UNIVERSITY OF ILLINOIS

COLLEGE OF ENGINEERING

ENGINEERING EXPERIMENT STATION

BULLETIN 478

ULTIMATE DESIGN OF PRESTRESSED CONCRETE BEAMS

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Price: $2.50

Edited by
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UNIVERSITY OF ILLINOIS BULLETIN
Volume 62, Number 79; April, 1965. Published nine times each month by the University of Illinois. Entered as second-class matter December 11, 1912, at the post office at Urbana, Illinois, under the Act of August 24, 1912. Office of Publication, 114 Altgeld Hall, Urbana, Illinois 61803.

ACKNOWLEDGMENT

The authors extend their thanks for the comments and suggestions of Professor K. Preiss, University of Illinois, Carl E. Thunman, Jr., Illinois Division of Highways, and Professor J. L. Merritt, University of Illinois, who reviewed the manuscript.

ABSTRACT

A GENERAL METHOD OF ANALYSIS OF PRESTRESSED CONCRETE BEAMS AT ULTIMATE IS PRESENTED. EQUATIONS FOR THE DETERMINATION OF ULTIMATE MOMENT ARE DERIVED. SIMPLIFIED EXPRESSIONS ARE PRESENTED FOR ULTIMATE MOMENT FOR USE IN DESIGN. A METHOD IS PRESENTED BY WHICH PRESTRESSED CONCRETE BEAMS CAN BE DESIGNED ON THE BASIS OF STRENGTH AND DUCTILITY. EXAMPLES OF DESIGN ARE CONSIDERED, IN WHICH ONLY THE REQUIRED DUCTILITY VARIES. A COMPARISON OF THE THREE EXAMPLES SHOWS THE INFLUENCE OF THE REQUIRED DUCTILITY ON THE DIMENSIONS OF THE BEAM, AND THE INFLUENCE OF COMPRESSION STEEL ON DUCTILITY. THE STRESSES AT TRANSFER AND AT SERVICE CONDITIONS ARE CALCULATED AND TABULATED.

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FIGURES

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NOTATION

a	=	distance from the neutral axis to the top fiber
A	=	gross cross-sectional area of the beam
A_s	=	area of prestressed steel
A'_s	=	area of non-prestressed compression steel
A_{sf}	=	area of prestressed steel that develops the rectangle $(b - b')t$ in the compression zone
A_{sr}	=	$A_s - A_{sf}$
b	=	width of compression zone or top flange
b'	=	web thickness
c_1	=	ratio of average to maximum stress for the flange
c_2	=	ratio of distance between the point of action of flange force and the top fiber, to the flange thickness
d	=	distance from the center of gravity of prestressed steel to the top fiber
d'	=	distance from the center of gravity of the non-prestressed compression steel to the top fiber
F	=	a compatibility factor
$F(\varepsilon_{su})$	=	f_{su}, equation of the stress-strain diagram of prestressed steel
$f(\varepsilon)$	=	stress in the concrete, equation of the stress-strain diagram of concrete
f'_c	=	cylinder strength of concrete at 28 days
f_{su}	=	stress in prestressed steel at failure
f'_s	=	ultimate strength of prestressing steel
f'_{su}	=	stress in non-prestressed compression steel at failure
f_y	=	yield point of non-prestressed compression steel
f^*_y	=	$f_y - 0.85\ f'_c$
$G(\varepsilon'_{su})$	=	f'_{su}, equation of the stress-strain diagram of non-prestressed compression steel
h	=	over-all depth of the beam
k	=	ratio of the width of bottom flange to that of top flange
k_1	=	ratio of the average to maximum stress in the compression zone

k_2 = ratio defining the position of the center of gravity of the compression force contributed by concrete; ak_2 is the distance of center of gravity from the top fiber

k_3 = ratio of strength of concrete in beam to that of cylinder

L = span length of a simply supported beam

M_g = moment due to the weight of the beam

M_ℓ = moment due to the live load

M_s = moment due to the superimposed dead load

M_u = required flexural strength of the beam

N_d = load factor for the dead load

N_ℓ = load factor for the live load

p = percentage of prestressed steel, A_s/bd

p' = percentage of non-prestressed compression steel, A'_s/bd

Q = $M_u/\phi bd^2 f'_c$

q = the ratio $p\ f_{su}/f'_c$

q' = the ratio $p'\ f^*_y/f'_c$

t = flange thickness

X = one half the theoretically required length of non-prestressed compression bars

γ = unit weight of concrete

ϵ = strain

ϵ_{ce} = strain in concrete at the level of steel due to effective prestress

ϵ_{se} = strain in prestressed steel due to effective prestress

$\epsilon_{s\ell}$ = limiting strain in prestressed steel

ϵ_{su} = strain in the prestressed steel at ultimate

ϵ'_{su} = strain in the non-prestressed compression steel at ultimate

ϵ_u = the ultimate strain of concrete in flexural compression

ϵ_y = strain at yield of non-prestressed steel

ϕ = a capacity reduction factor

φ = curvature of the section

ψ = a dimensionless shape factor, A/bh

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I. INTRODUCTION

Design of prestressed concrete beams is based upon two distinct concepts which lead to two design methods known as service load design or working stress design, and ultimate design.

In service load design the stresses in the beam are calculated on the basis of the assumption that concrete is an elastic material. These calculated stresses are to be less than, or equal to, certain limiting stresses known as allowable stresses. The allowable stresses are chosen so that the structure will perform its intended service satisfactorily under service conditions while providing indirectly for the safety of the beam.

In ultimate design the flexural strength or ultimate moment of the section is calculated based on the knowledge of behavior of the beam. The calculated ultimate moment is to be equal to, or greater than, the sum of moments of all forces each multiplied by a factor. These are known as load factors and are chosen so that the structure will be sufficiently safe under the service conditions. Moreover, ultimate design also requires certain ductility in the beam, so that prior to failure the beam will deform sufficiently. Ductility is measured by the deformation of the beam at failure.

In our present practice prestressed concrete beams are in most cases designed and proportioned by working stress design. The provisions of ultimate design are used to check the flexural strength of a section that has already been designed. Furthermore, the provisions for ultimate design in our present specifications are more suitable for calculating the flexural strength of a given section.

It should be pointed out that there is a relationship between working stress design and ultimate design. Although they have different bases, in fulfilling the objective of one, the objective of the other is satisfied to a certain extent.(1)*

It can be shown that the provisions of ultimate design can be used to proportion a section with a more rigorous control of ductility. The provisions of working stress design can then be used to check working stresses in the section so designed. Furthermore, rational design of a section is considerably simpler by ultimate design than by service load design.

The purpose of this study is to develop a method by which a prestressed concrete beam can be proportioned by the provisions of ultimate design. It is intended to show the importance of ductility in its influence upon the dimensions of the beam. In addition it is intended to study the influence of compression steel on ductility and the proportions of a section.

We will consider a simply supported bonded beam and assume that the strength of the beam is measured by flexure. We will assume that the only loads acting, in addition to the prestressing force, are the weight of the beam, the superimposed dead load, and live load.

*The numbers in parentheses refer to the entries in Chapter VII, References.

II. CALCULATION OF ULTIMATE MOMENT

A. ASSUMPTIONS

For the sake of simplicity the determination of ultimate moment is discussed here for prestressed concrete beams which have an idealized section as shown in Figure 1. The section considered is flanged; the prestressed steel is assumed to be bonded to concrete, and in addition to prestressed steel the section is assumed to have non-prestressed compression steel.

Practical sections sometimes contain non-prestressed tensile reinforcement which increase the flexural strength of the section and reduces its ductility. Non-prestressed tensile reinforcement is not considered here since it does not contribute to our primary purpose which is to develop a method of design which leads to the lightest section for a given strength and ductility.

The calculation of the ultimate moment is usually based upon the following assumptions:

1. The strain distribution in concrete varies linearly with depth in the compression zone of the beam.

2. The stress-strain diagrams for the prestressed as well as non-prestressed reinforcement are known.

3. Failure occurs when the strain in concrete at the top fiber reaches a limiting value.

4. The strain in non-prestressed compression steel is equal to the strain in concrete at the level of compression steel.

5. The average strain in steel is not greatly different from the maximum strain.

In addition to the above assumptions, the tension contributed by concrete is usually neglected since it is small at ultimate.

The neutral axis at failure may be either in the flange or below the flange depending upon the dimensions of the beam, the amount of steel and the properties of both steel and concrete. We will first consider the case in which the neutral axis falls in the flange.

B. FLEXURAL STRENGTH OF SECTIONS IN WHICH THE NEUTRAL AXIS AT ULTIMATE FALLS IN THE FLANGE

In this case the width of the compression flange is constant and is equal to b, as shown in Figure 2.

We will take $f = f(\varepsilon)$ as the equation for the stress-strain diagram for concrete and assume that this equation is the same for all the fibers in the compression zone of the beam. Since the width of the compression flange is constant and the strain distribution in the compression zone is assumed to be linear with depth, the ultimate moment can be written as follows:

$$M_u = \frac{a^2 b}{\varepsilon_u^2} \int_0^{\varepsilon_u} \varepsilon f(\varepsilon)\, d\varepsilon + A_s f_{su}(d - a)$$

$$+ A'_s f'_{su}\ (a - d') \qquad (1)$$

where a = distance from neutral axis to the top fiber

ε_u = the limiting strain

f_{su} = stress in prestressed steel

at failure

f'_{su} = stress in non-prestressed compression steel at failure

b = width of compression zone or top flange

d = distance from the center of gravity of prestressed steel to the top fiber

d' = distance from the center of gravity of the non-prestressed compression steel to the top fiber

A_s = area of prestressed steel

A'_s = area of non-prestressed compression steel

The above equation shows the sum of moment of all forces about the neutral axis which is the same as the bending moment at the section due to the external loads. The first term on the right side of the above equation is the compression force contributed by concrete and does not take into account the area of concrete replaced by compression steel. This effect is usually small, but it can be included by subtracting the term $A'_s f(\frac{\epsilon_u}{a}(a - d'))(a - d')$ from the right side of the equation.

It can be seen from Equation 1 that for a given section M_u depends on three quantities that are not known, namely a, the depth to the neutral axis; f_{su}, the stress in the prestressed steel; and f'_{su}, the stress in the non-prestressed compression steel -- if $f = f(\epsilon)$ the stress-strain diagram for concrete were known. Therefore, it is necessary to obtain other relations in order to be able to compute these unknown quantities.

From the equilibrium of horizontal forces in the section we have the following equation:

$$\frac{ab}{\epsilon_u}\int_0^{\epsilon_u} f(\epsilon)\, d\epsilon + A'_s f'_{su} = A_s f_{su}. \qquad (2)$$

As before, in the above equation the effect of the area of concrete replaced by compression steel is neglected. It may be included by adding the term $(A'_s f(\frac{\epsilon_u}{a}(a - d'))$ to the right side of the equation.

The strain in prestressed steel at failure can be expressed as the sum of the following quantities:

1. The strain in prestressed steel due to effective prestress designated as ϵ_{se}. Effective prestress is the magnitude of prestress after losses, or at the time of determination of ultimate moment.

2. The additional strain in steel that is induced by sufficient load to make the strain in concrete at the level of prestressed steel equal to zero. It can be shown that in bonded beams this addition to the strain in steel is equal to ϵ_{ce}, the strain in concrete at the level of the prestressed steel due to effective prestress.

Hence the total strain in steel at the load corresponding to zero strain in concrete at the level of steel is

$$\epsilon_{se} + \epsilon_{ce}.$$

3. Additional strain in steel from the load corresponding to zero strain in concrete at the level of steel to ultimate. This addition can be expressed as follows:

$$\frac{\epsilon_u}{a}(d - a)\, F.$$

The quantity F in the above expression is a compatibility factor. If concrete were not cracked and were bonded with steel its magnitude would be one. Since at ultimate, concrete at the level of prestressed steel is cracked, the value of F in a particular section depends on the position of crack and condition of bond. Its magnitude is usually less than one at ultimate, however it is difficult to predict its value for a particular section in a given beam. A detailed discussion of its magnitude is available.[2]

Hence ε_{su}, the strain in prestressed steel at failure, can be expressed as the sum of the quantities listed above

$$\varepsilon_{su} = \varepsilon_{se} + \varepsilon_{ce} + \frac{\varepsilon_u}{a} (d - a)F.$$

Since ε_{ce} is small in comparison with

$$\varepsilon_{se} + \frac{\varepsilon_u}{a} (d - a)F$$

it is often neglected.

In the following discussions the value of F will be taken as 1.0 for bonded beams. Hence the expression for the strain in prestressed steel at failure can be written as follows:

$$\varepsilon_{su} = \varepsilon_{se} + \varepsilon_{ce} + \frac{\varepsilon_u}{a} (d - a). \qquad (3)$$

Since we have assumed that the strain in concrete at the level of non-prestressed compression steel is equal to the strain in compression steel, we have

$$\varepsilon'_{su} = \frac{\varepsilon_u}{a} (a - d') \qquad (4)$$

where ε'_{su} = the strain in the non-prestressed compression steel at ultimate

We have also assumed that the stress-strain diagrams for prestressed steel and non-prestressed compression steel are known. The equations for these diagrams are designated as $F(\varepsilon_{su})$ and $G(\varepsilon'_{su})$ respectively.

$$f_{su} = F(\varepsilon_{su}) \qquad (5)$$

$$f'_{su} = G(\varepsilon'_{su}) \qquad (6)$$

In order to obtain M_u it is necessary to solve Equations 1,2,3,4,5, and 6 simultaneously for the six unknowns M_u, a, ε_{su}, f_{su}, ε'_{su}, and f'_{su}.

In a special case in which the section has no compression steel, ε'_{su} and f'_{su} as well as Equations 4 and 6, vanish and the problem is reduced to the solution of four equations for four unknowns. In this case Equations 1, 2, 3, and 5 are the relations, and M_u, a, f_{su}, and ε_{su} are the unknowns.

C. FLEXURAL STRENGTH OF SECTION IN WHICH THE NEUTRAL AXIS AT ULTIMATE FALLS BELOW THE FLANGE

When the neutral axis at ultimate falls below the flange the expression for M_u is the following:

$$M_u = \frac{b'a^2}{\varepsilon^2} \int_0^{\varepsilon_u} \varepsilon f(\varepsilon)\, d\varepsilon + \frac{(b - b')a^2}{\varepsilon_u^2} \int_{\varepsilon_u(a-t)/a}^{\varepsilon_u} \varepsilon f(\varepsilon)\, d\varepsilon + A_s f_{su} (d - a) + A'_s f'_{su} (a - d') \qquad (7)$$

where b' = the web thickness

t = the flange thickness.

The first term on the right side of the above equation is moment due to the compression force developed by a rectangle of width b' and depth a about the neutral axis. The second term is the moment of the compression force contributed by a rectangle of width $(b - b')$ and depth t about the neutral axis. Figure 3 shows all the forces of a section in which the neutral axis at ultimate is below the flange. As before, the area of concrete replaced by compression steel is neglected.

The expression $f(\varepsilon)$, the stress-strain diagram in concrete, is assumed known and applicable to each compressed fiber of the beam. In Equation 7 there are quantities a, f_{su}, f'_{su} which are unknown and should be determined by other relations.

From the equilibrium of horizontal forces we have:

$$\frac{b'a}{\epsilon_u}\int_0^{\epsilon_u} f(\epsilon)\, d\epsilon + \frac{(b - b')a}{\epsilon_u}\int_{\epsilon_u(a-t)/a}^{\epsilon_u} f(\epsilon)\, d\epsilon$$

$$+ A'_s f'_{su} = A_s f_{su}. \qquad (8)$$

As before the area of concrete replaced by the compression steel has been neglected.

Equations 3, 4, 5, and 6 are equally applicable in this case. Hence in order to find M_u it is necessary to solve Equations 7, 8, 3, 4, 5, and 6 simultaneously for the six unknowns M_u, a, f_{su}, ϵ_{su}, f'_{su}, and ϵ'_{su}.

D. SUMMARY OF GENERAL EQUATIONS

The ultimate moment in a section in which the neutral axis at failure is in the flange, or when $a < t$, is obtained by the simultaneous solutions of the following equations:

$$M_u = \frac{a^2 b}{\epsilon_u^2}\int_0^{\epsilon_u} \epsilon f(\epsilon)\, d\epsilon + A_s f_{su}\,(d - a)$$

$$+ A'_s f'_{su}\,(a - d') \qquad (1)$$

$$\frac{ab}{\epsilon_u}\int_0^{\epsilon_u} f(\epsilon)\, d\epsilon + A'_s f'_{su} = A_s f_{su} \qquad (2)$$

$$\epsilon_{su} = \epsilon_{se} + \epsilon_{ce} + \frac{\epsilon_u}{a}(d - a) \qquad (3)$$

$$\epsilon'_{su} = \frac{\epsilon_u}{a}(a - d') \qquad (4)$$

$$f_{su} = F(\epsilon_{su}) \qquad (5)$$

$$f'_{su} = G(\epsilon'_{su}). \qquad (6)$$

The ultimate moment for a section in which the neutral axis at failure is below the flange, i.e., when $t < a$, is obtained by the simultaneous solution of the following equations:

$$M_u = \frac{b'a^2}{\epsilon_u^2}\int_0^{\epsilon_u} \epsilon f(\epsilon)\, d\epsilon$$

$$+ \frac{(b - b')a^2}{\epsilon_u^2}\int_{\epsilon_u(a-t)/a}^{\epsilon_u} \epsilon f(\epsilon)\, d\epsilon$$

$$+ A_s f_{su}\,(d - a) + A'_s f'_{su}\,(a - d') \qquad (7)$$

$$\frac{b'a}{\epsilon_u}\int_0^{\epsilon_u} f(\epsilon)\, d\epsilon + \frac{(b - b')a}{\epsilon_u}\int_{\epsilon_u(a-t)/a}^{\epsilon_u} f(\epsilon)\, d\epsilon$$

$$+ A'_s f'_{su} = A_s f_{su} \qquad (8)$$

$$\epsilon_{su} = \epsilon_{se} + \epsilon_{ce} + \frac{\epsilon_u}{a}(d - a) \qquad (3)$$

$$\epsilon'_{su} = \frac{\epsilon_u}{a}(a - d') \qquad (4)$$

$$f_{su} = F(\epsilon_{su}) \qquad (5)$$

$$f'_{su} = G(\epsilon'_{su}). \qquad (6)$$

It should be remembered that in Equations 1 and 2 as well as 7 and 8 the area of concrete replaced by compression steel is not taken into account.

E. CALCULATION OF ULTIMATE MOMENT BY THE PROPERTIES OF STRESS BLOCK

The procedure for the calculation of ultimate moment discussed in the preceding sections is the most general formulation for the problem of computing ultimate moment and is consistent with the assumptions made. These assumptions are reasonable and are in agreement with observations.

Practically, however, there are difficulties in solving the problem by this procedure. Equations 1 and 7 for ultimate moment, as well as Equations 2 and 8 for equilibrium of horizontal forces, depend upon $f(\varepsilon)$ the stress-strain diagram for concrete. The stress-strain diagram for concrete is non-linear and may change shape as concrete strength changes. In addition the stress-strain diagram obtained from a concrete cylinder may not truly represent this relationship for all the fibers of the beam in the compression zone, and obtaining a stress-strain diagram for the concrete in the beam by testing a beam is a lengthy procedure. Even if a reasonable expression for $f(\varepsilon)$ were established from tests, the non-linear nature of the expressions for the compression force and moment contributed by concrete, make the solution of the non-linear simultaneous equations tedious.

Since strain varies linearly with depth in the compression zone, by adopting a stress-strain diagram for concrete, the stress distribution with depth will be defined. From the shape of stress-distribution in the compression zone both the force and the moment contributed by concrete can be calculated for any section even if the width of the section is variable. The shape of the stress distribution in the compressed zone of concrete is called the stress block.

To simplify the problem for a practical solution a somewhat different approach is introduced. In the equations of equilibrium of horizontal forces and moments, it is only necessary to know the compression force and moment contributed by concrete. If we can somehow find a way of estimating the force and moment contributed by concrete without knowing the actual distribution of stress in the compressed zone, there would be no need in having the stress-strain diagram for concrete.

This can be achieved if the average stress in the section and the point of action of the compression force contributed by concrete were known. That is, if the area and centroid of the stress block were known, the ultimate moment could be determined.

Let us first consider the case in which the width of compression zone is constant, i.e., the neutral axis at ultimate falls in the flange. In this case let us express the force and moment contributed by concrete as follows:

$$\frac{ab}{\varepsilon_u} \int_0^{\varepsilon_u} f(\varepsilon)\, d\varepsilon = k_1 k_3 f'_c ab \qquad (9)$$

$$\frac{a^2 b}{\varepsilon_u^2} \int_0^{\varepsilon_u} \varepsilon f(\varepsilon)\, d\varepsilon = k_1 k_3 f'_c ab\,(a - ak_2) \qquad (10)$$

where k_1 = ratio of the average to maximum stress in the compression zone

k_3 = ratio of strength of concrete in beam to that of cylinder

k_2 = a ratio defining the position of the center of gravity of the compression force contributed by concrete; ak_2 is the distance of center of gravity from the top fiber.

Equation 9 gives the compression force contributed by concrete, and Equation 10 gives the moment of this force about the neutral axis. Figure 4 shows the forces in the section. If the ratios k_1, k_3, and k_2 can be determined, this procedure is very convenient.

Substituting Equation 9 for the compression force contributed by concrete in Equation 2 we have

$$k_1 k_3 f'_c ab + A'_s f'_{su} = A_s f_{su} \qquad (2a)$$

or $$k_1\, k_3 = \frac{A_s f_{su} - A'_s f'_{su}}{f'_c\, ab}. \qquad (11)$$

Equation 11 indicates that $k_1\, k_3$ can be measured from tests, since all the quantities at the right side are either known or can be measured. The quantity $k_1\, k_3$ for the materials commonly used is in the neighborhood of 0.7. In a similar fashion k_2 can be evaluated since the moment in the section can be measured. The quantity k_2 varies around 0.42 and has little influence on the ultimate moment.(3)

This approach is convenient in cases where the width of compression zone is constant. Equation 1 can now be written as follows:

$$M_u = k_1\, k_3\, f'_c\, ab\, (a - ak_2) + A_s f_{su}\, (d - a)$$

$$+ A'_s f'_{su}\, (a - d').$$

This equation represents the sum of moment of all forces in the section about the neutral axis. In this case it can also be written in the following convenient form by taking moments about the centroid of the compression stress block:

$$M_u = A_s f_{su}(d - ak_2) + A'_s f'_{su}(ak_2 - d'). \qquad (1a)$$

Equation 2 for the equilibrium of horizontal forces in the section becomes the same as Equation 2a.

This method is inadequate when the width of the compression flange becomes non-uniform. This condition occurs when the neutral axis at ultimate falls below the flange. For the section shown in Figure 5 let us express the compression force developed by the rectangle $b'a$ as

$$\frac{b'a}{\varepsilon_u} \int_0^{\varepsilon_u} f(\varepsilon)\, d\varepsilon = k_1\, k_3\, f'_c\, a\, b'$$

and the compression force developed by the rectangle $(b - b')t$ as

$$\frac{(b - b')a}{\varepsilon_u} \int_{\varepsilon_u (a-t)/a}^{\varepsilon_u} f(\varepsilon)\, d\varepsilon = c_1 f'_c\, (b-b')t,$$

where c_1 is the ratio of average to maximum stress for the flange. Since the stress distribution is undefined, the quantity c_1 is undefined. It can vary widely between 0.7 and 1.00.

The moments of these compression forces about the neutral axis are

$$\frac{b'a^2}{\varepsilon_u^2} \int_0^{\varepsilon_u} f(\varepsilon)\varepsilon\, d\varepsilon$$

$$= k_1\, k_3\, f'_c\, ab'\, (a - ak_2)$$

and $$\frac{(b - b')a^2}{\varepsilon_u^2} \int_{\varepsilon_u (a-t)/a}^{\varepsilon_u} \varepsilon f(\varepsilon)\, d\varepsilon$$

$$= c_1\, f'_c\, (b - b')\, t\, (a - c_2 t),$$

where c_2 is the ratio of the distance between the point of action of flange force and the top fiber, to the flange thickness. The quantity c_2 is undefined; however, it does not influence the moment appreciably. It is in the neighborhood of 0.5. Figure 5 shows the forces in the section.

Equation 7 can now be written:

$$M_u = k_1\, k_3\, f'_c\, ab'\, (a - ak_2)$$

$$+ c_1\, f'_c\, (b - b')\, t\, (a - c_2 t)$$

$$+ A_s f_{su}\, (d - a) + A'_s f'_{su}\, (a - d').$$

The equation gives the sum of moments of all forces in the section about the neutral axis. Often it is more convenient to take moments about the center of gravity of prestressed steel. In this case we have

$$M_u = k_1\ k_3\ f'_c\ ab'\ (d - k_2 a)$$
$$+ c_1\ f'_c\ (b - b')\ t\ (d - c_2 t)$$
$$+ A'_s f'_{su}\ (d - d'). \qquad (7a)$$

It is also possible to take moments about the compression force contributed by the web. In this case the ultimate moment can be expressed:

$$M_u = A_s f_{su}\ (d - k_2 a)$$
$$+ c_1\ f'_c\ (b - b')\ t\ (k_2 a - c_2 t)$$
$$+ A'_s f'_{su}\ (k_2 a - d'). \qquad (7b)$$

Equation 8 for the equilibrium of horizontal forces in the section can be written:

$$k_1\ k_3\ f'_c\ a\ b' + c_1\ f'_c\ (b - b')\ t$$
$$+ A'_s f'_{su} = A_s f_{su}. \qquad (8a)$$

In summary, when the neutral axis at failure is in the flange, or when $t > a$, the ultimate moment in the section may be obtained by a simultaneous solution of the following equations:

$$M_u = A_s f_{su}\ (d - ak_2)$$
$$+ A'_s f'_{su}\ (ak_2 - d') \qquad (1a)$$

$$k_1\ k_3\ f'_c\ ab + A'_s f'_{su} = A_s f_{su} \qquad (2a)$$

$$\varepsilon_{su} = \varepsilon_{se} + \varepsilon_{ce} + \frac{\varepsilon_u}{a}\ (d - a) \qquad (3)$$

$$\varepsilon'_{su} = \frac{\varepsilon_u}{a}\ (a - d') \qquad (4)$$

$$f_{su} = F(\varepsilon_{su}) \qquad (5)$$

$$f'_{su} = G(\varepsilon'_{su}). \qquad (6)$$

When the neutral axis is below the flange, or when $t < a$, the ultimate moment may be calculated by a simultaneous solution of the following six equations.

$$M_u = k_1\ k_3\ f'_c\ ab'\ (d - k_2 a)$$
$$+ c_1\ f'_c\ (b - b')\ t\ (d - c_2 t)$$
$$+ A'_s f'_{su}\ (d - d') \qquad (7a)$$

$$k_1\ k_3\ f'_c\ ab' + c_1\ f'_c\ (b - b')\ t$$
$$+ A'_s f'_{su} = A_s f_{su} \qquad (8a)$$

$$\varepsilon_{su} = \varepsilon_{se} + \varepsilon_{ce} + \frac{\varepsilon_u}{a}\ (d - a) \qquad (3)$$

$$\varepsilon'_{su} = \frac{\varepsilon_u}{a}\ (a - d') \qquad (4)$$

$$f_{su} = F(\varepsilon_{su}) \qquad (5)$$

$$f'_{su} = G(\varepsilon'_{su}) \qquad (6)$$

As before, the area of concrete replaced by compression steel has been neglected in Equations 1a, 2a, 7a, and 8a.

The present thinking and practice in prestressed concrete is based upon the above equations. Instead of defining the stress-strain diagram the properties of the stress block k_1, k_3, k_2, c_1, and c_2 are specified. It should be pointed out that while $k_1\ k_3$ and

FIGURES and TABLES

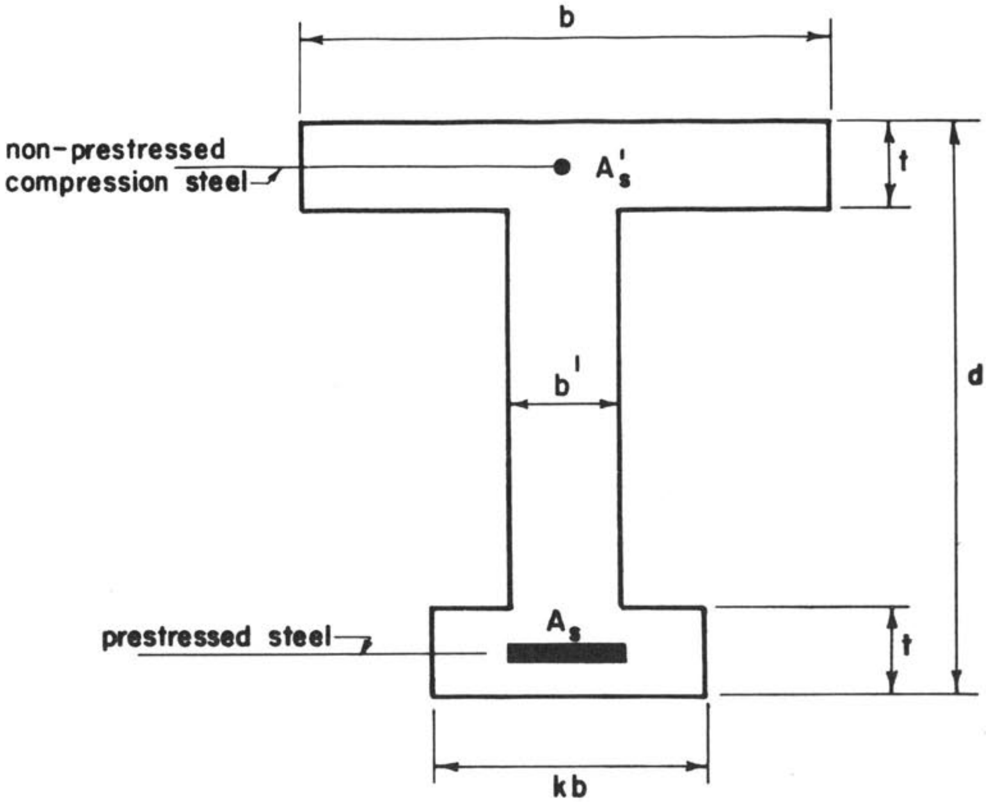

FIGURE 1. IDEALIZED I-SECTION

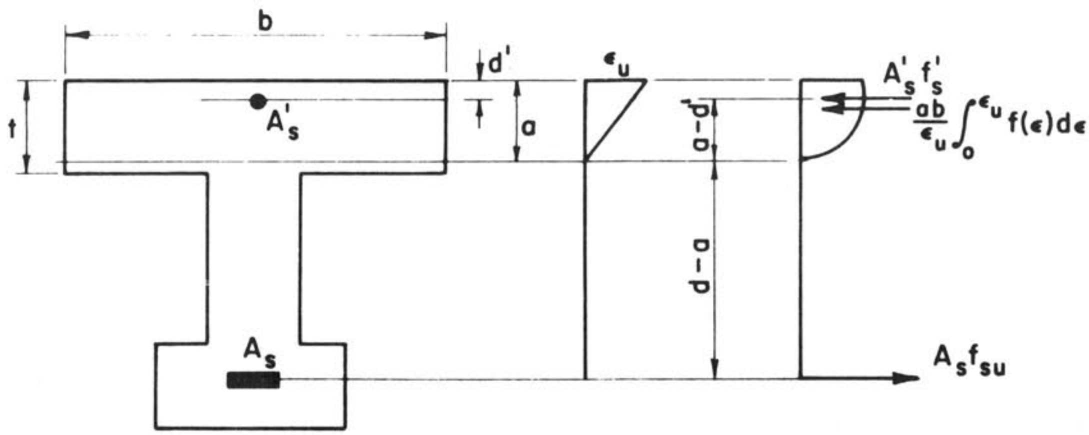

FIGURE 2. FLANGED SECTION, NEUTRAL AXIS IN THE FLANGE

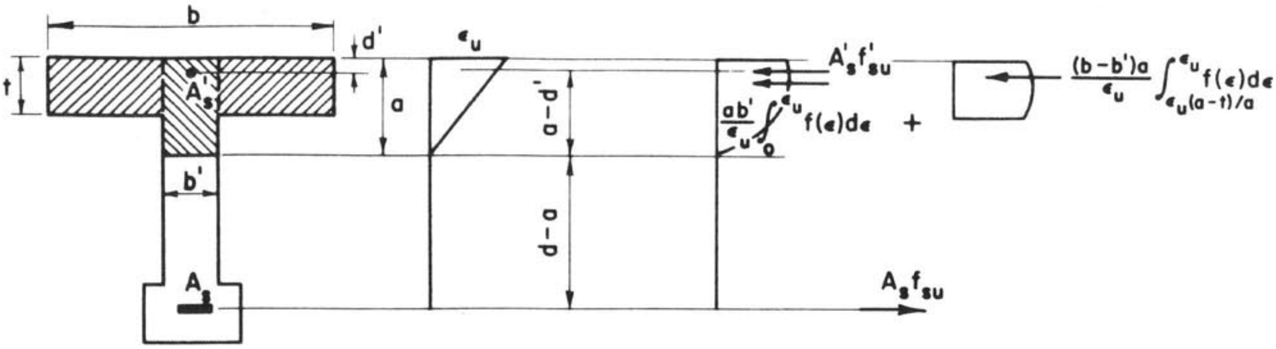

FIGURE 3. FLANGED SECTION, NEUTRAL AXIS BELOW THE FLANGE

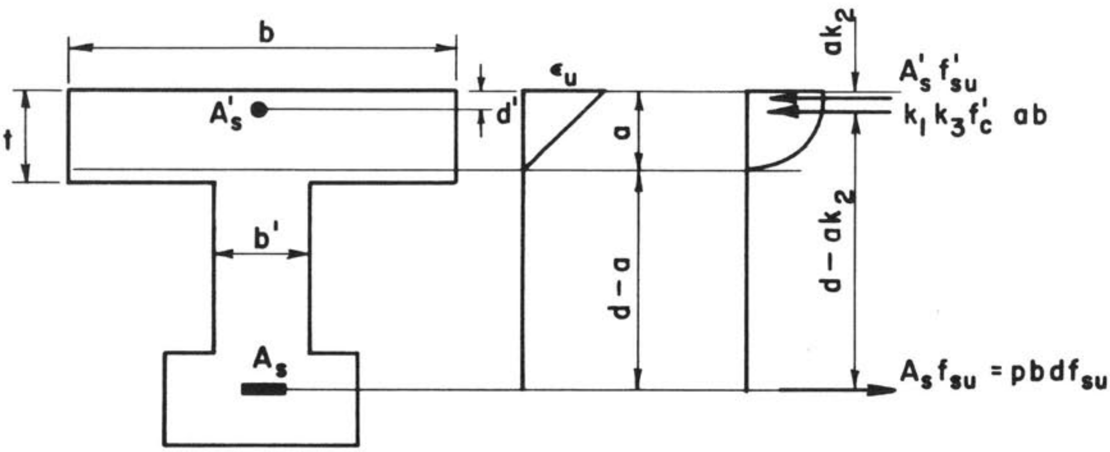

FIGURE 4. FLANGED SECTION, NEUTRAL AXIS IN THE FLANGE

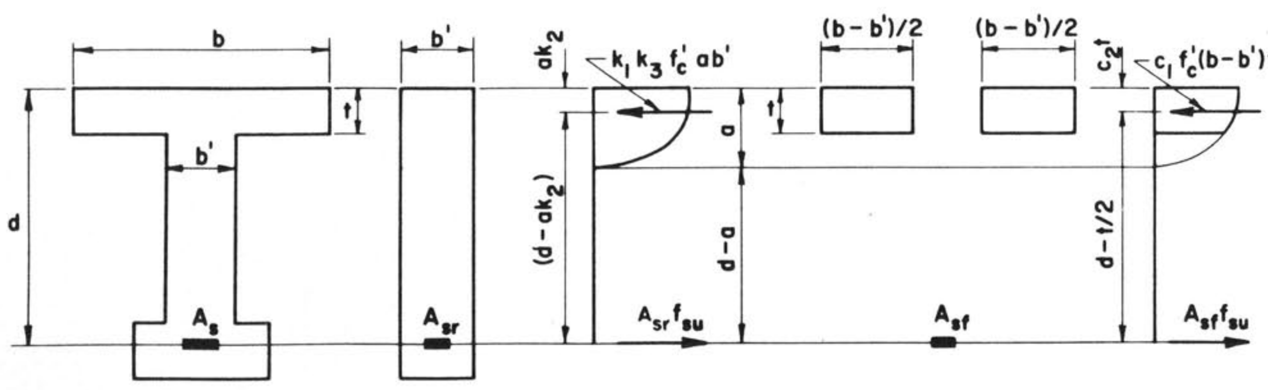

FIGURE 5. FLANGED SECTION, NEUTRAL AXIS BELOW THE FLANGE

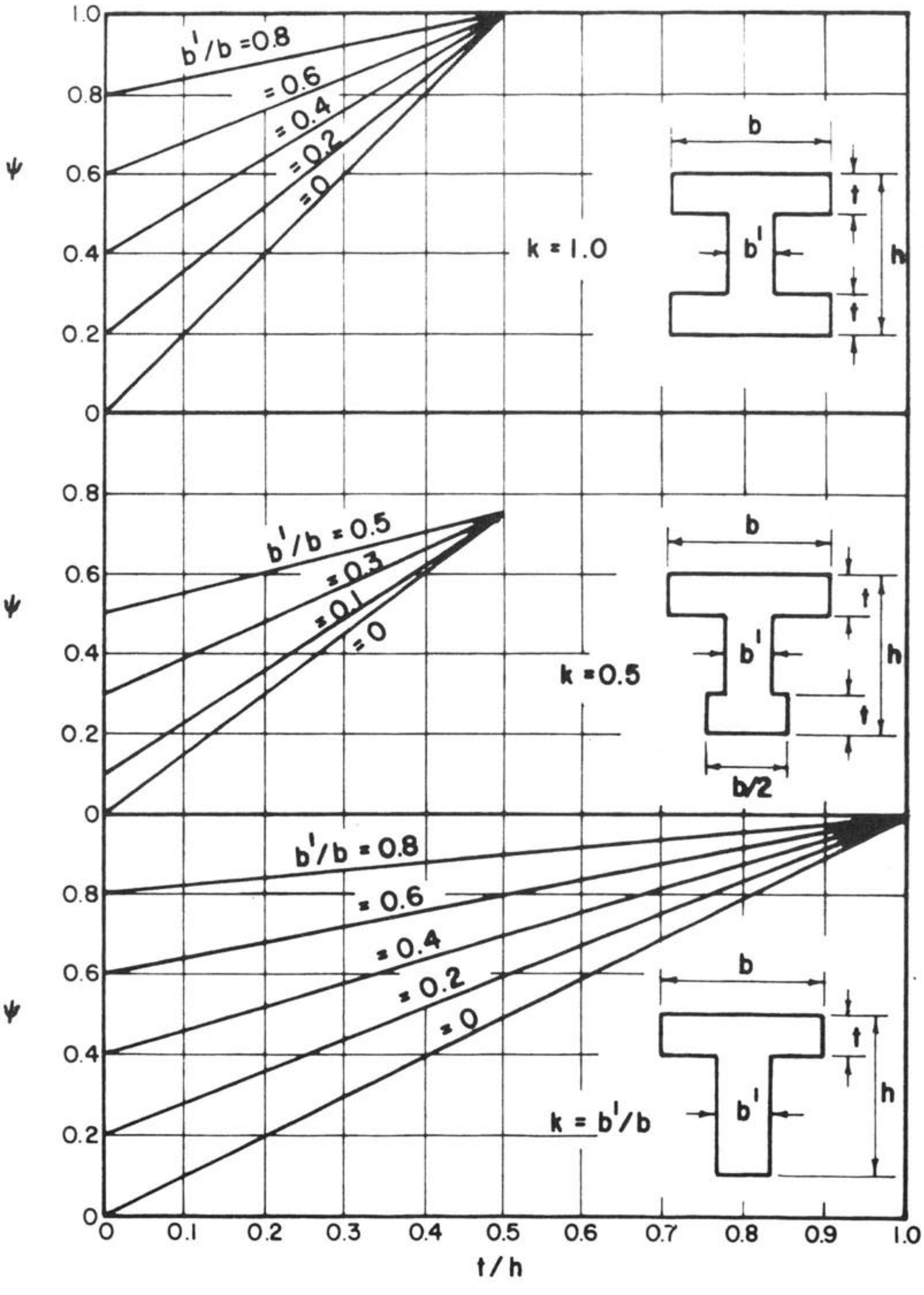

FIGURE 6. RELATIONSHIP BETWEEN ψ AND GEOMETRIC PARAMETERS OF THE SECTION

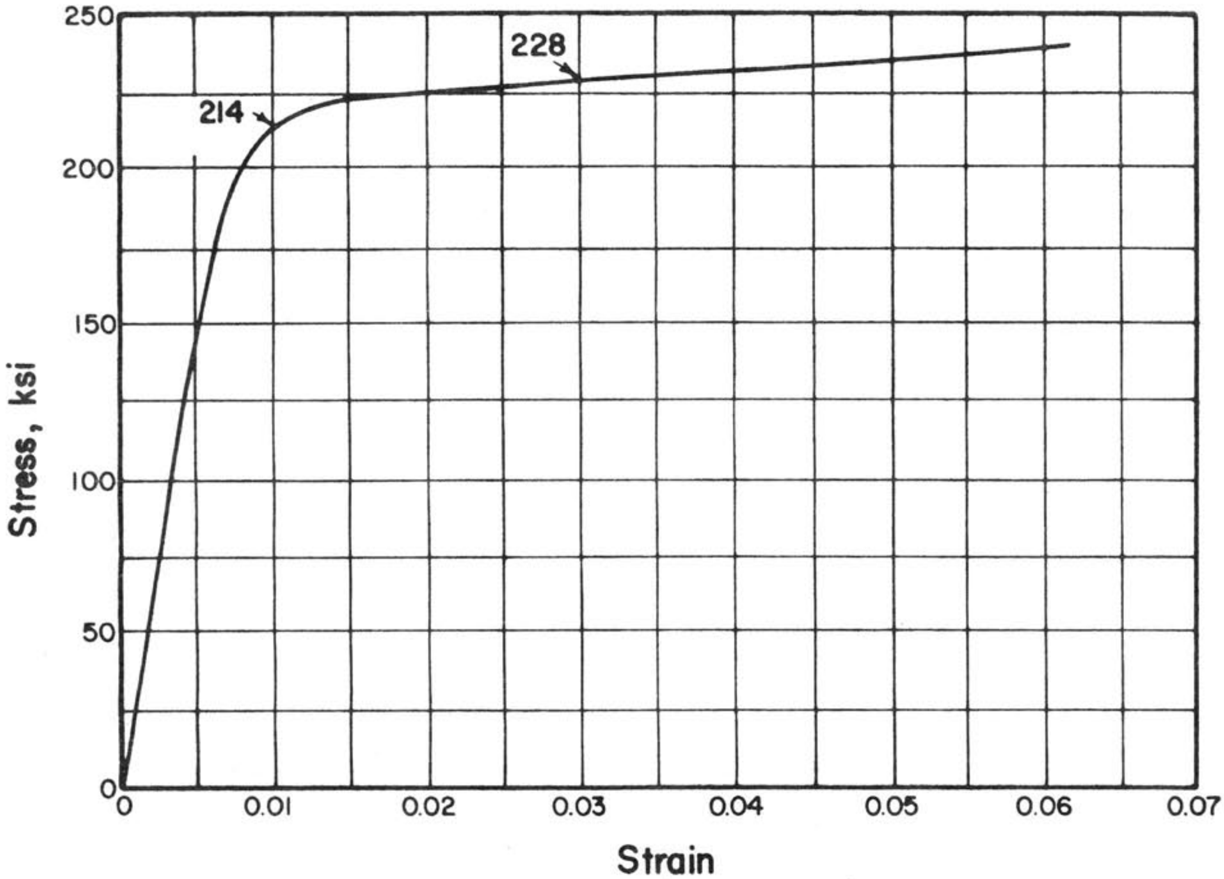

FIGURE 7. STRESS-STRAIN DIAGRAM FOR STEEL

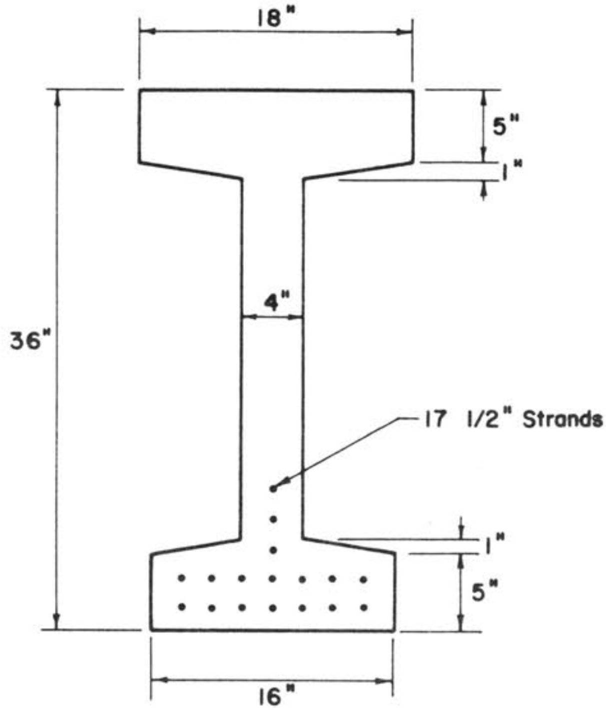

FIGURE 8. SECTION OF A BEAM WITH LOW DUCTILITY ε_{su} = 0.01 — WEIGHT OF BEAM = 300 LB/FT

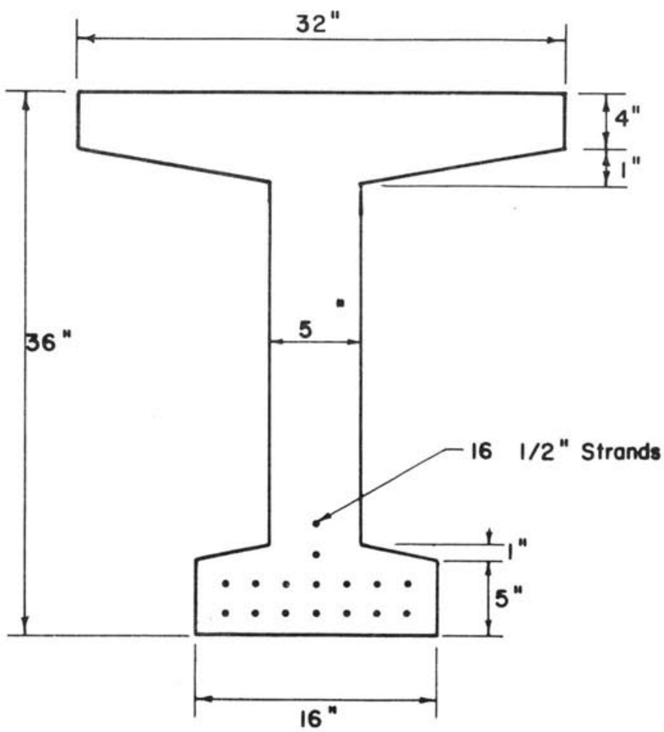

FIGURE 9. SECTION OF A BEAM WITH VERY HIGH DUCTILITY (ε_{su} = 0.03) — WEIGHT OF BEAM = 377 LB/FT

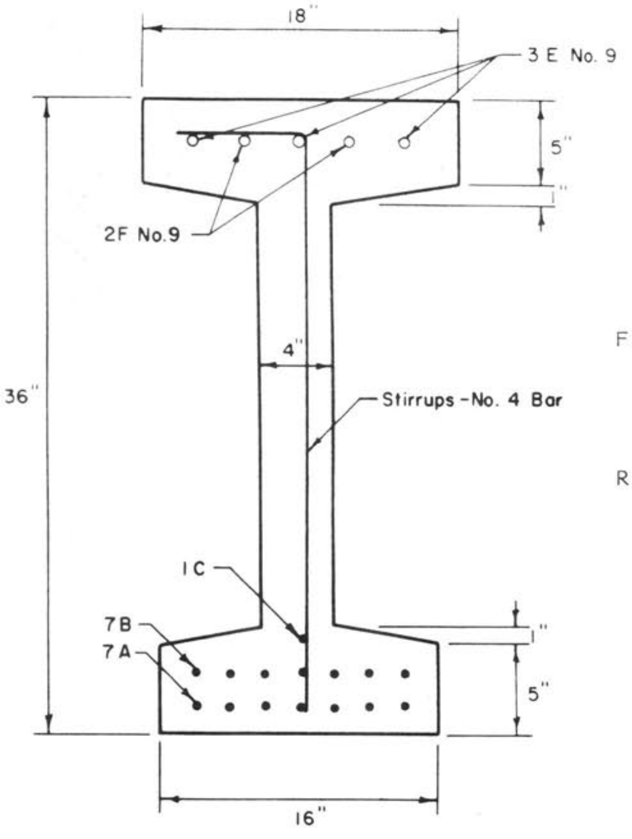

FIGURE 10. SECTION OF A BEAM IN WHICH THE HIGH REQUIRED DUCTILITY ($\varepsilon_{su} = 0.03$) IS OBTAINED BY THE USE OF COMPRESSION REINFORCEMENT — WEIGHT OF BEAM = 300 LB/FT

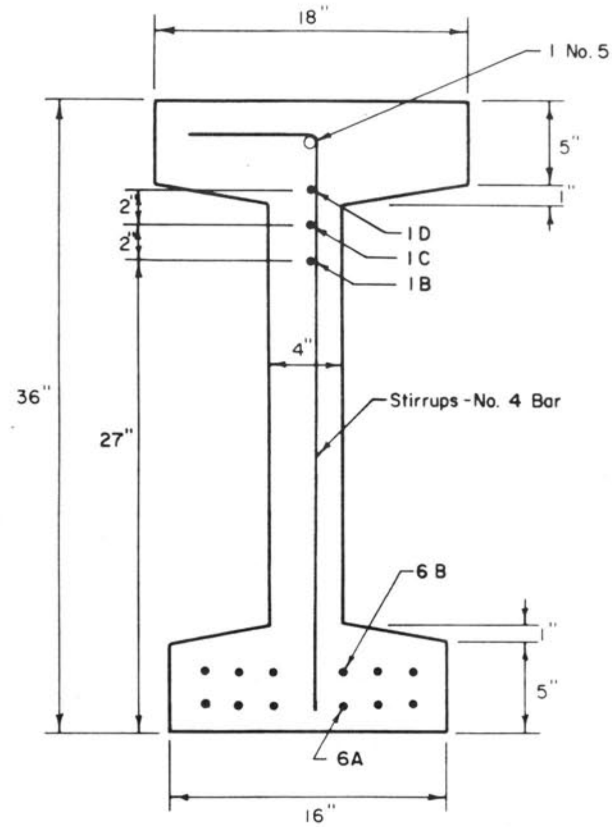

FIGURE 11. END SECTION OF THE BEAM WITH COMPRESSION REINFORCEMENT

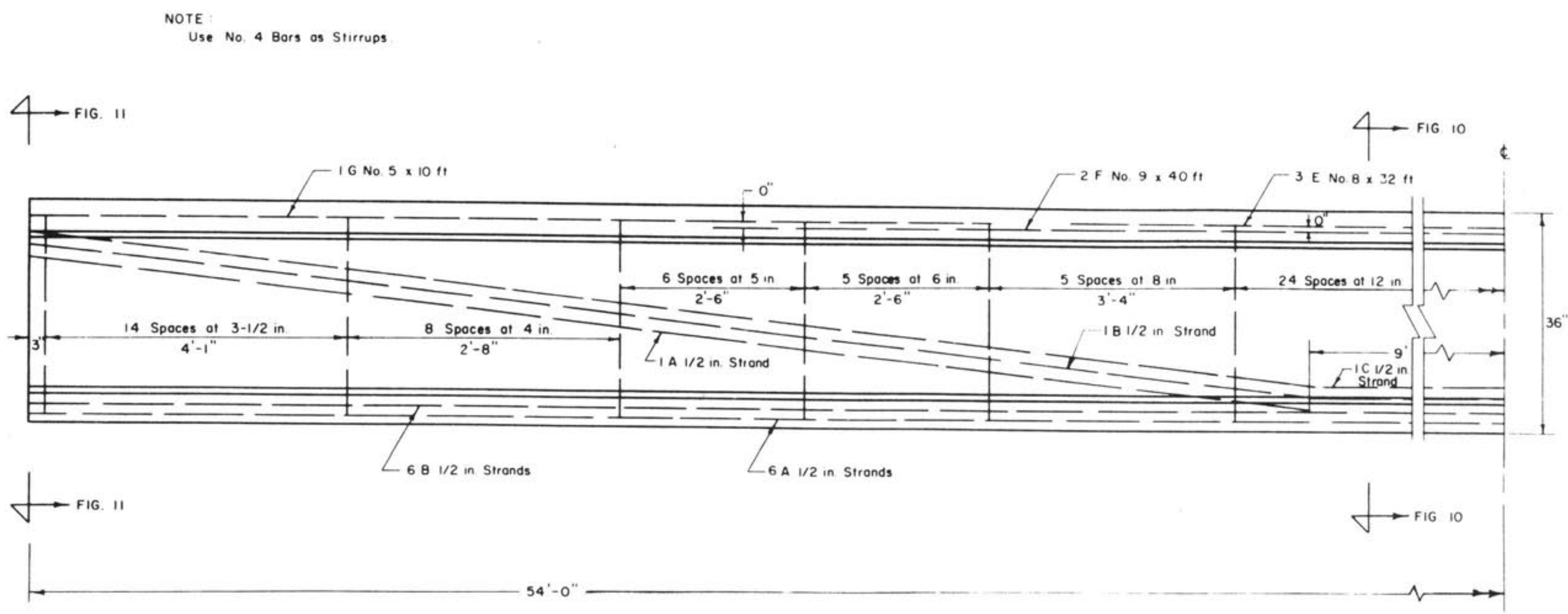

FIGURE 12. ELEVATION OF BEAM OF EXAMPLE 3

TABLE I SUMMARY OF SECTION PROPERTIES AND STRESSES FOR SECTIONS OBTAINED BY ULTIMATE DESIGN

(All section properties are based upon the transformed section assuming n = 7)
(Negative stresses are tensile)

Section	A in^2	y_t in	y_b in	I in^4	A_s in^2	A'_s in^2	Weight lb/ft	Stress Before Losses (transfer) ksi: top (tens)	Stress Before Losses (transfer) ksi: bottom (comp)	Stress After Losses ksi: top (comp)	Stress After Losses ksi: bottom (tens)
Illustrative Example 1a $\epsilon_{su} = 0.01$	302	18.14	17.86	52,190	2.44	-	300	-0.14	2.56	2.38	-0.29
Illustrative Example 1b $\epsilon_{su} = 0.03$	376	15.90	20.10	65,360	2.29	-	377	-0.09	2.20	1.68	-0.36
Illustrative Example 2 $\epsilon_{su} = 0.03$	330	16.62	19.38	59,150	2.15	5.00	300	-0.14	2.29	1.90	-0.41
Allowable Stresses American Concrete Code (318-63) (f'_c = 5 ksi; f'_{ci} = 4 ksi)								-0.19	2.40	2.25	-0.42

A = transformed area, y_t = distance from centroidal axis to top fiber; y_b = distance from centroidal axis to bottom fiber; I = moment of inertia; A_s = area of prestressed steel; A'_s = area of non-prestressed compressive steel; n = modular ratio for both types of steel; f'_c = strength of concrete; f'_{ci} = strength of concrete at transfer; prestress at transfer 150 ksi; effective prestress after losses 128 ksi.

k_2 can be estimated from tests with a sufficient degree of precision, there is no way to estimate c_1 and c_2 generally, so that they will be applicable to all sections. In spite of this weakness our specifications at present assign specific values to these coefficients.

F. PROVISIONS OF THE AMERICAN CONCRETE INSTITUTE BUILDING CODE

The provisions of the American Concrete Institute Code (318-63) do not consider the effect of compression steel on ultimate moment or strain in steel, and instead of giving the stress-strain diagram for concrete, the coefficients of the stress-block are specified.[4] In the following paragraphs the expressions given in the Code will be derived and discussed.

When the neutral axis is in the flange and there is no non-prestressed compression steel from the preceding section, the ultimate moment may be obtained by a simultaneous solution of the following equations.

$$M_u = A_s f_{su} (d - ak_2) \quad (1b)$$

$$k_1 k_3 f'_c ab = A_s f_{su} \quad (2b)$$

$$\epsilon_{su} = \epsilon_{se} + \epsilon_{ce} + \frac{\epsilon_u}{a} (d - a) \quad (3)$$

$$f_{su} = F(\epsilon_{su}) \quad (5)$$

Elimination of a between Equations 1b and 2b and between Equations 2b and 3 permits the reduction of the above four equations to three equations in which M_u, f_{su}, and ϵ_{su} are the unknowns. We will introduce $p = \frac{A_s}{bd}$, as percentage of steel. From Equations 1b and 2b we have:

$$M_u = A_s f_{su} d \left(1 - \frac{k_2}{k_1 k_3} p \frac{f_{su}}{f'_c}\right) \quad (13)$$

from Equations 2b and 3:

$$p \frac{f_{su}}{f'_c} = \frac{k_1 k_3 \epsilon_u}{\epsilon_{su} - \epsilon_{se} - \epsilon_{ce} + \epsilon_u} \quad (14)$$

$$f_{su} = F(\epsilon_{su}). \quad (5)$$

The American Concrete Institute Code (318-63) gives $\frac{k_2}{k_1 k_3} = 0.59$ and requires that the expression for the ultimate moment be multiplied by a capacity reduction factor as follows:

$$M_u = [A_s f_{su} d (1 - 0.59 p \frac{f_{su}}{f'_c})]\phi \quad (15)$$

where ϕ is a capacity reduction factor, taken as 0.9.

The Code implies that when the stress strain diagram for steel is available, f_{su} and ϵ_{su} can be obtained by a simultaneous solution of Equations 14 and 5; if not, f_{su} in Equation 15 may be calculated by the following expression:

$$f_{su} = f'_s (1 - 0.5 p f'_s/f'_c), \quad (16)$$

where f'_s is the ultimate strength of prestressing steel.

The Code controls the ductility of the beam by requiring that the ratio $p f_{su}/f'_c$ be less than or equal to 0.3. In cases in which $p f_{su}/f'_c$ is greater than 0.3, the ultimate moment may be calculated by Equation 15 provided that $p f_{su}/f'_c$ is taken as 0.3. This results in the following equation:

$$M_u = \phi (0.25 f'_c bd^2). \quad (17)$$

By limiting the value of $p f_{su}/f'_c$ the Code in effect requires that the strain in steel at failure be not less than a limiting value, since from Equation 14 we have

$$p \frac{f_{su}}{f'_c} = \frac{k_1 k_3 \epsilon_u}{\epsilon_{su} - \epsilon_{se} - \epsilon_{ce} + \epsilon_u} \leq 0.3.$$

The above requirement sets an acceptable range for ϵ_{su}, roughly in the neighborhood of 0.01, below which it cannot go as far as calculation of M_u is concerned.

For sections in which the neutral axis at ultimate falls below the flange and there is no non-prestressed compression steel, from the preceding section the ultimate moment may be obtained by a simultaneous solution of the following equations:

$$M_u = k_1 k_3 f'_c ab' (d - k_2 a) + c_1 f'_c (b - b') t (d - c_2 t) \quad (7b)$$

$$k_1 k_3 f'_c ab' + c_1 f'_c (b-b') t = A_s f_{su} \quad (8b)$$

$$\epsilon_{su} = \epsilon_{se} + \epsilon_{ce} + \frac{\epsilon_u}{a} (d - a) \quad (3)$$

$$f_{su} = F(\epsilon_{su}). \quad (5)$$

Elimination of a between Equations 7b and 8b and between Equations 8b and 3 will result in three equations in which M_u, f_{su}, and ϵ_{su} are the unknowns. We have

$$M_u = d[A_s f_{su} - c_1 f'_c (b - b') t] \left[1 - \frac{A_s f_{su} - c_1 f'_c (b - b')t}{k_1 k_3 f'_c b'd} k_2\right] + c_1 f'_c (b - b') t (d - c_2 t) \quad (18)$$

$$p \frac{f_{su}}{f'_c} = \frac{k_1 k_3 \epsilon_u}{\epsilon_{su} - \epsilon_{se} - \epsilon_{ce} + \epsilon_u} \left(\frac{b'}{b}\right) + c_1 \left(1 - \frac{b'}{b}\right) \frac{t}{d} \quad (19)$$

$$f_{su} = F(\epsilon_{su}). \quad (5)$$

The American Concrete Institute Code (318-63) designates the quantity

$$A_{sr} f_{su} = k_1 k_3 f'_c ab'$$

as that part of the force contributed by the prestressed steel that develops the rectangle $a\ b'$ in the compression zone of a flanged section. The quantity

$$A_{sf} f_{su} = c_1 f'_c (b - b') t$$

is designated as that part of steel that develops the rectangle $(b - b') t$ in the compression zone.

Since $A_s = A_{sr} + A_{sf}$ we have

$$A_s f_{su} - c_1 f'_c (b - b') t = A_{sr} f_{su}.$$

By substituting the above quantity in Equation 18 we have

$$M_u = A_{sr} f_{su} d \left(1 - \frac{k_2}{k_1 k_3} \frac{A_{sr}}{b'd} \frac{f_{su}}{f'_c}\right) + c_1 f'_c (b - b') t (d - c_2 t).$$

The Code gives $k_2/k_1 k_3 = 0.59$, $c_1 = 0.85$, and $c_2 = 0.50$ and requires that the expression for ultimate moment be multiplied by a capacity reduction factor as follows:

$$M_u = \phi \left[A_{sr} f_{su} d \left(1 - 0.59 \frac{A_{sr}}{b'd} \frac{f_{su}}{f'_c}\right) + 0.85 f'_c (b - b') t (d - 0.5t)\right]. \quad (20)$$

The arbitrary assumption of $c_1 = 0.85$ and $c_2 = 0.5$ by the Code may result in some inaccuracies when the neutral axis is very slightly below the bottom fiber of the flange. For sections of small b'/b the assumption of $c_1 = 0.85$ may even lead to a negative quantity for A_{sr}.

The Code implies that when the stress-strain diagram for the prestressed steel is available f_{su} and ϵ_{su} can be obtained by a simultaneous solution of Equations 19 and 5. If the stress-strain diagram is not available the Code permits calculation of f_{su} by Equation 16.

As before, the Code limits the quantity $\frac{A_{sr}}{b'd}\frac{f_{su}}{f'_c}$ to 0.3 or less. If this ratio is more than 0.3, the ultimate moment may be calculated by Equation 20 provided that $A_{sr}\ f_{su}/f'd\ f'_c$ is taken as 0.3. This results in the following equation:

$$M_u = \phi\ [0.25\ f'_c\ b'\ d^2 + 0.85\ f'_c\ (b - b')\ t\ (d - 0.5t)] \qquad (21)$$

since we have

$$A_{sr}\ f_{su}/b'\ d\ f'_c = \frac{k_1\ k_3\ \varepsilon_u}{\varepsilon_{su} - \varepsilon_{se} - \varepsilon_{ce} + \varepsilon_u} \leq 0.3.$$

Limitation of this quantity is equivalent to requiring a minimum value for ε_{su}, which as before, is in the neighborhood of 0.01.

It should be pointed out that the limitations on the ductility of the beam as given by the American Concrete Institute Code do not include the effect of any compression steel that may be present at the top of the beam. It will be shown in Chapter V, Section C that this effect is not negligible.

The neutral axis may either fall in the flange or below the flange. The distinction between the two cases is made according to the following inequalities:

If $t > a = p\ f_{su}\ d/k_1\ k_3\ f'_c$ the neutral axis falls in the flange. Since the American Concrete Institute Code gives $k_1\ k_3 = 0.7$ this condition can be restated as follows:

When $t > 1.4\ p\ \frac{f_{su}\ d}{f'_c}$, the neutral axis falls in the flange.

When $t < 1.4\ p\ \frac{f_{su}\ d}{f'_c}$, the neutral axis falls below the flange.

III. PROVISIONS FOR SAFETY AND DUCTILITY IN ULTIMATE DESIGN

In ultimate design a section is proportioned in such a way that the ultimate moment is greater than the moment developed under service loads by a prescribed quantity, and that it deforms a certain amount before it fails.

These concepts may be stated in the form of the following requirements:

$$M_u \geq N_d \, (M_g + M_s) + N_\ell \, M_\ell \tag{22}$$

and

$$\epsilon_{su} \geq \epsilon_{s\ell} \tag{23}$$

where M_u = the required flexural strength of the beam

N_d = the load factor for the dead load

M_g = moment due to the weight of the beam

M_s = moment due to the superimposed dead load

N_ℓ = the load factor for the live load

M_ℓ = moment due to the live load

ϵ_{su} = strain in steel at ultimate

$\epsilon_{s\ell}$ = limiting strain in steel

Expression 22 states that the required flexural strength of the beam should be at least equal to $N_d(M_g + M_s) + N_\ell M_\ell$, which is a requirement for the strength of the beam. For the type of loads considered here, the American Concrete Institute Code (318-63) gives $N_d = 1.5$ and $N_\ell = 1.8$.

Expression 23 states that the ductility of the beam should be large enough so that the strain in steel at ultimate will be at least equal to a given limiting value designated as $\epsilon_{s\ell}$.

From the discussions in the preceding section we know that the prediction of failure on the basis of moment depends upon the limiting strain ϵ_u, i.e., a value for ϵ_u is required for an unambiguous definition of failure.

There are many ways that ductility of the section can be measured. In Chapter II, Section F, it was shown that the American Concrete Institute Code defines ductility by the following quantity

$$\frac{k_1 \, k_3 \, \epsilon_u}{\epsilon_{su} - \epsilon_{se} - \epsilon_{ce} + \epsilon_u} .$$

Ductility may also be measured by the curvature at ultimate which may be defined as follows:

$$\varphi = \frac{\epsilon_u}{a} = \frac{\epsilon_{su} - \epsilon_{se} - \epsilon_{ce} + \epsilon_u}{d}$$

where φ is the curvature of the section.

Both of these measures of ductility are based upon the magnitude of ϵ_{su} which is the strain in prestressed steel at failure. Hence ϵ_{su} may be used as a measure of ductility.

This method is based upon the assumptions that we have a value for ϵ_u which defines flexural failure, and we have a minimum limit-quantity for ε_{su}, designated as $\varepsilon_{s\ell}$. In addition we have the stress-strain diagram for prestressed as well as non-prestressed compression steel.

IV. ULTIMATE DESIGN OF SECTIONS WITHOUT COMPRESSION REINFORCEMENT

A. METHOD OF ANALYSIS USED

The design procedure developed here does not depend on the method of analysis. The following are adopted.

When the neutral axis is in the flange, Equations 15, 14, and 5 will be adopted for the calculation of ultimate moment. Equation 15 is the expression for the ultimate moment given by the American Concrete Institute Code (318-63). We will further assume $k_1 k_3 = 0.70$.

These equations are rewritten here for convenient reference when $t > 1.4\, p \frac{f_{su} d}{f'_c}$:

$$M_u = \phi \left[A_s f_{su}\, d \left(1 - 0.59\, p \frac{f_{su}}{f'_c}\right)\right] \quad (15)$$

$$p \frac{f_{su}}{f'_c} = \frac{0.7\, \epsilon_u}{\epsilon_{su} - \epsilon_{se} - \epsilon_{ce} + \epsilon_u} \quad (14a)$$

$$f_{su} = F(\epsilon_{su}). \quad (5)$$

When the neutral axis is below the flange, Equations 20, 19, and 5 will be adopted for calculation of the ultimate moment. Equation 20 is the expression of the ultimate moment given by the American Concrete Institute Code. We will assume $k_1 k_3 = 0.70$, $c_1 = 0.85$, and $c_2 = 0.5$.

When $t < 1.4\, p \frac{f_{su}\, d}{f'_c}$ we have:

$$M_u = \phi \left[A_{sr}\, f_{su}\, d \left(1 - 0.59 \frac{A_{sr}}{b'd} \frac{f_{su}}{f'_c}\right) + 0.85\, f'_c\, (b - b')\, t\, (d - 0.5\, t)\right] \quad (20)$$

$$p \frac{f_{su}}{f'_c} = \frac{0.7\, \epsilon_u}{\epsilon_{su} - \epsilon_{se} - \epsilon_{ce} + \epsilon_u} \left(\frac{b'}{b}\right) + 0.85 \left(1 - \frac{b'}{b}\right) \frac{t}{d} \quad (19a)$$

$$f_{su} = F(\epsilon_{su}). \quad (5)$$

In addition to the above equations we also know that $\epsilon_{su} \geq \epsilon_{s\ell}$. This condition eliminates the need of considering Equations 17 and 21.

Introducing $q = pf_{su}/f'_c$, the above expressions may be written in dimensionless form as follows:

When $q < 0.7\ t/d$:

$$\frac{M_u}{\phi b d^2 f'_c} = q\ (1 - 0.59q) \quad (15a)$$

where $$q = \frac{0.7\, \epsilon_u}{\epsilon_{su} - \epsilon_{se} - \epsilon_{ce} + \epsilon_u}. \quad (14a)$$

When $q > 0.7\ t/d$:

$$\frac{M_u}{\phi b d^2 f'_c} = q \left[1 + \frac{t}{d}\left(\frac{b}{b'} - 1\right) - 0.59 \frac{b}{b'} q\right] - \frac{0.85}{2} \left(\frac{t}{d}\right)^2 \left(\frac{b}{b'} - 1\right) \quad (20a)$$

where $$q = \frac{0.7\, \epsilon_u (b'/b)}{\epsilon_{su} - \epsilon_{se} - \epsilon_{ce} + \epsilon_u} + 0.85\ (1 - b'/b)\ t/d. \quad (19a)$$

B. DETERMINATION OF AREA OF THE BEAM

Let us write $M_u/\phi bd^2 f'_c = Q$. Expression 22 can then be written as an equation in the following form:

$$Q\phi bd^2 f'_c = N_d (M_g + M_s) + N_\ell M_\ell .$$

Substituting $A/h\psi$ for b where A is the gross cross-sectional area of the beam, h is the overall depth, and ψ is a dimensionless shape factor, we have:

$$M_g = \frac{\gamma AL^2}{8} = Q\phi \frac{A}{h\psi} d^2 f'_c \frac{1}{N_d} - \frac{N_\ell}{N_d} M_\ell - M_s$$

or

$$A = \frac{M_s + \frac{N_\ell}{N_d} M_\ell}{\frac{d^2 f'_c Q\phi}{h\psi N_d} - \frac{\gamma L^2}{8}} \qquad (24)$$

where γ is the unit weight of concrete.

For the idealized I-section shown in Figure 1 the following general expression may be used for ψ:

$$\psi = \frac{A}{bh} = \frac{t}{h} (1 + k) + \frac{b'}{b} (1 - 2 \frac{t}{h}). \qquad (25)$$

The quantity k in the above equation is the ratio of the width of bottom flange to that of the top flange. Equation 25 is plotted in Figure 6 for a few typical cases. A study of Equation 24 shows that for a given depth and type of concrete A depends upon ψ and Q only. It can be seen that A decreases with Q and increases with ψ. In order to decrease the area of the beam it is necessary to increase Q and decrease ψ.

Since both Q and ψ are functions of t/h, b'/b, and d/h, in order to study their variation with A it will be more convenient to study the variation of $Q' = \frac{d}{h}\frac{Q}{\psi}$ with A.

In order to obtain the least area it is necessary to make Q' as large as possible. The quantities t/d and b'/b decrease with Q' hence they should be made as small as possible without causing the dimensions of the beam to become unreasonably thin.

From the expression for Q' and Equation 25, it can be seen that Q' decreases with k. However, since the bottom flange of the beam should be large enough to permit the placing of steel, k cannot be reduced indefinitely. It should be made as small as possible.

The quantity d/h should be made as large as possible; however, it is doubtful that in most practical cases it can be made greater than 0.9.

Since Q' increases with q it is desirable to make q as large as possible; however, Expression 23 sets the upper limit for q.

Since Expression 23 sets the required minimum ductility of the beam as a strain in steel equal to $\varepsilon_{s\ell}$, the required maximum q consistent with the required ductility can be computed from Equations 14a and 19a as follows:

$$q_{max} = \frac{0.7\,\varepsilon_u}{\varepsilon_{s\ell} - \varepsilon_{se} - \varepsilon_{ce} + \varepsilon_u} \qquad (14b)$$

or

$$q_{max} = \frac{0.7\,\varepsilon_u}{\varepsilon_{s\ell} - \varepsilon_{se} - \varepsilon_{ce} + \varepsilon_u} \left(\frac{b'}{b}\right) + 0.85 \left(1 - \frac{b'}{b}\right) \left(\frac{t}{d}\right) \qquad (19b)$$

whichever applies.

It should be pointed out that Equations 14b and 19b contain the additional parameter ε_{se}, the strain in steel due to effective prestress. It can be seen that since ε_{se} increases with q_{max} it should be taken as large as practicable. The practical upper limit for ε_{se} for the materials used in pretensioned construction is about 0.005.

C. ILLUSTRATIVE EXAMPLE I.

The following example is presented to

illustrate the procedure for the ultimate design of a prestressed concrete beam and to show the influence of the required ductility on the dimensions of the beam so designed.

It is necessary to design a simply supported beam of 54 foot span subjected to a superimposed dead load of 1.0 kips per linear foot (klf) and a live load of 0.6 klf. The load factors are given as $N_d = 1.5$ and $N_\ell = 1.8$, and the capacity reduction factor is $\phi = 0.9$. Design the section for: 1. a minimum ductility corresponding to $\epsilon_{su} = \epsilon_{s\ell} = 0.01$, and 2. a minimum ductility corresponding to $\epsilon_{su} = \epsilon_{s\ell} = 0.03$.

The effective prestress may be taken as the prestress after losses which in this problem is given as 128 kips per square inch (ksi). This value corresponds to a prestress of 150 ksi at transfer if the effectiveness is taken as 0.85. The strain due to effective prestress is $\epsilon_{se} = 0.0044$, and ϵ_{ce} may be approximated as 0.0006. This approximation may be verified after the section is designed. Also for the purposes of this problem assume $f'_c = 5$ ksi, $\epsilon_u = 0.004$, $\gamma = 0.15$ kips per cubic foot (kcf), and h = 36 in. The stress strain diagram for steel is shown in Figure 7.

1. Section with Minimum Required Ductility Corresponding to $\epsilon_{su} = 0.01$

In Chapter IV, Section B it was shown that the quantities t/h, b'/b, and k increase with A, hence they should be taken as small as possible. Here they will be taken as t/h = 1/6, b'/b = 1/4, and k = 0.8.

The shape factor of the section ψ is obtained using Equation 25 as follows:

$$\psi = \frac{1}{6}\ (1 + 0.8) + \frac{1}{4}\ (1 - 2\ \frac{1}{6}) = 0.467.$$

Assuming d/h = 0.9, for h = 36 inches, we obtain d = 32.4 inches and t/d = 0.185. The values of q_{max} and Q can be computed from Equations 19b and 20a since in this case q > 0.7 t/d and the neutral axis at ultimate is below the flange. From Equation 19b we have:

$$q_{max} = \frac{0.7 \times 0.004 \times 1/4}{0.010 - 0.0044 - 0.0006 + 0.004}$$

$$+ 0.85\ (1 - \frac{1}{4})\ 0.185 = 0.196$$

and from Equation 20a:

$$Q = \frac{M_u}{\phi b d^2 f'_c} = 0.196\ [1 + 0.185\ (4-1)$$

$$- 0.59 \times 4 \times 0.196] - \frac{0.85}{2}\ (0.185)^2 (4-1)$$

$$= 0.170.$$

The area A of the section can be obtained using Equation 24 with the following values:

$$M_s = \frac{1}{8} \times 54^2 \times 1.0 \times 12 = 4370 \text{ k-in}$$

$$M_\ell = \frac{1}{8} \times 54^2 \times 0.6 \times 12 = 2620 \text{ k-in}$$

$$\frac{N_\ell}{N_d} = 1.2$$

$$\frac{d^2 f'_c\ Q\phi}{h_\psi N_d} = \frac{(32.4)^2 \times 5 \times 0.170 \times 0.90}{36 \times 0.467 \times 1.5}$$

$$= 31.8 \text{ k-in}$$

$$\frac{\gamma L^2}{8} = \frac{.15 \times 54^2}{8 \times 12} = 4.6 \text{ k-in.}$$

Therefore

$$A = \frac{4370 + 1.2 \times 2620}{31.8 - 4.6} = 276 \text{ in}^2$$

and:

$$b = \frac{A}{\psi h} = \frac{276}{0.467 \times 36} = 16.4 \text{ in}$$

$$kb = 0.8 \times 16.4 = 13.1 \text{ in}$$

$$b' = \frac{1}{4} \times 16.4 = 4.1 \text{ in.}$$

The stress in the steel at ultimate can be found from the stress-strain diagram for steel shown in Figure 7.

$$f_{su} = 214 \text{ ksi.}$$

The amount of prestressing can be found from the definition of q to be

$$p = \frac{0.196 \times 5}{214} = 0.00458$$

$$A_s = 0.00458 \times 16.4 \times 32.4 = 2.44 \text{ sq. in.}$$

(Use seventeen 1/2-inch strands)

A total of seventeen 1/2-inch strands are needed. Each 1/2-inch strand has an area of 0.1438 square inch. The final dimensions of the section in this solution are shown in Figure 8. The bottom flange has been widened to accommodate the reinforcement. Both top and bottom flanges are tapered to facilitate construction. The properties of the transformed section as well as the stresses at the top and bottom fibers before and after losses are listed in Table 1.

2. Section with Minimum Required Ductility Corresponding to $\epsilon_{su} = 0.03$

The ultimate strain in the steel required for this example is very large and is not used frequently in actual practice. It has been selected to show that direct design for the largest levels of ductility is possible and to study how it affects the shape of the section.

All the quantities are the same as in part 1 of this example except that in this case $\epsilon_{su} = 0.03$, and k, the ratio of the width of bottom flange to width of top flange, is different. The bottom flange needs only be large enough to accommodate the reinforcement. Due to the fact that the higher the ductility the wider the top flange has to be to provide the required area under compression, it is necessary to select k small enough so that the bottom flange is not overdesigned. A value of $k = 0.5$ is selected. From Equations 24 and 25 it can be seen that the area of the section decreases with the web thickness. However, the web thickness cannot be reduced indefinitely, since the cover requirement for the draped reinforcement and the shearing strength determine the minimum thickness. In this case let $b'/b = 1/6$.

In view of the large ductility required the neutral axis is bound to be closer to the top fiber than in the preceding example. Determination of this position affects the selection of the thickness of the top flange. The value of a can be obtained from Equation 3 as follows:

$$0.030 = 0.0044 + 0.0006 + \frac{0.004}{a}(32.4-a)$$

from which $a = 4.47$ inches. Use of $t > 4.47$ inches would result in an overdesigned top flange, the bottom fibers of which would not be subjected to compressive stresses at ultimate. The value of $t = 4.5$ inches is selected as a practical dimension. Then $t/h = 0.125$ and Equation 25 yields the value of ψ as follows

$$\psi = (0.125)(1 + 0.5) + (1/6)\,[1-(2)(0.125)]$$

$$= 0.312.$$

The values of q and Q can be obtained from Equations 14a and 15a as follows:

$$q = \frac{(0.7)(0.004)}{0.03 - 0.0044 - 0.0006 + 0.004} = 0.0965$$

$$Q = 0.0965\,[1 - (0.59)(0.0965)] = 0.091$$

The above equations are applicable because $q < 0.7\ t/d = 0.0973$, and the neutral axis is in the flange of the resulting section.

The area of the section can be computed

from Equation 24 using the known values of case 1 and 25.5 k/in as the modified value of $d^2 f'_c Q\phi/h\psi N_d$.

Therefore,

$$A = \frac{4370 + (1.2)(2620)}{25.5 - 4.6} = 360 \text{ in}^2$$

and

$$b = \frac{A}{\psi h} = \frac{360}{(0.312)(36)} = 32 \text{ in}$$

$$kb = (0.5)(32) = 16 \text{ in}$$

$$b' = \frac{32}{6} = 5.34 \text{ in.}$$

From Figure 7, the stress strain diagram for steel, f_{su} may be obtained

$$f_{su} = 228 \text{ ksi}$$

and

$$p = \frac{(0.0965)(5)}{228} = 0.00212$$

or

$$A_s = 2.20 \text{ in}^2.$$

(Use sixteen 1/2-inch strands)

Figure 9 shows the final section of the beam. The dimension of the bottom flange is the minimum required to accommodate the prestressing steel at the required depth. It coincides with the calculated value of kb thereby requiring no adjustments. If kb turns out to be larger than necessary, only the minimum required should be used, as the bottom flange contributes nothing to ductility and strength. If the adjustment of the dimensions is large, recalculation may be necessary to improve the shape of the section. The properties of the transformed section and the stresses before and after losses for this part are also given in Table 1.

A comparison of Figures 8 and 9 indicates that a large ductility results in a heavy section. In this particular example increasing the required ϵ_{su} from 0.01 to 0.03 causes the weight of the beam to increase by 26 per cent. There is a 6 per cent saving in the amount of prestressing steel as the more ductile section requires one 1/2-inch strand less. This is because the larger stress in the steel at ultimate not only compensates the additional weight of the heavier section but also results in less required area of steel.

V. ULTIMATE DESIGN OF SECTIONS WITH NON-PRESTRESSED COMPRESSION STEEL

A. METHOD OF ANALYSIS USED

Determination of flexural strength of prestressed concrete beams with non-prestressed compression steel was discussed in Chapter II, Sections D and E. It was shown that the ultimate moment of a given section in which the neutral axis falls below the flange can be calculated by a simultaneous solution of Equations 7a, 8a, 3, 4, 5, and 6.

For design purposes the ultimate moment will be computed by Equation 7b assuming that the stress in compression steel has reached the yield point, and taking $c_1 = 0.85$, $c_2 = 0.5$, and $k_2 = 0.42$. In addition the expression for the ultimate moment will be multiplied by the capacity reduction factor ϕ.

$$M_u = \phi\,[A_s f_{su}\,(d - 0.42a) + 0.85\,f'_c\,(b - b')\,t\,(0.42a - 0.5t) + A'_s\,f^*_y\,(0.42a - d')] \quad (26)$$

where $f^*_y = f_y - 0.85\,f'_c$, and f_y is the yield point of non-prestressed compression steel.

Here it is assumed that non-prestressed compression steel is American Society for Testing and Materials Billet Steel A-15 with a flat stress-strain diagram beyond the yield point. The stress in the area of concrete replaced by compression steel is taken into account by the term $0.85\,f'_c$ which is an approximation.

In Equation 26 the quantities a and f_{su} are unknowns, and for their determination we need Equations 8a, 3, and 5. In Equation 8a we will take $k_1 k_3 = 0.7$ and $c_1 = 0.85$.

$$0.7\,f'_c\,a\,b' + 0.85\,f'_c\,(b - b')t + A'_s\,f^*_y = A_s f_{su} \quad (8b)$$

$$\epsilon_{su} = \epsilon_{se} + \epsilon_{ce} + \frac{\epsilon_u}{a}\,(d - a) \quad (3)$$

$$f_{su} = F(\epsilon_{su}) \quad (5)$$

Elimination of a between Equations 8b and 3 will result in the following:

$$f_{su} = \frac{0.70\,f'_c\,\epsilon_u\,(b'/b)}{p\,[\epsilon_{su} - \epsilon_{se} - \epsilon_{ce} + \epsilon_u]} + 0.85\,\frac{f'_c}{p}\,(1 - \frac{b'}{b})\,\frac{t}{d} + f^*_y\,\frac{A'_s}{A_s}. \quad (27)$$

The condition that the compression steel has yielded is satisfied by the following inequality:

$$\epsilon_{su} = \frac{\epsilon_u}{a}\,(a - d') \geq \epsilon_y.$$

We will substitute for a from Equation 8b in the above inequality and rearrange it to arrive at:

$$d' \leq (1 - \frac{\epsilon_y}{\epsilon_u})\left[\frac{A_s f_{su}}{0.7\,f'_c\,b'} - \frac{A'_s\,f^*_y}{0.7\,f'_c\,b'} - 1.21\,(\frac{b}{b'} - 1)\,t\right]. \quad (28)$$

Hence for the solution of unknowns M_u, f_{su}, and ϵ_{su} we have available Equations 26, 5, and 27.

Equation 26 may conveniently be expressed in the dimensionless form:

$$\frac{M_u}{\phi\ bd^2 f'_c} = q\left\{1 + (1 - \frac{q'}{q})\left[\frac{t}{d}\ (\frac{b}{b'} - 1) - 0.59\ q\ (1 - \frac{q'}{q})\ \frac{b}{b'}\right] - \frac{q'}{q}\ \frac{d'}{d}\right\} - \frac{0.85}{2}\ (\frac{t}{d})^2\ (\frac{b}{b'} - 1) \qquad (26a)$$

where $q = p\,\frac{f_{su}}{f'_c}$

$q' = \frac{A'_s}{bd}\,\frac{f^*_y}{f'_c}$

and

$$q = q' + \frac{0.7\ \epsilon_u\ (b'/b)}{\epsilon_{su} - \epsilon_{se} - \epsilon_{ce} + \epsilon_u} + 0.85\ (1 - \frac{b'}{b})\ \frac{t}{d}\ . \qquad (27a)$$

Expression 28 can similarly be presented in a dimensionless form as follows:

$$d' \leq d\ (1 - \frac{\epsilon_y}{\epsilon_u})[(q - q')\ \frac{b}{0.7\ b'} - 1.21\ \frac{t}{d}\ (\frac{b}{b'} - 1)]. \qquad (28a)$$

The expression for the required maximum value of q consistent with the required ductility corresponding to $\epsilon_{su} = \epsilon_{s\ell}$ is given by the following:

$$q_{max} = q' + \frac{0.7\ \epsilon_u}{\epsilon_{s\ell} - \epsilon_{se} - \epsilon_{ce} + \epsilon_u}\ (\frac{b'}{b}) + 0.85\ (1 - \frac{b'}{b})\ \frac{t}{d} \qquad (27b)$$

when $q' = 0$, the beam has no non-prestressed compressive reinforcement, and Equations 26a, 27a, and 27b become identical with Equations 20a, 19a, and 19b respectively.

The preceding equations were developed for the case of a T flanged section in which the neutral axis falls in the web. This condition can be stated

$$t > 1.4\ (q - q')\ d.$$

When the above inequality is not satisfied, the neutral axis falls in the flange and the flanged section becomes a rectangular section. For this case, $b' = b$ and Equations 26a, 27a, and 28a yield

$$\frac{M_u}{\phi bd^2 f'_c} = Q = q\ [1 - 0.59q\ (1 - \frac{q'}{q})^2 - \frac{q'}{q}\ \frac{d'}{d}] \qquad (26b)$$

$$q = q' + \frac{0.7\ \epsilon_u}{\epsilon_{su} - \epsilon_{se} - \epsilon_{ce} + \epsilon_u} \qquad (27c)$$

$$q\ max = q' + \frac{0.7\ \epsilon_u}{\epsilon_{s\ell} - \epsilon_{se} - \epsilon_{ce} + \epsilon_u} \qquad (27d)$$

$$d' < 1.4d\ (1 - \frac{\epsilon_y}{\epsilon_u})\ (q - q'). \qquad (28b)$$

Equation 27a implies that for a large required ductility corresponding to $\epsilon_{su} > \epsilon_{s\ell}$ it is possible to increase q, hence to decrease the area of the beam, by increasing q'. This relationship is very useful when the required ductility is high.

B. ILLUSTRATIVE EXAMPLE 2

In order to show that the non-prestressed compressive reinforcement increases the ductility without increasing the area of the section, the following example is presented.

It is required to design the section in "Illustrative Example 1" in such a way that for a ductility corresponding to $\epsilon_{su} = 0.03$, the area of the section will be the same as that for a ductility corresponding to $\epsilon_{su} = 0.01$.

The yield point stress of the compressive reinforcement may be assumed as $f_y = 50$ ksi.

The section designed in part 1 of

"Illustrative Example 1" has a ductility corresponding to $\varepsilon_{su} = 0.01$. The problem is to determine how much compressive steel of the type given should be placed, so that the ductility of the section will reach that corresponding to $\varepsilon_{su} = 0.03$.

The neutral axis was determined in Example 1, part 2 for the same required ε_{su} as being at a distance from the top fiber given by $a = 4.47$ in. In this case $a < t = 6$ in. and the T section behaves as a rectangular beam. Using Equations 27c and 26b with $Q = 0.170$ as in Example 1, part 1, and $d' = 2$ in. the following independent relations between q and q' are obtained:

$$q = q' + \frac{(0.7)(0.004)}{0.03 - 0.0044 - 0.0006 + 0.004}$$

$$0.170 = q - 0.59\ (q - q')^2 - q'\ \frac{2}{32.4}\ .$$

Solution of the above equations yields $q = 0.181$ and $q' = 0.084$.

From Figure 7, the stress-strain diagram for steel, $\varepsilon_{su} = 0.03$ corresponds approximately to $f_{su} = 228$ ksi. We have

$$p = q\ f'_c/f_{su} = \frac{(0.181)(5)}{228} = 0.00397$$

or

$$A_s = (0.00397)(32.4)(16.3) = 2.10 \text{ sq. in.}$$

(Use fifteen 1/2-inch strands)

Also

$$p' = q'\ f'_c/f^*_y = \frac{(0.084)(5)}{50-(0.85)(5)} = 0.0093$$

or

$$A'_s = (0.0093)(32.4)(16.3) = 4.9 \text{ in.}^2$$

(Use five #9 bars of hard grade steel)

The distance of these bars from the top must be such that $\varepsilon'_s > \varepsilon_y$, if yielding of the compressive reinforcement is to occur at ultimate. This condition can be checked using Expression 28b:

$$d' = 2 \text{ in.} < (1.4)\ (32.4)\ (1 - \frac{0.00167}{0.004})$$

$$(0.181 - 0.084) = 2.6 \text{ in.}$$

If d' greater than 2.6 inches had been selected, compressive steel at ultimate would not yield, requiring use of the actual value of stress in compression steel which is less than the yield stress.

The length of these non-prestressed bars need not be the total span of the beam. Theoretically they are not needed at a section where the required q is that of the section without the compression reinforcement. Assuming the distribution of the required q to be the same as the distribution of bending moments, the theoretical section at which the bars are no longer needed can be determined by the distance X_1 from the center line as follows:

$$X_1 = \frac{54}{2}\sqrt{\frac{0.084}{0.181}} = 18.4 \text{ ft.}$$

Further economy can be achieved if the non-prestressed compression bars are separated in two groups. A group of three short bars representing a q of .050 could then be cut at a section theoretically at a distance $X_2 =$ 14.2 feet from the center line. Taking into consideration the additional length required to develop bond the non-prestressed bars may be specified as 2 # 9 x 40' and 3 # 9 x 32'.

Figures 10 and 11 show the section of the beam at midspan and at the end respectively. Figure 12 shows the profile of the prestressed and non-prestressed steel. Three web strands have been draped to prevent overstressing of the end sections of the beam. In addition to the 5 # 9 non-prestressed top reinforcement an end # 5 has been added for practical construction purposes. Stirrups have been designed according to American Concrete Institute Code (318-63). As before the properties of the transformed section as well as the stresses before and after losses are given in the table.

A reduction in the amount of non-prestressed compression reinforcement is possible with a section having a wider top flange. The parameter q' is related to q by Equation 27b. Selection of a smaller value of q' or q'/q would fix q and permit the determination of the required Q by Equation 26b. The area of the section and its final shape can be determined as usual from Equation 24. If the proper values of t/h, b'/b, and k were selected the new section will present a flange wider than that of Example 1, part 1, but not as large as that of Example 1, part 2. Also the compressive reinforcement required will be smaller than that of Example 2. This solution would show that to obtain high ductility a compromise section can be obtained if some increment of weight is tolerated with a smaller amount of non-prestressed compression steel.

C. COMPARISON OF THE THREE SOLUTIONS

It has been shown that ultimate strength design provides a convenient procedure which leads to well proportioned sections. The desired ductility and strength were used as the fundamental constraints for proportioning the sections, while the stresses at transfer and under service loads were checked.

An examination of Table 1 shows interesting details. The beam of Example 1, part 1 with a required ductility corresponding to $\epsilon_{su} = 0.01$ required more prestressing steel (17 strands) than the beams of Example 1, part 2 and Example 2 with a required ductility corresponding to $\epsilon_{su} = 0.03$.

For the stress-strain diagram of prestressed steel adopted in these examples, any increase in ductility is accompanied with an increase in stress in steel at ultimate. For the larger ductility considered here the stress in steel increases at ultimate from 214 ksi to 228 ksi. This increase in steel stress causes a decrease in the required area of prestressing steel.

The beam of Example 1, part 2 shows that by increasing the width of the top flange and thereby adding concrete area to the compression zone, high ductility can be obtained. This, however, increases the weight of the section by 26 per cent, but decreases the amount of prestressing steel to 16 strands. The increase in stress in steel at ultimate not only supports the additional weight of the beam, but also permits a reduction in the required area of steel. Under the service loads this beam shows, however, a tendency for a large tensile stress at the bottom fiber due to the smaller amount of prestressing force.

The beam of Example 2 shows a different way of obtaining high ductility. Five #9 bars are added to the top flange of the low ductility section of Example 1, part 1. This increment in compression area raises the neutral axis and increases the lever arm of the resisting couple by approximately five per cent. In addition the stress in the steel at ultimate is increased from 214 ksi to 228 ksi, approximately seven per cent. These two factors combined explain the 12 per cent reduction in the number of prestressing strands (from 17 to 15), since the required tensile force at ultimate can be obtained with less area of steel at a higher stress and a larger lever arm. The non-prestressed bars also provide additional tensile strength for the top part of the beam at transfer and during handling operations. Furthermore, they have a tendency to reduce the inelastic deflections due to creep.

VI. SUMMARY AND CONCLUSIONS

The work reported here presents a study of ultimate design of prestressed concrete beams. It consists of a detailed discussion of various methods for calculating ultimate moment of practical sections including sections with non-prestressed compression reinforcement. A method is presented by which a prestressed concrete beam can be proportioned by ultimate design. Particular emphasis has been placed upon the requirement of ductility and its influence upon the dimensions of the section. The design examples presented show the actual method of proportioning as well as the influence of ductility on the dimensions of the beam.

The following conclusions may be drawn from the study presented in this work.

1. Methods with varying degrees of accuracy can be developed for the determination of ultimate moment in terms of the properties of the beam section. For design purposes the ultimate moment may be expressed conveniently in a dimensionless form.

2. The expressions for the calculation of the ultimate moment and ductility given in the American Concrete Institute Code (318-63) do not include the effect of non-prestressed compression steel. The influence of compression steel on the flexural strength is small and may be ignored. However, neglecting the effect of compression steel upon the ductility of the section is unreasonable. Compression steel contributes appreciably to the ductility of the section and should be taken into account. "Illustrative Example 2" shows that the most expeditious way for increasing the ductility of a section is by placing non-prestressed compression reinforcement as near the top fiber as possible.

3. A prestressed concrete beam can be proportioned for a given required minimum flexural strength and ductility. The stresses at transfer and at service conditions may be checked in a section thus obtained.

4. The dimensions of a section are influenced greatly by the required ductility. An increase in the required ductility results in an increase in the required area of the section, unless compression steel is provided.

5. For a large required ductility considerable saving in the area of the beam may be effected by use of non-prestressed compression steel. Compression steel has additional advantages such as its contribution to the crack stability of top fiber, its use as spacer for the web reinforcement and its function in providing more safety for the beam during transportation and erection.

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