b+it-2k,\chi_{-4})(4x+2k+1)^{b+it}}{(b+it)L(2b+2it-4k,\chi_0)} idt.$$

Using the fact that $|b+it| \geq t$, $|b+it| \geq b$ and Corollary 4.6, we have

$$V = O\left(\int_{-T}^{-1} \frac{|t|^{\frac{1}{2}}\log(|t|+2)}{t} x^b dt\right) + O\left(\int_{-1}^{1} \frac{|t|^{\frac{1}{2}}\log(|t|+2)}{b} x^b dt\right) + O\left(\int_{1}^{T} \frac{|t|^{\frac{1}{2}}\log(|t|+2)}{t} x^b dt\right).$$

We see that

$$O\left(\int_{-1}^{1} \frac{|t|^{\frac{1}{2}}\log(|t|+2)}{b} x^b dt\right) = O(x^b).$$

So we have

$$V = O\left(T^{1/2} x^b \log T\right). \tag{4.18}$$

Combining (4.16), (4.17) and (4.18), we have

$$F_2(4x+2k+1;\chi_0) = \frac{L^2(2k+1,\chi_{-4})L(4k+1,\chi_0)(4x+2k+1)^{4k+1}}{2(4k+1)L(4k+2,\chi_0)} + O\left(\frac{x^c}{T} + \frac{x^c \log T}{T^{1/2}} + T^{1/2}x^b \log T\right). \quad (4.19)$$

Say $\log x = O(x^{\varepsilon_2})$ for some $\varepsilon_2 > 0$ and set $b = 4k$, $c = 4k+1+\varepsilon_1$, $T = x$ and $\varepsilon = \varepsilon_1 + \varepsilon_2$ to get

$$F_2(4x+2k+1;\chi_0) = \frac{L^2(2k+1,\chi_{-4})L(4k+1,\chi_0)(4x+2k+1)^{4k+1}}{2(4k+1)L(4k+2,\chi_0)} + O\left(x^{4k+1/2+\varepsilon}\right).$$

The proofs of the estimates $(i),(ii)$ and (iv) follow similarly. □

For the twisted sums $G_i(x,\psi)$, we have the following estimates.

Proposition 4.8. *For the twisted sums $G_i(x;\psi)$ for Dirichlet characters $\psi \in \{\chi_0,\chi_1\}$ and $i = 1,2$. We have, for any $\varepsilon > 0$, the following estimates:*

$$(i)\ \ G_1(2x+k;\chi_0) = \frac{L(2k,\chi_0)(2x+k)^{2k}}{4k} + O\left(x^{2k+1/2+\varepsilon}\right).$$

$$(ii)\ \ G_1(2x+k;\chi_1) = \frac{L(2k,\chi_0)(2x+k)^{2k}}{2k} + O\left(x^{2k+1/2+\varepsilon}\right).$$

$$(iii)\ \ G_2(2x+k;\chi_0) = \frac{L^2(2k,\chi_0)L(4k-1,\chi_0)(2x+k)^{4k-1}}{2(4k-1)L(4k,\chi_0)} + O\left(x^{4k+1/2+\varepsilon}\right).$$

$$(iv)\ \ G_2(2x+k;\chi_1) = \frac{L^2(2k,\chi_0)L(4k-1,\chi_0)(2x+k)^{4k-1}}{(4k-1)L(4k,\chi_0)} + O\left(x^{4k+1/2+\varepsilon}\right).$$

4.3 The first moment

In this section, we use the asymptotic formulas obtained in the previous section to estimate the first moments of $\delta_{4k}(n)$ and $\delta_{4k+2}(n)$.

Theorem 4.9. *For an even value of $N > 2$ and any $\varepsilon > 0$, we have*

$$\sum_{n\le x} \delta_N(n) = \frac{\pi^{N/2}}{2^{N/2}\Gamma(N/2+1)} x^{N/2} + O\left(x^{(N-1)/2+\varepsilon}\right).$$

Proof. We split the proof in two cases depending on whether N is of the form $4k$ or $4k+2$. Suppose $N=4k$ and $0<\varepsilon_1<1$. Then using (4.10) we have,

$$\sum_{n\le x}\delta_{4k}(n)=\frac{1}{d_k}\sum_{n\le x}\sigma_{2k-1,\chi_1,\chi_0}(2n+k)+O\left(x^{k+1/2+\varepsilon_1}\right),$$

where d_k is defined in (3.8). Thus, from the definition of $G_{i,a}(x)$, we have

$$\sum_{n\le x}\delta_{4k}(n)=\frac{1}{d_k}G_{1,k}(2x+k)+O\left(x^{k+1/2+\varepsilon_1}\right). \tag{4.20}$$

From (4.13), recall that

$$G_{1,k}(2x+k)=\begin{cases} G_1(2x+k;\chi_o) & \text{if } k \text{ is odd,}\\ G_1(2x+k;\chi_1)-G_1(2x+k,\chi_0) & \text{if } k \text{ is even.}\end{cases} \tag{4.21}$$

Employing Proposition 4.8 and (4.21) in (4.20), we get

$$\sum_{n\le x}\delta_{4k}(n)=\frac{L(2k,\chi_0)(2x+k)^{2k}}{4kd_k}+O\left(x^{2k+1/2+\varepsilon}\right)$$

for any $\varepsilon>0$, irrespective of whether k is even or odd. By [5, Theorem 3.3.15], we have

$$L(2k,\chi_0)=(-1)^{k+1}\frac{(4^k-1)\pi^{2k}B_{2k}}{2(2k)!}. \tag{4.22}$$

Substituting the values of d_k and $L(2k,\chi_0)$, we get our result when N is of the form $4k$.

Now suppose, $N=4k+2$ and $0<\varepsilon_1<1$. Then by (4.11), we have

$$\sum_{n\le x}\delta_{4k+2}(n)=\frac{-(2k+1)}{2^{4k}B_{2k+1,\chi_{-4}}}\sum_{n\le x}\sigma_{2k,\chi_{-4},\chi_1}(4n+2k+1)+O\left(x^{k+1+\varepsilon_1}\right).$$

From the definition of $F_{i,a}(x)$, we deduce

$$\sum_{n\leq x}\delta_{4k+2}(n)=\frac{-(2k+1)}{2^{4k}B_{2k+1,\chi_{-4}}}F_{1,2k+1}(4x+2k+1)+O\left(x^{k+1+\varepsilon_1}\right).$$

Using (4.12) and Proposition 4.7 yields

$$F_{1,2k+1}(4x+2k+1)=\frac{\chi_{-4}(2k+1)L(2k+1,\chi_{-4})2^{4k}x^{2k+1}}{2k+1}+O\left(x^{2k+1/2+\varepsilon}\right)$$

for any $\varepsilon>0$. Hence,

$$\sum_{n\leq x}\delta_{4k+2}(n)=\frac{(-1)^{k+1}L(2k+1,\chi_{-4})x^{2k+1}}{B_{2k+1,\chi_{-4}}}+O\left(x^{2k+1/2+\varepsilon}\right).$$

Substituting the value of $L(2k+1,\chi_{-4})$ from (3.17) in the above formula yields our result.

□

4.4 The second moment

Theorem 4.10. *For an even value of $N>2$ and any $\varepsilon>0$, we have*

$$\sum_{n\leq x}\delta_N^2(n)=Y_Nx^{N-1}+O\left(x^{N-1/2+\varepsilon}\right),$$

where

$$Y_N=\frac{\pi^N}{2^N(N-1)\Gamma^2\left(\frac{N}{2}\right)}\frac{L(N-1,\chi_0)}{L(N,\chi_0)}.$$

Proof. We start by considering the case $N=4k$. Let $0<\varepsilon_1<1$. Then from (4.10), we have

$$\delta_{4k}(n)=\frac{1}{d_k}\sigma_{2k-1,\chi_1,\chi_0}(2n+k)+O\left(n^{k-1/2+\varepsilon_1}\right),$$

where d_k is given in (3.8). Squaring both sides and summing over n yields

$$\sum_{n\le x}\delta_{4k}^2(n)=\frac{1}{d_k^2}\sum_{n\le x}\sigma_{2k-1,\chi_1,\chi_0}^2(2n+k)+O\left(x^{2k+2\varepsilon_1}\right)$$
$$+O\left(x^{k-1/2+\varepsilon_1}\sum_{n\le x}\sigma_{2k-1,\chi_1,\chi_0}(2n+k)\right).$$

(Here, we used that fact that $\sigma_{2k-1,\chi_1,\chi_0}(2n+k)$ is always positive.) From the estimate for the first moment obtained in Theorem 4.9, we have

$$\sum_{n\le x}\delta_{4k}^2(n)=\frac{1}{d_k^2}\sum_{n\le x}\sigma_{2k-1,\chi_1,\chi_0}^2(2n+k)+O\left(x^{3k-1/2+\varepsilon_1}\right).$$

By employing the definition of $G_{i,a}(x)$, the above formula can be written as

$$\sum_{n\le x}\delta_{4k}^2(n)=\frac{1}{d_k^2}G_{2,k}(2x+k)+O\left(x^{3k-1/2+\varepsilon_1}\right). \tag{4.23}$$

From (4.13), recall that

$$G_{2,k}(2x+k)=\begin{cases}G_2(2x+k;\chi_o) & \text{if } k \text{ is odd,}\\ G_2(2x+k;\chi_1)-G_2(2x+k,\chi_0) & \text{if } k \text{ is even.}\end{cases}$$

By Proposition 4.8, we have that for any $\varepsilon>0$,

$$G_{2,k}(2x+k)=\frac{L^2(2k,\chi_0)L(4k-1,\chi_0)(2x+k)^{4k-1}}{2(4k-1)L(4k,\chi_0)}+O\left(x^{4k+1/2+\varepsilon}\right).$$

Applying this formula in (4.23), we get

$$\sum_{n\le x}\delta_{4k}^2(n)=\frac{L^2(2k,\chi_0)L(4k-1,\chi_0)(2x+k)^{4k-1}}{2(4k-1)d_k^2L(4k,\chi_0)}+O\left(x^{4k+1/2+\varepsilon}\right).$$

Substituting the values of d_k and $L(2k,\chi_0)$ from (3.8) and (4.22), we get the desired result for $N=4k$.

Next Suppose $N = 4k+2$. Then, by Theorem 3.12, we have

$$\delta_{4k+2}(n) = \frac{-(2k+1)}{2^{4k}B_{2k+1,\chi_{-4}}}\sigma_{2k,\chi_{-4},\chi_1}(4n+2k+1) + t(4n+2k+1),$$

where $\sum_{n=0}^{\infty} t(n)q^n$ is a cusp form of weight $2k+1$, level 8 and character χ_{-4}. Squaring both sides, we get

$$\begin{aligned}\delta_{4k+2}^2(n) = &\frac{(2k+1)^2}{2^{8k}B_{2k+1,\chi_{-4}}^2}\sigma_{2k,\chi_{-4},\chi_1}^2(4n+2k+1) \\ &- \frac{(2k+1)}{2^{4k-1}B_{2k+1,\chi_{-4}}}t(4n+2k+1)\sigma_{2k,\chi_{-4},\chi_1}(4n+2k+1) + t^2(4n+2k+1).\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{n\le x}\delta_{4k+2}^2(n) = &\frac{(2k+1)^2}{2^{8k}B_{2k+1,\chi_{-4}}^2}\sum_{n\le x}\sigma_{2k,\chi_{-4},\chi_1}^2(4n+2k+1) \\ &- \frac{(2k+1)}{2^{4k-1}B_{2k+1,\chi_{-4}}}\sum_{n\le x}t(4n+2k+1)\sigma_{2k,\chi_{-4},\chi_1}(4n+2k+1) + \sum_{n\le x}t^2(4n+2k+1). \quad (4.24)\end{aligned}$$

Using Cauchy-Schwarz inequality

$$\begin{aligned}&\sum_{n\le x}t(4n+2k+1)\sigma_{2k,\chi_{-4},\chi_1}(4n+2k+1) \\ &\le \left(\sum_{n\le x}t^2(4n+2k+1)\right)^{1/2}\left(\sum_{n\le x}\sigma_{2k,\chi_{-4},\chi_1}^2(4n+2k+1)\right)^{1/2}. \quad (4.25)\end{aligned}$$

We know from the Ramanujan-Peterson conjecture [5, Section 9.2.3] that $t(4n+2k+1) = O\left(n^{k+\varepsilon_1}\right)$ for any $0 < \varepsilon_1 < 1$. Hence,

$$\left(\sum_{n\le x}t^2(4n+2k+1)\right)^{1/2} = O\left(x^{\frac{2k+1}{2}+\varepsilon_1}\right). \quad (4.26)$$

Using (4.12) and Proposition 4.7, we have

$$F_{2,2k+1}(4x+2k+1) = \frac{2^{8k}L^2(2k+1,\chi_{-4})L(4k+1,\chi_0)x^{4k+1}}{(4k+1)L(4k+2,\chi_0)} + O\left(x^{4k+1/2+\varepsilon}\right) \quad (4.27)$$

for any $\varepsilon > 0$. Hence,

$$\left(\sum_{n\leq x} \sigma^2_{2k,\chi_{-4};\chi_1}(4n+2k+1)\right)^{1/2} = O\left(F_{2,2k+1}(4x+2k+1)\right)^{1/2} = O(x^{2k+1/2}). \quad (4.28)$$

Substituting (4.26) and (4.28) in (4.25) yields

$$\sum_{n\leq x} t(4n+2k+1)\sigma_{2k,\chi_{-4};\chi_1}(4n+2k+1) = O\left(x^{3k+3/4+\varepsilon_1}\right). \quad (4.29)$$

We then apply (4.26) and (4.29) in (4.24) to obtain

$$\sum_{n\leq x} \delta^2_{4k+2}(n) = \frac{(2k+1)^2}{2^{8k}B^2_{2k+1,\chi_{-4}}} \sum_{n\leq x} \sigma^2_{2k,\chi_{-4};\chi_1}(4n+2k+1) + O\left(x^{3k+3/4+\varepsilon_1}\right). \quad (4.30)$$

Observe that for a fixed k,

$$\sum_{n\leq x} \sigma^2_{2k,\chi_{-4};\chi_1}(4n+2k+1) = F_{2,2k+1}(4x+2k+1) + O(1). \quad (4.31)$$

Then, an application of (4.31) and (4.27) in (4.30) yields

$$\sum_{n\leq x} \delta^2_{4k+2}(n) = \frac{(2k+1)^2L^2(2k+1,\chi_{-4})L(4k+1,\chi_0)x^{4k+1}}{(4k+1)B^2_{2k+1,\chi_{-4}}L(4k+2,\chi_0)} + O\left(x^{4k+1/2+\varepsilon}\right).$$

Finally, by substituting the value of $L(2k+1,\chi_{-4})$ from (3.17), we get the claimed result.

□

Chapter 5

Future Works

Here we list a few possible continuations of the results of this thesis.

1. In Chapter 2, we studied the representation problem for binary quadratic forms of discriminant $-D$ where $\mathbb{Q}(\sqrt{-D})$ has class number 3. It is natural to study an analogous result for imaginary quadratic fields with class number 2. There are 18 such fields. It would be interesting to investigate whether the cusp part in any of these 18 cases has an eta product representation.

2. In Lemma 2.9 of Chapter 2, we showed that if a modular form f has a zero on the upper half plane, then $c(n)$'s in the expansion

$$f(q) = \prod_{n=1}^{\infty} (1-q^n)^{c(n)}$$

are unbounded. It would be worthwhile to investigate the converse of this Lemma. More specifically, if $c(n)$'s in the above expression are unbounded, then what can be said about the zeroes of f. A result in this direction is proved in [17].

3. In Chapter 3, we proved formulas for $\delta_k(n)$ for even values of k using the theory of integral weight modular forms. It is natural to investigate whether formulas for odd values of k can be obtained by studying modular forms of half integral weight. Another approach towards this investigation would be by using the Singular series obtained from the circle method of Hardy and Littlewood.

4. In Chapter 4, we proved asymptotic formulas for the first and the second moments of $\delta_k(n)$ for even values of k. The formulas for the second moment gives an analogue of the Wagon's conjecture for $\delta_k(n)$. One can also study the second moment of $\delta_k(n)$ for odd values of k. In particular, for odd k, we can investigate on the truth of the asymptotic formula

$$\sum_{n \leq x} \delta_k^2(n) = Y_k x^{k-1} + O\left(x^{k-1/2+\varepsilon}\right),$$

where $\varepsilon > 0$ and

$$Y_k = \frac{\pi^k}{2^k(k-1)\Gamma^2\left(\frac{k}{2}\right)} \frac{L(k-1,\chi_0)}{L(k,\chi_0)}.$$

We can also study whether the error term for the first and the second moment can be improved.

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Appendix A

q-expansions of some eta products

Here we record the q-expansions of some eta products that were used in the proof of Theorem 3.18. The computations are done using the Maple package q-series in https://qseries.org/fgarvan/qmaple/qseries/index.html.

In $S_5(\Gamma_0(8), \chi_{-4})$

$$(i)\ \eta^4(z)\eta^2(2z)\eta^4(4z) = q - 4q^2 + 16q^4 - 14q^5 - 64q^8 + 81q^9 + 56q^{10} - 238q^{13} + 256q^{16} + 322q^{17} - 324q^{18} - 224q^{20} + O\left(q^{21}\right)$$

$$(ii)\ \eta^4(2z)\eta^2(4z)\eta^4(8z) = q^2 - 4q^4 + 16q^8 - 14q^{10} - 64q^{16} + 81q^{18} + 56q^{20} + O\left(q^{21}\right)$$

In $S_6(\Gamma_0(4))$

$$(iii)\ \eta^{12}(2z) = q - 12q^3 + 54q^5 - 88q^7 - 99q^9 + 540q^{11} - 418q^{13} - 648q^{15} + 594q^{17} + 836q^{19} + O\left(q^{21}\right)$$

In $S_7(\Gamma_0(8), \chi_{-4})$

$$(iv)\ \frac{\eta^{12}(z)\eta^4(4z)}{\eta^2(2z)} = q - 12q^2 + 56q^3 - 112q^4 + 10q^5 + 352q^6 - 560q^7 + 320q^8 - 231q^9 - 120q^{10} + 1736q^{11} - 2176q^{12} + 1466q^{13}3520q^{14} + 560q^{15} + 8448q^{16} - 4766q^{17} + 2772q^{18} - 13608q^{19} - 1120q^{20} + O\left(q^{21}\right)$$

$$(v)\ \eta^8(z)\eta^2(4z)\eta^4(8z) = q^2 - 8q^3 + 20q4 - 72q^6 + 80q^7 + 16q^8 + 10q^{10} - 248q^{11} + 96q^{12} + 720q^{14} - 80q^{15} - 1216q^{16} - 231q^{18} - 1944q^{19} + 200q^{20} + O\left(q^{21}\right)$$

$$(vi)\ \eta^4(z)\eta^2(2z)\eta^8(8z) = q^3 - 4q^4 + 16q^6 - 10q^7 - 16q^8 + 31q^{11} + 32q^{12} - 160q^{14} + 10q^{15} + 192q^{16} - 243q^{19} - 40q^{20} + O\left(q^{21}\right)$$

$$(vii)\ \frac{\eta^4(2z)\eta^{12}(8z)}{\eta^2(4z)} = q^4 - 4q^6 + 4q^8 - 8q^{12} + 40q^{14} - 48q^{16} + 10q^{20} + O\left(q^{21}\right)$$

In $S_8(\Gamma_0(4))$

$$(viii)\ \frac{\eta^{32}(2z)}{\eta^8(z)\eta^8(4z)} = q + 8q^2 + 12q^3 - 64q^4 - 210q^5 + 96q^6 + 1016q^7 + 512q^8 - 2043q^9 - 1680q^{10} + 1092q^{11} - 768q^{12} + 1382q^{13} + 8128q^{14} - 2520q^{15} - 4096q^{16} + 14706q^{17} - 16344q^{18} - 39940q^{19} + 13440q^{20} + O\left(q^{21}\right)$$

$$(ix)\ \eta^8(2z)\eta^8(4z) = q^2 - 8q^4 + 12q^6 + 64q^8 - 210q^{10} - 96q^{12} + 1016q^{14} - 512q^{16} - 2043q^{18} + 1680q^{20} + O\left(q^{21}\right)$$

In $S_9(\Gamma_0(8), \chi_{-4})$

$$(x)\ \frac{\eta^{20}(z)\eta^4(4z)}{\eta^6(2z)} = q - 20q^2 + 176q^3 - 880q^4 + 2658q^5 - 4544q^6 + 2464q^7 + 6336q^8 - 15711q^9 + 20568q^{10} - 32560q^{11} + 45824q^{12} - 11614q^{13} - 63616q^{14} + 107360q^{15} - 163584q^{16} + 241026q^{17} - 128148q^{18} - 34320q^{19} - 12512q^{20} + O\left(q^{21}\right)$$

$$(xi)\ \frac{\eta^{16}(z)\eta^2(4z)\eta^4(8z)}{\eta^4(2z)} = q^2 - 16q^3 + 108q^4 - 384q^5 + 688q^6 - 224q^7 - 1392q^8 + 2304q^9 - 1438q^{10} + 2960q^{11} - 6592q^{12} + 1152q^{13} + 9632q^{14} - 9760q^{15} + 16576q^{16} - 31488q^{17} + 8865q^{18} + 3120q^{19} + 33112q^{20} + O\left(q^{21}\right)$$

$$(xii)\ \frac{\eta^{12}(z)\eta^8(8z)}{\eta^2(2z)} = q^3 - 12q^4 + 56q^5 - 112q^6 + 14q^7 + 304q^8 - 336q^9 - 128q^{10} - 185q^{11} + 1216q^{12} - 168q^{13} - 1568q^{14} + 610q^{15} - 1984q^{16} + 4592q^{17} + 768q^{18} - 195q^{19} - 9368q^{20} + O\left(q^{21}\right)$$

$$(xiii)\ \frac{\eta^8(z)\eta^{12}(8z)}{\eta^2(4z)} = q^4 - 8q^5 + 20q^6 - 68q^8 + 48q^9 + 96q^{10} - 272q^{12} + 24q^{13} + 280q^{14} + 336q^{16} - 656q^{17} - 576q^{18} + 2146q^{20} + O\left(q^{21}\right)$$

$$(xiv)\ \frac{\eta^4(z)\eta^2(2z)\eta^{16}(8z)}{\eta^4(4z)} = q^5 - 4q^6 + 16q^8 - 6q^9 - 32q^{10} + 64q^{12} - 3q^{13} - 56q^{14} - 64q^{16} + 82q^{17} + 192q^{18} - 512q^{20} + O\left(q^{21}\right)$$

$$(xv)\ \frac{\eta^4(2z)\eta^{12}(20z)}{\eta^6(4z)} = q^6 - 4q^8 + 8q^{10} - 16q^{12} + 14q^{14} + 16q^{16} - 48q^{18} + 128q^{20} + O\left(q^{21}\right)$$