Extremal results in hypergraph theory via the absorption method

by

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Abstract

The so-called "absorbing method" was first introduced in a systematic way by Rödl, Ruciński and Szemerédi in 2006, and has found many uses ever since. Speaking in a general sense, it is useful for finding spanning substructures of combinatorial structures. We establish various results of different natures, in both graph and hypergraph theory, most of them using the absorbing method:

(i) We prove an asymptotically best-possible bound on the strong chromatic number with respect to the maximum degree of the graph. This establishes a weak version of a conjecture of Aharoni, Berger and Ziv.

(ii) We determine asymptotic minimum codegree thresholds which ensure the existence of tilings with tight cycles (of a given size) in uniform hypergraphs. Moreover, we prove results on coverings with tight cycles.

(iii) We show that every 2-coloured complete graph on the integers contains a monochromatic infinite path whose vertex set is sufficiently "dense" in the natural numbers. This improves results of Galvin and Erdős and of DeBiasio and McKenney.

Dedication

Al petoto.

Vieja pared del arrabal, tu sombra fue mi compañera...

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Introduction

1.1 The absorption method

Some of the most important results in graph theory take the following form: the presence of some 'local' sufficient condition ensures the existence of a 'global' structure. Perhaps the archetypical example of a result of this type is Dirac's theorem [16], which states that every graph on $n \geq 3$ vertices whose minimum degree is at least $n/2$ contains a Hamiltonian cycle, that is, a cycle which visits every vertex of the graph exactly once. The method of "absorption" was developed during the search for generalisations of Dirac's theorem, but, in a general sense, it is helpful for showing the existence of spanning substructures in combinatorial structures. We use this technique to tackle problems of different natures in both graph and hypergraph theory.

The absorption method was first introduced in a systematic way by Rödl, Ruciński and Szemerédi [63] in the proof of a generalisation of Dirac's theorem to uniform hypergraphs. It must be remarked that similar ideas were used before by Krivelevich [49] to study triangle tilings in random graphs; and even earlier by Erdős, Gyárfás and Pyber [19] to find monochromatic partitions using cycles in complete edge-coloured graphs.

To discuss the absorption method we introduce the setting of hypergraphs, as it will form the basis for most of the results that follow. A *hypergraph* $H =$

$(V(H), E(H))$ consists of a *vertex set* $V(H)$ and an *edge set* $E(H)$, where each edge $e \in E(H)$ is a subset of $V(H)$. Given $k \geq 0$, a *k-uniform hypergraph* (or *k-graph* for short) is a hypergraph where every edge has size exactly k.

As discussed previously, the absorbing method gives a recipe to find spanning substructures. Perhaps the easiest example of a spanning structure in the setting of hypergraphs is that of a perfect matching. A *matching* M in a hypergraph H is a collection of pairwise disjoint edges $M \subseteq E(H)$. A matching M in H is said to be *perfect* if the union of its edges covers the whole vertex set of H. Note that if H is a k-graph on a vertex set of size n and has a perfect matching, then it must hold that n is divisible by k.

Suppose $\varepsilon > 0$ is given and small, and H is a k-graph on n vertices, with n divisible by k. The absorbing method applied to the problem of finding a perfect matching in a hypergraph H would consist in the following three steps.

(i) **Find an absorbing set**. Find a set $A \subseteq V(H)$ with the following property: for each subset $S \subseteq V(H)$, disjoint from A, which has size at most εn and $|S \cup A|$ is divisible by k, there exists a perfect matching in $H[S \cup A]$. Here $H[S \cup A]$ is the induced k-graph obtained by restricting the vertex set to $S \cup A$ and keeping the edges entirely contained within that set.

(ii) **Find an almost-perfect matching**. In the induced k-graph $H[V(H) \setminus A]$, find a matching M' that covers all but at most εn vertices.

(iii) **Absorb**. Let $S = V(H) \setminus (A \cup \bigcup_{e \in M'} e)$. By the choice of M', S has size at most εn. Since M' is a matching consisting of disjoint edges of size k, and $n = |V(H)|$ is divisible by k, it follows that $|S \cup A|$ is divisible by k. By the choice of A, there exists a perfect matching M'' in $H[S \cup A]$. Then $M := M' \cup M''$ is a perfect matching in H.

Of course, for this approach to work, we need sufficient structure in H to be able to carry out steps (i) and (ii). In typical applications of this strategy, to show that (i) holds, it is necessary to describe an ad-hoc construction, which is then found in H using probabilistic methods. To show that (ii) holds, there are different techniques already existing in the literature to find "almost-perfect" matchings, which are applicable if H satisfies certain appropriate requirements. We will encounter different instances of this approach in the following chapters.

1.2 Asymptotic upper bounds for the strong chromatic number

The first problem we consider is related to graph colouring. This is joint work with Allan Lo, and has been published in Combinatorics, Probability and Computing [54].

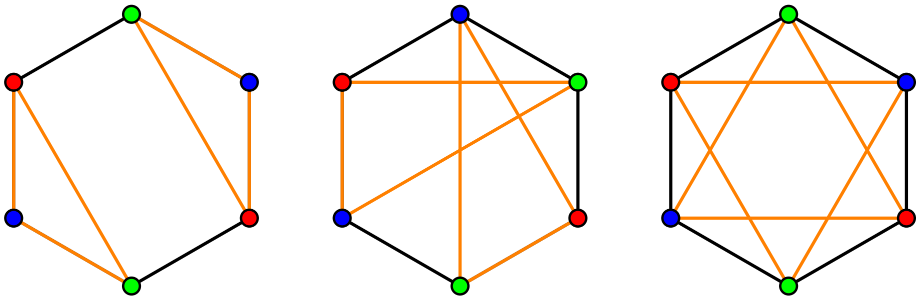

Figure 1.1: The cycle C_6 has strong chromatic number equal to 3. The picture shows C_6 using black edges, together with all of the possible ways (up to symmetry) to choose a spanning collection T of copies of K_3 on $V(C_6)$, using orange edges in each case. In each case, a 3-colouring for $C_6 \cup T$ is shown. This demonstrates that C_6 is 3-strongly colourable.

On the other hand, note that C_6 is not 2-strong colourable, since there exists a spanning collection M of copies of K_2 on $V(C_6)$ (i.e., a perfect matching) such that $C_6 \cup M$ is not 2-colourable. Indeed, it is not difficult to find a matching M such that $C_6 \cup M$ contains a triangle, and thus is not 2-colourable.

Let r be a positive integer. Let G be a graph on n vertices, where r divides n. We say that G is *strongly r-colourable* if it can be properly r-coloured after taking the union of G with any collection of spanning disjoint copies of K_r in the same vertex set. Equivalently, G is strongly r-colourable if for every partition $\{V_1, \dots, V_k\}$ of $V(G)$ with classes of size r, there is a proper vertex colouring of G using r colours with the additional property that every V_i receives all of the r colours. If r does not divide n, we say that G is *strongly r-colourable* if the graph obtained by adding $r\lceil n/r\rceil - n$ isolated vertices to G is r-strongly colourable. The *strong chromatic number* $\chi_s(G)$ of G is the minimum r such that G is r-strongly colourable.

The notion of strong chromatic number was introduced independently by Alon [4] and Fellows [24], although with a slightly different definition. In their version, a graph G is r-strongly-colourable if for every partition $\{V_1, \dots, V_k\}$ of $V(G)$ into sets of size at most r, the graph G' obtained by adding a clique in each of the sets V_i is r-colourable. For this different notion of "strong chromatic number" it is known that if a graph is r-strong-colourable, then it is $(r+1)$-strong-colourable [24, Theorem 6]. Instead, our notion of "strong chromatic number" follows closely that of Haxell [38] and Axenovich and Martin [8], where no such monotonicity is known to hold.

One of the first problems related to the strong chromatic number was the cycles-plus-triangles problem of Erdős (see [26]), who asked (in an equivalent form) if $\chi_s(C_{3m}) \leq 3$, where C_{3m} is the cycle on $3m$ vertices. This was answered affirmatively by Fleischner and Stiebitz [25] and independently by Sachs [67]. Figure 1.1 shows a particular instance of this problem using C_6.

It is an open problem to find the best bound on $\chi_s(G)$ in terms of $\Delta(G)$. Alon [5] proved that $\chi_s(G) \leq c\Delta(G)$ for some constant $c > 0$. Haxell [38] showed that $c = 3$ suffices and later [39] that $c \leq 11/4 + \varepsilon$ suffices given $\Delta(G)$ is large enough with respect to ε. On the other hand, there are examples showing $c \geq 2$ is necessary, as described, for instance, by Axenovich and Martin [8]. Such an example is shown in Figure 1.2.

It is conjectured (first explicitly stated by Aharoni, Berger and Ziv [2, Conjecture 5.4]) that this lower bound is also tight.

Conjecture 1.2.1. *For every graph G, $\chi_s(G) \leq 2\Delta(G)$.*

Conjecture 1.2.1 is known to be true for graphs G on n vertices with $\Delta(G) \geq n/6$. This was proven by Axenovich and Martin [8] and independently by Johansson,

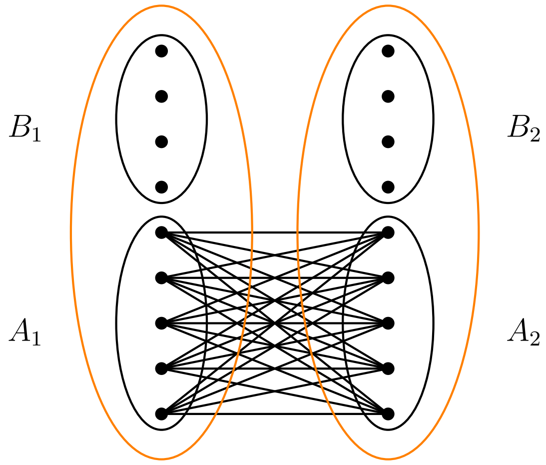

Figure 1.2: An example of a graph where $\chi_s(G) \geq 2\Delta(G)$. For any $\Delta \geq 1$, consider the graph G on $A_1 \cup B_1 \cup A_2 \cup B_2$ where all these sets are pairwise disjoint, $|A_i| = \Delta$ and $|B_i| = \Delta - 1$ for all $i \in \{1,2\}$, and add every possible edge between A_1 and A_2. Then $\Delta(G) = \Delta$ and $|V(G)| = 2(2\Delta-1)$. The partition $\{A_1 \cup B_1, A_2 \cup B_2\}$ shows that G is not strongly $(2\Delta-1)$-colourable. It is not possible to colour G with $2\Delta-1$ colours using every colour exactly once both in $A_1 \cup B_1$ and $A_2 \cup B_2$, since under that restriction the vertices in $A_1 \cup A_2$ would require at least 2Δ different colours.

Johansson and Markström [42].

In Chapter 3, we prove that Conjecture 1.2.1 is asymptotically true if $\Delta(G)$ is linear in $|V(G)|$.

Theorem 1.2.2. *For all $c, \varepsilon > 0$, there exists $n_0 = n_0(c,\varepsilon)$ such that the following holds: if G is a graph on $n \geq n_0$ vertices with $\Delta(G) \geq cn$, then $\chi_s(G) \leq (2+\varepsilon)\Delta(G)$.*

To see where the absorption technique appears in the proof of this result, we introduce some terminology. Let G be a graph and let $\mathcal{P} = \{V_1, \ldots, V_k\}$ be a partition of $V(G)$. A subset $S \subseteq V(G)$ is *$\mathcal{P}$-partite* if $|S \cap V_i| \leq 1$ for every $i \in [k]$. A *transversal of $\mathcal{P}$* is a $\mathcal{P}$-partite set of cardinality $|\mathcal{P}|$. An *independent transversal of $\mathcal{P}$* is a transversal of $\mathcal{P}$ which is also an independent set in G. We will write *transversal* and *independent transversal* if G and $\mathcal{P}$ are clear from the context.

The crucial observation is that having fixed a graph G and a partition $\mathcal{P} = \{V_1, \ldots, V_k\}$ of $V(G)$ into parts of size r, to find a proper vertex-colouring such that each V_i receives r colours; it is equivalent to find a vertex-partition of the graph into independent transversals with respect to G and $\mathcal{P}$. Indeed, if we have a vertex-partition of the graph into independent transversals, colouring each independent transversal with a different colour results in a colouring where each V_i receives r colours. In the other direction, each colour class of a proper colouring induces an independent set, hence, if each V_i receives r colours, each of the colour classes induced by the colouring must be an independent transversal, and together they partition the vertex set.

By doing this, we have shown that to prove Theorem 1.2.2 it is enough to find a perfect matching (a spanning collection of disjoint edges) in the hypergraph whose edges correspond to the independent transversals of G and $\mathcal{P}$. This allows the problem to be attacked using the absorption technique.

Being more precise, to prove Theorem 1.2.2 it suffices to show that given any partition $\mathcal{P}$ of $V(G)$ with classes of size $r \geq (2+\varepsilon)\Delta(G)$, $V(G)$ can be partitioned into independent transversals of $\mathcal{P}$. The assumption of $\Delta(G) \geq cn$ in Theorem 1.2.2 implies that the partitions $\mathcal{P}$ we need to consider will have a bounded number of classes, independent of n (namely, at most $1/2c$). Since Conjecture 1.2.1 is known to be true for graphs on n vertices with $\Delta(G) \geq n/6$, we can restrict ourselves to study graphs with $\Delta(G) \leq n/6$, and in such graphs any partition $\mathcal{P}$ of $V(G)$ with parts of size $r = (2+\varepsilon)\Delta(G) < 3\Delta(G)$ will have at least 3 classes. Thus Theorem 1.2.2 is implied by the following theorem, whose proof is the main objective of Chapter 3.

Theorem 1.2.3. *For all integers $k \geq 3$ and $\varepsilon > 0$, there exists $r_0 = r_0(k,\varepsilon)$ such that the following holds for all $r \geq r_0$: if G is a graph and $\mathcal{P}$ is a partition of $V(G)$ with k classes of size $r \geq (2+\varepsilon)\Delta(G)$, then there exists a partition of $V(G)$ into independent transversals of $\mathcal{P}$.*

By considering the complement graph, Theorem 1.2.3 immediately yields the following tiling-type result as a corollary. A *perfect K_k-tiling* of a graph G is a spanning subgraph of G with components which are complete graphs on k vertices.

Corollary 1.2.4. *For all integers $k \geq 3$ and $\varepsilon > 0$, there exists $n_0 = n_0(k, \varepsilon)$ such that the following holds: if $n \geq n_0$ and G is a k-partite graph with classes of size n and $\delta(G) \geq (k - 3/2 + \varepsilon)n$, then G has a perfect K_k-tiling.*

We remark that Corollary 1.2.4 is best possible up to the error term εn. To see this, consider a bipartite graph with two classes V_1, V_2 of size n each, with minimum degree $\lceil n/2 \rceil - 1$ and without a perfect matching. This can be done, for instance, by selecting for each $i \in [2]$ a subset $V_i' \subseteq V_i$ of size exactly $\lceil n/2 \rceil - 1$, and adding only the edges between V_1 and V_2 which intersect $V_1' \cup V_2'$. The union of the neighbourhoods of the vertices in $V_2 \setminus V_2'$ is exactly V_1', but $|V_1'| < n/2 < |V_2 \setminus V_2'|$. This shows (by Hall's condition) that a perfect matching does not exist.

Now, add disjoint vertex classes $V_3, \ldots, V_k$ of size n each, and, for every $i \in \{3, \ldots, k\}$, join every vertex of V_i to every vertex in the other classes. The final graph G satisfies $\delta(G) = (k-2)n + \lceil n/2 \rceil - 1$, but it does not have a perfect K_k-tiling since the existence of one would imply the existence of a perfect matching in $G[V_1 \cup V_2]$.

The proof of Theorem 1.2.3 is presented in Chapter 3.

1.3 Covering and tiling hypergraphs with tight cycles

The second problem we consider deals with the notions of covering and tiling hypergraphs. This is joint work with Jie Han and Allan Lo, and has been accepted for publication in Combinatorics, Probability and Computing [32].

We start by describing the problem of finding tilings in the setting of graphs. Let H and F be graphs. An *F-tiling* in H is a set of pairwise vertex-disjoint copies of F. An F-tiling is *perfect* if it spans the vertex set of H. Note that a perfect F-tiling is also known as an *F-factor* or a *perfect F-matching*. The following question in extremal graph theory has a long and rich history: given F and n, what is the maximum δ such that there exists a graph H on n vertices with minimum degree at least δ without a perfect F-tiling? We call such δ the *tiling* *degree threshold* for F and denote it by $t(n, F)$. Note that if $n \not\equiv 0 \bmod |V(F)|$ then a perfect F-tiling cannot exist, so this case is not interesting. Hence we will always assume that $n \equiv 0 \bmod |V(F)|$ whenever we discuss $t(n, F)$.

A first result in this sense comes from the celebrated theorem of Dirac [16] on Hamiltonian cycles, which easily shows that $t(n, K_2) = \lceil n/2 \rceil - 1$. Corrádi and Hajnal [11] proved that $t(n, K_3) = \lceil 2n/3 \rceil - 1$, and Hajnal and Szemerédi [31] generalized this result for complete graphs of any size, showing that $t(n, K_t) = \lceil (1 - 1/t)n \rceil - 1$. For a general graph F, Kühn and Osthus [51] determined $t(n, F)$ up to an additive constant depending only on F. This improved previous results due to Alon and Yuster [7], Komlós, Sarközy and Szemerédi [48] and Komlós [47].

The same type of problems can be studied in the setting of hypergraphs, as soon as we have selected a notion of "minimum degree". To do precisely that, we introduce the following definitions. Given a hypergraph H on vertex set $V = V(H)$, and a set $S \subseteq V$, let the *neighbourhood* $N_H(S)$ *of* S be the set $\{T \subseteq V \setminus S : T \cup S \in E\}$, and let the *degree* $\deg_H(S)$ *of* S be $\deg_H(S) = |N_H(S)|$, i.e., the number of edges of H containing S. If $w \in V$, then we also write $N_H(w)$ for $N_H(\{w\})$. We will omit the subscript if H is clear from the context. We denote by $\delta_i(H)$ the *minimum* i*-degree of* H, that is, the minimum of $\deg_H(S)$ over all i-element sets $S \in \binom{V}{i}$. Note that $\delta_0(H)$ is equal to the number of edges of H. Given a k-graph H, $\delta_{k-1}(H)$ and $\delta_1(H)$ are referred to as the *minimum codegree* and the *minimum vertex degree* of H, respectively.

Using these notions of codegree, we will generalise the "tiling thresholds" to the hypergraph case, and investigate their behaviour when we look for tilings made up of "tight cycles", which correspond to a generalisation of the notion of cycles in graphs.

1.3.1 Tiling thresholds in hypergraphs

Let H and F be k-graphs. An F*-tiling* in H is a set of pairwise vertex-disjoint copies of F. An F-tiling is *perfect* if it spans the vertex set of H. Note that a perfect F-tiling is also known as *F-factor* and *perfect F-matching*. For a k-graph F, define the *codegree tiling threshold* $t(n,F)$ to be the maximum of $\delta_{k-1}(H)$ over k-graphs H on n vertices without a perfect F-tiling.

Similarly to the case of graphs, note that if $n \not\equiv 0 \bmod |V(F)|$ then a perfect F-tiling cannot exist and, so, $t(n,F) = n-k+1$. Hence we will always assume that $n \equiv 0 \bmod |V(F)|$ whenever we discuss $t(n,F)$.

To describe the known results on tiling thresholds for k-graphs, when $k \geq 3$, we need some definitions and notation. Let K_t^k denote the complete k-graph on t vertices. We say that a k-graph H is *t-partite* (or that H is a *(k,t)-graph*, for short) if $V(H)$ has a partition $\mathcal{P} = \{V_1, \dots, V_t\}$ consisting of t clusters such that $|e \cap V_i| \leq 1$ for all edges $e \in E$ and all $1 \leq i \leq t$. That is, every edge of H is $\mathcal{P}$-partite. A (k,t)-graph H is *complete* if $E(H)$ consists of all k-sets e such that $|e \cap V_i| \leq 1$, for all $1 \leq i \leq t$. Equivalently, H consists precisely of all the $\mathcal{P}$-partite edges of size k.

For $k \geq 3$, Kühn and Osthus [51] determined $t(n, K_k^k)$ asymptotically and Rödl, Ruciński and Szemerédi [62] determined the exact value for sufficiently large n. Lo and Markström [53] determined $t(n, K_4^3)$ asymptotically, and independently, Keevash and Mycroft [44] determined $t(n, K_4^3)$ exactly for sufficiently large n. Mycroft [58] determined the asymptotic value of $t(n,K)$ for all complete (k,k)-graphs K. However, much less is known for non-k-partite k-graphs. For more results on tiling thresholds for k-graphs, see the survey of Zhao [75].

1.3.2 Covering thresholds

Now we introduce "coverings", which can be seen as a relaxation of the notion of "tilings". Given a k-graph F, an *F-covering in H* is a spanning set of copies of F. That is, we require the copies of F to cover every vertex of H, but, in contrast to an F-tiling, we do not insist that the copies of F are pairwise vertex-disjoint. Define the *codegree covering threshold* $c(n,F)$ of F to be the maximum of $\delta_{k-1}(H)$ over all k-graphs H on n vertices not containing an F-covering.

Trivially, a perfect F-tiling is an F-covering, and an F-covering has a copy of F. Thus,

$$\mathrm{ex}_{k-1}(n,F) \le c(n,F) \le t(n,F),$$

where $\mathrm{ex}_{k-1}(n,F)$ is the *codegree Turán threshold*, that is, the maximum of $\delta_{k-1}(H)$ over all F-free k-graphs H on n vertices. In this sense, the covering problem is an intermediate problem between the Turán and the tiling problems.

As observed by Han, Zang, and Zhao [33], for any non-empty (2-)graph F, we have $c(n,F) = \left(\frac{\chi(F)-2}{\chi(F)-1} + o(1)\right)n$, where $\chi(F)$ is the chromatic number of F. The same group of people studied the vertex-degree variant of the covering problem, for complete $(3,3)$-graphs K. Falgas-Ravry and Zhao [23] studied $c(n,F)$ when F is K_4^3, K_4^3 with one edge removed, K_5^3 with one edge removed, and other small 3-graphs, obtaining partial, exact and asymptotic results.

1.3.3 Cycles in hypergraphs

Given $1 \le \ell < k$, we say that a k-graph on more than k vertices is an *ℓ-cycle* if every vertex lies in some edge, there is a cyclic ordering of the vertices such that every edge consists of k consecutive vertices under this order, and every two consecutive edges (under the ordering of the vertices) intersect in exactly ℓ vertices. Note that an ℓ-cycle on s vertices can exist only if $k-\ell$ divides s. If $\ell = 1$ we call the cycle *loose*, if $\ell = k-1$ we call the cycle *tight*. We write C_s^k for the k-uniform tight cycle on s vertices.

When $k = 2$, ℓ-cycles reduce to the usual notion of cycles in graphs. Corrádi and Hajnal determined $t(n, C_3^2)$ and Wang determined $t(n, C_4^2)$ and $t(n, C_5^2)$ [73, 74]. Furthermore, El-Zahar [17] gave the following conjecture on cycle tilings.

Conjecture 1.3.1 ([17]). *Let G be a graph on n vertices and let $n_1, \dots, n_r \ge 3$ be integers such that $n_1 + \dots + n_r = n$. If $\delta(G) \ge \sum_{i=1}^{r} \lceil n_i/2 \rceil$, then G contains r vertex-disjoint cycles of lengths $n_1, \dots, n_r$ respectively.*

The bound on the minimum degree in Conjecture 1.3.1, if true, would be best

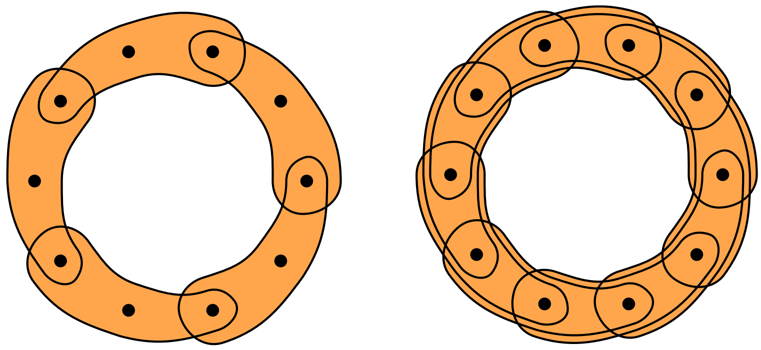

Figure 1.3: A 3-uniform loose cycle on 10 vertices, and a 3-uniform tight cycle on 10 vertices.

possible. In particular, the conjecture would imply that $t(n, C_s^2) = \lceil s/2 \rceil n/s - 1$. The conjecture was verified for $r = 2$ by El-Zahar and a proof (for large n) was announced by Abbasi [1] as well as by Abbasi, Khan, Sárközy and Szemerédi (see [71]).

Given integers ℓ, k such that $1 \le \ell \le (k-1)/2$, it is easy to see that a k-uniform ℓ-cycle on s vertices C satisfies $c(n, C) \le s + 1$ (by constructing C greedily). If $s \equiv 0 \bmod k$, then the tight cycle C_s^k is k-partite. For all $t \ge 1$, let $K^k(t)$ denote the complete (k, k)-graph whose vertex classes each have size t. Note that C_s^k is a spanning subgraph of $K^k(s/k)$. Erdős [20] proved the following result, which implies an upper bound on the Turán number of C_s^k.

Theorem 1.3.2 (Erdős [20])**.** *For all $k \ge 2$ and $s > 1$, there exists $n_0 = n_0(k, s)$ such that $\mathrm{ex}(n, K^k(s)) < n^{k-1/s^{k-1}}$ for all $n \ge n_0$.*

Our first original result surrounding this problem is a sublinear upper bound for $c(n, C_s^k)$ when $s \equiv 0 \bmod k$.

Proposition 1.3.3. *For all $2 \le k \le s$ with $s \equiv 0 \bmod k$, there exist $n_0(k, s)$ and $c = c(k, s)$ such that $c(n, C_s^k) \le cn^{1-1/s^{k-1}}$ for all $n \ge n_0$.*

There are some previously known results for tiling problems regarding ℓ-cycles. Whenever C is a 3-uniform loose cycle, $t(n, C)$ was determined exactly by Czygrinow [13]. For general loose cycles C in k-graphs, $t(n, C)$ was determined asymptotically by Mycroft [58] and exactly by Gao, Han and Zhao [28]. For tight cycles C_s^k with $s \equiv 0 \bmod k$, Mycroft [58] proved that $t(n, C_s^k) = (1/2 + o(1))n$. Notice that all mentioned cycle tiling results correspond to cases where the cycles are k-partite (since k-uniform loose cycles are k-partite for $k \geq 3$).

We now focus on the covering and tiling problems for the tight cycle C_s^k, for all integers k, s which do not necessarily make C_s^k a (k,k)-graph. We show that a minimum codegree of $(1/2 + o(1))n$ suffices to find a C_s^k-covering.

Theorem 1.3.4. *Let $k, s \in \mathbb{N}$ with $k \geq 3$ and $s \geq 2k^2$. For all $\gamma > 0$, there exists $n_0 = n_0(k, s, \gamma)$ such that for all $n \geq n_0$, $c(n, C_s^k) \leq (1/2 + \gamma)n$.*

Moreover, this result is asymptotically tight if k and s satisfy the following divisibility conditions.

Definition 1.3.5. *Let $2 \leq k < s$ and let $d = \gcd(k, s)$. We say that the pair (k, s) is* admissible *if $d = 1$ or k/d is even.*

Note that an admissible pair (k, s) satisfies $s \not\equiv 0 \bmod k$.

Proposition 1.3.6. *Let $3 \leq k < s$ be such that (k, s) is admissible. Then $c(n, C_s^k) \geq \lfloor n/2 \rfloor - k + 1$. Moreover, if k is even, then $\mathrm{ex}_{k-1}(n, C_s^k) \geq \lfloor n/2 \rfloor - k + 1$.*

Notice that if (k, s) is admissible, $k \geq 3$ is even and $s \geq 2k^2$, then Theorem 1.3.4 and Proposition 1.3.6 imply that $\mathrm{ex}_{k-1}(n, C_s^k) = (1/2 + o(1))n$.

We also have the following lower bounds for $\mathrm{ex}_{k-1}(n, C_s^k)$ (and hence, also for $c(n, C_s^k)$ and $t(n, C_s^k)$) which hold in all the cases where s is not divisible by k.

Proposition 1.3.7. *Let $k \geq 2$ and $s > k$ not divisible by k. Let p be a divisor of k which does not divide s. Then $\mathrm{ex}_{k-1}(n, C_s^k) \geq \lfloor n/p \rfloor - k + 2$. In particular, $\mathrm{ex}_{k-1}(n, C_s^k) \geq \lfloor n/k \rfloor - k + 2$.*

We also study the tiling problem corresponding to C_s^k. We give some lower bounds on $t(n, C_s^k)$. Notice that the following bound is significantly higher if (k, s) is admissible.

Proposition 1.3.8. *Let $2 \le k < s \le n$ with n divisible by s. Then*

$$t(n, C_s^k) \ge \lfloor n/2 \rfloor - k.$$

Moreover, if (k,s) is admissible, then

$$t(n, C_s^k) \ge \begin{cases} \left\lfloor \left(\frac{1}{2} + \frac{1}{2s} \right) n \right\rfloor - k & \text{if } k \text{ is even,} \\ \left\lfloor \left(\frac{1}{2} + \frac{k}{4s(k-1)+2k} \right) n \right\rfloor - k & \text{if } k \text{ is odd.} \end{cases}$$

On the other hand, recall that the case $s \equiv 0 \bmod k$ was solved asymptotically by Mycroft [58]; thus we study the complementary case. We prove an upper bound on $t(n, C_s^k)$ which is valid whenever $s \not\equiv 0 \bmod k$ and $s \ge 5k^2$. Note that the bound is asymptotically sharp if k is even and (k,s) is admissible.

Theorem 1.3.9. *Let $3 \le k < s$ be such that $s \ge 5k^2$ and $s \not\equiv 0 \bmod k$. Then, for all $\gamma > 0$, there exists $n_0 = n_0(k, s, \gamma)$ such that for all $n \ge n_0$ with $n \equiv 0 \bmod s$,*

$$t(n, C_s^k) \le \left(\frac{1}{2} + \frac{1}{2s} + \gamma \right) n.$$

We separate the proof of these results between Chapters 4 and 5. Chapter 4 contains the proof of the lower bounds for both problems (Propositions 1.3.6–1.3.8) and the upper bounds for the covering thresholds (Propositions 1.3.3 and Theorem 1.3.4). Chapter 5 contains the proof of Theorem 1.3.9.

1.4 Dense monochromatic infinite paths

The third problem we consider deals with paths in infinite complete graphs. This is joint work with Allan Lo and Guanghui Wang, and has been published in The Electronic Journal of Combinatorics [55].

A 2-*edge-colouring* of a graph G is an assignment of colours to the edges of G, such that every edge receives exactly one of two possible colours. We will always assume that these colours are “red” and “blue”. We say that G is *monochromatic* if all the edges of G are coloured with the same colour.

What is the length of the longest monochromatic path we can find as a subgraph of K_n, no matter which 2-edge-colouring we consider? This was answered by Gerencsér and Gyárfás [29], who proved that every 2-edge-coloured K_n contains a monochromatic path of length at least $2n/3$. This result is sharp, since there exist colourings of K_n where every monochromatic path has length at most $2n/3$.

Now consider the infinite complete graph $K_{\mathbb{N}}$ on the vertex set $\mathbb{N}$. For any subset $A \subseteq \mathbb{N}$, the *upper density* $\overline{d}(A)$ of A is defined as

$$\overline{d}(A) := \limsup_{n\to\infty} \frac{|A \cap \{1,\dots,n\}|}{n}.$$

Given a subgraph H of $K_{\mathbb{N}}$, we define the *upper density* $\overline{d}(H)$ *of* H to be that of $V(H)$. Aiming to generalise the results known in the finite case, it is natural to ask what are the densest paths which can be found in any 2-edge-coloured $K_{\mathbb{N}}$. This problem was considered first by Erdős and Galvin [22]. Other variants of this problem have been studied as well. For example, it is possible to consider other monochromatic subgraphs rather than paths, edge-colourings with more than two colours, different notions of density or monochromatic sub-digraphs of infinite edge-coloured digraphs, etc. Results along these lines have been obtained by Erdős and Galvin [21, 22], DeBiasio and McKenney [15] and Bürger, DeBiasio, Guggiari and Pitz [9].

We focus on the case of monochromatic paths in 2-edge-coloured complete graphs. By a classical result of Ramsey Theory, any 2-edge-colouring of $K_{\mathbb{N}}$ contains a monochromatic infinite complete graph, and, therefore, also a monochromatic infinite path P. However, this argument alone cannot guarantee a monochromatic path with positive upper density, as it was shown by Erdős [18] that there exist 2-edge-colourings of the infinite complete graph where every infinite monochromatic complete subgraph has upper density zero. Rado [61] showed that in every r-edge-coloured $K_{\mathbb{N}}$ there are r monochromatic paths, of distinct colours, which partition the vertex set. This immediately implies that every 2-edge-coloured $K_{\mathbb{N}}$ contains an infinite monochromatic path P with $\overline{d}(P) \geq 1/2$.

Erdős and Galvin [22] proved that for every 2-edge-colouring of $K_{\mathbb{N}}$ there exists a monochromatic path P with $\overline{d}(P) \geq 2/3$ and exhibited an example of a 2-edge-colouring of $K_{\mathbb{N}}$ such that every monochromatic path satisfies $\overline{d}(P) \leq 8/9$. DeBiasio and McKenney [15] improved the lower bound and showed that for every 2-edge-colouring of $K_{\mathbb{N}}$, there exists a monochromatic path P with $\overline{d}(P) \geq 3/4$.

Our result is an improvement to the lower bound on $\overline{d}(P)$.

Theorem 1.4.1. *Every 2-edge-colouring of $K_{\mathbb{N}}$ contains a monochromatic path P with $\overline{d}(P) \geq (9+\sqrt{17})/16 \approx 0.82019$.*

We remark that after this work was submitted, Corsten, DeBiasio, Lamaison and Lang [12], in independent work, obtained a stronger version of Theorem 1.4.1. They showed that every 2-edge-colouring of $K_{\mathbb{N}}$ contains a monochromatic path P with $\overline{d}(P) \geq (12+\sqrt{8})/17 \approx 0.87226$ and that this bound cannot be further improved.

Theorem 1.4.1 is proven in Chapter 6.

1.5 Outline of the thesis

In Chapter 2 we set up the notation to be used for the rest of the thesis, introduce preliminary concepts, and quote useful results.

Each remaining is dedicated to the study of a different problem. In Chapter 3 we investigate the notion of "strong chromatic number" for graphs. We use the technique of absorption to prove an asymptotically best possible upper bound for the strong chromatic number in terms of the maximum degree of a graph. In Chapter 4 we study the problem of finding coverings with tight cycles in uniform hypergraphs, and determine asymptotic minimum codegree conditions which ensure the existence of such spanning structures. Similarly, we dedicate

Chapter 5 to the investigation of tilings with tight cycles. In Chapter 6 we study the problem of finding "dense" monochromatic paths in infinite 2-edge-coloured complete graphs.

We finish in Chapter 7 with remarks and further directions for future research.

— 2 —

Preliminary concepts and results

2.1 Notation and basic definitions

Throughout the remainder of this document, we will use the following notation. Given reals a, b, c with $c > 0$, $a = b \pm c$ means that $b - c \leq a \leq b + c$. We write $x \ll y$ to mean that for all $y \in (0,1]$ there exists $x_0 \in (0,1)$ such that for all $x \leq x_0$ the following statements hold. Hierarchies with more constants are defined in a similar way and are to be read from the right to the left. It is implicitly understood that the appearance of $1/t$ in such a hierarchy implies that t is a positive integer.

We gather the notation and definitions for hypergraphs here. For an hypergraph $H = (V(H), E(H))$, we will simply write V and E for $V(H)$ and $E(H)$, respectively, if it is clear from the context. Given a set V and a positive integer k, $\binom{V}{k}$ denotes the set of subsets of V with size exactly k. Thus, a hypergraph $H = (V, E)$ is a k-graph if $E \subseteq \binom{V}{k}$.

For a hypergraph H and $S \subseteq V$, we let $H[S]$ be the subgraph of H induced on S, that is, $V(H[S]) = S$ and $E(H[S]) = \{e \in E : e \subseteq S\}$. Let $H \setminus S = H[V \setminus S]$. For hypergraphs H and G, let $H - G$ be the subgraph of H obtained by removing all edges in $E(H) \cap E(G)$.

For all k-graphs H and all $x \in V$, define the *link* $(k-1)$-*graph* $H(x)$ *of* x *in* H to be the $(k-1)$-graph with $V(H(x)) = V \setminus \{x\}$ and $E(H(x)) = N_H(x)$. Given integers $a_1, \dots, a_t \geq 1$, let $K^k(a_1, \dots, a_t)$ denote a complete (k,t)-graph with vertex partition $V_1, \ldots, V_t$ such that $|V_i| = a_i$ for all $1 \le i \le t$.

For a family $\mathcal{F}$ of k-graphs, an *$\mathcal{F}$-tiling* is a set of pairwise vertex-disjoint copies of (not necessarily identical) members of $\mathcal{F}$.

For a sequence of distinct vertices $v_1, \ldots, v_s$ in a k-graph H, we say $P = v_1 \cdots v_s$ is a *tight path* if all sets of k consecutive vertices form an edge. Note that all tight paths have an associated ordering of vertices. Hence, $v_1 \cdots v_s$ and $v_s \cdots v_1$ are assumed to be different tight paths, even if the corresponding subgraphs they define are the same.

Suppose that two tight paths P_1, P_2 in a graph H are vertex-disjoint and the juxtaposition of the vertices of P_1 followed by the vertices of P_2 (using the respective orderings in each case) results in a sequence of vertices that also defines a tight path in H. In that case, we call that path the *concatenation of P_1 and P_2* and we denote it by $P_1 P_2$. Note that $P_1 P_2$ contains more edges than $P_1 \cup P_2$. We naturally extend this definition (whenever it makes sense) to the concatenation of a sequence of paths $P_1, \ldots, P_r$, and we denote the resulting path by $P_1 \cdots P_r$. For two tight paths P_1 and P_2, we say that P_2 *extends* P_1, if $P_2 = P_1 P'$ for some tight path P' (where we might have $|V(P')| < k$, so that P' contains no edge). Also, we may define a tight cycle C by writing $C = v_1 \cdots v_s$, whenever $v_i \cdots v_s v_1 \cdots v_{i-1}$ is a tight path for all $1 \le i \le s$.

2.2 Probabilistic tools

We gather some useful statements about probability that we will use later. We start by recalling the classic Markov's inequality for non-negative random variables.

Lemma 2.2.1 (Markov's inequality)**.** *Let X be a non-negative random variable and $a > 0$. Then*

$$\mathbf{Pr}[X \ge a] \le \frac{\mathbf{E}[X]}{a}.$$

Next, we recall the following versions of the Chernoff inequalities.

Lemma 2.2.2 (Chernoff's inequalities (see, e.g., [41, Theorem 2.8]))**.** *Let X be a generalised binomial random variable, that is, X is the sum of independent Bernoulli random variables, possibly with different parameters. For every $0 < \lambda \leq \mathbf{E}[X]$,*

$$\mathbf{Pr}[|X - \mathbf{E}[X]| > \lambda] \leq 2\exp\left(-\frac{\lambda^2}{4\mathbf{E}[X]}\right). \tag{2.2.1}$$

Also, for every $\lambda > 0$,

$$\mathbf{Pr}[X - \mathbf{E}[X] > \lambda] \leq \exp\left(-\frac{\lambda^2}{2(\mathbf{E}[X] + \lambda/3)}\right). \tag{2.2.2}$$

Remark 2.2.3. A non-trivial fact from probability theory says that every hypergeometric distribution can be written as a sum of independent Bernoulli variables (see, e.g., [41, Theorem 2.10]). This implies that the inequalities (2.2.1) and (2.2.2) also hold when X is a sum of independent hypergeometric variables.

The following lemma is a convenient interpretation of the same Chernoff's inequalities in the hypergeometric case.

Lemma 2.2.4. *Let $\mu, \gamma > 0$ with $\mu + \gamma < 1$. Suppose that $S \subseteq [n]$ and $|S| \geq (\mu + \gamma)n$. Then*

$$\left|\left\{M \in \binom{[n]}{m} : |M \cap S| \leq \mu m\right\}\right| \leq \binom{n}{m} e^{-\frac{\gamma^2 m}{3(\mu+\gamma)}} \leq \binom{n}{m} e^{-\gamma^2 m/3}.$$

2.3 Hypergraph regularity

In Chapter 5 we will use the techniques of hypergraph regularity. The celebrated Regularity Lemma of Szemerédi [69, 70] states that every graph can be decomposed into "random-like" subgraphs. The simple structure which those "random-like" subgraphs have makes them easy to analyse, and provides an invaluable tool to tackle problems in extremal graph theory. After a lot of effort by various researchers, the Regularity Lemma was generalised to the setting of uniform hypergraphs [30, 59, 64–66, 72], enabling the use of "regularity techniques" for problems in extremal hypergraph theory. As our main tool, we use the notion of "regular slices" given by Allen, Böttcher, Cooley and Mycroft [3]. This is a structure obtained from the Hypergraph Regularity Lemma which is much simpler but retains many of its useful properties.

In the following subsections we introduce the notation and main results concerning hypergraph regularity.

2.3.1 Regular complexes

Let $\mathcal{P}$ be a partition of V into vertex classes $V_1, \ldots, V_s$. Recall that a subset $S \subseteq V$ is *$\mathcal{P}$-partite* if $|S \cap V_i| \leq 1$ for all $1 \leq i \leq s$. A hypergraph is *$\mathcal{P}$-partite* if all of its edges are $\mathcal{P}$-partite, and it is *s-partite* if it is $\mathcal{P}$-partite for some partition $\mathcal{P}$ with $|\mathcal{P}| = s$.

A hypergraph H is a *complex* if whenever $e \in E(H)$ and e' is a non-empty subset of e we have that $e' \in E(H)$. All the complexes considered in this thesis have the property that all vertices are contained in some edge. For a positive integer k, a complex H is a *k-complex* if all the edges of H consist of at most k vertices. The edges of size i are called i-edges of H. Given a k-complex H, for all $1 \leq i \leq k$ we denote by H_i the underlying i-graph of H: the vertices of H_i are those of H and the edges of H_i are the i-edges of H. Given $s \geq k$, a *(k,s)-complex* H is an s-partite k-complex.

Let H be a $\mathcal{P}$-partite k-complex. For $i \leq k$ and $X \in \binom{\mathcal{P}}{i}$, we write H_X for the subgraph of H_i induced by the set $\bigcup X$ (i.e., the union of the vertex classes which are the members of X). Note that H_X is an (i,i)-graph. In a similar manner we write $H_{X^<}$ for the hypergraph on the vertex set $\bigcup X$, whose edge set is $\bigcup_{X' \subsetneq X} H_{X'}$. Note that if H is a k-complex and X is a k-set, then $H_{X^<}$ is a $(k-1,k)$-complex.

Given $i \geq 2$, consider an (i,i)-graph H_i and an $(i-1,i)$-graph H_{i-1}, on the same vertex set, which are i-partite with respect to the same partition $\mathcal{P}$. We write $\mathcal{K}_i(H_{i-1})$ for the family of all $\mathcal{P}$-partite i-sets that form a copy of the complete

$(i-1)$-graph K_i^{i-1} in H_{i-1}. We define the *density of H_i with respect to H_{i-1}* to be

$$d(H_i|H_{i-1}) = \frac{|\mathcal{K}_i(H_{i-1}) \cap E(H_i)|}{|\mathcal{K}_i(H_{i-1})|} \quad \text{if} \quad |\mathcal{K}_i(H_{i-1})| > 0,$$

and $d(H_i|H_{i-1}) = 0$ otherwise. More generally, if $\mathbf{Q} = (Q_1, \dots, Q_r)$ is a collection of r subhypergraphs of H_{i-1}, we define $\mathcal{K}_i(\mathbf{Q}) := \bigcup_{j=1}^r \mathcal{K}_i(Q_j)$ and

$$d(H_i|\mathbf{Q}) = \frac{|\mathcal{K}_i(\mathbf{Q}) \cap E(H_i)|}{|\mathcal{K}_i(\mathbf{Q})|} \quad \text{if} \quad |\mathcal{K}_i(\mathbf{Q})| > 0,$$

and $d(H_i|\mathbf{Q}) = 0$ otherwise.

We say that H_i is *(d_i, ε, r)-regular with respect to H_{i-1}* if for all r-tuples $\mathbf{Q}$ with $|\mathcal{K}_i(\mathbf{Q})| > \varepsilon |\mathcal{K}_i(H_{i-1})|$ we have $d(H_i|\mathbf{Q}) = d_i \pm \varepsilon$. Instead of $(d_i, \varepsilon, 1)$-regularity we simply refer to *(d_i, ε)-regularity.* We also say simply that H_i is (ε, r)-regular with respect to H_{i-1} to mean that there exists some d_i for which H_i is (d_i, ε, r)-regular with respect to H_{i-1}. Given an i-graph G whose vertex set contains that of H_{i-1}, we say that G is *(d_i, ε, r)-regular with respect to H_{i-1}* if the i-partite subgraph of G induced by the vertex classes of H_{i-1} is (d_i, ε, r)-regular with respect to H_{i-1}.

Given $3 \le k \le s$ and a (k, s)-complex H with vertex partition $\mathcal{P}$, we say that *H is $(d_k, d_{k-1}, \dots, d_2, \varepsilon_k, \varepsilon, r)$-regular* if the following conditions hold:

(i) For all $2 \le i \le k-1$ and $A \in \binom{\mathcal{P}}{i}$, H_A is (d_i, ε)-regular with respect to $(H_{A^<})_{i-1}$, and

(ii) for all $A \in \binom{\mathcal{P}}{k}$, the induced subgraph H_A is (d_k, ε_k, r)-regular with respect to $(H_{A^<})_{i-1}$.

Sometimes we denote $(d_k, \dots, d_2)$ by $\mathbf{d}$ and write $(\mathbf{d}, \varepsilon_k, \varepsilon, r)$-regular to mean $(d_k, \dots, d_2, \varepsilon_k, \varepsilon, r)$-regular.

We will need the following "regular restriction lemma" which states that the restriction of regular complexes to a sufficiently large set of vertices in each vertex class is still regular, with somewhat degraded regularity properties. This is a well-known property of "regular partitions" in the setting of graph regularity. For hypergraphs, a version of this lemma (together with a sketch of proof) appears in [50, Lemma 4.1]. We use the statement of [3, Lemma 28].

Lemma 2.3.1 (Regular restriction lemma [3, Lemma 28]). *Let $k, m \in \mathbb{N}$ and $\beta, \varepsilon, \varepsilon_k, d_2, \dots, d_k$ be such that*

$$\frac{1}{m} \ll \varepsilon \ll \varepsilon_k, d_2, \dots, d_{k-1} \qquad \text{and} \qquad \varepsilon_k \ll \beta, \frac{1}{k}.$$

Let $r, s \in \mathbb{N}$ and $d_k > 0$. Set $\mathbf{d} = (d_k, \dots, d_2)$. Let G be a $(\mathbf{d}, \varepsilon_k, \varepsilon, r)$-regular (k, s)-complex with vertex classes $V_1, \dots, V_s$ each of size m. Let $V_i' \subseteq V_i$ with $|V_i'| \geq \beta m$ for all $1 \leq i \leq s$. Then the induced subcomplex $G[V_1' \cup \dots \cup V_s']$ is $(\mathbf{d}, \sqrt{\varepsilon_k}, \sqrt{\varepsilon}, r)$-regular.

2.3.2 Statement of the regular slice lemma

In this section we state the version of the hypergraph regularity lemma (Theorem 2.3.4) due to Allen, Böttcher, Cooley and Mycroft [3], which they call the *regular slice lemma*. A similar lemma was previously applied by Haxell, Łuczak, Peng, Rödl, Ruciński, Simonovits and Skokan in the case of 3-graphs [34, 35]. This lemma says that all k-graphs G admit a regular slice $\mathcal{J}$. In a rough sense, a regular slice is a regular multipartite $(k-1)$-complex whose vertex classes have equal size, with the crucial property that for most k-sets X of vertex classes of $\mathcal{J}$, the k-graph formed by the X-partite edges of G are "regular" with respect to the X-partite $(k-1)$-edges of $\mathcal{J}$.

Let $t_0, t_1 \in \mathbb{N}$ and $\varepsilon > 0$. We say that a $(k-1)$-complex $\mathcal{J}$ is (t_0, t_1, ε)-*equitable* if it has the following two properties:

(i) There exists a partition $\mathcal{P}$ of $V(\mathcal{J})$ into t parts of equal size, for some $t_0 \leq t \leq t_1$, such that $\mathcal{J}$ is $\mathcal{P}$-partite. We refer to $\mathcal{P}$ as the *ground partition* of $\mathcal{J}$, and to the parts of $\mathcal{P}$ as the *clusters* of $\mathcal{J}$.

(ii) There exists a *density vector* $\mathbf{d} = (d_{k-1}, \dots, d_2)$ such that, for all $2 \leq i \leq k-1$, we have $d_i \geq 1/t_1$ and $1/d_i \in \mathbb{N}$, and the $(k-1)$-complex $\mathcal{J}$ is $(\mathbf{d}, \varepsilon, \varepsilon, 1)$-

regular.

Let $X \in \binom{\mathcal{P}}{k}$. We write $\hat{\mathcal{J}}_X$ for the $(k-1,k)$-graph $(\mathcal{J}_{X^<})_{k-1}$. A k-graph G on $V(\mathcal{J})$ is (ε_k, r)-*regular with respect to* $\hat{\mathcal{J}}_X$ if there exists some d such that G is (d, ε_k, r)-regular with respect to $\hat{\mathcal{J}}_X$. We also write $d^*_{\mathcal{J},G}(X)$ for the density of G with respect to $\hat{\mathcal{J}}_X$, or simply $d^*(X)$ if $\mathcal{J}$ and G are clear from the context.

Definition 2.3.2 (Regular slice). *Given $\varepsilon, \varepsilon_k > 0$, $r, t_0, t_1 \in \mathbb{N}$, a k-graph G and a $(k-1)$-complex $\mathcal{J}$ on $V(G)$, we call $\mathcal{J}$ a $(t_0, t_1, \varepsilon, \varepsilon_k, r)$*-regular slice for G *if $\mathcal{J}$ is (t_0, t_1, ε)-equitable and G is (ε_k, r)-regular with respect to all but at most $\varepsilon_k \binom{t}{k}$ of the k-sets of clusters of $\mathcal{J}$, where t is the number of clusters of $\mathcal{J}$.*

Given a regular slice $\mathcal{J}$ for a k-graph G, we keep track of the relative densities $d^*(X)$ for k-sets X of clusters of $\mathcal{J}$, which is done via a weighted k-graph.

Definition 2.3.3. *Given a k-graph G and a (t_0, t_1, ε)-equitable $(k-1)$-complex $\mathcal{J}$ on $V(G)$, we let $R_{\mathcal{J}}(G)$ be the complete weighted k-graph whose vertices are the clusters of $\mathcal{J}$, and where each edge X is given weight $d^*(X)$. When $\mathcal{J}$ is clear from the context we write $R(G)$ instead of $R_{\mathcal{J}}(G)$.*

The regular slice lemma (Theorem 2.3.4) guarantees the existence of a regular slice $\mathcal{J}$ with respect to which $R(G)$ resembles G in various senses. In particular, $R(G)$ inherits the codegree condition of G in the following sense.

Let G be a k-graph on n vertices. Given a set $S \in \binom{V(G)}{k-1}$, recall that $\deg_G(S)$ is the number of edges of G which contain S. The *relative degree* $\overline{\deg}(S; G)$ *of* S *with respect to* G is defined to be

$$\overline{\deg}(S; G) = \frac{\deg_G(S)}{n-k+1},$$

Thus, $\overline{\deg}(S; G)$ is the proportion of k-sets of vertices in G extending S which are in fact edges of G. To extend this definition to weighted k-graphs G with weight function d^*, we define

$$\overline{\deg}(S;G) = \frac{\sum_{e \in E(G): S \subseteq e} d^*(e)}{n-k+1}.$$

Finally, for a collection $\mathcal{S}$ of $(k-1)$-sets in $V(G)$, the *mean relative degree* $\overline{\deg}(\mathcal{S};G)$ *of* $\mathcal{S}$ *in* G is defined to be the mean of $\overline{\deg}(S;G)$ over all sets $S \in \mathcal{S}$.

We will need an additional property of regular slices. Suppose G is a k-graph, $\mathcal{S}$ is a $(k-1)$-graph on the same vertex set, and $\mathcal{J}$ is a regular slice for G on t clusters. We say $\mathcal{J}$ is $(\eta,\mathcal{S})$-*avoiding* if for all but at most $\eta\binom{t}{k-1}$ of the $(k-1)$-sets Y of clusters of $\mathcal{J}$, it holds that $|\mathcal{J}_Y \cap \mathcal{S}| \leq \eta|\mathcal{J}_Y|$.

We can now state the version of the regular slice lemma that we will use.

Theorem 2.3.4 (Regular slice lemma [3, Lemma 6]). *Let $k \in \mathbb{N}$ with $k \geq 3$. For all $t_0 \in \mathbb{N}$, $\varepsilon_k > 0$ and all functions $r : \mathbb{N} \to \mathbb{N}$ and $\varepsilon : \mathbb{N} \to (0,1]$, there exist $t_1, n_1 \in \mathbb{N}$ such that the following holds for all $n \geq n_1$ which are divisible by $t_1!$. Let G be a k-graph on n vertices, and let $\mathcal{S}$ be a $(k-1)$-graph on the same vertex set with $|E(\mathcal{S})| \leq \theta\binom{n}{k-1}$. Then there exists a $(t_0, t_1, \varepsilon(t_1), \varepsilon_k, r(t_1))$-regular slice $\mathcal{J}$ for G such that, for all $(k-1)$-sets Y of clusters of $\mathcal{J}$, we have $\overline{\deg}(Y;R(G)) = \overline{\deg}(\mathcal{J}_Y;G) \pm \varepsilon_k$, and furthermore $\mathcal{J}$ is $(3\sqrt{\theta},\mathcal{S})$-avoiding.*

We remark that the original statement of [3, Lemma 6] includes many other properties which are satisfied by the regular slice $\mathcal{J}$, but we only state the properties we need. On the other hand, the original statement did not include the "avoiding" property with respect to a fixed $(k-1)$-graph $\mathcal{S}$. This, however, can be obtained easily from their proof, as we sketch now.

Proof (sketch). The original proof of the Regular Slice Lemma can be summarised as follows (we refer the reader to [3] for the precise definitions). First, they obtain an "equitable family of partitions" $\mathcal{P}^*$ from (a strengthened version of) the Hypergraph Regularity Lemma. This can be used to find suitable complexes in the following way: first, for each pair of clusters of $\mathcal{P}^*$, select a 2-cell uniformly at random. Then, for each triple of clusters of $\mathcal{P}^*$ select a 3-cell uniformly at random which is supported on the corresponding previously selected 2-cells; and so on, until we select $(k-1)$-cells. This will always output a (t_0, t_1, ε)-equitable $(k-1)$-complex $\mathcal{J}$, and the task is to check that, with positive probability, $\mathcal{J}$ is actually a $(t_0, t_1, \varepsilon, \varepsilon_k, r)$-regular slice satisfying the "desired properties" with respect to the reduced k-graph.

Having selected $\mathcal{J}$ at random as before, the most technical part of the proof is to show that the "desired properties" of the reduced k-graph (labelled (a), (b) and (c) in [3, Lemma 10]) hold with probability tending to 1 whenever n goes to infinity. Thankfully, that part of the proof does not require any modification for our purposes. Moreover, the selected $\mathcal{J}$ will be a $(t_0, t_1, \varepsilon, \varepsilon_k, r)$-regular slice with probability at least 1/2. This is shown by proving an upper bound on the expected number of k-sets of clusters of $\mathcal{J}$ for which G is not (ε_k, r)-regular, and an application of Markov's inequality (cf. [3, pp. 65–66]). It is a natural adaptation of this method that will show that $\mathcal{J}$ is also $(3\theta^{1/2}, \mathcal{S})$-avoiding with probability at least 2/3.

Let $\mathcal{S}$ be a $(k-1)$-graph on $V(G)$ of size at most $\theta\binom{n}{k-1}$. We only need to consider the edges of $\mathcal{S}$ which are $\mathcal{P}$-partite. Every $\mathcal{P}$-partite edge of $\mathcal{S}$ is supported in exactly one $(k-1)$-cell of the family of partitions $\mathcal{P}^*$, which by [3, Claim 32] is present in $\mathcal{J}$ with probability $p = \prod_{i=2}^{k-1} d_i^{\binom{k-1}{j}}$. Thus the expected size of $|E(\mathcal{S}) \cap E(\mathcal{J}_{k-1})|$ is at most $|E(\mathcal{S})|p \leq \theta p\binom{n}{k-1}$. By Markov's inequality (Lemma 2.2.1), with probability at least 2/3 we have $|E(H) \cap E(\mathcal{J}_{k-1})| \leq 3\theta p\binom{n}{k-1}$. By the previous discussion, with positive probability $\mathcal{J}$ satisfies all of the properties of [3, Lemma 10] and also that $|E(\mathcal{S}) \cap E(\mathcal{J}_{k-1})| \leq 3\theta p\binom{n}{k-1}$. Thus we may assume $\mathcal{J}$ satisfies all of the previous properties simultaneously, and it is only necessary to check that $\mathcal{J}$ is $(3\theta^{1/2}, \mathcal{S})$-avoiding.

Let t be the number of clusters of $\mathcal{P}$ and m the size of a cluster in $\mathcal{P}$. For each $(k-1)$-set of clusters Y, $\mathcal{J}_Y$ has $(1 \pm \varepsilon_k/10)pm^{k-1}$ edges (see [3, Fact 7]). We say a $(k-1)$-set of clusters Y is *bad* if $|\mathcal{J}_Y \cap E(\mathcal{S})| > \sqrt{6\theta}|\mathcal{J}_Y|$ and let $\mathcal{Y}$ be the set of bad $(k-1)$-sets. Then

$$3\theta p\binom{n}{k-1} \geq \sum_{Y} |\mathcal{J}_Y \cap E(\mathcal{S})| \geq |\mathcal{Y}|\sqrt{6\theta}(1-\varepsilon_k/10)pm^{k-1},$$

which implies $|\mathcal{Y}| \leq 3\theta^{1/2}\binom{t}{k-1}$. Then $\mathcal{J}$ is $(3\theta^{1/2}, \mathcal{S})$-avoiding, as desired. ∎

2.3.3 The d-reduced k-graph

Once we have a regular slice $\mathcal{J}$ for a k-graph G, we would like to work within k-tuples of clusters with respect to which G is both regular and dense. To keep track of those tuples, we introduce the following definition.

Definition 2.3.5 (The d-reduced k-graph). *Let G be a k-graph and $\mathcal{J}$ be a $(t_0, t_1, \varepsilon, \varepsilon_k, r)$-regular slice for G. Then for $d > 0$ we define the* d-reduced k-graph *$R_d(G)$ of G to be the k-graph whose vertices are the clusters of $\mathcal{J}$ and whose edges are all k-sets of clusters X of $\mathcal{J}$ such that G is (ε_k, r)-regular with respect to X and $d^*(X) \geq d$. Note that $R_d(G)$ depends on the choice of $\mathcal{J}$ but this will always be clear from the context.*

The next lemma states that for regular slices $\mathcal{J}$ as in Theorem 2.3.4, the codegree conditions are also preserved by $R_d(G)$.

Lemma 2.3.6 ([3, Lemma 8]). *Let $k, r, t_0, t \in \mathbb{N}$ and $\varepsilon, \varepsilon_k > 0$. Let G be a k-graph and let $\mathcal{J}$ be a $(t_0, t_1, \varepsilon, \varepsilon_k, r)$-regular slice for G. Then for all $(k-1)$-sets Y of clusters of $\mathcal{J}$, we have*

$$\overline{\deg}(Y; R_d(G)) \geq \overline{\deg}(Y; R(G)) - d - \zeta(Y),$$

where $\zeta(Y)$ is defined to be the proportion of k-sets Z of clusters with $Y \subseteq Z$ that are not (ε_k, r)-regular with respect to G.

For $0 \leq \mu, \theta \leq 1$, we say that a k-graph H on n vertices is (μ, θ)-*dense* if there exists $\mathcal{S} \subseteq \binom{V(H)}{k-1}$ of size at most $\theta\binom{n}{k-1}$ such that, for all $S \in \binom{V(H)}{k-1} \setminus \mathcal{S}$, we have $\deg_H(S) \geq \mu(n-k+1)$. In particular, if H has $\delta_{k-1}(H) \geq \mu n$, then it is $(\mu, 0)$-dense. By using Lemma 2.3.6, we show that $R_d(G)$ 'inherits' the property of being (μ, θ)-dense, albeit with some degraded parameters.

Lemma 2.3.7. *Let $1/n \ll 1/t_1 \leq 1/t_0 \ll 1/k$ and $\mu, \theta, d, \varepsilon, \varepsilon_k > 0$. Suppose that G is a k-graph on n vertices, that G is (μ, θ)-dense and let $\mathcal{S}$ be the $(k-1)$-graph on $V(G)$ whose edges are precisely $\{S \in \binom{V(G)}{k-1} : \deg_G(S) < \mu(n-k+1)\}$. Let $\mathcal{J}$ be a $(t_0, t_1, \varepsilon, \varepsilon_k, r)$-regular slice for G such that for all $(k-1)$-sets Y of clusters of $\mathcal{J}$, we have $\overline{\deg}(Y; R(G)) = \overline{\deg}(\mathcal{J}_Y; G) \pm \varepsilon_k$, and furthermore $\mathcal{J}$ is $(3\sqrt{\theta}, \mathcal{S})$-avoiding. Then $R_d(G)$ is $((1-\sqrt{\theta})\mu - d - \varepsilon_k - \sqrt{\varepsilon_k}, 3\sqrt{\theta} + 3\sqrt{\varepsilon_k})$-dense.*

Proof. Let $\mathcal{P}$ be the ground partition of $\mathcal{J}$ and $t = |\mathcal{P}|$. Let $m = n/t$. Clearly $|V| = m$ for all $V \in \mathcal{P}$. Let $\mathcal{Y}_1$ be the set of all $Y \in \binom{\mathcal{P}}{k-1}$ such that $|\mathcal{J}_Y \cap \mathcal{S}| \geq 3\sqrt{\theta}|\mathcal{J}_Y|$. Since $\mathcal{J}$ is $(3\sqrt{\theta}, \mathcal{S})$-avoiding, $|\mathcal{Y}_1| \leq 3\sqrt{\theta}\binom{t}{k-1}$.

For all $Y \in \binom{\mathcal{P}}{k-1}$, let $\zeta(Y)$ be defined as in Lemma 2.3.6. Let $\mathcal{Y}_2$ be the set of all $Y \in \binom{\mathcal{P}}{k-1}$ with $\zeta(Y) > \sqrt{\varepsilon_k}$. Since G is (ε_k, r)-regular with respect to all but at most $\varepsilon_k\binom{t}{k}$ of the k-sets of clusters of $\mathcal{P}$, it follows that $|\mathcal{Y}_2|\sqrt{\varepsilon_k}(t-k+1)/k \leq \varepsilon_k\binom{t}{k}$, namely, $|\mathcal{Y}_2| \leq \sqrt{\varepsilon_k}\binom{t}{k-1}$.

Then it follows that $|\mathcal{Y}_1 \cup \mathcal{Y}_2| \leq 3(\sqrt{\theta} + \sqrt{\varepsilon_k})\binom{t}{k-1}$. We will show that all $Y \in \binom{\mathcal{P}}{k-1} \setminus (\mathcal{Y}_1 \cup \mathcal{Y}_2)$ will have large codegree in $R_d(G)$, thus proving the lemma.

Consider any $Y \in \binom{\mathcal{P}}{k-1} \setminus (\mathcal{Y}_1 \cup \mathcal{Y}_2)$. Since $Y \notin \mathcal{Y}_2$, $\zeta(Y) \leq \sqrt{\varepsilon_k}$. By Lemma 2.3.6, we have

$$\begin{aligned}\overline{\deg}(Y; R_d(G)) &\geq \overline{\deg}(Y; R(G)) - d - \zeta(Y)\\ &\geq \overline{\deg}(Y; R(G)) - d - \sqrt{\varepsilon_k}\\ &\geq \overline{\deg}(\mathcal{J}_Y; G) - \varepsilon_k - d - \sqrt{\varepsilon_k}.\end{aligned}$$

So it suffices to show that $\overline{\deg}(\mathcal{J}_Y; G) \geq (1-3\sqrt{\theta})\mu$. Recall that $\overline{\deg}(\mathcal{J}_Y; G)$ is the mean of $\overline{\deg}(S; G)$ over all $S \in \mathcal{J}_Y$. Since $Y \notin \mathcal{Y}_1$, $|\mathcal{J}_Y \cap \mathcal{S}| \leq \sqrt{\theta}|\mathcal{J}_Y|$. By definition, for all $S \in \mathcal{J}_Y \setminus \mathcal{S}$, $\deg_G(S) \geq \mu(n-k+1)$. Thus $\overline{\deg}(\mathcal{J}_Y; G) \geq (1-\sqrt{\theta})\mu$, as required. ∎

2.3.4 The embedding lemma

We will need a version of an "embedding lemma" which gives sufficient conditions to find a copy of a (k,s)-graph H in a regular (k,s)-complex G.

Suppose that G is a (k,s)-graph with vertex classes $V_1,\dots,V_s$, which all have size m. Suppose also that H is a (k,s)-graph with vertex classes $X_1,\dots,X_s$ of size at most m. We say that a copy of H in G is *partition-respecting* if for all $1 \le i \le s$, the vertices corresponding to those in X_i lie within V_i.

Given a k-graph G and a $(k-1)$-graph J on the same vertex set, we say that G *is supported on* J if for all $e \in E(G)$ and all $f \in \binom{e}{k-1}$, $f \in E(J)$.

There are various results in the literature which enable us to count the number of subgraphs with bounded number of vertices inside appropriate regular complexes, and in particular, to ensure existence of a single copy. This is usually known as the "counting lemma for hypergraphs", and it has appeared in various slightly different versions, as done by Gowers [30], Nagle, Rödl and Schacht [59], or Cooley, Fountoulakis, Kühn and Osthus [10].

The version that we use can be easily deduced from a lemma stated by Cooley, Fountoulakis, Kühn and Osthus [10, Lemma 4] or, alternatively, from a lemma used by Allen, Böttcher, Cooley and Mycroft [10, Lemma 27]. We discuss the differences between those statements and Lemma 2.3.8 afterwards.

Lemma 2.3.8 (Embedding lemma). *Let $k,s,r,t,m_0 \in \mathbb{N}$ and let $d_2,\dots,d_{k-1},d,\varepsilon,\varepsilon_k > 0$ be such that $1/d_i \in \mathbb{N}$ for all $2 \le i \le k-1$, and*

$$\frac{1}{m_0} \ll \frac{1}{r}, \varepsilon \ll \varepsilon_k, d_2, \dots, d_{k-1} \qquad \text{and} \qquad \varepsilon_k \ll d, \frac{1}{t}, \frac{1}{s}.$$

Then the following holds for all $m \ge m_0$. Let H be a (k,s)-graph on t vertices with vertex classes $X_1,\dots,X_s$. Let $\mathcal{J}$ be a $(d_{k-1},\dots,d_2,\varepsilon,\varepsilon,1)$-regular $(k-1,s)$-complex with vertex classes $V_1,\dots,V_s$ all of size m. Let G be a k-graph on $\bigcup_{1\le i\le s} V_i$ which *is supported on* $\mathcal{J}_{k-1}$ *such that for all* $e \in E(H)$ *intersecting the vertex classes* $\{X_{i_j} : 1 \leq j \leq k\}$*, the* k*-graph* G *is* (d_e, ε_k, r)*-regular with respect to the* k*-set of clusters* $\{V_{i_j} : 1 \leq j \leq k\}$*, for some* $d_e \geq d$ *depending on* e*. Then there exists a partition-respecting copy of* H *in* G*.*

There are small differences between the statements of Lemma 2.3.8 and both of [10, Lemma 4] and [10, Lemma 27], which we discuss. First, the two cited lemmas are stronger in the sense that they give an approximate count on the number of partition-respecting copies of H in G, where as we only need the simple consequence of the existence of a single copy. Also, Lemma 4 in [10] allows H to be a complex (instead of a k-graph) and counts the number of copies of H in $\mathcal{J} \cup G$.

The main technical difference between Lemma 2.3.8 and Lemma 4 in [10] is that their lemma asks for the stronger condition that, for all $e \in E(H)$ intersecting the vertex classes $\{X_{i_j} : 1 \leq j \leq k\}$, the k-graph G should be (d, ε_k, r)-regular with respect to the k-set of clusters $\{V_{i_j} : 1 \leq j \leq k\}$, such that the value d does not depend on e, and $1/d \in \mathbb{N}$, whereas we allow G to be (d_e, ε_k, r)-regular for some $d_e \geq d$ depending on e and not necessarily satisfying $1/d_e \in \mathbb{N}$. By the discussion after Lemma 4.6 in [50], we can reduce to that case by working with a sub-k-complex of $\mathcal{J} \cup G$ which is $(d, d_{k-1}, d_{k-2}, \ldots, d_2, \varepsilon_k, \varepsilon, r)$-regular, whose existence is guaranteed by an application of the "slicing lemma" [10, Lemma 8].

On the other hand, the main technical difference between our Lemma 2.3.8 and [10, Lemma 27] is that their lemma only allows us to count k-graphs H whose associated partition $X_1, ..., X_s$ is such that each class has size exactly one, i.e., every vertex of H is embedded in a different cluster of the regular complex. However, that strengthening can also be obtained from their proof (see the discussion in [10, Appendix A]).

— 3 —

Asymptotic bounds for the strong chromatic number

The goal of this chapter is the proof of Theorem 1.2.3, which ensures the existence of a vertex-partition of a partite graph into independent transversals, under certain conditions. As explained in Section 1.2, Theorem 1.2.2 and Corollary 1.2.4 follow easily from Theorem 1.2.3.

This chapter is organised as follows. In Section 3.1 we prove an "absorbing lemma" for the independent transversals in a given partition (Lemma 3.1.5). More precisely, we find a small absorbing set, that is, given a partition $\mathcal{P}$ we find a small vertex set $A \subseteq V(G)$ which is balanced (i.e., it intersects each class of $\mathcal{P}$ in the same number of vertices) with the property that for every small balanced set $S \subseteq V(G)$, $A \cup S$ can be partitioned into independent transversals. Thus the problem of finding a partition into independent transversals is reduced to the problem of finding a collection of disjoint independent transversals covering almost all of the vertices. This is achieved by Lemma 3.2.1, which is proven in Section 3.2. The pieces of the proof are then put together in Section 3.3.

3.1 Absorption for independent transversals

The aim of this section is to prove Lemma 3.1.5, which gives the existence of an absorbing set. First we need the following simple lemma.

Lemma 3.1.1. *Let G be a graph and let $\mathcal{P} = \{V_1, \dots, V_k\}$ be a partition of $V(G)$ such that $|V_i| > 2\Delta(G)$ for all $i \in [k-1]$. Then for any $v_k, v'_k \in V_k$, there exists an independent transversal T of $\{V_1, \dots, V_{k-1}\}$ such that $T \cup \{v_k\}$ and $T \cup \{v'_k\}$ are independent transversals of $\mathcal{P}$.*

To prove Lemma 3.1.1 we will use the following result of Haxell [37], which was first proved equivalently in [36]. For more details and variations, see the discussion after Corollary 15 in [40].

Lemma 3.1.2 (Haxell [37, Theorem 3])**.** *Let G be a graph and let $\mathcal{P} = \{V_1, \dots, V_k\}$ be a partition of $V(G)$. If each $I \subseteq [k]$ satisfies $|\bigcup_{i \in I} V_i| > (2|I| - 2)\Delta(G)$, then there exists an independent transversal of $\mathcal{P}$.*

Now we prove Lemma 3.1.1.

Proof of Lemma 3.1.1. For every $i \in [k-1]$, let $V'_i := V_i \setminus (N(v_k) \cup N(v'_k))$. Let $\mathcal{P}' := \{V'_1, \dots, V'_{k-1}\}$ and $G' := G[\bigcup_{i \in [k-1]} V'_i]$. Clearly it is enough to find an independent transversal of $\mathcal{P}'$ in G'. For every non-empty $I \subseteq [k-1]$, we have that

$$\begin{aligned}
\left|\bigcup_{i \in I} V'_i\right| &\geq (2\Delta(G) + 1)|I| - |N(v_k) \cup N(v'_k)| \\
&\geq (2\Delta(G) + 1)|I| - 2\Delta(G) \\
&> (2|I| - 2)\Delta(G) \geq (2|I| - 2)\Delta(G').
\end{aligned}$$

By Lemma 3.1.2, G' has an independent transversal of $\mathcal{P}'$, as desired. ∎

Next, we prove that appropriate random subgraphs respecting a given partition preserve the relative maximum degree. The proof is achieved by an application of concentration inequalities.

Proposition 3.1.3. *Suppose $1/r \leq 1/m \ll \varepsilon, 1/k$. Let G be a graph and let $\mathcal{P} = \{V_1, \dots, V_k\}$ be a partition of $V(G)$ with classes of size $r \geq (2 + \varepsilon)\Delta(G)$. For each*

$i \in [k]$, let $R_i \subseteq V_i$ be an m-set chosen independently and uniformly at random. Let $v \in V(G)$ and $d'_v := |N_G(v) \cap (R_1 \cup \dots \cup R_k)|$. Then $2d'_v \le (1 - \varepsilon/12)m$ with probability at least $1 - \exp(-\varepsilon^2 m/288)$.

Proof. Fix $v \in V(G)$, and let $d := \deg_G(v)$. For each $i \in [k]$, let $d_i := |N_G(v) \cap V_i|$ and $d'_i := |N_G(v) \cap R_i|$. First, note that $d_1 + \dots + d_k = d \le \Delta(G) \le r/(2+\varepsilon)$. Secondly, note that, for each $i \in [k]$, d'_i is a hypergeometric random variable with expectation $d_i m/r$, and the random variables $d'_1, \dots, d'_k$ are independent. Since $d'_v = d'_1 + \dots + d'_k$, we deduce that d'_v is a sum of independent hypergeometric random variables with expectation $\mathbf{E}[d'] = dm/r \le m/(2+\varepsilon) \le m(1/2 + \varepsilon/6)$. This implies that $m/2 - \mathbf{E}[d'] \ge \varepsilon m/6$. Using the Chernoff inequality (2.2.2), we get

$$\begin{aligned}\mathbf{Pr}[2d' > (1-\varepsilon/12)m] = \mathbf{Pr}\left[d' - \mathbf{E}[d'] > \frac{m}{2} - \mathbf{E}[d'] - \frac{\varepsilon m}{12}\right] &\le \mathbf{Pr}\left[d' - \mathbf{E}[d'] > \frac{\varepsilon m}{12}\right] \\ \le \exp\left(-\frac{\varepsilon^2 m^2}{144 \times 2(m + \varepsilon m/36)}\right) &\le \exp\left(-\frac{\varepsilon^2 m}{288}\right),\end{aligned}$$

which proves the proposition. ∎

Using a supersaturation argument, Lemma 3.1.1 implies the following corollary. Its proof follows from quite standard methods (see, e.g., [57, Section 2]) which can be sketched as follows. After fixing v_k, v'_k, we will select at random equal-sized subsets of each cluster V_i. By the previous lemma, for almost all of the possible random choices the induced subgraph in the union of the selected subsets keeps the relative degree conditions. Then we can find a transversal in each of these induced subgraphs by using Lemma 3.1.1. Finally, we correct for the possible over counting.

Corollary 3.1.4. *Suppose $1/r, \eta \ll \varepsilon, 1/k$. Let G be a graph and let $\mathcal{P} = \{V_1, \dots, V_k\}$ be a partition of $V(G)$ with classes of size $r \ge (2+\varepsilon)\Delta(G)$. Then, for any two vertices $v_k, v'_k \in V_k$, there exist at least ηr^{k-1} independent transversals T of $\{V_1, \dots, V_{k-1}\}$ such that $T \cup \{v_k\}$ and $T \cup \{v'_k\}$ are independent transversals of $\mathcal{P}$.*

Proof. Without loss of generality, assume each V_i induces an independent set in G. Select m such that $1/r, \eta \ll 1/m \ll \varepsilon, 1/k$. Note that the choice of m depends on ε and k only.

Fix $v_k, v'_k \in V_k$. For each $i \in [k-1]$ select a random m-set $R_i \subseteq V_i$, and also select a random $(m-2)$-set $R'_k \subseteq V_k \setminus \{v_k, v'_k\}$, with all of these choices done independently. Let $R_k := R'_k \cup \{v_k, v'_k\}$, and let $G' := G[R_1 \cup R_2 \cup \cdots \cup R_k]$ be the induced subgraph of G according to this random choice of subsets. Let us say that a choice of $R_1, \ldots, R_{k-1}, R'_k$ is *valid* if $2\Delta(G') < m$. By Proposition 3.1.3, it is not difficult to deduce that the number of choices of $R_1, \ldots, R'_k$ such that $2\Delta(G') \geq m$ is at most

$$\binom{r-2}{m-2}\binom{r}{m}^{k-1} mk \exp\left(-\Omega_\varepsilon(m)\right) \leq \frac{1}{2}\binom{r-2}{m-2}\binom{r}{m}^{k-1},$$

where the inequality follows from the choice of m. Thus there are at least $\binom{r-2}{m-2}\binom{r}{m}^{k-1}/2$ valid choices for $R_1, \ldots, R'_k$.

By Lemma 3.1.1, each valid choice yields one set $T \subseteq V_1 \cup \cdots \cup V_{k-1}$ such that both $T \cup \{v_k\}$ and $T \cup \{v'_k\}$ are independent transversals. Since each such T can be yielded by at most $\binom{r-2}{m-2}\binom{r-1}{m-1}^{k-1}$ different choices of $R_1, \ldots, R'_k$, correcting for the overcount the number of different such sets T we get is at least

$$\frac{\binom{r-2}{m-2}\binom{r}{m}^{k-1}/2}{\binom{r-2}{m-2}\binom{r-1}{m-1}^{k-1}} = \frac{1}{2}\left(\frac{r}{m}\right)^{k-1} \geq \eta r^{k-1},$$

where the last inequality follows from $\eta \ll 1/m$. This gives the desired result. ∎

We can now state and prove our absorbing lemma. Given a graph G and a partition $\mathcal{P} = \{V_1, \ldots, V_k\}$ of $V(G)$, a subset $S \subseteq V(G)$ is $\mathcal{P}$-*balanced* (or just *balanced*, if $\mathcal{P}$ is clear from the context) if $|S \cap V_i| = |S \cap V_j|$ for every $i, j \in [k]$.

Lemma 3.1.5 (Absorbing lemma). *Let $k \geq 3$ and r be positive integers, and $\varepsilon, \gamma > 0$ be reals such that $0 < 1/r \ll \gamma \ll \varepsilon, 1/k$. Let $\alpha = \gamma/(8k^2)$ and $\beta = \gamma^2/(64k^3)$.*

Let G be an n-vertex graph and let $\mathcal{P} = \{V_1, \dots, V_k\}$ be a partition of $V(G)$ with classes of size $r \geq (2+\varepsilon)\Delta(G)$. Then there exists a $\mathcal{P}$-balanced set $A \subseteq V(G)$ of size at most αn such that, for every $\mathcal{P}$-balanced set $S \subseteq V(G)$ of size at most βn, $A \cup S$ can be partitioned into independent transversals of $\mathcal{P}$.

Proof. Take η such that $\gamma \ll \eta \ll 1/k, \varepsilon$. Let $m := k^2$. Given a balanced k-subset S of $V(G)$, an *absorbing set A for S* is an m-subset of $V(G)$, disjoint from S, such that both $G[A]$ and $G[A \cup S]$ can be partitioned into independent transversals of $\mathcal{P}$. For any balanced k-subset S, let $\mathcal{L}(S)$ be the family of absorbing sets for S.

Claim 3.1.6. *For each balanced k-subset S of $V(G)$, $|\mathcal{L}(S)| \geq \gamma\binom{r}{k}^k$.*

Proof of the claim. Let $S = \{s_1, \dots, s_k\}$ with $s_i \in V_i$ for every $i \in [k]$.

A tuple $(T, U_1, \dots, U_k)$ is *good* if $S, T, U_1, \dots, U_k$ are pairwise disjoint, $T = \{t_1, \dots, t_k\}$ is an independent transversal of $\mathcal{P}$ and, for every $i \in [k]$ and $t_i \in V_i$, both $U_i \cup \{s_i\}$ and $U_i \cup \{t_i\}$ are independent transversals of $\mathcal{P}$. Clearly, if $(T, U_1, \dots, U_k)$ is good, then $A = T \cup \left(\bigcup_{i\in[k]} U_i\right)$ is an absorbing set for S.

Let t_1 be an arbitrary vertex of $V_1 \setminus \{s_1\}$, for which we have at least $r - 1 \geq r/2$ different possible choices. By Corollary 3.1.4, there exist ηr^{k-1} independent transversals T' of $\{V_2, \dots, V_k\}$ such that both $T' \cup \{s_1\}$ and $T' \cup \{t_1\}$ are independent transversals of $\mathcal{P}$. By ignoring those T' which have non-empty intersection with S, we have at least $\eta r^{k-1}/2$ different possible choices for T'. Set $T = \{t_1\} \cup T'$. Repeating the same argument with s_i, t_i we can find $\eta r^{k-1}/2$ choices for U_i, for every $i \in [k]$. Therefore, there are at least $(\eta^{k+1}/2^{k+2})r^m$ good tuples. Using $\gamma \ll \eta \ll 1/m$, we find these good tuples yield at least $\gamma\binom{r}{k}^k$ different absorbing sets for S, as desired. □

Recall that $m = k^2$ and choose a family $\mathcal{F}$ of balanced m-sets by including each one of the $\binom{r}{k}^k$ balanced m-sets independently at random with probability

$$p := \frac{\gamma r}{16k^3\binom{r}{k}^k}.$$

By Chernoff's inequality (2.2.1), with probability $1 - o(1)$ we have that

$$|\mathcal{F}| \leq \frac{\gamma r}{8k^3}, \tag{3.1.1}$$

and, for every balanced k-set S,

$$|\mathcal{L}(S) \cap \mathcal{F}| \geq \frac{\gamma^2 r}{32k^3}. \tag{3.1.2}$$

We say a pair (A_1, A_2) of m-sets is *intersecting* if $A_1 \neq A_2$ and $A_1 \cap A_2 \neq \varnothing$. We say $\mathcal{F}$ *contains* a pair (A_1, A_2) if $A_1, A_2 \in \mathcal{F}$. The expected number of intersecting pairs contained in $\mathcal{F}$ is at most

$$\binom{r}{k}^k k^2 \binom{r}{k-1} \binom{r}{k}^{k-1} p^2 = \frac{\gamma^2 k^2 r^2 \binom{r}{k-1}}{(2^4 k^3)^2 \binom{r}{k}} = \frac{\gamma^2 r^2}{2^8 k^4} \frac{k}{r-k+1} \leq \frac{\gamma^2 r}{2^7 k^3}.$$

By Markov's inequality (Lemma 2.2.1), with probability at least 1/2 the number of intersecting pairs contained in $\mathcal{F}$ is at most $\gamma^2 r/(2^6 k^3)$. Therefore, with positive probability $\mathcal{F}$ satisfies (3.1.1) and (3.1.2) and contains at most $\gamma^2 r/(2^6 k^3)$ intersecting pairs.

By removing one m-set of every intersecting pair in $\mathcal{F}$, we obtain a family $\mathcal{F}'$ of pairwise disjoint balanced m-sets such that for every balanced k-set S,

$$|\mathcal{L}(S) \cap \mathcal{F}'| \geq \frac{\gamma^2 r}{32k^3} - \frac{\gamma^2 r}{2^6 k^3} = \frac{\gamma^2 r}{64k^3}.$$

Let $A := \bigcup_{F \in \mathcal{F}'} F$. Recall that $\alpha = \gamma/(8k^2)$ and $\beta = \gamma^2/64k^3$. Note that $V(A)$ has size at most $k^2|\mathcal{F}'| \leq k^2|\mathcal{F}| \leq \alpha n$, by (3.1.1) and $n = kr$. For every balanced $S \subseteq V(G)$ of size at most βn, we can partition it into at most $\beta r \leq \gamma^2 r/(64k^3)$ balanced k-sets, so it is possible to greedily assign a different absorbing m-set in $\mathcal{F}'$ to each one of these sets. Hence, $G[A \cup S]$ can be partitioned into independent transversals of $\mathcal{P}$, as desired. ∎

3.2 Partial strong colourings

Let G be a graph and $\mathcal{P} = \{V_1, \ldots, V_k\}$ be a partition of $V(G)$ with classes of size r. A *t-partial strong colouring* of G with respect to $\mathcal{P}$ is a collection of t disjoint independent transversals of $\mathcal{P}$ in G. Note that, if $\chi_s(G) = r$, then there exists an r-partial strong colouring of G with respect to $\mathcal{P}$. The aim of this section is to show the existence of $(1-\delta)r$-partial strong colourings of $\mathcal{P}$.

Lemma 3.2.1. *For each integer $k \geq 3$ and reals $\delta, \varepsilon > 0$, there exists $r_0 = r_0(k, \delta, \varepsilon)$ such that the following holds for all $r \geq r_0$. Let G be a graph and $\mathcal{P}$ be a partition of $V(G)$ with k classes of size $r \geq (2+\varepsilon)\Delta(G)$. Then there exists a $(1-\delta)r$-partial strong colouring of G with respect to $\mathcal{P}$.*

We need two extra ingredients to prove Lemma 3.2.1. The first will follow from a fractional version of Conjecture 1.2.1 which was proven by Aharoni, Berger and Ziv [2]. We say that a graph on n vertices is *fractionally strongly r-colourable* if, after adding $r\lceil n/r \rceil - n$ isolated vertices and taking the union with any collection of spanning copies of K_r in the same vertex set, the graph is fractionally r-colourable.

Theorem 3.2.2 (Aharoni, Berger and Ziv [2]). *Every graph G is fractionally strongly r-colourable, for every $r \geq 2\Delta(G)$.*

Recall that a fractional colouring of a graph G is a function w that assigns weights in $[0,1]$ to the independent sets of G, with the condition that for every vertex $v \in V(G)$, $\sum_{I \ni v} w(I) = 1$. The fractional chromatic number of G is the minimum of $\sum_I w(I)$ over all fractional colourings of G, where the sum ranges over all independent sets of G. Note that if a graph G is fractionally strongly r-colourable, then for every partition $\mathcal{P}$ of $V(G)$ with classes of size r, every optimal fractional colouring w of $\mathcal{P}$ is supported precisely on independent transversals of $\mathcal{P}$ and, for every vertex $v \in V(G)$, $\sum_{I \ni v} w(I) = 1$. Thus we have the following corollary of Theorem 3.2.2.

Corollary 3.2.3. *Let G be a graph and $\mathcal{P}$ a partition of $V(G)$ with classes of size $r \geq 2\Delta(G)$. Let $\mathcal{T}$ be the set of all independent transversals of $\mathcal{P}$. Then there exists $w : \mathcal{T} \to [0,1]$ such that $\sum_{T \ni v, T \in \mathcal{T}} w(T) = 1$ for all $v \in V(G)$.*

The second ingredient we need is a result that guarantees the existence of large matchings in uniform hypergraphs satisfying certain regularity conditions. We use the following result of Pippenger [60] (see [43, Theorem 1.1]), which is an instance of the "nibble method" pioneered by Rödl (see, e.g. [27]) to find large matchings in uniform hypergraphs.

Theorem 3.2.4 (Pippenger [60])**.** *For all integers $k \geq 2$ and $\delta \geq 0$, there exists $D_0 = D_0(k, \delta)$ and $\tau = \tau(k, \delta)$ such that the following is true for all $D \geq D_0$. If H is a k-uniform hypergraph on n vertices which satisfies*

(i) $\deg(v) = (1 \pm \tau)D$ *for all* $v \in V(H)$, *and*

(ii) $\deg(u, v) < \tau D$, *for all distinct* $u, v \in V(H)$,

then H contains a matching M covering all but at most δn vertices.

We now prove Lemma 3.2.1, whose proof is based on previous work of Lo and Markström [52, Lemma 3.5] and Alon, Frankl, Huang, Rödl, Ruciński and Sudakov [6, Claim 4.1]. The idea is to define a k-uniform hypergraph H whose edges correspond to independent transversals of $\mathcal{P}$, and moreover H satisfies the conditions of Theorem 3.2.4. Then, we can find a matching in H covering all but at most δn vertices, which yields a $(1-\delta)r$-partial strong colouring of $\mathcal{P}$.

We will define H using two rounds of randomness. First, we select a family of subgraphs of G, each one selected at random, so that they preserve the relative degree conditions, and such that no transversal appears in more than one of these subgraphs. Secondly, we find a "fractional matching" in each of these subgraphs using Corollary 3.2.3. Finally, we define H by selecting each edge to be present in H with probability proportional to its weight in some of the fractional matchings (should it appear in at least one).

Proof of Lemma 3.2.1. Fix $k \geq 3$ and $\delta, \varepsilon > 0$. Without loss of generality, suppose $\varepsilon \leq 1$. Choose r_0 such that $1/r_0 \ll 1/k, \delta, \varepsilon$. Now consider any $r \geq r_0$ and a graph G on $n = rk$ vertices with $r \geq (2+\varepsilon)\Delta(G)$. Fix a partition $\mathcal{P}$ of $V(G)$ with classes of size r. Note that $k = |\mathcal{P}| \geq 3$.

We select constants that will define the parameters for the first round of randomness. Fix $\eta_1, \eta_2 \in (0,1)$ such that

$$2\eta_1 > \eta_2 > \eta_1, \text{ and}$$
$$1 > 9\eta_1.$$

(For concreteness, $(\eta_1, \eta_2) = (0.1, 0.15)$ works.) Let $m := \lfloor r^{\eta_1} \rfloor$ and $t := \lfloor r^{1+\eta_1} \rfloor$.

Claim 3.2.5. *There exist t vertex sets $R(1), \ldots, R(t)$ such that*

(i) *for every $j \in [t]$, $R(j)$ is a balanced mk-set,*
(ii) *every $v \in V(G)$ is in $r^{2\eta_1} \pm 2r^{\eta_2}$ many sets $R(j)$,*
(iii) *every $\mathcal{P}$-partite 2-set is in at most two sets $R(j)$,*
(iv) *every $\mathcal{P}$-partite 3-set is in at most one set $R(j)$,*
(v) *for every $j \in [t]$, $m \geq 2\Delta(G[R(j)])$.*

Proof of the claim. To prove Claim 3.2.5, we will see that properties (i)–(v) hold with probability close to 1 if each $R(j)$ is a random balanced mk-set, chosen uniformly and independently, and r is large. Now we detail the necessary calculations.

Fix $\eta_3 \in (0,1)$ such that

$$\eta_2 > \eta_1 + \eta_3, \text{ and}$$
$$1 > 9\eta_1 + \eta_3.$$

(For concreteness, if $(\eta_1, \eta_2) = (0.1, 0.15)$, then $\eta_3 = 0.02$ works.) Recall that $m = \lfloor r^{\eta_1} \rfloor$, and let $p := m/r$. For every $i \in [k]$ and $j \in [t]$, choose R_i^j to be a subset of V_i of size m, chosen independently and uniformly at random. Let $R(j) := \bigcup_{i\in[k]} R_i^j$. Clearly, (i) holds. Now we show that each of (ii)–(v) holds with probability which goes to 1 (as r goes to infinity). Since we assume r is sufficiently large, this will show that the desired sets $R(1), \dots, R(t)$ exist.

Consider $j \in [t]$. Proposition 3.1.3 and a union bound over the vertices in $R(j)$, implies that $m/2 < \Delta(G[R(j)])$ with probability at most $mk\exp(-cm)$, for some $c > 0$. Another union bound over the $t \le r^2$ possible choices of j implies that (v) fails with probability at most $r^2 mk\exp(-cm)$, which goes to zero when r (and thus, $m = \lfloor r^{\eta_1} \rfloor$) goes to infinity.

Now we check (ii)–(iv). Note that for every $v \in V(G)$ and every $j \in [t]$, $\mathbf{Pr}[v \in R(j)] = p$. For a $\mathcal{P}$-partite subset $S \subseteq V(G)$, let

$$Y_S := |\{j : S \subseteq R(j)\}|.$$

Since the probability that a particular $R_i \subseteq V_i$ intersects S is p, we have

$$\mathbf{Pr}[S \subseteq R(j)] = p^{|S|} \tag{3.2.1}$$

From linearity of expectation we deduce $\mathbf{E}[Y_S] = tp^{|S|}$. Now, from the definition of p and t we get that $p \le r^{-1+\eta_1}$ and $t \le r^{1+\eta_1}$. Thus we deduce that, for every $\mathcal{P}$-partite subset $S \subseteq V(G)$,

$$\mathbf{E}[Y_S] = tp^{|S|} \le r^{1+\eta_1-(1-\eta_1)|S|}. \tag{3.2.2}$$

Moreover, since $p \ge r^{\eta_1-1} - r^{-1}$ and $t \ge r^{1+\eta_1} - 1$, if $|S| = 1$ we have

$$\mathbf{E}[Y_S] = tp^{|S|} \ge (r^{1+\eta_1} - 1)(r^{\eta_1-1} - r^{-1}) \ge r^{2\eta_1} - r^{\eta_1-1} - r^{\eta_1} \ge r^{2\eta_1} - r^{\eta_2}, \tag{3.2.3}$$

where the last bound holds since $\eta_2 > \eta_1$ and r is large.

In particular, for every $v \in V(G)$, from (3.2.3) and (3.2.2) we get that $r^{2\eta_1} - r^{\eta_2} \le$

$\mathbf{E}[Y_{\{v\}}] \leq r^{2\eta_1}$. By Lemma 2.2.2, we have that

$$\begin{aligned}\Pr\left[|Y_{\{v\}} - r^{2\eta_1}| > 2r^{\eta_2}\right] &\leq \Pr\left[|Y_{\{v\}} - \mathbf{E}[Y_{\{v\}}]| > r^{\eta_2}\right] \\ &\leq 2\exp\left(-\frac{r^{2\eta_2}}{4r^{2\eta_1}}\right) \\ &\leq 2\exp\left(-\frac{r^{2\eta_3}}{4}\right),\end{aligned}$$

where in the last inequality we have used that $\eta_3 < \eta_2 - \eta_1$. Thus, an union bound over all the possible choices of $v \in V(G)$, implies that (ii) fails with probability at most $2rk\exp\left(-\frac{r^{2\eta_3}}{4}\right)$, which goes to zero as r goes to infinity.

Let $Z_2 := |\{S \in \binom{V(G)}{2} : S \text{ is } \mathcal{P}\text{-partite}, Y_S \geq 3\}|$. Using (3.2.1), $p \leq r^{-1+\eta_1}$ and $t \leq r^{1+\eta_1}$ we observe that

$$\mathbf{E}(Z_2) < \binom{k}{2} r^2 t^3 p^6 \leq k^2 r^{-1+9\eta_1} \leq r^{-\eta_3}, \tag{3.2.4}$$

where in the last inequality we have used that $\eta_3 + 9\eta_1 < 1$ and that r is large. Let $Z_3 := |\{S \in \binom{V(G)}{3} : S \text{ is } \mathcal{P}\text{-partite}, Y_S \geq 2\}|$. Bounding the terms similarly, observe that

$$\mathbf{E}(Z_3) < \binom{k}{3} r^3 t^2 p^6 \leq k^3 r^{-1+8\eta_1} \leq r^{-\eta_3}. \tag{3.2.5}$$

Together with Markov's inequality (Lemma 2.2.1), (3.2.4) and (3.2.5) imply that (iii) and (iv) fail with probability which goes to zero as r goes to infinity.

Since each of the properties (ii)–(v) fail with probability going to zero as r goes to infinity, for r sufficiently large all the properties hold simultaneously with positive probability. In particular, this proves the desired $R(1), \dots, R(t)$ exist. □

Let $R(1),\dots,R(t)$ be given by Claim 3.2.5. By (v) and Corollary 3.2.3, for each $j \in [t]$ there exists a function w^j that assigns weights in $[0,1]$ to the independent transversals of $\mathcal{P}$ contained in $G[R(j)]$, such that for every $v \in V(G[R(j)])$, $\sum_{T \ni v} w^j(T) = 1$. Now we construct a random k-uniform graph H on $V(G)$ such that each independent transversal T of $\mathcal{P}$ is randomly and independently chosen as an edge of H with

$$\mathbf{Pr}[T \in H] = \begin{cases} w^{j_T}(T) & \text{if } T \subseteq G[R(j_T)] \text{ for some } j_T \in [r^{1+\eta_2}], \\ 0 & \text{otherwise.} \end{cases}$$

Note that j_T is unique by (iv) as $k \geq 3$, so H is well-defined. For each $v \in V(G)$, let $J_v = \{j : v \in R(j)\}$ so that $|J_v| = r^{2\eta_1} \pm 2r^{\eta_2}$ by (ii). For each $v \in V(G)$, let E_v^j be the set of independent transversals in $G[R(j)]$ containing v. Thus, for $v \in V(G)$, $\deg_H(v)$ is a generalised binomial random variable with expectation

$$\mathbf{E}[\deg_H(v)] = \sum_{j \in J_v} \sum_{T \in E_v^j} w^j(T) = |J_v| = r^{2\eta_1} \pm 2r^{\eta_2}.$$

Similarly, for every $\mathcal{P}$-partite 2-set $\{u, v\}$,

$$\mathbf{E}[\deg_H(u,v)] = \sum_{j \in J_u \cap J_v} \sum_{T \in E_u^j \cap E_v^j} w^j(T) \leq |J_u \cap J_v| \leq 2$$

by (iii). For every 2-set $\{u, v\}$ that is not $\mathcal{P}$-partite, $\deg_H(u, v) = 0$. Fix $\eta_4 \in (0, 1)$ such that $2\eta_1 > \eta_4 > \eta_2$. By using Chernoff's inequality (2.2.1), we may assume that, for every $v \in V(G)$ and every 2-set $\{u, v\} \subseteq V(G)$,

$$\deg_H(v) = r^{2\eta_1} \pm r^{\eta_4}, \qquad \deg_H(u,v) < r^{\eta_1}.$$

Thus H satisfies the hypothesis of Theorem 3.2.4 (with $D = r^{2\eta_1}$ and $\tau = \max\{r^{\eta_4 - 2\eta_1}, r^{-\eta_1}\}$) and the proof is completed. ■

3.3 Asymptotic bounds for the strong chromatic number

Now we have the tools to present the proof of the main result of this section, Theorem 1.2.3.

Proof of Theorem 1.2.3. Let r_0 and γ be such that $1/r_0 \ll \gamma \ll 1/k, \varepsilon$, and let

$r \geq r_0$. Consider a graph G on $n := rk$ vertices and a partition $\mathcal{P} = \{V_1, \ldots, V_k\}$ with classes of size $r \geq (2+\varepsilon)\Delta(G)$.

Let $\alpha := \gamma/(8k^2)$ and $\beta := \gamma^2/(64k^3)$. By Lemma 3.1.5, there exists a balanced set A of size at most αn such that, for every balanced set S of size at most βn, $G[A \cup S]$ can be partitioned into independent transversals of $\mathcal{P}$. Remove A from G to obtain a graph G', together with a partition $\mathcal{P}' = \{V_1', \ldots, V_k'\}$ obtained from $V_i' = V_i \setminus A$ for each $i \in [k]$. Note that $\Delta(G') \leq \Delta(G)$ and $r' := |V_i'| \geq (1-\alpha)r$ and, therefore, $r' \geq (1-\alpha)(2+\varepsilon)\Delta(G) \geq (2+\varepsilon/2)\Delta(G')$.

By Lemma 3.2.1, we obtain a $(1-\beta)r'$-partial strong colouring of G' with respect to $\mathcal{P}'$. This gives a collection $\mathcal{T}'$ of disjoint independent transversals of $\mathcal{P}$ that covers every vertex of G' except for a set S of size at most $\beta r' \leq \beta r$. Then $G[A \cup S]$ can be covered by a collection $\mathcal{T}$ of disjoint independent transversals of $\mathcal{P}$. Therefore, $\mathcal{T} \cup \mathcal{T}'$ is a spanning collection of disjoint independent transversals of $\mathcal{P}$, as desired. ∎

— 4 —

Covering hypergraphs with tight cycles

In this chapter we prove results about covering thresholds using tight cycles on s vertices in k-graphs. We also prove lower bounds for the existence, covering and tiling thresholds for tight cycles, and provide constructions that will also be used during the investigation of the tiling threshold with cycles (which is done in Chapter 5).

This chapter is organised as follows. In Section 4.2 we prove Propositions 1.3.6–1.3.8, which give lower bounds for the tiling threshold and covering threshold in various cases depending on k and s. In Section 4.4 we prove Proposition 1.3.3 and Theorem 1.3.4, which give upper bounds for covering thresholds.

4.1 Notation and sketch of proofs

4.1.1 Notation

We will use the following notation during this section. For all $k \in \mathbb{N}$, let $[k] := \{1, \ldots, k\}$. Let S_k be the symmetric group of all permutations of the set $[k]$, with the composition of functions as the group operation. Let $\mathrm{id} \in S_k$ be the *identity function* that fixes all elements in $[k]$. We use the standard "cycle notation" for permutations, which can be described as follows. Given distinct $i_1, \ldots, i_r \in [k]$, the *cyclic permutation* $(i_1 i_2 \cdots i_r) \in S_k$ is the permutation that maps i_j to i_{j+1} for all $1 \leq j < r$ and i_r to i_1, and fixes all the other elements; we say that such a cyclic permutation has *length* r. A well-known fact is that all permutations $\sigma \in S_k$ can be written as a composition of cyclic permutations $\sigma_1 \cdots \sigma_t$ such that these cyclic permutations are *disjoint*, meaning that there are no common elements between any pair of these different cyclic permutations.

Let H be a k-graph, $V_1, \dots, V_k$ be disjoint vertex sets of V and let $\sigma \in S_k$. We say that a tight path $P = v_1 \cdots v_\ell$ in H has *end-type σ with respect to* $V_1, \dots, V_k$ if for all $2 \leq i \leq k$, $v_{\ell-k+i} \in V_{\sigma(i)}$. Similarly, we say P has *start-type σ with respect to* $V_1, \dots, V_k$ if $v_i \in V_{\sigma(i)}$ for all $1 \leq i \leq k-1$. If H and $V_1, \dots, V_k$ are clear from the context, we simply say that P has *end-type σ* and *start-type σ*, respectively.

4.1.2 Sketch of proofs

We now sketch the proof of Theorem 1.3.4. Let H be a k-graph on n vertices with $\delta_{k-1}(H) \geq (1/2 + \gamma)n$. Consider any vertex $x \in V(H)$. We can show that, for some appropriate value of t, x is contained in some copy K of $K_k^k(t)$ with vertex classes $V_1, \dots, V_k$. Suppose that $s \equiv r \not\equiv 0 \bmod k$ with $1 \leq r < k$. Suppose $P = v_1 \cdots v_k$ is a tight path in K such that $v_i \in V_i$ for all $1 \leq i \leq k$ and $v_1 = x$. By wrapping around K, we may find a tight path $P_2 = v_1 \cdots v_\ell$ which extends P_1, but if we only use vertices and edges of K, then we have $v_j \in V_\ell$ where $j \equiv \ell \bmod k$, for all $j \in [\ell]$. To break this pattern, we will use some gadgets (see Section 4.3 for a formal definition). Roughly speaking, a gadget is a k-graph on $V(K)$ and some extra vertices of H. Using these gadgets we can extend P to a tight path P' with end-type σ, for an arbitrary $\sigma \in S_k$ (see Lemma 4.3.2). Having done that (and choosing σ appropriately), then it is easy to extend P' into a copy of C_s^k (by wrapping around $V_1, \dots, V_k$).

4.2 Lower bounds

In this section, we construct k-graphs which give lower bounds for the codegree Turán numbers and covering and tiling thresholds for tight cycles. These constructions will imply Proposition 1.3.6 and Proposition 1.3.8. A different construction will prove Proposition 1.3.7 and we postpone it to the last part of the section.

We remark that the bounds that we obtain are not best possible for all values of n, s and k. Some improvements can be made by considering the same examples but being more careful with the calculations of the minimum degree of each k-graph. Most of the times, this can be done by separating the analysis in cases depending on divisibility conditions of n, s and k (compare, for instance, with the extremal examples for perfect matchings in k-graphs [75, Construction 1.1]). This changes the lower bounds, in some of the cases, by an additive constant. We did not pursue this direction to simplify the presentation, since our main interest is in the asymptotics of each threshold function.

Let A and B be disjoint vertex sets. Define $H_0^k = H_0^k(A, B)$ to be the k-graph on $A \cup B$ such that the edges of H_0^k are exactly the k-sets e of vertices that satisfy $|e \cap B| \equiv 1 \bmod 2$ (see Figure 4.1). Note that $\delta_{k-1}(H_0^k) \geq \min\{|A|, |B|\} - k + 1$.

Recall the definition of admissible pairs (Definition 1.3.5): for two positive integers k, s which satisfy $2 \leq k < s$ and $d = \gcd(k, s)$, we say that the pair (k, s) is admissible if $d = 1$ or k/d is even.

Proposition 4.2.1. *Let $3 \leq k \leq s$ and $d = \gcd(k, s)$. Let A and B be disjoint vertex sets. Suppose that $H_0^k(A, B)$ contains a tight cycle C_s^k on s vertices with $V(C_s^k) \cap A \neq \varnothing$. Then $|V(C_s^k) \cap A| \equiv 0 \mod s/d$ and (k, s) is not an admissible pair.*

Proof. Let $C_s^k = v_1 \cdots v_s$. For all $1 \leq i \leq s$, let $\phi_i \in \{A, B\}$ be such that $v_i \in \phi_i$ and let $\phi_{s+i} = \phi_i$. If two edges e and e' in $E(H_0^k(A, B))$ satisfy $|e \cap e'| = k - 1$, then $|e \cap A| = |e' \cap A|$ by construction. Thus $\phi_{i+k} = \phi_i$ for all $1 \leq i \leq s$. Therefore, $\phi_{i+d} = \phi_i$ for all $1 \leq i \leq s$. Hence, $|V(C_s^k) \cap A| \equiv 0 \bmod s/d$.

Let $r := |\{v_1, \ldots, v_k\} \cap A| = |\{i : 1 \leq i \leq k, \phi_i = A\}|$. Note that $r > 0$ and $r \in \{k/d, 2k/d, \ldots, k\}$. Since $\{v_1, \ldots, v_k\}$ is an edge in $H_0^k(A, B)$, it follows that $k - r \equiv 1 \bmod 2$ and so, $r \not\equiv k \bmod 2$. This implies $d \geq 2$ and k/d is odd, i.e., (k, s) is not an admissible pair. ∎

Now we use Proposition 4.2.1 to prove Propositions 1.3.6 and 1.3.8.

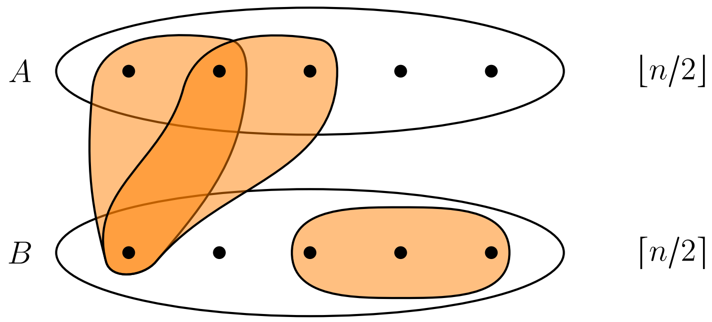

Figure 4.1: Example of the construction in the proof of Proposition 1.3.6, for $k = 3$ and $n = 10$. All of the triples which intersect A in exactly 0 or 2 vertices are present as edges, we only draw some of them for clarity.

Proof of Proposition 1.3.6. Let A and B be disjoint vertex sets of sizes $|A| = \lfloor n/2 \rfloor$ and $|B| = \lceil n/2 \rceil$. Consider the k-graph $H_0 = H_0^k(A, B)$. By Proposition 4.2.1, no vertex of A can be covered with a copy of C_s^k. Then $c(n, C_s^k) \geq \delta_{k-1}(H_0) \geq \lfloor n/2 \rfloor - k + 1$.

Moreover, if k is even, then $H_0^k(A, B) = H_0^k(B, A)$. So no vertex of B can be covered by a copy of C_s^k. Hence H_0 is C_s^k-free. Therefore, $\mathrm{ex}_{k-1}(n, C_s^k) \geq \delta_{k-1}(H_0) \geq \lfloor n/2 \rfloor - k + 1$. ∎

Proof of Proposition 1.3.8. To see the first part of the statement, let $d := \gcd(k, s)$ and $s' := s/d$. Note that $d \leq k < s$, thus $s' > 1$. Let A and B be disjoint vertex sets chosen such that $|A| + |B| = n$, $||A| - |B|| \leq 2$ and $|A| \not\equiv 0 \bmod s'$. Consider the k-graph $H_0 = H_0^k(A, B)$ and note that $\delta_{k-1}(H_0) \geq \min\{|A|, |B|\} - k + 1 \geq \lfloor n/2 \rfloor - k$ (see Figure 4.1). Proposition 4.2.1 implies that all copies C of C_s^k in H_0 satisfy $|V(C) \cap A| \equiv 0 \bmod s'$. Since $|A| \not\equiv 0 \bmod s'$, it is impossible to cover all the vertices in A with vertex-disjoint copies of C_s^k. This proves that $t(n, C_s^k) \geq \delta_{k-1}(H_0) \geq \lfloor n/2 \rfloor - k$, as desired.

Now suppose that (k, s) is an admissible pair. Let H be the k-graph on n vertices with a vertex partition $\{A, B, T\}$ with $|A| = \lceil (n - |T|)/2 \rceil$ and $|B| = \lfloor (n - |T|)/2 \rfloor$, where $|T|$ will be specified later. The edge set of H consists of all k-sets e such that $|e \cap B| \equiv 1 \bmod 2$ or $e \cap T \neq \varnothing$ (see Figure 4.2). Note that $\delta_{k-1}(H) \geq \min\{|A|, |B|\} + |T| - (k - 1) \geq \lfloor (n + |T|)/2 \rfloor - k + 1$. We separate the analysis into two cases depending on the parity of k.

Case 1: k even. Since $H[A \cup B] = H_0^k(A, B) = H_0^k(B, A)$, by Proposition 4.2.1, $H[A \cup B]$ is C_s^k-free. Thus, all copies of C_s^k in H must intersect T in at least one vertex. Hence, all C_s^k-tilings have at most $|T|$ vertex-disjoint copies of C_s^k. Taking $|T| = n/s - 1$ assures that H does not contain a perfect C_s^k-tiling. This implies that $t(n, C_s^k) \geq \lfloor (1/2 + 1/(2s))\, n \rfloor - k$.

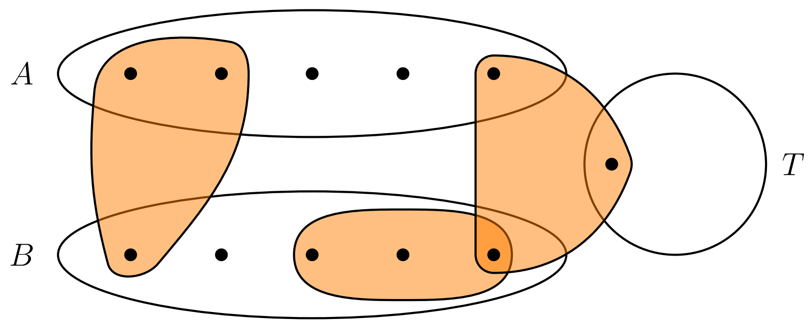

Figure 4.2: Example of the construction in the second part of the proof of Proposition 1.3.8, for $k = 3$. The edges correspond to: (i) triples in $A \cup B$ which intersect A in exactly 0 or 2 vertices, and (ii) the triples with non-empty intersection with T.

Case 2: k odd. Since $H[A \cup B] = H_0^k(A, B)$, by Proposition 4.2.1 no vertex in A can be covered by a copy of C_s^k. Hence, all copies of C_s^k in H with non-empty intersection with A must also have non-empty intersection with T. Moreover, all edges in H intersect A in at most $k-1$ vertices, so all copies of C_s^k in H intersect A in at most $s(k-1)/k$ vertices. Thus a perfect C_s^k-tiling would contain at most $|T|$ and at least $k|A|/(s(k-1))$ cycles intersecting A. Let $|T| = \lceil nk/(2s(k-1)+k) \rceil - 1$. Since $|T| < nk/(2s(k-1)+k)$ and $|A| \geq (n - |T|)/2$,

$$\frac{k|A|}{s(k-1)} \geq \frac{k(n-|T|)}{2s(k-1)} > \frac{nk}{2s(k-1)} \left(1 - \frac{k}{2s(k-1)+k}\right) > |T|,$$

and thus a perfect C_s^k-tiling in H cannot exist. This implies

$$t(n, C_s^k) \geq \delta_{k-1}(H) \geq \left\lfloor \frac{n+|T|}{2} \right\rfloor - k + 1 \geq \left\lfloor \left(\frac{1}{2} + \frac{k}{4s(k-1)+2k}\right) n \right\rfloor - k,$$

as desired. ∎

It remains to prove Proposition 1.3.7. We acknowledge and thank an anonymous referee who suggested this family of examples during the revision of the paper containing the research of this chapter [32]. We are not aware of the appearance of this example in the literature before, although it bears some resemblance to examples considered by Mycroft to give lower bounds for tiling problems [58, Section 2].

V_1

V_2

V_0

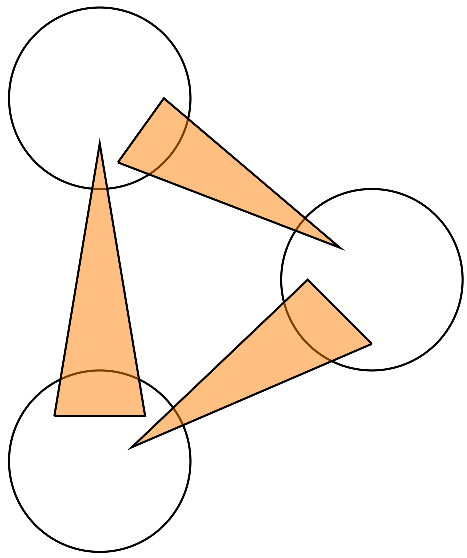

Figure 4.3: Example of the construction in the proof of Proposition 1.3.7, for $k = p = 3$. Every tight cycle in this hypergraph has length divisible by 3.

Proof of Proposition 1.3.7. Since p is a divisor of k which does not divide s, it holds that $1 < p \leq k$. Given a vertex set V of size n, partition it into p disjoint vertex sets $V_0, \dots, V_{p-1}$ of size as equal as possible, so $|V_i| \geq \lfloor n/p \rfloor$ holds for all $i \in \{0, \dots, p-1\}$. Assume that every $x \in V_i$ is labelled with i, for each $i \in \{0, \dots, p-1\}$. Let H be the k-graph on V where the edges are the k-sets such that the sum of the labels of its vertices is congruent to 1 modulo p (see Figure 4.3).

It is immediate to check that no edge of H is entirely contained in any sets V_i, and that, for every $(k-1)$-set S in V, $N_H(S) = V_j \setminus S$ for some $j \in \{0, \dots, p-1\}$. Thus H has codegree at least $\lfloor n/p \rfloor - k + 2$.

We show that H is C_s^k-free. Let C be a tight cycle on t vertices in H. It is enough to show that p divides t (since p does not divide s, it will follow that $t \neq s$). We double count the sum T of the labels of vertices, over all the edges of C. On one hand, $T \equiv 0 \bmod k$ since each vertex appears in exactly k edges of C and thus is counted k times. Since p divides k, $T \equiv 0 \bmod p$. On the other hand, the sum of the labels of a single edge is congruent to 1 modulo p and there are t of them, thus $T \equiv t \bmod p$. This implies that p divides t. ∎

4.3 G-gadgets

Throughout this section, let $\tau := (123\cdots k) \in S_k$ (recall that we are using the cyclic notation for permutations, so τ is a cyclic permutation of length k). Let H be a k-graph, and let K be a complete (k,k)-graph in H with its natural vertex partition $\{V_1, \dots, V_k\}$. Let P be a tight path in H with end-type $\pi \in S_k$. For $x \in V_{\pi(1)} \setminus V(P)$, Px is a tight path of H with end-type $\pi\tau$. We call such an extension a *simple extension of* P. By repeatedly applying r simple extensions (which is possible as long as there are available vertices), we may obtain an extension $Px_1\cdots x_r$ of P with end-type $\pi\tau^r$, using r extra vertices and edges in K.

In the same spirit, observe that if P_1 has end-type π and P_2 has start-type $\pi\tau$, then the sequence of ordered clusters corresponding to the last $k-1$ vertices of P_1 coincides with the corresponding sequence of the first $k-1$ vertices of P_2. Thus, by using one extra vertex $x \in V_{\pi(1)} \setminus (V(P_1) \cup V(P_2))$ we get that $P_1 x P_2$ is a tight path, i.e., we can join P_1 and P_2 into a single tight path by using one extra vertex

If P is a path with end-type π, we would like to find a path P' that extends P such that $|V(P')| \equiv |V(P)| \bmod k$ and P' has end-type σ, for arbitrary $\sigma \in S_k$. The goal of this section is to define and study 'G-gadgets', a tool which will allow us to do precisely that.

Let G be a 2-graph on $[k]$ and $S \subseteq V(H)$. We say $W_G \subseteq V(H)$ is a *G-gadget for K avoiding S* if there exists a family of pairwise-disjoint sets $\{W_{ij} : ij \in E(G)\}$ such that $W_G = \bigcup_{ij \in E(G)} W_{ij}$, and for all $ij \in E(G)$,

(W1) $|W_{ij}| = 2k - 1$,

(W2) $|W_{ij} \setminus V(K)| = 1$, $W_{ij} \cap S = \varnothing$ and, for all $1 \leq i' \leq k$,

$$|W_{ij} \cap V_{i'}| = \begin{cases} 1 & \text{if } i' \in \{i, j\}, \\ 2 & \text{otherwise,} \end{cases}$$

(W3) for all $\sigma \in S_k$ with $\sigma(1) \in \{i, j\}$, $H[W_{ij}]$ contains a spanning tight path with start-type $\sigma\tau$ and end-type $(ij)\sigma$.

If K is clear from the context, we will just say "a G-gadget avoiding S". For all edges $ij \in E(G)$, we write w_{ij} for the unique vertex in $W_{ij} \setminus V(K)$.

We emphasize that (W3) is the key property that allows us to obtain an extension of a path at the same time as we perform a change in the end-type. In words, (W3) says that given any $k - 1$ ordered clusters that miss V_i, there exists a tight path with vertex set W_{ij}, which starts with the same ordered $k - 1$ clusters and ends with the same ordered $k - 1$ clusters but with V_j replaced by V_i. In other words, W_{ij} allows us to "switch" the type of a path by replacing i by j. See Figure 4.4 for an example.

Recall that σ is a cyclic permutation if there exist distinct $i_1, \ldots, i_r \in [k]$ such that $\sigma = (i_1 i_2 \cdots i_r)$, that is, σ maps i_j to i_{j+1} for all $1 \leq j < r$ and i_r to i_1, and fixes all the other elements. Suppose P is a tight path with end-type π and σ is a cyclic permutation. In the next lemma, we show how to extend P into a tight path with end-type $\sigma\pi$ using a G-gadget, where G is a path.

Lemma 4.3.1. *Let $k \geq 3$ and $r \geq 2$. Let $\sigma = (i_1 i_2 \cdots i_r) \in S_k$ be a cyclic permutation. Let G be a 2-graph on $[k]$ containing the path $Q = i_1 i_2 \cdots i_r$. Let H be a k-graph containing a complete (k,k)-graph K with vertex partition $V_1, \ldots, V_k$. Suppose that P is a tight path in H with end-type $\pi \in S_k$ such that $\pi(1) = i_r$. Suppose W_G*

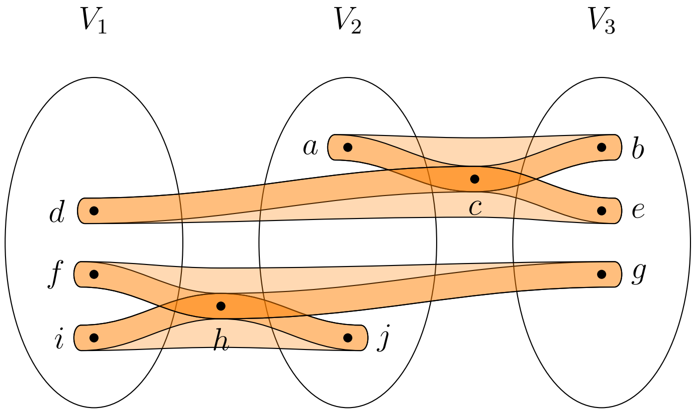

Figure 4.4: An example of a G-gadget in a 3-graph H. Let G be the graph on $[3]$ consisting of the edges 12 and 23. K is the complete $(3,3)$-graph with vertex partition V_1, V_2, V_3 (edges not shown) and W_G consists of the union of $W_{12} = \{a,b,c,d,e\}$ and $W_{23} = \{f,g,h,i,j\}$, including the edges $\{abc, bcd, cde, ace, fgh, ghi, hij, fhj\}$. The coloured edges are in $H \smallsetminus K$.
We show an example of (W3). Let $\sigma = \mathrm{id}$, and note that $\sigma(1) = 1$. In $H[W_{12}]$ we find the tight path $abcde$ on $5 = 2k-1$ vertices, whose start-type is $\sigma(123) = (123)$ and its end-type is $(12)\sigma = (12)$. This means the first two vertices of $abcde$ are in clusters V_2, V_3, and its last two vertices are in clusters V_1, V_3, respectively.

is a G-gadget avoiding $V(P)$ and $|V_i \smallsetminus V(P)| \geq 2|E(G)|$ for all $1 \leq j \leq k$. Then there exists an extension P' of P with end-type $\sigma\pi$ such that

(i) $|V(P')| = |V(P)| + 2k(r-1)$,

(ii) *for all* $1 \leq i \leq k$,

$$|V_i \cap (V(P') \smallsetminus V(P))| = \begin{cases} 2(r-1) - 1 & \text{if } i \in \{i_1, i_2, \ldots, i_{r-1}\}, \\ 2(r-1) & \text{otherwise,} \end{cases}$$

(iii) *there exists a* $(G-Q)$*-gadget* W_{G-Q} *for* K *avoiding* $V(P')$ *and*

(iv) $V(P') \smallsetminus V(P \cup K) = \{w_{i_j i_{j+1}} : 1 \leq j < r\}$.

Proof. We proceed by induction on r. First suppose that $r = 2$ and so $\sigma = (i_1 i_2)$. Consider a G-gadget W_G avoiding $V(P)$. Since $i_1 i_2 \in E(G)$, there exists a set $W_{i_1 i_2} \subseteq W_G$ disjoint from $V(P)$ such that $|W_{i_1 i_2}| = 2k-1$ and $H[W_{i_1 i_2}]$ contains a spanning tight path P'' with start-type $\pi\tau$ and end-type $(i_1 i_2)\pi = \sigma\pi$. Note that $|V_{i_2} \cap W_G| \leq 2|E(G)| - 1$, as $|V_{i_2} \cap W_{i_1 i_2}| = 1$. Hence $V_{i_2} \smallsetminus (V(P) \cup W_G) \neq \varnothing$. Take an arbitrary vertex $x_{i_2} \in V_{i_2} \setminus (V(P) \cup W_G)$ and set $P' = Px_{i_2}P''$. Since $\pi(1) = i_2$, it follows that P' is a tight path with end-type $\sigma\pi$, and P' satisfies properties (i), (ii) and (iv). Set $W_{G-i_1i_2} = W_G \setminus W_{i_1i_2}$. Then $W_{G-i_1i_2}$ is a $(G - i_1i_2)$-gadget for K avoiding $V(P')$, so P' satisfies property (iii), as desired.

Next, suppose $r > 2$. Let $\sigma' := (i_2i_3\cdots i_r)$ and note that $\sigma = (i_1i_2)\sigma'$. Then σ' is a cyclic permutation of length $r-1$, with $\pi(1) = i_r$ and the path $Q' = i_2\cdots i_{r-1}i_r$ is a subgraph of G. By the induction hypothesis, there exists an extension P'' of P with end-type $\sigma'\pi$ such that $|V(P'')| = |V(P)| + 2k(r-2)$ and, for all $1 \leq i \leq k$,

$$|V_i \cap (V(P'') \setminus V(P))| = \begin{cases} 2(r-2) - 1 & \text{if } i \in \{i_2, i_3, \ldots, i_{r-1}\}, \\ 2(r-2) & \text{otherwise.} \end{cases}$$

Moreover, there exists a $(G - Q')$-gadget $W_{G-Q'}$ avoiding $V(P'')$ and $V(P'') \setminus V(P \cup K) = \{w_{i_ji_{j+1}} : 2 \leq j < r\}$.

Note that $\sigma'\pi(1) = \sigma'(i_r) = i_2$ and $i_1i_2 \in E(G - Q')$. For all $1 \leq i \leq k$, $|V_i \setminus V(P')| \geq 2|E(G - Q')|$. Again by the induction hypothesis, there exists an extension P' of P'' with end-type $(i_1i_2)\sigma'\pi = \sigma\pi$ such that $|V(P')| = |V(P'')| + 2k = |V(P)| + 2k(r-1)$ and, for all $1 \leq i \leq k$,

$$|V_i \cap (V(P') \setminus V(P''))| = \begin{cases} 1 & \text{if } i = i_1, \\ 2 & \text{otherwise.} \end{cases}$$

and $V(P') \setminus (V(P'' \cup K)) = \{w_{i_1i_2}\}$, so P' satisfies properties (i), (ii) and (iv). Furthermore, set $W_{G-Q} = W_G - \bigcup_{j=1}^{r-1} W_{i_ji_{j+1}}$. Then W_{G-Q} is a $(G - Q)$-gadget for K avoiding $V(P')$, so P' satisfies property (iii) as well. ∎

In the next lemma, we show how to extend a path with end-type id to one with an arbitrary end-type. We will need the following definitions. Consider an arbitrary $\sigma \in S_k \setminus \{\mathrm{id}\}$. Write σ in its cyclic decomposition as

$$\sigma = (i_{1,1}i_{1,2}\cdots i_{1,r_1})(i_{2,1}i_{2,2}\cdots i_{2,r_2})\cdots(i_{t,1}i_{t,2}\cdots i_{t,r_t}),$$

where σ is a product of t disjoint cyclic permutations of respective lengths $r_1,\dots,r_t$, so that $r_j \geq 2$ and $i_{j,r_j} = \min\{i_{j,r'} : 1 \leq r' \leq r_j\}$ for all $1 \leq j \leq t$; and $i_{1,r_1} < i_{2,r_2} < \cdots < i_{t,r_t}$. Define $t(\sigma)$ to be the number t of cycles of σ in this decomposition, and we also define $m(\sigma) = i_{t(\sigma),r_{t(\sigma)}}$. On the other hand, if $\sigma = \mathrm{id}$, then define $t(\sigma) = 0$ and $m(\sigma) = 1$.

Define G_σ to be the 2-graph on $[k]$ consisting precisely of the (vertex-disjoint) paths $Q_j = i_{j,1}i_{j,2}\cdots i_{j,r_j}$ for all $1 \leq j \leq t(\sigma)$. So G_{id} is an empty 2-graph. Note that for all σ,

$$2|E(G_\sigma)| + t(\sigma) = 2\sum_{j=1}^{t(\sigma)} r_j - t(\sigma) \leq 2k - 1. \tag{4.3.1}$$

For $1 \leq i \leq k$ and $\sigma \in S_k \setminus \{\mathrm{id}\}$, set $X_{i,\sigma} = 1$ if $i \in \{i_{t',1},\dots,i_{t',r_{t'}-1}\}$ for some $1 \leq t' \leq t$, and $X_{i,\sigma} = 0$ otherwise. Also, for $1 \leq i \leq k$, set $Y_{i,\sigma} = 1$ if $i \in \{\sigma(j) : 1 \leq j < m(\sigma)\}$ and $Y_{i,\sigma} = 0$ otherwise. If $\sigma = \mathrm{id}$, then define $X_{i,\sigma} = Y_{i,\sigma} = 0$ for all $1 \leq i \leq k$.

Lemma 4.3.2. *Let $k \geq 3$. Let H be a k-graph containing a complete (k,k)-graph K with vertex partition $V_1,\dots,V_k$ and a tight path P with end-type* id. *Let $\sigma \in S_k$ and let G be a 2-graph on $[k]$ containing G_σ. Suppose that K has a G-gadget W_G avoiding $V(P)$, and $|V_i \setminus V(P)| \geq 2|E(G)| + 2$. Then there exists an extension P' of P with end-type $\sigma\tau^{m(\sigma)-1}$ such that*

(i) $|V(P')| = |V(P)| + 2k|E(G_\sigma)| + m(\sigma) - 1$,

(ii) *for all* $1 \leq i \leq k$, $|V_i \cap (V(P') \setminus V(P))| = 2|E(G_\sigma)| - X_{i,\sigma} + Y_{i,\sigma}$,

(iii) *K has a $(G - G_\sigma)$-gadget avoiding $V(P')$ and*

(iv) $V(P') \setminus V(P \cup K) = \{w_{ij} : ij \in E(G_\sigma)\}$.

Proof. Let

$$\sigma = (i_{1,1}i_{1,2}\cdots i_{1,r_1})(i_{2,1}i_{2,2}\cdots i_{2,r_2})\cdots(i_{t,1}i_{t,2}\cdots i_{t,r_t})$$

as defined above. We proceed by induction on $t = t(\sigma)$. If $t = 0$, then $\sigma = \mathrm{id}$ and $m(\sigma) = 1$, so the lemma holds by setting $P' = P$. Now suppose that $t \geq 1$ and the lemma is true for all $\sigma' \in S_k$ with $t(\sigma') < t$. Let

$$\sigma_1 := (i_{1,1}i_{1,2}\cdots i_{1,r_1})(i_{2,1}i_{2,2}\cdots i_{2,r_2})\cdots(i_{t-1,1}i_{t-1,2}\cdots i_{t-1,r_{t-1}})$$

and $\sigma_2 := (i_{t,1}i_{t,2}\cdots i_{t,r_t})$, so $\sigma_1\sigma_2 = \sigma_2\sigma_1 = \sigma$. For $1 \leq i \leq 2$, let $G_i := G_{\sigma_i}$ and $m_i := m(\sigma_i)$. Note that $G_\sigma = G_1 \cup G_2$. Let $G' := G - G_1$.

Since $t(\sigma_1) = t - 1$, by the induction hypothesis, there exists a path P_1 that extends P with end-type $\sigma_1\tau^{m_1-1}$ such that

(i') $|V(P_1)| = |V(P)| + 2k|E(G_1)| + m_1 - 1$,

(ii') for all $1 \leq i \leq k$, $|V_i \cap (V(P_1) \smallsetminus V(P))| = 2|E(G_1)| - X_{i,\sigma_1} + Y_{i,\sigma_1}$,

(iii') K has a G'-gadget $W_{G'}$ avoiding $V(P_1)$ and

(iv') $V(P_1) \smallsetminus V(P \cup K) = \{w_{ij} : ij \in E(G_1)\}$.

Note that for all $1 \leq i \leq k$,

$$|V_i \smallsetminus (V(P_1) \cup W_{G'})| \geq 2|E(G)| + 2 - (2|E(G_1)| + 1) - 2|E(G')| = 1.$$

We extend P_1 using $m_2 - m_1 > 0$ simple extensions, avoiding the set $V(P_1) \cup W_{G'}$ in each step, to obtain an extension P_2 of P_1 with end-type $\sigma_1\tau^{m_1-1}\tau^{m_2-m_1} = \sigma_1\tau^{m_2-1}$ such that

$$|V(P_2)| = |V(P_1)| + m_2 - m_1 = |V(P)| + 2k|E(G_1)| + m_2 - 1$$

and $W_{G'}$ is a G'-gadget for K that avoids $V(P_2)$. As P_1 has end-type $\sigma_1\tau^{m_1-1}$, $V(P_2) \smallsetminus V(P_1)$ contains precisely one vertex in V_i for all $i \in \{\sigma_1\tau^{m_1-1}(j) : 1 \leq j \leq$

$m_2 - m_1\} = \{\sigma_1(m_1), \dots, \sigma_1(m_2 - 1)\}$. Since $\sigma_1(i) = \sigma(i)$ for all $m_1 \le i < m_2$ and $m_2 = i_{t,r_t}$, together with (ii') we deduce that

$$|V_i \cap (V(P_2) \setminus V(P))| = 2|E(G_1)| - X_{i,\sigma_1} + Y_{i,\sigma}. \tag{4.3.2}$$

Note that $\sigma_1\tau^{m_2-1}(1) = \sigma_1(m_2) = \sigma_1(i_{t,r_t}) = i_{t,r_t}$. Since G' contains G_2, by Lemma 4.3.1 there exists an extension P' of P_2 with $|V(P')| = |V(P_2)| + 2k|E(G_2)|$ and P' has end-type $\sigma_2\sigma_1\tau^{m_2-1} = \sigma\tau^{m(\sigma)-1}$, as $m_2 = m(\sigma)$. Moreover, as $G' - G_2 = G - G_\sigma$, K has a $(G - G_\sigma)$-gadget avoiding $V(P')$, implying (iii). Similarly, (iv) holds. Note that

$$|V(P')| = |V(P_2)| + 2k|E(G_2)| = |V(P)| + 2k|E(G_\sigma)| + m(\sigma) - 1,$$

implying (i). Finally, for all $1 \le i \le k$, we have

$$|V_i \cap (V(P') \setminus V(P_2))| = \begin{cases} 2|E(G_2)| - 1 & \text{if } i \in \{i_{t,1}, \dots, i_{t,r_t-1}\}, \\ 2|E(G_2)| & \text{otherwise.} \end{cases}$$

So $|V_i \cap (V(P') \setminus V(P_2))| = 2|E(G_2)| - X_{i,\sigma_2}$. Note that $X_{i,\sigma} = X_{i,\sigma_1} + X_{i,\sigma_2}$ because σ_1 and σ_2 are disjoint. Thus, together with (4.3.2), (ii) holds. ■

Now we want to use the previous lemmas to find tight cycles of a given length. Let P be a tight path with start-type σ and end-type π. If $\pi = \sigma$, then there exists a tight cycle C containing P with $V(C) = V(P)$. Similarly if $\pi = \sigma\tau^{-r}$, then (by using r simple extensions) there exists a tight cycle C on $|V(P)| + r$ vertices containing P. In general, in order to extend P into a tight cycle we use Lemma 4.3.2 to first extend P to a path P' with end-type $\sigma\tau^{-r}$ for some suitable r, using the edges of K and a suitable G-gadget. The next lemma formalises this construction of the tight cycle C containing P and gives us precise bounds on the sizes of $V_i \cap (V(C) \setminus V(P))$ in the case where $\sigma = \pi$, which will be useful during

Section 5.3.

Lemma 4.3.3. *Let $k \geq 3$. Let $\sigma, \pi \in S_k$ and $0 \leq r < k$. Then there exists a 2-graph $G := G(\sigma, \pi, r)$ on $[k]$ consisting of a vertex-disjoint union of paths such that the following holds for all $s \geq k(2k-1)$ with $s \equiv r \bmod k$: let H be a k-graph containing a complete (k,k)-graph K with vertex partition $V_1, \dots, V_k$, and let P be a tight path with start-type σ and end-type π. Suppose W_G is a G-gadget for K avoiding $V(P)$ and $|V_i \setminus V(P)| \geq \lfloor s/k \rfloor + 1$. Then, there exists a tight cycle C on $|V(P)| + s$ vertices containing P, such that*

$$V(C) \setminus (V(P \cup K)) = \{w_{ij} : ij \in E(G)\}.$$

Moreover, if $\sigma = \pi$, then for all $1 \leq i, j \leq k$,

$$\big||V_i \cap (V(C) \setminus V(P))| - |V_j \cap (V(C) \setminus V(P))|\big| \leq 1.$$

Proof. Without loss of generality, we may assume that $\pi = \mathrm{id}$. Define $\sigma' = \sigma\tau^{-r} \in S_k$. Let $G := G_{\sigma'}$. Note that $|E(G)| \leq k-1$, $t(\sigma') \leq k/2$ and $2|E(G)| + t(\sigma') \leq 2k-1$ by (4.3.1). Let H, K, P be as defined in the lemma. By Lemma 4.3.2, there exists an extension P' of P with end-type $\sigma'\tau^{m(\sigma')-1}$ such that $|V(P')| = |V(P)| + 2k|E(G)| + m(\sigma') - 1$, for all $1 \leq i \leq k$,

$$|V_i \cap (V(P') \setminus V(P))| = 2|E(G)| - X_{i,\sigma'} + Y_{i,\sigma'}$$

and $V(P') \setminus (V(P \cup K)) = \{w_{ij} : ij \in E(G)\}$. We use $k - m(\sigma') + 1$ simple extensions to get an extension P'' of P' of order

$$|V(P'')| = |V(P')| + (k - m(\sigma') + 1) = |V(P)| + 2k|E(G)| + k.$$

Note that $V(P'') \setminus V(P')$ uses precisely one vertex in each of the clusters V_i for all $i \in \{\sigma'\tau^{m(\sigma')-1}(j) : 1 \leq j \leq k - m(\sigma') + 1\} = \{\sigma'(j) : m(\sigma') \leq j \leq k\} = \{j : Y_{j,\sigma'} = 0\}$.

It follows that for all $1 \le i \le k$,

$$|V_i \cap (V(P'') \setminus V(P))| = 2|E(G)| + 1 - X_{i,\sigma'}.$$

Note that P'' has end-type $\sigma'\tau^{m(\sigma')-1}\tau^{k-m(\sigma')+1} = \sigma' = \sigma\tau^{-r}$. For all $1 \le i \le k$ and $0 \le r < k$, set $Z_{i,\sigma,r} = 1$ if $i \in \{\sigma(j) : k-r+1 \le j \le k\}$, and set $Z_{i,\sigma,r} = 0$ otherwise. We use r more simple extensions to get an extension P''' of P with end-type $\sigma\tau^{-r}\tau^{r} = \sigma$ of order

$$|V(P''')| = |V(P'')| + r = |V(P)| + 2k|E(G)| + k + r,$$

such that for all $1 \le i \le k$,

$$|V_i \cap (V(P''') \setminus V(P))| = 2|E(G)| + 1 + Z_{i,\sigma,r} - X_{i,\sigma'}.$$

Since $|E(G)| \le k-1$ and $s \equiv r \bmod k$, it follows that $|V(P''')| \le |V(P)| + s$. Also, $|V(P''') \setminus V(P)| \equiv s \bmod k$. For all $1 \le i \le k$,

$$\begin{aligned}
|V_i \setminus V(P''')| &\ge |V_i \setminus V(P)| - 2|E(G)| - 1 + X_{i,\sigma'} - Z_{i,\sigma,r} \\
&\ge \lfloor s/k \rfloor - 2|E(G)| - 1 = \frac{1}{k}\left(k\lfloor s/k \rfloor - 2k|E(G)| - k\right) \\
&= \frac{1}{k}\left(s - r - 2k|E(G)| - k\right) = \frac{1}{k}\left(s - (|V(P''')| - |V(P)|)\right).
\end{aligned}$$

Since P''' has start-type σ and end-type σ, then we can easily extend P''' (using simple extensions) into a tight cycle C on $|V(P)| + s$ vertices. Note that $V(C) \setminus (V(P \cup K)) = \{w_{ij} : ij \in E(G)\}$, as desired.

Moreover, for all $1 \le i, j \le k$,

$$\begin{aligned}
&\big||V_i \cap (V(C) \setminus V(P))| - |V_j \cap (V(C) \setminus V(P))|\big| \\
&= \big||V_i \cap (V(P''') \setminus V(P))| - |V_j \cap (V(P''') \setminus V(P))|\big| \\
&= |(Z_{i,\sigma,r} - X_{i,\sigma'}) - (Z_{j,\sigma,r} - X_{j,\sigma'})|.
\end{aligned}$$

Suppose now that $\sigma = \pi = \mathrm{id}$. We will show that $-1 \le Z_{i,\sigma,r} - X_{i,\sigma'} \le 0$ for all $1 \le i \le k$, implying that, for all $1 \le i, j \le k$, $\big||V_i \cap (V(C) \setminus V(P))| - |V_j \cap (V(C) \setminus V(P))|\big| \le 1$. It suffices to show that if $Z_{i,\sigma,r} = 1$, then $X_{i,\sigma'} = 1$. If $r = 0$ then it is obvious, so suppose that $1 \le r < k$. Let $1 \le i \le k$ such that $Z_{i,\sigma,r} = 1$. Since $\sigma = \pi = \mathrm{id}$, then $\sigma' = \tau^{-r}$. So if $Z_{i,\sigma,r} = 1$, then $k - r + 1 \le i \le k$. To show that $X_{i,\tau^{-r}} = 1$, we need to show that i is not the minimal element in the cycle that it belongs in the cyclic decomposition of τ^{-r}, that is, there exists $m < i$ such that i is in the orbit of m under τ^{-r}. Let $d = \gcd(r,k)$. Choose $1 \le m \le d$ such that $m \equiv i \bmod d$. The order of τ^{-r} is exactly k/d and the orbit of m has exactly k/d elements. There are exactly k/d elements i' satisfying $1 \le i' \le k$ and $i' \equiv m \bmod d$, and all elements i' in the orbit of m also satisfy $i' \equiv m \bmod d$, so it follows that i is in the orbit of m under τ^{-r}. Finally, $m \le d \le k - r < i$. This proves that $X_{i,\tau^{-r}} = 1$, as desired. ∎

4.3.1 Finding G-gadgets in k-graphs with large codegree

We now turn our attention to the existence of G-gadgets. We prove that all large complete (k,k)-graphs contained in a k-graph H with $\delta_{k-1}(H)$ large have a G-gadget, for an arbitrary 2-graph G on $[k]$.

Lemma 4.3.4. *Let $0 < 1/n, 1/t_0 \ll \gamma, 1/k$. Let H be a k-graph on n vertices with $\delta_{k-1}(H) \ge (1/2 + \gamma)n$ containing a complete (k,k)-graph K with vertex partition $V_1, \dots, V_k$. Let $S \subseteq V(H)$ be a set of vertices such that $|V(K) \cup S| \le \gamma n/2$ and $|V_i \setminus S| \ge t_0$ for all $1 \le i \le k$. Let G be a 2-graph on $[k]$. Then there exists a G-gadget for K avoiding S.*

Proof. Choose $0 < 1/t \ll \gamma, 1/k$ and let $t_0 := t + k^2$. Suppose that $ij \in E(G)$ and $|V_\ell \setminus S| \ge t + 2|E(G)|$ for all $1 \le \ell \le k$. Let $U_\ell \subseteq V_\ell \setminus S$ with $|U_\ell| = t$ for all $1 \le \ell \le k$ and let $R = [k] \setminus \{i,j\}$. Let $U := \bigcup_{1 \le \ell \le k} U_\ell$ and

$$T := \left\{ A \in \binom{U}{k-1} : |A \cap U_r| = 1 \text{ for all } r \in R \text{ and } |A \cap (U_i \cup U_j)| = 1 \right\}.$$

Then T has size $2t^{k-1}$. By the codegree condition, all members in T have $(1/2 +$

$\gamma)n - |V(K) \cup S| \geq (1/2 + \gamma/2)n$ neighbours outside of $V(K) \cup S$ and by an averaging argument, there exists a vertex $w \notin V(K) \cup S$ such that $H(w)$ satisfies $|H(w) \cap T| \geq (1+\gamma)t^{k-1}$. For all $u \in U_i \cup U_j$, $N_{H(w)\cap T}(u)$ is a family of $(k-2)$-sets of $\bigcup_{r\in R} U_r$. We have that

$$\begin{aligned}
\sum_{(u_i,u_j)\in U_i\times U_j} &|N_{H(w)\cap T}(u_i) \cap N_{H(w)\cap T}(u_j)| \\
&\geq \sum_{(u_i,u_j)\in U_i\times U_j} \left(d_{H(w)\cap T}(u_i) + d_{H(w)\cap T}(u_j) - t^{k-2}\right) \\
&= t|H(w) \cap T| - t^k \geq t^k(1+\gamma) - t^k = \gamma t^k,
\end{aligned}$$

and by an averaging argument, there exists a pair $(x_i^*, x_j^*) \in U_i \times U_j$ such that $|N_{H(w)\cap T}(x_i^*) \cap N_{H(w)\cap T}(x_j^*)| \geq \gamma t^{k-2}$.

By the choice of t and by Theorem 1.3.2, we have that $N_{H(w)\cap T}(x_i^*) \cap N_{H(w)\cap T}(x_j^*)$ contains a copy K' of $K_{k-2}^{k-2}(2)$. Define $W_{ij} = V(K') \cup \{w, x_i^*, x_j^*\}$ and note that $|W_{ij}| = 2(k-2) + 3 = 2k - 1$.

We now check that (W3) holds for W_{ij}. Recall that, informally, this means that given any $k-1$ ordered clusters that miss V_i, there exists a tight path with vertex set W_{ij}, which starts with the same ordered $k-1$ clusters and ends with the same ordered $k-1$ clusters but with V_j replaced by V_i. Now we formalise this. For all $r \in R$, let $U_r \cap V(K') = \{x_r, x_r'\}$. Consider an arbitrary $\sigma \in S_k$ with $\sigma(1) = i$ and $\sigma(j') = j$. By construction, we have that

$$x_{\sigma(2)}x_{\sigma(3)}\cdots x_{\sigma(j'-1)}x_j^*x_{\sigma(j'+1)}x_{\sigma(j'+2)}\cdots x_{\sigma(k)}wx'_{\sigma(2)}x'_{\sigma(3)}\cdots x'_{\sigma(j'-1)}x_i^*x'_{\sigma(j'+1)}x'_{\sigma(j'+2)}\cdots x'_{\sigma(k)}$$

is a spanning tight path in $H[W_{ij}]$, of start-type $\sigma\tau$ and end-type $(ij)\sigma$. Clearly W_{ij} is an ij-gadget avoiding S.

Set $S' := S \cup W_{ij}$ and $G' := G - ij$. Repeating this construction for all edges in $E(G - ij)$ and using that $t_0 = t + k^2$, it is possible to conclude that K has a G-gadget avoiding S. ∎

4.3.2 Auxiliary k-graphs F_s

Given a tight cycle C_s^k, we would like to find a k-graph F_s such that $C_s^k \subseteq F_s$ and F_s is obtained from a complete (k,k)-graph by adding "few" extra vertices. This will be useful in Section 5.3.

Let K be a (k,k)-graph with vertex partition $V_1,\dots,V_k$. Consider a 2-graph G on $[k]$ with $E(G) = \{j_i j_i' : 1 \le i \le \ell\}$ and let $y_1,\dots,y_\ell$ be a set of ℓ vertices disjoint from $V(K)$. Let $W_G := \{y_1,\dots,y_\ell\}$. We define the *$G$-augmentation of K* to be the k-graph $F = F(K,G)$ such that

$$\begin{aligned} V(F) &= V(K) \cup W_G \text{ and} \\ E(F) &= E(K) \cup \bigcup_{1\le i\le \ell} (E(H(y_i,j_i)) \cup E(H(y_i,j_i'))), \end{aligned}$$

where $H(v,j)$ is a complete (k,k)-graph with partition

$$\{v\}, V_1, V_2, \dots, V_{j-1}, V_{j+1}, \dots, V_k.$$

The easy (but crucial) observation is that if $|V_i| \ge 2\ell$ for all $1 \le i \le k$, then the G-augmentation of K contains a G-gadget for K avoiding $\varnothing$. Using that, we can prove the following.

Proposition 4.3.5. *Let $k \ge 3$, $s \ge 2k^2$ and $s \not\equiv 0 \bmod k$. Then there exists a 2-graph G_s on $[k]$ that is a disjoint union of paths, and $a_{s,1},\dots,a_{s,k},\ell \in \mathbb{N}$ such that $|a_{s,i} - a_{s,j}| \le 1$ for all $i,j \in [k]$, $\ell = |E(G_s)| \le k-1$, and if $K = K^k(a_{s,1},\dots,a_{s,k})$, then F_s, the G_s-augmentation of K, contains a spanning copy of C_s^k and $|V(F_s) \setminus V(K)| = \ell$.*

Proof. Let $r \in \{1,\dots,k-1\}$ be such that $s \equiv r \bmod k$. Let G_s be the 2-graph obtained from Lemma 4.3.3 (with parameters $\sigma = \pi = \mathrm{id}$ and r). Note G_s is a disjoint union of paths and thus $\ell = E(G_s) \le k-1$.

Suppose that $V_1,\dots,V_k$ are disjoint sets of size $\lfloor s/k \rfloor + 1$ and let K' be the complete (k,k)-graph with partition $\{V_1,\dots,V_k\}$. For each $i \in [k]$, let $v_i \in V_i$ and consider the tight path $P = v_1 \cdots v_k$. Note that P has both start-type and end-type id. Let F' be the G_s-augmentation of K'. It is easily checked that $|V_i \setminus V(P)| \geq 2(k-1) \geq 2\ell$ and therefore there is a G_s-gadget for K' in F' avoiding $V(P)$. By the choice of G_s, F' contains a tight cycle C on s vertices containing P such that $V(C) \setminus V(K) = V(F') \setminus V(K') = W_{G_s}$ and, over the range $i \in [k]$, the values $|V(C) \cap V_i|$ differ at most by 1. It is easily checked that letting $a_{s,i} := |V(C) \cap V_i|$ we obtain the desired properties. ∎

4.4 Covering thresholds for tight cycles

In this section, we prove the upper bounds for the covering codegree threshold for tight cycles, proving Proposition 1.3.3 and Theorem 1.3.4. We first prove Proposition 4.4.2, which immediately implies Proposition 1.3.3 since $K^k(s)$ contains a $C^k_{s'}$-covering for all $s' \equiv 0 \bmod k$ with $s' \leq sk$. We will use the following classic result of Kővári, Sós and Turán [46].

Theorem 4.4.1 (Kővári, Sós and Turán [46])**.** *Let $z(m,n;s,t)$ denote the maximum possible number of edges in a bipartite 2-graph G with parts U and V for which $|U| = m$ and $|V| = n$, which does not contain a $K_{s,t}$ subgraph with s vertices in U and t vertices in V. Then*

$$z(m,n;s,t) < (s-1)^{1/t}(n-t+1)m^{1-1/t} + (t-1)m.$$

Proposition 4.4.2. *For all $k \geq 3$ and $s \geq 1$, let $n, c \geq 2$ such that $1/n, 1/c \ll 1/k, 1/s$. Then $c(n, K^k(s)) \leq cn^{1-1/s^{k-1}}$.*

Proof. Let H be a k-graph on n vertices with $\delta_{k-1}(H) \geq cn^{1-1/s^{k-1}}$. Fix a vertex $x \in V(H)$ and consider the link $(k-1)$-graph $H(x)$ of x. Let $U_1 := E(H(x))$. Note that

$$|U_1| \geq \frac{\binom{n-1}{k-2}\delta_{k-1}(H)}{k-1} \geq c^{1/2} n^{k-1-1/s^{k-1}}. \tag{4.4.1}$$

Let $U_2 := V(H) \setminus \{x\}$. Consider the bipartite 2-graph B with parts U_1 and U_2, where $e \in U_1$ is joined to $u \in U_2$ if and only if $e \cup \{u\} \in E(H)$. By the codegree condition of H, all $(k-1)$-sets $e \in U_1$ have degree at least $\delta_{k-1}(H) - 1$ in B. Hence

$$|E(B)| \geq |U_1|(\delta_{k-1}(H) - 1) \geq |U_1|(cn^{1-1/s^{k-1}} - 1). \tag{4.4.2}$$

We claim there is a $K_{n^{k-1-1/s^{k-2}},s-1}$ as a subgraph in B, with $n^{k-1-1/s^{k-2}}$ vertices in U_1 and $s-1$ vertices in U_2. Suppose not. Then, by Theorem 4.4.1,

$$\begin{aligned} |E(B)| &\leq z(|U_1|, n-1; n^{k-1-1/s^{k-2}}, s-1) \\ &< \left(n^{k-1-1/s^{k-2}}\right)^{\frac{1}{s-1}} n|U_1|^{1-\frac{1}{s-1}} + (s-1)|U_1| \\ &= |U_1|\left(n\left(\frac{n^{k-1-1/s^{k-2}}}{|U_1|}\right)^{\frac{1}{s-1}} + s - 1\right) \\ &\overset{(4.4.1)}{\leq} |U_1|\left(c^{-\frac{1}{2(s-1)}} n^{1-\frac{1}{s^{k-1}}} + s - 1\right) < |U_1| n^{1-\frac{1}{s^{k-1}}}. \end{aligned}$$

This contradicts (4.4.2).

Let K be a copy of $K_{n^{k-1-1/s^{k-2}},s-1}$ in B. Let $W := V(K) \cap U_1$ and $X := \{x_1, \ldots, x_{s-1}\} = V(K) \cap U_2$. Since $|W| = n^{k-1-1/s^{k-2}}$ and $1/n \ll 1/k, 1/s$, by Theorem 1.3.2, W contains a copy K' of $K^{k-1}(s)$. By construction, for all $y \in \{x\} \cup X$ and all $e \in E(K')$, $\{y\} \cup e \in E(H)$. Hence, $H[\{x\} \cup X \cup V(K')]$ contains a $K^k(s)$ covering x, as desired. ∎

We are ready to prove Theorem 1.3.4.

Proof of Theorem 1.3.4. Let $t \in \mathbb{N}$ be such that $1/n_0 \ll 1/t \ll \gamma, 1/s$. Let H be a k-graph on $n \geq n_0$ vertices with $\delta_{k-1}(H) \geq (1/2 + \gamma)n$. Fix a vertex x and a copy K of $K_k^k(t)$ containing x, which exists by Proposition 4.4.2. Let $V_1, \ldots, V_k$ be the vertex partition of K with $x \in V_1$. By the choice of t, $|V_i| \geq \max\{2k^2 + 2, \lfloor s/k \rfloor + 2\}$ for all $1 \leq i \leq k$.

Let $x_1 = x$ and select arbitrarily vertices $x_i \in V_i$ for $2 \leq i \leq k$. Now $P = x_1 \cdots x_k$ is a tight path on k vertices with both start-type and end-type id. Let G be a complete 2-graph on $[k]$. By Lemma 4.3.4, there exists a G-gadget for K avoiding $V(P)$. Thus, by Lemma 4.3.3, there exists a tight cycle C in $V(H)$ on s vertices containing P, and in turn, x. ■

— 5 —

Tiling hypergraphs with tight cycles

This chapter is dedicated to the investigation of tiling thresholds in hypergraphs, using k-uniform tight cycles on s vertices, which we denote by C_s^k. The main objective is the proof of Theorem 1.3.9, which gives an asymptotic upper bound for the tiling threshold of C_s^k.

This chapter is organised as follows. In Section 5.2 we prove the "absorbing lemma for C_s^k-tilings" (Lemma 5.2.4). Next, we ensure the existence of "almost perfect C_s^k-tilings". This is the content of Lemma 5.3.1, whose proof is done in several steps. First, in Section 5.3 we use the tools of hypergraph regularity and various reductions, to reduce the proof of Lemma 5.3.1 to a statement on weighted fractional matchings (Lemma 5.3.10). Then Lemma 5.3.10 is proven in Section 5.4.

5.1 Sketch of the proof

The proof of Theorem 1.3.9 uses the absorbing method. We first find a small vertex set $U \subseteq V(H)$ such that $H[U \cup W]$ has a perfect C_s^k-tiling for all small sets W with $|U| + |W| \equiv 0 \bmod s$. Thus the problem of finding a perfect C_s^k-tiling is reduced to finding a C_s^k-tiling in $H \setminus U$ covering almost all of the remaining vertices.

However, we do not find such a C_s^k-tiling directly. First, we use the results of Section 4.3.2 to find a k-graph F_s on s vertices which contains a C_s^k, and furthermore has a particularly useful structure: it is obtained from a complete (k,k)-graph by adding a few extra vertices.

Since F_s contains a spanning C_s^k, to find an almost perfect C_s^k-tiling it is enough to find an almost perfect F_s-tiling. Instead, we show that there exists an $\{F_s, E_s\}$-tiling $\mathcal{T}$ for some suitable k-graph E_s, subject to the minimisation of some objective function $\phi(\mathcal{T})$. To study this new problem, we consider the fractional relaxation and we call the objects under study "weighted fractional tilings" (see Section 5.3.3).

Our approach to link the "fractional" and "integral" tilings follows, at least in spirit, the work on Komlós on graph tilings [47], where an iterated regularity approach is used: "integral tilings" are used to find "good fractional tilings" and viceversa. In our particular setting, to argue about the existence of fractional tilings with good properties, we will use the existence of "integral" tilings, which we will transform to an improved "fractional tiling" by making a series of local modifications. Similarly, to translate the results about fractional tilings to integral tilings in graphs, we use the hypergraph regularity lemma in the form of the 'regular slice lemma' of Allen, Böttcher, Cooley and Mycroft [3].

5.2 Absorption for C_s^k-tilings

Now we begin our investigation of the tiling thresholds for tight cycles. We will proceed by an application of the absorption technique. The first step is to prove an absorbing lemma for tight cycle tilings, which in this case will correspond to Lemma 5.2.4. Proving this lemma is the goal of this section.

The results in this section are analogous to the lemmas used in Section 3.1 of Chapter 3, used to prove Lemma 3.1.5.

As a preliminary, we need the following "absorbing lemma", which is a special case of a lemma of Lo and Markström [53, Lemma 1.1].

Lemma 5.2.1 ([53, Lemma 1.1]). *Let $s \geq k \geq 3$ and $0 < 1/n \ll \eta, 1/s$ and $0 < \alpha \ll \mu \ll \eta, 1/s$. Suppose that H is a k-graph on n vertices and for all distinct vertices $x, y \in V(H)$ there exist ηn^{s-1} sets S of size $s-1$ such that both $H[S \cup \{x\}]$ and $H[S \cup \{y\}]$ contain a spanning C_s^k. Then there exists $U \subseteq V(H)$ of size $|U| \leq \mu n$ with $|U| \equiv 0 \bmod s$ such that there exists a perfect C_s^k-tiling in $H[U \cup W]$ for all $W \subseteq V(H) \setminus U$ of size $|W| \leq \alpha n$ with $|W| \equiv 0 \bmod s$.*

Thus, to find an absorbing set U, it is enough to find many $(s-1)$-sets S as above for each pair $x, y \in V(H)$. First we show that we can find one such S.

Lemma 5.2.2. *Let $s \geq 5k^2$ with $s \not\equiv 0 \bmod k$. Let $1/n \ll \gamma, 1/s$. Let H be a k-graph on n vertices with $\delta_{k-1}(H) \geq (1/2 + \gamma)n$. Then for all pair of distinct vertices $x, y \in V(H)$, there exists $S \subseteq V(H) \setminus \{x, y\}$ such that $|S| = s-1$ and both $H[S \cup \{x\}]$ and $H[S \cup \{y\}]$ contain a spanning C_s^k.*

Proof. Let $1/n \ll 1/t \ll \gamma, 1/s$. Consider the k-graph H_{xy} with vertex set $V(H_{xy}) = (V(H) \setminus \{x, y\}) \cup \{z\}$ (for some $z \notin V(H)$) and edge set

$$E(H_{xy}) = E(H \setminus \{x, y\}) \cup \{\{z\} \cup S : S \in N_H(x) \cap N_H(y)\}.$$

Note that $|V(H_{xy})| = n-1$ and $\delta_{k-1}(H_{xy}) \geq \gamma |V(H_{xy})|$. By Proposition 4.4.2, H_{xy} contains a copy K of $K_k^k(t)$ containing z. Let $V_1, \ldots, V_k$ be the vertex partition of K with $z \in V_1$.

Select arbitrarily vertices $v_i \in V_i$ for $2 \leq i \leq k$. Let $H' = H_{xy} \setminus \{z, v_2, \ldots, v_k\}$ and $K' = K \setminus \{z, v_2, \ldots, v_k\}$. Note that $\delta_{k-1}(H') \geq (1/2 + \gamma/2)|V(H')|$ and $K' \subseteq H'$. By Lemma 4.3.4 with H' and K' playing the roles of H and K respectively, there exists a K_k-gadget for K' in H'. Hence, there exists a K_k-gadget for K in H_{xy} avoiding $\{z, v_2, \ldots, v_k\}$.

Now we construct a copy of C_s^k in H_{xy} containing z. Note that $P = zv_2 \cdots v_k$ is a tight path on k vertices with start-type and end-type id. Since there exists a

K_k-gadget for K avoiding $V(P)$, by Lemma 4.3.3 H_{xy} contains a copy C of C_s^k containing z.

Finally, let $S = V(C) \setminus \{z\} \subseteq V(H)$. By construction, $|S| = s - 1$ and both $H[S \cup \{x\}]$ and $H[S \cup \{y\}]$ contain a spanning C_s^k in H, as desired. ∎

For the next lemma, we show that a large k-graph with large codegree has a lot of induced subgraphs which satisfy the assumptions of Lemma 5.2.2. To show this, we will use the Chernoff inequalities in the form of Lemma 2.2.4.

Lemma 5.2.3. *Let $k \geq 3$ and $0 < 1/m \ll \gamma, 1/k$. Let H be a k-graph on $n \geq m$ vertices with $\delta_{k-1}(H) \geq (1/2 + \gamma)n$. Let $x, y \in V(H)$ be distinct. Then the number of m-sets $R \subseteq V(H) \setminus \{x, y\}$ such that $\delta_{k-1}(H[R \cup \{x, y\}]) \geq (1/2 + \gamma/2)(m + 2)$ is at least $\binom{n-2}{m}/2$.*

Proof. Let T be a $(k-1)$-set in $V(H)$. Note that, since $1/n \leq 1/m \ll \gamma$,

$$|N_H(T) \setminus \{x, y\}| \geq \left(\frac{1}{2} + \gamma\right) n - 2 \geq \left(\frac{1}{2} + \frac{2}{3}\gamma\right)(n - 2).$$

We call an m-set $R \subseteq V(H) \setminus \{x, y\}$ *bad for T* if $|N_H(T) \cap R| \leq (1/2 + 3\gamma/5)m$. An application of Lemma 2.2.4 (with $1/2 + 3\gamma/5$, $\gamma/15$, $n - 2$, $N_H(T) \setminus \{x, y\}$ playing the roles of μ, γ, n and S, respectively) implies that the number of m-sets which are bad for T is at most

$$\left|\left\{R \in \binom{V(H) \setminus \{x, y\}}{m} : |N_H(T) \cap R| \leq (1/2 + 3\gamma/5)m\right\}\right| \leq \binom{n-2}{m} e^{-\gamma^2 m/675}.$$

Say an m-set $R \subseteq V(H) \setminus \{x, y\}$ is *good* if $\delta_{k-1}(R \cup \{x, y\}) > (1/2 + 3\gamma/5)m$ (and *bad*, otherwise). Note that for any good m-set R,

$$\delta_{k-1}(H[R \cup \{x, y\}]) > (1/2 + 3\gamma/5)m \geq (1/2 + \gamma/2)(m + 2),$$

thus it is enough to prove that there are at most $\binom{n-2}{m}/2$ bad m-sets. Note that R is bad if and only if there exists a $(k-1)$-set $T \subseteq R \cup \{x, y\}$ such that R is bad for T. Therefore, the number of bad sets is at most

$$\binom{m+2}{k-1}\binom{n-2}{m}e^{-\gamma^2 m/675} \le \frac{1}{2}\binom{n-2}{m},$$

where the inequality follows from the choice of m. ∎

With all of the tools at hand, we can now prove our "absorbing lemma", the goal of this section. We will apply the standard supersaturation trick to find many sets S as in the statement of Lemma 5.2.2. Then we can finish with Lemma 5.2.1.

Lemma 5.2.4. *Let $k \ge 3$ and $s \ge 5k^2$. Let $1/n \ll \alpha \ll \mu \ll \gamma, 1/s$. Let H be a k-graph on n vertices with $\delta_{k-1}(H) \ge (1/2+\gamma)n$. Then, there exists $U \subseteq V(H)$ of size $|U| \le \mu n$ with $|U| \equiv 0 \bmod s$ such that there exists a perfect C_s^k-tiling in $H[U \cup W]$ for all $W \subseteq V(H) \setminus U$ of size $|W| \le \alpha n$ with $|W| \equiv 0 \bmod s$.*

Proof. Let $\mu \ll \eta \ll 1/m \ll \gamma, 1/s$. Let x, y be distinct vertices in $V(H)$. By Lemma 5.2.3, at least $\binom{n-2}{m}/2$ of the m-sets $R \subseteq V(H) \setminus \{x,y\}$ are such that $\delta_{k-1}(H[R \cup \{x,y\}]) \ge (1/2+\gamma/2)(m+2)$. By Lemma 5.2.2, each one of these subgraphs contains a set $S \subseteq R$ of size $s-1$ such that $H[S \cup \{x\}]$ and $H[S \cup \{y\}]$ have spanning copies of C_s^k. Then the number of these sets S in H is at least

$$\frac{\frac{1}{2}\binom{n-2}{m}}{\binom{n-2-(s-1)}{m-(s-1)}} = \frac{\binom{n-2}{s-1}}{2\binom{m}{s-1}} \ge \eta n^{s-1}.$$

Then the result follows from Lemma 5.2.1. ∎

5.3 Almost perfect C_s^k-tilings

We continue with our investigation of C_s^k-tilings. Having already proved an absorbing lemma for tilings, the second step corresponds to a lemma which ensures the existence of an almost perfect C_s^k-tiling. The following lemma will play that role in the context of our application of the absorption technique.

Lemma 5.3.1. *Let $1/n \ll \alpha, \gamma, 1/s$, $k \geq 3$ and $s \geq 5k^2$ such that $s \not\equiv 0 \bmod k$. Let H be a k-graph on n vertices with $\delta_{k-1}(H) \geq (1/2 + 1/(2s) + \gamma)n$. Then H has a C_s^k-tiling covering at least $(1-\alpha)n$ vertices.*

The aim of this section is to prove Lemma 5.3.1, that is, finding an almost perfect C_s^k-tiling. Throughout this section, we fix $k \geq 3$ and $s \geq 5k^2$ with $s \not\equiv 0 \bmod k$. Let $G_s, W_{G_s}, a_{s,1}, \dots, a_{s,k}, \ell, F_s$ be given by Proposition 4.3.5.

Here we summarise some useful inequalities that will be used throughout the chapter. Let $M_s = \max_i a_{s,i}$ and $m_s = \min_i a_{s,i}$. We have

$$\ell + \sum_{i=1}^{k} a_{s,i} = s, \quad M_s \leq m_s + 1, \quad \text{and} \quad 1 \leq \ell \leq k-1. \tag{5.3.1}$$

From this, we can easily deduce

$$m_s + 1 \geq M_s \geq \frac{s-\ell}{k} \geq \frac{s-k+1}{k}. \tag{5.3.2}$$

5.3.1 Almost perfect $\{F_s, E_s\}$-tilings

As a first step, we introduce a family of auxiliary tilings. Recall that the k-graph in F_s contains a spanning C_s^k. Therefore, an F_s-tiling in H implies the existence of a C_s^k-tiling in H of the same size. We prove now that we can reduce the existence of almost perfect C_s^k-tilings (Lemma 5.3.1) to the proof of almost perfect tilings with some auxiliary graphs, including F_s (Lemma 5.3.2).

Define $E_s = K^k(M_s)$, the complete (k,k)-graph with each part of size M_s. Given an $\{F_s, E_s\}$-tiling $\mathcal{T}$ in H, let $\mathcal{F}_{\mathcal{T}}$ and $\mathcal{E}_{\mathcal{T}}$ be the set of copies of F_s and E_s in $\mathcal{T}$, respectively. Define

$$\phi(\mathcal{T}) = \frac{1}{n}\left(n - s\left(|\mathcal{F}_{\mathcal{T}}| + \frac{3}{5}|\mathcal{E}_{\mathcal{T}}|\right)\right).$$

The function was designed so that, over all $\{F_s, E_s\}$-tilings, it attains its lowest possible values whenever most of the edges are covered by copies of F_s. For instance, note that if $\mathcal{E}_{\mathcal{T}} = \varnothing$, then $\mathcal{T}$ is an F_s-tiling covering all but $\phi(\mathcal{T})n$ vertices. The value 3/5 was chosen with two criteria in mind: first, it should be less than one, so that if $\mathcal{T}'$ is obtained from $\mathcal{T}$ by removing one copy of E_s and replacing it with one copy of F_s, then $\phi(\mathcal{T}') < \phi(\mathcal{T})$. Secondly, it should be more than 1/2, so that if $\mathcal{T}'$ is obtained from $\mathcal{T}$ by removing one copy of F_s and replacing it with two copies of E_s, then $\phi(\mathcal{T}') < \phi(\mathcal{T})$. Roughly speaking, these two properties will allow us to gradually "improve" on a $\{F_s, E_s\}$-tiling (i.e., to decrease the value of ϕ) by doing local changes. This is an idealisation only: in the actual proof, this will be done in a fractional setting which will be described with detail afterwards.

Recall that (as defined in Section 2.3.3), for $0 \leq \mu, \theta \leq 1$, we say that a k-graph H on n vertices is (μ, θ)-dense if there exists $\mathcal{S} \subseteq \binom{V(H)}{k-1}$ of size at most $\theta\binom{n}{k-1}$ such that, for all $S \in \binom{V(H)}{k-1} \smallsetminus \mathcal{S}$, we have $\deg_H(S) \geq \mu(n-k+1)$.

Let $\phi(H)$ be the minimum of $\phi(\mathcal{T})$ over all $\{F_s, E_s\}$-tilings $\mathcal{T}$ in H. Given $n \geq k$ and $0 \leq \mu, \theta < 1$, let $\Phi(n, \mu, \theta)$ be the maximum of $\phi(H)$ over all (μ, θ)-dense k-graphs H on n vertices. Note that $\phi(H)$ and $\Phi(n, \mu, \theta)$ depend on k and s but they will be clear from the context.

The next lemma says that $\Phi(n, 1/2 + 1/(2s) + \gamma, \theta)$ can be made arbitrarily small by choosing θ small enough and n large. In other words, for this range of parameters, the k-graphs which are $(1/2 + 1/(2s) + \gamma, \theta)$-dense admit $\{F_s, E_s\}$-tilings $\mathcal{T}$ with $\phi(\mathcal{T})$ very close to zero.

Lemma 5.3.2. *Let $k \geq 3$ and $s \geq 5k^2$ with $s \not\equiv 0 \bmod k$. Let $1/n, \theta \ll \alpha, \gamma, 1/k, 1/s$. Then $\Phi(n, 1/2 + 1/(2s) + \gamma, \theta) \leq \alpha$.*

We now show that Lemma 5.3.2 implies Lemma 5.3.1. The idea behind is that if a k-graph H is such that $\phi(H)$ is very small, then (because of the definition of ϕ) this must because it admits a $\{F_s, E_s\}$-tiling where most of the vertices are covered by copies of F_s.

Proof of Lemma 5.3.1. Fix $\alpha, \gamma > 0$. Note that $|V(F_s)| = s$ and $|V(E_s)| = kM_s$. Using $s \geq 5k^2$ and (5.3.2), we deduce $kM_s/s \geq 1 - (k-1)/(5k^2)$. Since $k \geq 3$ by assumption, the right-hand size of the inequality is minimised over the possible choices of integer k precisely when $k = 3$, where it attains the value 43/45. Thus we deduce that $kM_s/s \geq 43/45$ holds. Let $\delta = (3s/5)/(kM_s)$. The previous calculations imply that $\delta \leq (3/5)/(43/45) < 1$.

Define $\alpha_1 = \alpha(1-\delta)$ and choose some $\theta \ll \alpha, \gamma, 1/k, 1/s$. Since $1/n \ll \alpha, \gamma, 1/k, 1/s$ as well, Lemma 5.3.2 (with α_1 in place of α) implies that $\Phi(n, 1/2 + 1/(2s) + \gamma, \theta) \leq \alpha_1$.

Let H be a k-graph on n vertices with $\delta_{k-1}(H) \geq (1/2 + 1/(2s) + \gamma)n$. Then $\phi(H) \leq \Phi(n, 1/2 + 1/(2s) + \gamma, 0) \leq \Phi(n, 1/2 + 1/(2s) + \gamma, \theta) \leq \alpha_1$. Let $\mathcal{T}$ be an $\{F_s, E_s\}$-tiling in H with $\phi(\mathcal{T}) \leq \alpha_1$. Hence,

$$1 - \alpha_1 \leq 1 - \phi(\mathcal{T}) \leq \frac{s}{n}\left(|\mathcal{F}_{\mathcal{T}}| + \frac{3}{5}|\mathcal{E}_{\mathcal{T}}|\right) = \frac{1}{n}\left(s|\mathcal{F}_{\mathcal{T}}| + \delta k M_s |\mathcal{E}_{\mathcal{T}}|\right),$$

where the last equality follows from the definition of δ. As $\mathcal{T}$ is a tiling, we have that $s|\mathcal{F}_{\mathcal{T}}| + kM_s|\mathcal{E}_{\mathcal{T}}| \leq n$. Hence, $1 - \alpha_1 \leq (1-\delta)s|\mathcal{F}_{\mathcal{T}}|/n + \delta$, and so

$$s|\mathcal{F}_{\mathcal{T}}| \geq \left(1 - \frac{\alpha_1}{1-\delta}\right)n = (1-\alpha)n.$$

Therefore H contains an $\mathcal{F}_s$-tiling $\mathcal{F}_{\mathcal{T}}$ covering all but at most αn vertices, implying the existence of a C_s^k-tiling of the same size. ■

5.3.2 Strongly dense k-graphs

In the previous subsection, the problem of finding an almost perfect tiling with tight cycles was reduced to that of proving Lemma 5.3.2. We will proceed by using regularity methods, and in this subsection we will prove a useful structural lemma to be used in the reduced graph which is obtained by the use of the regularity lemma.

Recall that, for $0 \leq \mu, \theta \leq 1$, we say that a k-graph H on n vertices is (μ, θ)-dense if there exists $\mathcal{S} \subseteq \binom{V(H)}{k-1}$ of size at most $\theta\binom{n}{k-1}$ such that, for all $S \in \binom{V(H)}{k-1} \setminus \mathcal{S}$, we have $\deg_H(S) \geq \mu(n-k+1)$.

We now strengthen this definition. For $0 \leq \mu, \theta \leq 1$, a k-graph H on n vertices is *strongly* (μ, θ)*-dense* if it is (μ, θ)-dense and, for all edges $e \in E(H)$ and all $(k-1)$-sets $X \subseteq e$, $\deg_H(X) \geq \mu(n-k+1)$. We prove that all (μ, θ)-dense k-graphs contain a strongly (μ', θ')-dense subgraph, for some degraded constants μ' and θ'. As mentioned before, this will be applied to the reduced graphs obtained from the regularity method later in the proof.

Lemma 5.3.3. *Let $n \geq 2k$ and $0 < \mu, \theta < 1$. Suppose that H is a k-graph on n vertices that is (μ, θ)-dense. Then there exists a sub-k-graph H' on $V(H)$ that is strongly $(\mu - 2^k\theta^{1/(2k-2)}, \theta + \theta^{1/(2k-2)})$-dense.*

Proof. Let $\mathcal{S}_1$ be the set of all $S \in \binom{V(H)}{k-1}$ such that $\deg_H(S) < \mu(n-k+1)$. Thus, $|\mathcal{S}_1| \leq \theta\binom{n}{k-1}$. Let $\beta = \theta^{1/(k-1)}$. Now, for all $j \in \{k-1, k-2, \dots, 1\}$ in turn we construct $\mathcal{A}_j \subseteq \binom{V(H)}{j}$ in the following way. Initially, let $\mathcal{A}_{k-1} = \mathcal{S}_1$. Given $j > 1$ and $\mathcal{A}_j$, we define $\mathcal{A}_{j-1} \subseteq \binom{V(H)}{j-1}$ to be the set of all $X \in \binom{V(H)}{j-1}$ such that there exist at least $\beta(n-j+1)$ vertices $w \in V(H)$ with $X \cup \{w\} \in \mathcal{A}_j$.

Claim 5.3.4. *For all $1 \leq j \leq k-1$, $|\mathcal{A}_j| \leq \beta^j\binom{n}{j}$.*

Proof of the claim. We prove it by induction on $k-j$. When $j = k-1$ it is immediate. Now suppose $2 \leq j \leq k-1$ and that $|\mathcal{A}_j| \leq \beta^j\binom{n}{j}$. By double counting the number of tuples (X, w) where X is a $(j-1)$-set in $\mathcal{A}_{j-1}$ and $X \cup \{w\} \in \mathcal{A}_j$ we have $|\mathcal{A}_{j-1}|\beta(n-j+1) \leq j|\mathcal{A}_j|$. By the induction hypothesis it follows that

$$|\mathcal{A}_{j-1}| \leq \frac{j}{\beta(n-j+1)}|\mathcal{A}_j| \leq \beta^{j-1}\binom{n}{j-1}. \qquad \square$$

For all $1 \leq j \leq k-1$, let F_j be the set of edges $e \in E(H)$ such that there exists $S \in \mathcal{A}_j$ with $S \subseteq e$, and let $F = \bigcup_{j=1}^{k-1} F_j$. Define $H' = H - F$. We will show that it satisfies the desired properties.

For each j-set, there are at most $\binom{n-j}{k-j}$ k-edges containing it. Thus, for all $1 \leq j \leq k-1$, the claim above implies that

$$|F_j| \leq |\mathcal{A}_j|\binom{n-j}{k-j} \leq \beta^j\binom{n}{j}\binom{n-j}{k-j} = \beta^j\binom{k}{j}\binom{n}{k}.$$

Therefore

$$|F| \leq \sum_{j=1}^{k-1}|F_j| \leq \binom{n}{k}\sum_{j=1}^{k-1}\binom{k}{j}\beta^j \leq 2^k\beta\binom{n}{k}.$$

Let $\mathcal{S}_2$ be the set of all $S \in \binom{V(H)}{k-1}$ contained in more than $2^k\sqrt{\beta}(n-k+1)$ edges of F. It follows that $|\mathcal{S}_2| \leq \sqrt{\beta}\binom{n}{k-1}$. This implies that $|\mathcal{S}_1 \cup \mathcal{S}_2| \leq (\theta + \sqrt{\beta})\binom{n}{k-1} = (\theta + \theta^{1/(2k-2)})\binom{n}{k-1}$. Now consider an arbitrary $S \in \binom{V(H)}{k-1} \setminus (\mathcal{S}_1 \cup \mathcal{S}_2)$. As $S \notin \mathcal{S}_1$, it follows that $\deg_H(S) \geq \mu(n-k+1)$. As $S \notin \mathcal{S}_2$, it follows that

$$\deg_{H'}(S) \geq \deg_H(S) - 2^k\sqrt{\beta}(n-k+1) \geq (\mu - 2^k\theta^{1/(2k-2)})(n-k+1).$$

Therefore, H' is $(\mu - 2^k\theta^{1/(2k-2)}, \theta + \theta^{1/(2k-2)})$-dense.

Let $e \in E(H')$ and let $X \in \binom{e}{k-1}$. It is enough to prove that $X \notin \mathcal{S}_1 \cup \mathcal{S}_2$. As $e \notin F_{k-1}$, it follows that $X \notin \mathcal{A}_{k-1} = \mathcal{S}_1$. So it is enough to prove that $X \notin \mathcal{S}_2$. Suppose the contrary, that $X \in \mathcal{S}_2$. Then X is contained in more than $2^k\sqrt{\beta}(n-k+1)$ edges $e' \in E(F)$. Let $W = N_F(X)$. For all $w \in W$, fix a set $A_w \in \bigcup_{j=1}^{k-1}\mathcal{A}_j$ such that $A_w \subseteq X \cup \{w\}$ and let $T_w = X \cap A_w$. If $A_w \subseteq X$ then $A_w \subseteq e \in E(H')$, a contradiction. Hence $w \in A_w$ for all $w \in W$, and therefore $|T_w| = |A_w| - 1 < |X|$ for all $w \in W$. We deduce $T_w \neq X$ for all $w \in W$. By the pigeonhole principle, there exists $T \subsetneq X$ and $W_T \subseteq W$ such that for all $w \in W_T$, $T_w = T$ and $|W_T| \geq |W|/(2^{k-1}) \geq 2\sqrt{\beta}(n-k+1) > \sqrt{\beta}n$.

Suppose $|T| = t \geq 1$. Then for all $w \in W_T$, $T \cup \{w\} = A_w \in \mathcal{A}_{t+1}$, so there are at least $\sqrt{\beta}n \geq \beta(n-t)$ vertices $w \in V(H)$ such that $T \cup \{w\} \in \mathcal{A}_{t+1}$. Therefore, $T \in \mathcal{A}_t$ and $T \subseteq X \subseteq e$, which is a contradiction because $e \notin F_t$. Hence, we may assume that $T = \varnothing$. Then for all $w \in W_T$, $\{w\} \in \mathcal{A}_1$. And so $|\mathcal{A}_1| \geq |W_T| > \sqrt{\beta}n$, contradicting the claim. ∎

5.3.3 Weighted fractional tilings

Now we return our focus to Lemma 5.3.2. The strategy we follow to prove Lemma 5.3.2 is to apply the Regular Slice Lemma (Theorem 2.3.4), and find a $\{F_s^*, E_s^*\}$-fractional tiling, for some some simpler k-graphs F_s^*, E_s^* in the reduced k-graph. By using the regularity methods, the fractional tiling in the reduced k-graph can be lifted to an actual tiling with copies of F_s, E_s in the original k-graph, which covers a similar proportion of vertices.

To define the k-graphs F_s^* and E_s^*, we use the notion of "G-augmentation" introduced in Section 4.3.2. Let K be a k-edge with vertices $\{x_1, \dots, x_k\}$. Let G_s be the 2-graph on $[k]$ given by Corollary 4.3.5. Let F_s^* be the G_s-augmentation of K (with respect to the vertex partition $V_i := \{x_i\}$ for all $i \in [k]$). Let $V(F_s^*) = \{x_1, \dots, x_k\} \cup \{y_1, \dots, y_\ell\}$, where $\ell = |E(G_s)|$. We refer to $c(F_s^*) = \{x_1, \dots, x_k\}$ as the set of *core vertices of* F_s^* and $p(F_s^*) = \{y_1, \dots, y_\ell\}$ as the set of *pendant vertices of* F_s^*. Define the function $\alpha : V(F_s^*) \to \mathbb{N}$ to be such that, for $u \in V(F_s^*)$,

$$\alpha(u) = \begin{cases} a_{s,i} & \text{if } u = x_i, \\ 1 & \text{if } u \in p(F_s^*). \end{cases}$$

Note that there is a natural k-graph homomorphism θ from F_s to F_s^* such that, for all $u \in V(F_s^*)$, $|\theta^{-1}(u)| = \alpha(u)$. Observe that (5.3.2), $s \geq 5k^2$ and $k \geq 3$ imply that $\alpha(u) = 1$ if and only if u is a pendant vertex.

Let $\mathcal{F}_s^*(H)$ be the set of copies of F_s^* in H. Given $v \in V(H)$ and $F^* \in \mathcal{F}_s^*(H)$, define

$$\alpha_{F^*}(v) = \begin{cases} \alpha(u) & \text{if } v \text{ corresponds to vertex } u \in V(F_s^*), \\ 0 & \text{otherwise.} \end{cases}$$

Given $v \in V(H)$ and $e \in E(H)$, define

$$\alpha_e(v) = \begin{cases} M_s & \text{if } v \in e, \\ 0 & \text{otherwise.} \end{cases}$$

We now define a *weighted fractional* $\{F_s^*, E_s^*\}$*-tiling of* H to be a function $\omega^* : \mathcal{F}_s^*(H) \cup E(H) \to [0,1]$ such that, for all vertices $v \in V(H)$,

$$\omega^*(v) := \sum_{F^* \in \mathcal{F}_s^*(H)} \omega^*(F^*)\alpha_{F^*}(v) + \sum_{e \in E(H)} \omega^*(e)\alpha_e(v) \le 1.$$

Note that if (contrary to our assumptions) $a_{s,1} = \cdots = a_{s,k} = 1$, then we would have $\alpha_{F^*}(v) = \mathbf{1}\{v \in V(F^*)\}$ for every $F^* \in \mathcal{F}_s^*(H)$ and $\alpha_e(v) = \mathbf{1}\{v \in e\}$ for every $e \in E(H)$, so ω^* would be the standard fractional $\{F_s, E_s\}$-tiling. Note that the definition depends on k and the functions α_{F^*}, α_e, but those will always be clear from the context.

Define the *minimum weight of* ω^* to be

$$\omega^*_{\min} = \min_{\substack{J \in \mathcal{F}_s^*(H) \cup E(H) \\ v \in V(H) \\ \omega^*(J)\alpha_J(v) \neq 0}} \omega^*(J)\alpha_J(v).$$

Analogously to $\phi(\mathcal{T})$, define

$$\phi(\omega^*) = \frac{1}{n}\left(n - s\left(\sum_{F^* \in \mathcal{F}_s^*(H)} \omega^*(F^*) + \frac{3}{5}\sum_{e \in E(H)} \omega^*(e)\right)\right).$$

Given $c > 0$ and a k-graph H, let $\phi^*(H, c)$ be the minimum of $\phi(\omega^*)$ over all weighted fractional $\{F_s^*, E_s^*\}$-tilings ω^* of H with $\omega^*_{\min} \ge c$. Note that $\phi^*(H, c)$ also depends on k, s, α_{F^*} and α_e, which will always be clear from the context.

Let $\mathcal{T}$ be an $\{F_s, E_s\}$-tiling. We say that a vertex v is *saturated under* $\mathcal{T}$ if it is covered by a copy of F_s and v corresponds to a vertex in W_{G_s} under that copy. Let $S(\mathcal{T})$ denote the set of all saturated vertices under $\mathcal{T}$. Define $U(\mathcal{T})$ as the set of all uncovered vertices under $\mathcal{T}$, that is, all vertices which do not belong to any copy of F_s or E_s of $\mathcal{T}$.

Analogously, given a weighted fractional $\{F_s^*, E_s^*\}$-tiling ω^*, we say that a vertex v is *saturated under* ω^* if

$$\sum_{\substack{F^* \in \mathcal{F}_s^*(H) \\ \alpha_{F^*}(v)=1}} \omega^*(F^*)\alpha_{F^*}(v) = 1,$$

that is, $\omega^*(v) = 1$ and all its weight comes from copies of F_s^* such that v corresponds to a pendant vertex. Let $S(\omega^*)$ be the set of all saturated vertices under ω^*. Also, define $U(\omega^*)$ as the set of all vertices $v \in V(H)$ such that $\omega^*(v) = 0$.

Proposition 5.3.5. *Let $k \geq 3$ and $s \geq 5k^2$ with $s \not\equiv 0 \bmod k$. Let H be a k-graph on n vertices. Let ω^* be a weighted fractional $\{F_s^*, E_s^*\}$-tiling in H. Then the following holds:*

(i) *$s \cdot \sum_{F^* \in \mathcal{F}_s^*} \omega^*(F^*) + k \cdot M_s \sum_{e \in E(H)} \omega^*(e) \leq n$. In particular, $\sum_{F^* \in \mathcal{F}_s^*} \omega^*(F^*) \leq n/s$ and $\sum_{e \in E(H)} \omega^*(e) \leq n/(kM_s)$,*

(ii) *$|S(\omega^*)| \leq \ell n/s$, and*

(iii) *if $S' \subseteq S(\omega^*)$ with $|S'| > n/s$, then there exists $F^* \in \mathcal{F}_s^*(H)$ with $\omega^*(F^*) > 0$ such that $|p(F^*) \cap S'| \geq 2$.*

Proof. For (i), note that

$$\begin{aligned} n &\geq \sum_{v \in V(H)} \sum_{F^* \in \mathcal{F}_s^*(H)} \omega^*(F^*)\alpha_{F^*}(v) + \sum_{v \in V(H)} \sum_{e \in E(H)} \omega^*(e)\alpha_e(v) \\ &= \sum_{F^* \in \mathcal{F}_s^*(H)} \omega^*(F^*) \sum_{v \in V(H)} \alpha_{F^*}(v) + \sum_{e \in E(H)} \omega^*(e) \sum_{v \in V(H)} \alpha_e(v) \\ &= s \sum_{F^* \in \mathcal{F}_s^*(H)} \omega^*(F^*) + kM_s \sum_{e \in E(H)} \omega^*(e). \end{aligned}$$

To prove (ii), recall that all of the vertices $v \in S(\omega^*)$ only receive weight from pendant vertices, and all copies of $F \in \mathcal{F}_s^*(H)$ have precisely ℓ pendant vertices, and therefore

$$|S(\omega^*)| = \sum_{v \in S(\omega^*)} \sum_{F^* \in \mathcal{F}_s^*(H)} \omega^*(F^*)\alpha_{F^*}(v) \leq \ell \sum_{F^* \in \mathcal{F}_s^*(H)} \omega^*(F^*) \overset{\text{(i)}}{\leq} \ell n/s.$$

Finally, for (iii), suppose the contrary, that, for all $F^* \in \mathcal{F}_s^*(H)$ with $\omega^*(F^*) > 0$, we have $\sum_{v \in S'} \alpha_{F^*}(v) = |p(F^*) \cap S'| \leq 1$. Then

$$\begin{aligned} |S'| &= \sum_{v \in S'} \sum_{F^* \in \mathcal{F}_s^*(H)} \omega^*(F^*)\alpha_{F^*}(v) = \sum_{F^* \in \mathcal{F}_s^*(H)} \omega^*(F^*) \sum_{v \in S'} \alpha_{F^*}(v) \\ &\leq \sum_{F^* \in \mathcal{F}_s^*(H)} \omega^*(F^*) \leq n/s, \end{aligned}$$

a contradiction. ∎

Note that F_s admits a natural perfect weighted fractional F_s^*-tiling, defined as follows. Let $a = \prod_{1 \leq i \leq k} a_{s,i}$. Let F be a copy of F_s and suppose that $V(F) = V_1 \cup \dots \cup V_k \cup W$, where $V_1, \dots, V_k$ forms a complete (k,k)-graph with $|V_i| = a_{s,i}$ for all $1 \leq i \leq k$ and $|W| = \ell$. Note that $a \leq M_s^k$. For all $(v_1, \dots, v_k) \in V_1 \times \dots \times V_k$, the vertices $\{v_1, \dots, v_k\} \cup W$ span a copy of F_s^*, where we identify $\{v_1, \dots, v_k\}$ with the core vertices of F_s^* and W with the pendant vertices of F_s^*. Define ω^* by assigning to all such copies the weight $1/a$. A similar method shows that E_s admits a perfect weighted fractional E_s^*-tiling, by setting $\omega^*(e) = M_s^{-k}$ for all $e \in E_s$.

We can naturally extend these constructions to find a weighted fractional $\{F_s^*, E_s^*\}$-tiling given an $\{F_s, E_s\}$-tiling, by repeating the above procedure over all copies of F_s and E_s. The following proposition collects useful properties of the obtained fractional tiling, for future reference. All of them are straightforward to check by using the construction outlined above, so we omit the proof.

Proposition 5.3.6. *Let $k \geq 3$ and $s \geq 5k^2$ with $s \not\equiv 0 \bmod k$. Let H be a k-graph and let $\mathcal{T}$ be an $\{F_s, E_s\}$-tiling in H. Then there exists a weighted fractional $\{F_s^*, E_s^*\}$-tiling ω^* such that*

(i) $\phi(\mathcal{T}) = \phi(\omega^*)$,

(ii) $|\mathcal{F}_{\mathcal{T}}| = \sum_{F^* \in \mathcal{F}_s^*(H)} \omega^*(F^*)$,

(iii) $|\mathcal{E}_{\mathcal{T}}| = \sum_{e \in E(H)} \omega^*(e)$,

(iv) $S(\omega^*) = S(\mathcal{T})$ *and* $U(\omega^*) = U(\mathcal{T})$,

(v) *for all* $F^* \in \mathcal{F}_s^*(H)$, $\omega^*(F^*) \in \{0, a^{-1}\}$, *where* $a = \prod_{1 \le i \le k} a_{s,i}$,

(vi) *for all* $e \in E(H)$, $\omega^*(e) \in \{0, M_s^{-k}\}$, *moreover if* $e \in E(E_s)$ *for some* $E_s \in \mathcal{E}_{\mathcal{T}}$, *then* $\omega^*(e) = M_s^{-k}$,

(vii) $\omega^*_{\min} \ge M_s^{-k}$, *and*

(viii) $\omega^*(v) \in \{0,1\}$ *for all* $v \in V(H)$.

The next lemma assures that if R is a reduced k-graph of H, then $\phi(H)$ is roughly bounded above by $\phi^*(R,c)$.

Lemma 5.3.7. *Let* $k \ge 3$ *and* $s \ge 5k^2$ *with* $s \not\equiv 0 \bmod k$. *Let* $c \ge \beta > 0$,

$$1/n \ll \varepsilon, 1/r \ll \varepsilon_k \ll 1/t_1 \le 1/t_0 \ll \beta, c, 1/s, 1/k, \qquad and \qquad \varepsilon_k \ll d.$$

Let H *be a* k*-graph on* n *vertices and* $\mathcal{J}$ *be a* $(t_0, t_1, \varepsilon, \varepsilon_k, r)$*-regular slice for* H*, and* $R = R_d(H)$ *be its* d*-reduced* k*-graph obtained from* $\mathcal{J}$*. Then* $\phi(H) \le \phi^*(R,c) + s\beta/c$.

Proof. Let ω^* be a weighted fractional $\{F_s^*, E_s^*\}$-tiling on R such that $\phi(\omega^*) = \phi^*(R,c)$ and $\omega^*_{\min} \ge c$. Let $t = |V(R)|$ and let $m = n/t$, so that each cluster in $\mathcal{J}$ has size m. Let n_F^* be the number of $F_s^* \in \mathcal{F}_s^*(R)$ with $\omega^*(F_s^*) > 0$ and n_E^* be the number of $E \in E(R)$ with $\omega^*(E) > 0$. Note that

$$n_F^* + n_E^* \le t/c.$$

For all clusters $U \in V(R)$, we subdivide U into disjoint sets $\{U_J\}_{J \in \mathcal{F}_s^*(R) \cup E(R)}$ of size $|U_J| = \lfloor \omega^*(J)\alpha_J(U)m \rfloor$.

In the next claim, we show that if $\omega^*(J) > 0$ for some $J \in \mathcal{F}_s^*(R) \cup E(R)$ then we can find a large F_s-tiling or a large E_s-tiling on $\bigcup_{U \in V(J)} U_J$.

Claim 5.3.8. *For all $J \in \mathcal{F}_s^*(R) \cup E(R)$ with $\omega^*(J) > 0$, $H\left[\bigcup_{U \in V(J)} U_J\right]$ contains*

(i) *an F_s-tiling $\mathcal{F}_J$ with $|\mathcal{F}_J| \geq m(\omega^*(J) - \beta)$ if $J \in \mathcal{F}_s^*(R)$; or*

(ii) *an E_s-tiling $\mathcal{E}_J$ with $|\mathcal{E}_J| \geq m(\omega^*(J) - \beta)$ if $J \in E(R)$.*

Proof of the claim. We will only consider the case when $J \in \mathcal{F}_s^*(R)$, as the case $J \in E(R)$ is proved similarly.

Suppose $c(J) = \{X_1, \ldots, X_k\}$ and $p(J) = \{Y_1, \ldots, Y_\ell\}$, so $V(J) = c(J) \cup p(J)$. We will first show that if $X_i' \subseteq X_i$ for all $1 \leq i \leq k$ and $Y_j' \subseteq Y_j$ for all $1 \leq j \leq \ell$ are such that $|X_i'| = |Y_j'| \geq \beta m$, then $H\left[\left(\bigcup_{1 \leq i \leq k} X_i'\right) \cup \left(\bigcup_{1 \leq j \leq \ell} Y_j'\right)\right]$ contains a copy F of F_s such that $|V(F) \cap X_i'| = a_{s,i}$ for all $1 \leq i \leq k$ and $|V(F) \cap Y_j'| = 1$ for all $1 \leq j \leq \ell$.

Indeed, take X_i', Y_j' as above and construct the subcomplex H' obtained by restricting H along with $\mathcal{J}$ to the subsets X_i', Y_j' and then deleting the edges in H not supported in k-tuples of clusters corresponding to edges in $E(J)$. Then H' is a $(k, k+\ell)$-complex. Since $\mathcal{J}$ is (t_0, t_1, ε)-equitable, there exists a density vector $\mathbf{d} = (d_{k-1}, \ldots, d_2)$ such that, for all $2 \leq i \leq k-1$, we have $d_i \geq 1/t_1$, $1/d_i \in \mathbb{N}$ and $\mathcal{J}$ is $(d_{k-1}, \ldots, d_2, \varepsilon, \varepsilon, 1)$-regular. As $J \subseteq R$, all edges e in $E(J) \cap E(R)$ induce k-tuples X_e of clusters in H with $d^*(X_e) = d_e \geq d$ and H is (d_e, ε_k, r)-regular with respect to X_e. By Lemma 2.3.1, the restriction of X_e to the subsets $\{X_1', \ldots, X_k', Y_1', \ldots, Y_\ell'\}$ is $(d_e, \sqrt{\varepsilon_k}, \sqrt{\varepsilon}, r)$-regular. Hence, by Lemma 2.3.8, there exists a partition-respecting copy F of F_s in H', that is, F satisfies $|V(F) \cap X_i'| = a_{s,i}$ for all $1 \leq i \leq s$ and $|V(F) \cap Y_j'| = 1$ for all $1 \leq j \leq \ell$, as desired.

Now consider the largest F_s-tiling $\mathcal{F}_J$ in $H\left[\bigcup_{U \in V(J)} U_J\right]$ such that all $F \in \mathcal{F}_J$ satisfy $|V(F) \cap X_i| = a_{s,i}$ for all $1 \leq i \leq k$ and $|V(F) \cap Y_j| = 1$ for all $1 \leq j \leq \ell$. Let $V(\mathcal{F}_J) = \bigcup_{F \in \mathcal{F}_J} V(F)$. By the discussion above, we may assume that $|U_J \setminus V(\mathcal{F}_J)| < \beta m$ for some $U \in V(J)$. A simple calculation shows that $|(Y_j)_J \setminus V(\mathcal{F}_J)| < \beta m$ for all $1 \leq j \leq \ell$ and $|(X_i)_J \setminus V(\mathcal{F}_J)| < a_{s,i}\beta m$ for all $1 \leq i \leq k$. Therefore, $\mathcal{F}_J$ covers at least $sm(\omega^*(J) - \beta)$ vertices and it follows that $|\mathcal{F}_J| \geq m(\omega^*(J) - \beta)$. □

Now consider the $\{F_s, E_s\}$-tiling $\mathcal{T} = \mathcal{F}_{\mathcal{T}} \cup \mathcal{E}_{\mathcal{T}}$ in H, where $\mathcal{F}_{\mathcal{T}} = \bigcup_{J \in \mathcal{F}_s^*(R)} \mathcal{F}_J$ and $\mathcal{E}_{\mathcal{T}} = \bigcup_{E \in E(R)} \mathcal{E}_J$ as given by the claim (and we take $\mathcal{F}_J = \mathcal{E}_J = \varnothing$ whenever $\omega^*(J) = 0$). Therefore

$$\begin{aligned}
|\mathcal{F}_{\mathcal{T}}| + \frac{3}{5}|\mathcal{E}_{\mathcal{T}}| &\geq \sum_{\substack{F_s^* \in \mathcal{F}_s^*(R) \\ \omega^*(F_s^*)>0}} m(\omega^*(F_s^*) - \beta) + \frac{3}{5} \sum_{\substack{E \in E(R) \\ \omega^*(E)>0}} m(\omega^*(E) - \beta) \\
&\geq m\left(\sum_{F_s^* \in \mathcal{F}_s^*(R)} \omega^*(F_s^*) + \frac{3}{5} \sum_{E \in E(R)} \omega^*(E) - \beta(n_F^* + n_E^*) \right) \\
&\geq m\left(\sum_{F_s^* \in \mathcal{F}_s^*(R)} \omega^*(F_s^*) + \frac{3}{5} \sum_{E \in E(R)} \omega^*(E) - \frac{\beta t}{c} \right) \\
&= mt\left(\frac{1 - \phi(\omega^*)}{s} - \frac{\beta}{c} \right) = \frac{n}{s}\left(1 - \phi(\omega^*) - \frac{\beta s}{c} \right).
\end{aligned}$$

Thus we have $\phi(H) \leq \phi(\mathcal{T}) \leq \phi(\omega^*) + s\beta/c = \phi^*(R, c) + s\beta/c$. ∎

5.3.4 Proof of Lemma 5.3.2

In this subsection we will prove Lemma 5.3.2. The proof can be sketched as follows. Suppose k, s and γ, as in the statement of the lemma are fixed. To find a contradiction, we will assume that there exists a value of $\alpha > 0$ such that, no matter how small $1/n$ and θ are chosen, $\Phi(n, 1/2 + 1/(2s) + \gamma, \theta) > \alpha$. We choose the supremum of all the possible values of α which make the statement fail. We consider a very large k-graph H which is strongly $(1/2 + 1/(2s) + \gamma, \theta')$-dense, with θ' very small. By applying the Regular Slice Lemma (Lemma 2.3.4), we obtain a reduced k-graph R, which is $(1/2 + 1/(2s) + \gamma', \theta'')$-strongy dense, with some slightly degraded parameters $1/2 + 1/(2s) + \gamma'$ and θ''. We can assume that $|V(R)|$ is large and θ'' is small.

Even though the "relative codegree" in R has decreased slightly with respect to the codegree of H, as a first step we show that we can relate the tilings in R with the tilings found in graphs with slightly larger codegree (Lemma 5.3.9). Next, since α was chosen as the supremum of all the values of α which make the statement fail, and since we assume that R is very large, in R we will find an (integral)

$\{F_s, E_s\}$-tiling $\mathcal{T}$ with $\phi(T)$ "close" to α. The next crucial step is to show that given an integral tiling in R with $\phi(T)$ close to α, we can find a fractional tiling showing that $\phi^*(R, c)$ is substantially smaller than α (Lemma 5.3.10). Finally, by using the tools from the previous section we can lift the fractional matching in the reduced graph to an integral tiling $\mathcal{T}'$ in H with $\phi(\mathcal{T}') < \alpha$, which will contradict the choice of α and thus prove the result.

The next lemma relates the value of $\Phi(n, \mu, \theta)$ to the value obtained over k-graphs with slightly better codegree properties. As explained before, this will be applied to a reduced graph.

Lemma 5.3.9. *Let $k \geq 3$ and $s \geq 5k^2$ with $s \not\equiv 0 \bmod k$. Let $\mu + \gamma/3 \leq 2/3$. Then $\Phi(n, \mu, \theta) \leq \Phi((1+\gamma)n, \mu + \gamma/3, \theta) + s\gamma$.*

Proof. Let H be a k-graph on n vertices that is (μ, θ)-dense. Consider the k-graph H' on the vertices $V(H) \cup A$ obtained from H by adding a set of $|A| = \gamma n$ vertices and adding all of the k-edges that have non-empty intersection with A. Since

$$\frac{\mu + \gamma}{1 + \gamma} \geq \mu + \gamma/3$$

as $\mu + \gamma/3 \leq 2/3$, H' is $(\mu + \gamma/3, \theta)$-dense.

Let $\mathcal{T}'$ be an $\{F_s, E_s\}$-tiling on H' satisfying $\phi(\mathcal{T}') = \phi(H')$. Consider the $\{F_s, E_s\}$-tiling $\mathcal{T}$ in H obtained from $\mathcal{T}'$ by removing all copies of F_s or E_s intersecting with A. It follows that

$$\begin{aligned} 1 - \phi(\mathcal{T}) = \frac{s}{n}\left(|\mathcal{F}_{\mathcal{T}}| + \frac{3}{5}|\mathcal{E}_{\mathcal{T}}|\right) &\geq \frac{s}{n}\left(|\mathcal{F}_{\mathcal{T}'}| + \frac{3}{5}|\mathcal{E}_{\mathcal{T}'}|\right) - s\gamma \\ &\geq \frac{s}{(1+\gamma)n}\left(|\mathcal{F}_{\mathcal{T}'}| + \frac{3}{5}|\mathcal{E}_{\mathcal{T}'}|\right) - s\gamma = 1 - \phi(\mathcal{T}') - s\gamma. \end{aligned}$$

Hence, $\phi(H) \leq \phi(\mathcal{T}) \leq \phi(\mathcal{T}') + s\gamma \leq \Phi((1+\gamma)n, \mu + \gamma/3, \theta) + s\gamma$. ■

The next lemma shows that given an $\{F_s, E_s\}$-tiling $\mathcal{T}$ of a strongly (μ, θ)-dense k-graph H with $\phi(T)$ "large", we can always find a better weighted fractional

$\{F_s^*, E_s^*\}$-tiling in terms of ϕ^*.

Lemma 5.3.10. *Let $k \geq 3$, $s \geq 5k^2$ with $s \not\equiv 0 \bmod k$, and $c = s^{-2k}$. For all $\gamma > 0$ and $0 \leq \alpha \leq 1$ there exists $n_0 = n_0(k, s, \gamma, \alpha) \in \mathbb{N}$ and $\nu = \nu(k, s, \gamma) > 0$ and $\theta = \theta(\alpha, k)$ such that following holds for all $n \geq n_0$. Let H be a k-graph on n vertices that is strongly $(1/2 + 1/(2s) + \gamma, \theta)$-dense and $\phi(H) \geq \alpha$. Then $\phi^*(H, c) \leq (1 - \nu)\phi(H)$.*

The proof of Lemma 5.3.10 is deferred to the next subsection, but we use it now to prove Lemma 5.3.2.

Proof of Lemma 5.3.2. Consider a fixed $\gamma > 0$. Suppose the result is false, that is, there exists $\alpha > 0$ such that for all $n \in \mathbb{N}$ and $\theta^* > 0$ there exists $n' > n$ satisfying $\Phi(n', 1/2 + 1/(2s) + \gamma, \theta^*) > \alpha$. Let α_0 be the supremum of all such α. Apply Lemma 5.3.10 (with parameters $\gamma/2, \alpha_0/2$ playing the roles of γ, α) to obtain $n_0 = n_0(k, s, \gamma/2, \alpha_0/2)$, $\nu = \nu(k, s, \gamma/2)$ and $\theta = \theta(\alpha_0/2, k)$. Let

$$0 < \eta \ll \nu, \gamma, \alpha_0, 1/s.$$

By the definition of α_0, there exists $\theta_1 > 0$ and $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$,

$$\Phi(n, 1/2 + 1/(2s) + \gamma, \theta_1) \leq \alpha_0 + \eta/2. \tag{5.3.3}$$

Now we prepare the setup to use the Regular Slice Lemma (Theorem 2.3.4). Let $c := s^{-k}$. Let $\beta, \varepsilon_k, \varepsilon, d, \theta^*, \theta', \gamma' > 0$ and $t_0, t_1, r, n_2 \in \mathbb{N}$ be such that

$$\begin{aligned}
&\gamma' \ll \eta, c, 1/s, 1/k, 1/n_0, 1/n_1,\\
&1/n_2 \ll \varepsilon, 1/r \ll \varepsilon_k, 1/t_1 \ll 1/t_0 \ll \beta \ll \gamma',\\
&\varepsilon_k \ll d \ll \gamma',\\
&\varepsilon_k \ll \theta' \ll \theta^* \ll \gamma', \theta, \theta_1,
\end{aligned}$$

and $n_2 \equiv 0 \bmod t_1!$.

Let H be a $(1/2 + 1/(2s) + \gamma, \theta')$-dense k-graph on $n \geq n_2$ vertices with

$$\phi(H) > \alpha_0 - \eta, \tag{5.3.4}$$

such H exists by the definition of α_0. By removing at most $t_1! - 1$ vertices we get a k-graph H' on at least n_2 vertices such that $|V(H')|$ is divisible by $t_1!$ and H' is $(1/2 + 1/(2s) + \gamma - \gamma', 2\theta')$-dense.

Let $\mathcal{S}$ be the set of $(k-1)$-tuples T of vertices of $V(H')$ such that $\deg_{H'}(T) < (1/2 + 1/(2s) + \gamma - \gamma')(|V(H')| - k + 1)$. Thus $|\mathcal{S}| \leq 2\theta' \binom{|V(H')|}{k-1}$. By Theorem 2.3.4, there exists a $(t_0, t_1, \varepsilon, \varepsilon_k, r)$-regular slice $\mathcal{J}$ for H' such that for all $(k-1)$-sets Y of clusters of $\mathcal{J}$, we have $\overline{\deg}(Y; R(H')) = \overline{\deg}(\mathcal{J}_Y; H') \pm \varepsilon_k$, and furthermore, $\mathcal{J}$ is $(3\sqrt{2\theta'}, \mathcal{S})$-avoiding.

Let $R = R_d(H')$ be the d-reduced k-graph obtained from H' and $\mathcal{J}$. Since $\theta', d, \varepsilon_k \ll \gamma'$, $\varepsilon_k \ll \theta'$ and $\mathcal{J}$ is $(3\sqrt{2\theta'}, \mathcal{S})$-avoiding, Lemma 2.3.7 implies that R is $(1/2 + 1/(2s) + \gamma - 2\gamma', 5\sqrt{\theta'})$-dense. By Lemma 5.3.3, there exists a subgraph $R' \subseteq R$, on the same vertex set, that is strongly $(1/2 + 1/(2s) + \gamma - 3\gamma', \theta^*)$-dense as $\theta' \ll \gamma', 1/k, \theta^*$. Since the vertices of R' are the clusters of $\mathcal{J}$, we note that $|V(R')| \geq t_0 \geq n_1$. By the fact that $\theta^* \leq \theta_1$, using Lemma 5.3.9 (with $9\gamma'$ playing the role of γ) and (5.3.3) we deduce

$$\begin{aligned}
\phi(R') &\leq \Phi(|V(R')|, 1/2 + 1/(2s) + \gamma - 3\gamma', \theta^*) \\
&\leq \Phi((1 + 9\gamma')|V(R')|, 1/2 + 1/(2s) + \gamma, \theta^*) + 9\gamma' s \\
&\leq \alpha_0 + \eta/2 + 9\gamma' s \leq \alpha_0 + \eta.
\end{aligned}$$

We further claim that $\phi^*(R', c) \leq \alpha_0 - 2\eta$. Note that $c = s^{-k}$ and $\alpha_0 \geq 4\eta$. Therefore, if $\phi(R') < \alpha_0/2$, then the claim holds by Proposition 5.3.6. Thus we may assume that $\phi(R') \geq \alpha_0/2$. Note that $|V(R')| \geq t_0 \geq n_0$, $\gamma - 3\gamma' \geq \gamma/2$, and

$\theta^* \leq \theta$. By the choice of n_0, ν, and θ (given by Lemma 5.3.10), we have

$$\phi^*(R', c) \leq (1-\nu)\phi(R') \leq (1-\nu)(\alpha_0 + \eta) \leq \alpha_0 - 2\eta,$$

where the last inequality holds since $\eta \ll \nu, \alpha_0$. Finally, recall that $\beta \ll \eta, c$, so an application of Lemma 5.3.7 implies that

$$\phi(H) \leq \phi^*(R, c) + s\beta/c \leq \phi^*(R', c) + s\beta/c \leq \alpha_0 - \eta,$$

contradicting (5.3.4). ■

5.4 Improving fractional matchings: Proof of Lemma 5.3.10

Before proceeding with the full details of the proof of Lemma 5.3.10, we first give a rough outline of the proof. Let $\mathcal{T}$ be an $\{F_s, E_s\}$-tiling of H satisfying $\phi(\mathcal{T}) = \phi(H)$. By Proposition 5.3.6, we obtain a weighted fractional $\{F_s^*, E_s^*\}$-tiling ω_0^* with $\phi(\omega_0^*) = \phi(\mathcal{T})$, $U(\omega_0^*) = U(\mathcal{T})$ and $(\omega_0^*)_{\min} \geq M_s^{-k}$. Our aim is to sequentially define weighted fractional $\{F_s^*, E_s^*\}$-tilings $\omega_1^*, \omega_2^*, \ldots, \omega_t^*$ such that $\phi(\omega_{j-1}^*) - \phi(\omega_j^*) \geq \nu_1/n$ for all $j \in [t]$, where ν_1 is a fixed positive constant. We will follow this procedure for $t = \Omega(n)$ steps, and we will show that ω_t^* satisfies the required properties.

Moreover, we will construct ω_{j+1}^* based on ω_j^* by changing the weights of $\mathcal{F}_s(H)$ and $E(H)$ on a small number of vertices, such that no vertex has its weight changed more than once during the whole procedure. Recall that $U(\mathcal{T})$ is the set of uncovered vertices under the "integral tiling" $\mathcal{T}$, which is equal to the set $U(\omega_0^*)$ of vertices which receive zero weight under ω_0^*. If $|U(\mathcal{T})|$ is large then we construct ω_{j+1}^* from ω_j^* via assigning weights to edges that contain at least $k-1$ vertices in $U(\mathcal{T})$. Suppose that $|U(\mathcal{T})|$ is small. Since $\phi(\mathcal{T}) \geq \alpha$, not all of the weight of ω_0^* can be used in copies of F_s^*. Thus there must exist edges $e \in E(H)$ with positive weight under ω_0^*. We use this to find $e \in E(H)$ with $\omega_j^*(e) > 0$. The crucial observation is that a copy of F_s^* can be obtained from an edge by adding a few extra vertices to it. We use this to obtain ω_{j+1}^* from ω_j^* by reducing the weight on e before assigning weight to some copy of F_s^* which originates from e. More care is needed to ensure that ω_{j+1}^* is indeed a weighted fractional $\{F_s^*, E_s^*\}$-tiling. Ideally we would like that the extra vertices which are added to e to form a copy of F_s^* are not saturated, if possible.

We summarise and recall the relevant properties of F_s^*, which was originally defined at the beginning of Section 5.3.3. There exists a 2-graph G_s on $[k]$ with $\ell \leq k-1$ edges which consists of a disjoint union of paths. Suppose $e_1, \dots, e_\ell$ is an enumeration of the edges of G_s and $e_i = j_i j_i'$ for all $i \in [s]$. If $X = \{x_1, \dots, x_k\}$, we can describe F_s^* as having vertices $V(F_s^*) = X \cup \{y_1, \dots, y_\ell\}$, and the edges of F_s^* are X together with $(X \setminus \{x_{j_i}\}) \cup \{y_i\}$ and $(X \setminus \{x_{j_i'}\}) \cup \{y_i\}$ for each $i \in [\ell]$. We call $c(F_s^*) = X$ and $p(F_s^*) = \{y_1, \dots, y_\ell\}$ the core and pendant vertices of F_s^*, respectively.

The following two lemmas are needed for the case when $U(\mathcal{T})$ is small. The idea is the following: suppose H is a k-graph on n vertices with $\delta_{k-1}(H) \geq (1/2 + 1/(2s) + \gamma)n$. If X is a k-edge in H, we would like to extend it into a copy F of F_s^* such that $c(F) = X$. Lemmas 5.4.1 and 5.4.2 will indicate where should we look for the vertices of $p(F)$.

Lemma 5.4.1. *Let $k \geq 3$, $s \geq 2k^2$ and $\ell \leq k-1$. Suppose that $N_i \subseteq [n]$ are such that $|N_i| \geq (1/2 + 1/(2s) + \gamma)n$ for all $1 \leq i \leq k$. Let G be a 2-graph on $\{N_1, \dots, N_k\}$ such that $N_i N_j \in E(G)$ if and only if $|N_i \cap N_j| \leq (\ell/s + \gamma)n$. Then G is bipartite.*

Proof. We will show that G does not have any cycle of odd length. It suffices to show that $N_{i_1} N_{i_{2j+1}} \notin E(G)$ for all paths $N_{i_1} \cdots N_{i_{2j+1}}$ in G on an odd number of vertices.

For any $S \subseteq [n]$, write $\overline{S} := [n] \setminus S$. First, note that if N_i is adjacent to N_j in G, then $|N_i \setminus \overline{N_j}| \leq (\ell/s + \gamma)\, n$ and $|\overline{N_j} \setminus N_i| \leq (n - |N_j|) - (|N_i| - |N_i \cap N_j|) \leq (\ell/s - \gamma)\, n$. For any three sets N_i, N_j, N_k it holds that $|N_i \setminus N_k| \leq |N_i \setminus \overline{N_j}| + |\overline{N_j} \setminus N_k|$. Hence, if $N_iN_jN_k$ is a path on three vertices in G, then

$$|N_i \setminus N_k| \le |N_i \setminus \overline{N_j}| + |\overline{N_j} \setminus N_k| \le 2\ell n/s.$$

Now consider a path in G on an odd number of vertices. Without loss of generality (after a suitable relabelling), we assume the path is given by $N_1N_2\cdots N_{2j+1}$ for some j which necessarily satisfies $2j+1 \le k$. By using the previous bounds repeatedly, we obtain

$$|N_1 \setminus N_{2j+1}| \le |N_1 \setminus N_3| + |N_3 \setminus N_5| + \cdots + |N_{2j-1} \setminus N_{2j+1}| \le \frac{2\ell jn}{s} \le \frac{\ell(k-1)n}{s}.$$

To conclude, since $\ell \le k-1$ and $s > 2k^2$ we obtain

$$|N_1 \cap N_{2j+1}| \ge |N_1| - \frac{\ell(k-1)n}{s} \ge \left(\frac{1}{2} + \frac{1}{2s} + \gamma\right)n - \frac{(k-1)^2 n}{s} > \left(\frac{\ell}{s} + \gamma\right)n.$$

Hence, $N_1N_{2j+1} \notin E(G)$, as desired. ∎

Lemma 5.4.2. *Let $k \ge 3, s \ge 5k^2$ with $s \not\equiv 0 \bmod k$. Let $\ell = |E(G_s)$. Let $1/n \ll \gamma, 1/k$ and $\theta > 0$. Let H be a strongly $(1/2 + 1/(2s) + \gamma, \theta)$-dense k-graph on n vertices. Let $X = \{x_1, \dots, x_k\}$ be an edge of H, and for all $1 \le i \le k$ let $N_i = N_H(X \setminus \{x_i\})$. Let $S \subseteq V(H)$ with $|S| \le (\ell/s + \gamma/3)n$ and $y_0 \in N_1 \cap N_2$. Suppose either $|N_1 \cap N_2| < (\ell/s + 2\gamma/3)n$ or $|N_i \cap N_j| \ge (\ell/s + 2\gamma/3)n$ for all $1 \le i, j \le k$. Then there exists a copy F^* of F_s^* such that $c(F^*) = X$ and $p(F^*) \cap (S \setminus \{y_0\}) = \varnothing$.*

Proof. Note that $|N_i| \ge (1/2 + 1/(2s) + \gamma)(n - k + 1) \ge (1/2 + 1/(2s) + 2\gamma/3)n$, for all $1 \le i \le k$. Let G be the 2-graph on $[k]$ such that $ij \in E(G)$ if and only if $|N_i \cap N_j| < (\ell/s + 2\gamma/3)n$. Note that if $ij \notin E(G)$, then $|N_i \cap N_j| \ge (\ell/s + 2\gamma/3)n \ge |S| + \ell$.

Recall that G_s, the 2-graph which defines F_s^*, is a disjoint union of paths. By our assumption, either $12 \in E(G)$ or G is empty. We claim that in either case, there exists a bijection $\phi : V(G_s) \to [k]$ such that $\{\phi(j_i)\phi(j_i') : j_ij_i' \in E(G_s)\} \cap E(G) \subseteq$

$\{12\}$. Indeed, if G is empty this is obvious. Otherwise, we use Lemma 5.4.1 which says that G is bipartite, say, with parts A and B such that $1 \in A$ and $2 \in B$. Since G_s is a union of paths, there exists a path $P \supseteq G_s$ on $V(G_s)$. We define a bijection $\phi : V(G_s) \to [k]$ by assigning the first $|A|$ vertices of P to A and ending with 1; and assigning the remaining $|B|$ vertices of P to B, beginning with 2. Then clearly $\{\phi(j_i)\phi(j_i') : j_i j_i' \in E(G_s)\} \cap E(G) \subseteq \{12\}$, as desired.

Let $e_1, \dots, e_\ell$ be an enumeration of the edges of $E(G_s)$. For a given $i \in [\ell]$ and $e_i = j_i j_i' \in E(G_s)$, if $\{\phi(j_i), \phi(j_i')\} = \{1, 2\}$, then let $y_i = y_0$. Otherwise, $\phi(j_i)\phi(j_i') \notin E(G)$ and therefore $|N_{\phi(j_i)} \cap N_{\phi(j_i')}| \geq |S| + \ell$, thus we can greedily pick $y_i \in (N_{\phi(j_i)} \cap N_{\phi(j_i')}) \setminus S$ such that $y_1, \dots, y_\ell$ are pairwise distinct. Then there exists a copy F^* of F_s^* with $c(F^*) = X$ and $p(F^*) = \{y_1, \dots, y_\ell\}$, which satisfies the required properties. ∎

Now we are ready to prove Lemma 5.3.10.

Proof of Lemma 5.3.10. We may assume that $\gamma \ll \alpha, 1/k, 1/s$. Recall that our aim is to define a sequence of fractional $\{F_s^*, E_s^*\}$-tilings $\omega_0^*, \dots, \omega_t^*$, for some $t \geq 0$. Let

$$\nu_1 = \frac{s}{25kM_s^k}, \quad \nu_2 = \frac{\gamma}{40k^3 s^k}, \quad \text{and} \quad \nu = \frac{\nu_1 \nu_2}{2}.$$

Choose $\theta \ll \alpha, 1/k$ and $1/n_0 \ll \alpha, \gamma, 1/k, 1/s$. Let H be a strongly $(1/2 + 1/(2s) + \gamma, \theta)$-dense k-graph on $n \geq n_0$ vertices with $\phi(H) \geq \alpha$. Choose $t = \lfloor \nu_2 \phi(H) n \rfloor$.

Recall that G_s, ℓ, F_s, m_s, M_s are given by Proposition 4.3.5 and they satisfy (5.3.1) and (5.3.2). Let $\mathcal{T}$ be an $\{F_s, E_s\}$-tiling on H with $\phi(\mathcal{T}) = \phi(H)$. Apply Proposition 5.3.6 and obtain a weighted fractional $\{F_s^*, E_s^*\}$-tiling w_0^* satisfying all the properties of the proposition.

Given that ω_j^* has been defined for some $0 \leq j \leq t$, define

$$A_j = \{v \in V(H) : \forall J \in \mathcal{F}_s^*(H) \cup E(H), \omega_j^*(J)\alpha_J(v) = \omega_0^*(J)\alpha_J(v)\}.$$

So A_j is the set of vertices such that ω_j^* is "identical to ω_0^*". Note that, by definition, if $v \in A_j$ then *every* copy of F_s^* or E_s^* which is incident with v must have the same weight as in ω_0^*; it is not enough that the sum of weights in v is identical both in ω_j^* and ω_0^*.

Note that by Proposition 5.3.6(viii), for all $v \in A_j$,

$$\omega_j^*(v) = \omega_0^*(v) \in \{0, 1\}. \tag{5.4.1}$$

Clearly we have $A_0 = V(H)$. Let $\mathcal{T}_0^+ = \{J \in \mathcal{F}_s^*(H) \cup E(H) : \omega_0^*(J) > 0\}$. The set A_j will indicate where we should look for graphs $J \in \mathcal{T}_0^+$ whose weight on ω_j^* is known (by knowing the weight on $J \in \omega_0^*$), and we will modify those to define the subsequent weighting ω_{j+1}^*.

By the definition of A_j, the following is true for all $J \in \mathcal{T}_0^+$: if $V(J) \cap A_j \neq \varnothing$, then $\omega_j^*(J) = \omega_0^*(J)$. Proposition 5.3.6 indicates the weights of ω_0^*, and using this together with the bounds of (5.3.1) we get that, for every $J \in \mathcal{T}_0^+$ such that $V(J) \cap A_j \neq \varnothing$, we have

$$\omega_j^*(J) - \frac{1}{M_s^{-k}} \begin{cases} = 0 & \text{if } J \in E(H) \text{ or } m_s = M_s, \\ \geq c & \text{otherwise.} \end{cases} \tag{5.4.2}$$

Now we turn to the task of making the construction of $\omega_1^*, \ldots, \omega_t^*$ explicit.

Claim 5.4.3. *There is a sequence of weighted fractional $\{F_s^*, E_s^*\}$-tilings $\omega_1^*, \ldots, \omega_t^*$ such that for all $1 \leq j \leq t$,*

(i) $A_j \subseteq A_{j-1}$ *and* $|A_j| \geq |A_{j-1}| - 5k^2$*;*

(ii) $(\omega_j^*)_{\min} \geq c$ *and*

(iii) $\phi(\omega_j^*) \leq \phi(\omega_{j-1}^*) - \nu_1/n$*.*

Note that Lemma 5.3.10 follows immediately from Claim 5.4.3 as $\phi(\omega_t^*) \leq \phi(H) - \nu_1 t/n \leq (1-\nu)\phi(H)$.

Proof of Claim 5.4.3. Suppose that, for some $0 \leq j < t$, we have already defined $\omega_0^*, \omega_1^*, \ldots, \omega_j^*$ satisfying (i)–(iii). We write $U_i = U(\omega_i^*)$, for each $i \in \{0, \ldots, j\}$.

Observe that $U_0 = U(\mathcal{T})$ by the choice of ω_0^* and Proposition 5.3.6(iv). Note that (i) implies that $|A_j| \geq |A_0| - 5k^2 j \geq n - 5k^2\nu_2\phi(H)n \geq n(1-\alpha\gamma/40)$, and therefore

$$n - |A_j| \leq \frac{\alpha\gamma}{40}n. \tag{5.4.3}$$

Now our task is to construct ω_{j+1}^*. We will use the following shorthand notation. For all $J \in \mathcal{F}_s^*(H) \cup E(H)$, if we have already defined ω_{j+1}^*, then we let

$$\partial(J) = \omega_{j+1}^*(J) - \omega_j^*(J).$$

Alternatively, we will define ω_{j+1}^* by setting the values $\partial(J)$ for every $J \in \mathcal{F}_s^*(H) \cup E(H)$.

The proof splits into two cases depending on the size of U_0. The idea behind each case is as follows. If U_0 is "large" then we will find disjoint $(k-1)$-sets in U_0 with large codegree. We can extend those $(k-1)$-sets into edges which receive weight zero in ω_j^* by construction. Then we will build ω_{j+1}^* by increasing the weight in those edges and decreasing the weight in some edges or copies of F_s^*, if necessary. Otherwise, if U_0 is small, then we find some $X \in E(H)$ with positive weight in ω_j^*, and we will find a copy F of F_s^* with X as "core vertices" by carefully choosing some "pendant vertices". Then we will construct ω_{j+1}^* from ω_j^* by increasing the weight in J, and decreasing the weight on X and some other edges and copies of F_s^* (which intersect with the "pendant vertices" of J).

Case 1: $|U_0| \geq 3\alpha n/4$**.** Note that $(U_0 \smallsetminus U_j) \cap A_j = \varnothing$, which implies that $A_j \cap U_0 \subseteq A_j \cap U_j$. Then, by (5.4.3), $|A_j \cap U_j| \geq |A_j \cap U_0| \geq |U_0| - \alpha\gamma n/40 \geq 3\alpha n/4 - \alpha\gamma n/40 \geq \alpha n/2$. Thus, together with $1/n \ll \alpha$ we get

$$\binom{|U_j \cap A_j|}{k-1} \geq \binom{\alpha n/2}{k-1} \geq \frac{\alpha^{k-1}}{2^k}\binom{n}{k-1} \geq \theta\binom{n}{k-1} + k^2\binom{n}{k-2},$$

as $\theta, 1/n \ll \alpha, 1/k$. Since H is strongly $(1/2+1/(2s)+\gamma,\theta)$-dense, we can (greedily) find k disjoint $(k-1)$-sets $W_1,\dots,W_k$ of $U_j \cap A_j$ such that $\deg(W_i) \geq (1/2+1/(2s)+\gamma)(n-k+1)$ for all $1 \leq i \leq k$. Define $N_i = N(W_i) \cap A_j$. Then

$$|N_i| \geq \left(\frac{1}{2}+\frac{1}{2s}+\gamma\right)(n-k+1)-(n-|A_j|) \overset{(5.4.3)}{\geq} \left(\frac{1}{2}+\frac{1}{2s}+\frac{\gamma}{2}\right)n. \tag{5.4.4}$$

Suppose that for some $1 \leq i \leq k$, there exists $x \in N_i \cap U_j$. Then $e = \{x\} \cup W_i \in E(H)$, so we can define $\omega^*_{j+1}(e) = 1$ and $\omega^*_{j+1}(J) = \omega^*_j(J)$ for all $J \in (\mathcal{F}^*_s(H) \cup E(H)) \setminus \{e\}$. In this case, $|A_{j+1}| = |A_j| - k \geq |A_j| - 5k^2$, $(\omega^*_{j+1})_{\min} = (\omega^*_j)_{\min} \geq c$ and $\phi(\omega^*_{j+1}) = \phi(\omega^*_j) - 3s/(5n) \leq \phi(\omega^*_j) - \nu_1/n$ so we are done. Thus, we may assume that

$$\bigcup_{1\leq i\leq k} N_i \subseteq A_j \setminus U_j. \tag{5.4.5}$$

For all $F^* \in \mathcal{F}^*_s(H)$, define

$$d_{F^*} = \sum_{i=1}^{k} |N_i \cap c(F^*)|.$$

Similarly, for all $e \in E(H)$, define

$$d_e = \sum_{i=1}^{k} |N_i \cap e|.$$

Case 1.1: there exists $F^* \in \mathcal{F}^*_s(H)$ with $\omega^*_j(F^*) > 0$ and $d_{F^*} \geq k+1$. In this case, we will construct ω^*_{j+1} from ω^*_j by replacing the weight of one copy of F^*_s and increasing the weight of two edges. This will turn out to be an improved fractional weighting by the definition of ϕ (here, the choice of the constant 3/5 in the function plays a crucial role).

By the case assumption, there exist distinct $i, i' \in \{1,\dots,k\}$ and distinct $x \in N_i \cap c(F^*)$, $x' \in N_{i'} \cap c(F^*)$ such that both $e_1 = W_i \cup \{x\}$ and $e_2 = W_{i'} \cup \{x'\}$ are edges in H. Note that since $x \in A_j$, by (5.4.2) we have $\omega^*_j(F^*) = \omega^*_0(F^*) \geq M_s^{-k}$.

Also, since $x, x' \in c(F^*)$, $\alpha_{F^*}(x), \alpha_{F^*}(x') \geq m_s$. Define ω^*_{j+1} to be such that

$$\partial(J) = \begin{cases} m_s M_s^{-(k+1)} & \text{if } J \in \{e_1, e_2\}, \\ -M_s^{-k} & \text{if } J = F^*, \\ 0 & \text{otherwise.} \end{cases}$$

Then ω^*_{j+1} is a weighted fractional $\{F^*_s, E^*_s\}$-tiling. First, note that $|A_{j+1}| = |A_j| - (3k + \ell - 2) \geq |A_j| - 5k^2$. Secondly, using (5.4.2) we have that $\omega^*_j(F^*)$ is either 0 or at least c. Thus we obtain

$$\begin{aligned} (\omega^*_{j+1})_{\min} &\geq \min\{(\omega^*_j)_{\min}, M_s \omega^*_{j+1}(e_1), c\} \\ &\geq \min\{c, m_s M_s^{-k}, c\} \geq c. \end{aligned}$$

Finally,

$$\phi(\omega^*_j) - \phi(\omega^*_{j+1}) = \frac{s}{n}\left(\partial(F^*) + \frac{3}{5}(\partial(e_1) + \partial(e_2))\right) = \frac{s}{nM_s^k}\left(\frac{6m_s}{5M_s} - 1\right).$$

Using (5.3.2), $s \geq 5k^2$, $\ell \leq k - 1$ and $k \geq 3$, we can bound m_s/M_s below by

$$\frac{m_s}{M_s} \geq \frac{M_s - 1}{M_s} \geq \frac{s - \ell - k}{s - \ell} = 1 - \frac{k}{s - \ell} \geq 1 - \frac{k}{5k^2 - k + 1} \geq \frac{40}{43}.$$

We deduce $\phi(\omega^*_j) - \phi(\omega^*_{j+1}) \geq 5s/(43M_s^k n) \geq \nu_1/n$, so we are done in this subcase.

Case 1.2: there exists $e \in E(H)$ with $\omega^*_j(e) > 0$ and $d_e \geq k + 1$. We will use a similar argument as the one used in Case 1.1, this time exchanging (the weight of) one edge by two edges. If this holds, then there exist distinct $i, i' \in \{1, \dots, k\}$ and distinct $x, x' \in e$ such that both $e_1 = W_i \cup \{x\}$ and $e_2 = W_{i'} \cup \{x'\}$ are edges in H. Since $x \in A_j$, Proposition 5.3.6(vi) and (5.4.2) implies that $\omega^*_j(e) = M_s^{-k}$.

Define ω^*_{j+1} to be such that

$$\partial(J) = \begin{cases} -M_s^{-k} & \text{if } J = e, \\ M_s^{-k} & \text{if } J \in \{e_1, e_2\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then ω^*_{j+1} is a weighted fractional $\{F^*_s, E^*_s\}$-tiling with $|A_{j+1}| = |A_j| - (3k-2) \geq |A_j| - 5k^2$. Note $\omega^*_{j+1}(e) = 0$ and $\omega^*_{j+1}(e_i) > \omega^*_j(e_i)$ for $i \in [2]$, so we have $(\omega^*_{j+1})_{\min} \geq (\omega^*_j)_{\min} \geq c$. Note that

$$\phi(\omega^*_j) - \phi(\omega^*_{j+1}) = \frac{3s}{5n}\left(\partial(e_1) + \partial(e_2) + \partial(e)\right) = \frac{3s}{5M_s^k n} \geq \frac{\nu_1}{n},$$

so this finishes the proof of this subcase.

Case 1.3: Both Case 1.1 and Case 1.2 do not hold. Thus $d_{F^*} \leq k$ for all $F^* \in \mathcal{F}^*_s(H)$ with $\omega^*_j(F^*) > 0$, and $d_e \leq k$ for all $e \in E(H)$ with $\omega^*_j(e) > 0$. Recall that $\alpha_{F^*}(v) \leq M_s$ if $v \in c(F^*)$ and $\alpha_{F^*}(v) = 1$ if $v \in p(F^*)$. Thus, for all $F^* \in \mathcal{F}^*_s(H)$ with $\omega^*_j(F^*) > 0$, we have

$$\begin{aligned} \sum_{i=1}^k \sum_{x \in N_i} \alpha_{F^*}(x) &\leq \sum_{i=1}^k \left(M_s |N_i \cap c(F^*)| + |N_i \cap p(F^*)|\right) \\ &= M_s d_{F^*} + \sum_{i=1}^k |N_i \cap p(F^*)| \leq k(M_s + \ell) \leq s + k^2. \end{aligned}$$

Therefore,

$$\sum_{F^* \in \mathcal{F}^*_s} \sum_{i=1}^k \sum_{x \in N_i} \omega^*_0(F^*) \alpha_{F^*}(x) \leq (s + k^2) \sum_{F^* \in \mathcal{F}^*_s(H)} w^*_0(F^*). \tag{5.4.6}$$

Similarly, for $e \in E(H)$ with $\omega^*_j(e) > 0$, we obtain

$$\sum_{i=1}^k \sum_{x \in N_i} \alpha_e(x) = \sum_{i=1}^k M_s |e \cap N_i| = M_s d_e \leq k M_s.$$

Hence,

$$\sum_{e\in E(H)}\sum_{i=1}^{k}\sum_{x\in N_i}\omega_0^*(e)\alpha_e(x) \le kM_s \sum_{e\in E(H)} w_0^*(e). \tag{5.4.7}$$

Combining everything, we deduce that

$$\begin{aligned}
\sum_{i=1}^{k}|N_i| = \sum_{i=1}^{k}\sum_{x\in N_i} 1 &\overset{(5.4.5),(5.4.1)}{=} \sum_{i=1}^{k}\sum_{x\in N_i}\omega_0^*(x) \\
&= \sum_{i=1}^{k}\sum_{x\in N_i}\sum_{J\in\mathcal{F}_s^*(H)\cup E(H)}\omega_0^*(J)\alpha_J(x) \\
&= \sum_{J\in\mathcal{F}_s^*(H)\cup E(H)}\sum_{i=1}^{k}\sum_{x\in N_i}\omega_0^*(J)\alpha_J(x) \\
&\overset{(5.4.6),(5.4.7)}{\le} (s+k^2)\sum_{F^*\in\mathcal{F}_s^*(H)} w_0^*(F^*) + kM_s\sum_{e\in E(H)} w_0^*(e) \\
&\overset{\text{Prop. 5.3.5(i)}}{\le} n + k^2\sum_{F^*\in\mathcal{F}_s^*(H)} w_0^*(F^*) \le n + \frac{k^2}{s}n \le \frac{6n}{5},
\end{aligned}$$

where the last inequality uses $s \ge 5k^2$. This contradicts (5.4.4) and finishes the proof of Case 1.

Case 2: $|U_0| < 3\alpha n/4$. Write $\mathcal{F}$, $\mathcal{E}$ for $\mathcal{F}_{\mathcal{T}}$, $\mathcal{E}_{\mathcal{T}}$, respectively; that is, $\mathcal{F}$ is the set of copies of F_s in the integer tiling $\mathcal{T}$ chosen at the beginning of the proof; similarly, $\mathcal{E}$ is the set of copies of E_s in $\mathcal{T}$. Note that $n = s|\mathcal{F}| + kM_s|\mathcal{E}| + |U_0|$. Hence,

$$\alpha \le \phi(\mathcal{T}) \le 1 - \frac{s}{n}|\mathcal{F}| \le \frac{1}{n}\left(kM_s|\mathcal{E}| + |U_0|\right) \le \frac{kM_s|\mathcal{E}|}{n} + \frac{3\alpha}{4}.$$

Using that $s \ge 5k^2$, $k \ge 3$, $1/n \ll \alpha$, $\gamma \le 1$ and (5.4.3), we have

$$|\mathcal{E}| \ge \frac{\alpha n}{4kM_s} \ge \frac{\alpha\gamma n}{40} + 1 \ge n - |A_j| + 1.$$

Hence there exists a copy of $E_s \in \mathcal{E}$ whose vertices are completely contained in A_j. Let $X = \{x_1, \dots, x_k\} \in E(H)$ be any edge belonging to that copy of E_s. By the choice of X and Proposition 5.3.6(vi), it holds that $X \subseteq A_j$ and

$$w_j^*(X) = w_0^*(X) = M_s^{-k}.$$

We would like to use Lemma 5.4.2 to find copies F of F_s^* with $c(F) = X$, and decrease the weight of X to be able to increase the weight of an appropriate copy of F_s^*. Recall that $S(\omega_j^*)$ is the set of saturated vertices with respect to ω_j^*. We write $S_j = S(\omega_j^*)$ and let $S' = S_j \cup (V(H) \setminus A_j)$. Proposition 5.3.5(ii) and (5.4.3) together imply that $|S'| \leq (\ell/s + \gamma/40)n$.

For all $1 \leq i \leq k$, let $N_i = N_H(X \setminus \{x_i\})$. We may assume (by relabelling) that either $|N_1 \cap N_2| < (\ell/s + 2\gamma/3)n$ or $|N_i \cap N_j| \geq (\ell/s + 2\gamma/3)$ for all $1 \leq i, j \leq k$.

Case 2.1: $(N_1 \cap N_2) \setminus S' \neq \varnothing$. In this case, select $y \in (N_1 \cap N_2) \setminus S'$ and apply Lemma 5.4.2 with S' and y playing the roles of S and y_0, respectively. We obtain a copy F_1 of F_s^* such that $c(F_1) = X$ and $p(F_1) \cap S' = \varnothing$. Then, $p(F_1) \subseteq A_j \setminus S_j$. Let $P_0 = p(F_1) \setminus U_j$. For each $p \in p(F_1) \cap U_j$, by (5.4.1), $\omega_j^*(p) = 0$. For every $p \in P_0$, by the definitions of A_j and U_j, there exists $J_p \in \mathcal{T}_0^+$ such that $p \in V(J_p)$, and since $p \notin S_j$ we also can choose J_p such that $\alpha_{J_p}(p) \geq m_s$. (The J_p might coincide for different $p \in P_0$.) Define ω_{j+1}^* to be such that

$$\partial(J) = \begin{cases} M_s^{-k} & \text{if } J = F_1, \\ -M_s^{-k} & \text{if } J = X, \\ -M_s^{-k}/m_s & \text{if } J = J_p \text{ for some } p \in P_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, ω_{j+1}^* is a weighted fractional $\{F_s^*, E_s^*\}$-tiling. First, note that $|A_{j+1}| \geq |A_j| - (|V(F_1)| + \sum_{p \in P_0} |V(J_p)|) \geq |A_j| - (2k + 2k^2) \geq |A_j| - 5k^2$. Secondly, (5.4.2) implies that $\omega_{j+1}^*(X) = 0$ and $\omega_{j+1}^*(F_1) \geq c$, and, moreover, for all $p \in P_0$, $\omega_{j+1}^*(J_p) \geq M_s^{-k}(1 - 1/m_s) \geq M_s^{-k-1} \geq c$. Thus, $(\omega_{j+1}^*)_{\min} \geq c$. Finally, since $|P_0| \leq |p(F_1)| = \ell$, we have

$$\phi(\omega_j^*) - \phi(\omega_{j+1}^*) \geq \frac{s}{n}\left(\partial(F_1) + \frac{3}{5}\partial(X) + \sum_{p \in P_0} \partial(J_p)\right) \geq \frac{s}{nM_s^k}\left(\frac{2}{5} - \frac{|P_0|}{m_s}\right)$$
$$\geq \frac{s}{nM_s^k}\left(\frac{2}{5} - \frac{\ell}{m_s}\right).$$

By (5.3.2), $\ell \leq k-1$ and $s \geq 5k^2$, we get

$$\frac{\ell}{m_s} \leq \frac{k-1}{M_s - 1} \leq \frac{k}{M_s} \leq \frac{k^2}{s-\ell} \leq \frac{k^2}{5k^2 - k + 1} \leq \frac{1}{4},$$

where the last inequality holds for every $k \geq 3$. Thus $\phi(\omega_j^*) - \phi(\omega_{j+1}^*) \geq 3s/(20nM_s^k) \geq \nu_1/n$ and we are done.

Case 2.2: $N_1 \cap N_2 \subseteq S'$**.** Since H is strongly $(1/2 + 1/(2s) + \gamma, \theta)$-dense and $1/n \ll \gamma, 1/k$, we deduce $|N_1 \cap N_2| \geq (1/s + \gamma)n$. Using $N_1 \cap N_2 \subseteq S'$ and (5.4.3), we have $|N_1 \cap N_2 \cap S_j \cap A_j| \geq (1/s + \gamma/2)n$. By Proposition 5.3.5(iii), there exists $F_2 \in \mathcal{F}_s^*(H) \cap \mathcal{T}_0^+$ and $|p(F_2) \cap N_1 \cap N_2 \cap S_j \cap A_j| \geq 2$. Let y_1', y_1'' be two distinct vertices in $p(F_2) \cap N_1 \cap N_2 \cap S_j \cap A_j$. We claim that

there exists $F_2' \in \mathcal{F}_s^*(H)$ such that $p(F_2') \setminus p(F_2) \subseteq A_j \setminus (S_j \cup X)$, their core vertices satisfy $c(F_2') = c(F_2)$, and $\{y_1', y_1''\} \setminus p(F_2') \neq \varnothing$. (5.4.8)

To see where we are heading, if we have found such F_2', then our aim will be to define ω_{j+1}^* by decreasing the weight of F_2 and X, which will allow us then to increase the weight of F_2' and a copy F_1' of F_s^* such that $c(F_1') = X$ and $\{y_1', y_1''\} \cap p(F_1') \neq \varnothing$.

Let us check (5.4.8) holds. Let $Z = c(F_2) = \{z_1, \ldots, z_k\}$ and for every $1 \leq i \leq k$ let $Z_i = N_H(Z \setminus \{z_i\})$. Since $y_1' \in p(F_2)$, without loss of generality (by relabelling) we may assume that $y_1' \in Z_1 \cap Z_2$. Suppose first that $(Z_1 \cap Z_2) \setminus (S' \cup X \cup V(F_2))$ is non-empty. Select any $y_1''' \in (Z_1 \cap Z_2) \setminus (S' \cup X \cup V(F_2))$. Thus there exists $F_2' \in \mathcal{F}_s^*(H)$ such that $c(F_2') = Z$, $p(F_2') = (p(F_2) \setminus \{y_1'\}) \cup \{y_1'''\}$, $p(F_2') \setminus p(F_2) = \{y_1'''\} \subseteq A_j \setminus (S_j \cup X)$ and $y_1' \in \{y_1', y_1''\} \setminus p(F_2')$, as desired. Hence, we may assume $Z_1 \cap Z_2 \subseteq S' \cup X \cup V(F_2)$. This implies that $|Z_1 \cap Z_2| \le |S' \cup X \cup V(F_2)| \le (\ell/s + \gamma/40)n + |X| + |V(F_2)| < (\ell/s + 2\gamma/3)n$. Apply Lemma 5.4.2 (with Z, Z_i, $S' \cup X \cup V(F_2)$, y_1' playing the roles of X, N_i, S and y_0, respectively) to obtain $F_2' \in \mathcal{F}_s^*$ such that $c(F_2') = Z$ and $p(F_2') \cap (S' \cup X \cup V(F_2) \smallsetminus \{y_1'\}) = \varnothing$. It is easily checked that F_2' satisfies (5.4.8).

Now take such an F_2' and assume (after relabelling, if necessary) that $y_1' \notin p(F_2')$. Apply Lemma 5.4.2 (with $X, N_i, S' \cup V(F_2'), y_1'$ playing the roles of X, N_i, S and y_0, respectively) to obtain F_1' such that $c(F_1') = X$ and $p(F_1') \cap (S' \smallsetminus \{y_1'\}) = \varnothing$.

Let $P' = (p(F_1') \smallsetminus \{y_1'\}) \cup (p(F_2') \smallsetminus p(F_2))$ and observe that $P' \subseteq A_j \smallsetminus S_j$. Let $P_0' = P' \smallsetminus U_j$. Arguing as in the previous case we see that for every $p \in P' \cap U_j$, $\omega_j^*(p) = 0$, and for every $p \in P_0'$ there exists $J_p \in \mathcal{T}_0^+$ such that $p \in V(J_p)$ and $\alpha_{J_p}(p) \ge m_s$.

Let ω_{j+1}^* be such that

$$\partial(J) = \begin{cases} M_s^{-k} & \text{if } J = F_1', \\ M_s^{-(k+1)} m_s & \text{if } J = F_2', \\ -M_s^{-k} & \text{if } J \in \{X, F_2\}, \\ -M_s^{-k}/m_s & \text{if } J = J_p \text{ for some } p \in P_0', \\ 0 & \text{otherwise.} \end{cases}$$

Since $p(F_1') \cup p(F_2') \subseteq P' \cup p(F_2)$, the decrease of weight in F_2 and the J_p implies that the vertices in $p(F_1') \cup p(F_2')$ get weight at most 1 under ω_{j+1}^*. Using that, it is not difficult to check that ω_{j+1}^* is indeed a weighted fractional $\{F_s^*, E_s^*\}$-tiling.

Note that $A_j \smallsetminus A_{j+1} \subseteq V(F_1') \cup V(F_2) \cup V(F_2') \cup \left(\bigcup_{p \in P_0'} V(J_p)\right)$ and $|P_0'| \le |p(F_1')| + |p(F_2')| = 2\ell$. Using that $\ell \le k-1$, we deduce $|A_{j+1}| \ge |A_j| - 3(k+\ell) - |P_0'|(k+\ell) \ge |A_j| - (3+2\ell)(k+\ell) \ge |A_j| - 5k^2$. Similarly as in the previous case, we deduce from (5.4.2) that $(\omega_{j+1}^*)_{\min} \ge c$.

Using that $|P_0'| \le 2\ell$, we deduce

$$\begin{aligned}\phi(\omega_j^*) - \phi(\omega_{j+1}^*) &\ge \frac{s}{n}\left(\partial(F_1') + \partial(F_2') + \partial(F_2) + \frac{3}{5}\partial(X) + \sum_{p\in P_0'} \partial(J_p)\right)\\ &= \frac{s}{nM_s^k}\left(1 + \frac{m_s}{M_s} - 1 - \frac{3}{5} - \frac{|P_0'|}{m_s}\right) \ge \frac{s}{nM_s^k}\left(\frac{m_s}{M_s} - \frac{3}{5} - \frac{2\ell}{m_s}\right).\end{aligned}$$

From (5.3.2), $s \ge 5k^2$ and $\ell \le k-1$, we deduce

$$\begin{aligned}\frac{m_s}{M_s} - \frac{3}{5} - \frac{2\ell}{m_s} &\ge \frac{2}{5} - \frac{1}{M_s} - \frac{2\ell}{m_s} \ge \frac{2}{5} - \frac{1+2\ell}{m_s} \ge \frac{2}{5} - \frac{k(1+2\ell)}{s-\ell-k}\\ &\ge \frac{2}{5} - \frac{2k^2-k}{5k^2-2k+1} = \frac{k+2}{25k^2-10k+5} \ge \frac{k+2}{25k^2} \ge \frac{1}{25k}.\end{aligned}$$

Thus we get $\phi(\omega_j^*) - \phi(\omega_{j+1}^*) \ge s/(25M_s^k kn) \ge \nu_1/n$ and we are done. This finishes the proof of Case 2.2 and of Claim 5.4.3. □

This concludes the proof of Lemma 5.3.10. ■

5.5 Tiling thresholds for tight cycles

Now we finalise our application of the absorbing method to prove our result on tiling thresholds, Theorem 1.3.9. We do so by applying the "absorbing lemma" (Lemma 5.2.4) in conjunction with the "almost perfect tiling lemma" (Lemma 5.3.1).

Proof of Theorem 1.3.9. Choose $1/n \ll \alpha \ll \mu \ll \gamma, 1/k, 1/s$. By Lemma 5.2.4 there exists $U \subseteq V(H)$ of size $|U| \le \mu n$ with $|U| \equiv 0 \bmod s$ such that there exists a perfect C_s^k-tiling in $H[U \cup W]$ for all $W \subseteq V(H) \setminus U$ of size $|W| \le \alpha n$ with $|W| \equiv 0 \bmod s$.

Define $H' = H \setminus U$. Then $\delta_{k-1}(H') \ge \delta_{k-1}(H) - |U| \ge (1/2 + 1/(2s) + \gamma/2)|V(H')|$. An application of Lemma 5.3.1 (with $\gamma/2, |V(H')|$ playing the roles of γ, n, respectively, and noting the hierarchy of constants in both lemmas are consistent) implies that there exists a C_s^k-tiling $\mathcal{T}'$ in H' covering at least $(1-\alpha)|V(H')|$ vertices. Let W be the set of uncovered vertices by $\mathcal{T}'$ in H'. Then $|W| \le \alpha n$ and

$|W| \equiv 0 \bmod s$. By the absorbing property of U, there exists a perfect C_s^k-tiling $\mathcal{T}''$ in $H[U \cup W]$. Then $\mathcal{T}' \cup \mathcal{T}''$ is a perfect C_s^k-tiling in H. ■

— 6 —

Dense monochromatic infinite paths

In this chapter we prove Theorem 1.4.1, which finds “dense” infinite monochromatic paths in complete 2-edge-coloured graphs over the natural numbers.

We start in Section 6.1 by setting the notation to be used during this chapter, and giving a short sketch of the proof. In Section 6.2 we state our main lemma (Lemma 6.2.1) and use it to deduce Theorem 1.4.1. In Section 6.3 we collect some useful miscellaneous tools that will be used during the proof of Lemma 6.2.1. In Section 6.4 we describe in detail the main tool used in this chapter, an algorithm (Algorithm 1). We also deduce crucial properties of its output. Finally, in Section 6.5 we use the outputs given by the algorithm to deduce Lemma 6.2.1.

6.1 Sketch of proof and notation

6.1.1 Sketch of proof

Our proof follows the strategy of Erdős and Galvin [22], where they reduce the problem of finding monochromatic paths to the problem of finding collections of monochromatic disjoint paths (“monochromatic path-forests”) satisfying certain conditions, which are then joined together to form an infinite path. Thus the problem is reduced to the search for monochromatic path-forests.

We tackle that problem by using an algorithm (Algorithm 1), which takes as its input a 2-edge-coloured complete graph on the positive integers, examines the vertices one by one in increasing order, and builds a sequence of increasing monochromatic path-forests, one for each colour. An analysis of the outputs of the algorithm will show that, in every colouring, one of these monochromatic path-forests must attain the desired density.

6.1.2 Notation

Given a graph G, we write $V(G)$ and $E(G)$ for its vertex and edge set, respectively, and let $e(G) \coloneqq |E(G)|$. Given $S \subseteq V(G)$, we write $G[S]$ for the subgraph of G induced by S. If $S, T \subseteq V(G)$ are disjoint, we write $G[S,T]$ for the bipartite graph with classes S and T consisting precisely of those edges in G with one endpoint in S and the other in T.

Let G be a 2-edge-coloured graph. Throughout this chapter, we assume its colours to be red and blue. For a vertex $x \in V(G)$ and a subset $S \subseteq V(G)$, we write the *red neighbourhood of* x *in* S for the set $N_G^R(x,S) \coloneqq \{y \in S : xy \text{ is coloured red}\}$, that is, the set of vertices in S connected to x with red edges. We define the *blue neighbourhood of* x *in* S analogously and we denote it by $N_G^B(x,S)$. For each $* \in \{R, B\}$, we also define $d_G^*(x,S) \coloneqq |N_G^*(x,S)|$ whenever $N_G^*(x,S)$ is finite, and $d_G^*(x,S) \coloneqq \infty$ otherwise.

For every $i \geq 0$, let $[i] \coloneqq \{1, \dots, i\}$ and $[i]_0 \coloneqq [i] \cup \{0\}$. If G is a graph and v is a vertex not in $V(G)$, define $G + \{v\}$ as the graph obtained by adding v to $V(G)$, and keeping the edges from G. If G is a graph and F is a set of edges (possibly joining vertices lying outside of $V(G)$), then define $G + F$ as the graph on the vertex set $V(G) \cup \left(\bigcup_{f \in F} f\right)$ whose edge set is $E(G) \cup F$.

6.2 Monochromatic path-forests

As mentioned before, we reduce the problem of finding monochromatic paths to the problem of finding collections of monochromatic disjoint paths satisfying certain conditions.

Consider a 2-edge-coloured $K_{\mathbb{N}}$. We say a vertex $x \in \mathbb{N}$ is *red* (or *blue*) if x has infinitely many red (or blue, respectively) neighbours in $K_{\mathbb{N}}$. Note that every vertex is red or blue, and also that it is possible for a vertex to be both red and blue. A 2-edge-colouring of $K_{\mathbb{N}}$ is *restricted* if there is no vertex that is both red and blue. We write R and B for the set of red and blue vertices of $K_{\mathbb{N}}$, respectively.

A *path-forest* is a collection of vertex-disjoint paths. Let $K_{\mathbb{N}}$ be a 2-edge-coloured graph. A path-forest F of $K_{\mathbb{N}}$ is said to be *red* if every edge of F is red, all endpoints of every path in F are red, and for every path P in F, its vertices $V(P)$ alternate between red and blue. Note that a red path-forest may contain isolated red vertices. A *blue path-forest* is defined similarly.

Our main lemma states that, given a restricted 2-edge-coloured $K_{\mathbb{N}}$, there exists a monochromatic path-forest F and an arbitrarily long interval $[t]$ such that $|V(F) \cap [t]|$ is large with respect to t. Recall that $(9+\sqrt{17})/16$ is approximately 0.82019.

Lemma 6.2.1. *Let $\varepsilon \in (0,1/2)$ and $k_0 \in \mathbb{N}$. For every restricted 2-edge-coloured $K_{\mathbb{N}}$, there exists an integer $t \geq k_0$ and red and blue path-forests F^R and F^B, respectively, of $K_{\mathbb{N}}$ such that*

$$\max\{|V(F^R) \cap [t]|, |V(F^B) \cap [t]|\} \geq ((9+\sqrt{17})/16 - \varepsilon)t.$$

We defer the proof of Lemma 6.2.1 to Section 6.5. We note that Lemma 6.2.1 implies the following corollary, which is valid for arbitrary 2-edge-colourings, not only restricted ones.

Corollary 6.2.2. *Let $\varepsilon \in (0,1/2)$ and $k_0 \in \mathbb{N}$. For every 2-edge-coloured $K_{\mathbb{N}}$, there exists an integer $t \geq k_0$ and red and blue path-forests F^R and F^B, respectively, such that*

$$\max\{|V(F^R) \cap [t]|, |V(F^B) \cap [t]|\} \geq ((9+\sqrt{17})/16 - \varepsilon)t.$$

Proof. Let W be the set of vertices which are simultaneously red and blue under the vertex-colouring of $K_{\mathbb{N}}$. Suppose first that $\mathbb{N} \setminus W$ is finite. Then for $t \geq k$ large enough, $|W \cap [t]| \geq (1-\varepsilon)t$. The vertices of $W \cap [t]$ form a monochromatic red path-forest F with $|V(F) \cap [t]| \geq (1-\varepsilon)t \geq ((9+\sqrt{17})/16-\varepsilon)t$, as desired.

Hence, we can suppose that $\mathbb{N} \setminus W$ is infinite. Suppose $\mathbb{N} \setminus W = \{v_1, v_2, \dots\}$ where $v_i < v_j$ for all $i < j$. Consider the induced subgraph of $K_{\mathbb{N}}$ on $\mathbb{N} \setminus W$, together with the inherited edge-colouring. Note that the induced edge-colouring in $\mathbb{N} \setminus W$ yields a vertex-colouring of $\mathbb{N} \setminus W$ which corresponds exactly to the restriction of the original vertex-colouring to $\mathbb{N} \setminus W$. In particular, the vertex-colouring in $\mathbb{N} \setminus W$ is restricted. Then Lemma 6.2.1 implies the existence of a (say) red path-forest F in $K_{\mathbb{N}}$ with $V(F) \cap W = \varnothing$ and $t \geq k$ such that $|V(F) \cap \{v_1, \dots, v_t\}| \geq ((9+\sqrt{17})/16-\varepsilon)t$. Then $F' := F \cup (W \cap [v_t])$ is a red path-forest in $K_{\mathbb{N}}$ with

$$\begin{aligned}|V(F') \cap [v_t]| = |W \cap [v_t]| + |V(F) \cap [v_t]| &\geq v_t - t + ((9+\sqrt{17})/16-\varepsilon)t \\ &\geq ((9+\sqrt{17})/16-\varepsilon)v_t,\end{aligned}$$

as desired. ∎

We use Corollary 6.2.2 now to deduce Theorem 1.4.1. The proof is based on the proof of [22, Theorem 3.5].

Proof of Theorem 1.4.1. Consider an arbitrary 2-edge-colouring of $K_{\mathbb{N}}$. Suppose that there exist two red vertices $x_1, x_2 \in \mathbb{N}$ and a finite subset S of $\mathbb{N}$ such that $K_{\mathbb{N}} \setminus S$ does not contain a red path between x_1 and x_2. For $i \in [2]$, let X_i be the set of vertices reachable from x_i using red paths in $\mathbb{N} \setminus S$. Let $X_3 = \mathbb{N} \setminus (X_1 \cup X_2 \cup S)$. Then X_1 and X_2 are infinite; X_1, X_2 and X_3 are pairwise disjoint and there are no red edges between any X_i, X_j for distinct $i, j \in [3]$. Thus there is an infinite blue path P on the vertex set $X_1 \cup X_2 \cup X_3 = \mathbb{N} \setminus S$. Since S is finite, $\overline{d}(P) = 1$, and thus we are done. An analogous argument is true if red is swapped with blue.

Hence, we can assume that

$$\text{for any two red (or blue) vertices } x_1, x_2 \text{ and any finite set } S \subseteq \mathbb{N} \smallsetminus \{x_1, x_2\}, \text{ there is a red (or blue, respectively) path joining } x_1 \text{ and } x_2 \text{ in } K_{\mathbb{N}} \smallsetminus S. \tag{6.2.1}$$

For all $i \in \mathbb{N}$, let $\varepsilon_i := 1/(2i)$. If the vertex 1 is red, set $P_1^R = (\{1\}, \varnothing)$ to be the red path with the vertex 1 and P_1^B to be empty. Otherwise, set P_1^R to be empty and $P_1^B = (\{1\}, \varnothing)$. Set $n_1 = 1$. Suppose that, for some $i \in \mathbb{N}$, we have already found an integer n_i and red and blue paths P_i^R and P_i^B, respectively, such that the endpoints of P_i^R are red, the endpoints of P_i^B are blue; and

$$\max\{|V(P_i^R) \cap [n_i]|, |V(P_i^B) \cap [n_i]|\} \geq ((9 + \sqrt{17})/16 - 2\varepsilon_i)n_i. \tag{6.2.2}$$

We construct n_{i+1}, P_{i+1}^R and P_{i+1}^B as follows. Let $r_i := \max\{V(P_i^R) \cup V(P_i^B) \cup \{n_i\}\}$ and $k_i := r_i/\varepsilon_{i+1} = 2(i+1)r_i$. Considering the induced subgraph of $K_{\mathbb{N}}$ on $\mathbb{N} \smallsetminus [r_i]$, Corollary 6.2.2 implies that there exists a monochromatic path-forest F_{i+1} and $t_i \geq k_i$ such that $|V(F_{i+1}) \cap \{r_i + 1, \ldots, r_i + t_i\}| \geq ((9 + \sqrt{17})/16 - \varepsilon_{i+1})t_i$. Let $n_{i+1} := r_i + t_i$. By the choice of k_i and since $t_i \geq k_i$,

$$\frac{t_i}{n_{i+1}} = \frac{t_i}{r_i + t_i} \geq \frac{k_i}{r_i + t_i} = \frac{r_i/\varepsilon_{i+1}}{r_i + r_i/\varepsilon_{i+1}} = \frac{1}{\varepsilon_{i+1} + 1} = 1 - \frac{\varepsilon_{i+1}}{1 + \varepsilon_{i+1}} \geq 1 - \varepsilon_{i+1},$$

so we deduce

$$|V(F_{i+1}) \cap [n_{i+1}]| \geq ((9 + \sqrt{17})/16 - \varepsilon_{i+1})t_i \geq ((9 + \sqrt{17})/16 - 2\varepsilon_{i+1})n_{i+1}.$$

Suppose F_{i+1} is red (if not, interchange the colours in what follows). Let $P_{i+1}^B := P_i^B$. Apply (6.2.1) repeatedly to join the endpoints of the paths in $P_i^R \cup F_i$ (while avoiding what is constructed so far) and obtain a red path P_{i+1}^R containing P_i^R and F_i with red vertices as endpoints.

By construction, we have $n_{i+1} > n_i$ and (6.2.2) holds for all $i \geq 1$. Without loss of generality, we may assume that $|V(P_i^R) \cap [n_i]| \geq ((9+\sqrt{17})/16 - 2\varepsilon_i)n_i$ for infinitely many values of i. Let $P := \bigcup_{i\geq 1} P_i^R$. Therefore, P is a monochromatic path and $\overline{d}(P) \geq (9+\sqrt{17})/16$. ■

6.3 Auxiliary results

In this section, we first consider two ways of extending a path forest.

Proposition 6.3.1. *Let G be a graph. Let $F \subseteq G$ be a path-forest and let $J \subseteq V(F)$ be such that every $j \in J$ has degree at most one in F. Let $x \in V(G) \setminus V(F)$ be such that $d_G(x, J) \geq 3$ and $j_1 \in N_G(x, J)$. Then there exists $j_2 \in J$ such that $F + \{xj_1, xj_2\}$ is a path-forest.*

Proof. Since $d_G(x, J) \geq 3$, there exist at least two neighbours of x in J, j_2 and j_3 say, which are distinct from each other and from j_1. In particular, one of them, say j_2, is not joined to j_1 via a path in F. This implies that $F + \{xj_1, xj_2\}$ is a path-forest, as required. ■

Proposition 6.3.2. *Let G be a graph and $F \subseteq G$ a path-forest. Let $Y \subseteq V(G) \setminus V(F)$ and $X \subseteq V(F)$. Suppose that*

(i) $\sum_{x\in X}(2 - d_F(x)) \geq 2|Y|$, *and*

(ii) *for every $x \in X$, $d_G(x, Y) \geq |Y| - 2$.*

Then there exists a path-forest $F' \subseteq G[X, Y]$ such that every path in F' has both endpoints in X, $F + F'$ is a path-forest, and $|V(F') \cap Y| \geq |Y| - 4$.

Proof. Without loss of generality, we may assume that $d_F(x) < 2$ for all $x \in X$. We proceed by induction on $|Y|$. It is trivial if $|Y| \leq 4$ (by setting F' to be empty). So we may assume that $|Y| \geq 5$. Note that $|X| \geq 5$ by (i). Let $x_1, x_2 \in X$ be such that x_1 and x_2 are not connected in F. By (ii) and $|Y| \geq 5$, there exists $y \in Y \cap N_G(x_1) \cap N_G(x_2)$. Set $F_1 := F + \{x_1y, x_2y\}$ and $Y' := Y \setminus \{y\}$. It is easy to check that F_1, X, Y' also satisfy the corresponding conditions (i) and (ii). Therefore, by our induction hypothesis, the proposition holds. ■

The next lemma is a useful statement about difference inequalities. We include its proof for completeness.

Lemma 6.3.3. *Let $\tau_1, \tau_2 > 0$, $c_0 \geq 0$ be given and let $s_0, s_1, \dots$ be a strictly increasing sequence of non-negative integers. Suppose there exists n_0 such that for every $n \geq n_0$,*

$$s_{n+1} \leq \tau_1 s_n - \tau_2 s_{n-1} + c_0. \tag{6.3.1}$$

Then $\tau_1^2 \geq 4\tau_2$.

Proof. Suppose $\tau_1^2 < 4\tau_2$. Choose $\delta \in (0,1)$ sufficiently small such that $\tau_1^2 < 4\tau_2(1-\delta)$ and let $\rho_1 := \tau_1/(1-\delta)$ and $\rho_2 := \tau_2/(1-\delta)$. Since $\{s_n\}_{n \in \mathbb{N}}$ is a strictly increasing sequence of non-negative integers, there exists $n_1 \geq n_0$ such that

$$\delta s_n \geq c_0 \text{ for every } n \geq n_1.$$

In particular, for $n \geq n_1$, $\delta s_{n+1} \geq c_0$ and together with (6.3.1) we deduce $s_{n+1} \leq \tau_1 s_n - \tau_2 s_{n-1} + \delta s_{n+1}$. Rearranging, we get that for every $n \geq n_1$,

$$s_{n+1} \leq \rho_1 s_n - \rho_2 s_{n-1}. \tag{6.3.2}$$

Consider the function $f : (-\infty, \rho_1) \to \mathbb{R}$ given by $f(x) = \rho_2/(\rho_1 - x)$. It is well-known that f is continuous. We claim that $x < f(x)$ for all $x < \rho_1$. Indeed, using the quadratic formula together with the condition $\rho_1^2 < 4\rho_2$, it is straightforward to deduce that the quadratic polynomial $x^2 - \rho_1 x + \rho_2$ has no real roots. Thus, for all real x it holds that $0 < x^2 - \rho_1 x + \rho_2 = x(x - \rho_1) + \rho_2$. In particular, if $x < \rho_1$ we can rearrange the last expression to get $x < f(x)$, as desired.

For every $n \geq n_1$, let $\beta_n := s_{n+1}/s_n$. From (6.3.2), for every $n \geq n_1$,

$$1 < \beta_n < \rho_1.$$

Using (6.3.2) it also follows that $\rho_1 s_n - \rho_2 s_{n-1} \geq \beta_n s_n$, which can be rearranged to get

$$\beta_{n-1} = \frac{s_n}{s_{n-1}} \geq \frac{\rho_2}{\rho_1 - \beta_n} = f(\beta_n) > \beta_n.$$

Since β_n is monotone decreasing and bounded, it converges to a limit $\beta \in [1, \rho_1)$. Moreover, the sequence $f(\beta_n)$ converges to the same limit. The continuity of f implies that $\beta = f(\beta) > \beta$, a contradiction. ∎

6.4 The path-forests algorithm

In this section we describe our main tool to prove Lemma 6.2.1, Algorithm 1. The algorithm will take as an input a red-blue edge-colouring of $K_{\mathbb{N}}$, and will output two sequences of monochromatic path-forests, $\{F_t^R\}_{t\geq 0}$ and $\{F_t^B\}_{t\geq 0}$, formed of red and blue path-forests, respectively. Before the t-th round of the algorithm, the algorithm will have already constructed the first t path-forests in both sequences, $\{F_i^R\}_{0\leq i\leq t-1}$ and $\{F_i^B\}_{0\leq i\leq t-1}$. During the t-th round, the algorithm will examine the vertex t, and by the end of the round it will output the new path-forests F_t^R and F_t^B. The sequences will be increasing, meaning that $F_{t-1}^R \subseteq F_t^R$ and $F_{t-1}^B \subseteq F_t^B$ for all $t \in \mathbb{N}$.

Unfortunately the algorithm and its analysis are quite involved, so we will give outlines of its description in increasing levels of detail. In Section 6.4.1 we give a rough outline of the algorithm, and we explain the intuition behind its design. Next, in Section 6.4.2 we describe the algorithm in detail, both by giving a formal description (Algorithm 1) and by explaining with detail the "meaning" of every step. Finally, in Section 6.4.3 we verify formally that the output of Algorithm 1 satisfies some useful properties.

6.4.1 Rough outline

As outlined above, the algorithm will output two sequences of monochromatic path-forests, $\{F_t^R\}_{t\geq 0}$ and $\{F_t^B\}_{t\geq 0}$, which are respectively red and blue. The role of the red and blue colours in the algorithm will be symmetric with respect to each other.

The first red path-forest F_0^R will contain every red vertex in $\mathbb{N}$ and no edges. Given F_{t-1}^R, we will build the next red path-forest F_t^R from F_{t-1}^R only by adding red edges which alternate between red and blue vertices. By doing this, the path-forests $\{F_i^R\}_{i\geq 0}$ form an increasing sequence of subgraphs, meaning that for every $i \leq j$, F_i^R will be a subgraph of F_j^R. The blue path-forests will evolve in a similar way.

Our algorithm is based on the following simple idea. For now, suppose that $t \in \mathbb{N}$ is a blue vertex and we have constructed red and blue path-forests F_{t-1}^R and F_{t-1}^B, respectively. Since t is blue, by construction we will have $t \in V(F_t^B)$ at the end of the round.

We would like to add t to the red path-forest F_{t-1}^R as well. That means, we should add the red edges tj_1, tj_2 to F_{t-1}^R, for some two different red vertices j_1 and j_2. We will only consider red vertices j_1, j_2 which satisfy either $j_1, j_2 > t$ or $j_1, j_2 < t$. The edges tj_1, tj_2 in the first case will be referred to as *forward edges*, in the second case we will call them *backward edges*. We remark that the red path-forest F_t^R (or the blue path-forest F_t^B, respectively) will contain all the red (or blue, respectively) vertices contained in $[t]$, but it might be possible that some vertices are never included in the path-forest of the opposite colour.

The algorithm will try to include the blue vertex t in the red path-forest F_{t-1}^R as follows. As a first step, we try to include t in the red path-forest by using red forward edges. That is, the algorithm will check if there exist red forward edges which allow t to be included in the red path forest, and will use them straight away (see Figure 6.1). If this fails, we will check the existence of red backward edges. Thus we would like to check the existence of previous "useful" red vertices $t' < t$ which connect to t with red edges. If they exist we could, in principle, use them to include t in the red path-forest.

The crucial observation is that the failure of t to be included in the red-path

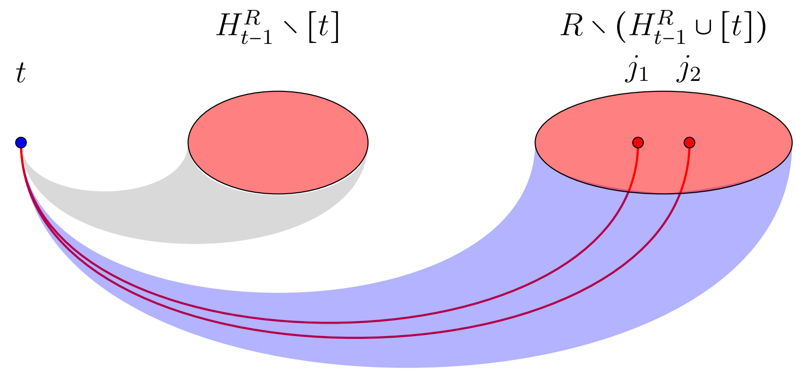

Figure 6.1: Step 1 of an iteration of Algorithm 1: "forward-succesful" case. At the beginning of round t, we have already constructed the monochromatic path-forests F^R_{t-1} and F^B_{t-1}, and we must examine vertex t to construct F^R_t and F^B_t. Suppose vertex t is blue. By construction, we will have $t \in V(F^B_{t-1})$. Next, we try to include t in F^R_{t-1} using red forward edges, that is, we look for red vertices $j_1, j_2 > t$ such that tj_1, tj_2 are red edges, and such that $F^R_{t-1} + \{tj_1, tj_2\}$ is a path-forest. In particular, we need to avoid the "forbidden" red vertices H^R_{t-1}, which correspond to red vertices which already have degree 2 in F^R_{t-1} and thus cannot be used. We will also define $\varphi(t) = j_1$, and it will hold that "most" of the red vertices in $R \smallsetminus (H^R_{t-1} \cup [\varphi(t)])$ will be connected to t with blue edges.

forest using forward edges implies that t is connected with "most" of the upcoming red vertices using only blue edges. Thus, in the future, when a red vertex $t'' > t$ fails to be included in the blue path-forest by using blue forward edges, the vertex t serves as a "useful" blue vertex, that is, endpoint of blue backward edges which connect with the red vertex t''. Thus, roughly speaking, the failure to include blue vertices in the red path-forest means that, in the future, it will be easier to include red vertices in the blue path-forest (and vice versa).

This is a simplified description of the algorithm. In the full description, the classification of vertices is more complicated. First, we handle the addition of backwards edges in a more careful way. Instead of joining every blue vertex which failed to be connected using red forward edges immediately with blue backward edges if they exist, we will proceed in batches. We will initially "collect" useful red vertices one by one in a set A^R_t, using the red vertices which failed to use blue backward edges. After the set reaches a desired size, we will set apart a set

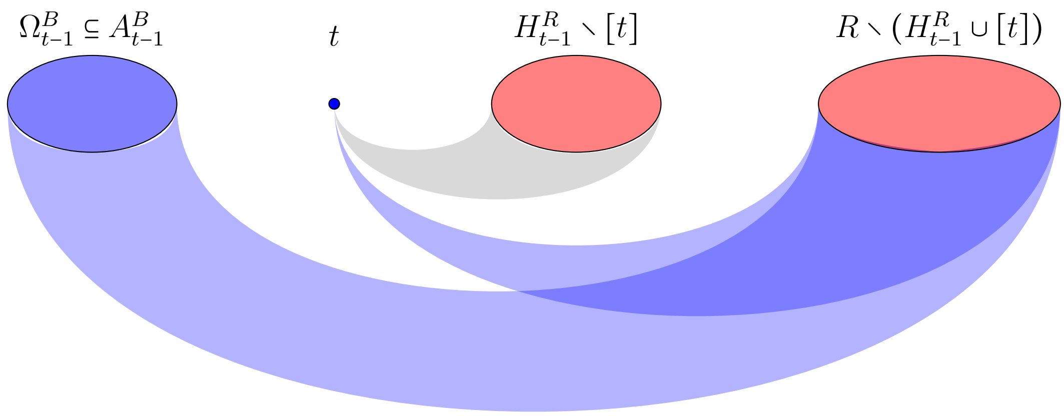

Figure 6.2: Step 2 of an iteration of Algorithm 1: if the "backwards-successful" case fails. In round t we examine vertex t, suppose vertex t is blue. If t does not have two red neighbours in $R \setminus (H^R_{t-1} \cup [t])$, then it will not be joined to F^R_{t-1} by using forward red edges. Note that t (and also the vertices in Ω^R_{t-1}) are joined mostly with blue edges with the vertices in $R \setminus (H^R_{t-1} \cup [t])$. Thus t will be added as an "useful" vertex in A^R_t. At a later round $t' \geq t$, the vertex t will become an "available" vertex in $\Omega^B_{t'}$. Moreover, if certain conditions are given (if "t is a backwards-successful" vertex) then t will belong to Y^B_t, which later could be included in the red path-forest using backward edges.

of "available" red vertices $\Omega^R_t \subseteq A^R_t$ which will deal with the next upcoming blue vertices. Only after that point, we will start collecting "waiting" blue vertices Γ^R_t which failed to use the red forward edges, without adding any red backward edges. Only when the set of "waiting" blue vertices reaches a certain size, we add the red edges between the "available" red vertices and the "waiting" blue vertices in a single step, and ensuring that the addition of the red edges guarantees we still have a red path-forest. The reason for proceeding in batches is that to find this set of backward edges we will use Proposition 6.3.2. This will ensure that all but a constant number of the blue waiting vertices will be included in the red path-forest after using the red backward edges. By choosing the size of these batches large enough, proportionally "most" of the blue waiting vertices will be covered by the red path-forest, which will be enough for our purposes.

Furthermore, we will exploit the fact that vertices which were connected using forward edges of the opposite colour can become "useful" but not immediately:

they will become useful at a later round. In fact, we will classify every vertex using four different possible "types", which will indicate whether (and how) the vertex can be included in the path-forest of the opposite colour, and the round which the vertex becomes "useful" (either immediately or at a later step).

6.4.2 Detailed outline

Now we describe the algorithm in full detail. We introduce the notation to describe the algorithm and its output. The formal description appears in Algorithm 1.

The algorithm receives as an input a restricted red-blue edge-colouring of $K^{\mathbb{N}}$ and an even integer ℓ (which controls the size of the "batches"). Let R, B be the red and blue vertices under the canonical vertex-colouring given by the edge-colouring, respectively.

The output of the algorithm will be

- two sequences of monochromatic path-forests, $\{F_t^R\}_{t\geq 0}$ and $\{F_t^B\}_{t\geq 0}$, which are red and blue monochromatic respectively,
- for each $t \geq 0$, sets $A_t^R, \Omega_t^R, \Gamma_t^R, W_t^R, X_t^R, Y_t^R, Z_t^R \subseteq R$,
- for each $t \geq 0$, sets $A_t^B, \Omega_t^B, \Gamma_t^B, W_t^B, X_t^B, Y_t^B, Z_t^B \subseteq B$,
- a function $\varphi : \mathbb{N} \to \mathbb{N}$.

Initially, F_0^R is defined as the red path-forest on vertex set R with no edges, and F_0^B is defined similarly. All of the "auxiliary sets" $A_0^*, \Omega_0^*, \Gamma_0^*, W_0^*, X_0^*, Y_0^*, Z_0^*$ are defined as empty, for all $* \in \{R, B\}$.

The algorithm will proceed in rounds, one for each $t \geq 1$, in increasing order. At the end of round t of the algorithm, the monochromatic path-forests F_i^* have been constructed, as well as the sets $A_i^*, \Omega_i^*, \Gamma_i^*, W_i^*, X_i^*, Y_i^*, Z_i^*$ for all $* \in \{R, B\}$ and $i \leq t$, and a partial function $\varphi : [t] \to \mathbb{N}$ (which we have obtained from the previous round by defining $\varphi(t)$).

We define

$$H_t^R := \{x \in R : \deg_{F_t^R}(x) = 2\},$$

that is, H_t^R is the set of the red-coloured vertices which have degree 2 in F_t^R. We will refer to H_t^R as the set of *forbidden red vertices at the end of round* t. Define the set H_t^B of forbidden blue vertices at the end of round t analogously with respect to the blue path-forest F_t^B. The idea is that these vertices no longer can be used to extend the path-forest of its colour by joining them to vertices of the opposite colour.

Now we further discuss the auxiliary sets $A_t^R, \Omega_t^R, \Gamma_t^R$, whose role was outlined in the previous discussion.

(i) $A_t^R \subseteq R \cap [t]$, the *red useful vertices at round* t,

(ii) $\Omega_t^R \subseteq R \cap [t]$, the *red available vertices at round* t, and

(iii) $\Gamma_t^R \subseteq R$, the *red waiting vertices at time* t.

We will also use the same names ("useful", "available", and "waiting") for the corresponding sets of blue vertices $A_t^B, \Omega_t^B, \Gamma_t^B \subseteq B \cap [t]$. Their role will be symmetric to those of $A_t^R, \Omega_t^R, \Gamma_t^R$, but for our discussion we will focus on the sets of red vertices only.

Useful vertices. The sets $\{A_t^R\}_{t\geq 0}$ will form an increasing family, i.e. $A_i^R \subseteq A_j^R \subseteq R$ for all $i < j$. Moreover, each of the sets A_t^R will be ordered, where the order might be different from the natural order induced by $\mathbb{N}$. These orders will always be increasing, meaning that A_{t+1}^R will be obtained from A_t^R always by appending some vertices "at the end" of the order given by A_t^R. The reason for this is as follows. It might happen that a vertex t is not included as an useful vertex after round t, but it becomes useful only at a later round $t' > t$ (i.e. $t \in A_{t'}^R \setminus A_{t'-1}^R$). Then the vertices of A_t^R will be ordered according to the first round in which they appeared as a useful vertex.

The role of the function $\varphi : \mathbb{N} \to \mathbb{N}$ is to describe and control the order of the sets A_t^R. More precisely, a red vertex t will be part of $A_{t'}^R$ only when $t' \geq \varphi(t)$, similarly with the blue vertices. Moreover, if the vertices of A_t^R appear in order as $(v_1, \dots, v_m)$, say, then $\varphi(v_1) \leq \varphi(v_2) \leq \dots \leq \varphi(v_m)$. Imprecisely speaking, for a red vertex t we would like $\varphi(t)$ to be "the last" of the blue vertices connected to t via forward blue edges (this makes sense since t is red, and the colouring is restricted, so t is connected with a finite number of blue forward edges). If no such blue vertices exist, then we define $\varphi(t) = t$. If $t' = \varphi(t)$ is chosen like this, then when the algorithm reaches step t', the red vertex t will now be connected to "most" of the upcoming blue vertices using only red edges, which makes t suitable to belong in $A^R_{t'}$.

Available vertices. Now we introduce the setup required to define the available vertices from the set of useful vertices. Given an ordered vertex set $V = \{v_i : i \in [n]\}$ and $* \in \{R, B\}$, define

$$\rho^*_t(V) := \sum_{v \in V} (2 - d_{F^*_t}(v)).$$

Since a path-forest has maximum degree at most 2, we can interpret $\rho^*_t(V)$ as the number of additional edges that we can theoretically add to V (joined with vertices outside of V) while keeping F^*_t as a path-forest.

The following properties are immediate from the definition of ρ^*_t, and therefore we omit the proof.

Proposition 6.4.1. *Let $S_1, S_2 \subseteq [n]$ be disjoint and $t \geq 0$. Then*

(i) $\rho^R_t(S_1) \leq 2|S_1|$;

(ii) $\rho^R_t(S_1 \cup S_2) = \rho^R_t(S_1) + \rho^R_t(S_2)$, *and*

(iii) *if* $d_{F^R_t}(s) = 2$ *for all* $s \in S_2$, *then* $\rho^R_t(S_1 \cup S_2) = \rho^R_t(S_1)$.

The corresponding statements hold with R replaced by B.

Suppose an even $\ell \in \mathbb{N}$ is given and $V = \{v_i : i \in [n]\}$ is an ordered set. If $\rho^*_t(V) \geq \ell$, then we define $\sigma^*_t(V)$ in the following way: let $s \in [n]$ be minimal such that $\rho^*_t(\{v_i : i \in [s]\}) \geq \ell$ and then select $V' \subseteq \{v_i : i \in [s]\} \subseteq V$ to be minimal with respect to inclusion such that $\rho^*_t(V') \geq \ell$, and let $\sigma^*_t(V) := V'$. Note that, by choice, $d_{F_t^*}(v) \leq 1$ for all $v \in V'$. Note as well that

$$\rho_t^*(\sigma_t^*(V)) \in \{\ell, \ell + 1\}. \tag{6.4.1}$$

This gives us the necessary tools to formalise the outline of the previous subsection, where we said that we define a "batch" of available red vertices among the "useful" vertices whenever it reached a certain "size". We do this as follows: if $\rho_t^R(A_t^R) \geq \ell$, we will define the available red vertices Ω_t^R as $\Omega_t^R = \sigma_t^R(A_t^R)$.

Waiting vertices. We have described the procedure to collect "useful" red and blue vertices. The set of "waiting" red vertices Γ_t^R will consist of red vertices which failed to use blue forward edges and thus are "waiting" to be included in the blue path-forest by using backward edges. As discussed before, we will work in batches. First the algorithm will collect useful and available vertices in A_t^B and Ω_t^B, as sketched. Only when Ω_{t-1}^B is non-empty at the start of round t we will start declaring red vertices as "waiting", in Γ_t^B. When the set of "waiting" vertices has reached a certain size, we will join the useful blue vertices in Ω_t^B with blue edges with (most of) the red waiting vertices in Γ_t^R. Then we will empty the set of red waiting vertices for the next round, and redefine the "available" blue vertices as the next batch of useful blue vertices, if it exists, or as empty, if there are not yet enough useful blue vertices.

To describe precisely the process which decides which vertices belong in Γ_t^R, we need to describe the role of the remaining red auxiliary sets $W_t^R, X_t^R, Y_t^R, Z_t^R$. At the end of round t, all the red vertices in $R \cap [t]$ will be partitioned in $\{W_t^R, X_t^R, Y_t^R, Z_t^R\}$. The classification of the red vertices into W_t^R, X_t^R, Y_t^R and Z_t^R will determine whether, and how, the vertices can be included in the blue path-forest, and when they can be included as "useful vertices" in A_t^R. The meaning of this classification goes as follows.

(i) $W_t^R \subseteq R \cap [t]$, the *red forward-successful* vertices which were included in the blue path-forest using blue forward edges.

If $t \in W_t^R$, then t forms part of the blue path-forest at round t, i.e., $t \in V(F_t^B)$. Thus, it will not form part of Γ_t^R. Furthermore, t will be included as an useful red vertex at the end of round $\varphi(t)$ (thus $t \in A_{\varphi(t)}^R$).

(ii) $X_t^R \subseteq R \cap [t]$, the *red backward-spoiled* vertices. A red vertex $x \le t$ will be included in X_t^R if it failed to use blue forward edges, but could not be declared as red waiting vertex because there were no blue available vertices when they were examined (i.e. $\Omega_{x-1}^B = \varnothing$).

If $t \in X_t^R$, then t will never be included in the blue path-forest (i.e., $t \notin \bigcup_{i\ge 0} V(F_i^B)$). But t will be included as a useful vertex at the end of its own round, i.e., we have $t \in A_t^R$.

(iii) $Y_t^R \subseteq R \cap [t]$, the *red backward-successful* vertices. A red vertex $x \le t$ will be included in X_t^R if it failed to use blue forward edges, and can be declared as a red waiting vertex: both because the set of blue available vertices is non-empty when x was examined (i.e. $\Omega_{x-1}^B \neq \varnothing$) and because x was not a "forbidden red vertex" when the last blue vertex of Ω_{x-1}^B was added as an useful vertex (i.e., for all $y \in \Omega_{x-1}^B$, $x \notin H_{\varphi(v)}^R$).

If $t \in Y_t^R$, then t will be included as a waiting vertex at this step (i.e., $t \in \Gamma_t^R$). It might be included in the blue path-forest (proportionally, most of the waiting vertices will be included in the blue path-forest). Furthermore, t will be included as an useful red vertex at the end of its own round, thus $\varphi(t) = t$ and $t \in A_t^R$.

(iv) $Z_t^R \subseteq R \cap [t]$, the *red forbidden-spoiled* vertices. A red vertex $x \le t$ will be included in Z_t^R if it failed to use blue forward edges, and the set of blue available vertices at the beginning of round x is not empty ($\Omega_{x-1}^B \neq \varnothing$), but x could not be declared as a red waiting vertex because x was a "forbidden vertex" when a vertex of Ω_{x-1}^B was added as useful (i.e., there exists $y \in \Omega_{x-1}^B$ such that $x \in H_{\varphi(v)}^R$).

If $t \in Z_t^R$, then its behaviour is similar to the "backward-spoiled" vertices. That is, t will never be included in the blue path-forest, and $t \in A_t^R$.

Sometimes we will refer to the vertices in $X_t^R \cup Z_t^R$ as the *red spoiled vertices*. Analogously, at the end of round t, the algorithm will output the corresponding partition $\{X_t^B, Y_t^B, Z_t^B, W_t^B\}$ of $B \cap [t]$.

We discuss a subtle distinction between the vertices included in Y_t^R and Z_t^R. In both cases, t failed to be included in the blue path-forest using forward edges, and the set of blue available vertices Ω_{t-1}^B is non-empty, but depending on the case t can be declared as "waiting" or not. Suppose t_Ω is the maximum of $\varphi(v)$ over all $v \in \Omega_{t-1}^B$. Thus all the available blue vertices collected at round t were declared as "useful" at time t_Ω, i.e., $\Omega_{t-1}^B \subseteq A_{t_\Omega}^B$. Let $x \in \Omega_{t-1}^B$, and suppose that x was included first as an useful vertex at round x', where $x \leq x' \leq t_\Omega < t$. The key property that characterises the useful vertices is that x is incident with blue edges to (almost) every non-forbidden upcoming red vertex *at the time when it was declared useful.* Thus, x will be "useful" to deal only with the red vertices which are not in $H_{x'}^R$. Since the red path-forests are increasing, the sets of forbidden red vertices can only increase, and thus we have $H_{t_\Omega}^R \supseteq H_{x'}^R$. For this reason, we can look at the set of forbidden red vertices at round t_Ω (instead of t) to decide if t can be declared as waiting or not.

Rounds and steps. Having defined the meaning of every object built by the algorithm, each round of the algorithm can be split naturally in four different "steps", as follows. These steps are written formally in Algorithm 1; here we give an equivalent but longer explanation in words, which (we hope) will be helpful to the reader.

For all $i < t$ and $* \in \{R, B\}$ the algorithm has constructed path-forests F_i^*, and sets $A_i^*, \Omega_i^*, \Gamma_i^*, W_i^*, X_i^*, Y_i^*, Z_i^*$. We also have defined the partial function $\varphi : [t-1] \to \mathbb{N}$. At the end of this round we must construct, for all $* \in \{R, B\}$ the path-forests F_t^*, the sets $A_t^*, \Omega_t^*, \Gamma_t^*, W_t^*, X_t^*, Y_t^*, Z_t^*$ and extend the partial

Algorithm 1: Path-forests algorithm

Input: An even integer $\ell \in \mathbb{N}$, and a restricted red-blue edge-colouring of $K^{\mathbb{N}}$.

Let $A_0^*, \Omega_0^*, \Gamma_0^*, \varphi, X_0^*, Y_0^*, Z_0^*, W_0^*$ be empty for all $* \in \{R, B\}$, let F_0^R be such that $V(F_0^R) = R$ and no edges, and F_0^B be such that $V(F_0^B) = B$ and no edges.

foreach $t \geq 1$ **do**

  if t *is red* **then**

    `/* Step 1: Adding` t `to the red path-forest */`

    $F_t^R \leftarrow F_{t-1}^R$.

    `/* Step 2: Classifying` t`. */`

    Let $J = N_{K_{\mathbb{N}}}^B(t, B) \setminus (H_{t-1}^B \cup [t])$, i.e., J is the set of future blue neighbours of t which are blue and not forbidden.

    $(W_t^R, X_t^R, Y_t^R, Z_t^R) \leftarrow (W_{t-1}^R, X_{t-1}^R, Y_{t-1}^R, Z_{t-1}^R)$.

    if $\Omega_{t-1}^B \neq \varnothing$ **then**

      Let $t_\Omega = \max\{\varphi(v) : v \in \Omega_{t-1}^B\}$ (it will hold that $t_\Omega < t$).

    end

    if $|J| \geq 3$ **then**

      $W_t^R \leftarrow W_t^R \cup \{t\}$.

    else if $|J| \leq 2$ *and* $\Omega_{t-1}^B = \varnothing$ **then**

      $X_t^R \leftarrow X_t^R \cup \{t\}$.

    else if $|J| \leq 2$, $\Omega_{t-1}^B \neq \varnothing$ *and* $t \notin H_{t_\Omega}^R$ **then**

      $Y_t^R \leftarrow Y_t^R \cup \{t\}$.

    else if $|J| \leq 2$, $\Omega_{t-1}^B \neq \varnothing$ *and* $t \in H_{t_\Omega}^R$ **then**

      $Z_t^R \leftarrow Z_t^R \cup \{t\}$.

    end

    `/* Step 3: Adding` t `to the blue path-forest. */`

    `/* Step 3A:` t `is forward-successful. */`

    case $t \in W_t^R$ **do**

      Let $j_1, j_2 \in J$ be such that $F_{t-1}^B + \{tj_1, tj_2\}$ is a blue path-forest and $\min\{j_1, j_2\}$ is maximised.

      $\varphi(t) \leftarrow \min\{j_1, j_2\}$.

      $F_t^B \leftarrow F_{t-1}^B + \{tj_1, tj_2\}$.

      $A_t^R \leftarrow A_{t-1}^R$.

      $\Omega_t^R \leftarrow \Omega_{t-1}^R$.

      $\Gamma_t^R \leftarrow \Gamma_{t-1}^R$.

    end

  ⋮ ⋮ ⋮

⋮

⋮

/* Step 3B: t is spoiled. */

case $t \in X_t^R \cup Z_t^R$ **do**

$\varphi(t) \leftarrow t$.

$F_t^B \leftarrow F_{t-1}^B$.

$A_t^R \leftarrow A_{t-1}^R \cup \{t\}$.

if $\rho_t^R(A_t^R) \geq \ell$ *and* $\Omega_{t-1}^R = \varnothing$ **then**

$\Omega_t^R \leftarrow \sigma_t^R(A_t^R)$.

else

$\Omega_t^R \leftarrow \Omega_{t-1}^R$.

$\Gamma_t^R \leftarrow \Gamma_{t-1}^R$.

/* Step 3C: t is backwards-successful. */

case $t \in Y_t^R$ **do**

Let $\varphi(t) = t$ and A_t^R, Ω_t^R as in the case $t \in X_t^R \cup Z_t^R$.

if $|\Gamma_{t-1}^R \cup \{t\}| < \ell/2$ **then**

$F_t^B \leftarrow F_{t-1}^B$.

$\Gamma_t^R \leftarrow \Gamma_{t-1}^R \cup \{t\}$.

else

/* Step 3D: Using backward edges. */

Let $F' \subseteq E(\Omega_{t-1}^B, \Gamma_{t-1}^R \cup \{t\})$ be such that $F_{t-1}^B + F'$ is a blue path-forest and $|(\Gamma_{t-1}^R \cup \{t\}) \setminus V(F_{t-1}^B + F')| \leq 4$.

$F_t^B \leftarrow F_{t-1}^B + F'$.

$\Gamma_t^R \leftarrow \varnothing$.

/* Step 4: Updating blue useful, available and waiting vertices */

$\Gamma_t^B \leftarrow \Gamma_{t-1}^B$.

$A_t^B \leftarrow A_{t-1}^B \cup (\varphi^{-1}(t) \cap B \cap [t-1])$.

if $\Omega_{t-1}^B = \varnothing$ *or Step 3D was executed* **then**

if $\rho_t^B(A_t^B) \geq \ell$ **then**

$\Omega_t^B \leftarrow \sigma_t^B(A_t^B)$.

else

$\Omega_t^B \leftarrow \varnothing$.

else

$\Omega_t^B \leftarrow \Omega_{t-1}^B$.

else if t *is blue* **then**

The same as the "t is red" case, exchanging the roles of R and B.

function to $\varphi : [t] \to \mathbb{N}$.

Suppose the current vertex under examination, t, is coloured red (otherwise, exchange the roles of red and blue in what follows).

Step 1: Adding t to the red path-forest (Line 4)

In this step, we define the current red path-forest F_t^R exactly as F_{t-1}^R. Note that $t \in R = V(F_0^R) \subseteq V(F_{t-1}^R) = V(F_t^R)$.

Step 2: Classifying t (Lines 5–18).

In this step, we include t in exactly one of the four possible sets W_t^R, X_t^R, Y_t^R, or Z_t^R, as sketched before. First we let J be the set of future non-forbidden blue vertices which are blue neighbours of t (Line 5). Then, if $\Omega_{t-1}^B \neq \varnothing$ we gather the value $t_\Omega = \max\{\varphi(v) : v \in \Omega_{t-1}^B\}$. (Lines 7–9). Intuitively, the available vertices of Ω_{t-1}^B can deal with all the waiting red vertices which were found after round t_Ω. It will hold that $t_\Omega < t$ (we will check this later in Lemma 6.4.5(x)).

If $|J| \geq 3$, then t has "blue forward" edges, and we include t in W_t^R as a "red forward-successful" vertex (Lines 10–11). Otherwise, $|J| \leq 2$. If the current set of available blue vertices that we can use is empty (i.e., $\Omega_{t-1}^B = \varnothing$), then there are no blue backward edges that we can use, and we include t in X_t^R as a "red backward-spoiled" vertex (Lines 12–13). Otherwise, $|J| \leq 2$ and $\Omega_{t-1}^B \neq \varnothing$. If t was not a forbidden red vertex at time t_Ω, we include $t \in Y_t^R$ as a "backward-successful" vertex; otherwise it is included in Z_t^R as a "forbidden-spoiled" vertex (Lines 14–18).

Step 3: Adding t to the blue path-forest (Lines 19–44)

In this step we update the blue path-forest. We try to include t in the blue path-forest, or as a waiting vertex, depending on its classification in the previous step. Also, we define the value of $\varphi(t)$, which defines the round when t will become a "useful" red vertex.

Step 3A: t **is forward-successful** (Lines 19–26).

If $t \in W_t^R$, we will add t to the blue path-forest using blue forward edges. We look for forward blue edges which are as far away as possible from t and such that its inclusion does not violate that F_{t-1}^B is a path-forest. To check that the algorithm is well-defined, we will verify later (Lemma 6.4.5) that it is always possible to find the required edges in this step.

As we will see later, Proposition 6.3.1 will tell us that we can always select two blue edges among the "last" three vertices of J which are connected to t using blue edges. This allows us to define $\varphi(t)$ as one of these last vertices. (The reason for doing this is that when t becomes useful at the end of round $\varphi(t)$, it will be connected with blue edges with at most two of the upcoming blue vertices.)

The sets $A_t^R, \Omega_t^R, \Gamma_t^R$ do not change from the previous round.

Step 3B: t **is spoiled** (Lines 27–35).

If $t \in X_t^R \cup Z_t^R$ then the red vertex is spoiled, and thus will not be included in the blue path-forest nor as a waiting vertex. Also, the blue path-forest will not change from the previous round. However, t will be included as an useful red vertex straight away (Line 30) and, as a consequence, we will need to update the current set of available red vertices (Lines 31–34).

Step 3C: t **is backwards-successful** (Lines 36–40).

If $t \in Y_t^R$, first we define $\varphi(t) = t$ and include t as an useful red vertex straight away. This means we have to update the set of available vertices as well, this is done as in the previous case (Line 36).

Next, we can include t as a waiting red vertex. If Γ_t^R "has not reached" the correct size after adding t, then we do not update the blue path-forest at this step (Lines 38–40). Otherwise, we can extend the blue path-forest using the red waiting vertices, which means we proceed to Step 3D.

Step 3D: Using the backward edges (Lines 41–44)

In this case, t is backward-successful and Γ_t^R has reached a size comparable with Ω_{t-1}^B and we can join Ω_{t-1}^B and Γ_t^R using backward blue edges.

We find a blue path-forest F' that joins these vertices and can be added to F_{t-1}^B, and covers all but at most 4 vertices of Γ_t^R (Lines 41–43). To check the algorithm is well-defined, we need to prove that in this step F' always exists, this will be done later in Lemma 6.4.5.

After this is done, we empty the set of red waiting vertices (Line 44).

Step 4: Update blue useful, available and waiting vertices (Lines 45–53).

In each of the possible scenarios of Step 3 we have defined the latest blue path-forest F_t^B. We need to update the remaining blue auxiliary sets. The set of waiting blue vertices will not change (Line 45).

Since some previous blue vertices $v < t$ might have $\varphi(v) = t$, we need to include them in the set A_t^R of useful vertices at round t (Line 46).

We need to define the definitive set of available blue vertices for the next round (Lines 47–53). This needs to be done in two cases: if $\Omega_{t-1}^B = \varnothing$ (since in the previous line we might have reached enough useful blue vertices) or if Step 3D was executed in this round, since in that case we have added blue edges to Ω_{t-1}^B and we need to get fresh available vertices for the following rounds. In this case, we update Ω_t^B as usual using ρ_t^B if the available vertices have reached the correct "size" (Line 48), otherwise we just declare the set of available vertices to be empty. If we are not in one of the cases where the set of blue available vertices might have changed, then we define it to be the set of the previous round (Line 53).

This finishes the description of the algorithm. In what follows, we will deduce properties of the objects constructed by the algorithm in every step.

6.4.3 Basic properties of the algorithm

In this subsection we will gather very basic properties of Algorithm 1, and then we will check that Algorithm 1 is well-defined. The only thing that needs any verification is that, during some parts of the algorithm, we claim we can find a certain set of edges that we add to F^R_{t-1} or F^B_{t-1}, such that the obtained graph is still a path-forest and satisfies certain properties.

This occurs twice during the algorithm: Lines 19–21 and Lines 41–43. To verify these steps can be executed, we will appeal to Propositions 6.3.1 and 6.3.2 respectively. Thus, to prove the correctness it is necessary to verify that the assumptions of those propositions hold when we invoke them. To do this, we first verify that the auxiliary sets and objects we have constructed satisfy certain desirable properties.

We begin by gathering very basic properties of the algorithm, which can be easily checked from Algorithm 1. We state them as a lemma for future reference.

Lemma 6.4.2. *Let $\ell \in \mathbb{N}$ be even. Suppose that $K_{\mathbb{N}}$ has a restricted 2-edge-colouring. Let F^*_t, Ω^*_t, Γ^*_t, A^*_t, φ, W^*_t, X^*_t, Y^*_t, Z^*_t be as defined by Algorithm 1, for all $* \in \{R, B\}$. Then the following holds for all $t \in \mathbb{N}$ (and similar statements hold if we interchange R and B):*

(i) $t = \sum_{*\in\{R,B\}} \left(|X^*_t| + |Y^*_t| + |Z^*_t| + |W^*_t|\right)$;

(ii) $\varphi(i) \geq i$ *for all* $i \in [t]$;

(iii) *for all* $i \in R \cap [t]$, $i \in W^R_i$ *if and only if* $\varphi(i) > i$. *Further, in this case,* $\varphi(i) \in B$;

(iv) $F^R_i \subseteq F^R_j$ *for all* $i, j \in [t]$ *with* $i \leq j$;

(v) *the sequence* $\{A^R_i\}_{i\in[t]}$ *is increasing, i.e.,* $A^R_i \subseteq A^R_j$ *for all* $i, j \in [t]$ *with* $i \leq j$. *Similarly,* $\{W^R_i\}_{i\in[t]}$, $\{X^R_i\}_{i\in[t]}$, $\{Y^R_i\}_{i\in[t]}$, $\{Z^R_i\}_{i\in[t]}$ *are increasing sequences;*

(vi) $\Gamma^R_t \subseteq Y^R_t$;

(vii) $R \cap [t] \subseteq V(F^R_t)$;

(viii) $A_t^R, \Omega_t^R, \Gamma_t^R \subseteq R \cap [t]$, $\Omega_t^R \subseteq A_t^R$;

(ix) $A_t^R = X_t^R \cup Z_t^R \cup Y_t^R \cup \{w \in W_t^R : \varphi(w) \leq t\}$;

(x) $\{\varphi(v) : v \in A_t^R\} \subseteq [t]$;

(xi) *if* $\Omega_t^R \neq \varnothing$, *then* $\rho_t^R(\Omega_t^R) \in \{\ell, \ell + 1\}$;

(xii) $|\Gamma_t^R| < \ell/2$;

(xiii) *if* $y > t$ *and* $y \in R$, *then* $y \notin V(F_t^B)$; *and*

(xiv) *if* $y > t$, $y \in R$ *and* $\deg_{F_t^R}(y) > 0$, *then* $N_{F_t^R}(y) \subseteq W_t^B$.

Proof. All the properties can be easily checked from Algorithm 1, using induction. Nevertheless, for completeness, we give a brief explanation and description for all of them.

Since the properties are easy or vacuously true for $t = 0$, we will assume that $t > 0$. Without loss of generality, we assume t is red.

1) Note (i) follows simply because every vertex in $[t]$ is classified in exactly one of the sets $X_t^R, X_t^B, Y_t^R, Y_t^B, Z_t^R, Z_t^B, W_t^R$, or W_t^B.
2) We check (ii) and (iii) now. By using induction, we can assume those statements hold for all $t' < t$, so it is only necessary to check them for t. There are two possible cases depending on the classification of t in Step 2. If t is red spoiled ($t \in X_t^R \cup Z_t^R$ in Step 3B) or red backwards-successful ($t \in Y_t^R$ in Step 3C) then it will hold that $\varphi(t) = t$. Otherwise, t is forward-successful ($t \in W_t^R$ in Step 3A) and $\varphi(t)$ will be such that $t\varphi(t)$ is a forward blue edge which joins t with a blue vertex. Thus $\varphi(t) > t$, and if this happens then the vertex $\varphi(t)$ is blue.
3) Items (iv)–(viii) just state simple facts from the algorithm. We build F_t^R from F_{t-1}^R only by adding edges and vertices (possibly none), we build A_t^R from A_{t-1}^R only by adding vertices (possibly none) and similarly with $W_t^R, X_t^R, W_t^R, Z_t^R$. Only vertices which are "backward-successful" are classified as "waiting" (Step 3C), thus $\Gamma_t^R \subseteq Y_t^R$. Every red vertex in $[t]$ forms part of F_t^R (Step 1). Every available red vertex is an useful red vertex, thus $\Omega_t^R \subseteq A_t^R$, and every

red available and red waiting vertex is red.

4) To see (ix), first note that every vertex $v \in [t]$ which is in $X_t^R \cup Z_t^R \cup Y_t^R$ is in A_t^R (Steps 3B or Step 3C). Thus it is only necessary to check that $A_t^R \cap W_t^R = \{w \in W_t^R : \varphi(w) \leq t\}$. Note that if $v \in W_t^R$ then $v \in W_v^R$. From (iii), we get $\varphi(v) > v$. Now, if $v \in W_v^R$, note that v is not added at A_v^R in its round (Step 3A). Now, let v' be the minimum $v' > v$ such that $v \in A_{v'}^R$. Note $v' \leq t$, since the sets $\{A_i^R\}_{i \geq 0}$ are increasing. The only line in the algorithm where $A_{v'}^R$ gets updated to include vertices from previous rounds is in Line 46, thus we deduce that v' is a blue vertex and (by minimality of v'), that $v \in \varphi^{-1}(v')$. Thus $\varphi(v) = v' \leq t$. To summarise, we have proven that for $v \in W_t^R$, $v \in A_t^R$ if and only if $\varphi(v) \leq t$, as desired.

5) We prove (x). Let $v \in A_t^R$. If $v \in W_t^R$, then from (ix) we deduce $\varphi(v) \leq t$. Otherwise, $v \in X_t^R \cup Z_t^R \cup Y_t^R$. From (iii) we get $\varphi(v) = v \leq t$.

6) To see (xi), note that if Ω_t^R is defined to be non-empty, then it is set to be equal to $\sigma_t^R(V)$ for some $V \subseteq \mathbb{N}$. Then the statement follows from (6.4.1).

7) We check (xii). By induction, we can assume that $|\Gamma_{t'}^R|, |\Gamma_{t'}^B| < \ell/2$ for all $t' < t$. Since we assume t is red, note that Γ_t^B is defined to be equal to Γ_{t-1}^B (in Step 4), and we are done by the inductive hypothesis. So it is only necessary to consider Γ_t^R. The set Γ_t^R is defined in Step 3. In Steps 3A and 3B $\Gamma_t^R = \Gamma_{t-1}^R$, and we are again done. Otherwise, we are in Step 3C. There are two cases: if $|\Gamma_{t-1}^R \cup \{t\}| < \ell/2$ then $\Gamma_{t-1}^R = \Gamma_t^R \cup \{t\}$ and the statement holds. Otherwise, $\Gamma_t^R = \varnothing$, and the statement again holds.

8) To see (xiii), let $y \in R \cap V(F_t^B)$. Note we only add red vertices to F_t^B in two ways: by using blue forward-edges or blue backward-edges. If y was added using blue forward-edges, then those edges were added in round y after vertex y was classified as a forward-successful vertex (in Step 3A). Thus $y \leq t$. Otherwise, if y was added using backward-edges, then $y \in Y_v^R$ (it was classified as a backward-successful vertex in round y), and the backward edges were added in Step 3C for some $y' \geq y$ (when the "waiting vertices"

$\Gamma^R_{y'}$ reached the correct size). We deduce that $y \le y' \le t$, as desired.

9) Finally, to see (xiv), let $y > t$ be a red vertex with $\deg_{F^R_t}(y) > 0$. Let $y' \in N_{F^R_t}(y)$. Since y' is a blue vertex with non-zero degree in F^R_t, by (xiii) we deduce $y' \le t < y$. In words, y' is connected to y with a red forward-edge. This implies that that y' was classified a forward-successful vertex at round y', and thus is in $W^B_{y'} \subseteq W^B_t$, where the inclusion follows from (v). Since y' was arbitrary, $N_{F^R_t}(y) \subseteq W^B_t$, as desired. ■

The following lemma will only be used to prove Lemma 6.4.5, i.e., that Algorithm 1 is well-defined; it will not be used for the posterior analysis. Essentially, it states that, at any given round, when red available vertices and blue waiting vertices are defined and non-empty, "most" of the edges between them are red (and the corresponding statements are true if the colours are interchanged).

Lemma 6.4.3. *Let $\ell \in \mathbb{N}$ be even. Suppose that $K_{\mathbb{N}}$ has a restricted 2-edge-colouring. Let F^*_t, Ω^*_t, Γ^*_t, A^*_t, φ, W^*_t, X^*_t, Y^*_t, Z^*_t be as defined by Algorithm 1, for all $* \in \{R, B\}$. Then, for all $t \in \mathbb{N}$, if $\Omega^R_{t-1}, \Gamma^B_{t-1} \neq \varnothing$ and $t \in Y^B_t$, then for all $v \in \Omega^R_{t-1}$, $d^B_{K_{\mathbb{N}}}(v, \Gamma^B_{t-1} \cup \{t\}) \le 2$. A similar statement holds if we interchange the roles of R and B.*

Proof. We begin by proving an auxiliary claim, which essentially makes precise that the vertices from Ω^R_t are all located "before" the vertices of Γ^B_t.

Claim 6.4.4. *For all $t \ge 0$, if $\Omega^R_t, \Gamma^B_t \neq \varnothing$, then* $\max \Omega^R_t \le \max\{\varphi(v) : v \in \Omega^R_t\} < \min \Gamma^B_t$ *(and the same is true with the roles of R and B interchanged).*

Proof of the claim. Let $t_\Omega = \max\{\varphi(v) : v \in \Omega^R_t\}$. Consider any $v \in \Omega^R_t$. We have $v \le \varphi(v) \le t_\Omega$ by Lemma 6.4.2(ii). Since v was arbitrary, we deduce $\max \Omega^R_t \le t_\Omega$.

Now consider $w = \min \Gamma^B_t$. When w was added to Γ^B_w in round w, it was classified as a backwards-successful vertex in Y^B_w. In particular, this could only have happened if $\Omega^R_{w-1} \neq \varnothing$. Since $w \in \Gamma^B_t$ and the available vertices only get updated when the set of waiting vertices is emptied (in Step 3D), it follows that the set of available vertices must have not changed between the end of round $w-1$ and the end of round t. In particular, $\Omega^R_{w-1} = \Omega^R_t$. Thus

$$\{\varphi(v) : v \in \Omega^R_t\} = \{\varphi(v) : v \in \Omega^R_{w-1}\} \subseteq \{\varphi(v) : v \in A^R_{w-1}\} \subseteq [w-1],$$

where the penultimate inclusion holds by Lemma 6.4.2(viii) and the last inclusion holds by Lemma 6.4.2(x). In particular, $[t_\Omega] \subseteq [w-1]$ and thus $t_\Omega < w = \min \Gamma^B_t$. This finishes the proof of the claim. $\square$

Let $t_\Omega = \max\{\varphi(v) : v \in \Omega^R_{t-1}\}$, and let $\Gamma' = \Gamma^B_{t-1} \cup \{t\}$. Using that $t \in Y^R_t$ and Lemma 6.4.2(vi), we get that $\Gamma' \subseteq Y^R_t$. Given any $v \in \Omega^R_{t-1}$, let $J_v := N^B_{K_\mathbb{N}}(v, B) \setminus (H^B_{v-1} \cup [v])$, which is J as defined at round number v (as in Line 5). For all $u \in \Gamma' \subseteq Y^B_t$, we must have $d_{F^B_{v-1}}(u) \leq d_{F^B_{t_\Omega}}(u) < 2$, where the first inequality follows from Lemma 6.4.2(iv) and the second one from the classification of $u \in Y^B_t$. Thus $\Gamma' \cap H^B_{v-1} = \varnothing$. From Claim 6.4.4 (with $t-1$ in place of t) we have $\min \Gamma' = \min \Gamma^B_{t-1} > \max \Omega^R_{t-1} \geq v$ and thus $\Gamma' \cap (H^B_{v-1} \cup [v]) = \varnothing$. We have shown that

$$N^B_{K_\mathbb{N}}(v, \Gamma') = J_v \cap \Gamma'. \tag{6.4.2}$$

Now we separate the analysis in cases, depending if $v \in \Omega^R_{t-1}$ is a forward-successful vertex or not. If $v \notin W^R_t$, then $d^B_{K_\mathbb{N}}(v, \Gamma') \leq |J_v| \leq 2$, where the first inequality follows from (6.4.2) and the second from $v \notin W^R_t$ and the classification in Step 2 in round v.

Thus, we can assume that $v \in W^R_t$. To find a contradiction, suppose that $d^B_{K_\mathbb{N}}(v, \Gamma') \geq 3$. In particular, using Claim 6.4.4 again we get $\varphi(v) < \min \Gamma'$ and therefore $\Gamma' \cap [\varphi(v)] = \varnothing$. From (6.4.2) we get $N^B_{K_\mathbb{N}}(v, \Gamma') \subseteq J_v \setminus [\varphi(v)]$. Together with $d^B_{K_\mathbb{N}}(v, \Gamma') \geq 3$, this means that there exist at least 3 vertices in $J_v \setminus [\varphi(v)]$ that are connected to v with blue edges. Let j_1 be the minimum of such vertices. By Proposition 6.3.1 (with F^R_{v-1}, v and j_1 playing the roles of F, x and j_1, respectively) there exists $j_2 > j_1$ is such that $F^R_{v-1} + \{vj_1, vj_2\}$ is a red path-forest. In Lines 20–21 of round v, we have set $\varphi(v)$ to be the minimum of two vertices a, b such that $F^B_{v-1} + \{ta, tb\}$ is a red path-forest; so this means that $\varphi(v) \geq j_1$. But $j_1 \in N^B_{K_{\mathbb{N}}}(v, \Gamma') \subseteq J_v \setminus [\varphi(v)]$ means that $j_1 > \varphi(v)$, a contradiction. ∎

Now we will use Lemmas 6.4.2 and 6.4.3 to check that Algorithm 1 is well-defined.

Lemma 6.4.5. *Algorithm 1 is well-defined.*

Proof. As discussed, we need to check that Propositions 6.3.1 and 6.3.2 can be invoked during Lines 19–21 and Lines 41–43 of the algorithm, respectively.

(i) Lines 19–21. We assume t is red and we are in Step 3A, which means that $t \in W^R_t$. Recall that J is defined as $N^B(t, B) \setminus (H^B_{t-1} \cup [t])$, and since $t \in W^R_t$, then $|J| \geq 3$. Since H^B_{t-1} is the set of degree-two vertices of F^R_{t-1}, we can apply Proposition 6.3.1 with F^B_{t-1}, J and t playing the roles of F, J and x to deduce there exist $j_1, j_2 \in J$ such that $F^B_{t-1} + \{tj_1, tj_2\}$ is a path-forest, as desired.

(ii) Lines 41–43. We want to apply Proposition 6.3.2 with the path-forest F^B_{t-1} playing the role of F (the path-forest we want to extend), $\Gamma^R_{t-1} \cup \{t\}$ playing the role of Y (the vertices which we want to cover), and Ω^B_{t-1} playing the role of $X \subseteq V(F^B_{t-1})$ (the vertices of the path-forest which will be connected with the vertices which we want to cover). Note that $\Omega^B_{t-1} \subseteq B \cap [t-1] \subseteq V(F^B_{t-1})$, where the first inclusion holds since t is red and Lemma 6.4.2(viii), and the second inclusion follows from Lemma 6.4.2(vii).

We need to verify that conditions (i)–(ii) from Proposition 6.3.2 hold. Note that $|\Gamma^R_{t-1} \cup \{t\}| \leq \ell/2$ by Lemma 6.4.2(xii) and $\sum_{x \in \Omega^B_{t-1}} (2 - d_{F^B_{t-1}}(x)) = \rho^B_{t-1}(\Omega^B_{t-1}) \geq \ell$, by Lemma 6.4.2(xi). Thus (i) holds. To check condition (ii) we need to check that for every $x \in \Omega^B_{t-1}$, $d^B_{K_{\mathbb{N}}}(x, \Gamma^R_{t-1} \cup \{t\}) \geq |\Gamma^R_{t-1} \cup \{t\}| - 2$, and this is given by Lemma 6.4.3. ∎

6.4.4 Properties of the algorithm

In this subsection, we will collect some useful information from the algorithm and its output. The idea is to gather the properties that we need from the output of Algorithm 1 so that the rest of the proof can be done just by referring to the lemmas (and Lemma 6.4.2), and not to the full description of the algorithm.

We make the following crucial definition. For all $* \in \{R, B\}$ and $t \in \mathbb{N}$, we define

$$c_t^* := |V(F_t^*) \cap [t]|.$$

Thus, for $t \geq 1$, c_t^R/t corresponds to the proportion of vertices in $[t]$ which are covered by $V(F_t^R)$, and similarly with c_t^B/t.

Lemma 6.4.6. *Let $\ell \in \mathbb{N}$ be even. Suppose that $K_{\mathbb{N}}$ has a restricted 2-edge-colouring. Let F_t^*, Ω_t^*, Γ_t^*, A_t^*, φ, W_t^*, X_t^*, Y_t^*, Z_t^* be as defined by Algorithm 1, for all $* \in \{R, B\}$. Then the following holds for all $t \in \mathbb{N}$ (and similar statements hold if we interchange R and B):*

(i) *if $v \in B$ with $d_{F_t^R}(v) > 0$, then $v \in W_t^B \cup Y_t^B$;*

(ii) *if $v \in W_t^B$, then $d_{F_t^R}(v) > 0$ and $N_{F_t^R}(v) \subseteq R \setminus [\varphi(v) - 1]$;*

(iii) *if $\rho_t^R(A_t^R) \geq \ell$, then $\Omega_t^R \neq \varnothing$;*

(iv) *if there exists $t' > t$ such that $\Omega_{t''}^B \neq \varnothing$ for all $t \leq t'' < t'$, then $X_t^R = X_{t'}^R$;*

(v) *if there exists $t' > t$ such that $\rho_{t'-1}^R(A_t^R) \geq \ell$, then $X_t^R = X_{t'}^R$;*

(vi) *$c_t^R \geq (1 - 8/\ell)(t - |Z_t^B| - |X_t^B|) - \ell/2$;*

(vii) *$2|Y_{t'}^B \setminus Y_t^B| + \ell \geq \rho_t^R(A_t^R) - \rho_{t'}^R(A_t^R)$ for $t' \geq t$;*

(viii) *if $\rho_{t'-1}^R(A_t^R) \geq \ell$ for some $t' \geq t$, then we have $\Omega_{t''}^R \subseteq A_t^R$ for all $t \leq t'' < t'$;*

(ix) *if $\rho_{t'-1}^R(A_t^R) \geq \ell$ for some $t' \geq t$, then for every $z \in Z_{t'}^B$, $\deg_{F_t^R}(z) = 2$ and $N_{F_t^R}(z) \subseteq W_t^R$.*

(x) *if $\rho_{t'-1}^R(A_t^R) \geq \ell$ for some $t' \geq t$, then $|Z_{t'}^B| \leq |W_t^R|$.*

Proof. We begin by checking (i). If $v \in B$ satisfies $d_{F_t^R}(v) > 0$, then in particular

$v \in B \cap V(F_t^R)$. By Lemma 6.4.2(xiii), we have $v \in B \cap [t]$. Now it is just a matter of looking at the classification of v in Step 2: only blue forward-successful vertices in W_v^B and blue backward-successful vertices in Y_v^B can be in $V(F_t^R)$, so we are done.

We check (ii). Let $v \in W_t^B$. Thus, at round v, the vertex v was classified as a "forward-successful vertex" (Step 2), and therefore joined to F_v^R using red forward edges. In particular, $0 < d_{F_v^R}(v) \leq d_{F_t^R}(v)$. Note that in Step 3A, $\varphi(v)$ is defined as the minimum of its two red neighbours in F_v^R, thus $N_{F_t^R}(v) = N_{F_v^R}(v) \subseteq R \setminus [\varphi(v) - 1]$ holds.

To see (iii), suppose its validity holds for all $t' < t$ by induction and that t is red. We need to check the steps where we define Ω_t^R and Ω_t^B. The red useful and available vertices A_t^R and Ω_t^R are defined during Step 3. In Step 3A the sets do not change from the previous round, so we are done by the induction hypothesis. In Step 3B and 3C we define $A_t^R = A_{t-1}^R \cup \{t\}$, and if $\rho_t^R(A_t^R) \geq \ell$ then Ω_t^R is defined as non-empty (Lines 31–34 and 36). The blue useful and available vertices are defined during Step 4. After A_t^B is defined, we analyse it and use it to define Ω_t^B (Lines 47–53). The only case where Ω_t^B is defined as empty happens when $\rho_t^B(A_t^B) < \ell$, and this finishes the proof of (iii).

Let us prove that (iv) holds. Suppose there exists $t'' > t$ such that $X_{t''}^R \neq X_t^R$. Let t^* be the minimum of such t'', and note that it is enough to show that $t^* > t'$. We have $t^* \in X_{t^*}^R$, and therefore during round t^*, the vertex t^* is in R and was classified as a "backwards-spoiled" vertex during Step 2. That could only happen if $\Omega_{t^*-1}^B = \varnothing$. Our assumption then implies that $t^* > t'$, as desired.

Now we check (v). Note that (iv) implies that it is enough to check that for all $t'' \in \{t, \ldots, t'-1\}$, $\Omega_{t''}^R \neq \varnothing$. Let $t'' \in \{t, ..., t'-1\}$ be arbitrary, and note that

$$\rho_{t''}^R(A_{t''}^R) \geq \rho_{t'-1}^R(A_{t''}^R) \geq \rho_{t'-1}^R(A_t^R) \geq \ell,$$

where the first inequality holds since $F_{t''}^R \subseteq F_{t'-1}^R$ (by Lemma 6.4.2(iv)), the second inequality holds since $A_t^R \subseteq A_{t''}^R$ (by Lemma 6.4.2(v)), and the third holds by assumption. Thus (iii) implies that $\Omega_{t''}^R \neq \varnothing$, as desired.

To prove (vi), we first estimate $V(F_t^R) \cap Y_t^B$. The vertices of Y_t^B might get included in the red path-forest whenever Step 3D is executed, that is, when $\Gamma_{t-1}^B \cup \{t\}$ is joined by red edges with vertices of Ω_{t-1}^R. We will need the following observations about the evolution of the sets Y_t^B and Γ_t^B. If t is red, then $Y_{t-1}^B = Y_t^B$. Otherwise, t is blue and Y_t^B differs from Y_{t-1}^B only if t is a backwards-successful vertex, i.e., $t \in Y_t^B$. If this happens, then Step 3C is executed in round t and we try to add t as a "waiting vertex" in Γ_t^B. There are two cases: in the first, $|\Gamma_{t-1}^B| < \ell/2 - 1$ and if this happens then $\Gamma_t^B = \Gamma_{t-1}^B \cup \{t\}$ and thus $|\Gamma_t^B| = |\Gamma_{t-1}^B| + 1$. Otherwise, $|\Gamma_{t-1}^B| = \ell/2 - 1$ and we execute Step 3D, which means Γ_t^B is defined as empty. We observe that Step 3D is executed exactly on those rounds $t > 0$ whenever Y_t^B reaches a non-zero size which is divisible by $\ell/2$, and at the end of those rounds it holds that $\Gamma_t^B = \varnothing$.

Now, given $t > 0$, partition Y_t^B into $\Gamma'_1, \Gamma'_2, \dots, \Gamma'_s, \Gamma'_{s+1}$ (with Γ'_{s+1} possibly empty) such that, for all $i \in [s]$, $|\Gamma'_i| = \ell/2$, $\max \Gamma'_i < \min \Gamma'_{i+1}$ and $|\Gamma'_{s+1}| < \ell/2$. In other words, $\Gamma'_1, \Gamma'_2, \dots, \Gamma'_s, \Gamma'_{s+1}$ is a partition of Y_t^B into sets of "consecutive" $\ell/2$ vertices. Now consider any $i \in [s]$ and let $t_i := \max \Gamma'_i$. By the previous observations, and since $t_i \in Y_{t_i}^B$, we have $|\Gamma_{t_i-1}^B| = \ell/2 - 1$, $\Gamma_{t_i-1}^B \cup \{t_i\} = \Gamma'_i$ and $\Gamma_{t_i}^B = \varnothing$. Moreover, all but at most 4 vertices of Γ'_i are added to F_t^R (whenever Step 3D is executed at round number t_i). Thus, we deduce

$$|V(F_t^R) \cap Y_t^R| \geq \sum_{i \in [s]} (|\Gamma'_i| - 4) = \sum_{i \in [s]} (1 - 8/\ell)|\Gamma'_i| = (1 - 8/\ell)|Y_t^B| - \ell/2. \tag{6.4.3}$$

Note that by our construction, we have $R \cap [t] \subseteq V(F_t^R)$. Furthermore, $W_t^B \subseteq V(F_t^R)$, since every vertex in W_t^B gets included in $V(F_t^R)$ (in Step 3A). Therefore, using (6.4.3),

$$c_t^R = |V(F_t^R) \cap [t]| \geq |R \cap [t]| + |W_t^B| + |Y_t^B|$$

$$= |R \cap [t]| + |W_t^B| + (1 - 8/\ell)|Y_t^B| - \ell/2$$
$$\geq (1 - 8/\ell)(t - |X_t^B| - |Z_t^B|) - \ell/2,$$

where the last line follows from Lemma 6.4.2(i). Hence (vi) holds.

To see (vii), note that $\rho_{t''}^R(A_t^R)$ is a decreasing sequence in $t'' \geq t$, because the path-forests $\{F_{t''}^R\}_{t'' \geq t}$ form an increasing sequence by Lemma 6.4.2(iv). The value $\rho_{t''}^R(A_t^R)$ decreases at some t'' if and only at round t'' we join some vertices of A_t^R to some vertices in $\Gamma_t^B \cup (Y_{t''}^B \setminus Y_t^B)$ with red edges to extend the red path-forest (in Step 3D). Each vertex of $y \in \Gamma_t^B \cup (Y_{t'}^B \setminus Y_t^B)$ reduces $\rho_t^R(A_t^R)$ by at most 2, thus we deduce $\rho_t^R(A_t^R) - \rho_{t'}^R(A_t^R) \leq 2(|\Gamma_t^B| + |Y_{t'}^B \setminus Y_t^B|)$. Further, by Lemma 6.4.2(xii), $|\Gamma_t^B| \leq \ell/2$, hence (vii) follows.

Now we check (viii). Suppose (viii) fails, thus there exists a minimal $t'' \in \{t, ..., t' - 1\}$ such that $\Omega_{t''}^R \nsubseteq A_t^R$. Note that $A_t^R \subseteq A_{t''}^R$ by Lemma 6.4.2(v), and the assumption implies that $\rho_{t''}^R(A_t^R) \geq \rho_{t'}^R(A_t^R) \geq \ell$. Let us examine round t'' of the algorithm. It must have happened that $\Omega_{t''}^R$ got redefined to be non-empty and different from $\Omega_{t''-1}^R$, which only leaves the possibility that $\Omega_{t''}^R = \sigma_{t''}^R(A_{t''}^R)$. Thus $\Omega_{t''}^R \subseteq A_{t''}^R$ is a minimal subset (according to the order of $A_{t''}^R$) such that $\rho_{t''}^R(\Omega_{t''}^R) \geq \ell$. But we know already that $A_t^R \subseteq A_{t''}^R$ and $\rho_{t''}^R(A_t^R) \geq \ell$, thus by the minimality condition it must hold that $\sigma_{t''}^R(A_{t''}^R) \subseteq A_t^R$, a contradiction.

We check (ix). Let $t'' \in \{t, \dots, t' - 1\}$. By (viii), $\Omega_{t''}^R \subseteq A_t^R$. Then, by Lemma 6.4.2(x),

$$\max\{\varphi(v) : v \in \Omega_{t''}^R\} \leq \max\{\varphi(v) : v \in A_t^R\} \leq t.$$

Consider any $z \in Z_{t'}^B$ and note from the previous observation that $\max\{\varphi(v) : v \in \Omega_{z-1}^R\} \leq t$. By the classification in Step 2 of Algorithm 1, $z \in Z_{t'}^B$ means that during its round, z was classified as a "backwards-spoiled" vertex. This implies that $2 = d_{F_{t_\Omega}^B}(z)$, where $t_\Omega = \max\{\varphi(v) : v \in \Omega_{z-1}^R\}$. Since $t_\Omega \leq t$, we deduce $2 = d_{F_t^B}(z)$. Moreover, since $t_\Omega < z$ we deduce that $N_{F_t^B}(z) = N_{F_{t_\Omega}^B}(z) \subseteq W_{t_\Omega}^R \subseteq W_t^R$, where the first inclusion follows from Lemma 6.4.2(xiv) and the second from Lemma 6.4.2(v). This holds for every $z \in Z_{t'}^B$, thus proving (ix).

Finally, to check (x), we use (ix) to estimate the number of edges in $F_t^B[Z_{t'}^B, W_t^R]$. We have

$$2|Z_{t'}^B| = e(F_t^B[Z_{t'}^B, W_t^R]) \leq 2|W_t^R|,$$

implying (x). ■

The next lemma collects more properties of the output given by the algorithm. Item (iv) is crucial: for every t, it states an inequality involving both c_t^R, c_t^B and $\rho_t^R(A_t^R), \rho_t^B(A_t^B)$. In a nutshell, the inequality says that not all of those values can be small. Since in our setting we will have $1/t \ll 1/\ell \ll 1$, the right-hand side of the inequality is essentially $2t$.

Lemma 6.4.7. *Let $\ell \in \mathbb{N}$ be even. Suppose that $K_\mathbb{N}$ has a restricted 2-edge-colouring. For all $t \in \mathbb{N}$ and $* \in \{R, B\}$, let $F_t^*, \Omega_t^*, \Gamma_t^*, A_t^*, \varphi, W_t^*, X_t^*, Y_t^*, Z_t^*$ be as defined by Algorithm 1. Then there exist $Y_t^* \subseteq D_t^* \subseteq W_t^* \cup Y_t^*$ for all $t \in \mathbb{N}$ and $* \in \{R, B\}$ such that*

(i) *$\rho_{t'}^R(A_{t'}^R) - \rho_t^R(A_t^R) \leq 2|D_{t'}^R \setminus D_t^R| + 2|X_{t'}^R \setminus X_t^R|$, for every $t' \geq t$;*

(ii) *$2|D_t^B| \geq 2|D_t^R \cup X_t^R \cup Z_t^R| - \rho_t^R(A_t^R)$;*

(iii) *if $\rho_{t'-1}^B(A_t^B) \geq \ell$ for some $t' \geq t$, then $2|D_t^B| \geq 2|D_t^R| + |X_{t'}^R \cup Z_{t'}^R| - \rho_t^R(A_t^R)$;*

(iv) *$c_t^R + c_t^B + \frac{1}{2}\rho_t^R(A_t^R) + \frac{1}{2}\rho_t^B(A_t^B) \geq 2(1 - 8/\ell)t - \ell$,*

and similar statements hold if we interchange R and B.

Proof. Let $U_t^R := \{w \in W_t^R : \varphi(w) \leq t\}$ and let $D_t^R := U_t^R \cup Y_t^R$. Intuitively speaking, D_t^R corresponds to the red vertices in $[t]$ which form part of the blue path-forest F_t^R and, at the same time, are already declared as useful vertices at round t. Note that $A_t^R = D_t^R \cup X_t^R \cup Z_t^R$, where this follows from Lemma 6.4.2(ix).

Hence, using Proposition 6.4.1(ii), we have

$$\rho_t^R(A_t^R) = \rho_t^R(D_t^R) + \rho_t^R(X_t^R) + \rho_t^R(Z_t^R) = \rho_t^R(D_t^R) + \rho_t^R(X_t^R). \tag{6.4.4}$$

as $d_{F_t^R}(z) = 2$ for all $z \in Z_t^R$.

Now let $t \le t'$. Note that $U_t^R \subseteq U_{t'}^R$ for $t \le t'$. Thus we can write $D_{t'}^R \cup X_{t'}^R$ as a disjoint union, as $(D_{t'}^R \setminus D_t^R) \cup (X_{t'}^R \setminus X_t^R) \cup (D_t^R \cup X_t^R)$. Hence, using Proposition 6.4.1(ii), we have

$$\begin{aligned}
\rho_{t'}^R(A_{t'}^R) &\overset{(6.4.4)}{=} \rho_{t'}^R(D_{t'}^R) + \rho_{t'}^R(X_{t'}^R) \\
&= \rho_{t'}^R(D_{t'}^R \setminus D_t^R) + \rho_{t'}^R(X_{t'}^R \setminus X_t^R) + \rho_{t'}^B(D_t^R \cup X_t^R) \\
&\overset{(6.4.4)}{=} \rho_{t'}^R(D_{t'}^R \setminus D_t^R) + \rho_{t'}^R(X_{t'}^R \setminus X_t^R) + \rho_{t'}^R(A_t^R) \\
&\le 2|D_{t'}^R \setminus D_t^R| + 2|X_{t'}^R \setminus X_t^R| + \rho_t^R(A_t^R),
\end{aligned}$$

where the last inequality follows from Proposition 6.4.1(i). This proves (i).

Now we check (ii). Let $G_t^R := F_t^R[\{1, \dots, t\}]$. Since F_t^R is a red path-forest, G_t^R is a bipartite graph. Since every edge of F_t^R joins a red vertex with a blue vertex, we might assume one of the bipartite classes of G_t^R consists only of red vertices, and the other one has only blue vertices. If $v \in B$ with $d_{F_t^R}(v) > 0$, then $v \in W_t^B \cup Y_t^B$ by Lemma 6.4.6(i). If $v \in W_t^B$ with $d_{G_t^R}(v) > 0$, then by Lemma 6.4.6(ii) we must have $\varphi(v) \le t$, and so $v \in U_t^B$. Hence if $v \in B$ with $d_{F_t^R}(v) > 0$, then $v \in D_t^B$. Therefore,

$$e(G_t^R) \le 2|D_t^B|. \tag{6.4.5}$$

Now we want to estimate $e(G_t^R)$ by counting $\sum_{u \in R \cap [t]} d_{G_t^R}(u)$. First, we claim that for every $u \in R \cap [t]$, $d_{G_t^R}(u) = d_{F_t^R}(u)$. Indeed, otherwise there exist $u \in R \cap [t]$ and $v \in B \setminus [t]$ such that uv is an edge in F_t^R. In particular, $v \in V(F_t^R)$, but this contradicts Lemma 6.4.2(xiii). Using this we can calculate

$$\begin{aligned} e(G_t^R) &= \sum_{u \in R \cap [t]} d_{G_t^R}(u) = \sum_{u \in R \cap [t]} d_{F_t^R}(u) \\ &\geq \sum_{u \in D_t^R \cup X_t^R \cup Z_t^R} d_{F_t^R}(u) = \sum_{u \in D_t^R \cup X_t^R \cup Z_t^R} d_{F_t^R}(u) \\ &= 2|D_t^R \cup X_t^R \cup Z_t^R| - \sum_{u \in D_t^R \cup X_t^R \cup Z_t^R} \big(d_{F_t^R}(u) - 2\big) \\ &= 2|D_t^R \cup X_t^R \cup Z_t^R| - \rho_t^R(A_t^R). \end{aligned}$$

Together with (6.4.5), we obtain (ii).

To see (iii) proceed similarly while considering the graph $F_t^R[\{1, \dots, t\} \cup X_{t'}^R \cup Z_{t'}^R]$. We use the assumption that $t' \geq t$ and $\rho_{t'-1}^B(A_t^B) \geq \ell$ to get $X_{t'}^R \smallsetminus X_t^R = \emptyset$ from Lemma 6.4.6(v). Thus $(X_{t'}^R \smallsetminus [t]) \cup Z_{t'}^R = Z_{t'}^R$. Lemma 6.4.6(ix) says that, for every $u \in (X_{t'}^R \smallsetminus [t]) \cup Z_{t'}^R = Z_{t'}^R$, $N_{F_t^R}(u) \subseteq W_t^B$ and $d_{F_t^R}(u) = 2$. Counting the edges of $F_t^R[\{1, \dots, t\} \cup X_{t'}^R \cup Z_{t'}^R]$ in two different ways, as before, gives the desired inequality.

By adding (ii) and its analogous version, we get

$$\frac{1}{2}\rho_t^R(A_t^R) + \frac{1}{2}\rho_t^B(A_t^B) \geq |X_t^R \cup Z_t^R \cup X_t^B \cup Z_t^B|. \tag{6.4.6}$$

Lemma 6.4.6(vi) implies that

$$c_t^R + c_t^B \geq 2(1 - 8/\ell)t - |X_t^R \cup Z_t^R \cup X_t^B \cup Z_t^B| - \ell,$$

which together with (6.4.6) implies (iv). ■

6.5 Proof of Lemma 6.2.1

6.5.1 Evolution of $\rho_t^R(A_t^R)$ and $\rho_t^B(A_t^B)$

To prove Lemma 6.2.1, we will consider the path-forests F_t^R, F_t^B for every $t \geq 1$, as constructed by Algorithm 1. If, given ε and k_0, for some $t \geq k_0$ we have

$\max\{c_t^R, c_t^B\} \geq ((9+\sqrt{17})/16 - \varepsilon)t$, then we are done. Therefore, assuming this is not the case, we will deduce information about the evolution of the parameters $\rho_t^R(A_t^R)$ and $\rho_t^B(A_t^B)$ as t increases, which we will use to finish the proof. We remark that for the rest of the proof it suffices to use the properties of F_t^R and F_t^B ensured by Lemmas 6.4.2, 6.4.6 and 6.4.7, instead of appealing to Algorithm 1.

First, we show that if $\rho_t^B(A_t^B) \geq \ell$ then there exists $t' > t$ such that $\rho_{t'}^B(A_t^B) < \ell$ (or we are already done). That is, almost all vertices A_t^B have degree 2 in the red path-forest at round t'. In words, the blue useful vertices which are defined at round t will eventually get used (i.e., joined with red waiting vertices) at some round in the future.

Lemma 6.5.1. *Let $\ell \in \mathbb{N}$ be even. Suppose that $K_{\mathbb{N}}$ has a restricted 2-edge-colouring. Let $F_t^*, \Omega_t^*, \Gamma_t^*, A_t^*, \varphi_t, W_t^*, X_t^*, Y_t^*, Z_t^*$ be as defined by Algorithm 1. Suppose $\rho_t^B(A_t^B) \geq \ell$. Then there exists $t' > t$ such that $\rho_{t'}^B(A_t^B) < \ell$ or $c_{t'}^B \geq (1-9/\ell)t'$.*

Proof. Suppose that $\rho_{t'}^B(A_t^B) \geq \ell$ for all $t' > t$ (or else we are done). Using Lemma 6.4.6(v) we deduce that $X_{t'}^R = X_t^R$ for all $t' > t$. Using Lemma 6.4.6(x), we deduce that $|Z_{t'}^R| \leq |W_t^B|$ for all $t' \geq t$. We have shown that for every $t' > t$,

$$|X_{t'}^R| + |Z_{t'}^R| \leq |X_t^R| + |W_t^B| \leq t, \tag{6.5.1}$$

where the last inequality follows from Lemma 6.4.6(i). Now choose $t^* = \ell(t+\ell/2)$, and note that $t^* \geq t$. Then Lemma 6.4.6(vi) implies that

$$\begin{aligned} c_{t^*}^B &\geq (1-8/\ell)t^* - |Z_{t^*}^R| - |X_{t^*}^R| - \ell/2 \\ &\overset{(6.5.1)}{\geq} (1-8/\ell)t^* - t - \ell/2 \\ &= (1-8/\ell)t^* - (t+\ell/2) = (1-9/\ell)t^*. \end{aligned}$$

This completes the proof. ■

Now, we show that if $\rho_t^B(A_t^B) \geq \ell$, then there exists a $t' > t$ such that $\rho_{t'}^B(A_{t'}^B) < \ell$ (or we are already done). This differs from Lemma 6.5.1 in the following: in the previous lemma we argued that, for a fixed t, there was another round t' in the future such that the useful blue vertices from A_t^B were "used" (i.e., $\rho_{t'}^B(A_t^B) < \ell$). Now, we show that there exists a round t' in the future where the set of available vertices *of that round* is empty (i.e., $\rho_{t'}^B(A_{t'}^B) < \ell$). Thus, there is always some round where "all" of the available vertices at that round are used.

Lemma 6.5.2. *Let $\ell \in \mathbb{N}$ be even and $1/t_0 \ll 1/\ell \ll \varepsilon \leq 1/2$. Suppose that $K_{\mathbb{N}}$ has a restricted 2-edge-colouring. Let F_t^*, Ω_t^*, Γ_t^*, A_t^*, φ_t, W_t^*, X_t^*, Y_t^*, Z_t^* be as defined by Algorithm 1. Suppose that $\rho_{t_0}^B(A_{t_0}^B) \geq \ell$. Then there exists $t' > t_0$ such that $\rho_{t'}^B(A_{t'}^B) < \ell$ or $\max\{c_{t'}^R, c_{t'}^B\} \geq (2\sqrt{2} - 2 - \varepsilon)t'$.*

Proof. Let $\alpha := 3 - 2\sqrt{2} \approx 0.1715$. Suppose the contrary, that is, for all $t > t_0$ we have

$$\rho_t^B(A_t^B) \geq \ell \text{ and } c_t^R, c_t^B \leq (1 - \alpha - \varepsilon)t. \tag{6.5.2}$$

Note that (6.5.2) together with Lemma 6.4.6(iii) implies that for all $t > t_0$, $\Omega_t^R \neq \varnothing$. Together with Lemma 6.4.6(iv) we deduce that, for all $t > t_0$,

$$X_t^R = X_{t_0}^R. \tag{6.5.3}$$

Since $1/\ell \ll \varepsilon$, from (6.5.2) we deduce that $\max\{c_t^B, c_t^R\} < (1 - 9/\ell)t$ holds for all $t \geq t_0$. Therefore, by Lemma 6.5.1, there must exist some $t > t_0$ such that $\rho_t^R(A_{t_0}^R) < \ell$.

We use similar arguments to define the following sequences of integers, t_i, t_i', t_i^R and t_i^B, for all $i > 0$. Given t_i, define t_{i+1}^R to be the minimum $t > t_i$ such that $\rho_t^R(A_{t_i}^R) < \ell$ (similarly as before, this exists by Lemma 6.5.1, (6.5.2) and $1/\ell \ll \varepsilon$). Analogously, define t_{i+1}^B. Define $t_{i+1} := \max\{t_{i+1}^R, t_{i+1}^B\}$ and $t_{i+1}' := \min\{t_{i+1}^R, t_{i+1}^B\}$.

This defines sequences t_i, t'_i such that, for all $i \geq 1$,

$$\begin{aligned} t_{i-1} < t'_i \leq t_i, \\ \min\{\rho^R_{t'_i}(A^R_{t_{i-1}}), \rho^B_{t'_i}(A^B_{t_{i-1}})\} < \ell, \\ \rho^R_{t_i}(A^R_{t_{i-1}}), \rho^B_{t_i}(A^B_{t_{i-1}}) < \ell. \end{aligned}$$

For convenience, let $t_{-1} := 0$ and, for every $i \geq 0$, let $I_i := \{t_{i-1}+1, \dots, t_i\}$. For every $i \geq 0$ and $* \in \{R, B\}$, let

$$x^*_i := |I_i \cap X^*_{t_i}| \text{ and } z^*_i := |I_i \cap Z^*_{t_i}|.$$

Lemma 6.4.6(vi) and (6.5.2) imply that

$$\begin{aligned} (1-\alpha-\varepsilon)t_i \geq c^R_{t_i} &\geq (1-8/\ell)t_i - |Z^B_{t_i}| - |X^B_{t_i}| - \ell/2 \\ &\geq (1-8/\ell)t_i - \sum_{j\in[i]_0}(x^B_j + z^B_j) - \ell/2, \end{aligned}$$

and a similar inequality also holds for $\sum_{j\in[i]_0}(x^R_j + z^R_j)$. In summary, using $1/t_i \leq 1/t_0 \ll 1/\ell \ll \varepsilon$, we have, for all $* \in \{R, B\}$,

$$\sum_{j\in[i]_0}(x^*_j + z^*_j) \geq (\alpha + \varepsilon/2)t_i. \tag{6.5.4}$$

Consider any $i \geq 1$. Write $T_i := \sum_{j\in[i]_0} t_j$. Lemma 6.4.2(v) and Lemma 6.4.6(vii) imply that

$$|Y^B_{t_i} \setminus Y^B_{t_{i-1}}| \geq |Y^B_{t^R_i} \setminus Y^B_{t_{i-1}}| \geq \frac{1}{2}(\rho^R_{t_{i-1}}(A^R_{t_{i-1}}) - \rho^R_{t^R_i}(A^R_{t_{i-1}}) - \ell) \geq \frac{1}{2}(\rho^R_{t_{i-1}}(A^R_{t_{i-1}}) - 2\ell)$$

and that a similar inequality holds for $|Y^R_{t_i} \setminus Y^R_{t_{i-1}}|$. Hence by combining both inequalities and using Lemma 6.4.7(iv), we have

$$|Y^B_{t_i} \setminus Y^B_{t_{i-1}}| + |Y^R_{t_i} \setminus Y^R_{t_{i-1}}| \geq \frac{1}{2}(\rho^R_{t_{i-1}}(A^R_{t_{i-1}}) + \rho^B_{t_{i-1}}(A^B_{t_{i-1}})) - 2\ell$$

$$
\begin{aligned}
&\overset{\text{Lem. 6.4.7(iv)}}{\geq} 2(1-8/\ell)t_{i-1} - c^R_{t_{i-1}} - c^B_{t_{i-1}} - 3\ell \\
&\overset{(6.5.2)}{\geq} 2(1-8/\ell)t_{i-1} - 2(1-\alpha-\varepsilon)t_{i-1} - 3\ell \\
&= 2(\alpha+\varepsilon)t_{i-1} - 16t_{i-1}/\ell - 3\ell \\
&\geq 2(\alpha+\varepsilon/2)t_{i-1},
\end{aligned}
$$

where the last inequality follows from $1/t_{i-1} \leq 1/t_0 \ll 1/\ell \ll \varepsilon$. Hence, for all $i \geq 0$, summing over all $j \leq i$ we get

$$
|Y^B_{t_i} \cup Y^R_{t_i}| \geq 2(\alpha+\varepsilon/2)T_{i-1}. \tag{6.5.5}
$$

Claim 6.5.3. *For all $i \geq 1$, $|W^R_{t_i} \cup W^B_{t_i} \cup X^B_{t_i} \cup Z^B_{t_i}| \geq (\alpha+\varepsilon/2)(T_{i+1} - T_{i-1}) - t_0$.*

Proof of the claim. Recall that for all $i \geq 0$, $T_i = \sum_{j\in[i]_0} t_j$. Thus $T_{i+1} - T_{i-1} = t_i + t_{i+1}$ for all $i \geq 1$. Hence, to prove the claim is equivalent to proving that $|W^R_{t_i} \cup W^B_{t_i} \cup X^B_{t_i} \cup Z^B_{t_i}| \geq (\alpha+\varepsilon/2)(t_{i+1}+t_i) - t_0$ holds for all $i \geq 1$. Let $i \geq 1$ be fixed for the rest of the proof.

We divide the proof into two cases. First suppose that $t^B_{i+1} \geq t^R_{i+1}$. Thus $t_{i+1} = t^B_{i+1}$, which by definition is the minimum $t' > t_i$ such that $\rho^B_{t'}(A^B_{t_i}) < \ell$. In particular, $\rho^B_{t_{i+1}-1}(A^B_{t_i}) \geq \ell$. By Lemma 6.4.6(x), we deduce $|W^B_{t_i}| \geq |Z^R_{t_{i+1}}| = \sum_{j\in[i+1]_0} z^R_j$. Hence

$$
\begin{aligned}
|W^R_{t_i} \cup W^B_{t_i} \cup X^B_{t_i} \cup Z^B_{t_i}| &\geq |W^B_{t_i}| + |X^B_{t_i} \cup Z^B_{t_i}| \geq \sum_{j\in[i+1]_0} z^R_j + \sum_{j\in[i]_0} (x^B_j + z^B_j) \\
&\overset{(6.5.3)}{=} \sum_{j\in[i+1]_0} (x^R_j + z^R_j) - x^R_0 + \sum_{j\in[i]_0} (x^B_j + z^B_j) \\
&\overset{(6.5.4)}{\geq} (\alpha+\varepsilon/2)(t_i + t_{i+1}) - t_0,
\end{aligned}
$$

so the claim holds in this case.

Now, suppose that $t^B_{i+1} < t^R_{i+1}$. By the choice of t^R_{i+1}, Lemma 6.4.6(x) implies that $|W^R_{t_i}| \geq |Z^B_{t_{i+1}}| = \sum_{j\in[i+1]_0} z^B_j$. By a similar argument together with Lemma 6.4.2(v), we get $|W^B_{t_i}| \geq |Z^R_{t^B_{i+1}}| \geq |Z^R_{t_i}| = \sum_{j\in[i]_0} z^R_j$. The choice of $t_{i+1} = t^R_{i+1}$ implies that

$\rho^R_{t_{i+1}-1}(A^R_{t_i}) \geq \ell$, thus Lemma 6.4.6(v) implies that $X^B_{t_{i+1}} = X^B_{t_i}$ and so $x^B_{i+1} = 0$. Using these bounds together, we get

$$\begin{aligned}
|W^R_{t_i} \cup W^B_{t_i} \cup X^B_{t_i} \cup Z^B_{t_i}| &\geq |W^R_{t_i}| + |W^B_{t_i}| + |X^B_{t_i}| \\
&\geq \sum_{j\in[i+1]_0} z^B_j + \sum_{j\in[i]_0} z^R_j + \sum_{j\in[i]_0} x^B_j \\
&\overset{(6.5.3)}{=} \sum_{j\in[i+1]_0} (x^B_j + z^B_j) + \sum_{j\in[i]_0} (x^R_j + z^R_j) - x^R_0 \\
&\overset{(6.5.4)}{\geq} (\alpha + \varepsilon/2)(t_i + t_{i+1}) - t_0.
\end{aligned}$$

This finishes the proof of the claim. □

Now we use Lemma 6.4.2(i), Claim 6.5.3 and (6.5.5), to get

$$\begin{aligned}
t_i - |Z^R_{t_i}| - |X^R_{t_i}| &\overset{\text{Lem. 6.4.2(i)}}{=} |Y^B_{t_i} \cup Y^R_{t_i}| + |W^R_{t_i} \cup W^B_{t_i} \cup X^B_{t_i} \cup Z^B_{t_i}| \\
&\overset{\text{Claim 6.5.3}}{\geq} |Y^B_{t_i} \cup Y^R_{t_i}| + (\alpha + \varepsilon/2)(T_{i+1} - T_{i-1}) - t_0 \\
&\overset{(6.5.5)}{\geq} 2(\alpha + \varepsilon/2)T_{i-1} + (\alpha + \varepsilon/2)(T_{i+1} - T_{i-1}) - t_0 \\
&= (\alpha + \varepsilon/2)(T_{i-1} + T_{i+1}) - t_0.
\end{aligned} \tag{6.5.6}$$

Now, by the definition of $T_i = \sum_{j\in[i]_0} t_j$ it follows that for all $i \geq 1$, $T_i - T_{i-1} = t_i$. Hence, (6.5.2) and Lemma 6.4.6(vi) imply that

$$\begin{aligned}
(1-\alpha)(T_i - T_{i-1}) = (1-\alpha)t_i &\overset{(6.5.2)}{\geq} c^B_{t_i} \\
&\overset{\text{Lem. 6.4.6(vi)}}{\geq} (1 - 8/\ell)(t_i - |Z^R_{t_i}| - |X^R_{t_i}|) - \ell/2 \\
&\overset{(6.5.6)}{\geq} (1 - 8/\ell)\left[(\alpha + \varepsilon/2)(T_{i-1} + T_{i+1}) - t_0\right] - \ell/2 \\
&\geq (1 - 8/\ell)(\alpha + \varepsilon/2)(T_{i-1} + T_{i+1}) - t_0 - \ell/2 \\
&\geq (\alpha + \varepsilon/4)(T_{i-1} + T_{i+1}) - t_0 - \ell/2,
\end{aligned}$$

where the last inequality follows from $1/\ell \ll \varepsilon$. Rearranging, we get $0 \geq (\alpha + \varepsilon/4)T_{i+1} - (1-\alpha)T_i + T_{i-1} - t_0 - \ell/2$. Therefore, Lemma 6.3.3 (and our choice of α)

implies

$$0 \le (1-\alpha)^2 - 4(\alpha + \varepsilon/4) < 1 - 6\alpha + \alpha^2 = 0,$$

a contradiction. ∎

Now we are ready to prove Lemma 6.2.1.

Proof of Lemma 6.2.1. Let $\alpha := (7-\sqrt{17})/16 \approx 0.27214$. Without loss of generality, we can assume $\varepsilon \ll \alpha$. Choose $\ell, k_0' \in \mathbb{N}$ such that ℓ is even, $k_0' \ge k_0$ and

$$0 < 1/k_0' \ll 1/\ell \ll \varepsilon \ll \alpha. \tag{6.5.7}$$

We will repeatedly use Lemma 6.4.2(i), which says that, for all $t \ge 0$

$$t = \sum_{* \in \{R,B\}} \left(|X_t^*| + |Y_t^*| + |Z_t^*| + |W_t^*|\right). \tag{6.5.8}$$

Lemma 6.4.6(vi) and (6.5.8) imply that for all $t \ge 0$,

$$c_t^R \ge (1 - 8/\ell)(|W_t^R \cup Y_t^R| + |W_t^B \cup Y_t^B| + |X_t^R \cup Z_t^R|) - \ell/2, \tag{6.5.9}$$

and a similar bound is true after replacing R by B.

We can suppose that for all $t \ge k_0$ we have

$$c_t^R, c_t^B \le (1 - \alpha - \varepsilon)t, \tag{6.5.10}$$

or else we are done. Now, suppose $t \ge k_0'$. From (6.5.10) and Lemma 6.4.7(iv), we get

$$2\left(1 - \frac{8}{\ell}\right) - \ell \le 2(1-\alpha-\varepsilon)t + \frac{1}{2}\left(\rho_t^R(A_t^R) + \rho_t^B(A_t^B)\right),$$

which, after rearranging, becomes

$$\rho_t^R(A_t^R) + \rho_t^B(A_t^B) \geq 4\left(\alpha + \varepsilon - \frac{8}{\ell}\right)t - 2\ell \geq 2\ell,$$

where the last inequality follows from (6.5.7) and $t \geq k_0'$. We deduce that

$$\rho_t^R(A_t^R) \geq \ell \text{ or } \rho_t^B(A_t^B) \geq \ell \qquad \forall t \geq k_0'. \tag{6.5.11}$$

Without loss of generality, we may assume that $\rho_{k_0'}^B(A_{k_0'}^B) \geq \ell$. Note

$$1 - \alpha = (9 + \sqrt{17})/16 \leq 0.82020 < 0.82842 \leq 2\sqrt{2} - 2,$$

thus, using (6.5.10) together with Lemma 6.5.2 and $\varepsilon \ll \alpha$, we deduce there exists $t > k_0'$ such that $\rho_t^B(A_t^B) < \ell$. Let t_0 be the minimum of all such possible t. Note that $\rho_{t_0}^R(A_{t_0}^R) \geq \ell$ by (6.5.11). Similarly, we can define t_1 to be the minimum $t > t_0$ such that $\rho_t^R(A_t^R) < \ell$. Now proceed in a slightly different way and define t_2 to be the minimum $t > t_1$ such that $\rho_t^B(A_{t_1}^B) < \ell$. Note that t_2 exists by Lemma 6.5.1 and (6.5.10), and that $t_0 < t_1 < t_2$.

Lemma 6.4.6(vi) and (6.5.10) imply that for all $* \in \{R, B\}$ and $i \in [2]$,

$$|X_{t_i}^* \cup Z_{t_i}^*| \geq (\alpha + \varepsilon/2)t_i. \tag{6.5.12}$$

Claim 6.5.4. *There exist*

$$H^R \subseteq Y_{t_1}^R \cup W_{t_1}^R \text{ and } H^B \subseteq Y_{t_1}^B \cup W_{t_1}^B \tag{6.5.13}$$

such that

$$|H^R| = |X_{t_1}^B \cup Z_{t_1}^B| - \ell, \qquad \text{and} \qquad |H^B| = |X_{t_1}^B \cup Z_{t_1}^B| + |X_{t_2}^R \cup Z_{t_2}^R| - \ell. \tag{6.5.14}$$

Proof of the claim. For every $* \in \{R, B\}$, consider $D_{t_1}^* \subseteq Y_{t_1}^* \cup W_{t_1}^*$ as given by

Lemma 6.4.7. Note that $\rho^B_{t_0}(A^B_{t_0}) \leq \ell$. Then Lemma 6.4.7(i) implies

$$\begin{aligned}\rho^B_{t_1}(A^B_{t_1}) - \ell \leq \rho^B_{t_1}(A^B_{t_1}) - \rho^B_{t_0}(A^B_{t_0}) &\leq 2|D^B_{t_1} \setminus D^B_{t_0}| + 2|X^B_{t_1} \setminus X^B_{t_0}| \\ &\leq 2|D^B_{t_1}| + 2|X^B_{t_1} \setminus X^B_{t_0}|.\end{aligned}$$

By the choice of t_0 and t_1, $\rho^R_{t'}(A^R_{t'}) \geq \ell$ for all $t_0 \leq t' < t_1$. Lemma 6.4.6(iii) then implies that $\Omega^R_{t''} \neq \varnothing$ for all $t'' \in \{t_0, \dots, t_1 - 1\}$. Therefore, Lemma 6.4.6(iv) implies that $X^B_{t_1} \setminus X^B_{t_0} = \varnothing$. Hence,

$$\rho^B_{t_1}(A^B_{t_1}) \leq 2|D^B_{t_1}| + \ell. \tag{6.5.15}$$

Lemma 6.4.7(ii) and (6.5.15) together imply that

$$2|D^R_{t_1}| \geq 2|D^B_{t_1}| + 2|X^B_{t_1} \cup Z^B_{t_1}| - \rho^B_{t_1}(A^B_{t_1}) \geq 2|X^B_{t_1} \cup Z^B_{t_1}| - \ell. \tag{6.5.16}$$

Recall that $\rho^R_{t_1}(A^R_{t_1}) \leq \ell$ and $\rho^B_{t_2-1}(A^B_{t_1}) \geq \ell$. By Lemma 6.4.7(iii),

$$\begin{aligned}2|D^B_{t_1}| &\geq 2|D^R_{t_1}| + 2|X^R_{t_2} \cup Z^R_{t_2}| - \rho^R_{t_1}(A^R_{t_1}) \geq 2|D^R_{t_1}| + 2|X^R_{t_2} \cup Z^R_{t_2}| - \ell \\ &\overset{(6.5.16)}{\geq} 2|X^B_{t_1} \cup Z^B_{t_1}| + 2|X^R_{t_2} \cup Z^R_{t_2}| - 2\ell.\end{aligned}$$

Thus $|D^B_{t_1}| \geq |X^B_{t_1} \cup Z^B_{t_1}| + |X^R_{t_2} \cup Z^R_{t_2}| - \ell$ and $|D^R_{t_1}| \geq |X^B_{t_1} \cup Z^B_{t_1}| - \ell$, which implies the existence of a set $H^* \subseteq D^*_{t_1} \subseteq Y^*_{t_1} \cup W^*_{t_1}$ of the desired size for each $* \in \{R, B\}$. □

Since $k'_0 \leq t_0 \leq t_1 \leq t_2$, we have $1/t_2, 1/t_1 \ll 1/\ell \ll \alpha, \varepsilon$. Let H^R and H^B be given by Claim 6.5.4. Let

$$\begin{aligned}a &:= |X^B_{t_1} \cup Z^B_{t_1}|, \\ b &:= |X^R_{t_1} \cup Z^R_{t_1}|, \\ c &:= |(X^B_{t_2} \cup Z^B_{t_2}) \setminus (X^B_{t_1} \cup Z^B_{t_1})|, \text{ and} \\ d &:= |(X^R_{t_2} \cup Z^R_{t_2}) \setminus (X^R_{t_1} \cup Z^R_{t_1})|.\end{aligned}$$

Thus, by 6.5.14, $|H^R| = a - \ell$ and $|H^B| = a + b + d - \ell$. Also note that, by (6.5.8) and (6.5.13), we have

$$\begin{aligned} t_1 &= |Y^B_{t_1} \cup W^B_{t_1}| + |Y^R_{t_1} \cup W^R_{t_1}| + |X^B_{t_1} \cup Z^B_{t_1}| + |X^R_{t_1} \cup Z^R_{t_1}| \\ &\geq |H^B| + |H^R| + |X^B_{t_1} \cup Z^B_{t_1}| + |X^R_{t_1} \cup Z^R_{t_1}| \\ &= (a + b + d - \ell) + (a - \ell) + a + b = 3a + 2b + d - 2\ell. \end{aligned} \tag{6.5.17}$$

Let $\delta := \varepsilon/2$ and $\rho := \alpha + \delta$. Since $\alpha = (7 - \sqrt{17})/16$ is the least real root of the polynomial $8x^2 - 7x + 1$ and $0 < \varepsilon < 1/2$, it follows that $1 \leq 7\rho - 8\rho^2$.

Now we use the previous bounds to get

$$\begin{aligned} 1 - \alpha - \varepsilon &\overset{(6.5.10)}{\geq} \frac{c^R_{t_1}}{t_1} \\ &\overset{(6.5.9)}{\geq} \frac{(1 - 8/\ell)(|W^R_{t_1} \cup Y^R_{t_1}| + |W^B_{t_1} \cup Y^B_{t_1}| + |X^R_{t_1} \cup Z^R_{t_1}|) - \ell/2}{t_1} \\ &\overset{(6.5.8)}{\geq} \frac{|W^R_{t_1} \cup Y^R_{t_1}| + |W^B_{t_1} \cup Y^B_{t_1}| + |X^R_{t_1} \cup Z^R_{t_1}| - \ell/2}{|W^R_{t_1} \cup Y^R_{t_1}| + |W^B_{t_1} \cup Y^B_{t_1}| + |X^R_{t_1} \cup Z^R_{t_1}| + |X^B_{t_1} \cup Z^B_{t_1}|} - \frac{8}{\ell}. \end{aligned}$$

Now, note that for $x \geq 0$ and $a \geq b \geq 0$ it holds that $(a+x)/(b+x) \geq a/b$. By (6.5.13) it holds that $|H^R| \leq |W^R_{t_1} \cup Y^R_{t_1}|$ and $|H^B| \leq |W^B_{t_1} \cup Y^B_{t_1}|$. Using this, we deduce

$$\begin{aligned} 1 - \alpha - \varepsilon &\geq \frac{|W^R_{t_1} \cup Y^R_{t_1}| + |W^B_{t_1} \cup Y^B_{t_1}| + |X^R_{t_1} \cup Z^R_{t_1}| - \ell/2}{|W^R_{t_1} \cup Y^R_{t_1}| + |W^B_{t_1} \cup Y^B_{t_1}| + |X^R_{t_1} \cup Z^R_{t_1}| + |X^B_{t_1} \cup Z^B_{t_1}|} - \frac{8}{\ell} \\ &\overset{(6.5.13)}{\geq} \frac{|H^R| + |H^B| + |X^R_{t_1} \cup Z^R_{t_1}| - \ell/2}{|H^R| + |H^B| + |X^R_{t_1} \cup Z^R_{t_1}| + |X^B_{t_1} \cup Z^B_{t_1}|} - \frac{8}{\ell} \\ &= \frac{2a + 2b + d - 5\ell/2}{3a + 2b + d - 2\ell} - \frac{8}{\ell} \geq \frac{2a + 2b + d}{3a + 2b + d} - \frac{\varepsilon}{2}, \end{aligned}$$

where the last line follows from (6.5.7), (6.5.12) and $1/t_1 \ll 1/\ell \ll \alpha, \varepsilon$. Rearranging, we get $\rho \leq a/(3a + 2b + d)$, and, recalling that $1 \leq 7\rho - 8\rho^2$, we have

$$3a + 2b + d \leq (7 - 8\rho)a. \tag{6.5.18}$$

A similar argument (by estimating $c^B_{t_1}/t_1$) shows that

$$3a + 2b + d \leq (7 - 8\rho)b. \tag{6.5.19}$$

Next, we would like to estimate $c^B_{t_2}/t_2$ and $c^R_{t_2}/t_2$. By the choice of t_1, Lemma 6.4.7(iv) and (6.5.10),

$$\begin{aligned}
\rho^B_{t_1}(A^B_{t_1}) &\geq 4(1 - 8/\ell)t_1 - 2\ell - 2(c^R_{t_1} + c^B_{t_1}) - \rho^R_{t_1}(A^R_{t_1}) \\
&\geq 4(1 - 8/\ell)t_1 - 2\ell - 4(1 - \alpha - \varepsilon)t_1 - \ell \\
&\geq 4(\alpha + 2\varepsilon/3)t_1,
\end{aligned}$$

where the last inequality follows from (6.5.7). Together with Lemma 6.4.6(vii) and the choice of t_2 we get

$$\begin{aligned}
2|Y^B_{t_2} \smallsetminus Y^B_{t_1}| &\geq \rho^B_{t_1}(A^B_{t_1}) - \rho^B_{t_2}(A^B_{t_1}) - \ell \\
&\geq 4(\alpha + 2\varepsilon/3)t_1 - 2\ell \\
&\overset{(6.5.17)}{\geq} 4(\alpha + 2\varepsilon/3)(3a + 2b + d - 2\ell) - 2\ell \\
&\geq 4\rho(3a + 2b + d),
\end{aligned}$$

where the last line uses $1/t_1 \ll 1/\ell \ll \varepsilon$. Thus we deduce

$$|Y^B_{t_2} \smallsetminus Y^B_{t_1}| \geq 2\rho(3a + 2b + d). \tag{6.5.20}$$

Using Claim 6.5.4, we get

$$\begin{aligned}
1 - \alpha - \varepsilon &\overset{(6.5.10)}{\geq} \frac{c^B_{t_2}}{t_2} \\
&\overset{(6.5.9)}{\geq} \frac{(1 - 8/\ell)(|W^R_{t_2} \cup Y^R_{t_2}| + |W^B_{t_2} \cup Y^B_{t_2}| + |X^B_{t_2} \cup Z^B_{t_2}|) - \ell/2}{t_2} \\
&\overset{(6.5.8)}{\geq} \frac{|W^R_{t_2} \cup Y^R_{t_2}| + |W^B_{t_2} \cup Y^B_{t_2}| + |X^B_{t_2} \cup Z^B_{t_2}| - \ell/2}{|W^R_{t_2} \cup Y^R_{t_2}| + |W^B_{t_2} \cup Y^B_{t_2}| + |X^B_{t_2} \cup Z^B_{t_2}| + |X^R_{t_2} \cup Z^R_{t_2}|} - \frac{8}{\ell}
\end{aligned}$$

$$
\begin{aligned}
&\overset{(6.5.13)}{\geq} \frac{|H^R| + |H^B| + |Y^B_{t_2} \smallsetminus Y^B_{t_1}| + |X^B_{t_2} \cup Z^B_{t_2}| - \ell/2}{|H^R| + |H^B| + |Y^B_{t_2} \smallsetminus Y^B_{t_1}| + |X^B_{t_2} \cup Z^B_{t_2}| + |X^R_{t_2} \cup Z^R_{t_2}|} - \frac{8}{\ell} \\
&\overset{(6.5.20)}{\geq} \frac{|H^R| + |H^B| + 2\rho(3a + 2b + d) + |X^B_{t_2} \cup Z^B_{t_2}| - \ell/2}{|H^R| + |H^B| + 2\rho(3a + 2b + d) + |X^B_{t_2} \cup Z^B_{t_2}| + |X^R_{t_2} \cup Z^R_{t_2}|} - \frac{8}{\ell} \\
&\overset{(6.5.14)}{\geq} \frac{2a + b + d + 2\rho(3a + 2b + d) + a + c - 3\ell/2}{2a + b + d + 2\rho(3a + 2b + d) + a + c + b + d - 2\ell} - \frac{8}{\ell} \\
&\geq \frac{3a + b + c + d + 2\rho(3a + 2b + d)}{3a + 2b + c + 2d + 2\rho(3a + 2b + d)} - \frac{\varepsilon}{2},
\end{aligned}
$$

where the last inequality follows from (6.5.7), (6.5.12) and $1/t_2 \ll 1/\ell \ll \alpha, \varepsilon$. Rearranging, we get $\rho \leq (b + d)/[(1 + 2\rho)(3a + 2b + d) + c + d]$. Recalling that $1 \leq 7\rho - 8\rho^2$, we get

$$(1 + 2\rho)(3a + 2b + d) + c + d \leq (7 - 8\rho)(b + d). \tag{6.5.21}$$

A similar argument (by estimating $c^R_{t_2}/t_2$) shows that

$$(1 + 2\rho)(3a + 2b + d) + c + d \leq (7 - 8\rho)(a + c). \tag{6.5.22}$$

By (6.5.18), (6.5.19), (6.5.21) and (6.5.22), we deduce that $Ax \leq 0$, where $x = (a, b, c, d)^t$ and

$$A = \begin{bmatrix} 8\rho - 4 & 2 & 0 & 1 \\ 3 & 8\rho - 5 & 0 & 1 \\ 7\rho - 2 & 1 + 2\rho & 4\rho - 3 & 1 + \rho \\ 3 + 6\rho & 12\rho - 5 & 1 & 10\rho - 5 \end{bmatrix}.$$

Now consider the column vector $y = (7 - 12\alpha, 2 - 4\alpha, 1, 3 - 4\alpha)^t$. Then $y \geq 0$ and $y^t A = ((81 - 120\alpha)\delta, (54 - 80\alpha)\delta, 4\delta, (31 - 40\alpha)\delta) \geq (\delta, \delta, \delta, \delta) > 0$. Since $Ax \leq 0$ and $x, y \geq 0$, we get

$$0 \geq (y^t A)x \geq (\delta, \delta, \delta, \delta)x = \delta(a + b + c + d) > 0,$$

which is a contradiction. ■

— 7 —

Further directions

In this chapter, we discuss possible directions for further research, based on the different topics contained in this thesis.

7.1 Asymptotic bounds for the strong chromatic number

Recall that the main result of Chapter 3 was Theorem 1.2.2, which established an asymptotically tight upper bound on the strong chromatic number, valid for graphs with linear maximum degree, thus obtaining a weak asymptotic version of a conjecture of Aharoni, Berger and Ziv (Conjecture 1.2.1).

The obvious open question is the validity of the whole conjecture.

Question 7.1.1. *Does every graph G satisfy $\chi_s(G) \leq 2\Delta(G)$?*

We suspect that the full answer is difficult and requires fundamentally new ideas. As a more attainable goal, it would be interesting to establish some strengthening of Theorem 1.2.2. Our proof works in the case where the graph G has a "linear" lower bound on the maximum degree, the number of vertices is large, and gives $\chi_s(G) \leq (2+\varepsilon)\Delta(G)$ as a result. Under the same hypotheses, is it possible to remove the error term in our result?

Question 7.1.2. *Let $c > 0$. Does every sufficiently large graph G on n vertices, with $\Delta(G) > cn$, satisfy $\chi_s(G) \leq 2\Delta(G)$?*

A possible strategy to tackle this question would be to use our method combined with some stability analysis of the extremal cases. However, we suspect this could be difficult, as the extremal examples can be very far from having "partial strong colourings". For instance, Szabó and Tardos [68] described examples of graphs G with maximum degree $\Delta(G) = \Delta$, together with a partition of $\mathcal{P}$ into clusters of size $2\Delta - 1$ such that $\mathcal{P}$ does not admit *any* independent transversal.

In a different direction, another possible strengthening would be to remove the lower bound on the maximum degree. We remark that, even with the error term in the conclusion, our proof collapses in this case, since we need some lower bound on $\Delta(G)$ to build the absorbing structures.

Question 7.1.3. *Let $\varepsilon > 0$. Does every sufficiently large graph G satisfy $\chi_{\mathrm{s}}(G) \leq (2 + \varepsilon)\Delta(G)$?*

Now we concentrate on tilings. Recall that we deduced Theorem 1.2.2 from a result on tilings with independent transversals (Theorem 1.2.3), which is easily seen to be equivalent to an statement on tilings with complete graphs in partite graphs (Corollary 1.2.4). More precisely, the corollary states that for any $\varepsilon > 0$, any positive integer $k \geq 3$, and n large enough, all k-partite graphs with classes of the same size n and minimum degree at least $(k - 3/2 + \varepsilon)n$ contain a K_k-tiling, which is best possible up to the error term εn.

This result is related, but different, to multipartite versions of the Hajnal-Szemerédi theorem, as obtained by Lo and Markström [52] and Keevash and Mycroft [44, 45]. In their results, they consider a "multipartite" notion of minimum degree. Assume that the partition of a k-partite graph G is given by $\{V_1, \ldots, V_k\}$. In their setting, they ask that for every distinct $i, j \in [k]$, the minimum degree of $G[V_i \cup V_j]$ is bounded below uniformly over the choice of i and j. In our case, the lower bound is on the minimum degree of G only.

It would be interesting to generalise these results to the setting of finding K_t^k-tilings (i.e., tilings with complete k-graphs on t vertices) in t-partite k-graphs.

It is also probably very challenging to do so, since for general values of $t \geq k \geq 3$ the values of $t(n, K_t^k)$ (i.e., the codegree threshold which guarantees perfect K_t^k-tilings) are not known. That is, there is no known analogue of the Hajnal-Szemerédi theorem for k-uniform graphs, let alone a "multipartite" version.

For K_t^k-tilings in general graphs, Lo and Zhao [56] proved that that there exist constants $c_1, c_2 > 0$, depending on k only, such that

$$\left(1 - c_2 \frac{\ln t}{t^{k-1}} + o(1)\right) n \leq t(n, K_t^k) \leq \left(1 - c_1 \frac{\ln t}{t^{k-1}} + o(1)\right) n.$$

Some results about K_t^k-tilings in t-partite k-graphs were obtained by Lo and Markström [52], which we proceed to describe. Let $t > k \geq 3$, and $\gamma > 0$, and suppose n is sufficiently large. Let H is a t-partite k-graph with a vertex partition $\mathcal{P} = \{V_1, \dots, V_t\}$ into clusters of size n each. Assume that for every $\mathcal{P}$-partite $(k-1)$-set S, and for every cluster V_i disjoint from S, the set S has at least

$$\left(1 - \frac{1}{\binom{t-1}{k-1} + 2\binom{t-2}{k-2}} + \gamma\right) n$$

neighbours in V_i. Then H has a K_t^k-tiling. It is not known if this value is best possible for general values of k and t, and further research would be interesting.

7.2 Covering and tiling hypergraphs with tight cycles

Now we discuss results on covering and tiling thresholds for tight cycles, as discussed in Chapters 4 and 5.

We start by considering covering thresholds. Let $s > k \geq 3$. Theorem 1.3.4 and Proposition 1.3.6 together imply that $c(n, C_s^k) = (1/2 + o(1))n$ for all admissible pairs (k, s) with $s \geq 2k^2$. A natural open question is to determine $c(n, C_s^k)$ for the non admissible pairs (k, s). The smallest case not covered by our constructions is when $(k, s) = (6, 8)$, and Proposition 1.3.7 implies that $c(n, C_8^6) \geq \lfloor n/3 \rfloor - 4$.

Question 7.2.1. *Is the lower bound for $c(n, C_s^k)$ given by Proposition 1.3.7* *asymptotically tight, for non admissible pairs* (k,s)*? In particular, is* $c(n, C_8^6) = (1/3 + o(1))n$*?*

Now, we consider the Turán thresholds. Theorem 1.3.4 and Proposition 1.3.6 also show that $\mathrm{ex}_{k-1}(n, C_s^k) = (1/2 + o(1))n$ for k even, $s \geq 2k^2$ and (k,s) is an admissible pair. We would like to know the asymptotic value of $\mathrm{ex}_{k-1}(n, C_s^k)$ in the cases not covered by our constructions. Proposition 1.3.7 implies that $\mathrm{ex}_{k-1}(n, C_s^k) \geq \lfloor n/k \rfloor - k + 2$ for s not divisible by k; but on the other hand, if $s \equiv 0 \bmod k$ then $\mathrm{ex}_{k-1}(n, C_s^k) = o(n)$, which follows easily from Theorem 1.3.2.

The simplest open case is when $k = 3$ and s is not divisible by 3. Note that $C_4^3 = K_4^3$, and the lower bound $\mathrm{ex}_2(n, C_4^3) \geq (1/2 + o(1))n$ holds in this case, as shown by Czygrinow and Nagle [14]. We conjecture that in the case $k = 3$, for $s > 4$ and not divisible by three, the lower bound given by Proposition 1.3.7 describes the correct asymptotic behaviour of $\mathrm{ex}_{k-1}(n, C_s^k)$.

Conjecture 7.2.2. $\mathrm{ex}_2(n, C_s^3) = (1/3 + o(1))n$ *for every* $s > 4$ *with* $s \not\equiv 0 \bmod 3$.

Finally, we discuss tiling thresholds. Let (k,s) be an admissible pair such that $s \geq 5k^2$. If k is even, then Theorem 1.3.9 and Proposition 1.3.8 imply that $t(n, C_s^k) = (1/2 + 1/(2s) + o(1))n$. We conjecture that for k odd, the bound given by Proposition 1.3.8 is asymptotically tight.

Conjecture 7.2.3. *Let* (k,s) *be an admissible pair such that* $k \geq 3$ *is odd and* $s \geq 5k^2$*. Then* $t(n, C_s^k) = (1/2 + k/(4s(k-1) + 2k) + o(1))n$*.*

Note that, for k odd, the extremal example given by Proposition 1.3.8 is an example of the so-called space barrier construction. However, it is different from the common construction which is obtained by attaching a new vertex set W to an F-free k-graph and adding all possible edges incident with W. On the other hand, for k even, it is indeed the common construction of a space barrier.

It also would be interesting to find bounds on the Turán, covering and tiling thresholds that hold whenever $k < s \leq 5k^2$. The known thresholds for these kind of k-graphs do not necessarily follow the pattern of the bounds we have found for longer cycles. For example, note that C^k_{k+1} is a complete k-graph on $k+1$ vertices, which suggests that for lower values of s the problem behaves in a different way. Concretely, when $(k,s) = (3,4)$, it is known that $t(n, C^3_4) = (3/4 + o(1))n$ [44, 53].

Question 7.2.4. *Given $k \geq 3$, what is the minimum s such that $t(n, C^k_s) \leq (1/2 + 1/(2s) + o(1))n$ holds?*

Finally, it would be interesting to see what is the analogous of "El-Zahar's conjecture" (Conjecture 1.3.1) in the case of k-uniform hypergraphs and tight cycles of length at least $k+1$ each.

Question 7.2.5. *Let $k \geq 3$. Let H be a k-graph on n vertices and let $n_1, \dots, n_r \geq k+1$ be integers such that $n_1 + \cdots + n_r = n$. What is the minimum value of $\delta_{k-1}(H)$, which ensures that H contains r vertex-disjoint tight cycles of lengths $n_1, \dots, n_r$ respectively?*

It is not clear what a plausible value for the best lower bound should be. We describe a family of extremal examples for the case of graphs, when $k = 2$. Given $n_1, \dots, n_r$ not all even, an example that shows tightness of the lower bound in Conjecture 1.3.1 consists on a complete tripartite graph where one cluster has size one less than the number of n_i which are odd, and the other two remaining clusters have size as equal as possible. Since every odd cycle in this graph must intersect all three clusters, it cannot contain vertex-disjoint cycles of the required lengths.

How should one adapt such a construction for the $k \geq 3$ case? The lower bounds given by the constructions in Propositions 1.3.6–1.3.8 deal with a single cycle length s, and the constructions themselves are usually different for each value of s. Thus it is not immediately clear how to adapt them to construct graphs with large codegree which avoid vertex-disjoint cycles of different lengths $n_1, \dots, n_r$.

7.3 Dense monochromatic infinite paths

We discuss possible further questions around Chapter 6, where we investigated monochromatic infinite paths.

The most natural question would be to ask for the best possible lower bound in Theorem 1.4.1. This has already been solved by Corsten, DeBiasio, Lamaison and Lang [12], who showed that every 2-edge-colouring of $K_{\mathbb{N}}$ contains a monochromatic path P with $\overline{d}(P) \geq (12 + \sqrt{8})/17 \approx 0.87226$ and that this constant cannot be further improved.

Perhaps the next natural question is to ask what happens if we consider r-edge-colourings of $K_{\mathbb{N}}$, for $r \geq 3$. Recall that by the result of Rado [61], every r-edge-colouring of $K_{\mathbb{N}}$ contains a monochromatic path P with $\overline{d}(P) \geq 1/r$. DeBiasio and McKenney [15, Example 3.4] show that if $r - 1$ is a prime power, then there exists a r-edge-colouring of $K_{\mathbb{N}}$ where every monochromatic connected subgraph H satisfies $\overline{d}(H) \leq 1/(r-1)$. They conjecture that this upper bound can be attained with monochromatic paths when $r = 3$.

Conjecture 7.3.1 ([15, Conjecture 8.2])**.** *Every 3-edge-colouring of $K_{\mathbb{N}}$ contains a monochromatic path P with $\overline{d}(P) \geq 1/2$.*

The use of prime powers in the previous upper bounds arises from the use of finite affine planes of order $r - 1$ to define the colourings, which are only known to exist for values which are powers of primes. By using these constructions, DeBiasio and McKenney [15, Corollary 3.5] show that for general $r \geq 3$, if q is the largest prime power satisfying $q \leq r-1$, then there exists a r-colouring of $K_{\mathbb{N}}$ where every monochromatic connected subgraph H satisfies $\overline{d}(H) \leq 1/q$. Elementary arguments about prime numbers show that this choice of q satisfies $1/q < 2/r$; but also that q/r approaches 1 whenever r goes to infinity. It would be interesting to establish further upper bounds for general r.

Question 7.3.2. *Given $r \geq 3$, what is the best possible value of $d > 0$ such that* *every r-edge-colouring of $K_{\mathbb{N}}$ contains a monochromatic connected subgraph with $\overline{d}(P) \geq d$?*

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