Analysis of models arising from heat conduction through fins using Lie symmetries and Tanh Method

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Declaration

I declare that this project is my own, unaided work. It is being submitted in fulfillment of the Degree of Master of Science at the University of the KwaZulu-Natal, Durban. It has not been submitted before for any degree or examination in any other University.

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Abstract

In this dissertation, we study heat transfer in longitudinal fins of rectangular and concave parabolic profiles. The thermal conductivity and heat transfer vary nonlinearly with temperature, and the power-law gives the temperature distribution. The step-change in base temperature and step change in base heat flow are used as boundary conditions. We employ Lie symmetry techniques in an attempt to solve initial and boundary value problems. We also use the Tanh method to analyze the heat transfer equation.

The governing equation for the fins of rectangular profile admitted three local symmetries, which reduced the original non-linear PDE into three non-linear ODEs. We analyzed steady-state as well as transient-state equations. Three cases were considered in the steady-state, and two cases were looked at for the transient state. The solutions in each case satisfied the step-change in base temperature and step change in base heat flow boundary conditions. The effects of the thermo-geometric fin parameter and the power-law exponents m and n, on temperature distribution are studied in all these problems. Furthermore, the fin efficiency and heat flow are analyzed.

The governing equation for the fins of the concave parabolic profile admitted two local symmetries, which reduced the original non-linear PDE into two non-linear ODEs. As we did in the fins of the rectangular profile, we analyzed steady-state as well as transient-state solutions. In the first three steady-state cases, the general solutions obtained did not satisfy the initial condition. In the fourth case, Abel's equation was obtained, and no further analysis was carried out. Using the differential invariants' method, the two non-linear ODEs

were further reduced into two first-order ODEs of Abel's second kind. At this stage, we considered a particular case when $m = 0$. This yielded an exact analytical solution that enabled us to investigate temperature and heat flux behavior in terms of increasing space, time, exponent n, and thermo-geometric fin parameter M. On the other hand, the second equation was transformed into a canonical form and solved, which yielded a particular function as tabled in the Appendix. There was no further analysis carried out.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Literature review

1.1.1 Rationale and motivation

Fins (or extended surfaces) play a vital role in enhancing the movement of heat between a surface and adjacent fluid. The study of heat transfer has become increasingly important because of its frequent applications in engineering, for instance, car radiators, refrigerators, power generators, air conditioning, air-cooled craft engines, and many other devices in which heat is generated and transferred. Problems describing heat transfer in extended surfaces are documented in a number of textbooks (see, e.g., [1, 2]).

Through the process of mathematical modeling, heat transfer problems are reduced to nonlinear differential equations. Steady-state differential equations have been extensively used in most applicable models describing heat transfer in fins. However, fins that are used in high-speed aircraft, electronic components, and automatic control equipment necessitate the understanding of transient heat transfer [3].

1.1.2 Recent developments

Researchers have spent considerable amount of time in formulating exact and approximate methods for solving differential equations, arising in heat conduction though the one-dimensional (see e.g. [4–9]) and the two-dimensional extended surfaces (see e.g. [10–19]).

The obtained solutions include series solutions [4, 6, 9], homotopy methods [8] and differential transformation method [20,21] and the Tanh method [22–24] in one-dimensional cases. Most existing solutions are constructed only for constant thermal conductivity and heat transfer coefficient.

The literature on exact solutions for one- and two-dimensional problems has been growing. Recently, an analysis of a steady nonlinear longitudinal fin of various profiles was given by Ndlovu and Moitsheki [25], Mhlongo and Moitsheki [26, 27]. Transient response of longitudinal rectangular fins to step-change in base temperature and in base heat flow conditions was studied by Mhlongo et al. [28].

Many authors [8, 9, 29–31] used symmetry analysis of the problem in [5] to determine all forms of the thermal conductivity and heat transfer coefficient for which the governing equation admits additional symmetries.

1.2 Aims and objectives of the project

The main goal of the project is to study heat transfer in longitudinal fins of rectangular and concave parabolic profiles and extend the work done in [28]. The thermal conductivity and heat transfer vary nonlinearly with temperature, and the temperature distribution is given by power law. Furthermore, we aim to analyze the problem subject to various boundary conditions, using step-change in base temperature and step change in base heat flow.

We employ techniques such as local symmetries and equivalent transformations. Symmetry reductions are performed, and group invariant solutions are constructed. The effects on temperature distribution of parameters in the dimensionless models are analyzed. The Tanh method will be used to compare results obtained using the symmetry method. We also make use of computer-aided procedures such as Maple [32], Mathematica [33], in particular, we use the freely available software DIMSYM [34] and REDUCE [35].

1.3 Outline of the dissertation

The dissertation is outlined as follows:

- Chapter 2 describes the mathematical formulation of the heat conduction problem.
- Symmetry techniques for differential equations are briefly discussed in chapter 3. The concepts, Lie point (local) symmetries, Lie algebras, invariant solution, and the optimal system will be discussed.
- In chapter 4, we discuss the transient response of longitudinal rectangular fin to step-change in base temperature and in base heat flow conditions using Lie symmetries and the Tanh Methods.
- In chapter 5, we consider a one-dimensional longitudinal fin of the concave parabolic profile subject to step-change in base temperature and base heat flow conditions using Lie symmetries.
- Lastly, we provide conclusions in chapter 6.

Chapter 2

Mathematical statement

2.1 Introduction

We briefly discuss the models representing heat transfer in longitudinal fins of different profiles in this chapter. The fins examined are given in terms of the characteristic length. This may be a determinant of why a few exact solutions exist. The mathematical formulation of the problem is given in Section 2.2 and is followed by a discussion of various fin profiles in Section 2.3. In Section 2.4, a brief description of heat transfer coefficient and thermal conductivity is given, followed by physical boundary conditions in Section 2.5. Fin efficiency and heat flux are discussed in Section 2.6. In Section 2.7, dimensionless variables are described. The concluding remarks are provided in section 2.8.

This chapter is a brief theoretical framework of heat transfer in longitudinal fins of different profiles. A complete theory is provided in texts such as [1].

2.2 The energy balance model

A one-dimensional longitudinal fin with a cross-sectional area A_c is shown in Fig. 2.1 with various components explained below.

- The perimeter of the fin is denoted by P,
- the length of the fin by L,
- the fin is attached to a fixed base surface of temperature T_b
- and extends into a fluid of temperature T_a.
- The fin profile is given by the function $F(X)$,
- fin thickness δ depends on the fin profile,
- and the fin thickness at the base is δ_b.

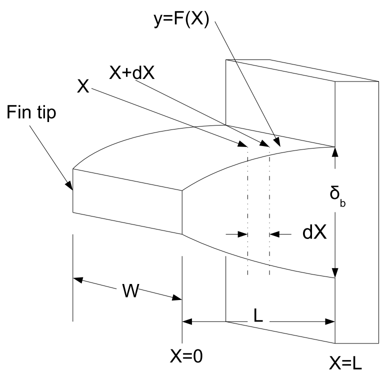

Figure 2.1: A graphical representation of a longitudinal fin of an arbitrary profile.

The energy balance for a longitudinal fin of an arbitrary profile is given by

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial X}\left(\frac{2}{\delta_b} F(X) K(T) \frac{\partial T}{\partial X}\right) - \frac{P}{A_c} H(T)\left(T - T_a\right), \quad 0 < X < L, \tag{2.1}$$

where

- ρ is the mass density,
- c is the specific heat,
- K and H are the non-uniform thermal conductivity and heat transfer coefficients depending on the temperature (see for example [7–9, 36]),
- T is the temperature distribution,
- t is time, and X is the spatial variable.

The property of a material's ability to conduct heat is called thermal conductivity. It is denoted by $K(T)$ and varies with temperature in many engineering applications. The heat transfer coefficient, denoted by $H(T)$, is the amount of heat passing through a unit area of a medium or system in a unit time when the temperature difference between the boundaries of the system is 1^0 Celsius. We the fin length from the tip to the base as shown in Fig. 2.1.

2.3 Fin profiles

Two fin profiles examined in this dissertation are presented in this section, these are, the rectangular and concave parabolic profiles. Fins with a parabolic profile contain less material and are more efficient, requiring minimum weight.

A longitudinal fin with a concave parabolic profile is costly and difficult to manufacture, but it provides the maximum heat dissipation for a given profile area [1]. The rectangular profile does not utilize the material efficiently, but it is preferred in most applications for the sake of simplicity in the fabrication [37].

The optimization of the design focuses on finding the shape and dimensions of the fins that would reduce the volume or mass for a given amount of heat dissipation, or alternatively, to maximize the dissipation of heat for a given volume or mass. One way to analyze the optimization problem is to select a suitable profile, then to determine the dimensions of the fins and to yield the maximum heat dissipation for a given volume and shape of the fin.

A general optimization problem of convective fins with constant thermal parameters was analyzed by Laor and Kalman [38].

The profile function $F(X)$ for longitudinal fins take the general form [1]

$$F(X) = \frac{\delta_b}{2}\left(\frac{X}{L}\right)^{\frac{1-2p}{1-p}}. \tag{2.2}$$

Here $p \in \mathbb{R},\ p \neq 1$.

2.3.1 Longitudinal fin of rectangular profile

For the longitudinal fin of rectangular profile shown in Fig. 2.2, the exponent on the general profile satisfies the geometry when $p = \frac{1}{2}$. The profile function for the fin then becomes [1]

$$F(X) = \frac{\delta_b}{2}. \tag{2.3}$$

2.3.2 Longitudinal fin of concave parabolic profile

If $p \to \infty$ we get a longitudinal fin of concave parabolic profile shown in Fig. 2.3. The profile function for the fin then becomes [1]

$$F(X) = \frac{\delta_b}{2}\left(\frac{X}{L}\right)^2. \tag{2.4}$$

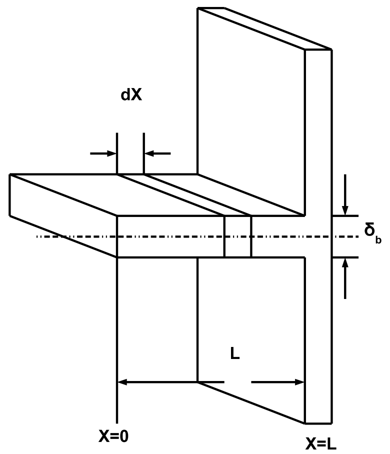

Figure 2.2: Schematic representation of a longitudinal fin of a rectangular profile.

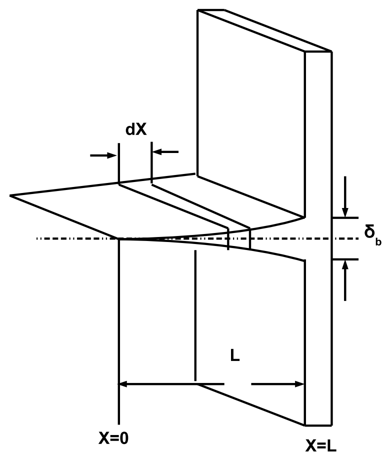

Figure 2.3: Schematic representation of a longitudinal fin of a concave parabolic profile.

2.4 Boundary conditions

In this section, we look at the boundary conditions. We assume that the fin is insulated at the tip; otherwise, the problem becomes overdetermined [39]. This boundary condition is noticed by Ünal for sufficiently long fins [40]. In this case, we have

$$\frac{\partial T(0,t)}{\partial X} = 0. \tag{2.5}$$

The step change in base temperature gives the boundary condition at the base of the fin [1]

$$T(L,t) = T_b, \tag{2.6}$$

in one case and by the step change in base heat flow

$$\frac{\partial T(L,t)}{\partial X} = \frac{q_b}{k_a A_c}, \tag{2.7}$$

in the other.

Here q_b is the base heat flux, k_a is thermal conductivity at the ambient temperature and other parameters have been defined earlier. The initial fin temperature is assumed to be

$$T(X,0) = 0. \tag{2.8}$$

2.5 Thermal conductivity and heat transfer coefficient

The thermal conductivity of a material in many engineering applications is given as a linear function and expressed as [5, 8, 41]

$$K(T) = k_a[1 + \beta(T - T_a)], \tag{2.9}$$

where β is the thermal conductivity gradient.

The thermal conductivity of the fin may be assumed to be a nonlinear function of the temperature, that is

$$K(T) = k_a \left(\frac{T - T_a}{T_b - T_a}\right)^m \quad \text{and} \quad k(T) = k_a \left(\frac{k_a A_c (T - T_a)}{q_b L}\right)^m, \tag{2.10}$$

for step-change in base temperature and step change in base heat flow conditions, respectively. The power-law is used to model the thermal conductivity of some materials such as Gallium Nitride (GaN) and Aluminium Nitride (AlN) (see e.g. [42]). Moreover, experimental data indicates that the exponent of the power-law for these materials is positive for lower temperatures and negative for at higher temperatures [43–45].

On the other hand, for most industrial applications the heat transfer coefficient may be given by the power law [40]

$$H(T) = h_b \left(\frac{T - T_a}{T_b - T_a}\right)^n, \tag{2.11}$$

given the step change in the base temperature and

$$h(T) = h_b \left(\frac{k_a A_c (T - T_a)}{q_b L}\right)^n \tag{2.12}$$

given the step change in the base heat flow. Here the exponent n and h_b are constants. The constant n ranges from 6 to 5, and in most practical applications, it lies between -3 and 3 [40]. The hypothetical boundary condition (i.e., insulation) at the tip of the fin is taken into account if the heat transfer coefficient is given by Eqs. (2.11) and (2.12) [40]. The transfer of heat through the outermost edge of the fin is negligible compared to that passing through the side [39]. When $n = -\frac{1}{4}$, the exponent represents laminar film boiling or condensation. The exponent represents laminar natural convection when $n = \frac{1}{4}$, turbulent natural convection when $n = \frac{1}{3}$, nucleate boiling when $n = 2$, radiation when $n = 4$, and $n = 0$ implies a constant heat transfer coefficient. If the thermal conductivity is a constant and $n = -1, 0, 1$ and 2, the exact solutions are constructed for the steady-state one-dimensional differential equation describing temperature distribution in a straight fin. [40]. In this dissertation, we attempt to construct exact steady-state solutions given nonconstant thermal conductivity.

2.6 Fin efficiency and heat flux

2.6.1 Fin efficiency

Newton's second law of cooling gives the heat transfer rate from a fin (see e.g. [1])

$$Q = \int_0^L PH(T)(T - T_a)dX. \tag{2.13}$$

Fin efficiency is the ratio of the fin heat transfer rate to the rate that would be if the entire fin were at the base temperature and is given by

$$\eta = \frac{Q}{Q_{ideal}} = \frac{\int_0^L PH(T)(T - T_a)dX}{Ph_b L(T - T_a)}. \tag{2.14}$$

2.6.2 Heat flux

Fourier's law gives the heat flux at the base of the fin (see e.g. [1])

$$q_b = A_c K(T)\frac{dT}{dX}. \tag{2.15}$$

The total heat flow of the fin is given by

$$q = \frac{q_b}{A_c H(T)(T - T_a)}. \tag{2.16}$$

2.7 Non-dimensionalization

We introduce the dimensionless variables and the dimensionless numbers given by

$$x = \frac{X}{L}, \quad \tau = \frac{k_a t}{\rho c L^2}, \quad k = \frac{K}{k_a}, \quad h = \frac{H}{h_b}, M^2 = \frac{P h_b L^2}{A_c k_a}, f(x) = \frac{2}{\delta_b} F(X), \tag{2.17}$$

with step change in base heat flow, the dimensionless temperature becomes [1]

$$\theta = \frac{k_a A_c (T - T_a)}{q_b L}, \tag{2.18}$$

and with the step change base temperature we have

$$\theta = \frac{T - T_a}{T_b - T_a}. \tag{2.19}$$

Equation (2.1) now reduces to

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left(f(x) k(\theta) \frac{\partial \theta}{\partial x} \right) - M^2 h(\theta)\theta, \quad 0 \leq x \leq 1. \tag{2.20}$$

The initial condition is given by

$$\theta(x, 0) = 0, \tag{2.21}$$

the step change in fin base temperature is given by

$$\theta(1, \tau) = 1, \tag{2.22}$$

the step change in fin base heat flux is given by

$$\left. \frac{\partial \theta}{\partial x} \right|_{x=1} = 1 \tag{2.23}$$

and fin tip boundary condition is given by

$$\left. \frac{\partial \theta}{\partial x} \right|_{x=0} = 0. \tag{2.24}$$

Here M is the thermo-geometric fin parameter, θ is the dimensionless temperature, x is the dimensionless spatial variable, $f(x)$ is the dimensionless fin profile, τ is dimensionless time, k is the dimensionless thermal conductivity, h is the dimensionless heat transfer coefficient, and h_b is the heat transfer coefficient at the fin base. The non-dimensional heat transfer coefficient and thermal conductivity of the fin are given by

$$h(\theta) = \theta^n, \quad k(\theta) = \theta^m$$

respectively.

2.8 Concluding remarks

In this chapter, we have discussed the mathematical formulation representing heat transfer in longitudinal fins of various profiles. We have taken into consideration the energy balance equation, the physical boundary conditions. Furthermore, the initial and boundary value problems are given in terms of non-dimensionless variables.

Chapter 3

Symmetry methods for differential equations

3.1 Introduction

We consider Lie symmetry techniques for differential equations in this chapter. A short description and theoretical background are given in Sections 3.1.1 and 3.1.2, respectively. In Section 3.2, we discuss the calculation of classical (local) Lie point symmetries. In Section 3.3, we discuss Lie Algebras. In Sections 3.4, we present the method for constructing an optimal system of subalgebras. In Section 3.5, we consider the basis of invariants. Methods of linearization and reductions of ordinary differential equations (ODEs) are discussed in Section 3.6, followed by a brief description of the Tanh method in Section 3.7. Concluding remarks are given in Section 3.8.

3.1.1 A short description

Sophus Lie introduced the theory of groups of transformation for differential equations (DEs) in the nineteenth century, aiming at extending various specialized methods for solving ODEs. The theory was influenced by the lectures given by Sylow on Galois theory and Abel's works. The order of an ODE according to Lie could be reduced by one, constructively, if it is invariant under a one-parameter Lie group of point transformations [46].

Various topics in determining the solutions of ODEs are required to fully understand Lie's work, including, among others, integrating factor, separable equation, homogeneous equation, reduction of order, the methods of undetermined coefficients and variation of parameters for linear equations, solution of the Euler equation, and the use of the Laplace transform. Lie also indicated that for linear PDEs, invariance under a Lie group leads directly to superpositions of solutions in terms of transforms [46].

3.1.2 A brief theoretical background of Lie symmetry analysis

In brief, a symmetry of a differential equation is an invertible transformation of dependent and independent variables that leave the form of the equation unchanged [46–51]. This point transformation, in Lie's view [52], forms a group that depends on a continuous parameter. The elementary examples of Lie groups include groups of translations, rotations, and scalings.

In his fundamental theorem, Lie showed that groups are characterized by infinitesimal generators. These infinitesimal generators can be extended to act on the space of independent and dependent variables and their derivatives up to any finite order. If the coefficients of governing differential equation (or a system of equations) are functions of independent and/or dependent variables, then the vector fields or symmetries admitted by the equation in question when these coefficients are arbitrary span the *principal Lie algebra* (PLA) (see, e.g., [53]). The consequence of the action of the infinitesimal generators on DEs is the reduction in the original equation as follows (but not limited to),

(i) In the case of the first order ODE equations, it reduces to a separable first-order ODE.

(ii) A second-order 1+1 dimensional PDE may be reduced to a second-order ODE.

(iii) A nonlinear second-order ODE may be reduced linear second-order ODE or to the first-order ODE or to the ODE with a cubic in the first derivative.

(iv) A PDE with n independent variables can be reduced to one with $n-1$ independent variables.

For further theory and applications of symmetry analysis, excellent text such as those of [46–51, 54–56] can be used.

3.2 Calculation of Lie point (local) symmetries

3.2.1 One-parameter group of transformations

Consider a set G_T of transformations

$$T_\epsilon : \overline{x}^i = f^i(x,u,\epsilon), \quad \overline{u}^\alpha = \phi^\alpha(x,u,\epsilon), i = 1,2\ldots,n;\ \alpha = 1,2,\ldots,m; \quad (3.1)$$

where ϵ is real parameter which is continuous in the neighborhood $\mathbf{D} \subset \mathbb{R}$ of $\epsilon = 0$ and f^i, ϕ^α are differentiable functions. G_T is a *continuous one-parameter (local) Lie group of transformations* in $\mathbb{R}^{n+m}$ if the following properties hold,

i) *Identity.* If $T_e \in G$ such that $T_e T_\epsilon = T_\epsilon T_e = T_\epsilon$, for any $\epsilon \in \mathbf{D}_* \subset \mathbf{D}$ and $T_\epsilon \in G_T$. T_e is the identity in G_T and e is the identity in $\mathbf{D}$.

ii) *Closure.* If T_ϵ, T_δ are in G_T and ϵ, δ in $\mathbf{D}_* \subset \mathbf{D}$, then $T_\epsilon T_\delta = T_\gamma \in G$, $\gamma = \psi(\epsilon, \delta) \in \mathbf{D}$.

iii) *Inverse element.* For $T_\epsilon, \epsilon \in \mathbf{D}_* \subset \mathbf{D}$, there exists $T_\epsilon^{-1} = T_{\epsilon^{-1}} \in G_T, \epsilon^{-1} \in \mathbf{D}$ such that $T_\epsilon T_{\epsilon^{-1}} = T_{\epsilon^{-1}} T_\epsilon = T_e$

iv) *Associativity.* If $T_\epsilon, T_\delta, T_\gamma$ are in G_T and ϵ, δ, γ in $\mathbf{D}_* \subset \mathbf{D}$, then $(T_\epsilon T_\delta) T_\gamma = T_\epsilon (T_\delta T_\gamma)$.

v) ϵ is a continuous parameter i.e. $\epsilon \in \mathbf{D}_*$, where $\mathbf{D}$ is an interval in $\mathbb{R}$,

vi) φ_i and ϕ_α are analytic,

vii) $\psi(\epsilon, \delta)$ is an analytic function of ϵ and δ.

According to the theory of Lie symmetries, the construction of a one-parameter group G_T is equivalent to the determination of the corresponding infinitesimal transformation generated by the infinitesimal generator. One-parameter groups are obtained by their corresponding generator either by Lie equations or by the exponential map.

3.2.2 The invariance criterion

consider an rth order PDE in s independent variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and one dependent variable u, given by

$$F(\mathbf{x}, u, u^{(1)}, \ldots, u^{(r)}) = 0, \tag{3.2}$$

where $u^{(r)}$, denotes the set of coordinates corresponding to all rth order partial derivatives of u with respect to $x_1, x_2, \ldots, x_n$. That is, a coordinate $u^{(r)}$ is denoted by

$$u_{j_1 j_2 \dots j_k} = \frac{\partial^r u}{\partial x_{j_1} \partial x_{j_2} \dots \partial x_{j_r}}, \tag{3.3}$$

with $j_q = 1, 2, \dots, n$ and $q = 1, 2, \dots, r$. Note that in case of the ODE, one considers one independent and one dependent variable. To determine the symmetry for the Eq. (3.2), one may seek transformations of the form

$$\overline{\mathbf{x}}_j = x_j + \epsilon \xi_j(\mathbf{x}, u) + O(\epsilon^2), \tag{3.4}$$

$$\overline{u} = u + \epsilon \eta(\mathbf{x}, u) + O(\epsilon^2) \tag{3.5}$$

called infinitesimal transformations. The coefficients ξ_j and η are the components of the infinitesimal generator acting on the $(\mathbf{x}, u)$ space given by

$$\Gamma = \xi_j(\mathbf{x}, u) \frac{\partial}{\partial x_j} + \eta(\mathbf{x}, u) \frac{\partial}{\partial u}, \tag{3.6}$$

which leaves the DE invariant. The action of Γ is extended in the governing equation through the rth prolongation given by

$$\Gamma^{[r]} = \Gamma + \eta_1(\mathbf{x}, u, u^{(1)}) \frac{\partial}{\partial u_j} + \dots + \eta_{j_1 j_2 \dots j_r}(\mathbf{x}, u, u^{(1)}, \dots, u^{(r)}) \frac{\partial}{\partial u_{j_1 j_2 \dots j_r}}, \tag{3.7}$$

where $r = 1, 2, \dots$

$$\eta_j = D_j(\eta) - u_k D_j(\xi_k), \; j = 1, 2, \dots, n; \tag{3.8}$$

$$\eta_{j_1 j_2 \dots j_r} = D_{jr}(\eta_{j_1 j_2 \dots j_{r-1}}) - u_{j_1 j_2 \dots j_{r-1} k} D_{jr}(\xi_k), \tag{3.9}$$

with $j_q = 1, 2, \dots, n$ and $q = 1, 2, \dots, r$, $r = 2, 3, \dots$ and D_j being the total x_j derivative operator defined by

$$D_j = \frac{\partial}{\partial x_j} + u_j \frac{\partial}{\partial u} + \dots + u_{j_1 j_2 \dots j_r} \frac{\partial}{\partial u_{j_1 j_2 \dots j_r}}. \tag{3.10}$$

The invariance criterion for symmetry determination is given by

$$\Gamma^{[r]} \left. (F = 0) \right|_{F=0} = 0. \tag{3.11}$$

Since the coefficients of Γ do not involve derivatives, we can separate (3.11) with respect to the derivatives of u and solve the resulting overdetermined system of linear homogeneous partial differential equations known as *determining equations.* The calculation is algorithmic and maybe facilitated by computer software such as REDUCE [35] or YaLie [57].

It may happen that the only solution to the overdetermined system of linear equations is trivial. When the general solution of the determining equations is nontrivial, two cases arise: (a) if the general solution contains a finite number, p, of essential arbitrary constants, then it corresponds to a p-parameter Lie algebra spanned by the base vectors (3.6); and (b) if the general solution cannot be expressed in terms of a finite number of essential constants, for example when it contains an arbitrary function of independent and/or dependent variables, then it corresponds to an infinite-parameter Lie group of transformations of the infinite-dimensional symmetry generator.

Illustrative example 3.1

As an illustrative example, we consider a nonlinear ODE

$$\frac{d}{dx}\left[\theta^m \frac{d\theta}{dx}\right] - M^2\theta^{n+1} = 0. \tag{3.12}$$

Given $n = -3m - 4$, and considering the transformation $y = \theta^{m+1}$, then Eq. (3.12) becomes the Ermakov-Pinney type equation [58]

$$y'' = (m+1)M^2y^{-3}. \tag{3.13}$$

The invariance criterion for symmetry determination is given by

$$\Gamma^{[2]} \quad \left(y'' - (m+1)M^2y^{-3}\right)\big|_{y''=(m+1)M^2y^{-3}} = 0, \tag{3.14}$$

where

$$\Gamma^{[2]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta_1\frac{\partial}{\partial y'} + \eta_2\frac{\partial}{\partial y''} \tag{3.15}$$

is the second prolongation of symmetry generator and the coefficients η_1 and η_2 are determined from the equations

$$\eta_1 = D_x(\eta) - y' D_x(\xi), \tag{3.16}$$

$$\eta_2 = D_x(\eta_1) - y'' D_x(\xi). \tag{3.17}$$

The total derivative D_x is given by

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \ldots. \tag{3.18}$$

The resulting determining equations are given by

$$\xi_{yy} = 0, \tag{3.19}$$

$$\eta_{yy} - 2\xi_{xy} = 0, \tag{3.20}$$

$$2y^3\eta_{xy} - y^3\xi_{xx} + 3(m+1)M^2\xi_y = 0, \tag{3.21}$$

$$y^4\eta_{xx} - (m+1)M^2 y\eta_y + 2(m+1)M^2 y\xi_x - 3(m+1)M^2\eta = 0. \tag{3.22}$$

Solving Eqs. (3.19)-(3.22), yield the symmetry generators

$$\Gamma_1 = \frac{\partial}{\partial x}, \tag{3.23}$$

$$\Gamma_2 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \tag{3.24}$$

$$\Gamma_3 = x\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y}. \tag{3.25}$$

One can easily show that symmetry generators (3.23)-(3.25) span the $SL(2,\mathbb{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R},\ ad - bc = 1\}$ Lie algebra (see e.g. [59]).

3.3 Lie algebras

Lie algebras are discussed in this section.. The reader is referred to Bluman *et al* [46, 47] for more details. Consider the infinitesimal generators

$$\Gamma_1 = \xi_1^i \frac{\partial}{\partial x^i} + \eta_1 \frac{\partial}{\partial u} \tag{3.26}$$

and

$$\Gamma_2 = \xi_2^i \frac{\partial}{\partial x^i} + \eta_2 \frac{\partial}{\partial u}. \tag{3.27}$$

The Lie bracket or the commutator of Γ_1 and Γ_2 is defined as

$$[\Gamma_1, \Gamma_2] = (\Gamma_1 \xi_2^i - \Gamma_2 \xi_1^i) \frac{\partial}{\partial x^i} + (\Gamma_1 \eta_2 - \Gamma_2 \eta_1) \frac{\partial}{\partial u} = \Gamma_1 \Gamma_2 - \Gamma_2 \Gamma_1. \tag{3.28}$$

The nonzero commutator of any two infinitesimal generator is also, an infinitesimal generator. It follows from (3.28) that the Lie bracket is skew-symmetric that is

$$[\Gamma_1, \Gamma_2] = -[\Gamma_2, \Gamma_1]. \tag{3.29}$$

Furthermore, any three infinitesimal generators $\Gamma_1, \Gamma_2, \Gamma_3$ satisfy Jacobi's identity

$$[[\Gamma_1, \Gamma_2], \Gamma_3] + [[\Gamma_2, \Gamma_3], \Gamma_1] + [[\Gamma_3, \Gamma_1], \Gamma_2] = 0. \tag{3.30}$$

A Lie algebra $\mathcal{L}$ is a vector space over some vector field $\mathcal{F}$ with an additional law of combination of elements in $\mathcal{L}$ satisfying the properties skew-symmetry and the Jacobi's identity. Furthermore, the following axioms hold;

the commutator is bilinear,

$$[a\Gamma_1 + b\Gamma_2, \Gamma_3] = a[\Gamma_1, \Gamma_3] + b[\Gamma_2, \Gamma_3] \tag{3.31}$$

$$[\Gamma_1, a\Gamma_2 + b\Gamma_3] = a[\Gamma_1, \Gamma_2] + b[\Gamma_1, \Gamma_3] \tag{3.32}$$

where a, b are arbitrary constants. The infinitesimal generators (or base vectors) span the Lie algebra.

It is more convenient to represent the Lie brackets in a commutator table. For example, considering the generators in (3.23)-(3.25), we construct the commutator Table. 3.1.

Table 3.1: Lie Bracket of the admitted symmetry algebra for (3.13)

$[\Gamma_i, \Gamma_j]$	Γ_1	Γ_2	Γ_3
Γ_1	0	$2\Gamma_3$	Γ_1
Γ_2	$-2\Gamma_3$	0	$-\Gamma_2$
Γ_3	$-\Gamma_1$	Γ_2	0

3.4 One-dimensional optimal system of subalgebras

Essentially there are two ways of constructing the one-dimensional optimal system of subalgebras (see, e.g., [60]), one is by Ovsiannikov [54, 61], based on determining the matrix of inner automorphism corresponding to the operators of the adjoint group of a given Lie algebra. The other method is by Olver [51] whereby the generator is simplified as much as possible by subjecting it to chosen adjoint transformation. Here we adopt and briefly discuss Olver's method.

Suppose that a PDE of the form (3.2) admits an s dimensional Lie algebra $\mathcal{L}_s$ viz., $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$. Reductions of independent variables by one is possible using any linear combination of base vectors

$$\Gamma = a_1\Gamma_1 + a_2\Gamma_2 + \ldots + a_s\Gamma_s \tag{3.33}$$

An optimal system [53, 54] is constructed to ensure that a minimal complete set of reduction is obtained from symmetries admitted by the governing equation. An optimal system of a Lie algebra is a set of r dimensional subalgebras such that every r dimensional subalgebra is equivalent to a unique element of the set under some element of the adjoint representation;

$$\mathrm{Ad}\left(e^{\epsilon\Gamma_i}\right)\Gamma_j = \sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}\left(\mathrm{Ad}\Gamma_i\right)^n\Gamma_j = \Gamma_j - \epsilon\left[\Gamma_i,\Gamma_j\right] + \frac{\epsilon^2}{2!}\left[\Gamma_i,\left[\Gamma_i,\Gamma_j\right]\right] - \ldots, \quad (3.34)$$

where $[\Gamma_i, \Gamma_j]$ is the commutator of Γ_i and Γ_j. Patera and Winternitz [62] constructed the optimal system of all one dimensional Lie subalgebras arising from three and four dimensional Lie algebras by comparing the Lie algebra with standard classifications previously evaluated. An alternative method developed by Olver [51] involves simplifying as much as possible the generator (3.35) by subjecting it to chosen adjoint transformations.

Illustrative example 3.2

Consider the three dimensional Lie algebra spanned by the base vectors (3.23)-(3.25). Their Lie brackets are shown in the commutator Table. 3.1. Reduction of order of Eq. (3.13) by one is possible using any linear combination

$$\Gamma = a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 \quad (3.35)$$

where a_1, a_2, a_3 are real constants. We need to simplify as much as possible the coefficients a_1, a_2, a_3 by carefully applying the adjoint maps to Γ. To compute the optimal system we first need to determine the Lie brackets as given for example, in Table. 3.2. Using (3.34) in conjunction with the commutator Table 3.1, for example

$$\mathrm{Ad}(\exp(\epsilon\Gamma_1))\Gamma_3 = \Gamma_3 - \epsilon[\Gamma_1,\Gamma_3] + \frac{\epsilon^2}{2!}[\Gamma_1,[\Gamma_1,\Gamma_3]] - \ldots = \Gamma_3 - \epsilon\Gamma_1$$

we construct the adjoint representation table shown in Table. 3.2.

Table 3.2: Adjoint representation table for (3.13)

$\mathrm{Ad}\,(\exp(\epsilon\Gamma_i))\,\Gamma_j$	Γ_1	Γ_2	Γ_3
Γ_1	Γ_1	$\Gamma_2 - 2\epsilon\Gamma_3 + \epsilon^2\Gamma_1$	$\Gamma_3 - \epsilon\Gamma_1$
Γ_2	$\Gamma_1 + 2\epsilon\Gamma_3 + \epsilon^2\Gamma_2$	Γ_2	$\Gamma_3 + \epsilon\Gamma_2$
Γ_3	$e^{\epsilon}\Gamma_1$	$e^{-\epsilon}\Gamma_2$	Γ_3

Starting with a nonzero vector (3.35) with $a_3 \neq 0$ and rescaling such that $a_3 = 1$, it follows from Table. 3.2 that acting on Γ by $\mathrm{Ad}\left(\exp\left(\frac{a_2}{2}\Gamma_2\right)\right)$, one obtains $\Gamma^I = \widetilde{a}_1\Gamma_1 + \Gamma_3$. Acting on Γ^I by $\mathrm{Ad}\,(\exp(c_2\Gamma_1))$ we get $\widetilde{a}_1 e^{-c_2}\Gamma_1 + \Gamma_3$. Depending on the sign of $\widetilde{a}_1$, the coefficient of Γ_1 can be assigned either $+1, -1$, or 0. Next, suppose that $a_3 = 0$, and assume that $a_2 \neq 0$ (say $a_2 = 1$ by rescaling); acting on the remaining vector by $\mathrm{Ad}\,(\exp(c_3\Gamma_2))$, we obtain $a_1 e^{c_3}\Gamma_1 + \Gamma_2$. Like before the coefficient of Γ_1 can be assigned either $+1, -1$ or 0. Eventually, we choose $a_1 \neq 0$ (say $a_1 = 1$ by rescaling) and this yield Γ_1. Thus the one dimensional optimal system is $\{\Gamma_3, \Gamma_3 \pm \Gamma_1, \Gamma_2 \pm \Gamma_1, \Gamma_2, \Gamma_1\}$.

3.5 Basis of invariants

In the method of variable reduction by invariants one seeks a compatible invariant solution expressed in the form

$$\mu(\varphi_1, \varphi_2, \ldots, \varphi_n) = 0; \tag{3.36}$$

where $\varphi_1, \varphi_2, \ldots, \varphi_n$ is a complete set of n independent invariants for a one-parameter Lie point transformation group (3.4) and (3.5).

The basis for invariants may be constructed by solving the characteristics equations in Pfaffian form, corresponding to the (3.6) and (3.11)

$$\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \ldots = \frac{du}{\eta} \tag{3.37}$$

A one-parameter group of transformations is a classical symmetry of (3.2), provided (3.4) and (3.5) leave (3.2) invariant. Moreover, an invariant solution of (3.2),

$$u = G(\mathbf{x}), \tag{3.38}$$

must satisfy the invariant surface condition (I.S.C.)

$$\sum_j \xi_j(\mathbf{x}, u)\frac{\partial u}{\partial x_j} = \eta(\mathbf{x}, u), \tag{3.39}$$

which follows from

$$\frac{d}{d\epsilon}(\overline{u} - G(\overline{\mathbf{x}})) = 0, \tag{3.40}$$

Illustrative example 3.3

The equation

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left[\theta^m \frac{\partial \theta}{\partial x}\right] - M^2\theta^{n+1} \tag{3.41}$$

admits among others symmetry generator

$$\Gamma = \left(\frac{m-n}{2}\right) x\frac{\partial}{\partial x} + \theta\frac{\partial}{\partial \theta} - n\tau\frac{\partial}{\partial \tau}. \tag{3.42}$$

By solving the characteristic equation

$$\frac{d\tau}{-n\tau} = \frac{dx}{\left(\frac{m-n}{2}\right)x} = \frac{d\theta}{\theta} \tag{3.43}$$

we obtain the basis of invariants

$$\gamma = x\tau^{\frac{m-n}{2n}} \qquad \text{and} \qquad \theta = \tau^{-\frac{1}{n}}G(\gamma). \tag{3.44}$$

Substituting (3.44) into the governing equation (3.41), one obtains the second order ODE

$$[G^m G']' + \frac{1}{n}G - \gamma G' - M^2 G^{n+1} = 0. \tag{3.45}$$

Although Eq. (3.45) may not be solved exactly, the application of the asymmetry generator has resulted in the reduction, which may be easier to solve numerically.

3.6 Methods of linearization and reductions of ODEs.

The second-order (in particular nonlinear) ODE admitting two, three, or eight-dimensional Lie algebra may be integrated completely using two-dimensional Lie (sub)algebra. The two admitted symmetries by a second-order ODE may either linearize the original equation or give rise to a second-order equation in terms of the canonical coordinates, with cubic as the highest degree of the first-order derivative. Note that any linear second order ODE admitting eight symmetries is equivalent to the simple equation $y'' = 0$.

Suppose that a given equation admits a non-Abelian two-dimensional algebra (subalgebra). That is, suppose the admitted Lie algebra is given by $[\Gamma_1, \Gamma_2] = \lambda_1 \Gamma_1, \quad \lambda_1 \in \mathbb{R}$. Following reduction of the original equation by Γ_1, Γ_2 in new variables is automatically admitted by the reduced equation. Such symmetries are referred to as *inherited symmetries.*

In the case where the admitted symmetry algebra is one-dimensional or when one considers the one-dimensional subalgebra, the order of equation may be reduced by one using the method of differential invariants discussed below.

3.6.1 Method of differential invariants

This method involves determining invariants from the first prolongation of the given symmetries. Given a second order ODE, then the order of this equation may be reduced by one upon determining the invariants from the first prolongation of the given symmetry generator. To illustrate, we consider the example below.

Illustrative example 3.4

Given the symmetry generator Γ_2 in (3.24), the first prolongation is given by

$$\Gamma_2^{[1]} = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + (y - xy')\frac{\partial}{\partial y'}. \tag{3.46}$$

The basis of invariants may be constructed by solving the equations.

$$\frac{dx}{x^2} = \frac{dy}{xy} = \frac{dy'}{y - xy'}. \tag{3.47}$$

The invariants are therefore

$$t = \frac{y}{x}, \quad u = y - xy'. \tag{3.48}$$

Writing $u = u(t)$, one obtains

$$u\frac{du}{dt} = \frac{(m+1)M^2}{t^3}, \tag{3.49}$$

which is an equation with separable variables. The exact solution is given by

$$\frac{u^2}{2} + \frac{(m+1)M^2}{2t^2} + k_1 = 0, \tag{3.50}$$

Where k_1 is an integral constant. We obtain, in terms of the original variables

$$y^2(y - xy')^2 + (m+1)M^2x^2 + K_1y^2 = 0. \tag{3.51}$$

where $K_1 = 2k_1$.

3.6.2 Lie's method of canonical coordinates

This method involves reduction of the second order ODE using two dimensional Lie (sub)algebra. Any two dimensional Lie algebra can be transformed using choice of basis and canonical variables t and u. Furthermore, (see e.g. [63])

(i) A second order ODE admitting a commutating pair of symmetries Γ_1 and Γ_2 that is, $[\Gamma_1, \Gamma_2] = 0$, such that a point transformation $t = \phi(x, y)$ and $u = \psi(x, y)$ which bring the canonical form to

(a)

$$\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial u} \quad \text{and}$$

(b)

$$\Gamma_1 = \frac{\partial}{\partial u}, \quad \Gamma_2 = t\frac{\partial}{\partial u}$$

reduce the original equation into

(a)

$$u'' = f(u') \quad \text{and}$$

(b)

$$u'' = f(u) \quad \text{respectively.}$$

(ii) A second order ODE admitting non-commutating symmetries Γ_1 and Γ_2 i.e. $[\Gamma_1, \Gamma_2] = \Gamma_1$, such that a point transformation $t = \phi(x, y)$ and $u = \psi(x, y)$ which bring the canonical form to

(a)

$$\Gamma_1 = \frac{\partial}{\partial u}, \quad \Gamma_2 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} \quad \text{and}$$

(b)

$$\Gamma_1 = \frac{\partial}{\partial u}, \quad \Gamma_2 = u\frac{\partial}{\partial u}$$

reduce the original equation into

(a)

$$u'' = \frac{1}{t} f(u') \quad \text{and}$$

(b)

$$u'' = f(t)u' \quad \text{respectively.}$$

Note that (a) is an equation that is at most cubic in the first derivative and (b) is linear. For a detailed account on reductions of ODEs, in particular second-order ODEs, by Lie point symmetries, the reader is referred to [63].

Illustrative example 3.5

As an illustration, we observed that Eq. (3.13) admits a three dimensional Lie algebra spanned by the base vectors given in (3.23)-(3.25). The non commuting pair of symmetries Γ_1 and Γ_3 leads to the canonical variables

$$t = y^2, \qquad u = x + y^2 \tag{3.52}$$

The corresponding canonical forms of Γ_1 and Γ_3 are

$$\Gamma_1^* = \frac{\partial}{\partial u}, \qquad \Gamma_3^* = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} \tag{3.53}$$

Writing $u = u(t)$ transforms Eq. (3.13) to

$$u'' = -\frac{1}{2}\frac{u' - 1}{t}\left[1 + 4(m+1)M^2(u' - 1)^2\right] \tag{3.54}$$

Here the prime denotes the total derivative with respect to t.

Solving equation Eq. (3.54), we obtain a solution that satisfies the Neumann boundary condition at $x = 0$, and the Dirichlet condition at $x = 1$, namely

$$\theta = \left[1 + (m+1)M^2(x^2 - 1)\right]^{\frac{1}{2(m+1)}}. \tag{3.55}$$

3.7 Tanh Method

We outline the fundamentals of Tanh method to solve nonlinear differential equations. For more details, we refer to [64–67]. A wave variable $\xi = k(x - Vt)$ changes one-dimensional nonlinear partial differential equation (PDE)

$$\theta_t = G(\theta, \theta_x, \theta_{xx}, \cdots) \tag{3.56}$$

or

$$\theta_{tt} = G(\theta, \theta_x, \theta_{xx}, \cdots) \tag{3.57}$$

into an ordinary differential equation (ODE)

$$-kV\frac{d\theta}{d\xi} = G\left(\theta, k\frac{d\theta}{d\xi}, k^2\frac{d^2\theta}{d\xi^2}, \cdots\right) \tag{3.58}$$

upon using a wave variable $\xi = x - Vt$. Eq.(2.20) is then integrated as long as all terms containing derivatives where integration constants are taken to be zeros. Introducing a new independent variable

$$Y = \tanh(\xi) \tag{3.59}$$

leads to the derivatives

$$\frac{d}{d\xi} = (1 - Y^2)\frac{d}{dY} \tag{3.60}$$

$$\frac{d^2}{d\xi^2} = (1 - Y^2)(-2Y\frac{d}{dY}) + (1 - Y^2)\frac{d^2}{dY^2} \tag{3.61}$$

It makes sense to attempt to find the solution(s) as a finite power series in Y,

$$F(Y) = \sum_{n=0}^{N} a_n Y^n \tag{3.62}$$

which include solitary-wave and shock-wave profiles. Integer N can be determined by considering homogeneous balance between the governing nonlinear term(s) and highest order derivatives of $\theta(\xi)$ in Eq.(2.20). The coefficients a_n follow from solving a nonlinear algebraic system. If, needed boundary conditions may be employed within the method. Having determined these coefficients, we get an analytic solution $\theta(x, t)$ in a closed-form.

3.8 Concluding Remarks

In this chapter, a brief outline of Lie symmetry technique was provided. Both historical and theoretical backgrounds of the field of symmetry analysis were discussed. A connection between the one-parameter group of transformations and corresponding infinitesimal transformations was provided. Furthermore, we discussed the determination of local and nonlocal symmetries, the notion of equivalence transformations, and Lie algebras. Also, we discussed the steps for the construction of optimal systems. The use of symmetries admitted by both the ODEs and PDEs are discussed and illustrated. Lastly, we gave a brief outline of the Tanh method for solving PDEs

Chapter 4

Analysis of heat transfer in a longitudinal fin of rectangular profile: Exact solutions

4.1 Introduction

We consider a longitudinal fin of a rectangular profile in this chapter. A mathematical model describing the problem is given in Section 4.2. In Section 4.3, we consider the classical Lie point symmetry analysis of a partial differential equation. We discuss exact analytical results in Section 4.4. In Section 4.5, we analyze the longitudinal fin using the Tanh method and give concluding remarks in Section 4.6.

4.2 Mathematical models

We consider a longitudinal one-dimensional fin as shown in Fig.2.2. The one dimensional heat balance equation is given by

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left[\theta^m \frac{\partial \theta}{\partial x}\right] - M^2\theta^{n+1}, \qquad 0 < x < 1, \tag{4.1}$$

with the initial and the step change in base temperature conditions

$$\theta(x,0) = 0, \quad \theta(1,\tau) = 1, \quad \left.\frac{\partial \theta}{\partial x}\right|_{x=0} = 0, \ \tau \geq 0, \tag{4.2}$$

and the step change in base heat flux conditions are given by

$$\theta(x,0) = 0, \quad \left.\frac{\partial \theta}{\partial x}\right|_{x=1} = 1, \quad \left.\frac{\partial \theta}{\partial x}\right|_{x=0} = 0, \ \tau \geq 0. \tag{4.3}$$

We solve (4.1) using Lie symmetries.

4.3 Classical Lie point symmetry analysis of a PDE

We determine symmetries for the governing equation (4.1), which implies seeking transformation of the form

$$\left.\begin{aligned} \overline{x} = &\ \ x + \epsilon\xi^1(\tau,x,\theta) + O(\epsilon^2) \\ \overline{\tau} = &\ \ \tau + \epsilon\xi^2(\tau,x,\theta) + O(\epsilon^2) \\ \overline{\theta} = &\ \ \theta + \epsilon\eta(\tau,x,\theta) + O(\epsilon^2) \end{aligned}\right\} \tag{4.4}$$

generated by the vector field

$$\Gamma = \xi^1(\tau,x,\theta)\frac{\partial}{\partial x} + \xi^2(\tau,x,\theta)\frac{\partial}{\partial \tau} + \eta(\tau,x,\theta)\frac{\partial}{\partial \theta}, \tag{4.5}$$

which leave the governing equation invariant. The invariance surface condition is given by

$$\Gamma^{[2]}\left(\text{Eq. (4.1)}\right)\Big|_{\text{Eq. (4.1)}} = 0. \tag{4.6}$$

Here $\Gamma^{[2]}$ is the second prolongation defined by

$$\Gamma^{[2]} = \Gamma + \zeta^{\tau}\frac{\partial}{\partial\theta_{\tau}} + \zeta^{x}\frac{\partial}{\partial\theta_{x}} + \zeta^{xx}\frac{\partial}{\partial\theta_{xx}}, \tag{4.7}$$

where the prolongation formulae are defined by

$$\begin{aligned}
\zeta^{\tau} &= D_{\tau}(\eta) - \theta_x D_{\tau}(\xi^1) - \theta_{\tau} D_{\tau}(\xi^2),\\
\zeta^{x} &= D_{x}(\eta) - \theta_x D_{x}(\xi^1) - \theta_{\tau} D_{x}(\xi^2),\\
\zeta^{xx} &= D_{x}(\zeta_x) - \theta_{xx} D_{x}(\xi^1) - \theta_{x\tau} D_{x}(\xi^2)
\end{aligned}$$

and D_x and D_τ are the operators of total differentiation with respect to x and τ, respectively.

The *determining equations* of (4.6) were calculated using the package Dimsym [34], which is a subprogram of Reduce [35].

4.3.1 Admitted Lie point symmetries

Three symmetries were obtained [28]

$$\Gamma_1 = \partial_\tau,\ \Gamma_2 = \partial_x,\ \Gamma_3 = \frac{1}{m}\Big\{ -n\tau\partial_\tau + \frac{m-n}{2}x\partial_x + \theta\partial_\theta\Big\} \tag{4.8}$$

We followed [28] and constructed the one dimensional optimal system of subalgebras and obtain the set

$$\{\Gamma_3,\ \Gamma_2 \pm \Gamma_1,\ \Gamma_2,\ \Gamma_1\} \tag{4.9}$$

The reductions by these elements of the one-dimensional optimal system are given in Table 4.1 [28].

Table 4.1: Reductions by elements of the optimal systems (5.5)

Symmetries	Reductions
Γ_3	$\gamma = x\tau^{(m-n)/2n}, \ \theta = \tau^{-1/n}H(\gamma)$ where
	H satisfies $-\frac{1}{n}H + \gamma H' = [H^m H']' - M^2 H^{n+1}$
$\Gamma_2 \pm \Gamma_1$	$\gamma = x \pm a\tau, \ \theta = H(\gamma)$ where H satisfies $\pm aH = [H^m H']' - M^2 H^{n+1}$

4.3.2 Some symmetry reductions

The admitted symmetries have reduced the number of independent variables of a partial differential equation by one (see Table 4.1). The reduced ODE may or may not be solved. Even when explicit exact analytical solutions may not be constructed, the reduction of variables has a number of advantages. The nonlinear boundary value problems may be easily solved by numerical methods. The steady-state and transient-state equations are solved, and Maple [32] used for graphical representation of solutions.

Example 4.1: steady state solutions for $n = m = -1$

The use of $\Gamma_1 = \partial_\tau$ yields a steady state heat transfer, which means that Eq. (4.1) is invariant under translation. The steady state equation with $n = m = -1$ is given by

$$\frac{d}{dx}\left[\theta^{-1}\frac{d\theta}{dx}\right] - M^2 = 0, \tag{4.10}$$

subject to

(i) the step change in base temperature conditions

$$\theta'(0) = 0, \ \theta(1) = 1 \text{ and} \tag{4.11}$$

(ii) the step change in base heat flow conditions

$$\theta'(0) = 0, \ \theta'(1) = 1 \tag{4.12}$$

The exact analytical solution to Eq. (4.10) subject to conditions (4.11) is

$$\theta = e^{\frac{M^2}{2}(x^2-1)}. \tag{4.13}$$

Solution (4.13) is depicted in Fig. 4.1. The exact analytical solution to Eq. (4.10) subject to conditions (4.12) is

$$\theta = \frac{1}{M^2} e^{\frac{M^2}{2}(x^2-1)} \tag{4.14}$$

Solution (4.14) is depicted in Fig. 4.2.

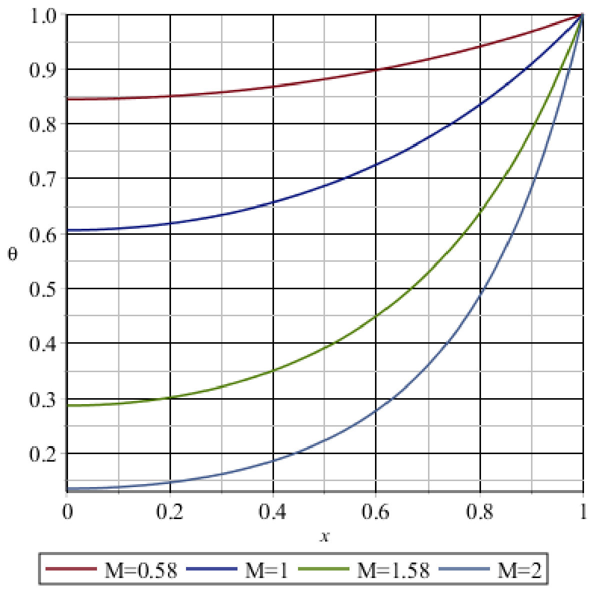

Figure 4.1: Exact analytical steady state solution for step change in base temperature condition for different M values $n = m = -1$.

Example 4.2: steady state solutions for $m \neq -1, n = -1$

The steady state equation with $m \neq -1, n = -1$ is given by

$$\frac{d}{dx}\left[\theta^m \frac{d\theta}{dx}\right] - M^2 = 0, \tag{4.15}$$

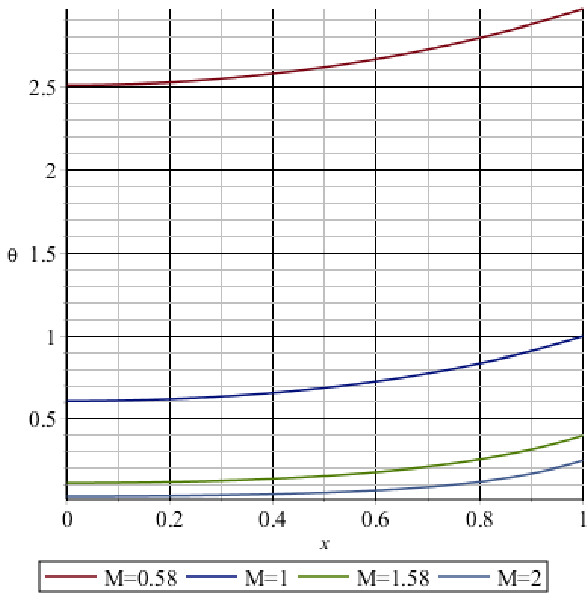

Figure 4.2: Exact analytical steady state solution for step change in base heat flow condition for different M values $n = m = -1$.

subject to (4.11) and (4.12). The exact analytical solution to Eq. (4.15) subject to conditions (4.11) is

$$\theta = \left[\frac{(m+1)M^2}{2}(x^2-1)+1\right]^{\frac{1}{m+1}}, \; m < -1 \tag{4.16}$$

Solution (4.16) for different values of m and M are depicted in Figs. 4.3 and 4.4 respectively. The exact analytical solution to Eq. (4.15) subject to conditions (4.12) is

$$\theta = \left[\frac{(m+1)M^2}{2}(x^2-1)+M^{2(m+1)/m}\right]^{\frac{1}{m+1}}, \; m < -1 \tag{4.17}$$

Solution (4.17) for different values of m and M are depicted in Figs. 4.5 and 4.6 respectively.

Example 4.3: steady state solutions for $n = m \neq -1$

The steady state equation with $n = m \neq -1$ is given by

$$\frac{d}{dx}\left[\theta^m \frac{d\theta}{dx}\right] - M^2\theta = 0, \tag{4.18}$$

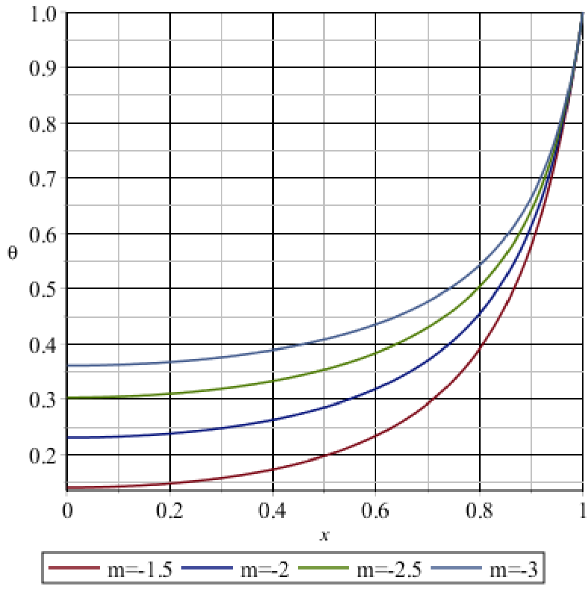

Figure 4.3: Exact analytical steady state solution for step change in base temperature condition for different m values $m \neq -1, n = -1$.

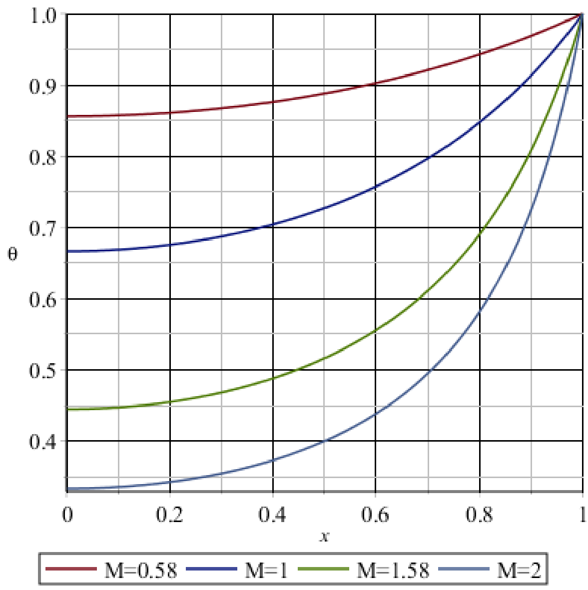

Figure 4.4: Exact analytical steady state solution for step change in base temperature condition for different M values $m \neq -1, n = -1$.

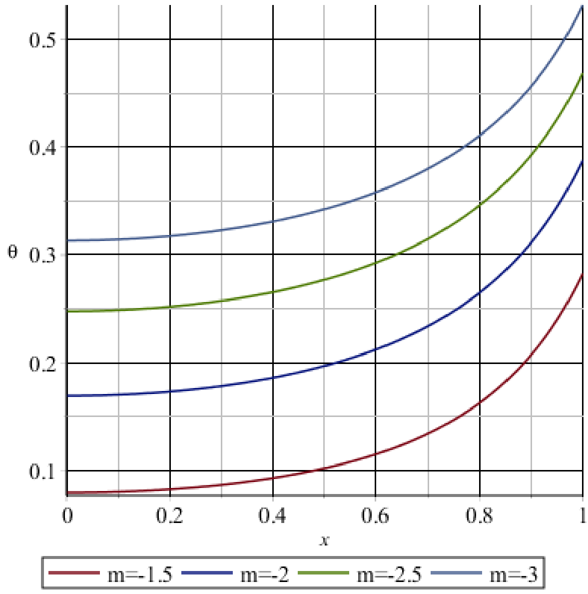

Figure 4.5: Exact analytical steady state solution for step change in base heat flow condition for different m values $m \neq -1, n = -1$.

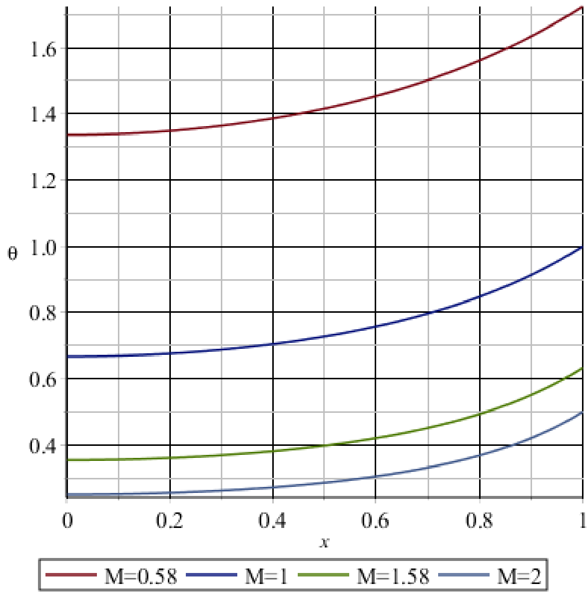

Figure 4.6: Exact analytical steady state solution for step change in base heat flow condition for different M values $m \neq -1, n = -1$.

subject to (4.11) and (4.12). If we substitute $y = \theta^{m+1}$ in (4.18) we get second order linear ODE

$$y'' - (m+1)M^2 y = 0. \tag{4.19}$$

The exact analytical solution to Eq. (4.18) subject to conditions (4.11) is

$$\theta = \left[\frac{\cosh\sqrt{m+1}Mx}{\cosh\sqrt{m+1}M}\right]^{\frac{1}{m+1}} \tag{4.20}$$

Solution (4.20) for different values of m and M are depicted in Figs. 4.7 and 4.8 respectively and Fig. 4.9 depicts fin efficiency. The exact analytical solution to Eq. (4.10) subject to conditions (4.12) is

$$\theta = \left[\frac{\cosh\sqrt{m+1}Mx}{\sqrt{m+1}M\sinh\sqrt{m+1}M}\right]^{\frac{1}{m+1}} \tag{4.21}$$

Solution (4.21) for different values of m and M are depicted in Figs. 4.10 and 4.11 respectively and Fig. 4.12 depicts fin efficiency.

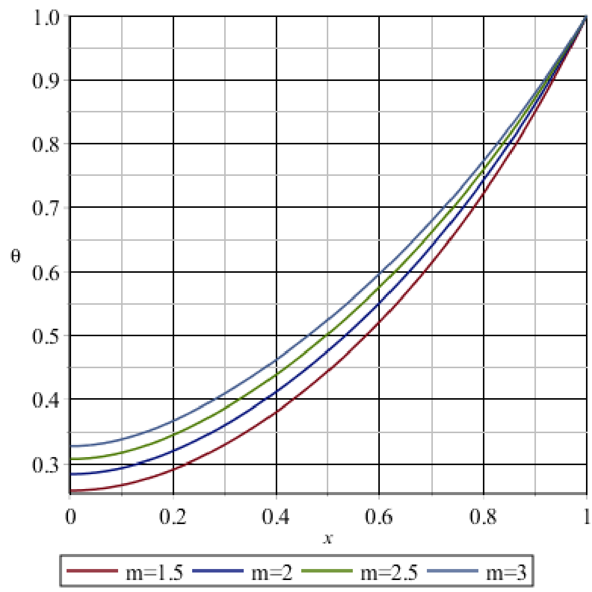

Figure 4.7: Exact analytical steady state solution for step change in base temperature condition for different m values $n = m \neq -1$.

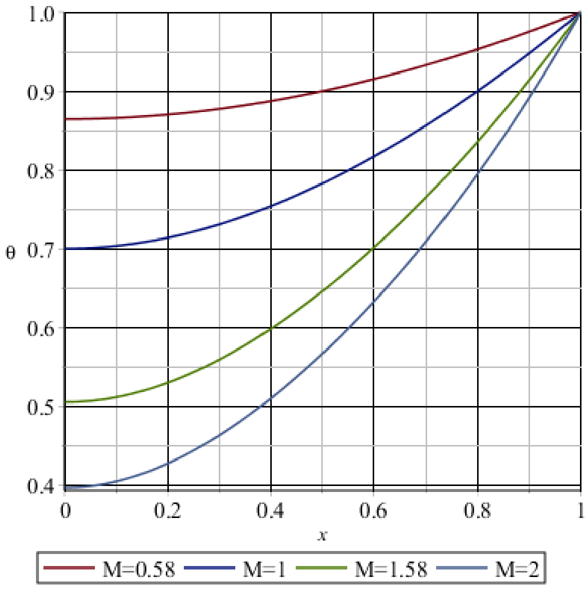

Figure 4.8: Exact analytical steady state solution for step change in base temperature condition for different M values $n = m \neq -1$.

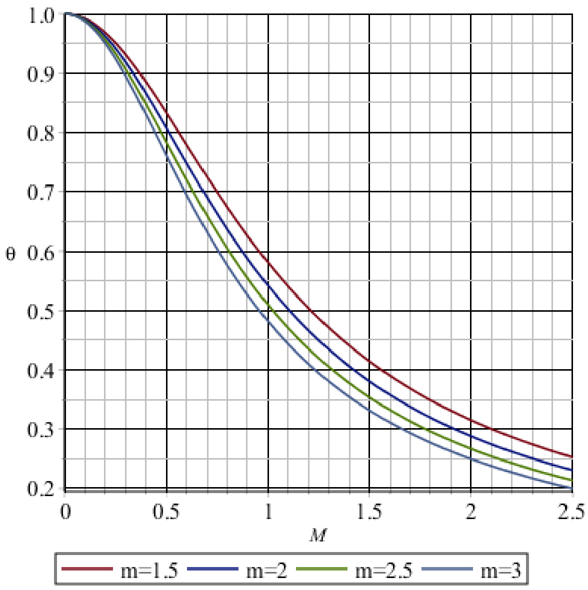

Figure 4.9: Efficiency for step change in base temperature condition for different m values $n = m \neq -1$.

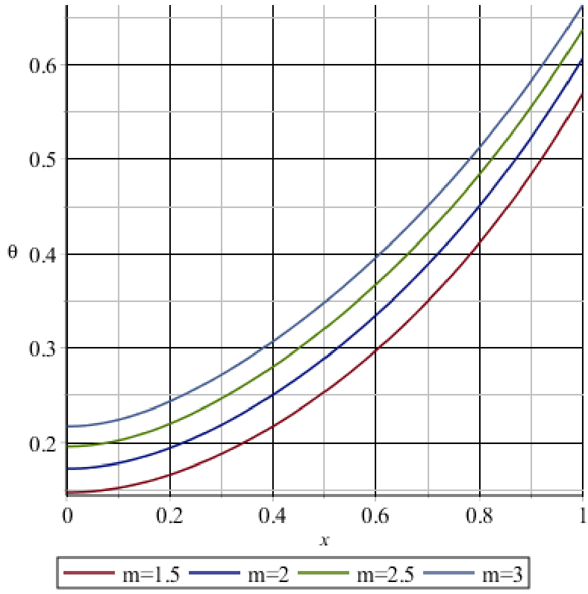

Figure 4.10: Exact analytical steady state solution for step change in base heat flow condition for different m values $n = m \neq -1$.

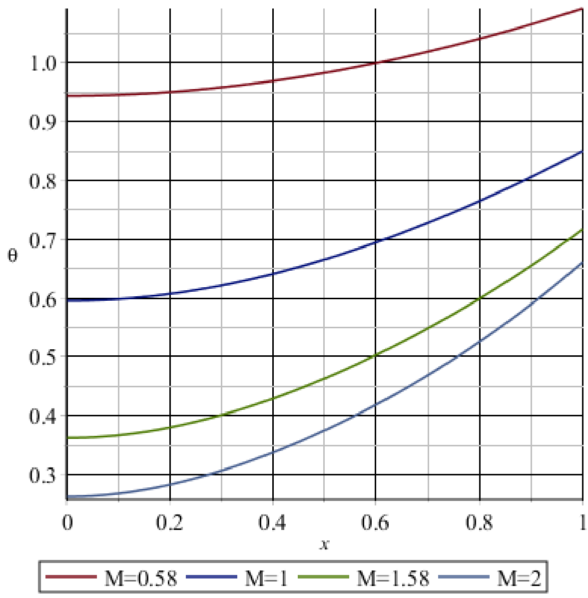

Figure 4.11: Exact analytical steady state solution for step change in base heat flow condition for different M values $n = m \neq -1$.

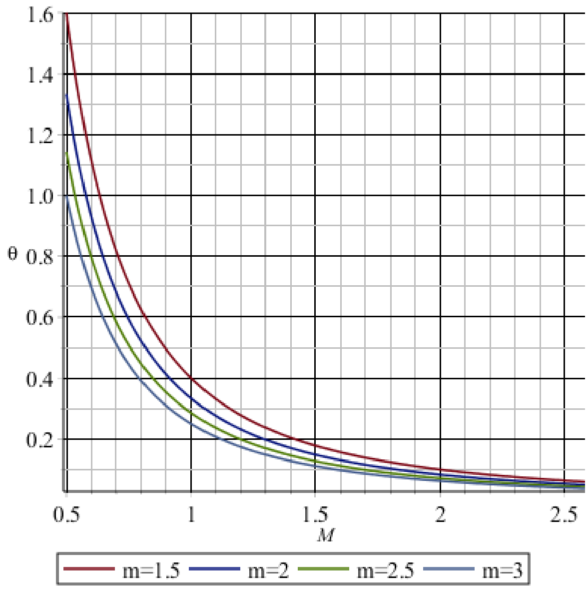

Figure 4.12: Efficiency for step change in base heat flow condition for different m values $n = m \neq -1$.

Example 4.4: transient heat flow $n = m = -1$

For $n = m = -1$, Γ_3 in (4.8) reduces (4.1) into a functional form of the exact solution

$$\theta = \tau H(\gamma)$$

where H satisfies an ODE

$$H + \gamma H' = [H^{-1}H']' - M^2 \tag{4.22}$$

The exact analytical solution to Eq. (4.22) subject to the conditions (4.11) is

$$\theta = \frac{\tau M^2}{\tau M^2 + e^{M^2/2} - e^{M^2 x^2/2}}. \tag{4.23}$$

Solution (4.23) for $M = 0.58$ and for different $M-$ values are shown in Figs. 4.13 and 4.14 respectively. The exact analytical solution to Eq. (4.22) subject to the conditions (4.12) is

$$\theta = \frac{\tau M^2}{\tau^{1/2} M^2 e^{M^2/4} + e^{M^2/2} - e^{M^2 x^2/2}} \tag{4.24}$$

Solution (4.24) for $M = 0.58$ and for different $M-$ values are shown in Figs. 4.15 and 4.16 respectively.

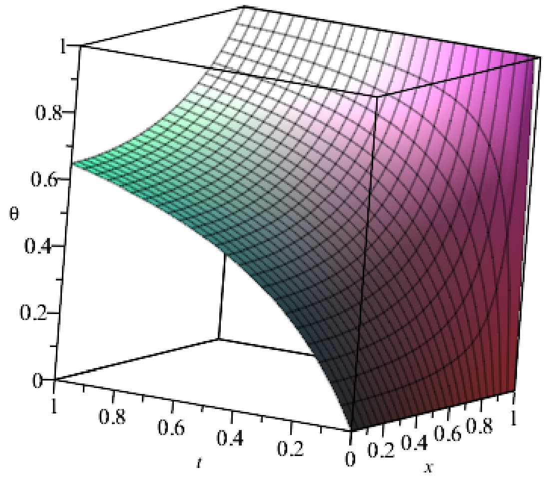

Figure 4.13: Exact analytical transient state solution for step change in base temperature condition $n = m = -1$ for $M = 0.58$.

Example 4.5: transient heat flow $n = m \neq -1$

We now consider reduction arising from $\Gamma_2 \pm \Gamma_1$ as shown in Table 4.1. If we substitute $y = H^{m+1}$ we get

$$\frac{d^2y}{d\gamma^2} - (m+1)M^2y = \pm(m+1)ay^{\frac{1}{m+1}}. \tag{4.25}$$

Eq. (4.25) is autonomous because it does not contain γ explicitly. If we let $y' = h(y)$ in (4.25) the equation is reduced to

$$h(y)h'(y) = \pm(m+1)ay^{\frac{1}{m+1}} + (m+1)M^2y, \tag{4.26}$$

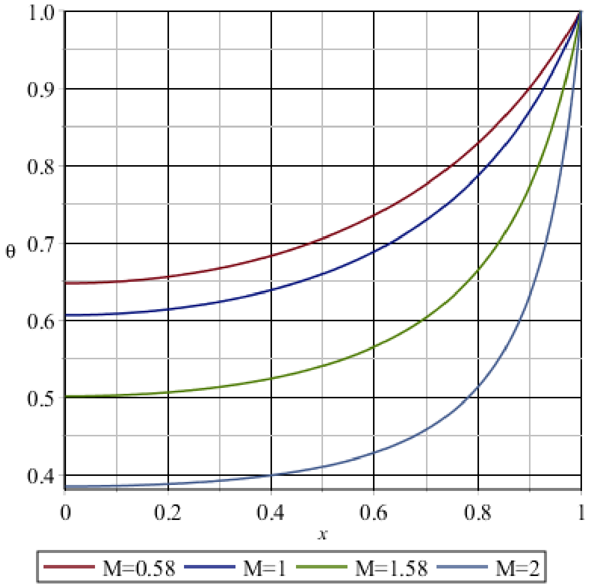

Figure 4.14: Exact analytical steady state solution for step change in base temperature condition for different m values $n = m = -1$ for different M-values.

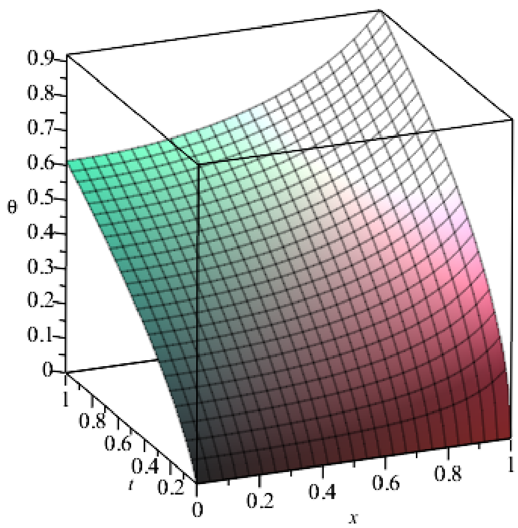

Figure 4.15: Exact analytical steady state solution for step change in base heat flow condition for different m values $n = m = -1$.

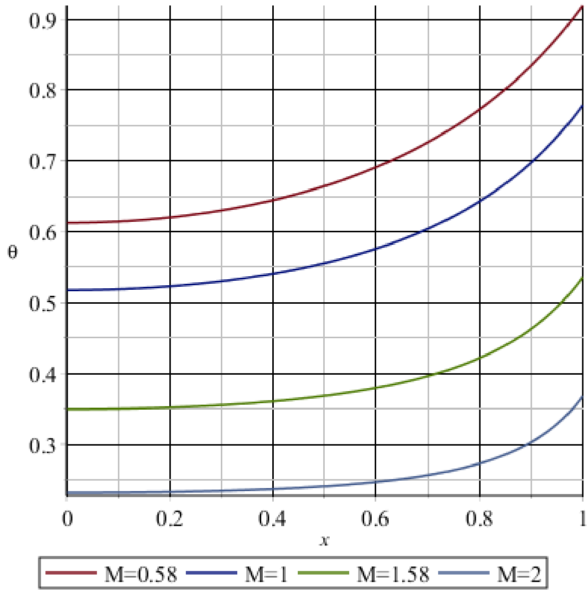

Figure 4.16: Exact analytical steady state solution for step change in base heat flow condition for different M values $n = m = -1$.

whose solution is

$$\frac{[h(y)]^2}{2} = \pm\frac{a(m+1)^2}{m+2}y^{\frac{m+2}{m+1}} + \frac{(m+1)M^2}{2}y^2 + C_1,$$

$$y'^2 = \pm\frac{2a(m+1)^2}{m+2}y^{\frac{m+2}{m+1}} + (m+1)M^2y^2 + 2C_1, \ m \neq -2. \tag{4.27}$$

We may, however, separate the variables and we need to carry with the integral of (4.27) yields

$$\int \frac{dy}{\sqrt{\pm\frac{2a(m+1)^2}{m+2}y^{\frac{m+2}{m+1}} + (m+1)M^2y^2 + 2C_1}} = \pm\int d\gamma. \tag{4.28}$$

If $m = -4/3$ we get

$$\int \frac{dy}{\sqrt{\pm\frac{a}{3}y^{-2} - \frac{M^2}{3}y^2 + 2C_1}} = \pm\int d\gamma, \tag{4.29}$$

we use the method of completing the square to get

$$\frac{\sqrt{3}}{M}\int \frac{ydy}{\sqrt{\left(\frac{9C_1^2 \pm aM^2}{M^4}\right) - \left(y^2 - \frac{3C_1}{M^2}\right)^2}} = \pm\int d\gamma. \tag{4.30}$$

Integration by trigonometric substitution yields

$$y^2 - \frac{3C_1}{M^2} = \sqrt{\frac{9C_1^2 \pm aM^2}{M^4}} \sin\Big[\frac{2M}{\sqrt{3}}(\pm\gamma + C_2)\Big], \tag{4.31}$$

and finally we make y the subject of the formula

$$y = \frac{1}{M}\sqrt{3C_1 + \sqrt{9C_1^2 \pm aM^2}\sin[\frac{2M}{\sqrt{3}} \pm (\gamma + C_2)]}. \tag{4.32}$$

We use the step change in temperature to conditions evaluate C_1 and C_2

$$y'(0) = 0, \text{ and } y(1) = 1 \tag{4.33}$$

$$\theta = \left[\frac{3C_1 + \sqrt{9C_1^2 \pm aM^2}\cos\Big[\pm\frac{2M}{\sqrt{3}}(x + at)\Big]}{M^2}\right]^{-3/2} \tag{4.34}$$

where $C_1 = \dfrac{M^2 + M\cos(\pm\frac{2M}{\sqrt{3}})\sqrt{M^2 \pm a\sin^2(\pm\frac{2M}{\sqrt{3}})}}{\sin^2(\pm\frac{2M}{\sqrt{3}})}$ and $C_2 = \frac{\pi\sqrt{3}}{4M}$.

4.4 Exact analytical results

We have obtained the exact analytical steady-state and transient-state solutions using the local symmetry techniques and the results are depicted in Figs. 4.1 - 4.16.

4.4.1 Steady-state results

In Fig. 4.1 - 4.8 we observe that the fin temperature increases with increasing spatial variable and magnitude of the exponent m. For step change in base temperature conditions depicted in Figs. 4.1, 4.3 - 4.4 and 4.7 - 4.8 the temperatures approach unity.

We also observe that the temperature decreases with increasing values of thermo-geometric fin parameter. $M = (Bi)^{1/2}E$, where $Bi = h_b\delta/k_a$ the Biot number and $E = L/\delta$ is the aspect ratio or the extension factor. Small values of M correspond to the relatively short and thick fins of high conductivity, and high values of M correspond to long and thin fins of poor conductivity [68]. A fin is an excellent dissipator at small values of M. We observe that temperature decreases as the values of the thermo-geometric fin parameter increase.

Lastly, Figs. 4.9 and 4.12 depict fin efficiency which decreases as the values of the thermo-geometric fin parameter increase. We also observe that the larger the magnitude of m, the smaller the fin efficiency.

4.4.2 Transient-state results

In Fig. 4.13 - 4.16 we observe that the fin temperature increases with increasing spatial variable.

Step change in base temperature

Fig. 4.13 depicts the temperature profile for increasing time and space. In Fig. 4.14 the temperature decreases with increasing values of the thermo-geometric fin parameter.

Step change in base heat flow

Fig. 4.15 depicts the temperature profile for increasing time and space. In Fig. 4.16 the temperature decreases with increasing values of the thermo-geometric fin parameter.

4.5 Analysis of longitudinal fin of rectangular profile using Tanh method

In this section, we apply the Tanh method to obtain solutions of nonlinear PDE with power-law thermal conductivity. A balancing procedure will be used to determine the degree of the power series, and boundary conditions will be imposed.

4.5.1 Solutions of nonlinear PDE with power-law thermal conductivity

We consider the nonlinear PDE with power-law thermal conductivity

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(\theta^m \frac{\partial \theta}{\partial x}\right) - M^2 \theta^{n+1} \tag{4.35}$$

We then transform the Eq.(4.35) using $\xi = k(x - Vt)$ into the following ODE:

$$-kV\frac{d\theta}{d\xi} = k^2 m\theta^{m-1}\left(\frac{d\theta}{d\xi}\right)^2 + k^2\theta^m\left(\frac{d^2\theta}{d\xi^2}\right) - M^2\theta^{n+1} \tag{4.36}$$

We introduce a new independent variable

$$Y = \tanh \xi \tag{4.37}$$

and find solution(s) as finite power series in Y of the form:

$$F(Y) = \sum_{n=0}^{N} a_N Y^N \tag{4.38}$$

Next, we replace Eq. (4.36) by

$$\begin{aligned} -kV(1-Y^2)\frac{dF(Y)}{dY} &= k^2 m F(Y)^{m-1}\left[(1-Y^2)\frac{dF(Y)}{dY}\right]^2 \\ &+k^2F(Y)^m\left[(1-Y^2)\frac{d}{dY}\left((1-Y^2)\frac{dF(Y)}{dY}\right)\right] - M^2F(Y)^{n+1} \end{aligned} \tag{4.39}$$

After substitution of Eq. (4.38) into Eq. (4.39), we verify that the highest powers of Y appear as $Y^{(m+1)N+2}$ in the third term and $Y^{(n+1)N}$ in the last term of Eq. (4.39). Balancing these then leads to:

$$(m+1)N+2=(n+1)N \tag{4.40}$$

$$N=\frac{2}{n-m} \tag{4.41}$$

We take $n=3$ and $m=2$, therefore we obtain $N=2$.

The next crucial step is to look for a solution of the form:

$$F(y)=a_0+a_1Y+a_2Y^2 \tag{4.42}$$

Substituting Eq. (4.42) into Eq. (4.39), we obtain:

$$\begin{aligned}-kV(1-Y^2)(a_1+2a_2Y)=2k^2(a_0+a_1Y+a_2Y^2)\Big[(1-Y^2)\\ (a_1+2a_2Y)\Big]^2+k^2(a_0+a_1Y+a_2Y)^2\Big[(1-Y^2)\\ (2a_2-2a_1Y-6a_2Y^2)\Big]-M^2(a_0+a_1Y+a_2Y^2)^4\end{aligned} \tag{4.43}$$

Equating the coefficients of every power of Y to zero, we obtain the following algebraic system:

$$Y^0: \quad 2a_0{a_1}^2k^2 + 2{a_0}^2a_2k^2 - {a_0}^2M^2 + a_1kV = 0$$

$$Y: \quad -2{a_0}^2a_1k^2 + 2{a_1}^3k^2 + 12a_0a_1a_2k^2 - 4{a_0}^3a_1M^2 + 2a_2kV = 0$$

$$Y^2: \quad -8a_0{a_1}^2k^2 - 8{a_0}^2a_2k^2 + 12a_0{a_2}^2k^2 - 6{a_0}^2{a_1}^2M^2 - 4{a_0}^3a_2M^2 - a_1kV = 0$$

$$Y^3: \quad 2{a_0}^2a_1k^2 - 6{a_1}^3k^2 - 36a_0a_1a_2k^2 + 20a_1{a_2}^2k^2 - 4a_0{a_1}^3M^2 - 12{a_0}^2a_1a_2M^2 - 2a_2kV = 0$$

$$Y^4: \quad 6a_0{a_1}^2k^2 + 6{a_0}^2a_2k^2 - 32{a_1}^2a_2k^2 - 32a_0{a_2}^2k^2 + 10{a_2}^3k^2 - {a_1}^4M^2 - 12a_0{a_1}^2a_2M^2 - 6{a_0}^2{a_2}^2M^2 = 0$$

$$Y^5: \quad 4{a_1}^3k^2 + 24a_0a_1a_2k^2 - 50a_1{a_2}^2k^2 - 4{a_1}^3a_2M^2 - 12a_0a_1{a_2}^2M^2 = 0$$

$$Y^6: \quad 20{a_1}^2a_2k^2 + 20a_0{a_2}^2k^2 - 24{a_2}^2k^2 - 6{a_1}^2{a_2}^2M^2 - 4a_0{a_2}^3M^2 = 0$$

$$Y^7: \quad 30a_0{a_2}^2k^2 - 4a_1{a_2}^3M^2 = 0$$

$$Y^8: \quad 14{a_2}^2k^2 - {a_2}^4M^2 = 0$$

Solving the algebraic system with the aid of MATHEMATICA yields [33]:

$$a_2 = \frac{14k^2}{M^2} \tag{4.44}$$

$$a_1 = 0 \tag{4.45}$$

$$a_0 = -\frac{28k^2}{3M^2} \tag{4.46}$$

Therefore, Eq. (4.42) becomes:

$$F(Y) = \frac{14k^2}{M^2}\left(Y^2 - \frac{2}{3}\right), \tag{4.47}$$

and in terms of the original variables

$$\theta(t,x) = \frac{14k^2}{M^2}\left(\tanh^2(k(x-Vt)) - \frac{2}{3}\right), \tag{4.48}$$

Applying the initial conditions $\theta(0,x) = 0$ and boundary conditions $\theta(t,1) = 1$ to Eq. (4.48), we obtain the values of k and V as:

$$k = \frac{1}{\tanh(x)}\left(\pm\sqrt{\frac{2}{3}}\right) \tag{4.49}$$

$$V = \frac{28t \pm \sqrt{14} \times \sqrt{56t^2 + 9M^2t^2\tanh^2 x^2}}{28t^2} \tag{4.50}$$

4.6 Concluding remarks

In this chapter, we considered a longitudinal fin with a rectangular profile. The classical Lie point symmetry analysis of the mathematical model was performed. The original PDE was transformed into a set of nonlinear second-order ODEs which resulted in steady- and transient-state heat flow equations. The steady-state equations were solved using substitution and direct integration. The transient-state equation was transformed into an autonomous equation which was further reduced into a first-order ODE. The equation was then solved using integration. The solution was represented graphically, showing the effects of the thermo-geometric fin parameter M, the exponents m, and n and, in some cases, the fin efficiency. Lastly, the Tanh method was employed to analyze the governing equation.

Chapter 5

Analysis of heat transfer in a longitudinal fin of concave parabolic profile: Exact solutions

5.1 Introduction

In this chapter, we discuss a longitudinal fin of concave parabolic profile. A mathematical model describing the problem is given in Section 5.2. In Section 5.3, we consider the classical Lie point symmetry analysis of a partial differential equation. We discuss some symmetry reductions in Section 5.4. In Section 5.5, we analyze the exact analytical results and give concluding remarks in Section 5.6.

5.2 Mathematical models

We consider a longitudinal one-dimensional fin of concave parabolic profile as shown in Fig. 2.3. The heat balance equation is given by

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left(x^2\theta^m \frac{\partial \theta}{\partial x}\right) - M^2\theta^{n+1}, \quad 0 < x < 1, \tag{5.1}$$

with the initial and the step change in base temperature conditions

$$\theta(x,0) = 0, \quad \theta(1,\tau) = 1, \quad \left.\frac{\partial \theta}{\partial x}\right|_{x=0} = 0, \ \tau \geq 0, \tag{5.2}$$

and the step change in base heat flux conditions are given by

$$\theta(x,0) = 0, \quad \left.\frac{\partial \theta}{\partial x}\right|_{x=1} = 1, \quad \left.\frac{\partial \theta}{\partial x}\right|_{x=0} = 0, \ \tau \geq 0. \tag{5.3}$$

5.3 Lie point symmetry analysis of a PDE

We solve (5.1) by first using Lie symmetries. The resulting ODEs are analyzed using the method of differential invariants.

5.3.1 Admitted Lie point symmetries

Two symmetries were obtained

$$\Gamma_1 = \partial_\tau, \ \Gamma_2 = -x\partial_x \tag{5.4}$$

We constructed the one dimensional optimal system of subalgebras and obtain the set

$$\{\Gamma_1, \ \Gamma_2, \ \Gamma_1 \pm \Gamma_2\} \tag{5.5}$$

The reductions by these elements of the one-dimensional optimal system are given in Table 5.1.

Table 5.1: Reductions by elements of the optimal systems (5.5)

Symmetries	Reductions
$\Gamma_2 + \Gamma_1$	$\gamma = xe^{\tau},\ \theta = H(\gamma)$ where H satisfies $\gamma H' = [\gamma^2 H^m H']' - M^2 H^{n+1}$
$\Gamma_2 - \Gamma_1$	$\gamma = \frac{e^{\tau}}{x},\ \theta = H(\gamma)$ where H satisfies $\gamma H' = \gamma^2 [H^m H']' - M^2 H^{n+1}$

5.4 Some symmetry reductions

The admitted symmetries reduced the number of independent variables of a partial differential equation by one (see Table 5.1). The reduced ODE may or may be solved. Even when the explicit exact analytical solutions may not be constructed, the reduction of variables has a number of advantages. For example, the nonlinear boundary value problems may be easily solved by numerical methods.

In particular, $\Gamma_1 = \partial_\tau$ implies

$$\frac{d}{dx}\left[x^2\theta^m\frac{d\theta}{dx}\right] - M^2\theta^{n+1} = 0, \tag{5.6}$$

which is a steady state solved by [25].

5.4.1 Steady-state solutions

If we substitute $y = \theta^{m+1}$ on the reductions in (5.6) we get

$$x^2y'' + 2xy' - (m+1)M^2y^{\frac{n+1}{m+1}} = 0 \tag{5.7}$$

Example 5.1: steady state solutions for $m = n$

From Eq. (5.7), we get the Euler-Cauchy equation

$$x^2y'' + 2xy' - (m+1)M^2y = 0. \tag{5.8}$$

We expect a solution of the form x^r, where r is the solution of the auxiliary equation

$$r^2 + r - (m+1)M^2 = 0.$$

Thus the general solution is

$$y = C_3 x^{\frac{-1+\sqrt{1+4(m+1)M^2}}{2}} + C_4 x^{\frac{-1-\sqrt{1+4(m+1)M^2}}{2}}, \tag{5.9}$$

and finally

$$\theta = \left[C_3 x^{\frac{-1+\sqrt{1+4(m+1)M^2}}{2}} + C_4 x^{\frac{-1-\sqrt{1+4(m+1)M^2}}{2}}\right]^{\frac{1}{m+1}}, \tag{5.10}$$

The Eq. (5.10) satisfy $\theta(1) = 1$ and $\theta'(1) = 1$ of both step change in base temperature conditions and step change in base heat flux conditions, but it fails at $\theta'(0) = 0$.

Example 5.2: steady state solutions for $m = n = -1$

From Eq. (5.6), we obtain second order ODE

$$\frac{d}{dx}\left[x^2\theta^{-1}\frac{d\theta}{dx}\right] - M^2 = 0, \tag{5.11}$$

whose solution is

$$\theta = Ax^{M^2}e^{-\frac{B}{x}}. \tag{5.12}$$

We note again that the initial condition $\theta'(0) = 0$, is not satisfied.

Example 5.3: steady state solutions for $m \neq n = -1$

From Eq. (5.6), we obtain second order ODE

$$\frac{d}{dx}\left[x^2\theta^{m}\frac{d\theta}{dx}\right] - M^2 = 0, \tag{5.13}$$

whose solution is

$$\theta = \left[(m+1)(M^2\ln x - \frac{C_1}{x} + C_2)\right]^{\frac{1}{m+1}}. \tag{5.14}$$

We note again that the initial condition $\theta'(0) = 0$, is not satisfied.

Example 5.4: steady state solutions for $m \neq n$

We solve (5.7) by using Lie symmetries directly. Symmetry obtained $\Gamma_1 = -2x\partial_x$ where

$$\xi = -2x, \ \eta = 0 \tag{5.15}$$

We now proceed with the *method of differential invariants.* The group in (5.15), has the invariant $u = y$

$$\zeta^x = D(\eta) - D(\xi)y' \tag{5.16}$$

and

$$\zeta^{xx} = D(\zeta^x) - D(\xi)y'' \tag{5.17}$$

Hence

$$\begin{aligned} \zeta^x &= \left[\frac{\partial 0}{\partial x} + y'\frac{\partial 0}{\partial y} + \cdots\right] - y'\left[\frac{\partial(-2y)}{\partial x} + y'\frac{\partial(-2y)}{\partial y} + \cdots\right], \\ &= 2y', \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} \zeta^{xx} &= \left[\frac{\partial(2y')}{\partial x} + y'\frac{\partial(2y')}{\partial y} + \cdots\right] - y'\left[\frac{\partial(2y')}{\partial x} + y'\frac{\partial(2y')}{\partial y} + \cdots\right], \\ &= 2y'' - 2y''^2. \end{aligned} \tag{5.19}$$

The first and second prolongations are

$$\begin{aligned} \Gamma^{(1)} &= -2x\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + 2y'\frac{\partial}{\partial y'} \\ \Gamma^{(2)} &= -2x\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + 2y'\frac{\partial}{\partial y'} + (2y'' - 2y''^2)\frac{\partial}{\partial y''} \end{aligned} \tag{5.20}$$

From the characteristics equations

$$\frac{dx}{-2x} = \frac{dy}{0} = \frac{dy'}{2y'} = \frac{dy''}{2y'' - 2y''^2} \tag{5.21}$$

The first and second invariants are $u = y$ and $v = xy'$ respectively. Now

$$\begin{aligned}
\frac{dv}{du} = \frac{\frac{dv}{dx}}{\frac{du}{dx}} &= \frac{\frac{\partial v}{\partial x} + y'\frac{\partial v}{\partial y} + y''\frac{\partial v}{\partial y'}}{\frac{\partial u}{\partial x} + y'\frac{\partial u}{\partial y} + y''\frac{\partial u}{\partial y'}} \\
\frac{dv}{du} &= \frac{y' + y'0 + y''x}{0 + y'1 + y''0} \\
\frac{dv}{du} &= 1 + \frac{x}{y'}y'' \qquad (5.22)
\end{aligned}$$

Eq. (5.22) transforms (5.7) into

$$v\frac{dv}{du} + v = (m+1)M^2u^{\frac{n+1}{m+1}} \qquad (5.23)$$

5.4.2 Transient-state solutions

Example 5.5: transient state solution

If we substitute $y = H^{m+1}$ on the reductions in Table 5.1 we get

$$\gamma^2\frac{d^2y}{d\gamma^2} + (2 - y^{-\frac{m}{m+1}})\gamma\frac{dy}{d\gamma} - (m+1)M^2y^{\frac{n+1}{m+1}} = 0, \qquad (5.24)$$

$$\gamma^2\frac{d^2y}{d\gamma^2} - y^{-\frac{m}{m+1}}\gamma\frac{dy}{d\gamma} - (m+1)M^2y^{\frac{n+1}{m+1}} = 0. \qquad (5.25)$$

We determine symmetries for the second-order ODEs (5.24) and (5.25), which implies seeking transformation of the form

$$\overline{\gamma} = \gamma + \epsilon\xi(\gamma, y) + O(\epsilon^2) \qquad (5.26)$$

$$\overline{y} = y + \epsilon\eta(\gamma, y) + O(\epsilon^2)$$

generated by the vector field

$$\Gamma = \xi(\gamma, y)\frac{\partial}{\partial\gamma} + \eta(\gamma, y)\frac{\partial}{\partial y} \qquad (5.27)$$

which leave the governing equations invariant. The invariance surface conditions are given by

$$\Gamma^{[2]}\mathrm{Eq.}(5.24)|_{\ \mathrm{Eq.}(5.24)} = 0, \qquad (5.28)$$

$$\Gamma^{[2]}\mathrm{Eq.}(5.25)|_{\ \mathrm{Eq.}(5.25)} = 0, \qquad (5.29)$$

Here $\Gamma^{[2]}$ is the second prolongation defined by

$$\Gamma^{[2]} = \Gamma + \zeta^{\gamma}\frac{\partial}{\partial y'} + \zeta^{\gamma\gamma}\frac{\partial}{\partial y''} \tag{5.30}$$

where the prolongation formulae are defined by

$$\begin{aligned} \zeta^{\gamma} &= D_{\gamma}(\eta) - y' D_{\gamma}(\xi), \\ \zeta^{\gamma\gamma} &= D_{\gamma}(\zeta^{\gamma}) - y'' D_{\gamma}(\xi), \end{aligned} \tag{5.31}$$

with D_{γ} being the total derivative operator defined by

$$D_{\gamma} = \partial_{\gamma} + y'\partial_{y} + y''\partial_{y'} + ...,$$

The prime implies differentiation with respect to γ. The symmetries for (5.24) and (5.25) respectively obtained by using [34] and [35] are

$$\Gamma_1 = -2\gamma\partial_{\gamma},\ \Gamma_2 = \gamma\partial_{\gamma} \tag{5.32}$$

5.4.3 Method of differential invariants

We further reduce (5.24) and (5.25) to first-order ODEs by using the method of differential invariants [46, 48]. The invariant for both groups in (5.32) is $u = y$. The first and second infinitesimals are calculated and shown in Table 5.2 with their corresponding prolongations. Their characteristic equations, together with the first and second invariants arising from these prolongations, are given in Table 5.3.

The first and second invariants in Table 5.3 reduce equations (5.24) and (5.25) further into Abel's equations of the second kind given in Table 5.4 [69].

5.4.4 Special case: $m = 0$

We solved the two ODEs in Table 5.4 for $m = 0$ and obtained an exact analytical solution shown in Table 5.5 for the first equation. The analysis is carried

Table 5.2: The first and second infinitesimals and prolongations for (5.24) and (5.25)

Equation	Infinitesimals	Prolongations
(5.24)	$\zeta^{\gamma} = 2y'$	$\Gamma^{(1)} = -2\gamma\partial_{\gamma} + 0\partial_{y} + 2y'\partial_{y'}$
	$\zeta^{\gamma\gamma} = 2y'' - 2y''^{2}$	$\Gamma^{(2)} = -2\gamma\partial_{\gamma} + 0\partial_{y} + 2y'\partial_{y'} + (2y'' - 2y''^{2})\partial_{y''}$
(5.25)	$\zeta^{\gamma} = y'$	$\Gamma^{(1)} = -2\gamma\partial_{\gamma} + 0\partial_{y} + 2y'\partial_{y'}$
	$\zeta^{\gamma\gamma} = -y'' + y''^{2}$	$\Gamma^{(2)} = -2\gamma\partial_{\gamma} + 0\partial_{y} + 2y'\partial_{y'} + (y''^{2} - y'')\partial_{y''}$

Table 5.3: Characteristic equation, first and second invariants

Equation	Characteristic equation	first invariant	second invariant
(5.24)	$\frac{d\gamma}{-2\gamma} = \frac{dy}{0} = \frac{dy'}{2y'}$	$u = y$	$v = \gamma y'$
(5.25)	$\frac{d\gamma}{\gamma} = \frac{dy}{0} = \frac{dy'}{-y'}$	$u = y$	$v = \gamma y'$

Table 5.4: The first and second invariants with corresponding reductions for equations (5.24) and (5.25)

Equation	first invariant	second invariant	Reduction
(5.24)	$u = y$	$v = \gamma y'$	$v\frac{dv}{du} + (1 - u^{-\frac{m}{m+1}})v - (m+1)M^{2}u^{\frac{n+1}{m+1}} = 0$
(5.25)	$u = y$	$v = \gamma y'$	$v\frac{dv}{du} - (1 + u^{-\frac{m}{m+1}})v - (m+1)M^{2}u^{\frac{n+1}{m+1}} = 0$

out in section 4, and the graphs are depicted in Figs. 5.1 - 5.8. The second ODE was transformed into canonical form, solved, and a special function shown in Table 1 was obtained.

Table 5.5: The reduction if $m = 0$ and general exact solution for equation (5.25).

first invariant	second invariant	ODE	Solution
$u = y$	$v = \gamma y'$	$v\frac{dv}{du} - M^2u^{n+1} = 0$	$\theta(x,\tau) = \left[1 \pm \frac{M}{n}\sqrt{\frac{8}{\lvert n+2\rvert}}(\ln x + t)\right]^{-\frac{2}{n}}.$
$u = y$	$v = \gamma y'$	$v\frac{dv}{du} - 2v - M^2u^{n+1} = 0$	see Table 1

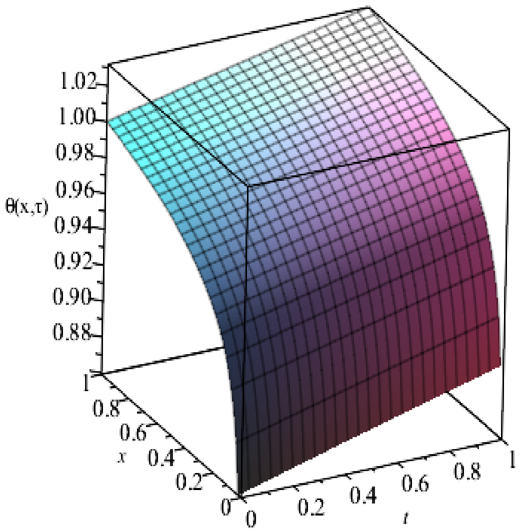

Figure 5.1: Graphical representation of temperature variations with increasing time and space for step change in base temperature conditions when $m = 0$, $M = 0.25$.

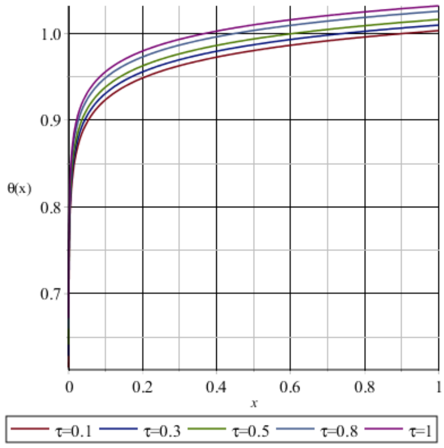

Figure 5.2: Graphical representation of temperature variations for different times with increasing space for step change in base temperature conditions when $m = 0,\ n = -5,\ M = 0.25$.

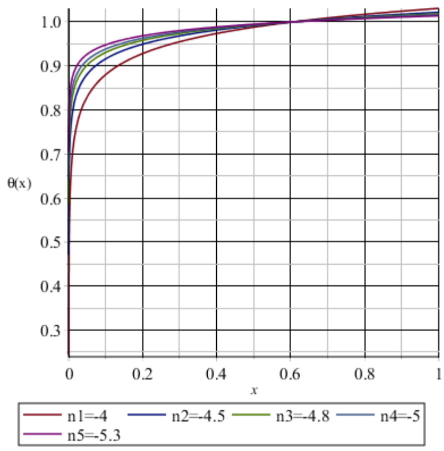

Figure 5.3: Graphical representation of temperature profiles for different n with increasing time for step change in base temperature conditions when $m = 0,\ \tau = 0.5,\ M = 0.25$.

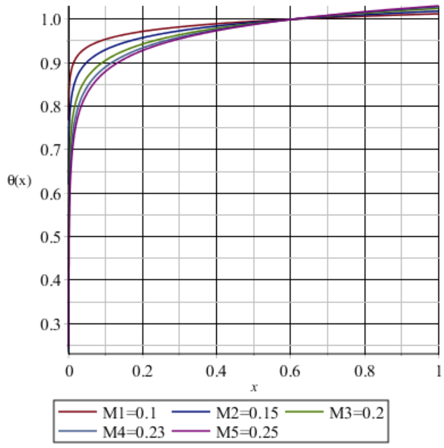

Figure 5.4: Graphical representation of temperature profiles for different M-values with increasing time for step change in base temperature conditions when $m = 0,\ n = -5,\ \tau = 0.5$.

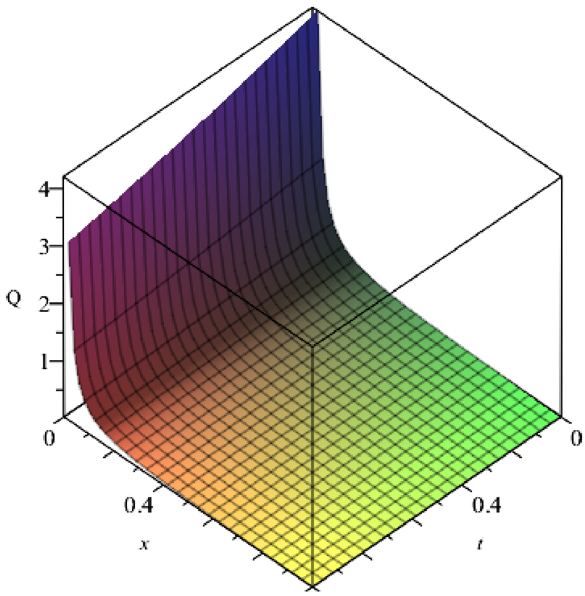

Figure 5.5: Graphical representation of heat flow with increasing time and space for step change in base temperature conditions when $m = 0,\ M = 0.25$.

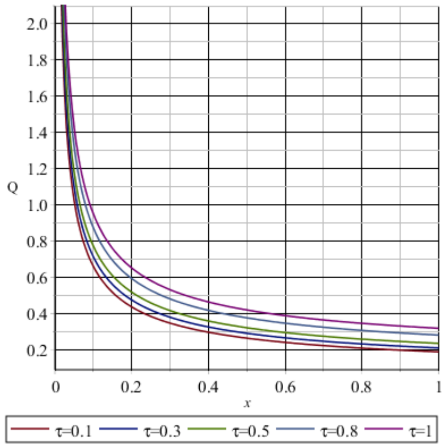

Figure 5.6: Graphical representation of heat flow for different times with increasing space for step change in base temperature conditions when $m = 0$, $n = -5$, $M = 0.25$.

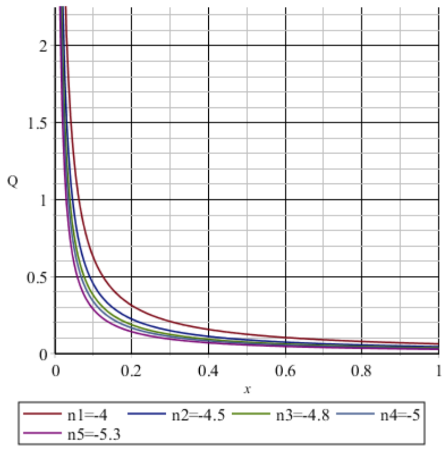

Figure 5.7: Graphical representation of heat flow for different n with increasing time for step change in base temperature conditions when $m = 0$, $\tau = 0.5$, $M = 0.25$.

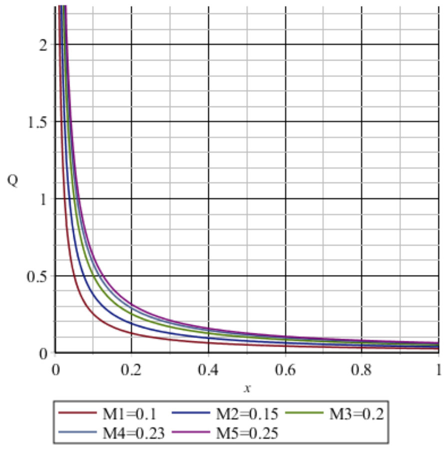

Figure 5.8: Graphical representation of heat flow for different M-Values with increasing time for step change in base temperature conditions when $m = 0,\ n = -5,\ \tau = 0.5$.

5.5 Exact analytical results

We have obtained the exact analytical solutions using the local symmetry techniques and some results are depicted in Figs. 5.1 - 5.8.

5.5.1 Steady-state results

In example 5.1, the Eq. (5.10) satisfy $\theta(1) = 1$ and $\theta'(1) = 1$ of both step change in base temperature conditions and step change in base heat flux conditions, but it fails at $\theta'(0) = 0$. In examples 5.2 - 5.4, general solutions were obtained but all of them did not satisfy the initial condition $\theta'(0) = 0$.

5.5.2 Transient-state results

In example 5.5, we obtained exact solution for $m = 0$ whose graphs are depicted in Figs. 5.1 - 5.8. Figs. 5.1 and 5.5 depict the temperature and heat flow profiles for increasing time and space obtained from step change in base temperature.

Temperature profiles with a step-change in base temperature

Fig. 5.1 depicts the temperature profile for increasing time and space obtained from step change in base temperature. In Fig. 5.2 we observe that temperature increases with time. We observe in Fig. 5.3 that the temperature decreases with increasing values of n. In Fig. 5.4 we observe that temperature decreases with increasing values of the thermo-geometric fin parameter M.

Heat flow profiles with a step-change in base temperature

Fig. 5.5 depicts the heat flow profile for increasing time and space obtained from step change in base temperature. In Fig. 5.6 we observe that heat flow increases with time. We observe in Fig. 5.7 that the heat flow increases with increasing values of n. In Figs. 5.8 we observe that heat flow decreases with increasing values of the thermo-geometric fin parameter M.

5.6 Concluding remarks

In this chapter, we considered a longitudinal fin of a concave parabolic profile. The classical Lie point symmetry analysis of the mathematical model was performed. The original PDE was transformed into nonlinear second-order ODEs, resulting in steady- and transient-state heat flow equations. The nonlinear second-order ODEs were further reduced to first-order ODEs by using the method of differential invariants and canonical variables. The general solutions for the steady-state satisfied some boundary conditions except the initial condition. The transient-state solutions for $m = 0$ were obtained. The effects of the thermo-geometric fin parameter M, the exponents m, and n and, in some cases, the fin efficiency on temperature were discussed. Lastly, the Tanh method was employed to solve the governing equation.

Chapter 6

Conclusions

In this dissertation, we investigated models describing heat transfer in longitudinal fins of rectangular and concave parabolic profiles. We employed symmetries to solve the initial and boundary value problems.

6.1 Longitudinal fins of rectangular profile

The governing equation admitted three local symmetries, which reduced the original non-linear PDE into three non-linear ODEs. We analyzed steady-state as well as transient-state solutions. We observed that as thermo-geometric fin pameter M increases the temperature decreases. Small values of M correspond to the relatively short and thick fins of high conductivity, and high values of M correspond to long and thin fins of poor conductivity [68]. A fin is an excellent dissipator at small values of M. We observe that temperature decreases as the values of the thermo-geometric fin parameter increase. Lastly, we also observe that the larger the magnitude of m, the smaller the fin efficiency.

6.2 Longitudinal fins of concave parabolic profile

The governing equation admitted two local symmetries, which reduced the original non-linear PDE into a transient-state equation and two non-linear ODEs. For the steady-state solutions, we obtained an Euler-Cauchy equation which yielded a general solution that did not satisfy the initial condition. By using the method of differential invariants, the non-linear ODEs were further reduced into two first-order ODE of Abel's second kind. At this stage, we considered a special case when $m = 0$. This yielded an exact analytical solution which enabled us to investigate the behavior of temperature and heat flux in terms of increasing space, time, exponent n, and thermo-geometric fin parameter M. On the other hand, the second equation was transformed into a canonical form and solved, which yielded a special function as tabled in the Appendix, and there was no further analysis on the case. Lastly, the assumption that temperature-dependent thermal conductivity may be given by the power law is physically realistic.

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Appendix A

The equation $v\frac{dv}{du} - (1 + u^{-\frac{m}{m+1}})v - (m+1)M^2 u^{\frac{n+1}{m+1}} = 0$ is transformed into a canonical form,

$$v\frac{dv}{dz} - v = \frac{M^2}{2^{n+2}} z^{n+1}, \text{where} \quad z = 2u \tag{A.1}$$

whose solution for different values of n is given by Table 1. Further analysis of (A.1) is omitted here.

Table A.1: Solutions of (A.1) for different values of n

n	$v = v(u)$
-1	$v = -2^{-n}\frac{M^2}{4}(W(-2^n \frac{e^{-1-\frac{2^n 4C_1}{M^2}-\frac{2^n 4u}{M^2}}}{M^2}) + 1)$
$-3/2$	$\frac{-\text{Ai}(1, -\frac{(-2^{1+n}u^{3/2}M)^{2/3}(v-2u)}{M^2 u}M - (-2^{1+n}u^{3/2}M)^{1/3}\text{Ai}-\frac{(-2^{1+n}u^{3/2}M)^{2/3}(v-2u)}{M^2 u})}{\text{Bi}(1, -\frac{(-2^{1+n}u^{3/2}M)^{2/3}(v-2u)}{M^2 u}M - (-2^{1+n}u^{3/2}M)^{1/3}\text{Bi}-\frac{(-2^{1+n}u^{3/2}M)^{2/3}(v-2u)}{M^2 u})} + C_1 = 0$
-2	$v = \frac{\sqrt{2}(IMe^{\frac{(v-2u)^2}{M^2}} + \sqrt{\pi}\text{erf}(Iv-2u)2u)}{2u} + C_1$

- $\text{Ai}(z)$ - Airy function
- $\text{Ai}(1, z)$ - Derivative of Airy function

- $\mathrm{Bi}(z)$ - Airy function
- $\mathrm{Bi}(1, z)$ - Derivative of Airy function
- W - Lambert function
- I - Imaginary number $\sqrt{-1}$
- $\mathrm{erf}(z)$ - Error function