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FALLING SHADOW THEORY WITH APPLICATIONS IN HOOPS

Abstract

The falling shadow theory is applied to subhoops and filters in hoops. The notions of falling fuzzy subhoops and falling fuzzy filters in hoops are introduced, and several properties are investigated. Relationship between falling fuzzy subhoops and falling fuzzy filters are discussed, and conditions for a falling fuzzy subhoop to be a falling fuzzy filter are provided. Also conditions for a falling shadow of a random set to be a falling fuzzy filter are displayed.

Keywords: Hoop, fuzzy subhoop, fuzzy filter, falling fuzzy subhoop, falling fuzzy filter.

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1. Introduction

In the study of a unified treatment of uncertainty modelled by meaning of combining probability and fuzzy set theory, Wang and Sanchez [17] introduced the theory of falling shadows which directly relates probability concepts with the membership function of fuzzy sets. Falling shadow representation theory shows us the way of selection relaid on the joint degrees distributions. It is reasonable and convenient approach for the theoretical

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development and the practical applications of fuzzy sets and fuzzy logics. Falling shadow representation theory shows us the way of selection relaid on the joint degrees distributions. It is reasonable and convenient approach for the theoretical development and the practical applications of fuzzy sets and fuzzy logics. The mathematical structure of the theory of falling shadows is formulated in [15]. After that many scholars applied it to (fuzzy) algebraic structures (see [13, 11, 10, 12, 19, 20, 21]). Hoops which are introduced by B. Bosbach in [6, 7] are naturally ordered commutative residuated integral monoids. In [1], Agliáno introduced a continuous t-norm which is a continuous map $*$ from $[0,1]^2$ into $[0,1]$ such that $\langle [0,1], *, 1\rangle$ is a commutative totally ordered monoid. Since the natural ordering on $[0,1]$ is a complete lattice ordering, each continuous t-norm induces naturally a residuation $\rightarrow$ and $\langle [0,1], *, \rightarrow, 1\rangle$ becomes a commutative naturally ordered residuated monoid, also called a hoop. The variety of basic hoops is precisely the variety generated by all algebras $\langle [0,1], *, \rightarrow, 1\rangle$, where $*$ is a continuous t-norm. In [1], they investigated the structure of the variety of basic hoops and some of its subvarieties. In particular, they provided a complete description of the finite subdirectly irreducible basic hoops, and they showed that the variety of basic hoops is generated as a quasivariety by its finite algebras. They extended these results to Hájeks BL-algebras, and gived an alternative proof of the fact that the variety of BL-algebras is generated by all algebras arising from continuous t-norms on $[0,1]$ and their residua. Also, they in [2], overviewed recent results about the lattice of subvarieties of the variety BL of BL-algebras and the equational definition of some families of them. Kondo [14] considered fundamental properties of some types of (implicative, positive implicative and fantastic) filters of hoops, and R. A. Borzooei and M. Aaly Kologani [4] investigated some properties and equivalent definitions of these filters on hoops. Also, they studied the relation between these filters and found that under which conditions they are equivalent. Borzooei et al. studied fuzzy set theory of subhoops and filters in hoops (see [3, 5]).

In this paper, we apply the falling shadow theory to subhoops and filters in hoops. We introduce the notions of falling fuzzy subhoops and falling fuzzy filters in hoops, and investigate several properties. We consider relationship between falling fuzzy subhoops and falling fuzzy filters. We provide conditions for a falling fuzzy subhoop to be a falling fuzzy filter. We also provide conditions for a falling shadow of a random set to be a falling fuzzy filter. Also, we show that every fuzzy filter of a hoop is a falling fuzzy filter and falling fuzzy subhoop and we prove that under which conditions a falling shadow can be a falling fuzzy filter of a hoop.

2. Preliminaries

By a *hoop* we mean an algebra $(H, \odot, \rightarrow, 1)$ in which $(H, \odot, 1)$ is a commutative monoid and, for any $x, y, z \in H$, the following assertions are valid.

(H1) $x \rightarrow x = 1$,

(H2) $x \odot (x \rightarrow y) = y \odot (y \rightarrow x)$,

(H3) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$.

We define a relation "$\leq$" on a hoop H by

$$(\forall x, y \in H)(x \leq y \;\Leftrightarrow\; x \rightarrow y = 1). \tag{2.1}$$

It is easy to see that $(H, \leq)$ is a poset. A nonempty subset S of H is called a *subhoop* of H if it satisfies:

$$(\forall x, y \in S)(x \odot y \in S,\; x \rightarrow y \in S). \tag{2.2}$$

Note that every subhoop contains the element 1.

Proposition 2.1 ([8]). Let $(H, \odot, \rightarrow, 1)$ be a hoop. For any $x, y, z \in H$, the following conditions hold:

($a1$) $(H, \leq)$ is a meet-semilattice with $x \wedge y = x \odot (x \rightarrow y)$.

($a2$) $x \odot y \leq z$ if and anly if $x \leq y \rightarrow z$.

($a3$) $x \odot y \leq x, y$ and $x^n \leq x$, for any $n \in \mathbb{N}$.

($a4$) $x \leq y \rightarrow x$.

($a5$) $1 \rightarrow x = x$ and $x \rightarrow 1 = 1$.

($a6$) $x \odot (x \rightarrow y) \leq y$, $x \odot y \leq x \wedge y \leq x \rightarrow y$.

($a7$) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$.

($a8$) $x \leq y$ implies $x \odot z \leq y \odot z$, $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$.

($a9$) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z)$.

A nonempty subset F of a hoop H is called a *filter* of H (see [8]) if the following assertions are valid.

$$(\forall x, y \in H)(x, y \in F \;\Rightarrow x \odot y \in F), \tag{2.3}$$

$$(\forall x, y \in H)(x \in F,\; x \leq y \;\Rightarrow\; y \in F). \tag{2.4}$$

Note that the conditions (2.3) and (2.4) means that F is closed under the operation $\odot$ and F is upward closed, respectively.

Note that a subset F of a hoop H is a filter of H if and only if the following assertions are valid (see [8]):

$$1 \in F, \tag{2.5}$$

$$(\forall x, y \in H)\,(x \to y \in F,\; x \in F \;\Rightarrow\; y \in F)\,. \tag{2.6}$$

A fuzzy set μ in a hoop H is called a *fuzzy subhoop* of H if it satisfies:

$$\begin{aligned}(\forall x, y \in H)(\mu(x \odot y) \geq \min\{\mu(x), \mu(y)\},\\ \mu(x \to y) \geq \min\{\mu(x), \mu(y)\}).\end{aligned} \tag{2.7}$$

A fuzzy set μ in a hoop H is called a *fuzzy filter* of H (see [3]) if the following assertions are valid.

$$(\forall x \in H)\,(\mu(x) \leq \mu(1))\,, \tag{2.8}$$

$$(\forall x, y \in H)\,(\mu(y) \geq \min\{\mu(x), \mu(x \to y))\,. \tag{2.9}$$

Given a fuzzy set μ in H and $t \in [0, 1]$, the set

$$\mu_t := \{x \in H \mid \mu(x) \geq t\} \tag{2.10}$$

is called the *t-level set* of μ in H.

We now display the basic theory on falling shadows. We refer the reader to the papers [9, 15, 16, 18, 17] for further information regarding the theory of falling shadows.

Given a universe of discourse U, let $\mathcal{P}(U)$ denote the power set of U. For each $u \in U$, let

$$\ddot{u} := \{E \mid u \in E \text{ and } E \subseteq U\}, \tag{2.11}$$

and for each $E \in \mathcal{P}(U)$, let

$$\ddot{E} := \{\ddot{u} \mid u \in E\}. \tag{2.12}$$

An ordered pair $(\mathcal{P}(U), \mathcal{B})$ is said to be a hyper-measurable structure on U if $\mathcal{B}$ is a σ-field in $\mathcal{P}(U)$ and $\ddot{U} \subseteq \mathcal{B}$. Given a probability space $(\mho, \mathcal{A}, P)$ and a hyper-measurable structure $(\mathcal{P}(U), \mathcal{B})$ on U, a random set on U is defined to be a mapping $\eta : \mho \to \mathcal{P}(U)$ which is $\mathcal{A}$-$\mathcal{B}$ measurable, that is,

$$(\forall C \in \mathcal{B})\, (\eta^{-1}(C) = \{\varepsilon \mid \varepsilon \in \mho \text{ and } \eta(\varepsilon) \in C\} \in \mathcal{A}). \tag{2.13}$$

Suppose that η is a random set on U. Let

$$\tilde{f}(u) := P(\varepsilon \mid u \in \eta(\varepsilon)) \text{ for each } u \in U.$$

Then $\tilde{f}$ is a kind of fuzzy set in U. We call $\tilde{f}$ a falling shadow of the random set η, and η is called a cloud of $\tilde{f}$.

For example, $(\mho, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$, where $\mathcal{A}$ is a Borel field on $[0, 1]$ and m is the usual Lebesgue measure. Let $\tilde{f}$ be a fuzzy set in U and $\tilde{f}_t := \{u \in U \mid \tilde{f}(u) \geq t\}$ be a t-cut of $\tilde{f}$. Then

$$\eta : [0, 1] \to \mathcal{P}(U),\ t \mapsto \tilde{f}_t$$

is a random set and η is a cloud of $\tilde{f}$. We shall call η defined above as the cut-cloud of $\tilde{f}$ (see [9]).

3. Falling fuzzy subhoops and filters

In what follows, let H denote a hoop unless otherwise specified.

Definition 3.1. Let $(\mho, \mathcal{A}, P)$ be a probability space, and let

$$\eta : \mho \to \mathcal{P}(H)$$

be a random set. If $\eta(\varepsilon)$ is a filter (resp. a subhoop) of H for any $\varepsilon \in \mho$ with $\eta(\varepsilon) \neq \emptyset$, then the falling shadow $\tilde{f}$ of the random set η, i.e.,

$$\tilde{f}(x) = P(\varepsilon \mid x \in \eta(\varepsilon)) \tag{3.1}$$

is called a *falling fuzzy filter* (resp. *falling fuzzy subhoop*) of H.

Example 3.2. Consider a hoop $(H, \odot, \to, 1)$ in which $H = \{0, a, b, 1\}$ is a set with Cayley tables (Tables 1 and 2). Let $(\mho, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and consider a mapping

Table 1. Cayley table for the binary operation "$\odot$"

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Table 2. Cayley table for the binary operation "$\rightarrow$"

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

$$\eta : [0,1] \rightarrow \mathcal{P}(H),\ t \mapsto \begin{cases} \{1\} & \text{if } t \in [0, 0.3), \\ \{1, a\} & \text{if } t \in [0.3, 0.7], \\ \{1, b\} & \text{if } t \in (0.7, 1] \end{cases} \tag{3.2}$$

Then $\eta(t)$ is both a subhoop and a filter of H for all $t \in [0,1]$. Thus the falling shadow $\tilde{f}$ of η is both a falling fuzzy subhoop and a falling fuzzy filter of H, and it is given as follows:

$$\tilde{f}(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x = 1, \\ 0.4 & \text{if } x = a, \\ 0.3 & \text{if } x = b. \end{cases} \tag{3.3}$$

Example 3.3. Consider a hoop $(H, \odot, \rightarrow, 1)$ in which $H = [0,1]$ is the unit interval in $\mathbb{R}$ and $\odot$ and $\rightarrow$ are given by $a \odot b = \min\{a, b\}$ and

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{if } a > b \end{cases} \tag{3.4}$$

for all $a, b \in H$. Let $(\mho, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and let $\eta : [0,1] \rightarrow \mathcal{P}(H)$ be defined by

$$\eta(t) = \begin{cases} [\frac{2}{3}, 1] & \text{if } t \in [0.6, 1], \\ [\frac{1}{2}, 1] & \text{if } t \in [0, 0.6] \end{cases} \tag{3.5}$$

Then $\eta(t)$ is a filter of H for all $t \in [0, 1]$. Thus the falling shadow $\tilde{f}$ of η is a falling fuzzy filter of H, and it is given as follows:

$$\tilde{f}(x) = \begin{cases} 0.4 & \text{if } x \in [\frac{2}{3}, 1], \\ 1 & \text{if } x \in [\frac{1}{2}, 1], \\ 0 & \text{if } x \in [0, \frac{1}{2}). \end{cases} \tag{3.6}$$

Example 3.4. Given a probability space $(\mho, \mathcal{A}, P)$, let $\mathcal{H}$ denote the set of all mappings from $\mho$ to a hoop H, that is,

$$\mathcal{H} := \{h \mid h : \mho \to H \ \text{ is a mapping}\}. \tag{3.7}$$

Let $\boxdot$ and $\twoheadrightarrow$ be binary operations on $\mathcal{H}$ defined by

$$(\forall \varepsilon \in \mho) \begin{pmatrix} (f \boxdot g)(\varepsilon) = f(\varepsilon) \odot g(\varepsilon) \\ (f \twoheadrightarrow g)(\varepsilon) = f(\varepsilon) \to g(\varepsilon) \end{pmatrix} \tag{3.8}$$

for all $f, g \in \mathcal{H}$. Also, we define a mapping

$$\mathbf{1} : \mho \to H, \ \varepsilon \mapsto 1. \tag{3.9}$$

It is routine to verify that $(\mathcal{H}, \boxdot, \twoheadrightarrow, \mathbf{1})$ is a hoop. For any subhoop and/or filter F of H and $h \in \mathcal{H}$, let

$$F_h := \{\varepsilon \in \mho \mid h(\varepsilon) \in F\} \tag{3.10}$$

and

$$\eta : \mho \to \mathcal{P}(\mathcal{H}), \ \varepsilon \mapsto \{h \in \mathcal{H} \mid h(\varepsilon) \in F\}. \tag{3.11}$$

Then $F_h \in \mathcal{A}$ and $\eta(\varepsilon) = \{h \in \mathcal{H} \mid h(\varepsilon) \in F\}$ is a subhoop and/or filter of $\mathcal{H}$. Since

$$\eta^{-1}(\ddot{h}) = \{\varepsilon \in \mho \mid h \in \eta(\varepsilon)\} = \{\varepsilon \in \mho \mid h(\varepsilon) \in F\} = F_h \in \mathcal{A}, \tag{3.12}$$

we know that η is a random set of $\mathcal{H}$. Let

$$\tilde{G}(h) = P(\varepsilon \mid h(\varepsilon) \in F).$$

Then $\tilde{G}$ is a falling fuzzy subhoop and/or filter of $\mathcal{H}$.

A *BE-algebra* is an algebra$(A, \rightsquigarrow, 1)$ of the type $(2,0)$ such that for all $x, y, z \in A$ the following axioms are fulfilled:

$(BE1)$ $x \rightsquigarrow x = 1$,

$(BE2)$ $x \rightsquigarrow 1 = 1$,

$(BE3)$ $1 \rightsquigarrow x = x$,

$(BE4)$ $x \rightsquigarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightsquigarrow z)$.

COROLLARY 3.5. (i) The algebraic structure $(\mathcal{H}, \boxdot, \twoheadrightarrow, \mathbf{1})$ is a BCK-algebra. (ii) The algebraic structure $(\mathcal{H}, \twoheadrightarrow, \mathbf{1})$ is a BE-algebra.

PROOF: The proof is straightforward. $\square$

THEOREM 3.6. *Every fuzzy filter* (resp. *fuzzy subhoop*) *of H is a falling fuzzy filter* (resp. *falling fuzzy subhoop*) *of H.*

PROOF: Let $\tilde{f}$ be a fuzzy filter (resp. fuzzy subhoop) of H. Then $\tilde{f}_t$ is a filter (resp. subhoop) of H for all $t \in [0,1]$. Define a random set as follows:

$$\eta : [0,1] \to \mathcal{P}(H),\ t \mapsto \tilde{f}_t.$$

Then $\tilde{f}$ is a falling fuzzy filter (resp. falling fuzzy subhoop) of H. $\square$

The converse of Theorem 3.6 is not true, in general. In fact, the falling fuzzy filter $\tilde{f}$ in Example 3.2 is not a fuzzy filter of H since $\tilde{f}(0) = 0 < 0.3 = \min\{\tilde{f}(a), \tilde{f}(a \to 0)\}$.

THEOREM 3.7. *Every falling fuzzy filter is a falling fuzzy subhoop.*

PROOF: Straightforward. $\square$

COROLLARY 3.8. Every fuzzy filter is a falling fuzzy subhoop.

The following example shows that the converse of Theorem 3.7 and Corollary 3.8 are not true in general.

Example 3.9. Consider a hoop $(H, \odot, \to, 1)$ in which $H = \{0, a, b, 1\}$ is a set with Cayley tables (Tables 3 and 4).

Table 3. Cayley table for the binary operation "$\odot$"

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Table 4. Cayley table for the binary operation "$\rightarrow$"

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Let $(\mho, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and consider a mapping

$$\eta : [0,1] \rightarrow \mathcal{P}(H),\ t \mapsto \begin{cases} \{1, a, 0\} & \text{if } t \in [0, 0.4), \\ \{1, a\} & \text{if } t \in [0.4, 0.75], \\ \{1, b, 0\} & \text{if } t \in (0.75, 1] \end{cases} \tag{3.13}$$

Then $\eta(t)$ is a subhoop H for all $t \in [0,1]$. Thus the falling shadow $\tilde{f}$ of η is a falling fuzzy subhoop of H which is given as follows.

$$\tilde{f}(x) = \begin{cases} 0.65 & \text{if } x = 0, \\ 1 & \text{if } x = 1, \\ 0.75 & \text{if } x = a, \\ 0.25 & \text{if } x = b. \end{cases} \tag{3.14}$$

But $\eta(t) = \{1, a, 0\}$ is not a filter of H for $t \in [0, 0.4)$ since $a \in \eta(t)$ and $a \rightarrow b = 1 \in \eta(t)$, but $b \notin \eta(t)$. Hence $\tilde{f}$ is not a falling fuzzy filter of H. Since

$$\tilde{f}(b) = 0.25 < 0.75 = \min\{\tilde{f}(a \rightarrow b), \tilde{f}(a)\},$$

we know that $\tilde{f}$ is not a fuzzy filter of H.

We provide a condition for a falling fuzzy subhoop to be a falling fuzzy filter.

Theorem 3.10. *Given a falling fuzzy subhoop $\tilde{f}$ of H, the following are equivalent.*

(1) *$\tilde{f}$ is a falling fuzzy filter of H.*

(2) *For each $\varepsilon \in \mho$, the following is valid.*

$$(\forall x, y \in H)\,(x \in \eta(\varepsilon), y \in H \setminus \eta(\varepsilon) \;\Rightarrow\; x \to y \in H \setminus \eta(\varepsilon))\,. \tag{3.15}$$

Proof: Assume that $\tilde{f}$ is a falling fuzzy filter of H. Then $\eta(\varepsilon)$ is a filter of H for all $\varepsilon \in \mho$. Let $x, y \in H$ be such that $x \in \eta(\varepsilon)$ and $y \in H \setminus \eta(\varepsilon)$. If $x \to y \in \eta(\varepsilon)$, then $y \in \eta(\varepsilon)$ which is a contradiction. Hence $x \to y \in H \setminus \eta(\varepsilon)$. Let $\tilde{f}$ be a falling fuzzy subhoop of H in which (2) is true. Then $\eta(\varepsilon)$ is a subhoop of H for all $\varepsilon \in \mho$. Thus $1 \in \eta(\varepsilon)$. Let $x, y \in H$ be such that $x \in \eta(\varepsilon)$ and $x \to y \in \eta(\varepsilon)$. If $y \in H \setminus \eta(\varepsilon)$, then $x \to y \in H \setminus \eta(\varepsilon)$ by (3.15). This is a contradiction, and so $y \in \eta(\varepsilon)$. Therefore $\eta(\varepsilon)$ is a filter of H for all $\varepsilon \in \mho$, and thus $\tilde{f}$ is a falling fuzzy filter of H. □

Given a probability space $(\mho, \mathcal{A}, P)$ and a falling shadow $\tilde{f}$ of a random set η on H, consider the set

$$\mho(x; \eta) := \{\varepsilon \in \mho \mid x \in \eta(\varepsilon)\} \tag{3.16}$$

for $x \in H$. Then $\mho(x; \eta) \in \mathcal{A}$.

Proposition 3.11. If $\tilde{f}$ is a falling fuzzy filter of H, then

$$(\forall x, y \in H)\,(x \leq y \;\Rightarrow\; \mho(x; \eta) \subseteq \mho(y; \eta))\,, \tag{3.17}$$

$$(\forall x, y \in H)\,(\mho(x \to y; \eta) \cap \mho(x; \eta) \subseteq \mho(y; \eta))\,, \tag{3.18}$$

$$(\forall x \in H)\,(\mho(x; \eta) \subseteq \mho(1; \eta))\,, \tag{3.19}$$

$$(\forall x, y \in H)\,(\mho(y; \eta) \subseteq \mho(x \to y; \eta))\,. \tag{3.20}$$

$$(\forall x, y, z \in H)\,(x \odot y \leq z \;\Rightarrow\; \mho(x; \eta) \cap \mho(y; \eta) \subseteq \mho(z; \eta))\,. \tag{3.21}$$

Proof: Let $\tilde{f}$ be a falling fuzzy filter of H. Then $\eta(\varepsilon)$ is a filter of H for all $\varepsilon \in \mho$. Let $x, y \in H$ be such that $x \leq y$ and let $\varepsilon \in \mho(x; \eta)$. Then $x \to y = 1 \in \eta(\varepsilon)$ and $x \in \eta(\varepsilon)$. Thus $y \in \eta(\varepsilon)$, that is, $\varepsilon \in \mho(y; \eta)$. Hence $\mho(x; \eta) \subseteq \mho(y; \eta)$. Let $\varepsilon \in \mho(x \to y; \eta) \cap \mho(x; \eta)$ for all $x, y \in H$. Then

$x \to y \in \eta(\varepsilon)$ and $x \in \eta(\varepsilon)$. Since $\eta(\varepsilon)$ is a filter of H, we have $y \in \eta(\varepsilon)$, and so $\varepsilon \in \mho(y;\eta)$. This shows that (3.18) is valid. Since $x \leq 1$ for all $x \in H$, it follows from (3.17) that (3.19) holds. Since $y \leq x \to y$ for all $x, y \in H$, it follows from (3.17) that (3.20) holds. Let $x, y, z \in H$ be such that $x \odot y \leq z$. Then $x \leq y \to z$, i.e., $x \to (y \to z) = 1$. It follows from (3.18) and (3.19) that

$$\begin{aligned}\mho(z;\eta) &\supseteq \mho(y \to z;\eta) \cap \mho(y;\eta)\\ &\supseteq \mho(x;\eta) \cap \mho(x \to (y \to z);\eta) \cap \mho(y;\eta)\\ &= \mho(x;\eta) \cap \mho(1;\eta) \cap \mho(y;\eta)\\ &= \mho(x;\eta) \cap \mho(y;\eta).\end{aligned}$$

Hence (3.21) is valid. □

PROPOSITION 3.12. If $\tilde{f}$ is a falling fuzzy subhoop of H, then

$$(\forall x, y \in H)\left(\begin{array}{c}\mho(x;\eta) \cap \mho(y;\eta) \subseteq \mho(x \odot y;\eta)\\ \mho(x;\eta) \cap \mho(y;\eta) \subseteq \mho(x \to y;\eta)\end{array}\right). \tag{3.22}$$

PROOF: If $\tilde{f}$ is a falling fuzzy subhoop of H, then $\eta(\varepsilon)$ is a subhoop of H for all $\varepsilon \in \mho$. Let $\varepsilon \in \mho(x;\eta) \cap \mho(y;\eta)$. Then $x \in \eta(\varepsilon)$ and $y \in \eta(\varepsilon)$. It follows that $x \odot y \in \eta(\varepsilon)$, that is, $\varepsilon \in \mho(x \odot y;\eta)$. Hence $\mho(x;\eta) \cap \mho(y;\eta) \subseteq \mho(x \odot y;\eta)$. Similarly, we get $\mho(x;\eta) \cap \mho(y;\eta) \subseteq \mho(x \to y;\eta)$. □

COROLLARY 3.13. Every falling fuzzy filter $\tilde{f}$ of H satisfies the condition (3.22).

PROPOSITION 3.14. If $\tilde{f}$ is a falling fuzzy filter of H, then

$$(\forall x, y \in H)\,(\mho(x \odot y;\eta) = \mho(x;\eta) \cap \mho(y;\eta)), \tag{3.23}$$

$$(\forall x, y, z \in H)\,(\mho((x \to y) \to z;\eta) \cap \mho(y;\eta) \subseteq \mho(x \to z;\eta)). \tag{3.24}$$

PROOF: Since $x \odot y \leq x$ and $x \odot y \leq y$ for all $x, y \in H$, it follows from (3.17) that $\mho(x \odot y;\eta) \subseteq \mho(x;\eta)$ and $\mho(x \odot y;\eta) \subseteq \mho(y;\eta)$ Hence $\mho(x;\eta) \cap \mho(y;\eta) \supseteq \mho(x \odot y;\eta)$ for all $x, y \in H$. Combining this and Proposition 3.12 induces (3.23). Since

$$y \odot ((x \to y) \to z) \leq y \odot (y \to z) \leq z \leq x \to z$$

for all $x, y, z \in H$, we have

$$\mho(x \to z;\eta) \supseteq \mho(y \odot ((x \to y) \to z);\eta) = \mho(y;\eta) \cap \mho((x \to y) \to z;\eta)$$

by (3.17) and (3.23). □

PROPOSITION 3.15. If $\tilde{f}$ is a falling fuzzy subhoop of H, then

$$(\forall x, y \in H)\left(\tilde{f}(x \odot y) \geq \tilde{f}(x) + \tilde{f}(y) - 1,\ \tilde{f}(x \to y) \geq \tilde{f}(x) + \tilde{f}(y) - 1\right). \tag{3.25}$$

PROOF: Assume that $\tilde{f}$ is a falling fuzzy subhoop of H. Then $\eta(\varepsilon)$ is a subhoop of H for all $\varepsilon \in \mho$. Hence

$$\{\varepsilon \in \mho \mid x \in \eta(\varepsilon)\} \cap \{\varepsilon \in \mho \mid y \in \eta(\varepsilon)\} \subseteq \{\varepsilon \in \mho \mid x \odot y \in \eta(\varepsilon)\}$$

and

$$\{\varepsilon \in \mho \mid x \in \eta(\varepsilon)\} \cap \{\varepsilon \in \mho \mid y \in \eta(\varepsilon)\} \subseteq \{\varepsilon \in \mho \mid x \to y \in \eta(\varepsilon)\}.$$

for any $x, y \in H$, and so

$$\begin{aligned}
\tilde{f}(x \odot y) &= P(\varepsilon \mid x \odot y \in \eta(\varepsilon)) \\
&\geq P(\varepsilon \mid x \in \eta(\varepsilon)) \cap P(\varepsilon \mid y \in \eta(\varepsilon)) \\
&\geq P(\varepsilon \mid x \in \eta(\varepsilon)) + P(\varepsilon \mid y \in \eta(\varepsilon)) - P(\varepsilon \mid x \in \eta(\varepsilon) \ \text{ or } \ y \in \eta(\varepsilon)) \\
&= \tilde{f}(x) + \tilde{f}(y) - 1
\end{aligned}$$

and

$$\begin{aligned}
\tilde{f}(x \to y) &= P(\varepsilon \mid x \to y \in \eta(\varepsilon)) \\
&\geq P(\varepsilon \mid x \in \eta(\varepsilon)) \cap P(\varepsilon \mid y \in \eta(\varepsilon)) \\
&\geq P(\varepsilon \mid x \in \eta(\varepsilon)) + P(\varepsilon \mid y \in \eta(\varepsilon)) - P(\varepsilon \mid x \in \eta(\varepsilon) \ \text{ or } \ y \in \eta(\varepsilon)) \\
&= \tilde{f}(x) + \tilde{f}(y) - 1.
\end{aligned}$$

This completes the proof. □

PROPOSITION 3.16. If $\tilde{f}$ is a falling fuzzy filter of H, then

$$\tilde{f}(y) \geq \tilde{f}(x \to y) + \tilde{f}(x) - 1 \tag{3.26}$$

for all $x, y \in H$ with $\tilde{f}(x \to y) + \tilde{f}(x) \geq 1$.

PROOF: If $\tilde{f}$ is a falling fuzzy filter of H, then $\eta(\varepsilon)$ is a filter of H for all $\varepsilon \in \mho$. For any $x, y \in H$, if $\tilde{f}(x \to y) + \tilde{f}(x) \geq 1$, then

$$\{\varepsilon \in \mho \mid x \to y \in \eta(\varepsilon)\} \cap \{\varepsilon \in \mho \mid x \in \eta(\varepsilon)\} \subseteq \{\varepsilon \in \mho \mid y \in \eta(\varepsilon)\},$$

and so

$$\begin{aligned}
\tilde{f}(y) &= P(\varepsilon \mid y \in \eta(\varepsilon)) \\
&\geq P(\varepsilon \mid x \to y \in \eta(\varepsilon)) \cap P(\varepsilon \mid x \in \eta(\varepsilon)) \\
&\geq P(\varepsilon \mid x \to y \in \eta(\varepsilon)) + P(\varepsilon \mid x \in \eta(\varepsilon)) - P(\varepsilon \mid x \to y \in \eta(\varepsilon) \text{ or } x \in \eta(\varepsilon)) \\
&= \tilde{f}(x \to y) + \tilde{f}(x) - 1.
\end{aligned}$$

This completes the proof. □

THEOREM 3.17. *For any falling shadow $\tilde{f}$ of the random set η, if two conditions (3.18) and (3.19) are valid, then $\tilde{f}$ is a falling fuzzy filter of H.*

PROOF: Assume that $\eta(\varepsilon)$ is nonempty for all $\varepsilon \in \mho$. Then there exists $x \in \eta(\varepsilon)$ and so $\varepsilon \in \mho(x; \eta) \subseteq \mho(1; \eta)$. Thus $1 \in \eta(\varepsilon)$. Let $x, y \in H$ be such that $x \to y \in \eta(\varepsilon)$ and $x \in \eta(\varepsilon)$. Then $\varepsilon \in \mho(x \to y; \eta)$ and $\varepsilon \in \mho(x; \eta)$. It follows from (3.18) that

$$\varepsilon \in \mho(x \to y; \eta) \cap \mho(x; \eta) \subseteq \mho(y; \eta).$$

Thus $y \in \eta(\varepsilon)$, and hence $\eta(\varepsilon)$ is a filter of H. Therefore the falling shadow $\tilde{f}$ of the random set η is a falling fuzzy filter of H. □

THEOREM 3.18. *If a falling shadow $\tilde{f}$ of the random set η satisfies (3.17), (3.19) and (3.23), then $\tilde{f}$ is a falling fuzzy filter of H.*

PROOF: Let $x, y \in H$. Since $x \odot (x \to y) \leq y$, it follows from (3.17) and (3.23) that

$$\mho(y; \eta) \supseteq \mho(x \odot (x \to y); \eta) = \mho(x; \eta) \cap \mho(x \to y; \eta).$$

Therefore $\tilde{f}$ is a falling fuzzy filter of H by Theorem 3.17. □

PROPOSITION 3.19. If a falling shadow $\tilde{f}$ of the random set η satisfies (3.17) and (3.23), then

$$(\forall x, y, z \in H)\, (\mho(x \to y; \eta) \cap \mho(y \to z; \eta) \subseteq \mho(x \to z; \eta)), \tag{3.27}$$

$$(\forall x, y, z \in H)\, (\mho(x \odot z; \eta) \cap \mho(x \to y; \eta) \subseteq \mho(y \odot z; \eta)). \tag{3.28}$$

Proof: Since $(x \to y) \odot (y \to z) \leq x \to z$ for all $x, y, z \in H$, the condition (3.27) is induced by (3.17) and (3.23). Since $(z \odot x) \odot (x \to y) \leq z \odot y$ for all $x, y, z \in H$, the condition (3.28) is induced by (3.17) and (3.23). □

Since every falling fuzzy filter $\tilde{f}$ of H satisfies two conditions (3.17) and (3.23), we have the following corollary.

Corollary 3.20. Every falling fuzzy filter $\tilde{f}$ of H satisfies the conditions (3.27) and (3.28).

Theorem 3.21. *If a falling shadow $\tilde{f}$ of the random set η satisfies (3.19) and (3.21), then $\tilde{f}$ is a falling fuzzy filter of H.*

Proof: Let $\tilde{f}$ be a falling shadow of the random set η satisfying (3.19) and (3.21). Since $x \odot (x \to y) \leq y$ for all $x, y \in H$, we have $\mho(x;\eta) \cap \mho(x \to y;\eta) \subseteq \mho(y;\eta)$. Using Theorem 3.17, we know that $\tilde{f}$ is a falling fuzzy filter of H. □

Theorem 3.22. *If a falling shadow $\tilde{f}$ of the random set η satisfies (3.19) and (3.24), then $\tilde{f}$ is a falling fuzzy filter of H.*

Proof: Let $\tilde{f}$ be a falling shadow of the random set η satisfying (3.19) and (3.24). Taking $x = 1$, $y = x$ and $z = y$ in (3.24) induces the condition (3.18). Therefore $\tilde{f}$ is a falling fuzzy filter of H by Theorem 3.17. □

Theorem 3.23. *If a falling shadow $\tilde{f}$ of the random set η satisfies (3.19) and (3.28), then $\tilde{f}$ is a falling fuzzy filter of H.*

Proof: Let $\tilde{f}$ be a falling shadow of the random set η satisfying (3.19) and (3.28). Taking $z = 1$ in (3.28) induces the condition (3.18). Therefore $\tilde{f}$ is a falling fuzzy filter of H by Theorem 3.17. □

4. Conclusions and future work

The falling shadow theory is applied to subhoops and filters in hoops. The notions of falling fuzzy subhoops and falling fuzzy filters in hoops are introduced, and several properties are investigated. Relationship between falling fuzzy subhoops and falling fuzzy filters are discussed, and conditions for a falling fuzzy subhoop to be a falling fuzzy filter are provided. Also conditions for a falling shadow of a random set to be a falling fuzzy filter are displayed. On the basis of these results, we will apply the theory of falling shadows to the another type of ideals and filters in hoops and investigate some properties and equali definition of them and study the relation between them in future study.

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