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Invariant Scalar Product and Associated Structures for Tachyonic Klein–Gordon Equation and Helmholtz Equation

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Abstract: Although describing very different physical systems, both the Klein–Gordon equation for tachyons ($m^2 < 0$) and the Helmholtz equation share a remarkable property: a unitary and irreducible representation of the corresponding invariance group on a suitable subspace of solutions is only achieved if a non-local scalar product is defined. Then, a subset of oscillatory solutions of the Helmholtz equation supports a unirrep of the Euclidean group, and a subset of oscillatory solutions of the Klein–Gordon equation with $m^2 < 0$ supports the scalar tachyonic representation of the Poincaré group. As a consequence, these systems also share similar structures, such as certain singularized solutions and projectors on the representation spaces, but they must be treated carefully in each case. We analyze differences and analogies, compare both equations with the conventional $m^2 > 0$ Klein–Gordon equation, and provide a unified framework for the scalar products of the three equations.

Keywords: unitary and irreducible representation; Poincaré group; Euclidean group; non-local scalar product; tachyonic scalar field

1. Introduction

Recently, it was shown by the authors of [1] that the scalar tachyonic representation of the Poincaré group can be realized unitarily and irreducibly on suitable, oscillatory solutions of the Klein–Gordon equation with negative squared mass, $m^2 = -\kappa^2 < 0$:

$$\partial_\mu \partial^\mu \phi - \kappa^2 \phi = 0\,, \tag{1}$$

where the usual signature $(+, -, -, -)$ and the units in which $\hbar = c = 1$ are assumed. The key ingredient in order to achieve that is the non-local scalar product given by the following:

$$\langle \phi, \varphi \rangle_t = \frac{\kappa^2}{4\pi} \int d^3x \int d^3x' \left(\phi(x)^* k_2^t(|\vec{x} - \vec{x}'|) \varphi(x') + \partial_0 \phi(x)^* k_1^t(|\vec{x} - \vec{x}'|) \partial_{0'} \varphi(x') \right). \tag{2}$$

Here, a particular Cauchy hypersurface at $x^0 = 0$ is chosen. The integration kernels are given by the following:

$$k_2^t(|\vec{x} - \vec{x}'|) = \frac{\tilde{Y}_2(\kappa|\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|^2}\,, \qquad k_1^t(|\vec{x} - \vec{x}'|) = -\frac{Y_1(\kappa|\vec{x} - \vec{x}'|)}{\kappa|\vec{x} - \vec{x}'|}\,, \tag{3}$$

where $Y_j(x)$ are Bessel functions of the second kind. An adequate regularization for k_2^t is indicated by the tilde.

There is a sharp contrast between (2) and the more familiar, standard scalar product for the $m^2 > 0$ Klein–Gordon equation.

$$\langle \phi, \varphi \rangle_{st} = i \int d^3x \left(\phi(x)^* \partial_0 \varphi(x) - \partial_0 \phi(x)^* \varphi(x) \right). \tag{4}$$

For the tachyonic case, both scalar products are invariant under the Poincaré group transformations. However, while (2) is unitarily equivalent to the scalar product of the tachyonic representation in momentum space, (4) cannot be, given that it is indefinite (this can be traced back to the absence of a gap in the energy p_0). The scalar product (2) in momentum space is given by the following:

$$\langle \phi, \varphi \rangle_m = \int_{|\vec{p}|>\kappa} \frac{d^3p}{\sqrt{\vec{p}^{\,2} - \kappa^2}} \left(\phi^+(\vec{p})^* \varphi^+(\vec{p}) + \phi^-(\vec{p})^* \varphi^-(\vec{p}) \right), \tag{5}$$

which is obviously positive definite. Here, we adopt a Cartesian projection on $p_0 = \pm \sqrt{\vec{p}^{\,2} - \kappa^2} = 0$. Note that the ball of momenta $|\vec{p}| < \kappa$ is cut out. Both components of the wave functions for positive and negative energies $(\phi^+(\vec{p}), \phi^-(\vec{p}))$ are needed in the *same* irreducible representation with support on the one-sheet hyperboloid: the sets of the form $(\phi^+(\vec{p}), 0)$ or $(0, \phi^-(\vec{p}))$ are not invariant subspaces under Lorentz transformations.

The correspondence between the momentum and the configuration space realizations of the representation requires the definition of a generalized Fourier transform, which maps initial conditions $\phi(\vec{x}) \equiv \phi(x)|_{x_0=0}$ and $\dot{\phi}(\vec{x}) \equiv \partial_0 \phi(x)|_{x_0=0}$ to $(\phi^+(\vec{p}), \phi^-(\vec{p}))$ and vice versa:

$$\phi(\vec{x}) = \frac{1}{\sqrt{2(2\pi)^3}} \int_{|\vec{p}|>\kappa} \frac{d^3p}{|p_0|} \left(\phi^+(\vec{p}) + \phi^-(\vec{p}) \right) e^{i\vec{p}\cdot\vec{x}}, \tag{6}$$

$$\dot{\phi}(\vec{x}) = \frac{-i}{\sqrt{2(2\pi)^3}} \int_{|\vec{p}|>\kappa} d^3p \left(\phi^+(\vec{p}) - \phi^-(\vec{p}) \right) e^{i\vec{p}\cdot\vec{x}}, \tag{7}$$

$$\phi^{\pm}(\vec{p}) = \frac{1}{\sqrt{2(2\pi)^3}} \int d^3x \left((|p_0|)_+ \phi(\vec{x}) \pm i(|p_0|)^0_+ \dot{\phi}(\vec{x}) \right) e^{-i\vec{p}\cdot\vec{x}}, \tag{8}$$

where $(|p_0|)_+ = |p_0|$ for $|\vec{p}| \geq \kappa$, $(|p_0|)_+ = 0$ for $|\vec{p}| < \kappa$, $(|p_0|)^0_+ = 1$ for $|\vec{p}| \geq \kappa$ and $(|p_0|)^0_+ = 0$ for $|\vec{p}| < \kappa$.

The kind of non-local expression (2) for a scalar product is not new. The subset of oscillatory solutions of the Helmholtz equation in four dimensions (for instance),

$$\frac{\partial^2 \phi}{\partial x_0^2} + \frac{\partial^2 \phi}{\partial \vec{x}^{\,2}} + \kappa^2 \phi = 0, \tag{9}$$

supports a unitary and irreducible representation of the $4D-$Euclidean group, but a similar, non-local scalar product is required as follows:

$$\langle \phi, \varphi \rangle_h = \frac{\kappa^2}{4\pi} \int_{\mathbb{R}^3} d^3x \int d^3x' \left(\phi(x)^* k_2^h(|\vec{x} - \vec{x}'|) \varphi(x') + \partial_0 \phi(x)^* k_1^h(|\vec{x} - \vec{x}'|) \partial_{0'} \varphi(x') \right) \tag{10}$$

with

$$k_2^h(|\vec{x} - \vec{x}'|) = \frac{J_2(\kappa|\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|^2}, \qquad k_1^h(|\vec{x} - \vec{x}'|) = \frac{J_1(\kappa|\vec{x} - \vec{x}'|)}{\kappa|\vec{x} - \vec{x}'|}, \tag{11}$$

where $J_\alpha(z)$ is the Bessel functions of the first kind. Although the Helmholtz equation can describe different physical phenomena, both in configuration and momentum space, we use x to denote the independent variables to simplify the notation, singularizing a spatial component as x^0 and the other three as $\vec{x}$. We note that, in this case, only the Fourier variables within a ball with radius κ are allowed (see later discussion).

The Klein–Gordon equation with $m^2 < 0$ might, in principle, limit its applications to describe tachyon-like unstable modes in different models in supersymmetry [2], cosmol-

ogy [3,4], quantum field theory in curved spacetime [5] or string theory [6,7]. However, the new framework, with the scalar product (2) and a proper use of the generalized Fourier transform, could potentially shed new light in the analysis of those (and other) physical models and their unitarity. Beyond that, the role of the Hilbert space of oscillatory solutions, together with the scalar product (2), as the one-particle Hilbert space of a quantum field theory, is to be analyzed, and further fundamental applications are yet to be developed as well.

The physical significance of the Helmholtz equation is quite diverse, ranging from wave optics [8,9] to a momentum space representation for a particle moving freely on a sphere [10].

It is interesting to remark that this form of a scalar product is not as exceptional as it might seem. In fact, there is an uncommon, non-local form of the usual scalar product for the $m^2 > 0$ Klein–Gordon equation solutions, as we show in Section 3. Now, we turn to briefly review the definition of the space of oscillatory solutions for the $m^2 < 0$ Klein–Gordon and Helmholtz equations, the projections onto those spaces, and the unitarity of the representations.

2. Space of Solutions for the Tachyonic Klein–Gordon and Helmholtz Equations

Let us unify some notation in order to simplify the discussion. KGt will stand for the $m^2 < 0$ Klein–Gordon equation, KG for the usual one, and H for the Helmholtz equation. Both KGt and H are to be written as the following:

$$\sigma \frac{\partial^2 \phi}{\partial x_0^2} + \frac{\partial^2 \phi}{\partial \vec{x}^2} + \kappa^2 \phi = 0, \tag{12}$$

where $\sigma = +1$ for H and $\sigma = -1$ for KGt. px indicates $p_0 x^0 + \sigma \vec{p} \cdot \vec{x}$. Obviously, σ should also be present in the structure constants and generators of the Lie Algebras of the corresponding symmetry group in each case, but we do not need to make it explicit now.

2.1. Invariant and Irreducible Subspaces of Solutions

Equation (12) has elementary solutions given by the following:

$$\psi_{\vec{p}\pm}(x) = \frac{1}{\sqrt{2(2\pi)^3}} e^{-ipx}, \quad p_0 = \pm\sqrt{\sigma(-\vec{p}^2 + \kappa^2)}. \tag{13}$$

If translations generated by $i\partial_\mu$ are to be unitary on the space made of the closure of the linear span of solutions, such as (13), $\vec{p}$ must be real. p_0 must also be real; hence, $\vec{p}$ must lie within a ball of radius κ for H and outside it for KGt. As a consequence of p_0 being real, solutions spanned by (13) are oscillatory in x^0, but given that the $\vec{p}$ space is incomplete, any superposition of (13) is guaranteed to be spatially oscillatory. That is why we call these spaces of solutions $\mathcal{H}_{osc}$. Note that Poincaré and Euclidean transformations preserve the definition of $\mathcal{H}_{osc}$ in each case: p belongs to the mass one-sheet hyperboloid or the sphere, which are the orbits of the Lorentz group or the group $SO(4)$, respectively.

A more rigorous and complete classification of the solutions of H (which extends trivially to KGt) can be found in [10] (Appendix B). We remark here that the initial value problem for (12) is well-posed only for the oscillatory initial conditions: the solutions' behavior changes smoothly for small perturbations when real exponential and linear solutions (the limiting case) are excluded.

2.2. Projectors onto the Irreducible Subspaces

Observing the generalized Fourier transform (6)–(8), it can be seen that it cuts out the spatial momenta inside (KGt) or outside (H) the ball of radius κ. This means that initial conditions must be restricted in order to render the transformation unitary: they must be oscillatory in the above sense. This restriction to (or, better, projection onto) suitable

initial conditions for functions in configuration space can be obtained by inserting (8) in (6) and (7) for KGt:

$$\frac{1}{2(2\pi)^3} \int d^3x' \left\{ \int_{|\vec{p}|>\kappa} d^3p\, 2(|\vec{p}|)^0_+ e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} \right\} \phi(\vec{x}')$$

$$= \phi(\vec{x}) - \int d^3x' \left(\frac{\kappa}{2\pi}\right)^{\frac{3}{2}} \frac{J_{\frac{3}{2}}(\kappa|\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|^{\frac{3}{2}}} \phi(\vec{x}') \equiv \tilde{\phi}(\vec{x}),$$

where we have split the integral as $\int_{|\vec{p}|>\kappa} = \int_{\mathbb{R}^3} - \int_{|\vec{p}|<\kappa}$ and obtained a Dirac delta and a kernel with a Bessel function of the first kind, $k_{3/2}(r) = \left(\frac{\kappa}{2\pi}\right)^{\frac{3}{2}} \frac{J_{3/2}(r)}{r^{3/2}}$ for each term (see also Equation (3.5) in [11]). Obviously, for H we only have $k_{3/2}(r)$. The convolution with $k_{3/2}(r)$ gives zero for functions $\phi(\vec{x})$ and $\dot{\phi}(\vec{x})$, whose ordinary Fourier spectrum is outside the ball, and the identity for the spectrum is inside the ball. These projectors (either $\delta^{(3)}(\vec{x}-\vec{x}') - k_{3/2}(\kappa|\vec{x}-\vec{x}'|)$ or $k_{3/2}(\kappa|\vec{x}-\vec{x}'|)$) are orthogonal projectors with respect to the corresponding non-local scalar products. They constitute a non-trivial operator in $L^2(\mathbb{R}^3)$, and commute with the operators of the corresponding representation, but they become the identity in $\mathcal{H}_{osc}$. This is a manifestation of Schur's lemma of irreducible representations.

2.3. Unitarity

The $\mathcal{H}_{osc}$ spaces are irreducible spaces in the sense that they can be mapped unitarily to the representation space of unitary and irreducible representations of the corresponding symmetry groups. This is made by the generalized Fourier transform (6)–(8) in the case of KGt. Let us show that, for KGt, the scalar product (2) is crucial to achieve unitarity. The corresponding H version is obtained by changing $|\vec{p}| > \kappa$ by $|\vec{p}| < \kappa$ (see [10]).

The momentum space wave functions corresponding to (13), eigenstates of momentum $\vec{p}'$, can be obtained by using the new Fourier Transform (8) (recall that $|\vec{p}'| \geq \kappa$ is required): $\psi_{\vec{p}'+}(\vec{p}) = (|p_0|\delta^{(3)}(\vec{p}-\vec{p}'), 0)$ (for positive energy states), and $\psi_{\vec{p}'-}(\vec{p}) = (0, |p_0|\delta^{(3)}(\vec{p}-\vec{p}'))$ (for negative energy states). We can insert them in the scalar product for momentum space (5) and obtain the following:

$$\langle \psi_{\vec{p}\lambda}, \psi_{\vec{p}'\lambda'} \rangle_m = \delta_{\lambda\lambda'} |p_0| \delta^{(3)}(\vec{p}-\vec{p}'), \tag{14}$$

where $\lambda, \lambda' = \pm$. Now, using that $\psi_{\vec{p}\lambda}(\vec{x}) = \frac{1}{\sqrt{2(2\pi)^3}} e^{i\vec{p}\cdot\vec{x}}$ and $\dot{\psi}_{\vec{p}\lambda}(\vec{x}) = \frac{-i\lambda|p_0|}{\sqrt{2(2\pi)^3}} e^{i\vec{p}\cdot\vec{x}}$, let us obtain (14) in configuration space, where (2) is needed:

$$\langle \psi_{\vec{p}\lambda}, \psi_{\vec{p}'\lambda'} \rangle_t =$$

$$= \frac{\kappa^2}{4(2\pi)^4} \int d^3x' \left\{ \int d^3x\, k_2^t(|\vec{x}-\vec{x}'|) e^{-i\vec{p}\cdot(\vec{x}-\vec{x}')} \right.$$

$$\left. + \lambda\lambda'|p_0||p_0'| \int d^3x\, k_1^t(|\vec{x}-\vec{x}'|) e^{-i\vec{p}\cdot(\vec{x}-\vec{x}')} \right\} e^{-i(\vec{p}-\vec{p}')\cdot\vec{x}'}$$

$$= \frac{1}{2(2\pi)^3} \int d^3x' \left\{ (|p_0|)_+ + \lambda\lambda'|p_0'|(|p_0|)^0_+ \right\} e^{-i(\vec{p}-\vec{p}')\cdot\vec{x}'} \tag{15}$$

$$= \frac{1}{2}(1+\lambda\lambda')(|p_0|)_+ \delta^{(3)}(\vec{p}-\vec{p}') = \delta_{\lambda\lambda'}(|p_0|)_+ \delta^{(3)}(\vec{p}-\vec{p}').$$

We insert $e^{i\vec{p}\cdot\vec{x}'} e^{-i\vec{p}\cdot\vec{x}'}$ in the first step and recall that $(|p_0|)_+$ and $(|p_0|)^0_+$ are zero if $|\vec{p}| < \kappa$. Additionally, we use $\int d^3x\, e^{-i\vec{p}\cdot\vec{x}} = (2\pi)^3 \delta^{(3)}(\vec{p})$ and $\int d^3x\, k_1^t(|\vec{x}|) e^{-i\vec{p}\cdot\vec{x}} = \frac{4\pi}{\kappa^2} \frac{(|p_0|)^0_+}{|p_0|}$. The integral containing k_2^t does not converge: a regularization is required, as we have already mentioned. Using dimensional regularization in dimension d and performing the analytical continuation to dimension 3, we find $\int_{\mathbb{R}^3} d^3x\, k_2^t(|\vec{x}|) e^{-i\vec{p}\cdot\vec{x}} = \frac{4\pi}{\kappa^2} (|p_0|)_+$.

Equations (14) and (15) are in agreement within the Hilbert space of oscillatory solutions, which confirms the unitary equivalence of the representations in configuration

and momentum space. Had we employed the usual scalar product (4), we would have arrived at $\langle \psi_{\vec{p}\lambda}, \psi_{\vec{p}'\lambda'} \rangle = \frac{1}{2}(\lambda + \lambda')|p_0|\delta^{(3)}(\vec{p} - \vec{p}')$, which is incompatible with the positive-definiteness of the scalar product (5) in momentum space representation, as well as with the restriction to $\mathcal{H}_{osc}$ in the configuration space representation.

3. A Note on Non-Local Scalar Products for the ($M^2 > 0$) Klein–Gordon Equation with Positive Energy

The usual scalar product (4) for the $m^2 > 0$ Klein–Gordon equation is indefinite for general solutions. This is due to the fact that the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow$ is represented reducibly. However, if we restrict to positive-energy solutions, the representation is irreducible and the usual scalar product turns out to be positive-definite, although this is not explicit, given the minus sign in (4). In such a case, the time derivative can be replaced by the action of the operator P_0, $\partial_0\phi(x) = -iP_0\phi(x)$, with $P_0 > 0$. Then, given that the representation is unitary, we can obtain the following non-local form:

$$\langle \phi, \varphi \rangle_{st} = \int d^3x \Big(\phi(x)^* P_0 \varphi(x) + P_0\phi(x)^* \varphi(x) \Big) = 2 \int d^3x \, \phi(x)^* P_0 \varphi(x)$$
$$= \int d^3x \int d^3x' \phi(x)^* k_2^s(|\vec{x} - \vec{x}'|)\varphi(x') \,. \tag{16}$$

The action of P_0 is obtained *formally* by means of the following convolution:

$$P_0\varphi(x) = \int d^3x' \Big(-8m^2 \frac{\tilde{K}_2(m|\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|^2} \Big) \varphi(x') \equiv \int d^3x' k_2^s(|\vec{x} - \vec{x}'|)\varphi(x') \,, \tag{17}$$

where $K_\alpha(z)$ is the modified Bessel functions of the second kind, and the tilde indicates a certain regularization. We insist that this is only possible because the sign of P_0 is fixed in the irreducible representation. An expression of this kind for the scalar product for functions with positive energy can also be found in [12] (substituting there Equation (5.127) in (5.131)). As in the case of $m^2 < 0$, the convolution kernel turns out to be singular and needs to be regularized in a proper way.

Is should be remarked that, even though the $m^2 > 0$ Klein–Gordon equation is second-order in time, the initial conditions $\varphi(x)|_{x_0=0}$ and $\partial_0\varphi(x)|_{x_0=0}$ are not independent when restricted to this representation: they are related by P_0. As a consequence, yet another (non-local) form of the scalar product is possible as well in terms of just the time derivatives at the Cauchy surface:

$$\langle \phi, \varphi \rangle_{st} = \int d^3x \int d^3x' \partial_0\phi(x)^* k_1^s(|\vec{x} - \vec{x}'|)\partial_{0'}\varphi(x') \,, \tag{18}$$

with $k_1^s(r) = 8m\frac{K_1(mr)}{r}$ and where the kernel k_1^s now encodes the action of the inverse of the P_0 operator.

The three forms for the scalar product, (4), (16) and (18), are equivalent. Thus, a general form can be cast in an arbitrary combination of those three forms.

4. Unified Description of Scalar Products for the Klein-Gordon and Helmholtz Equations

As commented in the introduction, non-local scalar products, such as (2), are not new; they already appeared in the pioneer study [13] of scalar products on spaces of solutions of Klein–Gordon ($m^2 > 0$) and Helmholtz equations. In [13], a general scalar product of the following form was proposed:

$$\langle \phi, \varphi \rangle_{\text{gen}} = \int d^3x \int d^3x' (\phi(x)^*, \partial_0\phi(x)^*) \begin{pmatrix} M_{11}(\vec{x}, \vec{x}') & M_{12}(\vec{x}, \vec{x}') \\ M_{21}(\vec{x}, \vec{x}') & M_{22}(\vec{x}, \vec{x}') \end{pmatrix} \begin{pmatrix} \varphi(x') \\ \partial_{0'}\varphi(x') \end{pmatrix} \tag{19}$$

involving the values of the functions $\phi(x)$ and their first derivatives $\partial_0\phi(x)$ at some Cauchy hypersurface ($x_0 \equiv t = 0$ for the KG equation and $x_0 \equiv x_k = 0$ for some spatial coordinate x_k for the Helmholtz equation). The functions $M_{ij}(\vec{x}, \vec{x}')$ are integral kernels to be determined by imposing invariance of the scalar product (19) under the corresponding symmetry group of the equation (the Poincaré group for the KG equation and the Euclidean group for the Helmholtz equation). In addition, the scalar product (19) was requested to be bounded and positive definite when restricted to a suitable invariant and irreducible (under the corresponding group) subspace of solutions (positive energy solutions for the $m^2 > 0$ KG equation and oscillatory solutions for the Helmholtz equation). These conditions lead to the scalar products (4) and (10) for the $m^2 > 0$ KG and the Helmholtz case, respectively.

Pursuing further the ideas in [13], we searched in [1] for an invariant scalar product corresponding to the tachyonic $m^2 < 0$ KG equation, but we had to relax the boundedness condition of the integral kernels $M_{ij}(\vec{x}, \vec{x}')$, allowing for divergent, though regularizable, kernels such as those appearing in (2). In addition, a restriction to the subspace of oscillatory solutions of the $m^2 < 0$ KG equation is required in order to achieve irreducibility and positive definiteness.

Additionally, we were able to obtain in [1] a general form for the integral kernels that derive from a unique function $K(x)$ and that allows to write the scalar product, in the case of the $m^2 < 0$ KG equation, in a covariant fashion:

$$\langle \phi, \varphi \rangle_{\text{cov}} = \frac{\kappa^2}{4\pi} \int_\Sigma d\sigma^\mu \int_{\Sigma'} d\sigma^{\nu'} \left(\phi(x)^* \partial_\mu \partial_{\nu'} K(x - x') \varphi(x') + \partial_\mu \phi(x)^* K(x - x') \partial_{\nu'} \varphi(x') \right), \quad (20)$$

where $K(x) = -\frac{Y_1(\kappa s)}{\kappa s}$ with $s = \sqrt{-x^2}$, and $Y_1(x)$ is a Bessel function of the second kind. Here, Σ (and Σ' for the primed coordinates) is a Cauchy hypersurface and $d\sigma^\mu$ is its area element.

This strongly suggests that the scalar product (19) proposed in [13] can be restricted, without losing generality, to the following:

$$\langle \phi, \varphi \rangle_{\text{gen}} = \int d^3x \int d^3x' (\phi(x)^*, \partial_0\phi(x)^*) \begin{pmatrix} \partial_0\partial_{0'} M(x, x') & \partial_0 M(x, x') \\ \partial_{0'} M(x, x') & M(x, x') \end{pmatrix} \begin{pmatrix} \varphi(x') \\ \partial_{0'}\varphi(x') \end{pmatrix}, \quad (21)$$

with $M(x, x') = \Delta(\sqrt{-(x - x')^2})$ and $\Delta(x)$ being an invariant and irreducible solution of the homogeneous KG equation (either for $m^2 < 0$ or $m^2 > 0$) or the Helmholtz equation. An invariant solution here means a scalar solution with respect to Poincaré or Euclidean transformations, i.e., a solution which is a function of x^2. Irreducibility means that it belongs to the irreducible subspace of solutions supporting the considered irreducible and unitary representation of the group (either Poincaré or Euclidean group). Due to the invariance of this scalar product, we can restrict to $x_0 = 0$, resulting in the following:

$$\langle \phi, \varphi \rangle_{\text{gen}} = \int d^3x \int d^3x' (\phi(\vec{x})^*, \dot{\phi}(\vec{x})^*) \begin{pmatrix} \ddot{K}(|\vec{x} - \vec{x}'|) & \dot{K}(|\vec{x} - \vec{x}'|) \\ -\dot{K}(|\vec{x} - \vec{x}'|) & K(|\vec{x} - \vec{x}'|) \end{pmatrix} \begin{pmatrix} \varphi(\vec{x}') \\ \dot{\varphi}(\vec{x}') \end{pmatrix}, \quad (22)$$

where $\ddot{K} \equiv \partial_0\partial_{0'}\Delta|_{x_0=x_0'=0}$, $\dot{K} \equiv \partial_0\Delta|_{x_0=x_0'=0} = -\partial_{0'}\Delta|_{x_0=x_0'=0}$, and $K = \Delta|_{x_0=x_0'=0}$.

The invariant solutions for the homogeneous KG equation are well-known, both for $m^2 > 0$ and $m^2 < 0$.

4.1. Case $m^2 > 0$

We have the following:

$$\Delta_\pm(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{d^3p}{\sqrt{\vec{p}^2 + m^2}} e^{\mp ipx} \quad (23)$$

which are two invariant and irreducible solutions of the homogeneous KG equation, associated with the irreducible and unitary representations of the (connected) proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow$ with positive and negative energies, respectively.

Computing the scalar product (22) with the functions $\Delta_{\pm}$ we obtain the following:

$$\langle \phi, \varphi \rangle_{\pm} = \int d^3x \int d^3x' (\phi(\vec{x})^*, \dot{\phi}(\vec{x})^*) \begin{pmatrix} k_2^s(|\vec{x} - \vec{x}'|) & \pm i\delta(\vec{x} - \vec{x}') \\ \mp i\delta(\vec{x} - \vec{x}') & k_1^s(|\vec{x} - \vec{x}'|) \end{pmatrix} \begin{pmatrix} \varphi(\vec{x}') \\ \dot{\varphi}(\vec{x}') \end{pmatrix}. \quad (24)$$

These scalar products are definite positive in their respective subspaces of positive and negative energy solutions.

In the literature, it is common to consider linear combinations of these invariant functions, which are no longer irreducible under $\mathcal{P}_+^{\uparrow}$ (although still invariant), but that are invariant and irreducible under a unitary and irreducible representation of $\mathcal{P}_+^{\uparrow} \cup \mathcal{P}_+^{\downarrow}$. These are (see, for instance, [14]) the usual Pauli–Jordan function:

$$\Delta_{\text{PJ}}(x) = -\frac{i}{2}(\Delta^+ - \Delta^-) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{d^3p}{\sqrt{\vec{p}^2 + m^2}} e^{i\vec{p}\cdot\vec{x}} \sin\left(x_0\sqrt{\vec{p}^2 + m^2}\right), \quad (25)$$

and the Δ_1 function:

$$\Delta_1(x) = \frac{1}{2}(\Delta^+ + \Delta^-) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{d^3p}{\sqrt{\vec{p}^2 + m^2}} e^{i\vec{p}\cdot\vec{x}} \cos\left(x_0\sqrt{\vec{p}^2 + m^2}\right). \quad (26)$$

Computing the scalar product (22) with the Pauli–Jordan function Δ_{PJ}, the standard scalar product (4) is recovered. The problem is that it is not positive definite since Δ_{PJ} is only irreducible under $\mathcal{P}_+^{\uparrow} \cup \mathcal{P}_+^{\downarrow}$ and contains a minus sign. It is possible to restrict it to positive energy solutions, but then the transformations of Section 3 apply, rendering it into a non-local scalar product.

For the case of the function Δ_1, the associated scalar product (22) is the following:

$$\langle \phi, \varphi \rangle_1 = \frac{1}{(2\pi)^3} \int d^3x \int d^3x' \left(\phi(x)^* k_2^s(|\vec{x} - \vec{x}'|) \varphi(x') + \dot{\phi}(x)^* k_1^s(|\vec{x} - \vec{x}'|) \dot{\varphi}(x') \right), \quad (27)$$

which is the sum of (16) and (18). This scalar product is positive definite for both the positive and negative energy representations of $\mathcal{P}_+^{\uparrow}$. Note that, in this form, the scalar product for the cases $m^2 > 0$ (27), $m^2 < 0$ (2) and Helmholtz equation are identical, simply changing Bessel K functions by Bessel Y functions or Bessel J functions. In addition, in the KG cases, a proper regularization in computing the integration kernels appearing in (22) is required.

4.2. Case $m^2 < 0$

In this case,

$$\Delta_t(x) = \frac{1}{(2\pi)^3} \int_{|\vec{p}| \geq \kappa} \frac{d^3p}{\sqrt{\vec{p}^2 - \kappa^2}} e^{i\vec{p}\cdot\vec{x}} \cos\left(x_0\sqrt{\vec{p}^2 - \kappa^2}\right) \quad (28)$$

is an invariant and irreducible solution of the $m^2 < 0$ KG equation (see [15–17]).

Computing the scalar product (22) with the solution Δ_t, the scalar product (2) is recovered, as expected.

4.3. Case of Helmholtz Equation

For the case of Helmholtz equation the invariant and irreducible function is given by the following:

$$\Delta_h(x) = \frac{1}{(2\pi)^3} \int_{|\vec{p}| \leq \kappa} \frac{d^3p}{\sqrt{\kappa^2 - \vec{p}^2}} e^{i\vec{p}\cdot\vec{x}} \cos\left(x_0\sqrt{\kappa^2 - \vec{p}^2}\right). \quad (29)$$

Computing the scalar product (22) with the solution Δ_h, the scalar product (10) is recovered, as expected.

5. Outlook

We have achieved a generalized class of non-local scalar products in configuration space, which seems to be appropriate for both Klein–Gordon solutions with $m^2 > 0$ and $m^2 < 0$, as well as for solutions of the Helmholtz equation. This last case was already found in [13], and actually inspired us to generalize the standard scalar product for Klein–Gordon fields. For the sake of completeness, the case of scalar solutions with zero mass should have been included in the present analysis but, from the physical point of view, the corresponding representation of the Poincaré group can formally be obtained as a $c \to \infty$ limit of the $m^2 > 0$ representations; from the mathematical aspects, the situation is much more involved and will require a separate publication.

The covariant expression (20) suggests a sounder analysis of the equivalence of the quantizations associated with different choices of the Cauchy surface Σ. It is clear from the early stage of quantum (field) theory [18] that different choices of the Cauchy surface related by the action of the Poincaré group on a given irreducible representation lead to the same quantum theory. A limit situation appeared long ago, in high-energy hadronic physics, prior to chromodynamics, when interactions were essentially mediated by currents, which had to be kept finite at the ∞-momentum frame. There, the quantization was achieved on a null Cauchy surface, giving rise to so-called light-cone quantization [19,20], a process which also required some sort of (non-local) renormalization of the physical states. Even though the cross sections turned out to be equivalent to those obtained in the standard quantization, a mathematical proof of the equivalence was (and is at present, to our understanding) lacking. The situation in the case of the Helmholtz equation is more flexible since the rotation group relates all possible Cauchy hyperplanes.

We also may question the opportunity for the use of $m^2 < 0$ of any of the Cauchy surfaces related to $x^0 = 0$ by Lorentz transformations, that is to say, a typical Cauchy surface for ordinary matter. The reason for that is the existence of what would be tachyonic particles traveling along an $\vec{x}$-direction while keeping x^0 constant. As regards unitarity of the theory, the situation is somehow similar to that created in non-globally hyperbolic spaces (case of anti-de Sitter space-time). It is clear that the less natural choice of $x^0 = 0$ is recommended, however, by the necessity of dealing simultaneously with ordinary matter.

An algebraic association of Cauchy surfaces with irreducible orbits of the Poincaré group (symmetry group, in general) can be realized on the grounds of a group quantization scheme (see [21] and references therein), where quantizations are nothing other than unitary and irreducible representations of a central extension of the classical symmetry group driving the phase invariance in quantum mechanics. In the case of the Poincaré group, with lie algebra,

$$[M_{\mu\nu}, P_\rho] = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu$$

is extended by the central generator Ξ of $U(1)$ as the following:

$$[\widetilde{M}_{\mu\nu}, \widetilde{P}_\rho] = \eta_{\nu\rho} \widetilde{P}_\mu - \eta_{\mu\rho} \widetilde{P}_\nu - (\lambda_\mu \eta_{\nu\rho} - \lambda_\nu \eta_{\mu\rho}) \Xi \equiv C_{\mu\nu,\ \rho}^{\sigma} \widetilde{P}_\sigma + C_{\mu\nu,\ \rho}^{\Xi} \Xi \ ,$$

where

$$C_{\mu\nu,\ \rho}^{\Xi} \equiv \lambda_\nu \eta_{\mu\rho} - \lambda_\mu \eta_{\nu\rho} \ ,$$

and λ_μ is a vector in the Poincaré co-algebra belonging to a given co-adjoint orbit.

The vector λ ultimately characterizes the proper Cauchy surface Σ, driving the evolutive structure of the (classical variational calculus, leading to the field equation and) quantization; λ is normal to Σ. In fact, the (left) canonical one-form on the extended Poincaré group in the direction of the generator Ξ is the quantization form generalizing the Poincaré–Cartan form Θ_{PC}; the kernel of $d\Theta_{PC}$ constitutes the evolution of the system.

In particular, for ordinary matter, λ is a time-like vector so that the Cauchy surface is spatial; for photons, λ is a null vector and Σ, a null surface; and for tachyons, Σ is, naturally, a (Lorentzian) surface containing the x^0 axis.

It is suspected that the use of a Cauchy surface containing the x^0 axis requires a specific regularization procedure for propagators (invariant Pauli–Jordan functions) related to a possible convenient renormalization of physical states in momentum space.

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