Exponential synchronization of a complex dynamical network with piecewise-homogeneous Markovian jump structure and coupling delay

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Abstract—In this paper, exponential synchronization problem is studied for a complex dynamical network (CDN) with a class of Markovian jump structure with coupling-time delay. The CDN under consideration has a piecewise-Markovian jump structure with piecewise-constant transition rates (TRs). First piecewise-homogeneous Markovian process is modeled by two Markov chains then, the synchronization problem of the CDN is inspected by constructing Lyapunov-Krasovskii function with Markov dependent matrices. Ultimately, the controller gain matrices for guaranteeing the synchronization problem in terms of linear matrix inequalities (LMIs) are derived. A CDN with a Chua circuit dynamic for each node is given as a numerical example to show the effectiveness of the theoretical results.

Index Terms—Exponential synchronization, Complex dynamical network, Lyapunov-Krasovskii theory, Piecewise-homogeneous Markovian parameters, Coupling delay

I. Introduction

Complex dynamical networks consist of a large number of interconnected nodes, in which each node stands for an individual unit with specific content and edges represent the relation between the coupled nodes. Applications of CDNs are abundant in various fields ranging from biology, physic to engineering [1]. Word wide web, neural networks, electric power grids and multi-agent systems can be referred to complex dynamical networks in the engineering field [2]–[4]. Synchronization problem as one of the most considerable properties in the CDNs has attracted great attention in recent studies [5], [6]. The synchronization means that the trajectories of all interconnected dynamical nodes (slaves) can track an isolated node's trajectory (master). Time-delay exists in real-systems due to traffic congestion or limited speed in transmitting of information. Time delay leads to instability, oscillation or chaos that should be modeled to exhibit the reality of the system much better. One of the time delay types that exists in many real models is coupling time delay [7]. The coupling time delay is caused by the exchange of data among units in a network.

On the other hand, some networks in real-world systems may be faced to jump or switch in their structure. Switching can occur in connection topology of a network due to link failure or new creation such as arbitrary or stochastic switching in an electrical power grid. If the reason of switching is stochastic factors such as sudden environmental changes [8], random component failure, or link failure, then switching is stochastic process which is usually modeled by Markov Chain. The corresponding systems are called Markovian jump systems (MJSs). Markov chains are suitable for modeling stochastic processes which are memoryless. The most important factor of a MJS is the TR of the jumping process which characterizes the behavior of the system. Numerous studies focused on the synchronization problem of Markovian jump CDNs (MJCDNs), however in these studies Markov structures have been analyzed traditionally. In traditional analysis and synthesis of MJLSs, the TRs are assumed to be constant over time [9], [10]. Although it is very idealistic to get the accurate and complete TR information due to the difficulty or the costs of measuring the TRs [11]. One of the popular solution for solving Markov process with time varying TRs (non-homogeneous Markov process) is to consider TRs as piecewise-constant. In other words, in this way, the non-homogeneous Markov process can be considered as piecewise-homogeneous Markov process. Modeling of the piecewise-homogeneous Markov structure is performed through two Markov chains; (i) a Markov chain as a low level switching signal that manages the switching between the system dynamics or coupling topologies with piecewise-constant time varying TRs. The piecewise-constant time-varying TRs mean TRs are time-varying but invariant during an interval, (ii) another Markov chain as a high level signal that causes stochastic variations among possible TR matrices of previous Markov process. Many papers have investigated the synchronization problem of CDNs with a homogeneous Markov structure (Markov process with constant TR over time) [14]–[16]. To the best of the authors' knowledge, [13] and [12] are the only references about synchronization of CDNs with piecewise-homogeneous Markovian jump parameters. However, in those studies one or two of the coefficient matrices are mode-dependent and all the other parameters in [12], [13] are considered constant at all times.

Hence contributions of this paper compared with existing literatures are as follows:

- All coefficient matrices are piecewise-homogeneous Markovian mode-dependent. The synchronization problem for such a MJCDN with piecewise-constant TRs has not been investigated so far.
- The Marovian structure is assumed to be piecewise-homogeneous that is modeled by two level Markov signals. Such a two level Markovian structure is appropriate for representing time varying TRs.

The aim of this paper is to address synchronization and synchronizability criteria for the CDN and design the mode-dependent state feedback controller for each node of the network.

Notations. Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ is the set of real $n\times m$ matrices. The notation $\mathbf{X} > \mathbf{Y}$ ($\mathbf{X} \geq \mathbf{Y}$), where $\mathbf{X}$ and $\mathbf{Y}$ are symmetric matrices, means that $\mathbf{X}-\mathbf{Y}$ is positive definite (positive semi definite). The matrices $\mathbf{I}$ and $\mathbf{0}$ are identity and zero matrices with compatible dimensions, respectively. The superscript "T" stands for the transpose and diag$\{.\}$ stands for block diagonal matrix. $\|.\|$ represents the Euclidean norm of a vector and its induced norm of the matrix. $\Pr\{\alpha\}$ means the occurrence probability of the event α, and $\Pr\{\alpha|\beta\}$ means the occurrence probability of the event conditional on β. $\mathbb{E}\{x\}$ ($\mathbb{E}\{x|y\}$) represents the expectation of the stochastic variable x (conditional on y). The kronecker product of matrices $\mathbf{R}\in\mathbb{R}^{m\times n}$ and $\mathbf{Q}\in\mathbb{R}^{p\times q}$ is a matrix in $\mathbb{R}^{mp\times nq}$, which is denoted as ($\mathbf{R}\otimes\mathbf{Q}$). Symmetric terms in a symmetric matrix are denoted by $*$. $\lambda_{max}(.)$ and $\lambda_{min}(.)$ denote to be the largest and smallest eigenvalues of a given matrix.

II. Problem formulation and preliminaries

Consider the CDN with piecewise-homogeneous Markovian structure which in dynamical equation of each node, the coupling delay and the coefficient matrices are supposed to be mode-dependent:

$$\begin{aligned}\dot{\mathbf{x}}_b(t) &= \mathbf{A}(r_t)\mathbf{x}_b(t)+\mathbf{B}(r_t)\mathbf{f}(\mathbf{x}_b(t))\\ &+\sum_{d=1}^{N} g_{bd}(r_t)\mathbf{\Gamma}(r_t)\mathbf{x}_d(t-\tau)+\mathbf{u}_b(t) \qquad (1)\\ &\qquad b=1,2,\dots,N,\end{aligned}$$

where $\mathbf{x}_b(t)=[x_{b1}(t),x_{b2}(t),...,x_{bn}(t)]^T\in\mathbb{R}^n$ denotes the state vector of the b^{th} node at time t, $\mathbf{f}(\mathbf{x}_b(t))$ and $\mathbf{u}_b(t)$ are a continuous nonlinear vector function and the control input of the node b, respectively. $\mathbf{A}(r_t),\mathbf{B}(r_t)\in\mathbb{R}^{n\times n}$ are the mode dependent matrices. $\mathbf{\Gamma}(r_t)=[\gamma_{bd}(r_t)]_{n\times n}$ indicates inner-coupling between the elements of the node itself, $\mathbf{G}(r_t)=[g_{bd}(r_t)]_{N\times N}$ denotes outer-coupling between the nodes of the whole network and represents the topological structure of the network. If there is a connection between node b and node d, then $g_{bd}(r_t)=g_{db}(r_t)\neq 0$, $b\neq d$, otherwise $g_{bd}(r_t)=g_{db}(r_t)=0$, $b\neq d$. It is assumed that the sum of each row of $\mathbf{G}(r_t)$ is zero, i.e. $\sum_{d=1,b\neq d}^{N} g_{bd}(r_t)=-g_{bb}(r_t)$, $b=1,2,\dots,N$. The scalar τ represents coupling delay. The process $\{r_t,t\geq 0\}$ is a continuous-time non-homogeneous Markov process which takes its values in the finite set $\mathcal{W}=\{1,2,...,w\}$ that describes switching between different modes. The probability function for the procedure $\{r_t,t\geq 0\}$ with time-varying TR matrix $\Pi^{\sigma_{t+\Delta t}}=[\pi_{ij}^{\sigma_{t+\Delta t}}]_{w\times w}$ is defined by

$$\Pr\{r_{t+\Delta t}=j|\mathrm{r}_t=i\}=\begin{cases}\pi_{ij}^{\sigma_{t+\Delta t}}\Delta t+o(\Delta t), & i\neq j,\\ 1+\pi_{ii}^{\sigma_{t+\Delta t}}\Delta t+o(\Delta t), & i=j,\end{cases} \qquad (2)$$

where $\Delta t>0$, $\lim_{\Delta t\to 0} o(\Delta t)/\Delta t=0$, and $\pi_{ij}^{\sigma_{t+\Delta t}}\geq 0$ for $i\neq j$ denotes the TR from mode i at time t to mode j at time $t+\Delta t$ in TR matrix with TR matrix $\Pi^{\sigma_{t+\Delta t}}=[\pi_{ij}]_{w\times w}$ with the following condition

$$\pi_{ii}^{\sigma_{t+\Delta t}}=-\sum_{j=1,i\neq j}^{w}\pi_{ij}^{\sigma_{t+\Delta t}}, \qquad (3)$$

Similar to previous Markov process, the process $\{\sigma_t,t\geq 0\}$ is continuous-time Markov process with its value in the finite set $\mathcal{V}=\{1,2,...,v\}$. This process is homogeneous and time invariant. The probability function with TR matrix $\Lambda=[\rho_{mn}]_{v\times v}$ is given by:

$$\Pr\{\sigma_{t+\Delta t}=n|\sigma_t=m\}=\begin{cases}\rho_{mn}\Delta t+o(\Delta t), & m\neq n,\\ 1+\rho_{mm}\Delta t+o(\Delta t), & m=n,\end{cases} \qquad (4)$$

where $\Delta t>0$, $\lim_{\Delta t\to 0} o(\Delta t)/\Delta t=0$, and $\rho_{mn}\geq o$ for $m\neq n$ denotes the TR from mode m at time t to mode n at time $t+\Delta t$ with the following condition

$$\rho_{mm}=-\sum_{n=1,m\neq n}^{v}\rho_{mn}, \qquad (5)$$

The high-level signal $\{\sigma_t,t\geq 0\}$ determines the switching TR matrix for the low level signal $\{r_t,t\geq 0\}$ and the low level signal $\{r_t,t\geq 0\}$ specifies the switching dynamic or topology structure modes.

Assumption 1. The function $\mathbf{f}:\mathbb{R}^n\to\mathbb{R}^n$ in the system (1) has a sector-bounded property which satisfies the following condition

$$[\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})-\mathbf{U}(\mathbf{x}-\mathbf{y})]^T[\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})-\mathbf{V}(\mathbf{x}-\mathbf{y})]\leq 0, \qquad (6)$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{U}$,$\mathbf{V}$ are known constant matrices with appropriate dimensions.

Let define the error vector $\mathbf{e}_b(t) = \mathbf{x}_b(t) - \mathbf{s}(t)$, where the state trajectories of the isolated node (or uncoupled node) $\mathbf{s}(t)$ is

$$\dot{\mathbf{s}}(t) = \mathbf{A}(r_t)\mathbf{s}(t) + \mathbf{B}(r_t)\mathbf{f}(\mathbf{s}(t)), \tag{7}$$

where $\mathbf{s}(t)$ is a special solution of the system (12). The error dynamic system can be obtained as follow:

$$\begin{aligned}\dot{\mathbf{e}}_b(t) &= \mathbf{A}(r_t)\mathbf{e}_b(t) + \mathbf{B}(r_t)\mathbf{g}(\mathbf{e}_b(t)) \\ &+ \sum_{d=1}^{N} g_{bd}(r_t)\mathbf{\Gamma}(r_t)\mathbf{e}_d(t-\tau) + \mathbf{u}_b(t) \\ & \qquad b = 1, 2, \ldots, N,\end{aligned} \tag{8}$$

where $\mathbf{g}(\mathbf{e}_b(t)) = \mathbf{f}(\mathbf{e}_b(t) + \mathbf{s}(t)) - \mathbf{f}(\mathbf{s}(t))$. Note that the above equation is right, while the condition $g_{kk}(r_t) = -\sum_{j=1,k\neq j}^{N} g_{kj}(r_t)$ holds for $k = 1, \ldots, N$.

The following feedback controllers are assumed as

$$\mathbf{u}_b(t) = \mathbf{K}_b(r_t, \sigma_t)\mathbf{e}_b(t), \quad b = 1, .., N, \tag{9}$$

where $\mathbf{K}_b(r_t, \sigma_t) \in \mathbb{R}^{n\times n}$ is the mode-dependent feedback gain matrix to be determined for each node in the network, $w \times v$ gain matrices should be designed. By substituting (9) into (8), we obtain

$$\dot{\mathbf{e}}(t) = \bar{\mathbf{A}}(r_t)\mathbf{e}(t) + \bar{\mathbf{B}}(r_t)\bar{\mathbf{g}}(\mathbf{e}(t)) + \bar{\mathbf{G}}(r_t)\mathbf{e}(t-\tau) + \mathbf{K}(r_t, \sigma_t)\mathbf{e}(t), \tag{10}$$

where $\mathbf{e}(t) = [\mathbf{e}_1^T(t), \mathbf{e}_2^T(t), \ldots, \mathbf{e}_N^T(t)]^T$, $\bar{\mathbf{g}}(\mathbf{e}(t)) = [\mathbf{g}^T(\mathbf{e}_1(t)), \mathbf{g}^T(\mathbf{e}_2(t)), \ldots, \mathbf{g}^T(\mathbf{e}_N(t))]^T$, $\bar{\mathbf{A}}(r_t) = \mathbf{I}_N \otimes \mathbf{A}(r_t)$, $\bar{\mathbf{B}}(r_t) = \mathbf{I}_N \otimes \mathbf{B}(r_t)$, $\bar{\mathbf{G}}(\mathbf{r_t}) = (\mathbf{G}(r_t) \otimes \mathbf{\Gamma}(r_t))$ and

$$\mathbf{K}(r_t, \sigma_t) = \mathrm{diag}\{\mathbf{K}_1(r_t, \sigma_t), \mathbf{K}_2(r_t, \sigma_t), ..., \mathbf{K}_N(r_t, \sigma_t)\}. \tag{11}$$

Definition 1. The CDN (1) is said to be exponentially mean square synchronized if there exist scalars $\alpha > 0$ and $\beta > 0$ as decay-rate and decay-coefficient, respectively, such that:

$$\mathbb{E}\{\|\mathbf{e}(t)\|^2\} \leq \beta e^{-\alpha t} \sup_{-\tau\leq\theta\leq 0} \{\|\mathbf{e}(\theta)\|^2, \|\dot{\mathbf{e}}(\theta)\|^2\}, \forall t > 0. \tag{12}$$

Lemma 1: (Jensen inequality). For any matrix $\mathbf{\Delta} \geq 0$, scalars η_1, η_2, $(\eta_2 > \eta_1)$ and a vector function $\varphi : [\eta_1, \eta_2] \to \mathbb{R}^n$ such that integrations concerned are well defined, then:

$$\begin{aligned}&(\eta_2 - \eta_1)\int_{\eta_1}^{\eta_2} \varphi^T(\alpha)\mathbf{\Delta}\varphi(\alpha)d\alpha \\ &\geq \left[\int_{\eta_1}^{\eta_2} \varphi(\alpha)d\alpha\right]^T \mathbf{\Delta} \left[\int_{\eta_1}^{\eta_2} \varphi(\alpha)d\alpha\right].\end{aligned} \tag{13}$$

III. Main results

In this section, two theorems are presented to guarantee the synchronization and to design the controller gains. In order to simplify of notation in the whole paper, it is regarded $\{r_t = i\}$ and $\{\sigma_t = m\}$.

Theorem 1: For given matrices $\mathbf{K}_{i,m}$, and scalars τ, α, the complex dynamical network (1) is exponentially mean square synchronized, if there exist symmetric positive definite matrices $\mathbf{P}_{i,m}$, $\mathbf{Q}$, $\mathbf{R}$ and scalars $\lambda_{i,m} > 0$, such that for any $i \in \mathcal{W}$, $m \in \mathcal{V}$, the following LMIs hold,

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{\Xi}_{11} & \mathbf{\Xi}_{12} & \mathbf{\Xi}_{13} & \tau(\bar{\mathbf{A}}_i + \mathbf{K}_{i,m})^T\mathbf{R} \\ * & \mathbf{\Xi}_{22} & 0 & \tau\bar{\mathbf{G}}_i^T\mathbf{R} \\ * & * & \mathbf{\Xi}_{33} & \tau\bar{\mathbf{B}}_i^T\mathbf{R} \\ * & * & * & -\mathbf{R} \end{bmatrix} < 0, \tag{14}$$

$\mathbf{\Xi}_{11} = 2\alpha\mathbf{P}_{i,m} + \mathbf{P}_{i,m}\mathbf{K}_{i,m} + \mathbf{K}_{i,m}^T\mathbf{P}_{i,m} + \mathbf{P}_{i,m}\bar{\mathbf{A}}_i + \bar{\mathbf{A}}_i^T\mathbf{P}_{i,m} - \lambda_{i,m}\bar{\mathbf{U}} - e^{-2\alpha\tau}\mathbf{R} + \mathbf{Q} + \sum_{n\in\mathcal{V}} \rho_{mn}\mathbf{P}_{i,n} + \sum_{j\in\mathcal{W}} \pi_{ij}^m\mathbf{P}_{j,m}$, $\mathbf{\Xi}_{12} = \mathbf{P}_{i,m}\bar{\mathbf{G}}_i + e^{-2\alpha\tau}\mathbf{R}$, $\mathbf{\Xi}_{13} = \mathbf{P}_{i,m}\bar{\mathbf{B}}_i - \lambda_{i,m}\bar{\mathbf{V}}$, $\mathbf{\Xi}_{22} = -e^{2\alpha\tau}\mathbf{Q} - e^{-2\alpha\tau}\mathbf{R}$, $\mathbf{\Xi}_{33} = -\lambda_{i,m}\mathbf{I}$,

$$\bar{\mathbf{U}} = \frac{(\mathbf{I}_N \otimes \mathbf{U})^T(\mathbf{I}_N \otimes \mathbf{V})}{2} + \frac{(\mathbf{I}_N \otimes \mathbf{V})^T(\mathbf{I}_N \otimes \mathbf{U})}{2},$$

$$\bar{\mathbf{V}} = -\frac{(\mathbf{I}_N \otimes \mathbf{U})^T + (\mathbf{I}_N \otimes \mathbf{V})^T}{2},$$

Proof. Consider the following Lyapunov-Krasovskii functional candidate for the error system (10)

$$V(\mathbf{e}_t, r_t, \sigma_t) = \sum_{k=1}^{3} V_k(\mathbf{e}_t, r_t, \sigma_t), \tag{15}$$

where

$$\begin{aligned}V_1(\mathbf{e}_t, r_t, \sigma_t) &= e^{2\alpha t}\mathbf{e}^T(t)\mathbf{P}_{r_t,\sigma_t}\mathbf{e}(t), \\ V_2(\mathbf{e}_t, r_t, \sigma_t) &= \int_{t-\tau}^{t} e^{2\alpha s}\mathbf{e}^T(s)\mathbf{Q}\mathbf{e}(s)ds, \\ V_3(\mathbf{e}_t, r_t, \sigma_t) &= \tau\int_{-\tau}^{0}\int_{t+\theta}^{t} e^{2\alpha s}\dot{\mathbf{e}}(s)^T(s)\mathbf{R}\dot{\mathbf{e}}(s)dsd\theta.\end{aligned}$$

Define $\mathcal{L}$ weak infinitesimal generator of the Markov process along the Lyapunov-Krasovskii functional $V(\mathbf{e}_t, r_t, \sigma_t)$ as follows:

$$\begin{aligned}\mathcal{L}V(\mathbf{e}_t, r_t, \sigma_t) = \lim_{h\to 0}\frac{1}{h}\{&\mathbb{E}\{V(\mathbf{e}_{t+h}, r_{t+h}, \sigma_{t+h})\,|\mathbf{e}_t\,, \\ &r_t = i, \sigma_t = m\} - V(\mathbf{e}_t, r_t = i, \sigma_t = m)\}.\end{aligned} \tag{16}$$

Then, it can be calculated that [17]

$$\begin{aligned}\mathcal{L}V_1(\mathbf{e}_t, r_t, \sigma_t) &= 2\alpha e^{2\alpha t}\mathbf{e}^T(t)\mathbf{P}_{i,m}\mathbf{e}(t) \\ &+ 2e^{2\alpha t}\mathbf{e}^T(t)\mathbf{P}_{i,m}\dot{\mathbf{e}}(t) \\ &+ e^{2\alpha t}\mathbf{e}^T(t)[\sum_{n\in\mathcal{V}} \rho_{mn}\mathbf{P}_{i,n} + \sum_{j\in\mathcal{W}} \pi_{ij}^m\mathbf{P}_{j,m}]\mathbf{e}(t),\end{aligned} \tag{17}$$

$$\begin{aligned}\mathcal{L}V_2(\mathbf{e}_t, r_t, \sigma_t) &= e^{2\alpha t}\mathbf{e}^T(t)\mathbf{Q}\mathbf{e}(t) \\ &- e^{2\alpha(t-\tau)}\mathbf{e}^T(t-\tau)\mathbf{Q}\mathbf{e}(t-\tau) \\ &+ \int_{t-\tau}^{t} e^{2\alpha s}\mathbf{e}^T(s)[\sum_{n\in\mathcal{V}} \rho_{mn}\mathbf{Q} + \sum_{j\in\mathcal{W}} \pi_{ij}^m\mathbf{Q}]\mathbf{e}(s)ds,\end{aligned} \tag{18}$$

$$\mathcal{L}V_3(\mathbf{e}_t, r_t, \sigma_t) = \tau^2 e^{2\alpha t}\mathbf{e}^T(t)\mathbf{R}\mathbf{e}(t) - \tau\int_{t-\tau}^{t} e^{2\alpha s}\dot{\mathbf{e}}^T(s)\mathbf{R}\dot{\mathbf{e}}(s)ds + \tau\int_{-\tau}^{0}\int_{t+\theta}^{t} e^{2\alpha s}\dot{\mathbf{e}}^T(s)[\sum_{n\in\mathcal{V}}\rho_{mn}\mathbf{R} + \sum_{j\in\mathcal{W}}\pi_{ij}^m\mathbf{R}]\dot{\mathbf{e}}(s)ds. \quad (19)$$

Based on Lemma 1 the integral $-\tau\int_{t-\tau}^{t} e^{2\alpha s}\dot{\mathbf{e}}^T(s)\mathbf{R}\dot{\mathbf{e}}(s)ds$ in (19) will be written as

$$-\tau\int_{t-\tau}^{t} e^{2\alpha s}(\dot{\mathbf{e}}(s)^T)\mathbf{R}(\dot{\mathbf{e}}(s))ds \leqslant -e^{2\alpha t}\left[\int_{t-\tau}^{t}(\dot{\mathbf{e}}(s))ds\right]^T(e^{-2\alpha\tau}\mathbf{R})\left[\int_{t-\tau}^{t}(\dot{\mathbf{e}}(s))ds\right]. \quad (20)$$

It is obvious that

$$\int_{t-\tau}^{t}\dot{\mathbf{e}}(s)ds = \mathbf{e}(t) - \mathbf{e}(t-\tau). \quad (21)$$

By substituting (21) into (20), one can obtain

$$-\tau\int_{t-\tau}^{t} e^{2\alpha s}(\dot{\mathbf{e}}(s)^T)\mathbf{R}(\dot{\mathbf{e}}(s))ds \leqslant -e^{2\alpha(t-\tau)}\begin{bmatrix}\mathbf{e}(t)\\ \mathbf{e}(t-\tau)\end{bmatrix}^T\begin{bmatrix}\mathbf{R} & -\mathbf{R}\\ -\mathbf{R} & \mathbf{R}\end{bmatrix}\begin{bmatrix}\mathbf{e}(t)\\ \mathbf{e}(t-\tau)\end{bmatrix}. \quad (22)$$

Also, based on the equations (3) and (5), for the integrals in (18) and (19) we have:

$$\begin{cases}\int_{t-\tau}^{t} e^{2\alpha s}\mathbf{e}^T(s)[\sum_{n\in\mathcal{V}}\rho_{mn}\mathbf{Q} + \sum_{j\in\mathcal{W}}\pi_{ij}^m\mathbf{Q}]\mathbf{e}(s)dsd\theta = 0,\\ \int_{-\tau}^{0}\int_{t+\theta}^{t} e^{2\alpha s}\mathbf{e}^T(s)[\sum_{n\in\mathcal{V}}\rho_{mn}\mathbf{R} + \sum_{j\in\mathcal{W}}\pi_{ij}^m\mathbf{R}]\mathbf{e}(s)dsd\theta = 0.\end{cases} \quad (23)$$

For any scalars $\lambda_{i,m} > 0$, $(i \in \mathcal{W}, m \in \mathcal{V})$, it can be found from (6) that the (24) holds

$$\lambda_{i,m}\begin{bmatrix}\mathbf{e}_i(t)\\ \mathbf{g}(\mathbf{e}_i(t))\end{bmatrix}^T\psi\begin{bmatrix}\mathbf{e}_i(t)\\ \mathbf{g}(\mathbf{e}_i(t))\end{bmatrix} \leqslant 0, \quad (24)$$

where

$$\psi = \begin{bmatrix}\frac{\mathbf{U}^T\mathbf{V}+\mathbf{V}^T\mathbf{U}}{2} & -\frac{\mathbf{U}^T+\mathbf{V}^T}{2}\\ * & \mathbf{I}\end{bmatrix}.$$

The above inequality is equivalent to

$$\mathbf{\Theta}(t) = e^{2\alpha t}\lambda_{i,m}\begin{bmatrix}\mathbf{e}(t)\\ \bar{\mathbf{g}}(\mathbf{e}(t))\end{bmatrix}^T\begin{bmatrix}\bar{\mathbf{U}} & \bar{\mathbf{V}}\\ * & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{e}(t)\\ \bar{\mathbf{g}}(\mathbf{e}(t))\end{bmatrix} \leqslant 0. \quad (25)$$

Then we conclude that

$$\begin{aligned}\mathcal{L}V(\mathbf{e}_t, r_t, \sigma_t) &\leqslant \sum_{k=1}^{3}\mathcal{L}V_k(\mathbf{e}_t, r_t, \sigma_t) - \Theta(t)\\ &\leqslant 2\alpha e^{2\alpha t}\mathbf{e}^T(t)\mathbf{P}_{i,m}\mathbf{e}(t) + 2e^{2\alpha t}\mathbf{e}^T(t)\mathbf{P}_{i,m}\dot{\mathbf{e}}(t)\\ &+e^{2\alpha t}\mathbf{e}^T(t)[\sum_{n\in\mathcal{V}}\rho_{mn}\mathbf{P}_{i,n} + \sum_{j\in\mathcal{W}}\pi_{ij}^m\mathbf{P}_{j,m}]\mathbf{e}(t) + e^{2\alpha t}\mathbf{e}^T(t)\mathbf{Q}\mathbf{e}(t)\\ &- e^{2\alpha(t-\tau)}\mathbf{e}^T(t-\tau)\mathbf{Q}\mathbf{e}(t-\tau) + \tau^2 e^{2\alpha t}\dot{\mathbf{e}}^T(t)\mathbf{R}\dot{\mathbf{e}}(t)\\ &-e^{2\alpha t}\begin{bmatrix}\mathbf{e}(t)\\ \mathbf{e}(t-\tau)\end{bmatrix}^T\begin{bmatrix}e^{-2\alpha\tau}\mathbf{R} & -e^{-2\alpha\tau}\mathbf{R}\\ -e^{-2\alpha\tau}\mathbf{R} & e^{-2\alpha\tau}\mathbf{R}\end{bmatrix}\begin{bmatrix}\mathbf{e}(t)\\ \mathbf{e}(t-\tau)\end{bmatrix}\\ &- e^{2\alpha t}\lambda_{i,m}\begin{bmatrix}\mathbf{e}(t)\\ \bar{\mathbf{g}}(\mathbf{e}(t))\end{bmatrix}^T\begin{bmatrix}\bar{\mathbf{U}} & \bar{\mathbf{V}}\\ * & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{e}(t)\\ \bar{\mathbf{g}}(\mathbf{e}(t))\end{bmatrix}.\end{aligned} \quad (26)$$

In other words, it can be derived that

$$\mathcal{L}V(\mathbf{e}_t, r_t, \sigma_t) \leqslant e^{2\alpha t}\xi^T(t)\Xi\xi(t),$$

where $\xi(t) = [\ \mathbf{e}^T(t), \mathbf{e}^T(t-\tau), \bar{\mathbf{g}}^T(\mathbf{e}(t))]^T$ and

$$\Xi = \begin{bmatrix}\mathbf{\Xi}_{11} & \mathbf{\Xi}_{12} & \mathbf{\Xi}_{13}\\ * & \mathbf{\Xi}_{22} & 0\\ * & * & \mathbf{\Xi}_{33}\end{bmatrix} + \begin{bmatrix}\tau(\mathbf{K}_{i,m}^T + \bar{\mathbf{A}}_i^T)\\ \tau\bar{\mathbf{G}}_i^T\\ \tau\bar{\mathbf{B}}_i^T\end{bmatrix}\mathbf{R}\begin{bmatrix}\tau(\mathbf{K}_{i,m}^T + \bar{\mathbf{A}}_i^T)\\ \tau\bar{\mathbf{G}}_i^T\\ \tau\bar{\mathbf{B}}_i^T\end{bmatrix}^T. \quad (27)$$

Based on the Schur complement, the equation (27) is equivalent to (14). If $\phi < 0$ then

$$\mathcal{L}V(\mathbf{e}_t, r_t, \sigma_t) < 0. \quad (28)$$

According to Dynkin's formula we have

$$\begin{aligned}\mathrm{E}\{V(\mathbf{e}_0, r_0, \sigma_0)\} &= \sum_{k=1}^{3}\mathrm{E}\{V_k(\mathbf{e}_0, r_0, \sigma_0)\}\\ &\leqslant \lambda_{\max}(\mathbf{P}_{i,m})\|\mathbf{e}(0)\|^2\\ &+ \lambda_{\max}(\mathbf{Q})\sup_{-\tau\leqslant\theta\leqslant 0}\|\mathbf{e}(\theta)\|^2\int_{-\tau}^{0}e^{2\alpha x}dx\\ &+ \lambda_{\max}(\mathbf{R})\sup_{-\tau\leqslant\theta\leqslant 0}\|\dot{\mathbf{e}}(\theta)\|^2\int_{-\tau}^{0}\int_{-s}^{0}e^{2\alpha x}dxds\\ &\leqslant a\sup_{-\bar{\tau}\leqslant\theta\leqslant 0}\|\mathbf{e}(\theta)\|^2 + b\sup_{-\bar{\tau}\leqslant\theta\leqslant 0}\|\dot{\mathbf{e}}(\theta)\|^2,\end{aligned} \quad , \quad (29)$$

where

$$a = \lambda_{\max}(\mathbf{P}_{i,m})\frac{1 - e^{-2\alpha\tau}}{2\alpha}\lambda_{\max}(\mathbf{Q}).$$

$$b = \frac{2\alpha\tau - 1 + e^{-2\alpha\tau}}{4\alpha^2}\lambda_{\max}(\mathbf{R}).$$

Hence

$$\mathrm{E}\{\|\mathbf{e}(t)\|^2\} \leqslant \frac{a+b}{\lambda_{\min}(\mathbf{P}_{i,m})} e^{-2\alpha t} \sup_{-\bar{\tau}\leqslant\theta\leqslant 0} \{\|\mathbf{e}(\theta)\|^2, \|\dot{\mathbf{e}}(\theta)\|^2\}. \tag{30}$$

The CDN (1) is exponential synchronized to the isolated node (9) by Definition 1. This completes proof.

In the following theorem, the mode-dependent controller of the form (9) has been designed.

Theorem 2: For given positive scalars α, τ the error system (10) is exponentially stabled in the mean square, if there exist symmetric positive definite matrices $\mathbf{P}_{i,m} = \mathrm{diag}\{\mathbf{P}^1_{i,m}, \mathbf{P}^2_{i,m}, ..., \mathbf{P}^N_{i,m}\}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{X}_{i,m} = \mathrm{diag}\{\mathbf{X}^1_{i,m}, \mathbf{X}^2_{i,m}, ..., \mathbf{X}^N_{i,m}\}$, $\mathbf{S}$, and scalars $\lambda_{i,m} > 0$ for any $i \in \mathcal{W}$, $m \in \mathcal{V}$, such that the following LMI hold:

$$\mathbf{\Phi} = \begin{bmatrix} \breve{\mathbf{\Xi}}_{11} & \mathbf{\Xi}_{12} & \mathbf{\Xi}_{13} & \tau\mathbf{X}^T_{i,m} + \tau\bar{\mathbf{A}}^T_i\mathbf{P}_{i,m} \\ * & \mathbf{\Xi}_{22} & 0 & \tau\bar{\mathbf{G}}^T_i\mathbf{P}_{i,m} \\ * & * & \mathbf{\Xi}_{33} & \tau\bar{\mathbf{B}}^T_i\mathbf{P}_{i,m} \\ * & * & * & \mathbf{RI} - 2\mathbf{P}_{i,m} \end{bmatrix} < 0, \tag{31}$$

where $\breve{\mathbf{\Xi}}_{11} = 2\alpha\mathbf{P}_{i,m} + \mathbf{X}_{i,m} + \mathbf{X}^T_{i,m} + \mathbf{P}_{i,m}\bar{\mathbf{A}}_i + \bar{\mathbf{A}}^T_i\mathbf{P}_{i,m} - \lambda_{i,m}\bar{\mathbf{U}} - e^{-2\alpha\tau}\mathbf{R} + \mathbf{Q} + \sum_{n\in\mathcal{V}} \rho_{mn}\mathbf{P}_{i,n} + \sum_{j\in\mathcal{W}} \pi^m_{ij}\mathbf{P}_{j,m}$ and the other parameters are the same as in Theorem 1. Moreover, the controller gain matrices are determined by:

$$\mathbf{K}_{i,m} = \mathbf{P}^{-1}_{i,m}\mathbf{X}_{i,m}. \tag{32}$$

Proof. Pre- and post-multiplying both sides of (18) by $\mathbf{J}^T = \mathrm{diag}\{\mathbf{I}, \mathbf{I}, \mathbf{I}, \mathbf{I}, \mathbf{P}_{i,m}\mathbf{R}^{-1}\}$ and $\mathbf{J}$, respectively, and substituting $\mathbf{X}_{i,m} = \mathbf{P}_{i,m}\mathbf{K}_{i,m}$, obtains

$$\mathbf{\Phi} = \begin{bmatrix} \breve{\mathbf{\Xi}}_{11} & \mathbf{\Xi}_{12} & \mathbf{\Xi}_{13} & \tau\mathbf{X}^T_{i,m} + \tau\bar{\mathbf{A}}^T_i\mathbf{P}_{i,m} \\ * & \mathbf{\Xi}_{22} & 0 & \tau\bar{\mathbf{G}}^T_i\mathbf{P}_{i,m} \\ * & * & \mathbf{\Xi}_{33} & \tau\bar{\mathbf{B}}^T_i\mathbf{P}_{i,m} \\ * & * & * & -\mathbf{P}_{i,m}\mathbf{R}^{-1}\mathbf{P}_{i,m} \end{bmatrix} < 0, \tag{33}$$

It is clear that the nonlinear term $-\mathbf{P}_{i,m}\mathbf{R}^{-1}\mathbf{P}_{i,m}$ leading to the (33) would not to be a LMI. Regarding that $\mathbf{R} > 0$ and $\mathbf{P}_{i,m}\mathbf{R}^{-1}\mathbf{P}_{i,m} - \mathbf{P}_{i,m}\mathbf{R}^{-1}\mathbf{P}_{i,m} \geqslant 0$, hence by the Schur complement, the (34) holds

$$\begin{bmatrix} \mathbf{P}_{i,m}\mathbf{R}^{-1}\mathbf{P}_{i,m} & -\mathbf{P}_{i,m} \\ * & \mathbf{RI} \end{bmatrix} \geqslant 0, \tag{34}$$

corresponding to [13], it is proved that:

$$-\mathbf{P}_{i,m}\mathbf{R}^{-1}\mathbf{P}_{i,m} \leqslant \mathbf{RI} - 2\mathbf{P}_{i,m}. \tag{35}$$

If (31) holds, then (33) holds. Thus, this completes the proof.

IV. Numerical example

In this section, the following example is provided to demonstrate the effectiveness of the drived results. The given scalars are $\alpha = 0.2$ and $\tau = 0.5$. The MJCDN (1) with three identical nodes, where state dynamics of each node is Chua's circuit described by

$$\mathbf{f}(\mathbf{x}_i(t)) = \begin{bmatrix} -a(h(\mathbf{x}_{i1}(t))) \\ 0 \\ 0 \end{bmatrix},$$

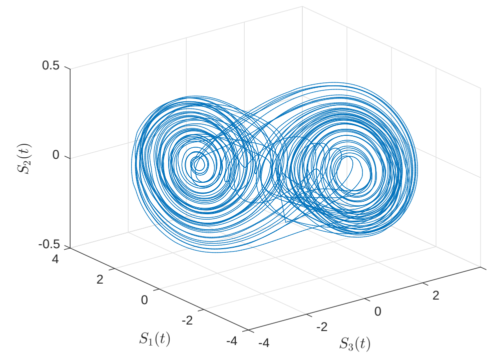

Fig. 1. The chaotic behavior of (7)

where $h(\mathbf{x}_{i1}(t)) = -0.5(m_1 - m_0)(|\mathbf{x}_{i1}(t)+1| - |\mathbf{x}_{i1}(t)-1|)$, $a = 9$, $m_0 = \frac{-1}{7}$, and $m_1 = \frac{2}{7}$.

Two following matrices satisfy the sector-bounded condition in (7)

$$\mathbf{U} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{V} = \begin{bmatrix} 9(\frac{3}{7}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

Consider $\mathcal{W} = \{1, 2\}$ for low-level signal $\{r_t, t \geqslant 0\}$, therefore the mode-dependent coefficient matrices are considered as

$$\mathbf{A}_1 = \begin{bmatrix} -9(\frac{2}{7}) & 9 & 0 \\ 1 & -1 & 1 \\ 0 & -14.286 & 0 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} -9.1(\frac{2}{7}) & 9.2 & 0 \\ 1.1 & -0.9 & 1.1 \\ 0 & -14.286 & 0.1 \end{bmatrix},$$

$$\mathbf{B}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 0.9 \end{bmatrix},$$

Fig. 1 represents the chaotic behavior (double scroll) of the isolated node (7).

Moreover, inner coupling matrices for each mode $\{r_t, t \geqslant 0\}$ are $\mathbf{\Gamma}_1 = \mathbf{I}_{3\times3}$ and $\mathbf{\Gamma}_2 = 0.8\mathbf{\Gamma}_1$, outer coupling matrices are respectively as

$$\mathbf{G}_1 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \quad \mathbf{G}_2 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix},$$

The $\mathcal{V} = \{1, 2, 3\}$ for the high level signal $\{\sigma_t, t \geq 0\}$ is regarded, so the piecewise-homogeneous transition rate matrices of low-level Markov signal $\{r_t, t \geq 0\}$ for each modes in $\mathcal{V}$ are given as

$$\Pi^1 = \begin{bmatrix} -4.5 & 4.5 \\ 3.75 & -3.75 \end{bmatrix}, \Pi^2 = \begin{bmatrix} -2.75 & 2.75 \\ 3 & -3 \end{bmatrix},$$

$$\Pi^3 = \begin{bmatrix} -4 & 4 \\ 1.5 & -1.5 \end{bmatrix}.$$

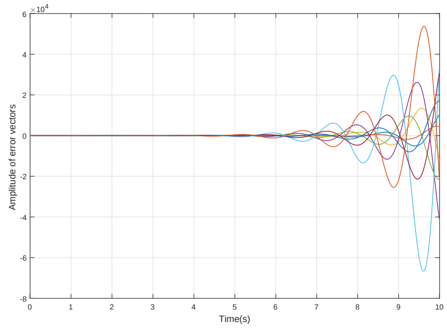

Fig. 2. Synchronization errors for the MJCDN without control input

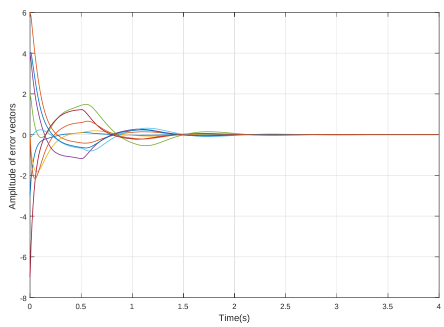

Fig. 3. Synchronizing control effort for MJCDN with control input

and the transition rate matrix of the high signal is as follows:

$$\Lambda = \begin{bmatrix} -4.5 & 2 & 2.5 \\ 4.5 & -7.5 & 3 \\ 1 & 2 & -3 \end{bmatrix}.$$

The state trajectories of the synchronization errors without control input and the state trajectories of the controlled synchronization errors are shown in Fig. 2 and Fig.3 respectively. The initial state values for the isolated node are assumed as $\mathbf{s}(0) = [-0.2, -0.3, 0.2]^T$. The nodes of the network have initial values as $\mathbf{x}_1(0) = [-3, 6, 0]^T$, $\mathbf{x}_2(0) = [4, 2, 0]^T$, and $\mathbf{x}_3(0) = [-7, 4, 0]^T$. The upper bounds on the time delay for different values of the decay rate α are listed in Table 1.

TABLE I
Upper bound of τ for different decay rate α

Upper bound of delay τ	0.81	0.75	0.7	0.6
Decay rate α	0.2	0.8	1.2	1.8

V. Conclusion

In this paper, the exponential mean-square synchronization has been studied for the piecewise-homogeneous MJCDN 1 with coupling delay. All of the coefficient matrices in our model are considered to be mode-dependent. The piecewise-homogeneous Markovian jump structure is suitable for modeling of the Markovian process with time-varying TRs. By using Lyapunov-Krasovskii approach, delay-dependent criteria have been derived to guarantee the exponential synchronization problem. Based on the obtained results, the synchronization controllers are then designed in terms of LMIs. Finally one numerical example has been proposed to demonstrate the usefulness the proposed results.

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