Journal of Optimization Theory and Applications (2020) 186:375–397
https://doi.org/10.1007/s10957-020-01714-w

TECHNICAL NOTE

Necessary and Sufficient Conditions for Robust Minimal Solutions in Uncertain Vector Optimization

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Received: 6 November 2019 / Accepted: 3 July 2020 / Published online: 21 July 2020

Abstract
We introduce a new notion of a vector-based robust minimal solution for a vector-valued uncertain optimization problem, which is defined by means of some open cone. We present necessary and sufficient conditions for this kind of solution, which are stated in terms of some directional derivatives of vector-valued functions. To prove these results, we apply the methods of set-valued analysis. We also study relations between our definition and three other known optimality concepts. Finally, for the case of scalar optimization, we present two general algorithm models for computing vector-based robust minimal solutions.

Keywords Uncertain optimization · Robust minimal solutions · Set-valued analysis · Radial derivatives

Mathematics Subject Classification 49J52 · 49J53 · 49N30 · 90C31

1 Introduction

In many optimization problems, one has to deal with some uncertainty of the data. Mathematically, this can be described by an additional parameter, which influences

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Communicated by Anita Schöbel.

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either the objective function (as in [1]), or the functions defining constraints (as in [2]), or both. The exact value of this parameter is unknown at the moment of decision, but it can be assumed that the parameter values lie in a given *uncertainty set*.

The theory of uncertain optimization (also called robust optimization) for multiobjective problems is a relatively new direction of research: the authors of paper [3], submitted in 2014, write that it “has been started only within the last 2 years”. One possible approach to uncertain multiobjective optimization is to interpret an uncertain optimization problem as a special set-valued optimization problem and then apply the methods of set-valued analysis; see, e.g., [4, Section 3.1] and [1, Section 5]. In this paper, we follow [1] regarding the formulation of a set-valued problem associated with an uncertain vector optimization problem. We study the notion of Q-minimality (where Q is an open cone) in the context of uncertain vector optimization. We define four types of robust Q-minimal solutions, where the first one is new (a *vector-based robust Q-minimal solution*; see Definition 3.2(a)), while the other three are variants of some definitions known from the literature. The paper is devoted to studying relations between these four types of solutions, proving some optimality conditions for vector-based robust Q-minimal solutions and constructing algorithm models for finding them.

The organization of this paper is as follows: In Sect. 2, we briefly discuss Q-minimal solutions for set-valued optimization problems. In Sect. 3, we formulate an uncertain vector optimization problem and construct the associated set-valued optimization problem. We also define four concepts of robust Q-minimal solutions and examine relations between them. Section 4 provides one more relation for the particular case of scalar optimization. In Sect. 5, we prove a characterization of a vector-based robust Q-minimal solution of an uncertain optimization problem in terms of a radial derivative of some vector-valued function. Since this characterization may be difficult to apply in practice, in the next two sections we present other optimality conditions (necessary in Sect. 6 and sufficient in Sect. 7), which have simpler forms but are not characterizations. In Sect. 8, we discuss two general algorithm models for finding vector-based robust Q-minimal solutions for the case of scalar optimization with a finite number of scenarios. Finally, in Sect. 9, we present a computational example.

2 Q-Minimal Solutions in Set-Valued Optimization

Let X, Y be normed spaces, S be a nonempty subset of X, and $F : X \rightrightarrows Y$ be a set-valued map. We define the graph of F as follows:

$$\operatorname{graph} F := \{(x, y) \in X \times Y : y \in F(x)\}.$$

We denote by F_S the restriction of F to S defined by

$$F_S(x) := \begin{cases} F(x), & \text{if } x \in S, \\ \emptyset, & \text{if } x \notin S \end{cases} \tag{1}$$

(see [5, p. 132]). Let Q be an arbitrary open cone in Y, which is nonempty and different from Y. We remind that an *open cone* Q is an open set satisfying the condition $\lambda y \in Q$ for all $y \in Q$ and $\lambda > 0$. We consider the following set-valued optimization problem:

$$\text{minimize } F(x) \text{ subject to } x \in S, \tag{2}$$

where the minimization is understood with respect to the cone Q, according to the following definition.

Definition 2.1 Let $(\bar{x}, \bar{y}) \in \operatorname{graph} F_S$. We say that $(\bar{x}, \bar{y})$ is a Q-minimal solution of problem (2), if

$$(F(S) - \bar{y}) \cap (-Q) = \emptyset, \tag{3}$$

where

$$F(S) := \bigcup_{x \in S} F(x).$$

We introduce the following relation $\prec$ in Y:

$$(y_1 \prec y_2) :\Leftrightarrow (y_2 - y_1 \in Q)\,. \tag{4}$$

In particular, if the cone Q is convex, then the relation $\prec$ is transitive.

Remark 2.1 It is easy to see that $(\bar{x}, \bar{y})$ is a Q-minimal solution of problem (2), if and only if $y \not\prec \bar{y}$ for all $y \in F(S)$.

The notion of a Q-minimal solution has been introduced in [6]. It includes several types of solutions, known from the literature, as particular cases:

(i) a strong (or ideal) efficient point of $F(S)$,
(ii) a weak efficient point of $F(S)$,
(iii) a positive-properly efficient point of $F(S)$,
(iv) a Geoffrion-properly efficient point of $F(S)$,
(v) a Borwein-properly efficient point of $F(S)$,
(vi) a Henig-properly efficient point of $F(S)$,
(vii) a strong Henig-properly efficient point of $F(S)$,
(viii) a super efficient point of $F(S)$;

the details are described in [7, Prop. 1.2] and [6, Thm. 21.7].

For other solution concepts in set-valued optimization, see [8, Section 2.6].

A particular case of problem (2) is the vector optimization problem:

$$\text{minimize } f(x) \text{ subject to } x \in S, \tag{5}$$

where $f : X \to Y$ is a single-valued map.

Definition 2.2 Let $\bar{x} \in S$. We say that $\bar{x}$ is a Q-minimal solution of problem (5), if $(\bar{x}, f(\bar{x}))$ is a Q-minimal solution of problem (2) with F defined by $F(x) := \{f(x)\}$.

Remark 2.2 Obviously, $\bar{x}$ is a Q-minimal solution of problem (5), if and only if

$$(f(S) - f(\bar{x})) \cap (-Q) = \emptyset. \tag{6}$$

3 An Uncertain Vector Optimization Problem

In this section, we formulate an uncertain vector optimization problem as in [1, Section 5], define four types of its robust Q-minimal solutions, and discuss the relationships between them.

Let X, Y, Z be normed spaces, let S and $\mathcal{U}$ be nonempty subsets of X and Z, respectively, and let $f : X \times \mathcal{U} \to Y$.

Definition 3.1 An uncertain vector optimization problem $P(\mathcal{U})$ is defined as the family

$$(P(\xi), \xi \in \mathcal{U}) \tag{7}$$

of vector optimization problems

$$P(\xi) : \quad \text{minimize } f(x, \xi) \text{ subject to } x \in S. \tag{8}$$

For each $x \in X$, we denote

$$F(x) := \{f(x, \xi) : \xi \in \mathcal{U}\} \subseteq Y. \tag{9}$$

Then, $F : X \rightrightarrows Y$ is a set-valued map. In this way, we can construct a set-valued optimization problem of the form (2), associated with the uncertain vector optimization problem (7).

Definition 3.2 Let $\bar{x} \in S$, and let F be defined by (9). We say that

(a) $\bar{x}$ is a vector-based robust Q-minimal solution of $P(\mathcal{U})$, if there exists $\bar{y} \in F(\bar{x})$ such that $(\bar{x}, \bar{y})$ is a Q-minimal solution of (2);
(b) $\bar{x}$ is a flimsily robust Q-minimal solution of $P(\mathcal{U})$, if it is a Q-minimal solution of $P(\xi)$ for at least one $\xi \in \mathcal{U}$;
(c) $\bar{x}$ is a highly robust Q-minimal solution of $P(\mathcal{U})$, if it is a Q-minimal solution of $P(\xi)$ for all $\xi \in \mathcal{U}$;
(d) $\bar{x}$ is a set-based robust Q-minimal solution of $P(\mathcal{U})$, if there exists no $x \in S$ such that $F(x) \subseteq F(\bar{x}) - Q$.

Remark 3.1 Part (a) of Definition 3.2 is new. Parts (b) and (c) are introduced here by analogy with Definitions 4 and 5, respectively, in [3], where the usual efficiency instead of Q-minimality was used. Part (d) is analogous to Definition 3.2 in [9]. For other concepts of robust solutions in uncertain optimization and relations between them, see [10]. The motivation for using the vector-based approach in this paper was to obtain some intermediate concept between definitions (b) and (c), which, however, proved successful in the scalar-valued case only; see Sect. 4. We will try to extend our results to vector-valued problems in a further research.

The proposition below clarifies the relation between Definitions 2.2 and 3.2(a).

Proposition 3.1 *A point $\bar{x} \in S$ is a vector-based robust Q-minimal solution of $P(\mathcal{U})$, if and only if there exists $\bar{\xi} \in \mathcal{U}$ such that $(\bar{x}, \bar{\xi})$ is a Q-minimal solution of the following vector optimization problem:*

$$\text{minimize } f(x, \xi) \text{ subject to } (x, \xi) \in S \times \mathcal{U}. \tag{10}$$

Proof By Definition 3.2(a), formula (9) and Remark 2.2 (where S should be replaced by $S \times \mathcal{U}$), we have the following chain of equivalences:

$$\begin{aligned}
&\bar{x} \text{ is a vector-based robust } Q\text{-minimal solution of } P(\mathcal{U})\\
\Leftrightarrow\ & \exists \bar{y} \in F(\bar{x}),\ (F(S) - \bar{y}) \cap (-Q) = \emptyset\\
\Leftrightarrow\ & \exists \bar{\xi} \in \mathcal{U},\ \big(F(S) - f(\bar{x}, \bar{\xi})\big) \cap (-Q) = \emptyset\\
\Leftrightarrow\ & \exists \bar{\xi} \in \mathcal{U},\ \big(f(S \times \mathcal{U}) - f(\bar{x}, \bar{\xi})\big) \cap (-Q) = \emptyset\\
\Leftrightarrow\ & \exists \bar{\xi} \in \mathcal{U},\ (\bar{x}, \bar{\xi}) \text{ is a } Q\text{-minimal solution of (10).}
\end{aligned} \tag{11}$$

□

Corollary 3.1 *A point $\bar{x} \in X$ is a vector-based robust Q-minimal solution of $P(\mathcal{U})$, if and only if there exists $\bar{\xi} \in \mathcal{U}$ such that*

$$f(x, \xi) \not\leq f(\bar{x}, \bar{\xi}) \text{ for all } (x, \xi) \in S \times \mathcal{U}.$$

Proof This follows easily from (4) and the fourth statement in (11). □

The rest of this section is devoted to studying relations between the different concepts of Q-minimality for $P(\mathcal{U})$ which are listed in Definition 3.2. Propositions 3.2 and 3.3 show that the implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (b) are always true. Examples 3.1–3.6 prove that in the general case, no other implication between definitions (a), (b), (c), (d) is valid. Later in Sect. 4, we will show that the implication (c) $\Rightarrow$ (a) holds for the particular case of a scalar uncertain optimization problem.

Proposition 3.2 *If $\bar{x}$ is a vector-based robust Q-minimal solution of $P(\mathcal{U})$, then it is a flimsily robust Q-minimal solution of $P(\mathcal{U})$.*

Proof By assumption, there exists $\bar{y} \in F(\bar{x})$ such that $(\bar{x}, \bar{y})$ is a Q-minimal solution of (2). Hence, for each $\xi \in \mathcal{U}$, we have

$$(f(S, \xi) - \bar{y}) \cap (-Q) \subseteq (F(S) - \bar{y}) \cap (-Q) = \emptyset. \tag{12}$$

The relation $\bar{y} \in F(\bar{x})$ implies that $\bar{y} = f(\bar{x}, \bar{\xi})$ for some $\bar{\xi} \in \mathcal{U}$. Of course, this $\bar{\xi}$ also satisfies (12). Therefore, we have

$$(f(S, \bar{\xi}) - f(\bar{x}, \bar{\xi})) \cap (-Q) = \emptyset,$$

which by Remark 2.2 is equivalent to $\bar{x}$ being a Q-minimal solution of $P(\bar{\xi})$. □

Proposition 3.3 *If $\bar{x}$ is a highly robust Q-minimal solution of $P(\mathcal{U})$, then it is a flimsily robust Q-minimal solution of $P(\mathcal{U})$.*

Proof This follows immediately from the definitions (see [3, Lemma 6]). □

Below, the symbol $[y, y']$ with y, y' belonging to Y (or another normed space), denotes the line segment, i.e., $[y, y'] = \{\lambda y + (1-\lambda)y' : 0 \leq \lambda \leq 1\}$.

Example 3.1 This example shows that (b) $\not\Rightarrow$ (a), (b) $\not\Rightarrow$ (c) and (d) $\not\Rightarrow$ (a).

Let $X = Y = Z = \mathbb{R}$, $Q =]0, \infty[$, $S = \mathcal{U} = [-1, 1]$, $f(x, \xi) = x^2\xi$. Then, $F(x) = [-x^2, x^2]$ for all $x \in \mathbb{R}$. Observe that for each $\xi \in [0, 1]$, the point $\bar{x} = 0$ is a Q-minimal solution of $P(\xi)$, and for $\xi \in [-1, 0[$, it is not a Q-minimal solution of $P(\xi)$. Thus, $\bar{x}$ is a flimsily robust (but not highly robust) Q-minimal solution of $P(\mathcal{U})$. However, $\bar{x}$ is not a vector-based robust Q-minimal solution of $P(\mathcal{U})$ because the only element $\bar{y} \in F(0)$ is $\bar{y} = 0$, and

$$F(S) \cap (-Q) = [-1, 1] \cap]-\infty, 0[= [-1, 0[\neq \emptyset.$$

We can also see that $\bar{x} = 0$ is a set-based robust Q-minimal solution of $P(\mathcal{U})$. Indeed, for each $x \in S$, we have

$$F(x) = [-x^2, x^2] \not\subseteq]-\infty, 0[= \{0\} -]0, \infty[= F(\bar{x}) - Q.$$

Example 3.2 This example shows that (a) $\not\Rightarrow$ (d) and (b) $\not\Rightarrow$ (d).

Let $X = Y = Z = \mathbb{R}$, $Q =]0, \infty[$, $S = \mathcal{U} = [-1, 1]$, $f(x, \xi) = \left(x^2 - 1\right)\xi$. Then, $F(x) = [x^2 - 1, -x^2 + 1]$ for all $x \in \mathbb{R}$. We can see that $\bar{x} = 0$ is a flimsily robust (but not highly robust) Q-minimal solution of $P(\mathcal{U})$ since it is a Q-minimal solution of $P(\xi)$ for $\xi \in [0, 1]$ and it is not a Q-minimal solution of $P(\xi)$ for $\xi \in [-1, 0[$. Moreover, $\bar{x}$ is a vector-based robust Q-minimal solution of $P(\mathcal{U})$ because, for $\bar{y} = -1$, we have

$$(F(S) - \bar{y}) \cap (-Q) = ([-1, 1] + 1) \cap]-\infty, 0[= [0, 2] \cap]-\infty, 0[= \emptyset.$$

However, $\bar{x} = 0$ is not a set-based robust Q-minimal solution of $P(\mathcal{U})$. Indeed, for each $x \in S \backslash \{\bar{x}\}$, we have

$$F(x) = [x^2 - 1, -x^2 + 1] \subseteq]-\infty, 1[= [-1, 1] -]0, \infty[= F(\bar{x}) - Q.$$

Example 3.3 This example shows that (d) $\not\Rightarrow$ (b) and (d) $\not\Rightarrow$ (c).

Let $X = Y = Z = \mathbb{R}$, $Q =]0, \infty[$, $S = [-1, 1]$, $\mathcal{U} = \{-0.5, 0.5\}$, $f(x, \xi) = (x - \xi)^2$. Then, $F(x) = \left\{(x - \xi)^2 : \xi \in \mathcal{U}\right\}$ for all $x \in \mathbb{R}$; in particular, $F(0) =$

$\{(-0.5)^2, (0.5)^2\} = \{0.25\}$. We will show that $\bar{x} = 0$ is a set-based robust Q-minimal solution of $P(\mathcal{U})$. Suppose that this is not true. Then, there exists $x \in S$ such that

$$F(x) \subseteq F(\bar{x}) - Q =]-\infty, 0.25[. \tag{13}$$

However, if $x \in [0, 1]$, then $(x + 0.5)^2 \in F(x)$ and $(x + 0.5)^2 \geq 0.25$, which contradicts (13). Similarly, if $x \in [-1, 0]$, then $(x - 0.5)^2 \in F(x)$ and $(x - 0.5)^2 \geq 0.25$, which also contradicts (13).

On the other hand, $\bar{x} = 0$ is not a flimsily robust Q-minimal solution of $P(\mathcal{U})$. This follows because the only solution of $P(-0.5)$ is equal to -0.5 with $f(-0.5, -0.5) = 0$, and similarly, the only solution of $P(0.5)$ is equal to 0.5 with $f(0.5, 0.5) = 0$, while at $\bar{x} = 0$, both minimized functions $f(\cdot, -0.5)$ and $f(\cdot, 0.5)$ have strictly positive values. Of course, by Proposition 3.3, $\bar{x}$ is also not a highly robust Q-minimal solution of $P(\mathcal{U})$.

Example 3.4 This example shows that (c) $\not\Rightarrow$ (a).

Let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $Q = \{(y_1, y_2) \in \mathbb{R}^2 : y_i > 0, i = 1, 2\}$, $S = [0, 3]$, $\mathcal{U} = \{1, 2\}$,

$$f(x, \xi) = \begin{cases} (0, 2x), & \text{if } \xi = 1,\ x \in [0, 3], \\ (-0.5x + 1, 0.5x), & \text{if } \xi = 2,\ x \in [0, 2], \\ (-0.5x + 1, -0.5x + 2), & \text{if } \xi = 2,\ x \in]2, 3]. \end{cases}$$

Let us note that

$$f(S, 1) = [(0, 0), (0, 6)],$$
$$f(S, 2) = [(1, 0), (0, 1)] \cup [(-0.5, 0.5), (0, 1)].$$

Let $\bar{x} = 1$. Observe that $\bar{x}$ is a highly robust Q-minimal solution of $P(\mathcal{U})$. Indeed, $f(\bar{x}, 1) = (0, 2)$, $f(\bar{x}, 2) = (0.5, 0.5)$,

$$f(S, 1) - f(\bar{x}, 1) = [(0, -2), (0, 4)],$$
$$f(S, 2) - f(\bar{x}, 2) = [(0.5, -0.5), (-0.5, 0.5)] \cup [(-1, 0), (-0.5, 0.5)].$$

Therefore,

$$(f(S, 1) - f(\bar{x}, 1)) \cap (-Q) = \emptyset,$$
$$(f(S, 2) - f(\bar{x}, 2)) \cap (-Q) = \emptyset,$$

which means that $\bar{x}$ is a Q-minimal solution of both vector optimization problems $P(\xi)$, $\xi = 1, 2$.

However, the point $\bar{x} = 1$ is not a vector-based robust Q-minimal solution of $P(\mathcal{U})$. Let us suppose the contrary. Then, there exists a point $\bar{y} \in F(\bar{x}) = \{(0, 2), (0.5, 0.5)\}$ such that

$$\left(\bigcup_{\xi\in\mathcal{U}} f(S,\xi) - \bar{y}\right) \cap (-Q) = \emptyset,$$

where

$$\bigcup_{\xi\in\mathcal{U}} f(S,\xi) = [(0,0),(0,6)] \cup [(1,0),(0,1)] \cup [(-0.5,0.5),(0,1)].$$

But $\bar{y}$ cannot be equal to $(0, 2)$, because

$$(-0.5, 0.5) = f(3,2) \in \bigcup_{\xi\in\mathcal{U}} f(S,\xi)$$

and

$$(-0.5, 0.5) - (0, 2) = (-0.5, -1.5) \in (-Q).$$

Similarly, $\bar{y}$ cannot be equal to $(0.5, 0.5)$, because

$$(0, 0) = f(0,1) \in \bigcup_{\xi\in\mathcal{U}} f(S,\xi)$$

and

$$(0, 0) - (0.5, 0.5) = (-0.5, -0.5) \in (-Q).$$

Therefore, we get a contradiction.

Example 3.5 This example shows that (a) $\not\Rightarrow$ (c).

Take the same data as in Example 3.4, except for the definition of f, which has now the form

$$f(x,\xi) = \begin{cases} (0, 2x), & \text{if } \xi = 1,\ x \in [0,3], \\ (-x+2, x), & \text{if } \xi = 2,\ x \in [0,1], \\ (-0.5x+1.5, -0.5x+1.5), & \text{if } \xi = 2,\ x \in\,]1,3]. \end{cases}$$

Let us note that

$$\begin{aligned} f(S,1) &= [(0,0),(0,6)], \\ f(S,2) &= [(2,0),(1,1)] \cup [(0,0),(1,1)]. \end{aligned}$$

Let $\bar{x} = 1$. Observe that $\bar{x}$ is not a highly robust Q-minimal solution of $P(\mathcal{U})$ (it is only flimsily robust). Indeed, $f(\bar{x},1) = (0,2)$, $f(\bar{x},2) = (1,1)$,

$$\begin{aligned} f(S,1) - f(\bar{x},1) &= [(0,-2),(0,4)], \\ f(S,2) - f(\bar{x},2) &= [(1,-1),(0,0)] \cup [(-1,-1),(0,0)]. \end{aligned}$$

Therefore,

$$(f(S,1) - f(\bar{x},1)) \cap (-Q) = \emptyset,$$
$$(f(S,2) - f(\bar{x},2)) \cap (-Q) \neq \emptyset,$$

which means that $\bar{x}$ is a Q-minimal solution of vector optimization problem $P(1)$ but is not a Q-minimal solution of vector optimization problem $P(2)$.

However, the point $\bar{x} = 1$ is a vector-based robust Q-minimal solution of $P(\mathcal{U})$. Indeed, taking $\bar{y} = (0,2) \in F(\bar{x})$, we obtain

$$\left(\bigcup_{\xi \in U} f(S,\xi) - \bar{y} \right) \cap (-Q) = \emptyset,$$

since

$$\bigcup_{\xi \in U} f(S,\xi) - \bar{y} = [(0,-2),(0,4)] \cup [(2,-2),(1,-1)] \cup [(0,-2),(1,-1)].$$

Example 3.6 This example shows that (c) $\not\Rightarrow$ (d).

Let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $Q = \{(y_1,y_2) \in \mathbb{R}^2 : y_i > 0, i = 1,2\}$, $S = [0,2]$, $\mathcal{U} = \{1,2\}$,

$$f(x,\xi) = \begin{cases} (2,x), & \text{if } \xi = 1,\ x \in [0,1], \\ (-1.5x+3.5,1), & \text{if } \xi = 1,\ x \in]1,2], \\ (x,2), & \text{if } \xi = 2,\ x \in [0,1], \\ (1,-1.5x+3.5), & \text{if } \xi = 2,\ x \in]1,2]. \end{cases}$$

We will show that $\bar{x} = 1$ is a highly robust Q-minimal solution of $P(\mathcal{U})$. Indeed, $\bar{x}$ is a Q-minimal solution of $P(1)$ because the set

$$\begin{aligned} f(S,1) - f(\bar{x},1) &= [(0.5,1),(2,1)] \cup [(2,0),(2,1)] - (2,1) \\ &= [(-1.5,0),(0,0)] \cup [(0,-1),(0,0)] \end{aligned}$$

has empty intersection with $-Q$. Similarly, $\bar{x}$ is a Q-minimal solution of $P(2)$ because the set

$$\begin{aligned} f(S,1) - f(\bar{x},2) &= [(1,0.5),(1,2)] \cup [(0,2),(1,2)] - (1,2) \\ &= [(0,-1.5),(0,0)] \cup [(-1,0),(0,0)] \end{aligned}$$

has empty intersection with $-Q$.

However, $\bar{x} = 1$ is not a set-based robust Q-minimal solution of $P(\mathcal{U})$. To see this, take $x = 2$. We have

$$F(\bar{x}) = \{f(1,1), f(1,2)\} = \{(2,1),(1,2)\},$$
$$F(x) = \{f(2,1), f(2,2)\} = \{(0.5,1),(1,0.5)\},$$

and

$$F(\bar{x}) - Q = \{(y_1, y_2) : y_1 < 1, y_2 < 2\} \cup \{(y_1, y_2) : y_1 < 2, y_2 < 1\}.$$

It follows that $F(x) \subseteq F(\bar{x}) - Q$, which contradicts Definition 3.2(d).

4 The Case of Scalar Optimization

In this section we consider the case where $Y = \mathbb{R}$ and $Q =]0, \infty[$. In this case, the relation $\prec$ may be replaced by the usual strict inequality $<$. We will show that in this case, one more relation between two parts of Definition 3.2 holds, which implies that a vector-based robust Q-minimal solution is an intermediate notion between a highly robust Q-minimal solution and a flimsily robust Q-minimal solution.

Theorem 4.1 *Let $Y = \mathbb{R}$ and $Q =]0, \infty[$. Suppose that the values of the set-valued map $F : X \rightrightarrows \mathbb{R}$ are closed and bounded from below. Then, every highly robust Q-minimal solution of $P(\mathcal{U})$ is a vector-based robust Q-minimal solution of $P(\mathcal{U})$.*

Proof Let $\bar{x} \in S$ be a highly robust Q-minimal solution of $P(\mathcal{U})$. Then, for each $\xi \in \mathcal{U}$, $\bar{x}$ is a Q-minimal solution of the scalar optimization problem $P(\xi)$, which means that $\bar{x}$ is a global minimum point of $f(\cdot, \xi)$ in the usual sense:

$$f(\bar{x}, \xi) \leq f(x, \xi) \text{ for all } x \in S. \tag{14}$$

Since the set $F(\bar{x}) = \{f(\bar{x}, \xi) : \xi \in \mathcal{U}\}$ is closed and bounded from below, there exists $\bar{\xi} \in \mathcal{U}$ such that

$$f(\bar{x}, \bar{\xi}) = \min \{f(\bar{x}, \xi) : \xi \in \mathcal{U}\}. \tag{15}$$

Conditions (14) and (15) imply that

$$f(\bar{x}, \bar{\xi}) \leq f(\bar{x}, \xi) \leq f(x, \xi) \text{ for all } (x, \xi) \in S \times \mathcal{U}.$$

Consequently, by Proposition 3.1, $\bar{x}$ is a vector-based robust Q-minimal solution of $P(\mathcal{U})$. □

5 A Characterization of Vector-Based Robust *Q*-Minimal Solutions

In this section, we present a characterization of a vector-based robust Q-minimal solution of $P(\mathcal{U})$ in terms of some radial derivative of the function f appearing in (8), restricted to $S \times \mathcal{U}$. By our knowledge, such results are not known even in the special scalar-valued case. It seems to be possible to derive similar characterizations for different solutions based on the set-based approach to uncertain optimization; for example, for part (d) of Definition 3.2. We plan to describe corresponding results in a subsequent paper.

First, we recall the definition of an outer radial derivative of an arbitrary set-valued mapping.

Definition 5.1 Let $F : X \rightrightarrows Y$, let $(\bar{x}, \bar{y}) \in \operatorname{graph} F$, and let m be a positive integer. The m-th order outer radial derivative of F at $(\bar{x}, \bar{y})$ is the set-valued map $\overline{D}_R^m F(\bar{x}, \bar{y}) : X \rightrightarrows Y$ defined by

$$\begin{aligned} &\overline{D}_R^m F(\bar{x}, \bar{y})(u) \\ &\quad := \left\{ v \in Y : \exists t_n > 0, \exists (u_n, v_n) \to (u, v), \forall n,\ \bar{y} + t_n^m v_n \in F(\bar{x} + t_n u_n) \right\}. \end{aligned} \tag{16}$$

The derivative $\overline{D}_R^1 F(\bar{x}, \bar{y})$ was first introduced in [5]; the derivative $\overline{D}_R^m F(\bar{x}, \bar{y})$ (for an arbitrary m) was defined in [7]. An interesting feature of radial derivatives is that contrary to classical derivatives, they lead to global sufficient conditions without any (generalized) convexity assumptions. This is due to the fact that we do not require that t_n converges to zero in (16).

In particular, if $f : X \to Y$ is a single-valued mapping, we will use the notation $\overline{D}_R^m f(\bar{x}; u)$ instead of $\overline{D}_R^m F(\bar{x}, f(\bar{x}))(u)$, where $F : X \rightrightarrows Y$ is the multifunction defined by $F(x) := \{f(x)\}$. Hence, it follows from (16) that

$$\begin{aligned} &\overline{D}_R^m f(\bar{x}; u) \\ &\quad = \left\{ v \in Y : \exists t_n > 0,\ \exists (u_n, v_n) \to (u, v),\ \forall n,\ f(\bar{x}) + t_n^m v_n = f(\bar{x} + t_n u_n) \right\} \\ &\quad = \left\{ v \in Y : \exists t_n > 0,\ \exists u_n \to u,\ t_n^{-m}(f(\bar{x} + t_n u_n) - f(\bar{x})) \to v \right\}. \end{aligned} \tag{17}$$

Proposition 5.1 *Let $f : X \to Y$, $\bar{x}, u \in X$, and let m be a positive integer. Then, $f(\bar{x} + u) - f(\bar{x}) \in \overline{D}_R^m f(\bar{x}; u)$.*

Proof It is sufficient to take the constant sequences $t_n \equiv 1$ and $(u_n, v_n) \equiv (u, v)$ in (17). □

We now return to the uncertain optimization problem $P(\mathcal{U})$. We will denote by $f_{S\times\mathcal{U}}$ the restriction of $f : X \times Z \to Y$ to $S \times \mathcal{U}$. Then, by analogy with (17), we can write, for any $(\bar{x}, \bar{\xi}) \in S \times \mathcal{U}$,

$$\begin{aligned} &\overline{D}_R^m f_{S\times\mathcal{U}}((\bar{x}, \bar{\xi}); (x, \xi)) \\ &\quad = \{y \in Y : \exists t_n > 0, \exists (x_n, \xi_n, y_n) \to (x, \xi, y), \forall n, \\ &\qquad f(\bar{x}, \bar{\xi}) + t_n^m y_n \in f_{S\times\mathcal{U}}(\bar{x} + t_n x_n, \bar{\xi} + t_n \xi_n)\} \\ &\quad = \{y \in Y : \exists t_n > 0, \exists (x_n, \xi_n, y_n) \to (x, \xi, y), \forall n,\ \bar{x} + t_n x_n \in S, \\ &\qquad \bar{\xi} + t_n \xi_n \in \mathcal{U},\ f(\bar{x}, \bar{\xi}) + t_n^m y_n = f(\bar{x} + t_n x_n, \bar{\xi} + t_n \xi_n)\}. \end{aligned} \tag{18}$$

We have the following counterpart of Proposition 5.1.

Proposition 5.2 *Let $f : X \times Z \to Y$, $(\bar{x}, \bar{\xi}) \in S \times \mathcal{U} \subseteq X \times Z$, and let m be a positive integer. Suppose that $x \in X$ and $\xi \in \mathcal{U}$ are such that $\bar{x} + x \in S$ and $\bar{\xi} + \xi \in \mathcal{U}$. Then,*

$$f(\bar{x} + x, \bar{\xi} + \xi) - f(\bar{x}, \bar{\xi}) \in \overline{D}_R^m f_{S\times\mathcal{U}}((\bar{x}, \bar{\xi}); (x, \xi)).$$

Proof It is sufficient to take $t_n \equiv 1$ and $(x_n, \xi_n, y_n) \equiv (x, \xi, y)$ in (18). □

Theorem 5.1 *A point $\bar{x} \in S$ is a vector-based robust Q-minimal solution of $P(\mathcal{U})$ if and only if*

$$\exists \bar{\xi} \in \mathcal{U},\ \overline{D}_R^m f_{S\times\mathcal{U}}((\bar{x}, \bar{\xi}); (S - \bar{x}, \mathcal{U} - \bar{\xi})) \cap (-Q) = \emptyset. \tag{19}$$

Proof Part "if". Suppose that $\bar{x}$ is not a vector-based robust Q-minimal solution of $P(\mathcal{U})$. Then, for each $\bar{y} \in F(\bar{x})$ (where F is given by (9)), the pair $(\bar{x}, \bar{y})$ is not a Q-minimal solution of (2). This is equivalent to

$$(F(S) - \bar{y}) \cap (-Q) \neq \emptyset \text{ for all } \bar{y} \in F(\bar{x}). \tag{20}$$

Since $\bar{y} \in F(\bar{x})$ is equivalent to $\bar{y} = f(\bar{x}, \bar{\xi})$ for some $\bar{\xi} \in \mathcal{U}$, we obtain from (20) that

$$\big(F(S) - f(\bar{x}, \bar{\xi})\big) \cap (-Q) \neq \emptyset \text{ for all } \bar{\xi} \in \mathcal{U}. \tag{21}$$

Take any $\bar{\xi} \in \mathcal{U}$. By (21), there exists $x \in S$ such that

$$\big(F(x) - f(\bar{x}, \bar{\xi})\big) \cap (-Q) \neq \emptyset.$$

Using the definition of F, we see that there exists $\xi \in \mathcal{U}$ such that

$$f(x, \xi) - f(\bar{x}, \bar{\xi}) \in -Q. \tag{22}$$

By defining $u := x - \bar{x} \in S - \bar{x}$ and $d := \xi - \bar{\xi} \in \mathcal{U} - \bar{\xi}$, we can rewrite (22) as

$$f(\bar{x} + u, \bar{\xi} + d) - f(\bar{x}, \bar{\xi}) \in -Q. \tag{23}$$

However, by Proposition 5.2 and the relations $\bar{x} + u = x \in S$, $\bar{\xi} + d = \xi \in \mathcal{U}$, we have

$$f(\bar{x} + u, \bar{\xi} + d) - f(\bar{x}, \bar{\xi}) \in \overline{D}_R^m f_{S\times\mathcal{U}}((\bar{x}, \bar{\xi}); (u, d)). \tag{24}$$

Combining (23) and (24), we get

$$\overline{D}_R^m f_{S\times\mathcal{U}}((\bar{x}, \bar{\xi}); (u, d)) \cap (-Q) \neq \emptyset. \tag{25}$$

We have thus verified that for each $\bar{\xi} \in \mathcal{U}$, there exist $u \in S - \bar{x}$ and $d \in \mathcal{U} - \bar{\xi}$ such that (25) holds. This contradicts (19).

Part "only if". Let $\bar{x} \in S$ be a vector-based robust Q-minimal solution of $P(\mathcal{U})$, then

$$\exists \bar{y} \in F(\bar{x}),\ (F(S) - \bar{y}) \cap (-Q) = \emptyset. \tag{26}$$

Hence, there exists $\bar{\xi} \in \mathcal{U}$ such that $\bar{y} = f(\bar{x}, \bar{\xi})$, and consequently,

$$\big(F(S) - f(\bar{x}, \bar{\xi})\big) \cap (-Q) = \emptyset.$$

We will show that

$$\overline{D}^m_R f_{S\times\mathcal{U}}((\bar{x}, \bar{\xi}); (S - \bar{x}, \mathcal{U} - \bar{\xi})) \cap (-Q) = \emptyset. \tag{27}$$

Suppose to the contrary that (27) is false, then there exist $x \in S$, $\xi \in \mathcal{U}$ and $y \in Y$ such that

$$y \in \overline{D}^m_R f_{S\times\mathcal{U}}((\bar{x}, \bar{\xi}); (x - \bar{x}, \xi - \bar{\xi})) \cap (-Q). \tag{28}$$

By (28) and (18), there exist sequences $t_n > 0$ and $(x_n, \xi_n, y_n) \to (x - \bar{x}, \xi - \bar{\xi}, y)$ such that for all n, we have

$$\bar{x} + t_n x_n \in S,\ \bar{\xi} + t_n \xi_n \in \mathcal{U} \text{ and } f(\bar{x}, \bar{\xi}) + t_n^m y_n = f(\bar{x} + t_n x_n, \bar{\xi} + t_n \xi_n). \tag{29}$$

Since Q is open and $y_n \to y \in -Q$, we have $y_n \in -Q$ for sufficiently large n. As Q is an open cone, the last relation implies $t_n^m y_n \in -Q$. From this and (29), we deduce

$$\begin{aligned} t_n^m y_n &\in \big\{ f(\bar{x} + t_n x_n, \bar{\xi} + t_n \xi_n) - \bar{y} \big\} \cap (-Q) \\ &\subseteq (f(S, \mathcal{U}) - \bar{y}) \cap (-Q) = (F(S) - \bar{y}) \cap (-Q), \end{aligned}$$

a contradiction to (26). □

The characterization given in Theorem 5.1 is difficult to apply in practice as it involves the restriction $f_{S\times\mathcal{U}}$, which is not easy to compute, especially if the constraint set S is defined by some functional conditions. Therefore, in the next two sections we present a necessary condition (Theorem 6.2) and a sufficient condition (Theorem 7.1), both for a vector-based robust Q-minimal solution of $P(\mathcal{U})$, which do not use this restricted function.

6 Necessary Optimality Conditions

The following derivative for a set-valued mapping $F : X \rightrightarrows Y$ was first defined in [11].

Definition 6.1 Let $(\bar{x}, \bar{y}) \in \operatorname{graph} F$, and let m be a positive integer. The m-th order outer contingent-type derivative of F at $(\bar{x}, \bar{y})$ is the set-valued map $\overline{d}^m F(\bar{x}, \bar{y}) : X \rightrightarrows Y$ defined by

$$\begin{aligned} &\overline{d}^m F(\bar{x}, \bar{y})(u) \\ &\quad := \{v \in Y : \exists h_n \downarrow 0, \exists u_n \to u, \exists v_n \to v, \forall n,\ f(\bar{x}, \bar{\xi}) + h_n^m v_n \in F(\bar{x} + h_n u_n)\}. \end{aligned} \tag{30}$$

We will also use the following derivative for a vector-valued map $f : X \to Y$ (if it exists):

$$d^m f(\bar{x}; u) := \lim_{\substack{t \downarrow 0 \\ w \to u}} t^{-m} \left(f(\bar{x} + tw) - f(\bar{x}) \right),$$

where m is a positive integer, and $u, w \in X$.

Definition 6.2 The contingent cone to S at $\bar{x} \in \mathrm{cl}S$ is defined as follows:

$$K(S, \bar{x}) := \{v \in X : \exists t_n \downarrow 0, \exists v_n \to v, \forall n,\ \bar{x} + t_n v_n \in S\}.$$

The following two theorems are, in view of Definition 3.2(a), necessary conditions for $\bar{x} \in S$ to be a vector-based robust Q-minimal solution of $P(\mathcal{U})$.

Theorem 6.1 *Let $(\bar{x}, \bar{y}) \in \mathrm{graph}F_S$, and let m be a positive integer. If $(\bar{x}, \bar{y})$ is a Q-minimal solution of problem (2), then*

$$\overline{d}^m F_S(\bar{x}, \bar{y})(X) \cap (-Q) = \emptyset. \tag{31}$$

Proof Suppose that $(\bar{x}, \bar{y})$ is a Q-minimal solution of (2) but condition (31) is false, then there exist vectors $u \in X$ and

$$v \in \overline{d}^m F_S(\bar{x}, \bar{y})(u) \cap (-Q).$$

By the definition of $\overline{d}^m F_S$, there exist sequences $t_n \to 0^+$ and $(u_n, v_n) \to (u, v)$ such that

$$\bar{y} + t_n^m v_n \in F_S(\bar{x} + t_n u_n),$$

which is equivalent to

$$\bar{x} + t_n u_n \in S \text{ and } \bar{y} + t_n^m v_n \in F(\bar{x} + t_n u_n). \tag{32}$$

Since Q is open and $v_n \to v \in -Q$, we have $v_n \in -Q$ for sufficiently large n. As Q is an open cone, the last relation implies $t_n^m v_n \in -Q$. From this and (32), we deduce

$$t_n^m v_n \in (F(\bar{x} + t_n u_n) - \bar{y}) \cap (-Q) \subseteq (F(S) - \bar{y}) \cap (-Q),$$

a contradiction to (3). □

Remark 6.1 Contrary to the other results of this paper, Theorem 6.1 remains valid even if F is an arbitrary set-valued map, not necessarily defined by formula (9).

Theorem 6.2 *Let F be given by (9), let $(\bar{x}, \bar{y}) \in \mathrm{graph}F_S$, and let m be a positive integer. Suppose that for each $\bar{\xi} \in \mathcal{U}$ satisfying the condition*

$$f(\bar{x}, \bar{\xi}) = \bar{y}, \tag{33}$$

and for each pair $(u, d) \in X \times Z$*, there exists the derivative* $d^m f((\bar{x}, \bar{\xi}); (u, d)) \in Y$*. If* $(\bar{x}, \bar{y})$ *is a* Q*-minimal solution of* (2) *(where* F *is given by* (9)*), then*

$$d^m f((\bar{x}, \bar{\xi}); (u, d)) \notin -Q \tag{34}$$

for all vectors $\bar{\xi} \in \mathcal{U}$ *satisfying* (33) *and for all*

$$(u, d) \in K(S \times \mathcal{U}, (\bar{x}, \bar{\xi})). \tag{35}$$

Proof Suppose that the desired conclusion is false, then there exist vectors $\bar{\xi} \in \mathcal{U}$ and $(u, d) \in X \times Y$ satisfying (33) and (35), respectively, such that

$$v := d^m f((\bar{x}, \bar{\xi}); (u, d)) \in -Q. \tag{36}$$

By (35), there exist sequences $t_n \to 0^+$ and $(u_n, d_n) \to (u, d)$ such that for all n,

$$(\bar{x}, \bar{\xi}) + t_n(u_n, d_n) \in S \times \mathcal{U}. \tag{37}$$

Let $\xi_n := \bar{\xi} + t_n d_n$. By (36) and the definition of $d^m f$, we have

$$v_n := t_n^{-m} \left(f(\bar{x} + t_n u_n, \xi_n) - f(\bar{x}, \bar{\xi}) \right) \to v. \tag{38}$$

It follows from (33) and (38) that

$$\bar{y} + t_n^m v_n = f(\bar{x} + t_n u_n, \xi_n). \tag{39}$$

By (37), we obtain $\bar{x} + t_n u_n \in S$ and $\xi_n \in \mathcal{U}$. These two relations, and conditions (9), (39) give

$$\bar{y} + t_n^m v_n \in F(\bar{x} + t_n u_n) = F_S(\bar{x} + t_n u_n).$$

We have thus verified that there exist sequences $t_n \to 0^+$, $u_n \to u$ and $v_n \to v$ such that $\bar{y} + t_n^m v_n \in F_S(\bar{x} + t_n u_n)$ for all n. This means that $v \in \overline{d}^m F_S(\bar{x}, \bar{y})(u)$. But this contradicts Theorem 6.1 because $v \in -Q$. □

Example 6.1 Let $X = Y = Z = \mathbb{R}$, $Q =]0, \infty[$, $S = [-1, 1]$, $\mathcal{U} = [0, 1]$, $f(x, \xi) = x^2 + \xi$. Then, $F(x) = [x^2, x^2 + 1]$ for all $x \in \mathbb{R}$. The point $\bar{x} = 0$ is a vector-based robust Q-minimal solution of $P(\mathcal{U})$ because there exist $\bar{y} = 0$ and $\bar{\xi} = 0 \in \mathcal{U}$ such that $\bar{y} = f(\bar{x}, \bar{\xi}) \in F(\bar{x})$ and

$$(F(S) - \bar{y}) \cap (-Q) = [0, 2] \cap] - \infty, 0[= \emptyset.$$

Observe that $\bar{\xi} = 0$ is the only element of $\mathcal{U}$ satisfying condition (33). We also have

$$K(S \times \mathcal{U}, (\bar{x}, \bar{\xi})) = K([-1, 1] \times [0, 1], (0, 0)) = \{(u, d) \in \mathbb{R}^2 : d \geq 0\}.$$

For such directions (u, d), we can compute

$$\begin{aligned} d^1 f((\bar{x}, \bar{\xi}); (u, d)) &= d^1 f((0,0); (u, d)) \\ &= \lim_{\substack{t \downarrow 0 \\ (w,h) \to (u,d)}} t^{-1} (f(tw, th) - f(0,0)) \\ &= \lim_{\substack{t \downarrow 0 \\ (w,h) \to (u,d)}} t^{-1} \left(t^2 w^2 + th\right) = d \geq 0. \end{aligned}$$

Since $d \notin -Q$, the necessary condition given in Theorem 6.2 is satisfied for $m = 1$. Note that for $m = 2$, we cannot apply Theorem 6.2 because the derivative $d^2 f((\bar{x}, \bar{\xi}); (u, d))$ (for $d > 0$) does not exist as an element of $\mathbb{R}$:

$$d^2 f((\bar{x}, \bar{\xi}); (u, d)) = \lim_{\substack{t \downarrow 0 \\ (w,h) \to (u,d)}} t^{-2} \left(t^2 w^2 + th\right) = +\infty.$$

Example 6.2 Take the same data as in Example 6.1, except for the definition of f which has now the form $f(x, \xi) = x^2 + \xi^2$. As before, we have $F(x) = [x^2, x^2 + 1]$ for all $x \in \mathbb{R}$. Moreover, $\bar{x} = 0$ is a vector-based robust Q-minimal solution of $P(\mathcal{U})$ with the same points $\bar{y} = 0$ and $\bar{\xi} = 0$. In this example, we can apply Theorem 6.2 both for $m = 1$ and $m = 2$ because

$$\begin{aligned} d^1 f((\bar{x}, \bar{\xi}); (u, d)) &= \lim_{\substack{t \downarrow 0 \\ (w,h) \to (u,d)}} t^{-1} \left(t^2 w^2 + t^2 h^2\right) = 0, \\ d^2 f((\bar{x}, \bar{\xi}); (u, d)) &= \lim_{\substack{t \downarrow 0 \\ (w,h) \to (u,d)}} t^{-2} \left(t^2 w^2 + t^2 h^2\right) = u^2 + d^2 \geq 0. \end{aligned}$$

Example 6.3 Let $X = Y = Z = \mathbb{R}$, $Q =]0, +\infty[$, $S = \mathcal{U} = [-1, 1]$, $f(x, \xi) = x^2 \xi$. Then, $F(x) = [-x^2, x^2]$ for all $x \in \mathbb{R}$. Observe that for each $\xi \in [0, 1]$, the point $\bar{x} = 0$ is a Q-minimal solution of $P(\xi)$, and for $\xi \in [-1, 0[$, it is not a Q-minimal solution of $P(\xi)$. Moreover, the point $\bar{x} = 0$ is not a vector-based robust Q-minimal solution of $P(\mathcal{U})$ because the only element $\bar{y} \in F(0)$ is $\bar{y} = 0$, and

$$F(S) \cap (-Q) = [-1, 1] \cap]-\infty, 0[= [-1, 0[\neq \emptyset.$$

We will show that applying Theorem 6.2, we can exclude the point 0 as a possible vector-based robust Q-minimal solution of $P(\mathcal{U})$. Take any point $\bar{\xi} \in \mathcal{U}$; it obviously satisfies condition (33) of the form $f(0, \bar{\xi}) = 0$. Since

$$K(S \times \mathcal{U}, (\bar{x}, \bar{\xi})) = K([-1, 1] \times [-1, 1], (0, 0)) = \mathbb{R}^2,$$

we can take any direction as (u, d) in (34). We can verify that

$$d^m f((0, \bar{\xi}); (u, d)) = \lim_{\substack{t \downarrow 0 \\ (w,h) \to (u,d)}} t^{-m} \left(t^2 w^2 (\bar{\xi} + th)\right) = \begin{cases} 0, & \text{for } m = 1, \\ u^2 \bar{\xi}, & \text{for } m = 2. \end{cases}$$

This result for $m = 2$ is negative if $u \neq 0$ and $\bar{\xi} < 0$. Hence, condition (34) does not hold for $m = 2$.

7 Sufficient Optimality Conditions

We will now prove a sufficient optimality condition for uncertain optimization.

Theorem 7.1 *Let F be given by (9), and let $\bar{x} \in S$. If there exists $\bar{\xi} \in \mathcal{U}$ such that*

$$\overline{D}^m_R f((\bar{x}, \bar{\xi}); (S - \bar{x}, \mathcal{U} - \bar{\xi})) \cap (-Q) = \emptyset, \tag{40}$$

then $\bar{x}$ is a vector-based robust Q-minimal solution of problem $P(\mathcal{U})$.

Proof Suppose that the desired conclusion is false. Then, for each $\bar{y} \in F(\bar{x})$, the pair $(\bar{x}, \bar{y})$ is not a Q-minimal solution of (2). By arguing as in the proof of Theorem 5.1 part "if", we can show that for each $\bar{\xi} \in \mathcal{U}$, there exist $u = x - \bar{x} \in S - \bar{x}$ and $d = \xi - \bar{\xi} \in \mathcal{U} - \bar{\xi}$ such that

$$f(\bar{x} + u, \bar{\xi} + d) - f(\bar{x}, \bar{\xi}) \in -Q. \tag{41}$$

However, by Proposition 5.1, we have

$$f(\bar{x} + u, \bar{\xi} + d) - f(\bar{x}, \bar{\xi}) \in \overline{D}^m_R f((\bar{x}, \bar{\xi}); (u, d)). \tag{42}$$

Combining (41) and (42), we get

$$\overline{D}^m_R f((\bar{x}, \bar{\xi}); (u, d)) \cap (-Q) \neq \emptyset. \tag{43}$$

We have thus verified that for each $\bar{\xi} \in \mathcal{U}$, there exist $u \in S - \bar{x}$ and $d \in \mathcal{U} - \bar{\xi}$ such that (43) holds. This contradicts the assumption of the theorem. □

Example 7.1 Let $X = Y = Z = \mathbb{R}$, $Q =]0, \infty[$, $S = [-1, 1]$, $\mathcal{U} = [0, 1]$, $f(x, \xi) = x^2 + \xi^2$, and $\bar{x} = 0$ (we have the same data as in Example 6.2). We will show that condition (40) holds for $\bar{\xi} = 0$. Indeed, for each $x \in S$ and $d \in \mathcal{U}$, we have

$$\begin{aligned}
&\overline{D}^m_R f((0, 0); (x, d)) \\
&\quad = \left\{v \in Y : \exists t_n > 0,\ \exists (x_n, d_n) \to (x, d),\ t_n^{-m}(f(t_n x_n, t_n d_n) - f(0, 0)) \to v\right\} \\
&\quad = \left\{v \in Y : \exists t_n > 0,\ \exists (x_n, d_n) \to (x, d),\ v_n := t_n^{-m}(t_n^2 x_n^2 + t_n^2 d_n^2) \to v\right\}.
\end{aligned} \tag{44}$$

Since each sequence $\{v_n\}$ in (44) is nonnegative, we have $\overline{D}^m_R f((0, 0); (x, d)) \subseteq [0, \infty[$, and consequently, $\overline{D}^m_R f((0, 0); (x, d)) \cap (-Q) = \emptyset$. But x and d are arbitrary points of S and $\mathcal{U}$, respectively, which implies that $\overline{D}^m_R f((0, 0); (S, \mathcal{U})) \cap (-Q) = \emptyset$. Thus, Theorem 7.1 can be applied to deduce that $\bar{x}$ is a vector-based robust Q-minimal solution of problem $P(\mathcal{U})$.

8 Construction of Algorithms for a Finite Set of Scenarios

In this section, we return to the case of scalar optimization considered in Sect. 4. We present two general algorithm models that can be useful for solving the particular case of problem $P(\mathcal{U})$ where the set $\mathcal{U}$ is finite: $\mathcal{U} = \{\xi_1, ..., \xi_m\}$ (we say that we have m different scenarios). This case is important for some practical applications; see, e.g., [3, Example 3].

Throughout this section, we assume that for each $i \in \{1, ..., m\}$, the function $f(\cdot, \xi_i) : X \to \mathbb{R}$ belongs to a fixed class $\mathcal{F}$ of functions. We also assume that there exists an algorithm $A(g, x_0)$ which, for a given function $g \in \mathcal{F}$ and a given starting point $x_0 \in S$, generates an infinite sequence $\{x_k\}$ converging to some point $\bar{x}$ which is a global minimizer for g on S :

$$\bar{x} = \lim_{k\to\infty} x_k \text{ and } g(\bar{x}) \le g(x) \text{ for all } x \in S. \tag{45}$$

The first algorithm model is valid under an additional assumption of regularity stated in Definition 8.1. This assumption helps to find a vector-based robust Q-minimal solution faster than in the general case that will be considered later.

Definition 8.1 We say that a finite set of scenarios $\mathcal{U}$ is regular, if it satisfies the following condition for each pair $i, j \in \{1, ..., m\}, i \neq j$:

$$\big(\exists x \in X : f(x, \xi_i) < f(x, \xi_j)\big) \Rightarrow \big(\forall u \in X : f(u, \xi_i) < f(u, \xi_j)\big). \tag{46}$$

Condition (46) means that strict inequalities between the values of f for different scenarios are preserved throughout the whole space X, and consequently, the graphs of $f(\cdot, \xi_i)$ for different values of ξ_i do not intersect.

The following algorithm can be used to find a vector-based robust Q-minimal solution of $P(\mathcal{U})$ in the case where the number of elements of $\mathcal{U}$ is relatively small.

Algorithm Model 1

Step 1. Choose a starting point $(x_0, \xi_0) \in S \times \mathcal{U}$.
Step 2. If there exists $\xi \in \mathcal{U}$ such that $f(x_0, \xi) < f(x_0, \xi_0)$, then set $\xi_0 := \xi$ and repeat Step 2. Otherwise, go to Step 3.
Step 3. Run the algorithm $A(f(\cdot, \xi_0), x_0)$, generating an infinite sequence $\{x_k\}$.

Theorem 8.1 *Suppose that the set $\mathcal{U}$ is regular. Then, the limit $\bar{x}$ of the sequence $\{x_k\}$ generated by Algorithm Model 1 is a vector-based robust Q-minimal solution of $P(\mathcal{U})$.*

Proof Suppose that the desired conclusion is false. Then, by Corollary 3.1, there exists a point $(x^*, \xi^*) \in S \times \mathcal{U}$ such that

$$f(x^*, \xi^*) < f(\bar{x}, \xi_0). \tag{47}$$

Since condition (45) holds for $g = f(\cdot, \xi_0)$, we have

$$f(\bar{x}, \xi_0) \le f(x, \xi_0) \text{ for all } x \in S. \tag{48}$$

In particular, taking $x = x^*$ in (48) and combining this inequality with (47), we obtain

$$f(x^*, \xi^*) < f(x^*, \xi_0). \tag{49}$$

Since $\mathcal{U}$ is regular, condition (49) implies that a similar inequality must also hold for x_0:

$$f(x_0, \xi^*) < f(x_0, \xi_0).$$

This, however, contradicts the construction of Algorithm Model 1 (when we go from Step 2 to Step 3, ξ_0 is the best scenario at the point x_0). □

Remark 8.1 If, in Algorithm Model 1, $A(f(\cdot, \xi_0), x_0)$ terminates after a finite number of steps, then we can still use Theorem 8.1 by assuming that the sequence $\{x_k\}$ is constant after it reaches a global minimizer $\bar{x}$ of $f(\cdot, \xi_0)$ on S.

We are now going to describe another algorithm model which does not require the regularity condition (46). On the other hand, we assume that $\mathcal{F}$ satisfies the following condition for each positive integer m:

$$f_1, ..., f_m \in \mathcal{F} \Rightarrow \min\{f_1, ..., f_m\} \in \mathcal{F}. \tag{50}$$

Algorithm Model 2

Step 1. Choose a starting point $x_0 \in S$.
Step 2. Define the function $g(x) := \min\{f(x, \xi) : \xi \in \mathcal{U}\}$ and run the algorithm $A(g, x_0)$, generating an infinite sequence $\{x_k\}$.

Remark 8.2 The description of Algorithm Model 2 is very simple. However, it may require computing the minimum of m values $f(x_k, \xi_1), ..., f(x_k, \xi_m)$ at each step of the algorithm $A(g, x_0)$. If the regularity condition (46) is not satisfied, we can have this minimum attained at different scenarios $\xi \in \mathcal{U}$ for different values of x_k.

Theorem 8.2 *Suppose that condition (50) is satisfied. Then, the limit $\bar{x}$ of the sequence $\{x_k\}$ generated by Algorithm Model 2 is a vector-based robust Q-minimal solution of $P(\mathcal{U})$.*

Proof By condition (45) and the definition of g, we have

$$g(\bar{x}) \leq g(x) \leq f(x, \xi) \text{ for all } (x, \xi) \in S \times \mathcal{U}. \tag{51}$$

Take any $\bar{\xi} \in \mathcal{U}$ such that $g(\bar{x}) = f(\bar{x}, \bar{\xi})$. This equality and (51) give that

$$f(\bar{x}, \bar{\xi}) \leq f(x, \xi) \text{ for all } (x, \xi) \in S \times \mathcal{U}.$$

Therefore, by Proposition 3.1, $\bar{x}$ is a vector-based robust Q-minimal solution of $P(\mathcal{U})$. □

9 A Computational Example

Algorithm Models 1 and 2, presented above, require applying some global minimization method for a given real-valued function. Such methods exist but, for possibly nonconvex functions, they are rather complicated. To illustrate the theory developed in the previous section, here we present a simple example of one-dimensional uncertain optimization problem, for which Algorithm Model 1 can be applied in combination with the Shubert optimization method described in [12]. The Shubert method is designed for seeking the global maximum of a function of one real variable. Below, we briefly present its version adapted for minimization.

Let $f : [a, b] \to \mathbb{R}$ be a real-valued function satisfying the Lipschitz condition, which means that there exists a constant $C \geq 0$ such that for each $x, y \in [a, b]$, the following inequality holds: $|f(x) - f(y)| \leq C\,|x - y|$.

We introduce the following notation:

$$\begin{aligned} \phi &:= \min\{f(x) : x \in [a, b]\}, \\ \Phi &:= \{x \in [a, b] : f(x) = \phi\}, \\ F_n(x) &:= \max\{f(x_k) - C\,|x - x_k| : k \in \{0, 1, .., n\}\}. \end{aligned}$$

The Shubert Algorithm

Step 1. Choose a starting point $x_0 \in [a, b]$. Set $n = 0$.

Step 2. Find a point x_{n+1}, at which the function F_n attains its minimum on $[a, b]$. Increase n by 1 and repeat Step 2.

The following theorem is a reformulation of a result from [12, p. 381] .

Theorem 9.1 *The Shubert algorithm generates an infinite sequence* $\{x_n\}$ *such that:*

(a) *the sequence* $\{f(x_n)\}$ *converges to* ϕ*;*
(b) *the sequence* $\{M_n\}$*, where* $M_n := \min\{F_n(x) : x \in [a, b]\}$*, is nondecreasing and converges to* ϕ*;*
(c) $\inf\{|x - x_n| : x \in \Phi\} \underset{n\to\infty}{\to} 0.$

In the following example, we have used Scientific WorkPlace 5.00 software for numerical computations.

Example 9.1 Let $X = Y = Z = \mathbb{R}$, $S = [0, 4]$, $\mathcal{U} = \{0, 1, 2, 3\}$. We define the function f as follows:

$$f(x, \xi) := x(x - 2)(x - 4) + \xi = x^3 - 6x^2 + 8x + \xi.$$

We want to apply Algorithm Model 1 to solve the problem $P(\mathcal{U})$, which is defined by (7)–(8). Obviously, the regularity condition (46) is satisfied. We proceed as follows:

1. We choose a starting point $(x_0, \xi_0) \in S \times \mathcal{U}$. For this example, let it be equal to $(1, 2)$.

2. We check if there exists $\xi \in \mathcal{U}$ such that

$$f(x_0, \xi) < f(x_0, \xi_0). \tag{52}$$

We see that there are two values for ξ that satisfy this inequality: $\xi = 0$ and $\xi = 1$. Let us choose the first one, and set $\xi_0 := 0$.

3. Since there is no ξ satisfying (52), we go to Step 3 of Algorithm Model 1, that is, we apply the Shubert algorithm to the function $g := f(\cdot, 0)$ with the starting point $x_0 = 1$. It is easy to show that g satisfies the Lipschitz condition on $[0, 4]$ with the constant $C = 8$. First, we construct the function F_0:

$$F_0(x) = g(x_0) - C\,|x - x_0| = g(1) - 8\,|x - 1| = 3 - 8\,|x - 1|\,.$$

4. We look for a point x_1, at which F_0 attains its minimum on $[0, 4]$. Since the graph of F_0 consists of two line segments, and its maximum is attained at x_0, the minimum must be attained at one of the endpoints of $[0, 4]$. Let us compute these values:

$$F_0(0) = 3 - 8\,|0 - 1| = 3 - 8 = -5,$$
$$F_0(4) = 3 - 8\,|4 - 1| = 3 - 24 = -21.$$

Hence, we accept $x_1 = 4$.

5. We construct the function F_1:

$$\begin{aligned} F_1(x) &= \max\{g(x_0) - C\,|x - x_0|\,, g(x_1) - C\,|x - x_1|\} \\ &= \max\{3 - 8\,|x - 1|\,, 0 - 8\,|x - 4|\}. \end{aligned}$$

6. We look for a point x_2, at which F_1 attains its minimum on $[0, 4]$. Observe that F_1 is a piecewise linear function, which can be described as follows:

$$F_1(x) = \begin{cases} 3 - 8(-x + 1) = 8x - 5, & \text{for } x \in [0, 1], \\ 3 - 8(x - 1) = -8x + 11, & \text{for } x \in [1, a], \\ 0 - 8(-x + 4) = 8x - 32, & \text{for } x \in [a, 4], \end{cases}$$

where a is the solution of equation $-8x + 11 = 8x - 32$, that is, $a = \frac{43}{16} = 2.6875$. The minimum can be attained at one of the points $0, a, 4$. We compute:

$$\begin{aligned} &F_1(0) = F_0(0) = -5, \\ &F_1(a) = F_0(a) = -8 \cdot \frac{43}{16} + 11 = -\frac{43}{2} + 11 = -10.5, \\ &F_1(4) = 8 \cdot 4 - 32 = 0. \end{aligned}$$

Hence, we can take 2.6875 as an exact value for x_2.

7. We construct the function F_2:

$$\begin{aligned} F_2(x) &= \max\{g(x_0) - C|x - x_0|, g(x_1) - C|x - x_1|, g(x_2) - C|x - x_2|\} \\ &= \max\{3 - 8|x - 1|, 0 - 8|x - 4|, -2.425 - 8|x - 2.6875|\} \\ &= \begin{cases} 8x - 5, & \text{for} \quad x \in [0, 1], \\ -8x + 11, & \text{for} \quad x \in [1, a_1], \\ 8x - 23.925, & \text{for} \quad x \in [a_1, 2.6875], \\ -8x + 19.075, & \text{for} \quad x \in [2.6875, a_2], \\ 8x - 32, & \text{for} \quad x \in [a_2, 4], \end{cases} \end{aligned}$$

   where a_1 is the solution of equation $-8x+11 = 8x-23.925$, that is, $a_1 \approx 2.1828$, and a_2 is the solution of equation $-8x + 19.075 = 8x - 32$, that is, $a_2 \approx 3.1922$.

8. We look for a point x_3, which minimizes F_2 on $[0, 4]$. It can be one of the points $0, a_1, a_2, 4$. We compute the corresponding values:

$$\begin{aligned} F_2(0) &= -5, \\ F_2(a_1) &= F_2(2.1828) = -6.4624, \\ F_2(a_2) &= F_2(3.1922) = -6.4624, \\ F_2(4) &= 0. \end{aligned}$$

   Since the values of F_2 at the points a_1 and a_2 are equal, we could accept each one of them as the next approximation x_3. However, only $a_2 \approx 3.1922$ is relatively close to the true global minimizer of g on $[0, 4]$, which can be found analytically: $2 + \frac{2}{3}\sqrt{3} \approx 3.1547$. We can see that the performance of the algorithm depends on the choice of minimizers for F_n at each iteration, which is called "sampling" in [12].

Acknowledgements The authors are grateful to the University of Łódź for providing necessary funds and conditions needed to complete this research.

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