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The shape of reasoning: Using geometry to promote the reasoning proficiency strand

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Derek Hurrell shares a number of geometry tasks designed to tease out students' reasoning. Concrete materials are used to initiate discussions and develop a culture of reasoning in the mathematics classroom.

To develop the capability of Reasoning, I can think of no more a powerful medium than geometry. Geometry is such a tactile and visual subject that it allows students to reason about what they can touch, see, and manipulate. It promotes conversations about shape and transformations and can be used to encourage students to think deeply about mathematics. It gives them the opportunity to discuss, debate, hypothesise, conjecture, justify and generalise, and what is more, it should be really good fun!

For some more than others, it has taken a little while to really embrace that the *Australian Curriculum* (AC) tells us more than just what to teach. It is true that the AC does contain the nouns of the teaching, that is, the content strands of Number and Algebra, Measurement and Geometry, and Statistics and Probability (Sullivan, 2011). Equally significantly, the AC also gives clear direction that this content should be enacted through the mathematical proficiencies of Fluency, Understanding, Reasoning, and Problem Solving, the verbs of the teaching (Sullivan, 2011; Burrows, Raymond, & Clarke, 2020).

One of the most common ways that the reasoning proficiency is illustrated in classrooms is to get the students to explain what they are doing, and why they are doing what they are doing. While this is in no way incorrect, it does not completely satisfy what is required from this strand. In conversations with teachers about how they promote reasoning in their mathematics teaching, the issue of encouraging productive and deep discussion, while covering the content is quite often raised. Teachers can sometimes have trouble finding the appropriate carriage within which to structure these discussions. One powerful way this can be achieved is by using geometry as the way to encourage students and create the habit of talking deeply about their mathematical thinking.

The Australian Curriculum describes reasoning as follows:

> Students develop an increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising. Students are reasoning mathematically when they explain their thinking, when they deduce and justify strategies used and conclusions reached, when they adapt the known to the unknown, when they transfer learning from one context to another, when they prove that something is true or false, and when they compare and contrast related ideas and explain their choices.
>
> (ACARA, 2020)

In order to provide opportunities for these actions (reasoning, deduction, analysis, evaluation, explanation etc.) to become necessary, and therefore observable, we need to provide activities and a pedagogy which allows all students to engage with the mathematics. The first step to this is to consider using the Concrete-Representational-Abstract (CRA) approach (Bruner, 1966). CRA is based around Piaget's seminal work in the 1960s regarding conceptual development through: an 'enactive' stage where learning should involve concrete experiences; an 'iconic' stage, where pictorial representations and other graphic representations are employed; and a 'symbolic' stage, where abstract symbols and notation are appropriate for the learner. Most importantly in developing a culture of reasoning in the mathematics classroom, the use of concrete materials initiates the discussions through allowing students an effective way in which to represent their thinking, in a manner which they and their teacher then can explore. We all know it is difficult in the first instance to discuss abstract ideas. Concrete materials give us something to look at, manipulate, and use to illustrate our thinking, in other words a starting place for reasoning.

Triangles task

Begin this task by asking students to identify a triangle from the shapes provided on the Identify the triangle(s) sheet (see Figure 1). If a triangle is not selected that immediately tells you something! Some students may only select the equilateral triangle as this tends to be the archetypal triangle, the triangle most often shown in books and in the media. If this occurs it may be that they are at the stage of their development which Van Hiele (1999) referred to as the Visualisation stage. These students simply recognise the shape without necessarily attending to the properties that make up that shape i.e., "I know it's a triangle because that's what a triangle looks like." Some other students may identify several of the shapes as being triangles. These students are starting to show a deepening awareness that two or more shapes can look a little different but still be the same in some respects. However, the question to the students who select one, or multiple versions of the triangle is the same, "How do you know it is a triangle?"

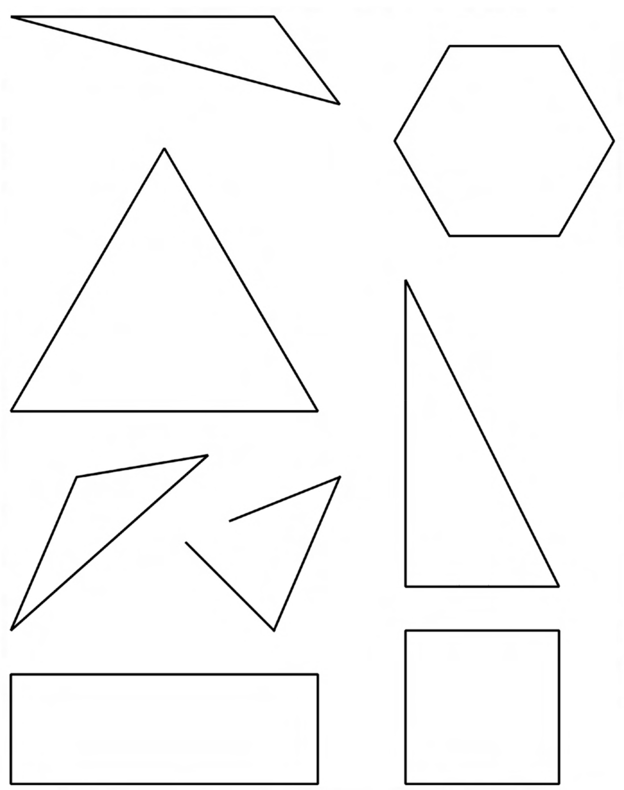


Figure 1. Identify the triangle(s).

This question is important. Recognising the shape is a start, but what are the properties that make up a triangle? The desire is that students recognise that a triangle is a closed shape with three sides and three angles. When they are aware of, can identify and articulate their understanding at this level they have reached Van Hiele's level 2, Analysis. At this level students have the opportunity to display their reasoning. This developing understanding of what makes a triangle a triangle can be extended through activities which require the students to sort and classify shapes in the environment. Arm the students with a digital camera and send them out on a scavenger hunt to take pictures of triangles outside of the classroom. The conversations that can ensue are quite remarkable. They can then turn some of these pictures into mini 'tutorials' by using something such as Explain Me, which is an uncomplicated app which allows them to easily annotate their photos or provide an audio description of their understanding. This provides not only a great presentation method but an artefact which can be used as assessment.

Further, there should be an investigation about why they did not choose the non-triangles. Using non-examples are a great way to reinforce the properties that make a triangle a triangle. The open 'shape', which is situated towards the centre of the sheet, often generates lots of discussion about whether it belongs with the triangles. The fact that it has a couple of the attributes of a triangle (i.e., three sides and three angles) brings into question the quite often-used definition of a triangle as simply a three-sided shape.

For those students who are ready for it, there is yet another line of inquiry about the types of triangles (i.e., equilateral, isosceles and scalene) that were represented on the sheet which could lead to sorting and naming these different triangles, whilst all the time reinforcing the properties of a triangle. Although formal classification of triangles is not expected in the *Australian Curriculum* until Year 7 (ACMMG165), most students seem to be able to name and define the triangles far earlier than this if they are provided with opportunities to explore this shape. This exploration is of course guided, and what we need to do, is keep asking two fundamental questions "What is the same?" and "What is different?".

These questions can be further explored by getting students to cut out the shapes from the sheet and doing some sorting and classifying of them. At this point the emphasis should be on students explaining their thinking and justifying any conclusions they reach. Menningsen and Stein (1997) conducted research in 68 classrooms and in those classrooms where mathematical reasoning was maintained, the researchers identified some common factors. The factors included building on students' prior knowledge, providing an appropriate amount of time and providing sustained emphasis on students to explain. Do not be surprised if this sustained emphasis is initially met with some reluctance on the part of the students. My experience is that it takes a while before they really understand what is expected of them. You should emphasise that you are not looking for what Kahneman (2011) called 'quick answers', that instead you are looking for 'slow answers', answers which are deliberate and engage them in mathematical reasoning.

Tri-angles

Another activity which can promote reasoning involves determining the sum of the angles in a triangle. The stimulus sheet can once again be used for this activity. This is an activity which is not expected in the *Australian Curriculum* until Year 7 (ACMMG166), but again, I have used it successfully with much younger students. The use of manipulative materials makes this problem for more accessible.

Once each student has a triangle, they are then asked to think of a way in which they can prove that the internal angles sum to 180°. It is important that a protractor is not used at this point, otherwise the activity becomes about student familiarity with a protractor. It is worth remembering that a protractor is a scaled instrument that is not always immediately accessible to many students, and therefore should never be introduced without some explicit instruction on how to use it properly. Further, using a protractor can turn this into an arithmetic activity (e.g., 75° + 35° + 70°) and a measuring activity. It is not that these are not worthy outcomes, but for this activity, these are not the main objective and should not be pursued at the expense of the reasoning that should underpin this activity.

Task: Cut out a triangle.

- Sean told me that when you add together the size of the corners (angles) of the triangle it is 180°.
- Sean tore his triangles to prove he was correct. Can you show what he might have done and what he said to prove he was right?

Figure 2. A triangles task.

In all my years of teaching, even when dealing with young students, there has always seemed to be some sort of innate understanding of the term 180 if not 180°. Students, because of their play on bikes and skateboards are aware that when you 'do' a 180 you are facing in one direction, you do the 180, and then you are facing in the opposite direction. This is by no means a robust understanding of the degrees of turn, but it does give a good place to start the conversation about the fact that 180 is actually 180°. You can also leverage the fact that many students have the knowledge that a right angle is 90°. By looking in the environment both in the class and outside of it you can pose the question of what happens when you push two objects such as Lego bricks together, or two tiles (Figure 3) that have 90° angles, to get the students to notice that a straight line is formed, and a straight line is 180°.

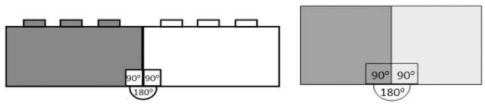
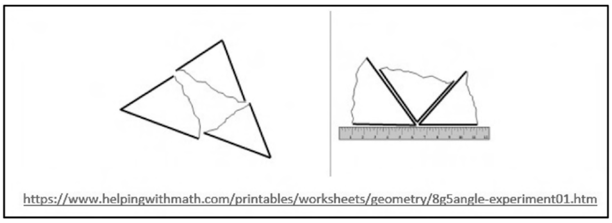


Figure 3. Lego bricks and tiles.

They are going to use this information that a straight line is 180° to adapt something known to something unknown and to prove that something is true or false.

Try to avoid giving a solution to the problem but once your teacher judgment leads you to believe that they are becoming frustrated you might want to give them the following instructions:



1. Carefully tear the triangle into three similar sizes as shown.
2. Line up the pieces along a straight line with the three vertices touching.
3. Tell yourself the story of how this proves the internal angles add to 180°.

As mentioned earlier there were a variety of triangle types on the triangle sheet, they were not all the rather archetypal equilateral triangle. This is so that attention can be drawn to an understanding that all triangles, regardless of their type, have internal angles that sum to 180° (Figure 4). We are generalising a rule, but a

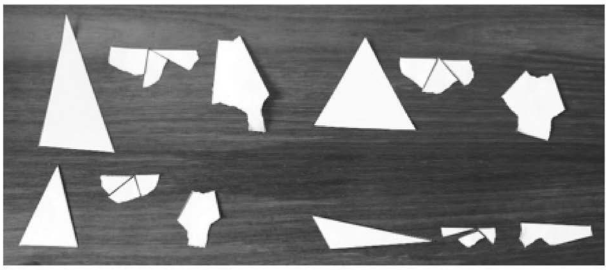


Figure 4. Ripped triangles.

rule we have proven, one that they should be able to tell the story about. We can then explore the idea of using a similar procedure in the completion of our task to answer the question of "How does knowing about the internal angles of triangles help you work out a rule the internal angles of quadrilaterals?" We are adapting the known to the unknown and transferring learning from one context to another.

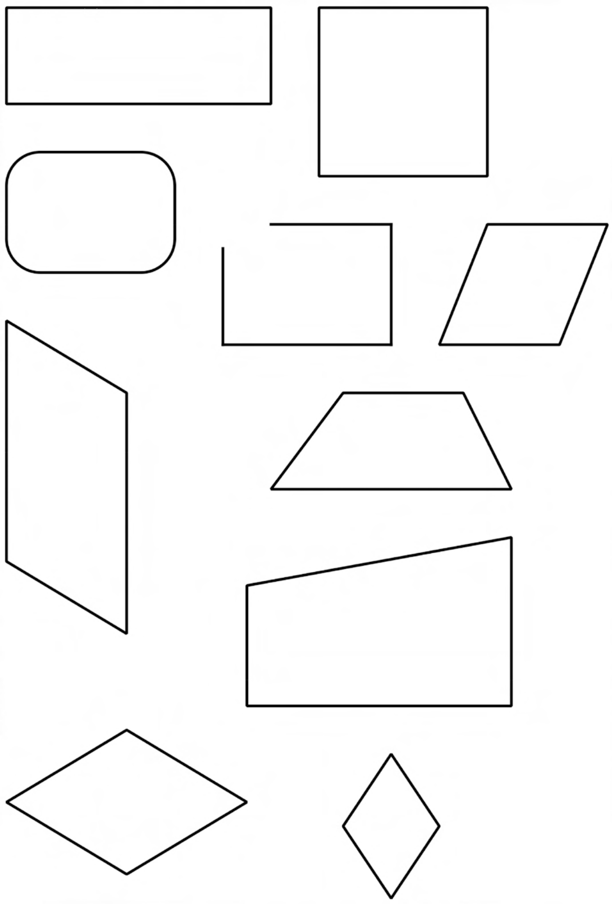


Figure 5. Identify the rectangle(s).

Provide the students with a sheet on which quadrilaterals are prominent (Figure 5) and ask similar and related questions used in exploring triangles to encourage the students to take what they know, adapt it to a new situation and then explain their reasoning. Students seem to get real pleasure from saying "Because of what I learned through using the triangles…". For those students who are ready to be further challenged, pentagons, hexagons, indeed any polygon, can be explored to see if generalisations about the sums of angles can be made.

Geometry is such a rich medium through which the proficiency of reasoning can be developed and encouraged, it is a pity that it is a strand of mathematics that is not more prominent in many classrooms. Geometry almost demands the use of manipulative materials, which in turn provides ample opportunity for meaningful reflection, discussion, analysis, explanation, justification and generalisation. It really does tick so many of the reasoning boxes!

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