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Results in Mathematics

On the Raşa Inequality for Higher Order Convex Functions

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Abstract. We study the following $(q-1)$th convex ordering relation for qth convolution power of the difference of probability distributions μ and ν

$$(\nu - \mu)^{*q} \geq_{(q-1)cx} 0, \quad q \geq 2,$$

and we obtain the theorem providing a useful sufficient condition for its verification. We apply this theorem for various families of probability distributions and we obtain several inequalities related to the classical interpolation operators. In particular, taking binomial distributions, we obtain a new, very short proof of the inequality given recently by Abel and Leviatan (2020).

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1. Introduction

Let $I \subset \mathbb{R}$ be an interval (finite or infinite). Recall that a function $\varphi\colon I \to \mathbb{R}$ is convex, if the inequality

$$\varphi\big(tx + (1-t)y\big) \leq t\varphi(x) + (1-t)\varphi(y)$$

holds for all $x, y \in I$ and for all $t \in [0,1]$.

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The Bernstein operator B_n associated with a continuous function $\varphi : [0,1] \to \mathbb{R}$ (see [10]) is defined by

$$B_n(\varphi)(x) = \sum_{i=0}^{n} p_{n,i}(x)\varphi\left(\frac{i}{n}\right), \quad x \in [0,1],$$

where

$$p_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad 0 \le j \le n.$$

Mrowiec et al. [11] proved the following theorem on inequality for Bernstein operators.

Theorem 1. *Let $n \in \mathbb{N}$ and $x, y \in [0,1]$. Then*

$$\sum_{i=0}^{n}\sum_{j=0}^{n} \left(p_{n,i}(x)p_{n,j}(x) + p_{n,i}(y)p_{n,j}(y) - 2p_{n,i}(x)p_{n,j}(y)\right)\varphi\left(\frac{i+j}{2n}\right) \ge 0 \quad (1)$$

for all convex functions $\varphi \in \mathcal{C}([0,1])$.

This inequality was stated by Ioan Raşa as an open problem about thirty years ago. During the Conference on Ulam's Type Stability (Rytro, Poland, 2014), Raşa [14] recalled his problem. Theorem 1 affirms the conjecture (see also [1–4,7,8,15] for further results on the I. Raşa problem).

The proof given by Mrowiec et al. [11] makes use of probability theory. Let μ and ν be two finite Borel measures on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \varphi(x)\mu(dx) \le \int_{\mathbb{R}} \varphi(x)\nu(dx) \quad \text{for all convex functions} \quad \varphi\colon \mathbb{R} \to \mathbb{R}$$

provided the integrals exist. Then μ is said to be smaller then ν in the convex order (denoted as $\mu \le_{cx} \nu$). In [11], the authors proved the following stochastic convex ordering relation for convolutions of binomial distributions $B(n,x)$ and $B(n,y)$ ($n \in \mathbb{N}$, $x, y \in [0,1]$):

$$B(n,x) * B(n,y) \le_{\text{cx}} \tfrac{1}{2}(B(n,x) * B(n,x) + B(n,y) * B(n,y)), \quad (2)$$

which is a probabilistic version of inequality (1).

In [6], we gave a generalization of inequality (2). We introduced and studied the following convex ordering relation

$$\mu * \nu \le_{\text{cx}} \tfrac{1}{2}(\mu * \mu + \nu * \nu), \quad (3)$$

where μ and ν are two probability distributions on $\mathbb{R}$. We note, that inequality (3) can be regarded as the Raşa type inequality. In [6], we proved Theorem 2.3 providing a very useful sufficient condition for verification that μ and ν satisfy (3). We applied Theorem 2.3 for μ and ν from various families of probability distributions. In particular, we obtained a new proof for binomial distributions, which is significantly simpler and shorter than that given in [11]. By (3), we can also obtain inequalities related to some approximation operators associated with μ and ν (such as Bernstein-Schnabl operators, Mirakyan-Szász operators, Baskakov operators and others, cf. [6]). Note, that in [8], we gave also necessary and sufficient condition for verification that μ and ν satisfy (3).

Recently, Abel and Leviatan [2] gave a generalization of the Raşa inequality (1) to q-monotone functions. Given a function $f\colon I \to \mathbb{R}$, denote

$$\begin{aligned}
&\Delta_h^1 f(x) = \Delta_h f(x) = f(x+h) - f(x),\\
&\Delta_h^{n+1} f(x) = \Delta_h^n(\Delta_h f(x)),\\
&\Delta_{h_1\ldots h_n h_{n+1}} f(x) = \Delta_{h_1}\ldots\Delta_{h_n}\Delta_{h_{n+1}} f(x) = \Delta_{h_1\ldots h_n}\big(\Delta_{h_{n+1}} f(x)\big),
\end{aligned}$$

for $n \in \mathbb{N}$, $x \in I$ and $h, h_1, \ldots, h_n, h_{n+1} \geq 0$ with all needed arguments belonging to I. Let $q \geq 1$. A function $f : I \to \mathbb{R}$ is *q-monotone* if $\Delta_h^q f(x) \geq 0$ for all $h \geq 0$ and $x \in \mathbb{R}$ such that x, $x + qh \in I$.

Theorem 2. [2] *Let q, $n \in \mathbb{N}$. If $f \in \mathcal{C}([0,1])$ is a q-monotone function, then*

$$\begin{aligned}
&sgn(x-y)^q \sum_{\nu_1,\ldots,\nu_q=0}^{n} \sum_{j=0}^{q} (-1)^{q-j}\binom{q}{j}\left(\prod_{i=1}^{j} p_{n,\nu_i}(x)\right)\left(\prod_{i=j+1}^{q} p_{n,\nu_i}(y)\right)\\
&\quad\times f\left(\frac{\nu_1+\ldots+\nu_q}{qn}\right) \geq 0 \qquad (4)
\end{aligned}$$

In this paper, we give a generalization of inequality (3) to higher order convex order. Let us review some notations. In the classical theory of convex functions their natural generalization are convex functions of higher-order.

Let $n \in \mathbb{N}$ and $x_0, \ldots, x_n$ be distinct points in I. Denote by $[x_0, \ldots, x_n; f]$ the divided difference of f at $x_0, \ldots, x_n$ defined by the recurrence

$$\begin{aligned}
[x_0; f] &= f(x_0),\\
[x_0, \ldots, x_n; f] &= \frac{[x_1, \ldots, x_n; f] - [x_0, \ldots, x_{n-1}; f]}{x_n - x_0} \quad \text{for} \quad n \geq 1.
\end{aligned}$$

Following Hopf [5] and Popoviciu [12,13], a function $f\colon I \to \mathbb{R}$ is called *convex of order n* (or *n-convex*) if

$$[x_0, \ldots, x_{n+1}; f] \geq 0$$

for all $x_0 < \ldots < x_{n+1}$ in I.

We list some properties of n-th order convexity (see [9]).

Proposition 1. *If the function $f\colon I \to \mathbb{R}$ is n-convex on I $(n \geq 1)$, then*

$$\Delta_{h_1}\ldots\Delta_{h_n}\Delta_{h_{n+1}} f(x) \geq 0$$

for all $x \in I$, $h_1, \ldots, h_n, h_{n+1} \geq 0$ such that $x + h_1 + \cdots + h_{n+1} \in I$.

Proposition 2. *If $I \subset \mathbb{R}$ is an open interval, then a function $f\colon I \to \mathbb{R}$ is n-convex on I $(n \geq 1)$ if, and only if, its derivative $f^{(n-1)}$ exists and is convex on I (with the convention $f^{(0)}(x) = f(x)$).*

Proposition 3. *Let $f \in \mathcal{C}(I)$ and $n \geq 1$. Then the following statements are equivalent.*

(a) *f is n-convex on I.*
(b) *f is $(n+1)$-monotone on I.*
(c) *$\Delta_h^{n+1} f(x) \geq 0$ for all $x \in I$ and $h \geq 0$ with $x + (n+1)h \in I$.*
(d) *$\Delta_{h_1} \ldots \Delta_{h_n} \Delta_{h_{n+1}} f(x) \geq 0$ for all $x \in I$ and $h_1, \ldots, h_n, h_{n+1} \geq 0$ with $x + h_1 + \cdots + h_{n+1} \in I$.*

Recall the definition of n-convex orders.

Definition 1. Let $n \geq 1$. Let μ and ν be two finite signed Borel measures on I such that

$$\int_I \varphi(x)\mu(dx) \leq \int_I \varphi(x)\nu(dx) \quad \text{for all } n\text{-convex functions } \varphi\colon I \to \mathbb{R} \tag{5}$$

provided the integrals exist. Then μ is said to be smaller then ν in the n-convex order (denoted as $\mu \leq_{n-cx} \nu$).

In particular, $\mu \leq_{1-cx} \nu$ coincides with $\mu \leq_{cx} \nu$. Observe, that this definition does not depend on the choice of the interval I. Indeed, let $I_1, I_2 \subset \mathbb{R}$ be two intervals and let μ, ν be two finite signed Borel measures on $I_1 \cap I_2$. Assume that (5) is satisfied for the interval I_1. Then for every n-convex function $\varphi : I_2 \to \mathbb{R}$ there exist n-convex functions $\psi_k : \mathbb{R} \to \mathbb{R}$ such that $\int_{I_1} \psi_k(x)\mu(dx) = \int_{I_2} \psi_k(x)\mu(dx) \underset{k\to\infty}{\longrightarrow} \int_{I_2} \varphi(x)\mu(dx)$ and $\int_{I_1} \psi_k(x)\nu(dx) = \int_{I_2} \psi_k(x)\nu(dx) \underset{k\to\infty}{\longrightarrow} \int_{I_2} \varphi(x)\nu(dx)$ (if $\int_{I_2} \varphi(x)\mu(dx)$ and $\int_{I_2} \varphi(x)\nu(dx)$ exist). Clearly, $\psi_k|_{I_1}$ are n-convex on I_1. It follows that (5) is satisfied for I_2. Similarly, if (5) is satisfied for I_2 then it is also satisfied for I_1. We do not give a precise proof, because we will not use this observation any further.

We study the following generalization of (3)

$$(\nu - \mu)^{*q} \geq_{(q-1)cx} 0, \quad q \geq 2, \tag{6}$$

where μ, ν are probability measures and $(\nu - \mu)^{*q}$ is the convolution $(\nu - \mu) * (\nu - \mu) * \cdots * (\nu - \mu)$ with q terms $(\nu - \mu)$. Note, that $(\nu - \mu)^{*2} = \nu * \nu - 2\nu * \mu + \mu * \mu$, thus (6) for $q = 2$ is equivalent to (3). Inequality (6) can be regarded as the Raşa type inequality.

In Theorem 4, we give a very useful sufficient condition for verification that μ and ν satisfy (6). In particular, by Theorem 4, taking binomial distributions (Theorem 10 (a)), we obtain

$$[sgn(x-y)]^q \left(B(n,x) - B(n,y)\right)^{*q} \geq_{(q-1)cx} 0, \quad q \geq 2,$$

which is equivalent to inequality (4). Consequently, we obtain a new proof of inequality (4) given by Abel and Leviatan [2], which is significantly simpler and shorter than that given in [2].

We apply Theorem 4 for μ and ν from various families of probability distributions. Using inequality (6), we can also obtain inequalities related to some approximation operators associated with μ and ν (such as Bernstein-Schnabl operators, Mirakyan-Szász operators, Baskakov operators and others).

2. The Raşa Type Inequality for $(q-1)$-Convex Orders

In the sequel, we make use of the theory of stochastic orders. Let us recall some basic notations and results (see [16]). Let μ be a probability distribution on $\mathbb{R}$. For $x \in \mathbb{R}$ the delta symbol δ_x denotes one-point probability distribution satisfying $\delta_x(\{x\}) = 1$. As usual, $F_\mu(x) = \mu((-\infty, x])$ $(x \in \mathbb{R})$ stands for the cumulative distribution function of μ. If μ and ν are two probability distributions such that $F_\mu(x) \geq F_\nu(x)$ for all $x \in \mathbb{R}$, then μ is said to be *smaller than* ν *in the usual stochastic order* (denoted by $\mu \leq_{\mathrm{st}} \nu$). An important characterization of the usual stochastic order is given by the following theorem.

Theorem 3. [16, p. 5] *Two probability distributions μ and ν satisfy $\mu \leq_{st} \nu$ if, and only if there exist two random variables X and Y defined on the same probability space, such that μ is the distribution of X, ν is the distribution of Y ($X \sim \mu$ and $Y \sim \nu$ for short) and $P(X \leq Y) = 1$.*

In the following theorem, we give a very useful sufficient condition that will be used for verification of some convex orders.

Theorem 4. *Let $q \geq 2$. Let μ and ν be two probability distributions on I, such that $\mu \leq_{st} \nu$. Then*

$$(\nu - \mu)^{*q} \geq_{(q-1)cx} 0.$$

Proof. Note that $(\nu - \mu)^{*q}$ is a signed measure on $qI = \{qx : x \in I\}$. Let $f\colon qI \to \mathbb{R}$ be a $(q-1)$-convex function on qI. Then, by Proposition 1,

$$\Delta_{h_q}\Delta_{h_{q-1}} \dots \Delta_{h_2}\Delta_{h_1} f(x) \geq 0 \tag{7}$$

for all $x \in qI$, $h_1, \dots, h_{q-1}, h_q \geq 0$ such that $x + h_1 + \cdots + h_q \in qI$.

Let $x_1, \dots, x_q, y_1, \dots, y_q \in I$ be such that $x_1 \leq y_1, \dots, x_q \leq y_q$. Then taking $h_i = y_i - x_i$ for $i = 1, \dots, q$ and $x = x_1 + \ldots + x_q$, by (7), we obtain

$$\begin{aligned} 0 &\leq \Delta_{y_q - x_q}\Delta_{y_{q-1}-x_{q-1}} \dots \Delta_{y_2 - x_2}\Delta_{y_1 - x_1} f(x_1 + \ldots + x_q) \\ &= \sum_{A \subset \{1,\dots,q\}} (-1)^{|A|} f(z_1 + \ldots + z_q), \end{aligned} \tag{8}$$

where $z_i = x_i$ if $i \in A$ and $z_i = y_i$ if $i \notin A$, $i = 1, \dots, q$.

Since $\mu \leq_{\mathrm{st}} \nu$, by Theorem 3, there exist independent random vectors (X_1, Y_1), (X_2, Y_2), $\dots$, (X_q, Y_q), such that

$$X_i \sim \mu,\ Y_i \sim \nu \text{ and } P(X_i \leq Y_i) = 1, \qquad i = 1, \dots, q. \tag{9}$$

Let $H_i = Y_i - X_i$, $i = 1, \dots, q$. Since the measures μ and ν are concentrated on I, from (9) it follows that

$$P(X_i \in I) = P(Y_i \in I) = 1 \text{ and } P(H_i \geq 0) = 1, \qquad i = 1, \dots, q.$$

Let $X = X_1 + \ldots + X_q$. Taking into account that $Y_1 + \ldots + Y_q = X + H_1 + \ldots + H_q$, by (9), we obtain

$$P(X \in qI) = P(X + H_1 + \ldots + H_q \in qI) = 1.$$

By inequality (8), with $x_i = X_i$ and $y_i = Y_i$, we obtain

$$P\left(\sum_{A\subset\{1,\ldots,q\}}(-1)^{|A|}f(Z_1+\ldots+Z_q)\geq 0\right)=1,$$

where $Z_i = X_i$ if $i \in A$ and $Z_i = Y_i$ if $i \notin A$, $i = 1,\ldots,q$, consequently, we have

$$\mathbb{E}\left(\sum_{A\subset\{1,\ldots,q\}}(-1)^{|A|}\;f(Z_1+\ldots+Z_q)\right)\geq 0. \tag{10}$$

We have

$$\begin{aligned}(\nu-\mu)^{*q}&=\sum_{k=0}^{q}\binom{q}{k}(-1)^k\mu^{*k}*\nu^{*(q-k)}\\&=\sum_{A\subset\{1,\ldots,q\}}(-1)^{|A|}\;\pi_1*\pi_2*\ldots*\pi_q,\end{aligned} \tag{11}$$

where $\pi_i = \mu$ if $i \in A$ and $\pi_i = \nu$ if $i \notin A$, $i = 1,\ldots,q$.

Then, by (10) and (11), we obtain

$$\int_{qI}f(x)(\nu-\mu)^{*q}(dx)=\int_{qI}f(x)\;\left(\sum_{A\subset\{1,\ldots,q\}}(-1)^{|A|}\;\pi_1*\pi_2*\ldots*\pi_q\right)(dx)$$

$$=\mathbb{E}\left(\sum_{A\subset\{1,\ldots,q\}}(-1)^{|A|}\;f(Z_1+\ldots+Z_q)\right)\geq 0.$$

The theorem is proved. □

Theorem 5. *Let $q \geq 2$.Let μ and ν be two probability distributions on I, such that $\mu \geq_{st} \nu$. Then*

$$(-1)^q(\nu-\mu)^{*q}\geq_{(q-1)cx}0.$$

Proof. If $\mu \geq_{st} \nu$, then $\nu \leq_{\mathrm{st}} \mu$. Then, applying Theorem 4, we obtain

$$(\mu-\nu)^{*q}\geq_{(q-1)cx}0.$$

Since $(\mu-\nu)^{*q} = (-1)^q(\nu-\mu)^{*q}$, the theorem is proved. □

We consider also some generalization of Theorem 4 to signed measures.

If μ and ν are two finite signed Borel measures on $\mathbb{R}$ such that $\mu(\mathbb{R}) = \nu(\mathbb{R})$ and $\mu((-\infty,x]) \geq \nu((-\infty,x])$ $(x \in \mathbb{R})$, then μ is said to be *smaller than ν in the usual stochastic order* (denoted by $\mu \leq_{\mathrm{st}} \nu$).

Theorem 6. *Let $q \geq 2$. Let μ and ν be two finite signed Borel measures on I such that $\mu(I) = \nu(I)$ and $\mu \leq_{st} \nu$. Then*

$$(\nu-\mu)^{*q}\geq_{(q-1)cx}0.$$

Proof. Let μ and ν be two finite signed Borel measures on I such that $\mu(I) = \nu(I)$ and $\mu \leq_{\mathrm{st}} \nu$. By the Hahn decomposition theorem, there exist two non-negative finite measures $(\nu-\mu)^+$ and $(\nu-\mu)^-$ (the positive part and negative part of $\nu-\mu$), such that $\nu-\mu = (\nu-\mu)^+ - (\nu-\mu)^-$. Since $\mu(I) = \nu(I)$, it follows that $(\nu-\mu)^+(I) = (\nu-\mu)^-(I)$, consequently, without loss of generality we may assume, that both $(\nu-\mu)^+$ and $(\nu-\mu)^-$ are probability measures. Moreover, the condition $\mu \leq_{\mathrm{st}} \nu$ is equivalent to $(\nu-\mu)((-\infty, x]) \leq 0$ $(x \in \mathbb{R})$, consequently, we have $(\nu-\mu)^+ \geq_{\mathrm{st}} (\nu-\mu)^-$. Then, by Theorem 4, we obtain

$$((\nu-\mu)^+ - (\nu-\mu)^-)^{*q} \geq_{(q-1)cx} 0. \tag{12}$$

Since $(\nu-\mu)^+ - (\nu-\mu)^- = \nu-\mu$, this completes the proof. □

As an immediate consequence of Theorem 6, we obtain the following theorem.

Theorem 7. *Let $q \geq 2$. Let τ be a finite signed Borel measure on I such that $\tau(I) = 0$ and $\tau \geq_{st} 0$. Then*

$$\tau^{*q} \geq_{(q-1)cx} 0.$$

Similarly to Theorem 4, the following theorems can be proved.

Theorem 8. *Let $q \geq 2$. Let $\mu_1, \ldots, \mu_q$ and $\nu_1, \ldots, \nu_q$ be probability distributions on I, such that $\mu_i \leq_{st} \nu_i$ for $i = 1, \ldots, q$. Then*

$$(\nu_1 - \mu_1) * \ldots * (\nu_q - \mu_q) \geq_{(q-1)cx} 0.$$

Theorem 9. *Let $q \geq 2$. Let $\tau_1, \ldots, \tau_q$ be finite signed Borel measures on I, such that $\tau_i(I) = 0$ and $\tau_i \geq_{st} 0$ for $i = 1, \ldots, q$. Then*

$$\tau_1 * \ldots * \tau_q \geq_{(q-1)cx} 0.$$

We will apply Theorems 4 and 5 for μ and ν from various families of probability distributions. As a result, we obtain new proofs of the results of Abel and Leviatan [2] and several new inequalities, which are analogues of (1).

The binomial distribution $B(n, x)$ $(n \in \mathbb{N},\ x \in [0, 1])$ is the probability distribution given by $B(n, x)(\{i\}) = p_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}$ for $i = 0, 1, \ldots, n$.

The Poisson distribution $Poiss(\lambda)$ $(\lambda > 0)$ is the probability distribution given by $Poiss(\lambda)(\{i\}) = s_i(\lambda) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$ for $i = 0, 1, \ldots$ ($Poiss(0) = \delta_0$ i.e. $s_0(0) = 1$, $s_i(0) = 0$ for $i = 1, 2, \ldots$).

The negative binomial distribution $NB(r, p)$ $(r > 0,\ 0 \leq p < 1)$ is the probability distribution given by

$$NB(r,p)(\{i\}) = nb_i(r,p) = \binom{i+r-1}{i} p^i (1-p)^r = \frac{\Gamma(i+r)}{\Gamma(r) \cdot i!} p^i (1-p)^r \quad \text{for} \quad i = 0, 1, \ldots$$

(if $r = 0$, then $NB(0, p) = \delta_0$ i.e., $nb_0(0, p) = 1$ and $nb_i(0, p) = 0$ for $i > 0$).

The gamma distribution $\Gamma(\alpha,\beta)$ ($\alpha,\beta>0$) is the distribution given by the density function $\gamma_{\alpha,\beta}(x)=\frac{\beta^\alpha x^{\alpha-1}e^{-x\beta}}{\Gamma(\alpha)}$ for $x>0$ and $\gamma_{\alpha,\beta}(x)=0$ for $x\le 0$. By convention, we define $\Gamma(0,\beta)=\delta_0$ for every $\beta>0$.

The beta distribution $Beta(\alpha,\beta)$ ($\alpha,\beta>0$) is the distribution given by the density function

$$b_{\alpha,\beta}(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\quad\text{for } x\in(0,1),$$

and $b_{\alpha,\beta}(x)=0$ for $x\notin(0,1)$. By convention, we define $Beta(0,\beta)=\delta_0$ and $Beta(\alpha,0)=\delta_1$ for every $\alpha,\beta>0$.

By $N(m,\sigma^2)$, we denote the normal (Gaussian) distribution with expected value m and variance σ^2.

Proposition 4. [6]

(a) *Let* $n\in\mathbb{N}$ *and* $x_1,x_2\in[0,1]$. *Then* $B(n,x_1)\le_{st}B(n,x_2)\ \Leftrightarrow\ x_1\le x_2$.
(b) *Let* $\lambda_1,\lambda_2\ge 0$. *Then* $Poiss(\lambda_1)\le_{st}Poiss(\lambda_2)\ \Leftrightarrow\ \lambda_1\le\lambda_2$.
(c) *Let* $r_1,r_2\ge 0$ *and* $p_1,p_2\in[0,1)$. *If* $r_1\le r_2$ *and* $p_1\le p_2$, *then* $NB(r_1,p_1)\le_{st} NB(r_2,p_2)$.
(d) *Let* $\alpha_1,\alpha_2\ge 0$ *and* $\beta_1,\beta_2>0$. *If* $\alpha_1\le\alpha_2$ *and* $\beta_1\ge\beta_2$, *then* $\Gamma(\alpha_1,\beta_1)\le_{st}\Gamma(\alpha_2,\beta_2)$.
(e) *Let* $\alpha_1,\beta_1,\alpha_2,\beta_2\ge 0$ *be such that* $\alpha_1+\beta_1>0$ *and* $\alpha_2+\beta_2>0$. *If* $\alpha_1\le\alpha_2$ *and* $\beta_1\ge\beta_2$, *then* $Beta(\alpha_1,\beta_1)\le_{st}Beta(\alpha_2,\beta_2)$.
(f) *Let* $m_1,m_2\in\mathbb{R}$ *and* $\sigma_1^2,\sigma_2^2>0$. *Then* $N(m_1,\sigma_1^2)\le_{st}N(m_2,\sigma_2^2)\Leftrightarrow(m_1\le m_2$ *and* $\sigma_1^2=\sigma_2^2)$.

From Theorems 4, 5 and Proposition 4, we obtain immediately the following theorem.

Theorem 10. (a) *Let* $n\in\mathbb{N}$ *and* $x,y\in[0,1]$. *Then*

$$[sgn(x-y)]^q\,(B(n,x)-B(n,y))^{*q}\ge_{(q-1)cx}0.$$

(b) *Let* $x,y\ge 0$. *Then*

$$[sgn(x-y)]^q\,(Poiss(x)-Poiss(y))^{*q}\ge_{(q-1)cx}0.$$

(c) *Let* $r_1,r_2\ge 0$ *and* $p_1,p_2\in[0,1)$. *If* $(r_1-r_2)(p_1-p_2)\ge 0$, *then*

$$\left[\frac{1}{2}\left(sgn(r_1-r_2)+sgn(p_1-p_2)\right)\right]^q\left(NB(r_1,p_1)-NB(r_2,p_2)\right)^{*q}\ge_{(q-1)cx}0.$$

(d) *Let* $\alpha_1,\alpha_2\ge 0$ *and* $\beta_1,\beta_2>0$ *satisfy* $(\alpha_1-\alpha_2)(\beta_1-\beta_2)\le 0$. *Then*

$$\left[\frac{1}{2}\left(sgn(\alpha_1-\alpha_2)+sgn(\beta_2-\beta_1)\right)\right]^q\left(\Gamma(\alpha_1,\beta_1)-\Gamma(\alpha_2,\beta_2)\right)^{*q}\ge_{(q-1)cx}0.$$

(e) *Let* $\alpha_1,\beta_1,\alpha_2,\beta_2>0$ *satisfy* $(\alpha_1-\alpha_2)(\beta_1-\beta_2)\le 0$. *Then*

$$\left[\frac{1}{2}\left(sgn(\alpha_1-\alpha_2)+sgn(\beta_2-\beta_1)\right)\right]^q\left(Beta(\alpha_1,\beta_1)-Beta(\alpha_2,\beta_2)\right)^{*q}$$
$$\ge_{(q-1)cx}0.$$

(f) *Let* $x, y \in \mathbb{R}$ *and* $\sigma^2 > 0$. *Then*

$$sgn(x-y)^q (N(x,\sigma^2) - N(y,\sigma^2))^{*q} \geq_{(q-1)cx} 0.$$

Note that Theorem 2 is an immediate consequence of Theorem 10 (a) and Definition 1. If $n \in \mathbb{N}$ and $x, y \in [0,1]$, then $B(n,x)$ and $B(n,y)$ are probability distributions on $[0,n]$. Let $q = 2, 3, \ldots$ and $f : [0,1] \to \mathbb{R}$ be a $(q-1)$-convex function. Then the function $\varphi : [0, qn] \to \mathbb{R}$ given by $\varphi(t) = f\left(\frac{t}{qn}\right)$ is a $(q-1)$-convex (q-monotone) function. By Theorem 10 (a) and Definition 1, we obtain (4). Since $B(n,x) * B(m,x) = B(n+m,x)$, (4) can be also written as

$$sgn(x-y)^q \sum_{k=0}^{q} \sum_{i=0}^{kn} \sum_{j=0}^{(q-k)n} (-1)^{q-k} \binom{q}{k} p_{kn,i}(x) p_{(q-k)n,j}(y) f\left(\frac{i+j}{qn}\right) \geq 0.$$

Theorem 2 is closely related to Bernstein operator (or Bernstein-Schnabl operator) and the binomial probability distribution. Theorem 10 allows to obtain similar results related to other distributions and other interpolation operators, such as Mirakyan-Szász operators S_t, Baskakov operators V_t, Bleimann-Butzer-Hahn operators L_n, gamma operators G_t, Müller gamma operators G_t^*, Lupaş beta operators $\mathcal{B}_t^*$, inverse beta operators T_t, Meyer–König–Zeller operators M_t and others.

Mirakyan-Szász operator $S_t\colon D_S \to \mathcal{C}([0,\infty))$ (where $t > 0$ and $D_S \subset \mathcal{C}([0,\infty))$ consists of functions of at most exponential growth) is related to the Poisson distribution and is given by

$$S_t(f)(x) = \sum_{i=0}^{\infty} s_i(tx) f\left(\tfrac{i}{t}\right) \quad \text{for } x \in [0,\infty),$$

where $s_i(x) = e^{-x}\frac{x^i}{i!}$. By Theorem 10 (b), we obtain immediately:

$$sgn(x-y)^q \sum_{\nu_1,\ldots,\nu_q=0}^{\infty} \sum_{j=0}^{q} (-1)^{q-j} \binom{q}{j} \left(\prod_{i=1}^{j} s_{\nu_i}(tx)\right) \left(\prod_{i=j+1}^{q} s_{\nu_i}(ty)\right) \times f\left(\frac{\nu_1 + \ldots + \nu_q}{qt}\right) \geq 0$$

$$\text{or} \qquad sgn(x-y)^q \sum_{k=0}^{q} \sum_{i=0}^{\infty} (-1)^{q-k} \binom{q}{k} s_i(ktx + (q-k)ty) f\left(\frac{i}{qt}\right) \geq 0$$

for every $t > 0$ and $(q-1)$-convex (q-monotone) function $f \in D_S$ (cf. [2, Theorem 3.1]).

Baskakov operator $V_t\colon D_V \to \mathcal{C}([0,\infty))$ (where $t > 0$ and $D_V \subset \mathcal{C}([0,\infty))$ consists of functions of at most polynomial growth) is related to the negative binomial distribution and is given by

$$V_t(f)(x) = \sum_{i=0}^{\infty} v_{t,i}(x) f\left(\tfrac{i}{t}\right) \quad \text{for } x \in [0,\infty),$$

where $v_{t,i}(x) = \binom{t+i-1}{i} \frac{x^i}{(1+x)^{t+i}} = nb_i\left(t, \frac{x}{x+1}\right)$. Theorem 10 (c) yields the inequality

$$sgn(x-y)^q \sum_{\nu_1,\ldots,\nu_q=0}^{\infty} \sum_{j=0}^{q} (-1)^{q-j} \binom{q}{j} \left(\prod_{i=1}^{j} v_{t,\nu_i}(x)\right) \left(\prod_{i=j+1}^{q} v_{t,\nu_i}(y)\right)$$
$$\times f\left(\frac{\nu_1+\ldots+\nu_q}{qt}\right) \geq 0$$

$$\text{or} \qquad sgn(x-y)^q \sum_{k=0}^{q} \sum_{i,j=0}^{\infty} (-1)^{q-k} \binom{q}{k} v_{kt,i}(x) v_{(q-k)t,j}(y) f\left(\frac{i+j}{qt}\right) \geq 0$$

for every $q = 2, 3, \ldots$, $t > 0$ and $(q-1)$-convex (q-monotone) function $f \in D_V$ (cf. [2, Theorem 3.2]).

Theorem 10(d)–(f) leads in the obvious way to other inequalities related to integral interpolation operators. These inequalities are integral counterparts of the above inequalities. E.g. Theorem 10(d) leads to

$$sgn(x-y)^q \sum_{k=0}^{q} \int_0^{\infty} (-1)^{q-k} \binom{q}{k} \gamma_{ktx+(q-k)ty,t}(u) f(u) du \geq 0,$$

which is related to the gamma distribution and the operator

$$G_t(f) := \int_0^{\infty} \gamma_{tx,t}(u) f(u) du = \int_0^{\infty} \gamma_{tx,1}(u) f\left(\frac{u}{t}\right) du = \int_0^{\infty} \frac{u^{tx-1} e^{-u}}{\Gamma(tx)} f\left(\frac{u}{t}\right) du.$$

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