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STRICTLY POSITIVE SOLUTIONS FOR ONE-DIMENSIONAL NONLINEAR ELLIPTIC PROBLEMS

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ABSTRACT. We study the existence and nonexistence of strictly positive solutions for the elliptic problems $Lu = m(x)u^p$ in a bounded open interval, with zero boundary conditions, where L is a strongly uniformly elliptic differential operator, $p \in (0,1)$, and m is a function that changes sign. We also characterize the set of values p for which the problem admits a solution, and in addition an existence result for other nonlinearities is presented.

1. INTRODUCTION

For $\alpha < \beta$, let $\Omega := (\alpha, \beta)$ and let $m \in L^2(\Omega)$ be a function that changes sign in Ω. Let $p \in (0,1)$ and let L be a one-dimensional strongly uniformly elliptic differential operator given by

$$Lu := -a(x)u'' + b(x)u' + c(x)u, \tag{1.1}$$

where $a, b \in C(\overline{\Omega})$, $0 \le c \in L^\infty(\Omega)$ and $a(x) \ge \lambda > 0$ for all $x \in \Omega$. Our aim in this article is to consider the existence and nonexistence of solutions for the problem

$$\begin{gathered} Lu = mu^p \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

The question of existence of strictly positive solutions for semilinear Dirichlet problems with indefinite nonlinearities as (1.2) is challenging and intriguing, and to our knowledge there are few results concerning this issue. In contrast to superlinear problems where any nonnegative (and nontrivial) solution is automatically positive (and in fact is in the interior of the positive cone under standard assumptions), for the analogous sublinear equations the situation is far less clear, even in the one-dimensional case. For instance, it is known that if m is smooth and $m^+ \not\equiv 0$ then for any $p \in (0,1)$ there exist nontrivial nonnegative solutions that actually vanish in a subset of Ω (see e.g. [1, 5]), and when $L = -u''$ one may also construct examples of strictly positive solutions that do not belong to the interior of the positive cone (see [6]).

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The problem (1.2) was considered recently in [6] for the laplacian operator, where several *non-comparable* sufficient conditions for the existence of solutions where proved under some evenness assumptions on m. In the present paper we shall adapt and extend the approach in [6] in order to derive our main results for a general operator. More precisely, in Section 3 we shall give two non-comparable sufficient conditions on m in the case $b \equiv 0$ (see Theorem 3.1 and Remark 3.2), and when $b \not\equiv 0$ we shall also exhibit sufficient conditions in Theorem 3.5 and Corollary 3.8. Let us mention that these last conditions are non-comparable between each other nor between the ones in Theorem 3.1. Moreover, one of them substantially improves the results known for $L = -u''$ (see Remarks 3.6, 3.7 and 3.9). Also, as a consequence of the aforementioned results we shall characterize the set of p's such that (1.2) admits a solution and we shall deduce an existence theorem for other nonlinearities (see Corollaries 3.10 and 3.13 respectively). Let us finally say that necessary conditions on m for the existence of solutions are stated in Theorem 3.11.

To relate our results to others already existing let us mention that to our knowledge no necessary condition on m is known in the case of a general operator (other than the obvious one derived from the maximum principle, i.e. $m^+ \not\equiv 0$), and the only sufficient condition we found in the literature is that the solution φ of $L\varphi = m$ in Ω, $\varphi = 0$ on $\partial\Omega$, satisfies $\varphi > 0$ in Ω (see [9, Theorem 4.4], [8, Theorem 10.6]). Let us note that although the above condition is even true for the n-dimensional problem, it is far from being necessary in the sense that there are examples of (1.2) having a solution but with the corresponding φ satisfying $\varphi < 0$ in Ω (cf. [6]). Concerning the laplacian operator, (1.2) was treated in [6, Theorem 2.1], and as we said before there are also further results there under different evenness assumptions on m. Let us finally mention that existence of solutions for problem (1.2) has also been studied when $L = -u''$ and $m \geq 0$ but assuming that $m \in C(\Omega)$ (see e.g. [11], [3] and the references therein), and some similar results to the ones that appear here have been obtained recently by the authors in [10] for some related problems involving quasilinear operators.

We would like to conclude this introduction with some few words on the corresponding n-dimensional problem. As we noticed in the above paragraph the condition in [9] is still valid in this case, and some of the techniques in [6] can be applied if $L = -\Delta$ (see Section 3 in [6] for the radial case, and also [7]). We are strongly convinced that some of the theorems presented here should still have some counterpart in n dimensions but we are not able to provide a proof.

2. Preliminaries

Since $a(x) \geq \lambda > 0$ for all $x \in \Omega$ and $a \in C(\overline{\Omega})$, from now on we consider without loss of generality that L is given by

$$Lu := -u'' + b(x)u' + c(x)u, \tag{2.1}$$

with b and c as in (1.1). For $f \in L^r(\Omega)$ with $r > 1$ we say that u is a (strong) solution of the problem $Lu = f$ in Ω, $u = 0$ in $\partial\Omega$, if $u \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ and the equation is satisfied *a.e.* $x \in \Omega$. Given $g : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function such that $g(., \xi) \in L^2(\Omega)$ for all ξ, we say that u is a (weak) subsolution of

$$\begin{aligned} Lu &= g(x,u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.2}$$

if $u \in W^{1,2}(\Omega)$, $u \le 0$ on $\partial\Omega$ and

$$\int_\Omega u'\phi' + bu'\phi + cu\phi \le \int_\Omega g(x,u)\phi \quad \text{for all } 0 \le \phi \in W_0^{1,2}(\Omega).$$

(Weak) supersolutions are defined analogously.

The following lemma is a direct consequence of the integration by parts formula (e.g. [2, Corollary 8.10]).

Lemma 2.1. *For $i : 1,\dots,n$, let $u_i \in W^{2,2}(x_i,x_{i+1})$ or $u_i \in C^2(x_i,x_{i+1}) \cap C^1([x_i,x_{i+1}])$ such that $u_i(x_{i+1}) = u_{i+1}(x_{i+1})$, $u_i'(x_{i+1}) \le u_{i+1}'(x_{i+1})$ and*

$$-u_i'' + bu_i' + cu_i \le g(x,u_i) \qquad a.e.\ x \in (x_i,x_{i+1}) \text{ for all } i : 1,\dots,n.$$

Let $\Omega := (x_1,x_{n+1})$ and set $u(x) := u_i(x)$ for all $x \in \Omega$. Then $u \in W^{1,2}(\Omega)$ and

$$\int_\Omega u'\phi' + bu'\phi + cu\phi \le \int_\Omega g(x,u)\phi \quad \text{for all } 0 \le \phi \in W_0^{1,2}(\Omega).$$

In particular, if also $u \le 0$ on $\partial\Omega$, then u is a subsolution of (2.2).

The next remark compiles some necessary facts about problem (1.2).

Remark 2.2. (i) It is immediate to check that (1.2) possesses a solution if and only if it has a solution with τm in place of m, for any $\tau > 0$.

(ii) Let us write as usual $m = m^+ - m^-$ with $m^+ = \max(m,0)$ and $m^- = \max(-m,0)$. It is also easy to verify that (1.2) admits arbitrarily large supersolutions (if $m^+ \not\equiv 0$; if $m^+ \equiv 0$ there is no solution by the maximum principle). Indeed, let $\varphi > 0$ be the solution of $L\varphi = m^+$ in Ω, $\varphi = 0$ on $\partial\Omega$. Let $k \ge (\|\varphi\|_\infty + 1)^{p/(1-p)}$. Then $k(\varphi+1)$ is a supersolution since

$$L(k(\varphi+1)) \ge kL\varphi \ge (k(\|\varphi\|_\infty+1))^p m^+ \ge (k(\varphi+1))^p m \quad \text{in } \Omega \tag{2.3}$$

and $\varphi = k > 0$ on $\partial\Omega$.

The two following lemmas provide some useful upper bounds for the L^∞-norm of the nonnegative subsolutions of (1.2). To avoid overloading the notation we write from now on

$$\overline{B}_\alpha(x) := e^{\int_\alpha^x b(r)dr}, \quad \underline{B}_\alpha(x) := e^{-\int_\alpha^x b(r)dr}.$$

Lemma 2.3. *Let $0 \le u \in W^{2,2}(\Omega)$ be such that $Lu \le mu^p$ in Ω. Then*

$$\|u\|_{L^\infty(\Omega)} \le \Big[\int_\alpha^\beta \overline{B}_\alpha(x)\|m^+\underline{B}_\alpha\|_{L^1(\alpha,x)}dx\Big]^{1/(1-p)}. \tag{2.4}$$

Proof. Since $\underline{B}_\alpha, u' \in W^{1,2}(\Omega)$, we may apply the product differentiation rule and hence

$$\begin{aligned} -(\underline{B}_\alpha u')' &\le -(\underline{B}_\alpha u')' + \underline{B}_\alpha cu \\ &= \underline{B}_\alpha(-u'' + bu' + cu) \\ &\le \underline{B}_\alpha m u^p \le \underline{B}_\alpha m^+ \|u\|^p_{L^\infty(\Omega)}. \end{aligned}$$

Integrating on (α,x) for $x \in (\alpha,\beta)$ (see e.g. [2, Theorem 8.2]) and noting that $\underline{B}_\alpha(\alpha)u'(\alpha) = u'(\alpha) \ge 0$ we obtain

$$-\underline{B}_\alpha(x)u'(x) \le \|u\|^p_{L^\infty(\Omega)} \int_\alpha^x \underline{B}_\alpha(t)m^+(t)dt.$$

Dividing by $\underline{B}_\alpha(x) > 0$ and integrating now on (y, β) for $y \in (\alpha, \beta)$, since $u(\beta) = 0$ we get

$$0 \le \frac{u(y)}{\|u\|^p_{L^\infty(\Omega)}} \le \int_y^\beta [\overline{B}_\alpha(x) \int_\alpha^x \underline{B}_\alpha(t) m^+(t) dt] dx \quad \text{for all } y \in (\alpha, \beta),$$

and the lemma follows. □

Let

$$M^+ := \{x \in \Omega : m \ge 0\}, \quad M^- := \{x \in \Omega : m < 0\}. \tag{2.5}$$

Lemma 2.4. *Let $0 \le u \in W^{2,2}(\Omega)$ be such that $Lu \le mu^p$ in Ω, and let M^+ be given by* (2.5)*. If $c > 0$ in M^+, then*

$$\|u\|_{L^\infty(\Omega)} \le [\sup_{x \in M^+} \frac{m^+(x)}{c(x)}]^{1/(1-p)}.$$

Proof. Without loss of generality we assume that $u \not\equiv 0$. Furthermore, let us suppose first that $\|u\|_{L^\infty(\Omega)} > 1$. Let $x_0 \in \Omega$ be a point where u attains its absolute maximum. There exists $\delta > 0$ such that $u \ge 1$ in $I_\delta(x_0) := (x_0 - \delta, x_0 + \delta)$. There also exist $x_1, x_2 \in I_\delta(x_0)$ satisfying $x_1 < x_0 < x_2$ and $u'(x_2) \le 0 \le u'(x_1)$. We have that

$$-(\underline{B}_\alpha u')' + \underline{B}_\alpha cu \le \underline{B}_\alpha mu^p \le \underline{B}_\alpha m^+ u^p \quad \text{in } \Omega$$

and so in $I_\delta(x_0)$ we get that (because $u \ge 1$ in $I_\delta(x_0)$) $-(\underline{B}_\alpha u')' \le \underline{B}_\alpha (m^+ - c)u$. Integrating on (x_1, x_2) we obtain

$$0 \le \underline{B}_\alpha(x_1)u'(x_1) - \underline{B}_\alpha(x_2)u'(x_2) = \int_{x_1}^{x_2} -(\underline{B}_\alpha u')' \le \int_{x_1}^{x_2} \underline{B}_\alpha (m^+ - c)u. \tag{2.6}$$

Since $u \ge 1$ in (x_1, x_2) and $\underline{B}_\alpha \ge e^{-\|b^+\|_\infty (x_2 - \alpha)}$ in (x_1, x_2), from (2.6) it follows that there exists $E \subset (x_1, x_2)$ with $|E| > 0$ (where $|E|$ denotes the Lebesgue measure of E) such that $m^+(x) \ge c(x)$ a.e. $x \in E$. Moreover, due to the fact that $c > 0$ a.e. $x \in M^+$ it must hold that $m^+ > 0$ a.e. $x \in E$. In particular, $E \subset M^+$ and therefore

$$1 \le \sup_{x \in E} \frac{m^+(x)}{c(x)} \le \sup_{x \in M^+} \frac{m^+(x)}{c(x)}. \tag{2.7}$$

Let u now be as in the statement of the lemma, and let $\varepsilon > 0$. Then

$$L \frac{u}{\|u\|_\infty - \varepsilon} \le \frac{m}{(\|u\|_\infty - \varepsilon)^{1-p}} \Big(\frac{u}{\|u\|_\infty - \varepsilon} \Big)^p.$$

Applying the first part of the proof with $m/(\|u\|_\infty - \varepsilon)^{1-p}$ and $u/(\|u\|_\infty - \varepsilon)$ in place of m and u respectively, from (2.7) we deduce that

$$\big(\|u\|_{L^\infty(\Omega)} - \varepsilon\big)^{1-p} \le \sup_{x \in M^+} \frac{m^+(x)}{c(x)}$$

and since ε is arbitrary this completes the proof of the lemma. □

We shall need the next result when we characterize the set of p's such that (1.2) admits a solution.

Lemma 2.5. *Suppose* (1.2) *has a solution $u \in W^{2,2}(\Omega)$, and let $q \in (p, 1)$. Then there exists $v \in W^{2,2}(\Omega)$ solution of* (1.2) *with q in place of p.*

Proof. Let $\gamma := (1-p)/(1-q)$. Let $0 \leq \phi \in C_c^\infty(\Omega)$, and let Ω' be an open set such that $\operatorname{supp}\phi \subset \Omega' \Subset \Omega$. One can check that $u^\gamma \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega')$. Furthermore, noticing that $\gamma > 1$ and $\gamma - 1 + p = \gamma q$ we find that

$$\begin{aligned}
L(u^\gamma) &= -\gamma(u''u^{\gamma-1} + (\gamma-1)u^{\gamma-2}(u')^2) + b\gamma u^{\gamma-1}u' + cu^\gamma \\
&\leq \gamma u^{\gamma-1}(-u'' + bu' + cu) \leq \gamma u^{\gamma-1} m u^p \\
&= \gamma m (u^\gamma)^q \quad \text{in } \Omega'.
\end{aligned}$$

Multiplying the above inequality by ϕ, integrating over Ω' and using the integration by parts formula we obtain that

$$\begin{aligned}
\int_\Omega (u^\gamma)'\phi' + b(u^\gamma)'\phi + cu^\gamma\phi &= \int_{\Omega'} [-(u^\gamma)'' + b(u^\gamma)' + cu^\gamma]\phi \\
&\leq \gamma \int_\Omega m(u^\gamma)^q \phi.
\end{aligned}$$

Now, let $0 \leq v \in W_0^{1,2}(\Omega)$. There exists $\{\phi_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\Omega)$ with $\phi_n \geq 0$ in Ω and such that $\phi_n \to v$ in $W^{1,2}(\Omega)$ (e.g. [4, p. 50]). Employing the above inequality with ϕ_n in place of ϕ and going to the limit we see that u^γ is a subsolution of (1.2) with γm in place of m. Thus, taking into account Remark 2.2 (i) and (ii) we get a solution $v \in W_0^{1,2}(\Omega)$ of (1.2), and by standard regularity arguments $v \in W^{2,2}(\Omega)$. ☐

3. Main results

We set

$$C_p := \frac{2(1+p)}{(1-p)^2}, \tag{3.1}$$

and for any interval I,

$$\lambda_1(m, I) := \text{the positive principal eigenvalue for } m \text{ in } I.$$

Theorem 3.1. *Assume $b \equiv 0$. Let $m \in L^2(\Omega)$ with $m^- \in L^\infty(\Omega)$ and suppose there exist $\alpha \leq x_0 < x_1 \leq \beta$ such that $0 \not\equiv m \geq 0$ in $I := (x_0, x_1)$. Let $\gamma := \max\{(\beta - x_0), (x_1 - \alpha)\}$ and let C_p be given by (3.1).*

(i) If it holds that

$$\frac{\|m^-\|_{L^\infty(\Omega)}}{\|c\|_{L^\infty(\Omega)}} \sinh^2\Big[\gamma\sqrt{\frac{\|c\|_\infty}{C_p}}\Big] \leq \frac{1}{\lambda_1(m, I)} \tag{3.2}$$

then there exists a solution $u \in W^{2,2}(\Omega)$ to problem (1.2).

(ii) If it holds that

$$\frac{\|m^-\|_{L^\infty(\Omega)}}{\|c\|_{L^\infty(\Omega)}} [\cosh(\gamma\sqrt{(1-p)\|c\|_{L^\infty(\Omega)}}) - 1] \leq \frac{1}{\lambda_1(m, I)} \tag{3.3}$$

then there exists a solution $u \in W^{2,2}(\Omega)$ to problem (1.2).

Proof. Recalling Remark 2.2 it suffices to construct a strictly positive (in Ω) subsolution u for (1.2) with τm in place of m, for some $\tau > 0$. Moreover, without loss of generality we may assume that $\alpha < x_0 < x_1 < \beta$ (in fact, it shall be clear from the proof how to proceed if either $x_0 = \alpha$ or $x_1 = \beta$). To provide such u we shall employ Lemma 2.1 with $n = 3$ and $g(x, \xi) = \tau m(x)\xi^p$.

We shall take $u_2 > 0$ with $\|u_2\|_{L^\infty(I)} = 1$ as the positive principal eigenfunction associated to the weight m in I, that is satisfying

$$\begin{gathered} Lu_2 = \lambda_1(m, I) m u_2 \quad \text{in } I \\ u_2 = 0 \quad \text{on } \partial I. \end{gathered}$$

Since $m \geq 0$ in I, for $\tau > 0$ we have that $Lu_2 = \lambda_1(m, I) m u_2 \leq \tau m u_2^p$ whenever

$$\lambda_1(m, I) \leq \tau. \tag{3.4}$$

On the other hand, suppose now that (3.2) holds and pick τ satisfying

$$\frac{\|m^-\|_{L^\infty(\Omega)}}{\|c\|_{L^\infty(\Omega)}} \sinh^2 \Big[\gamma \sqrt{\frac{\|c\|_\infty}{C_p}}\Big] \leq \frac{1}{\tau} \leq \frac{1}{\lambda_1(m, I)} \tag{3.5}$$

(in particular, (3.4) holds). Let $x \in [\alpha, x_1]$ and define

$$f(x) = \sqrt{\frac{\tau \|m^-\|_\infty}{\|c\|_\infty}} \sinh[\sqrt{\frac{\|c\|_\infty}{C_p}} (x - \alpha)].$$

A few computations show that $C_p (f')^2 - \|c\|_\infty f^2 = \tau \|m^-\|_\infty$ in (α, x_1). Moreover, $f(\alpha) = 0$, $f(x) > 0$ for $x \in (\alpha, x_1)$ and $f', f'' \geq 0$ for such x. Let us now fix $k := 2/(1-p)$. Then we have

$$kp = k - 2, \quad k(k-1) = C_p. \tag{3.6}$$

We set $u_1 := f^k$. Taking into account (3.6) and the above mentioned facts we find that

$$\begin{aligned} Lu_1 &= -k[(k-1) f^{k-2} (f')^2 + f^{k-1} f''] + c f^k \\ &\leq -C_p f^{k-2} (f')^2 + \|c\|_\infty f^k \\ &= -f^{k-2} \tau \|m^-\|_\infty \\ &\leq \tau m u_1^p \quad \text{in } (\alpha, x_1). \end{aligned} \tag{3.7}$$

Furthermore, since f is increasing we get that $\|u_1\|_\infty = [f(x_1)]^k$ and therefore using the first inequality in (3.5) and the fact that $x_1 - \alpha \leq \gamma$ one can verify that $\|u_1\|_\infty \leq 1$.

In a similar way, if for $x \in [x_0, \beta]$ we define $u_3 := g^k$ where g is given by

$$g(x) := \sqrt{\frac{\tau \|m^-\|_\infty}{\|c\|_\infty}} \sinh\Big[\sqrt{\frac{\|c\|_\infty}{C_p}} (\beta - x)\Big],$$

then $Lu_3 \leq \tau m u_3^p$ in (x_0, β), $\|u_3\|_\infty \leq 1$, $u_3(\beta) = 0$ and $u_3(x) > 0$ for $x \in (x_0, \beta)$.

We choose now

$$\begin{gathered} \underline{x}_0 := \sup\{x \in I : u_1(y) > u_2(y) \text{ for all } y \in (x_0, x]\}, \\ \overline{y} := \max\{x \in I : u_2(x) = 1\}, \\ \underline{y} := \min\{x \in I : u_2(x) = 1\}. \end{gathered}$$

We observe that $\underline{x}_0 \in I$ exists because $u_1(\alpha) = u_2(x_0) = 0$ and $u_1(x_1) \leq 1 = \|u_2\|_\infty$. Moreover, since u_1 and u_2 are C^1, by the definition of $\underline{x}_0$ we have that $u_1(\underline{x}_0) = u_2(\underline{x}_0)$ and $u_1'(\underline{x}_0) \leq u_2'(\underline{x}_0)$ (for the last inequality it is enough to note that

$$\frac{u_1(x) - u_1(\underline{x}_0)}{x - \underline{x}_0} < \frac{u_2(x) - u_2(\underline{x}_0)}{x - \underline{x}_0}$$

for every $x \in (x_0, \underline{x}_0)$), and also clearly $\underline{x}_0 < \underline{y}$. Analogously, there exists $\overline{x}_1 \in I$ such that $u_2(\overline{x}_1) = u_3(\overline{x}_1)$ and $u_2'(\overline{x}_1) \le u_3'(\overline{x}_1)$, and satisfying $\overline{x}_1 > \overline{y}$. In particular, $\underline{x}_0 < \overline{x}_1$. Hence, defining u by $u := u_1$ in $[\alpha, \underline{x}_0]$, $u := u_2$ in $[\underline{x}_0, \overline{x}_1]$ and $u := u_3$ in $[\overline{x}_1, \beta]$, we have that $u = 0$ on $\partial\Omega$ and u fulfills the hypothesis of Lemma 2.1 and as we said before this proves (i) (let us mention that if $x_0 = \alpha$ then in order to build u we only use u_2 and u_3, and if $x_1 = \beta$ then we do not need u_3).

Let us prove (ii). We shall take u_2 as above. We now fix τ such that

$$\frac{\|m^-\|_{L^\infty(\Omega)}}{\|c\|_{L^\infty(\Omega)}}[\cosh(\gamma\sqrt{(1-p)\|c\|_{L^\infty(\Omega)}}) - 1] \le \frac{1}{\tau} \le \frac{1}{\lambda_1(m, I)}. \tag{3.8}$$

We set $k := 1/(1-p)$, and for $x \in [\alpha, x_1]$ we define

$$f(x) := \frac{\tau\|m^-\|_\infty}{\|c\|_\infty}\Big[\cosh\Big(\sqrt{\frac{\|c\|_\infty}{k}}(x-\alpha)\Big) - 1\Big].$$

Then $f(\alpha) = 0$, $f > 0$ in (α, x_1) and $f' \ge 0$. Furthermore, by the first inequality in (3.8) $\|u_1\|_\infty \le 1$, and it can be seen that $kf'' - \|c\|_\infty f = \tau\|m^-\|_\infty$. Define now $u_1 := f^k$. Observing that $kp = k-1$ we derive that

$$\begin{aligned} Lu_1 &= -k[(k-1)f^{k-2}(f')^2 + f^{k-1}f''] + cf^k \\ &\le -kf^{k-1}f'' + \|c\|_\infty f^k = -f^{k-1}\tau\|m^-\|_\infty \\ &\le \tau m u_1^p \quad \text{in } (\alpha, x_1). \end{aligned}$$

In the same way, if for $x \in [x_0, \beta]$ we set $u_3 := g^k$ where g is given by

$$g(x) := \frac{\tau\|m^-\|_\infty}{\|c\|_\infty}\Big[\cosh\Big(\sqrt{\frac{\|c\|_\infty}{k}}(\beta - x)\Big) - 1\Big],$$

then $Lu_3 \le \tau m u_3^p$ in (x_0, β), $\|u_3\|_\infty \le 1$, $u_3(\beta) = 0$ and $u_3 > 0$ in (x_0, β). Now the proof of (ii) can be finished as in (i). $\square$

Remark 3.2. Let us mention that the inequalities in (i) and (ii) are not comparable. Indeed, we first check that for $p \approx 1$ (3.2) is better than (3.3). Let $\kappa := \gamma\sqrt{\|c\|_\infty}$. Since $\frac{1}{\sqrt{C_p}} = (1-p)\sqrt{\frac{1}{2(1+p)}}$, it is sufficient to observe that

$$0 \le \lim_{p\to 1^-} \frac{\sinh^2[\kappa(1-p)\sqrt{\frac{1}{2(1+p)}}]}{\cosh(\kappa\sqrt{1-p}) - 1} \le \lim_{p\to 1^-} \frac{\sinh^2(\kappa(1-p))}{\cosh(\kappa\sqrt{1-p}) - 1} = 0.$$

We now show that for $0 < p \approx 0$ (3.3) is better than (3.2). It suffices to prove this for $p = 0$ because the dependence on p in both inequalities is continuous. For $p = 0$ (3.2) and (3.3) become

$$\begin{gathered} \frac{\|m^-\|_\infty}{\|c\|_\infty}\sinh^2(\kappa/\sqrt{2}) \le \frac{1}{\lambda_1(m, I)} \\ \frac{\|m^-\|_\infty}{\|c\|_\infty}(\cosh\kappa - 1) \le \frac{1}{\lambda_1(m, I)} \end{gathered}$$

and so we only have to check that for every $x > 0$ it holds that $\sinh^2(x/\sqrt{2}) > \cosh x - 1$ which is easy to verify.

Remark 3.3. If in (3.2) we take limit as $\|c\|_{L^\infty(\Omega)} \to 0$ we arrive to the condition

$$\frac{\gamma^2}{C_p}\|m^-\|_{L^\infty(\Omega)} \le \frac{1}{\lambda_1(m,I)} \tag{3.9}$$

which is the one that appears for $L = -u''$ in [6, Theorem 2.1].

Remark 3.4. In the statement of Theorem 3.1 one can replace the condition (3.2) by

$$\frac{\|m^-\|_{L^\infty(\Omega)}}{\|c\|_{L^\infty(M^-)}}\sinh^2\Big[\gamma\sqrt{\frac{\|c\|_{L^\infty(M^-)}}{C_p}}\Big] \le \frac{1}{\lambda_1(m,I)}, \tag{3.10}$$

$$c \le m^+ \text{ in } M^+, \tag{3.11}$$

where M^+ and M^- are given by (2.5). Indeed, we first observe that if (3.10) holds then one can reason as in (3.7) and prove that $Lu_1 \le \tau m u_1^p$ in $(x_0,\beta)\cap M^-$. On the other side, if (3.11) is true then since in the proof of the theorem f is chosen satisfying $f'' \ge 0$ and $\|f^k\|_\infty \le 1$, then we also have

$$\begin{aligned} Lu_1 &= -k[(k-1)f^{k-2}(f')^2 + f^{k-1}f''] + cf^k \\ &\le cf^k \le m^+ f^k \\ &\le m^+ f^{kp} = m u_1^p \quad \text{in } (x_0,\beta)\cap M^+. \end{aligned}$$

The same reasoning can be done for u_3 and hence the proof can be continued as in the theorem. A similar observation is valid for (3.3).

Theorem 3.5. *Let $m \in L^2(\Omega)$ and suppose there exist $\alpha \le x_0 < x_1 \le \beta$ such that $0 \not\equiv m \ge 0$ in $I := (x_0,x_1)$. Let C_p be given by (3.1).*

(i) If $m^- \in L^\infty(\Omega)$ and it holds that

$$0 < \frac{(\gamma_b\|\underline{B}_\alpha\|_{L^\infty(\Omega)})^2}{C_p - \|c\|_{L^\infty(\Omega)}(\gamma_b\|\underline{B}_\alpha\|_{L^\infty(\Omega)})^2}\|m^-\|_{L^\infty(\Omega)} \le \frac{1}{\lambda_1(m,I)}, \tag{3.12}$$

where

$$\gamma_b := \max\{\|\overline{B}_\alpha\|_{L^1(\alpha,x_1)}, \|\overline{B}_\alpha\|_{L^1(x_0,\beta)}\},$$

then there exists a solution $u \in W^{2,2}(\Omega)$ of (1.2).

(ii) If $c \equiv 0$ and it holds that

$$(1-p)\mathcal{M} < \frac{1}{\lambda_1(m,I)} \tag{3.13}$$

where

$$\mathcal{M} := \max\Big\{\int_{x_0}^{\beta}\overline{B}_\alpha(x)\|m^-\underline{B}_\alpha\|_{L^1(x,\beta)}dx, \int_{\alpha}^{x_1}\overline{B}_\alpha(x)\|m^-\underline{B}_\alpha\|_{L^1(\alpha,x)}dx\Big\},$$

then there exists a solution $u \in W^{2,2}(\Omega)$ of (1.2).

Proof. The proof follows the lines of the proof of Theorem 3.1 and hence we omit the details. Let us prove (i). We take u_2 as in the aforementioned theorem, and we choose τ such that

$$\frac{(\gamma_b\|\underline{B}_\alpha\|_{L^\infty(\Omega)})^2}{C_p - \|c\|_{L^\infty(\Omega)}(\gamma_b\|\underline{B}_\alpha\|_{L^\infty(\Omega)})^2}\|m^-\|_{L^\infty(\Omega)} \le \frac{1}{\tau} \le \frac{1}{\lambda_1(m,I)}.$$

Let $x \in [\alpha, x_1]$ and define

$$u_1(x) := \Big(\sigma \int_\alpha^x \overline{B}_\alpha(y) dy\Big)^k,$$

where

$$\sigma := \Big[\frac{\|\underline{B}_\alpha\|^2_{L^\infty(\Omega)}(\tau\|m^-\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)})}{C_p}\Big]^{1/2}, \quad k := \frac{2}{1-p}.$$

We have that $u_1(\alpha) = 0$, $u_1 > 0$ in (α, x_1) and that u_1 is increasing. Moreover, after some computations one can check that $\|u_1\|_\infty \leq 1$ and

$$\begin{aligned}
-(\underline{B}_\alpha(x)u_1'(x))' &= -k(k-1)\sigma^2(\sigma \int_\alpha^x \overline{B}_\alpha(y)dy)^{k-2}\overline{B}_\alpha(x) \\
&\leq -\|\underline{B}_\alpha\|_{L^\infty(\Omega)}(\tau\|m^-\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)})(\sigma \int_\alpha^x \overline{B}_\alpha(y)dy)^{kp} \\
&\leq \underline{B}_\alpha(\tau m - c)u_1^p \leq \underline{B}_\alpha(\tau m u_1^p - cu_1);
\end{aligned}$$

that is, $Lu_1 \leq \tau m u_1^p$ in (α, x_1). The existence of u_3 follows similarly. Let us prove (ii). We pick τ satisfying

$$(1-p)\mathcal{M} < \frac{1}{\tau} < \frac{1}{\lambda_1(m, I)}. \tag{3.14}$$

For $x \in [\alpha, x_1]$ we define

$$u_1(x) := \Big(\sigma \int_\alpha^x \overline{B}_\alpha(y)\|m^-\underline{B}_\alpha + \varepsilon\|_{L^1(\alpha,y)}dy\Big)^k$$

where

$$\sigma := \tau(1-p), \quad k := \frac{1}{1-p}, \quad \varepsilon > 0.$$

Taking ε small enough and employing (3.14) one can see that $\|u_1\|_\infty \leq 1$. Also, a few computations yield

$$\begin{aligned}
-(\underline{B}_\alpha(x)u_1'(x))' &\leq -k\sigma^k\Big(\int_\alpha^x \overline{B}_\alpha(y)\|m^-\underline{B}_\alpha + \varepsilon\|_{L^1(\alpha,y)}dy\Big)^{k-1}(m^-(x)\underline{B}_\alpha(x) + \varepsilon) \\
&\leq -\tau m^-(x)\underline{B}_\alpha(x)\Big(\sigma \int_\alpha^x \overline{B}_\alpha(y)\|m^-\underline{B}_\alpha + \varepsilon\|_{L^1(\alpha,y)}dy\Big)^{kp} \\
&\leq \tau \underline{B}_\alpha m u_1^p.
\end{aligned}$$

Since u_3 can be defined analogously, this concludes the proof of (ii). $\square$

Remark 3.6. Let us note that the inequalities in (i) and (ii) are not comparable because one involves the L^∞-norm of m^- and the constant C_p, and the other one does not.

Remark 3.7. (i) It can be verified that (3.2) is better than (3.12) when $b \equiv 0$ (noting that in this case $\underline{B}_\alpha = \overline{B}_\alpha = 1$ and $\gamma_b = \gamma$ (γ as in the statement of Theorem 3.1)). If also $c \equiv 0$, (3.12) becomes exactly (3.9), that is, the condition deduced from the aforementioned theorem for the laplacian operator.
(ii) In the case $b \equiv 0$, (3.13) reads as

$$(1-p)\max\Big\{\int_{x_0}^\beta \|m^-\|_{L^1(t,\beta)}dt, \int_\alpha^{x_1} \|m^-\|_{L^1(\alpha,t)}dt\Big\} < \frac{1}{\lambda_1(m, I)} \tag{3.15}$$

which is substantially better than the condition stated in [6, Theorem 2.1], for $L = -u''$. Also, (3.15) is clearly not comparable (for the same reason as in the above remark) with the inequalities that can deduced from Theorem 3.1 in the case $c \equiv 0$ (i.e., as the one included in Remark 3.3).

Corollary 3.8. *Let*

$$K_b := \int_\alpha^\beta \overline{B}_\alpha(x) \|\underline{B}_\alpha\|_{L^2(\alpha,x)} dx.$$

If (3.13) *holds with* $m/(K_b\|m^+\|_{L^2(\alpha,\beta)}) - c$ *instead of* m*, then there exists a solution* $u \in W^{2,2}(\Omega)$ *of* (1.2).

Proof. Applying Hölder's inequality in (2.4) we see that $\|u\|_{L^\infty(\Omega)}^{1-p} \le \|m^+\|_{L^2(\alpha,\beta)} K_b$ for any nonnegative subsolution of (1.2). Now, let $\tau := 1/(K_b\|m^+\|_{L^2(\alpha,\beta)})$, and let u be the solution of (1.2) with $\tau m - c$ in place of m provided by Theorem 3.5 (ii). It follows that $\|u\|_\infty \le 1$ and thus

$$-u'' + bu' = (\tau m - c)u^p \le \tau m u^p - cu$$

and recalling once again Remark 2.2 the corollary follows. $\square$

Remark 3.9. (i) Given *any operator* L *and any* $m \in L^2(\Omega)$ *with* $0 \not\equiv m \ge 0$ *in some* $I \subset \Omega$, let us note that the above corollary implies that (1.2) has a solution if p is sufficiently close to 1.

(ii) Given any operator L and any $m \in L^2(\Omega)$ with $m^- \in L^\infty(\Omega)$ and $0 \not\equiv m \ge 0$ in some $I \subset \Omega$, let us observe that (3.12) says that (1.2) possesses a solution for $\overline{m} := m\chi_{\Omega - I} + km\chi_I$ if $k > 0$ is large enough.

The next result provides the structure of the set of p's such that (1.2) has a solution.

Corollary 3.10. *Let* $m \in C(M^+) \cap L^2(\Omega)$ *with* $m^+ \not\equiv 0$ *and let* $\mathcal{P}$ *be the set of* $p \in (0,1)$ *such that* (1.2) *admits some solution* $u \in W^{2,2}(\Omega)$*. Then* $\mathcal{P} = (0,1)$ *or either* $\mathcal{P} = (p,1)$ *or* $\mathcal{P} = [p,1)$ *for some* $p > 0$.

Proof. By Remark 3.9 (i) we have that $\mathcal{P} \ne \emptyset$. Let $p^* := \inf \mathcal{P}$. If $\mathcal{P} \ne (0,1)$, Lemma 2.5 implies that $p^* > 0$ and that (1.2) has a solution for every $p > p^*$. Therefore, either $\mathcal{P} = (p^*,1)$ or $\mathcal{P} = [p^*,1)$. $\square$

We write

$$\begin{gathered} I_R(x_0) := (x_0 - R, x_0 + R), \\ \mathfrak{I} := \{I_R(x_0) \subset \Omega : m \le 0 \text{ in } I_R(x_0)\}. \end{gathered} \tag{3.16}$$

Theorem 3.11. *Let* C_p *and* $\mathfrak{I}$ *be given by* (3.1) *and* (3.16) *respectively. Suppose there exists* $u \in W^{2,2}(\Omega)$ *solution of* (1.2)*. Then*

$$\sup_{I_R(x_0)\in\mathfrak{I}} \Big[\Big[\frac{\gamma_{b,R}}{\|\overline{B}_\alpha\|_{L^\infty(I_R(x_0))}} \Big]^2 \inf_{I_R(x_0)} m^- \Big] \le C_p \int_\alpha^\beta \overline{B}_\alpha(x) \|m^+ \underline{B}_\alpha\|_{L^1(\alpha,x)} dx, \tag{3.17}$$

where

$$\gamma_{b,R} := \min\Big\{ \int_{x_0}^{x_0+R} \overline{B}_\alpha(y)dy, \int_{x_0-R}^{x_0} \overline{B}_\alpha(y)dy \Big\}.$$

Let M^+ *be given by* (2.5)*. If also* $c > 0$ *in* M^+*, then* (3.17) *must also hold with* $C_p \sup_{x\in M^+} \frac{m^+(x)}{c(x)}$ *in the right side of the inequality.*

Proof. We proceed by contradiction. Suppose (3.17) is not true and let $I_R(x_0) \in \mathfrak{I}$ be such that

$$C_p \int_\alpha^\beta \overline{B}_\alpha(x) \|m^+ \underline{B}_\alpha\|_{L^1(\alpha,x)} dx \le \Big[\frac{\gamma_{b,R}}{\|\overline{B}_\alpha\|_{L^\infty(I_R(x_0))}}\Big]^2 \inf_{I_R(x_0)} m^-. \tag{3.18}$$

For $x \in \overline{I}_R(x_0)$, we define a function w as follows. If $x \in [x_0, x_0+R]$ we set

$$w(x) := \Big(\sigma \int_{x_0}^x \overline{B}_\alpha(y) dy\Big)^k,$$

where

$$\sigma := \Big[\frac{\inf_{I_R(x_0)} m^-}{C_p \|\overline{B}_\alpha\|^2_{L^\infty(I_R(x_0))}}\Big]^{1/2}, \quad k := \frac{2}{1-p},$$

and if $x \in [x_0 - R, x_0]$ we set $w(x) := (\sigma \int_x^{x_0} \overline{B}_\alpha(y) dy)^k$ with σ and k as above. In (x_0, x_0+R) we find that

$$\begin{aligned}
(\underline{B}_\alpha w')' - \underline{B}_\alpha c w &\le k(k-1)\sigma^2 \Big(\sigma \int_\alpha^x \overline{B}_\alpha(y) dy\Big)^{k-2} \overline{B}_\alpha \\
&\le \frac{\inf_{I_R(x_0)} m^-}{\|\overline{B}_\alpha\|_{L^\infty(I_R(x_0))}} \Big(\sigma \int_\alpha^x \overline{B}_\alpha(y) dy\Big)^{kp} \\
&\le \underline{B}_\alpha m^- w^p;
\end{aligned}$$

i.e., $Lw \ge -m^- w^p$, and the same is also valid in $(x_0 - R, x_0)$.

Let u be a solution of (1.2). We claim that $u \le w$ in $I_R(x_0)$. Indeed, if not, let $\mathcal{O} := \{x \in I_R(x_0) : w(x) < u(x)\}$. Since $Lu = -m^- u^p$ in $I_R(x_0)$, we have $L(w-u) \ge m^-(u^p - w^p) \ge 0$ in $\mathcal{O}$. Let $\overline{x} \in \partial\mathcal{O}$. Then $w(\overline{x}) = u(\overline{x})$ or either $\overline{x} = x_0 + R$ or $\overline{x} = x_0 - R$. If $\overline{x} = x_0 + R$, by Lemma 2.3 and (3.18) we obtain

$$\begin{aligned}
u(\overline{x})^{1-p} &\le \|u\|^{1-p}_{L^\infty(\Omega)} \\
&\le \int_\alpha^\beta \overline{B}_\alpha(x) \|m^+ \underline{B}_\alpha\|_{L^1(\alpha,x)} dx \\
&\le \Big[\frac{\int_{x_0}^{x_0+R} \overline{B}_\alpha(y) dy}{\|\overline{B}_\alpha\|_{L^\infty(I_R(x_0))}}\Big]^2 \frac{\inf_{I_R(x_0)} m^-}{C_p} = w(\overline{x})^{1-p},
\end{aligned}$$

and we arrive to the same inequality if $\overline{x} = x_0 - R$. Therefore the maximum principle says that $u \le w$ in $\mathcal{O}$ which is not possible. Thus, $u \le w$ in $I_R(x_0)$; but $u > 0$ in Ω and $w(x_0) = 0$. Contradiction.

To conclude the proof we note that the last statement of the theorem may be derived as above applying Lemma 2.4 instead of Lemma 2.3. □

Remark 3.12. (i) It follows from the above theorem that given b, m, p fixed, there exists $0 \le c_0 \in L^\infty(\Omega)$ such that for all $c \in L^\infty(\Omega)$ with $c \ge c_0$ the problem (1.2) *does not admit a solution.* Note that given L, m, p fixed with $0 \not\equiv m \le 0$ in some $I \subset \Omega$, *neither there is a solution for* $\underline{m} := m\chi_{\Omega - I} + km\chi_I$ *if* $k > 0$ *is large enough.*

(ii) We observe that (3.17) always is true if p is sufficiently close to 1. Let us mention that this must indeed occur by Remark 3.9.

As a consequence of the previous theorems we derive an existence result for problems of the form

$$
\begin{gathered}
Lu = mf(u) \quad \text{in } \Omega \\
u > 0 \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered}
\tag{3.19}
$$

for certain continuous functions $f : [0, \infty) \to [0, \infty)$. Now we state assumption

(H1) There exist $k_1, k_2 > 0$ and $p \in (0, 1)$ such that

$$k_1 \xi^p \leq f(\xi) \leq k_2 \xi^p \text{ for all } \xi \in [0, \underline{K}],$$

where

$$\underline{K} := \Big[k_1 \int_\alpha^\beta \overline{B}_\alpha(x) \| m^+ \underline{B}_\alpha \|_{L^1(\alpha,x)} dx \Big]^{1/(1-p)},$$

and $f(\xi) \leq k_3 \xi^q$ for all $\xi \in [\overline{K}, \infty)$ some $\overline{K}, k_3 > 0$ and $q \in (0, 1)$.

Note that we make no monotonicity nor concavity assumptions on f.

Corollary 3.13. *Let f satisfy* (H1) *and suppose* (1.2) *has a solution with $k_1 m^+ - k_2 m^-$ instead of m. Then there exists a solution $u \in W^{2,2}(\Omega)$ of* (3.19).

Proof. Let u be the solution of (1.2) with $k_1 m^+ - k_2 m^-$ in place of m. It follows from Lemma 2.3 that $\|u\|_\infty \leq \underline{K}$, and so from (H1) we deduce that

$$Lu = (k_1 m^+ - k_2 m^-) u^p \leq mf(u) \quad \text{in } \Omega.$$

On the other side, let $\varphi > 0$ be the solution of $L\varphi = m^+$ in Ω, $\varphi = 0$ on $\partial\Omega$, and let $k \geq \max\big\{\overline{K}, (k_3(\|\varphi\|_\infty + 1)^q)^{1/(1-q)}\big\}$. Recalling (H1) and reasoning as in (2.3) we see that

$$L(k(\varphi + 1)) \geq km^+ \geq k_3 (k(\varphi + 1))^q m^+ \geq mf(k(\varphi + 1)) \quad \text{in } \Omega$$

and the corollary is proved. $\square$

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