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ABSTRACT

This report treats the buckling and post-buckling behavior of a cylindrical shell that is subjected to uniform external pressure p on its lateral surface, and an axial compressive force F (Fig. A). The force F varies with the pressure p in such a way that $F = \lambda a^2 p$, in which a is the mean radius of the shell and λ is a dimensionless constant. If the shell is immersed in a fluid at constant pressure p and if the force F results only from the pressure p on the ends, $\lambda = \pi$.

The ends of the shell are assumed to provide simple support to the cylindrical wall. Accordingly, the radial and circumferential displacement components of the middle surface of the wall vanish at the ends. If the ends of the shell are free to warp, no other constraint is imposed on the deformation. If the ends of the shell are rigid, the axial displacement is constant at either end. Both of these cases were investigated. For generality, the shell was considered to be reinforced by several rings or hoops.

Only geometrically perfect shells were studied; that is, initial dents and out-of-roundness were not taken into account. Only shells with a linear stress-strain relation were considered.

If the axial force F is not too great, the shell assumes a fluted form when it buckles. This form is illustrated by Fig. B, which is a photograph of some of Sturm's test specimens [7]*. The number of flutes in the buckled form is influenced strongly by the ratio L/a, in which L is the length of the shell. Fig. C illustrates several forms of cross sections of cylindrical shells that have been buckled by external pressure.

When the axial force F predominates, the buckled shell assumes a form in which diamond-shaped facets occur (1. Art. 85). This type of buckling was not considered in the present study; the axial force F was assumed to be so small that the fluted pattern occurs. The admissible range of F was not determined, but the fluted pattern usually occurs if λ does not exceed π.

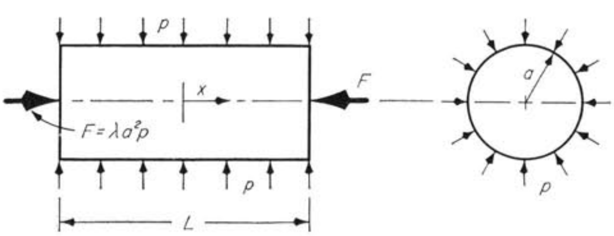

Fig. A. Forces Acting on Shell

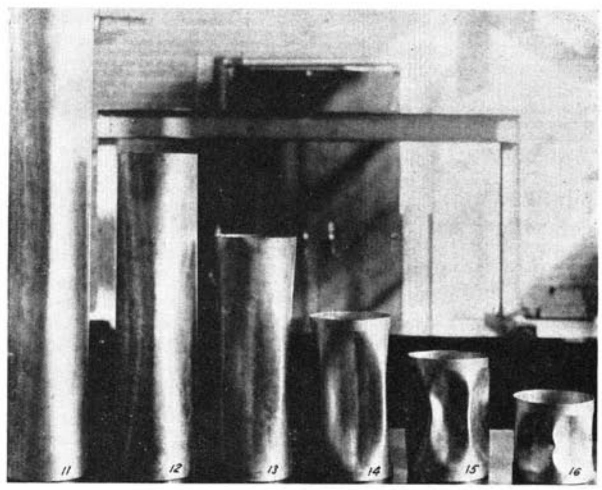

Fig. B. Front Views of Buckled Cylinders

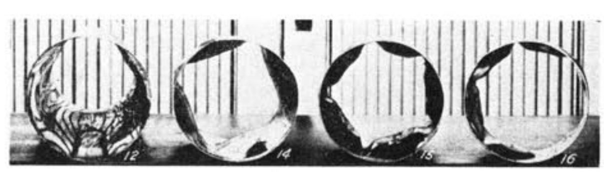

Fig. C. End Views of Buckled Cylinders

* Numbers in parentheses, unless otherwise identified, refer to the References at the end of this report.

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I. PRELIMINARY CONSIDERATIONS

1. Introduction

Experimental data on the collapsing pressures of cylindrical shells have been obtained by Fairbairn, Carman, Jasper and Sullivan, Saunders and Windenburg, Windenburg and Trilling, Sturm, and numerous other investigators.[2, 3, 4, 5, 6, 7] Theoretical studies of the problem have been performed by Bryan, Southwell, Cook, von Mises, Donnell, Sturm, and others.[8, 9, 10, 11, 12, 7] The history of these theories (to 1948) is contained in the work of Batdorf.[13]

Von Mises and most of the subsequent investigators implicitly based their analyses on the general principle that a motionless conservative mechanical system becomes unstable when the value of its total potential energy ceases to be a *relative* minimum. The theory of buckling based on this principle is sometimes called the "infinitesimal theory" since investigations of relative minima require only infinitesimal variations. The buckling load determined by the infinitesimal theory has been designed by Friedrichs[14] as the "Euler critical load," since Euler employed the infinitesimal theory in his study of columns. The Euler critical loads for elastic cylindrical shells that are subjected to external hydrostatic pressure are in close correlation with experimental data, provided that the shells are long in comparison with their diameters. However, the Euler critical loads are much too high for short thin shells.

In 1938, von Kármán and Tsien[15] called attention to the fact that an ideal shell can be in a state of weak stability, such that a small blow or other disturbance causes it to snap into a badly deformed shape. Simple examples of this type of equilibrium are common. The equilibrium of a coin that is balanced on its edge is stable, but such weak stability is usually unsuitable for engineering design. Similarly, if the center of gravity of a ship is so high that the slightest push will cause the ship to capsize, the border line of stability has been reached. However, this condition has no significance for the design of hulls. Analogously, the Euler critical load of a shell loses much of its significance when snap-through can occur. We merely know that the Euler critical load is the upper bound of the load that will actually cause failure.

In this investigation, the occurrence of weak stability is manifested by the conclusion that the pressure required to maintain a buckled form is frequently much less than the Euler critical pressure. A pressure-deflection curve for an ideal elastic cylindrical shell that is loaded by external pressure has the general form shown in Fig. 1. The falling part of the curve (dotted in Fig. 1) represents unstable equilibrium configurations. Also, the continuation of line OE (dotted) represents unstable unbuckled configurations. Actually, the shell snaps from some configuration A to another configuration B, as indicated by the dashed line in Fig. 1. Theoretically, point A coincides with the Euler critical pressure E, but initial imperfections and accidental disturbances prevent the shell from reaching this point. To some extent, point A is indeterminate, but it is presumably higher than the minimum point C, unless the shell has excessive initial dents or lopsidedness. In this report, a hypothesis of Tsien[19] is used for locating point A. The pressure at point C is the minimum pressure under which a buckled form can persist. Thus,

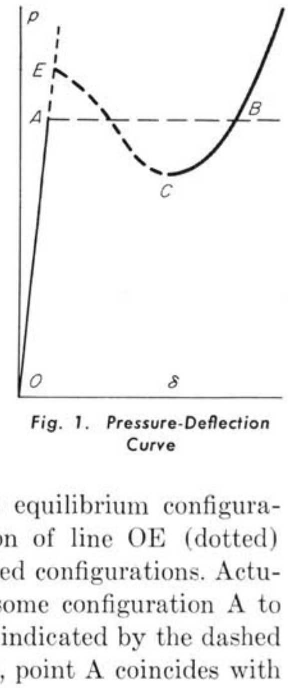


Fig. 1. Pressure-Deflection Curve

if the shell is in a buckled state, and if the external pressure is gradually relieved, the shell will snap back to the unbuckled form when the pressure at point C is reached.

An analysis of the post-buckling behavior of a structure determines the buckling load automatically. For example, an analysis of the form of a buckled column reveals that there is no real nonzero solution unless the load exceeds a certain value, the Euler critical load. Accordingly, in principle, the nonlinear theory of equilibrium obviates the need for a special theory of buckling. However, as a practical expedient, it is usually easier to determine the Euler buckling load of a structure by solving a linear eigenvalue problem than by calculating the bifurcation point of a curve in configuration space that represents all equilibrium configurations.

Problems of post-buckling behavior of elastic shells may be approached in two different ways. On the one hand, we may seek to solve the equilibrium equations and the compatibility equations, in consistency with given boundary conditions. However, in the large-deformation theory of elasticity, the compatibility equations are an extremely complicated set of differential equations, represented by the vanishing of a Riemann tensor.[20] As Dr. C. Lanczos once remarked, "We could not hope to solve the general compatibility equations, but fortunately we already know their general solution. It is merely an arbitrary displacement vector. We should be happy that we know this solution, and we should make every possible use of it."

When the components of the displacement vector are adopted as the dependent variables in a shell problem, only the equilibrium equations and the boundary conditions remain to be considered. The equilibrium equations may be derived by balancing forces on a differential element, but, in large-deformation theories, the rotations of the elements introduce a complexity into this procedure. Consequently, the equilibrium equations are obtained most readily in terms of the initial coordinates by applying the Euler equations of the calculus of variations to the potential energy integral. Unfortunately, in most shell problems, the equilibrium equations are too complicated to be solved rigorously. Instead of tackling the equilibrium equations directly, we may revert to the potential energy integral and apply approximation methods of the calculus of variations. This procedure was employed in this investigation. The theory is accordingly founded on the well-known principle that all states of equilibrium — stable and unstable — are determined by the stationary values of the potential energy. The stable states correspond to relative minima of potential energy.

The potential energy of the shell is the sum of four parts; namely, the membrane strain energy, the strain energy of bending, the strain energy of reinforcing rings, and the potential energy of external forces. Articles 3 to 13, inclusive, are devoted to the derivation of the potential energy expression.

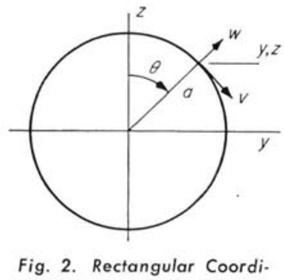


Fig. 2. Rectangular Coordinates of Shell

In the development of the theory, the axial, circumferential, and radial components of displacement of the middle surface (u, v, w) (Fig. 2) are approximated by three terms of Fourier series (Eq. 11). By using the assumption that the shell buckles without incremental hoop strain on the middle surface, the Fourier coefficients v_1, v_2, v_3, w_0, w_2, w_3 are all expressed as functions of w_1. Subsequently, w_1 is replaced by a more convenient parameter W, defined by $W = (n - 1/n)\ w_1/a$, where n is the number of waves in the periphery of the buckled shell. It is assumed that $W = W_0 \cos \pi x/L$, where x is an axial coordinate with origin at the center section of the shell, and W_0 is a constant that must eventually be chosen to minimize the buckling pressure. The Fourier coefficients u_0, u_1, u_2, u_3 are determined by the calculus of variations to minimize the buckling pressure. Accordingly, these are finally expressed as functions of W.

2. Notations

- a = mean radius of the shell
- L = length of the shell
- h = thickness of the shell
- $r = a/L$
- I = moment of inertia of the cross section of a reinforcing ring about its centroidal axis
- p = pressure on the lateral surface of the shell
- F = axial force that acts on the shell (Fig. A)
- λ = a constant, defined by $F = \lambda a^2 p$
- n = number of complete waves in a cross section of the buckled shell

E = Young's modulus
ν = Poisson's ratio
$\xi = \frac{n}{2r}\sqrt{\frac{1-\nu}{2}} = 0.295804nL/a$, if $\nu = 0.30$.
x = an axial coordinate with origin at the center section of the shell
θ = an angular coordinate (Fig. 2)
u,v,w = axial, circumferential and radial displacement components of the middle surface due to buckling (Fig. 2)
V = total potential energy of the shell (strain energy plus potential energy of external forces)
ΔV = increment of potential energy due to buckling (Eq. 100)
U_r = strain energy of a reinforcing ring (Eq. 99)
U_b = part of the strain energy of the shell that results from bending
K = constant in the buckling formula, $p_{cr} = KEh/a$
K_i = value of K determined by the infinitesimal theory of buckling
K_{st} = value of K determined by the snap-through theory of buckling (Tsien's theory)
$K_1, K_2, \ldots, K_{18}$ = functions of n and ν, defined by Eqs. (39), (47), (58), and (67), and tabulated in Table 1
$a_1,a_2,b_1,b_2,b_3,c_1,c_2,c_3$ = functions of n, ν, and r, defined by Eqs. (72) and (98), and tabulated in Tables 2-20
B_1, B_2, B_3 = constants defined by Eq. (101)
W_0 = A parameter defined by Eq. (36). W_0 is a measure of the deflection due to buckling.
$W = W_0 \cos\frac{\pi x}{L}$

Primes denote derivatives with respect to x.

II. POTENTIAL ENERGY OF A SHELL WITH FLEXIBLE ENDS

3. Membrane Strains

In this article expressions for the membrane strains of the shell in terms of the displacement components of the middle surface of the shell are derived.

The shell is referred to rectangular coordinates (x, y, z), such that the x-axis is the geometrical axis of the cylinder (Fig. 2). The positive x-axis in Fig. 2 is directed toward the reader. The circle in Fig. 2 represents a cross section of the middle surface of the unbuckled shell. The origin of x is taken to be the middle section of the shell.

When the shell buckles, the particle that lies at point (x, y, z) on the middle surface is displaced to the point (x^*, y^*, z^*). In terms of the axial, circumferential, and radial displacements components (u, v, w) and the angular coordinate θ (Fig. 2), the coordinates (x^*, y^*, z^*) are given by

$$\left.\begin{aligned} x^* &= x + u \\ y^* &= a\sin\theta + v\cos\theta + w\sin\theta \\ z^* &= a\cos\theta - v\sin\theta + w\cos\theta \end{aligned}\right\} \quad (1)$$

The displacement components (u, v, w) are functions of x and θ. In deriving Eq. (1), we have neglected the fact that the deformation before buckling alters the radius slightly.

If x and θ take infinitesimal increments dx and $d\theta$, the coordinates (x^*, y^*, z^*) take increments (dx^*, dy^*, dz^*). These increments are obtained by differentiation of Eq. (1); hence

$$\left.\begin{aligned} dx^* &= (1 + u_x)\,dx + u_\theta d\theta \\ dy^* &= (v_x\cos\theta + w_x\sin\theta)\,dx \\ &\quad + (a\cos\theta + v_\theta\cos\theta - v\sin\theta \\ &\quad + w_\theta\sin\theta + w\cos\theta)\,d\theta \\ dz^* &= (-v_x\sin\theta + w_x\cos\theta)\,dx \\ &\quad + (-a\sin\theta - v_\theta\sin\theta - v\cos\theta \\ &\quad + w_\theta\cos\theta - w\sin\theta)\,d\theta \end{aligned}\right\} \quad (2)$$

where subscripts x and θ denote partial derivates.

Consider two differential vectors (dx^*, dy^*, dz^*) and $(\delta x^*, \delta y^*, \delta z^*)$, the first being the increments of (x^*, y^*, z^*) when x alone receives an increment dx, and the second being the increments of (x^*, y^*, z^*) when θ alone receives an increment $d\theta$. Setting $d\theta = 0$ in Eq. (2), we obtain

$$\left.\begin{aligned} dx^* &= (1 + u_x)\,dx \\ dy^* &= (v_x\cos\theta + w_x\sin\theta)\,dx \\ dz^* &= (-v_x\sin\theta + w_x\cos\theta)\,dx \end{aligned}\right\} \quad (2')$$

Setting $dx = 0$ in Eq. (2), we obtain

$$\left.\begin{aligned} \delta x^* &= u_\theta d\theta \\ \delta y^* &= (a\cos\theta + v_\theta\cos\theta - v\sin\theta \\ &\quad + w_\theta\sin\theta + w\cos\theta)\,d\theta \\ \delta z^* &= (-a\sin\theta - v_\theta\sin\theta - v\cos\theta \\ &\quad + w_\theta\cos\theta - w\sin\theta)\,d\theta \end{aligned}\right\} \quad (2'')$$

The squares of the magnitudes of the vectors (dx^*, dy^*, dz^*) and $(\delta x^*, \delta y^*, \delta z^*)$ are

$$(ds^*)^2 = (dx^*)^2 + (dy^*)^2 + (dz^*)^2$$
$$(\delta s^*)^2 = (\delta x^*)^2 + (\delta y^*)^2 + (\delta z^*)^2$$

Accordingly, Eqs. (2′) and (2″) yield

$$(ds^*)^2 = \left[(1 + u_x)^2 + v_x^2 + w_x^2\right](dx)^2 \quad (3)$$

$$\begin{aligned} (\delta s^*)^2 &= \left[u_\theta^2 + a^2 + v_\theta^2 + v^2 + w_\theta^2 + w^2 \right. \\ &\quad \left. + 2\,av_\theta + 2aw + 2v_\theta w - 2vw_\theta\right](d\theta)^2 \end{aligned} \quad (4)$$

The initial magnitudes of the vectors (dx^*, dy^*, dz^*) and $(\delta x^*, \delta y^*, \delta z^*)$—that is, the magnitudes of the line elements before buckling—are

$$ds = dx, \qquad \delta s = ad\theta$$

Consequently,

$$\left(\frac{ds^*}{ds}\right)^2 = 1 + 2u_x + u_x^2 + v_x^2 + w_x^2 \quad (5)$$

$$\left(\frac{\delta s^*}{\delta s}\right)^2 = 1 + 2\left(\frac{v_\theta + w}{a}\right) + \left(\frac{u_\theta}{a}\right)^2 + \left(\frac{v_\theta + w}{a}\right)^2 + \left(\frac{v - w_\theta}{a}\right)^2 \quad (6)$$

Since the material will not admit large strains, the ratios ds^*/ds and $\delta s^*/\delta s$ are approximately equal to unity. Therefore, the additive terms involving, u, v, w on the right sides of Eqs. (5) and (6) are small compared to unity. Accordingly, ds^*/ds and $\delta s^*/\delta s$ are closely approximated by binomial expansions of the square roots of the right sides of Eqs. (5) and (6) in which only terms to the second degree are retained. Thus, we obtain

$$\frac{ds^*}{ds} = 1 + u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2$$
$$\frac{\delta s^*}{\delta s} = 1 + \frac{v_\theta + w}{a} + \frac{u_\theta^2}{2a^2} + \frac{1}{2}\left(\frac{v - w_\theta}{a}\right)^2 \quad (7)$$

The shell is already strained before it buckles. When buckling occurs, line elements in the x and θ directions receive incremental strains, $\Delta\epsilon_x$ and $\Delta\epsilon_\theta$. According to the customary definition of strain, these increments are

$$\Delta\epsilon_x = \frac{ds^* - ds}{ds}, \qquad \Delta\epsilon_\theta = \frac{\delta s^* - \delta s}{\delta s}$$

Consequently, by Eq. (7),

$$\Delta\epsilon_x = u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2$$
$$\Delta\epsilon_\theta = \frac{v_\theta + w}{a} + \frac{u_\theta^2}{2a^2} + \frac{1}{2}\left(\frac{v - w_\theta}{a}\right)^2 \quad (8)$$

The shearing strain $\gamma_{x\theta}$ is defined by $\gamma_{x\theta} = \cos\phi$, where ϕ is the angle between the vectors (dx^*, dy^*, dz^*) and $(\delta x^*, \delta y^*, \delta z^*)$. Therefore,

$$\gamma_{x\theta} = \frac{dx^*\delta x^* + dy^*\delta y^* + dz^*\delta z^*}{ds^*\delta s^*}$$

Thus, by Eqs. (2′) and (2″)

$$\begin{aligned}\gamma_{x\theta} = \frac{dxd\theta}{ds^*\delta s^*}&[(1 + u_x)\,u_\theta + (v_x\cos\theta + w_x\sin\theta)\\ &\cdot(a\cos\theta + v_\theta\cos\theta - v\sin\theta + w_\theta\sin\theta\\ &+ w\cos\theta) + (-v_x\sin\theta + w_x\cos\theta)\,(-a\sin\theta\\ &- v_\theta\sin\theta - v\cos\theta + w_\theta\cos\theta - w\sin\theta)]\end{aligned}$$

Since $ds = dx$ and $\delta s = ad\theta$, this equation reduces to

$$\begin{aligned}\gamma_{x\theta} = \frac{1}{a}\frac{ds}{ds^*}\frac{\delta s}{\delta s^*}&[(1 + u_x)\,u_\theta + av_x\\ &+ v_x(v_\theta + w) - w_x(v - w_\theta)]\end{aligned}$$

Expanding the reciprocals of the right sides of Eq. (7) by the binomial theorem, we obtain, to first degree terms,

$$ds/ds^* = 1 - u_x, \qquad \delta s/\delta s^* = 1 - \frac{v_\theta + w}{a}$$

Only the first degree terms are needed in these expansions, since the first degree terms lead to second degree terms in the preceding formula for $\gamma_{x\theta}$.

Eliminating ds/ds^* and $\delta s/ds^*$, we obtain, to second degree terms

$$\begin{aligned}\gamma_{x\theta} = &\frac{u_\theta}{a} + v_x - w_x\left(\frac{v - w_\theta}{a}\right)\\ &- u_xv_x - \frac{u_\theta}{a}\left(\frac{v_\theta + w}{a}\right)\end{aligned} \quad (9)$$

The axial displacement component u is evidently small compared to the radial component w. Consequently, the term $u_\theta^2/2a^2$ will be discarded from Eq. (8). Also, the terms u_xv_x and $u_\theta(v_\theta+w)/a^2$ will be discarded from Eq. (9), since they are small compared to the respective additive terms v_x and u_θ/a. A comparison of the relative magnitudes of v and w is difficult. It has been found that the quadratic terms in v exert a predominant effect in some problems of buckling of rings. Consequently, all the quadratic terms in v and w will be retained.

Eqs. (8) and (9) merely give the incremental strains due to buckling. The strains just before buckling are denoted by $\epsilon_x^{(0)}$ and $\epsilon_\theta^{(0)}$. The initial shearing strain is evidently zero. Consequently, when the quadratic terms containing u are neglected, the complete formulas for the strain components are

$$\left.\begin{aligned}\epsilon_x &= \epsilon_x^{(0)} + u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2\\ \epsilon_\theta &= \epsilon_\theta^{(0)} + \frac{v_\theta + w}{a} + \frac{1}{2}\left(\frac{v - w_\theta}{a}\right)^2\\ \gamma_{x\theta} &= \frac{u_\theta}{a} + v_x - w_x\left(\frac{v - w_\theta}{a}\right)\end{aligned}\right\} \quad (10)$$

4. Fourier Analysis of Displacement Components

Equations (10) express the membrane strains in terms of the displacement components of the middle surface of the shell. In this article, the displacement components (u, v, w) of the middle surface are expressed in the form of Fourier series in θ. Also, by the assumption that the shell buckles with zero incremental hoop strain, the coefficients in the series for the v and w displacement components are expressed in terms of a single parameter w_1.

In view of the fluted pattern that a buckled cylindrical shell adopts, the functions u, v, w may be represented by Fourier series, as follows:

$$\left.\begin{aligned}u &= u_0 + u_1\cos n\theta + u_2\cos 2n\theta\\ &\quad + u_3\cos 3n\theta + \ldots\\ v &= v_1\sin n\theta + v_2\sin 2n\theta + v_3\sin 3n\theta + \ldots\\ w &= w_0 + w_1\cos n\theta + w_2\cos 2n\theta\\ &\quad + w_3\cos 3n\theta + \ldots\end{aligned}\right\} \quad (11)$$

Here, n denotes the number of complete waves in the periphery. The coefficients u_i, v_i, w_i are functions of x alone. Only the terms to $3n\theta$ will be retained in Eq. (11).

The membrane strains that accompany buckling are small, since large membrane strains cause excessive strain energy. This fact is exemplified if we deform a piece of sheet metal in our hands. Although we can bend it easily, we cannot stretch it noticeably. This circumstance implies that the middle surface of a buckled cylindrical shell remains approximately developable, since a wide departure from a developable form would require large membrane strains. Loosely speaking, the "easiest" way for a shell to buckle is that which entails the smallest membrane strains. Consequently, we introduce the assumption that the incremental hoop strain $\Delta\epsilon_\theta$ that accompanies buckling is zero. This assumption does not exactly yield minimum strain energy, since the axial strain ϵ_x and the shearing strain $\gamma_{x\theta}$ are then too large in some regions—particularly the end regions of the shell. Consequently, the buckling pressure that is obtained with the assumption $\Delta\epsilon_\theta = 0$ is slightly too large, both for the infinitesimal theory and the snap-through theory. The termination of the series in Eq. (11) after the third terms also raises the computed buckling pressures, since this approximation, like the assumption $\Delta\epsilon_\theta = 0$, implies artificial constraints on the buckling pattern.

Eq. (11) yields

$$\left.\begin{aligned} \frac{v_\theta + w}{a} &= \frac{w_0}{a} + \alpha_1 \cos n\theta + \alpha_2 \cos 2n\theta \\ &\quad + \alpha_3 \cos 3n\theta \\ \frac{v - w_\theta}{a} &= \beta_1 \sin n\theta + \beta_2 \sin 2n\theta + \beta_3 \sin 3n\theta \end{aligned}\right\} \tag{12}$$

where

$$\alpha_i = \frac{inv_i + w_i}{a}, \qquad \beta_i = \frac{v_i + inw_i}{a} \tag{13}$$

As was remarked previously, the term u_θ^2 will be dropped from Eq. (8). Then Eqs. (8), (12), and (13) yield

$$\begin{aligned} \Delta\epsilon_\theta = \frac{w_0}{a} &+ \alpha_1 \cos n\theta + \alpha_2 \cos 2n\theta + \alpha_3 \cos 3n\theta \\ &+ \frac{1}{2}(\beta_1 \sin n\theta + \beta_2 \sin 2n\theta + \beta_3 \sin 3n\theta)^2 \end{aligned}$$

With the trigonometric identity,

$$\sin in\theta \sin jn\theta = \frac{1}{2}[\cos(i-j)n\theta - \cos(i+j)n\theta]$$

we obtain, after regrouping terms,

$$\left.\begin{aligned} \Delta\epsilon_\theta &= \frac{w_0}{a} + \frac{1}{4}(\beta_1^2 + \beta_2^2 + \beta_3^2) \\ &+ \frac{1}{2}(2\alpha_1 + \beta_1\beta_2 + \beta_2\beta_3)\cos n\theta \\ &+ \frac{1}{2}\left(2\alpha_2 - \frac{1}{2}\beta_1^2 + \beta_1\beta_3\right)\cos 2n\theta \\ &+ \frac{1}{2}(2\alpha_3 - \beta_1\beta_2)\cos 3n\theta \\ &- \frac{1}{2}\left(\frac{1}{2}\beta_2^2 + \beta_1\beta_3\right)\cos 4n\theta \\ &- \frac{1}{2}\beta_2\beta_3 \cos 5n\theta - \frac{1}{4}\beta_3^2 \cos 6n\theta \end{aligned}\right\} \tag{14}$$

Necessary and sufficient conditions for $\Delta\epsilon_\theta$ to vanish are that each coefficient in Eq. (14) vanish. Hence,

$$\begin{aligned} &\beta_3 = 0, \qquad \beta_2 = 0, \qquad \alpha_3 = 0, \qquad \alpha_1 = 0 \\ &\alpha_2 - \frac{1}{4}\beta_1^2 = 0, \qquad w_0 + \frac{1}{4}a\beta_1^2 = 0 \end{aligned}$$

These conditions yield

$$\left.\begin{aligned} v_1 &= -\frac{w_1}{n}, \qquad v_2 = -2nw_2, \qquad v_3 = 0 \\ w_0 &= -\frac{(n - 1/n)^2 w_1^2}{4a}, \\ w_2 &= \frac{w_0}{4n^2 - 1}, \qquad w_3 = 0 \end{aligned}\right\} \tag{15}$$

or

$$\left.\begin{aligned} v_1 &= -\frac{w_1}{n}, \qquad v_2 = \frac{n(n - 1/n)^2 w_1^2}{2a(4n^2 - 1)}, \\ v_3 &= 0 \\ w_0 &= -\frac{(n - 1/n)^2 w_1^2}{4a}, \\ w_2 &= \frac{-(n - 1/n)^2 w_1^2}{4a(4n^2 - 1)}, \qquad w_3 = 0 \end{aligned}\right\} \tag{16}$$

Eq. (16) expresses the coefficients in the v and w equations (Eq. 11) in terms of w_1. Since the curve of a buckled cross section cannot intersect itself, the admissible values of w_1 are restricted to a finite range. If w_1 lies outside of this range, θ ceases to be a regular parameter for the buckled cross section.

Eqs. (10), (11), and (16) yield the following expressions for the strains (where primes denote derivatives with respect to x):

$$\left.\begin{aligned} \epsilon_x &= \epsilon_x^{(0)} + u'_0 + u'_1 \cos n\theta + u'_2 \cos 2n\theta \\ &\quad + u'_3 \cos 3n\theta + \frac{1}{2}(v'_1 \sin n\theta \\ &\quad + v'_2 \sin 2n\theta)^2 + \frac{1}{2}(w'_0 \\ &\quad + w'_1 \cos n\theta + w'_2 \cos 2n\theta)^2 \\ \epsilon_\theta &= \epsilon_\theta^{(0)} \\ \gamma_{x\theta} &= -\frac{n}{a}u_1 \sin n\theta - \frac{2n}{a}u_2 \sin 2n\theta \end{aligned}\right\} \tag{17}$$

$$
\left.\begin{aligned}
&-\frac{3n}{a} u_3 \sin 3n\theta + v'_1 \sin n\theta \\
&+ v'_2 \sin 2n\theta - (w'_0 + w'_1 \cos n\theta \\
&+ w'_2 \cos 2n\theta)\left(\frac{v_1 + nw_1}{a}\right) \sin n\theta
\end{aligned}\right| \quad (17)
$$

With the trigonometric identities,

$$\cos n\theta \cos 2n\theta = \frac{1}{2}(\cos n\theta + \cos 3n\theta)$$

$$\sin n\theta \sin 2n\theta = \frac{1}{2}(\cos n\theta - \cos 3n\theta)$$

$$\sin n\theta \cos 2n\theta = \frac{1}{2}(\sin 3n\theta - \sin n\theta)$$

these equations yield

$$
\left.\begin{aligned}
\epsilon_x = &\left[\epsilon_x^{(0)} + u'_0 + \frac{1}{4} v'^2_1 + \frac{1}{4} v'^2_2 + \frac{1}{2} w'^2_0 + \frac{1}{4} w'^2_1 + \frac{1}{4} w'^2_2\right] \\
&+ \left[u'_1 + \frac{1}{2} v'_1 v'_2 + w'_0 w'_1 + \frac{1}{2} w'_1 w'_2\right] \cos n\theta \\
&+ \left[u'_2 - \frac{1}{4} v'^2_1 + \frac{1}{4} w'^2_1 + w'_0 w'_2\right] \cos 2n\theta \\
&+ \left[u'_3 - \frac{1}{2} v'_1 v'_2 + \frac{1}{2} w'_1 w'_2\right] \cos 3n\theta \\
&+ \left[-\frac{1}{4} v'^2_2 + \frac{1}{4} w'^2_2\right] \cos 4n\theta \\
\gamma_{x\theta} = &\left[-\frac{n}{a} u_1 + v'_1 - w'_0 \beta_1 + \frac{1}{2} w'_2 \beta_1\right] \sin n\theta \\
&+ \left[-\frac{2n}{a} u_2 + v'_2 - \frac{1}{2} w'_1 \beta_1\right] \sin 2n\theta \\
&- \frac{1}{2} w'_2 \beta_1 \sin 3n\theta - \frac{3n}{a} u_3 \sin 3n\theta
\end{aligned}\right\} \quad (18)
$$

Eqs. (18) are of the form,

$$
\left.\begin{aligned}
\epsilon_x &= \epsilon_x^{(0)} + C_0 + C_1 \cos n\theta + C_2 \cos 2n\theta \\
&\quad + C_3 \cos 3n\theta + C_4 \cos 4n\theta \\
\gamma_{x\theta} &= S_1 \sin n\theta + S_2 \sin 2n\theta + S_3 \sin 3n\theta
\end{aligned}\right\} \quad (19)
$$

where C and S indicate coefficients of cosine and sine terms respectively.

Eqs. (16) and (18) yield

$$
\left.\begin{aligned}
C_0 &= u'_0 + \frac{1}{4}(1 + 1/n^2) w'^2_1 + \frac{(n - 1/n)^4 w_1^2 w'^2_1}{16a^2}\left[2 + \frac{4n^2 + 1}{(4n^2 - 1)^2}\right] \\
C_1 &= u'_1 - \frac{(n - 1/n)^2}{4a}\left[2 + \frac{3}{4n^2 - 1}\right] w_1 w'^2_1 \\
C_2 &= u'_2 + \frac{1}{4}(1 - 1/n^2) w'^2_1 + \frac{(n - 1/n)^4 w_1^2 w'^2_1}{4a^2 (4n^2 - 1)} \\
C_3 &= u'_3 + \frac{(n - 1/n)^2 w_1 w'^2_1}{4a (4n^2 - 1)} \\
C_4 &= \frac{-(n - 1/n)^4 w_1^2 w'^2_1}{16a^2 (4n^2 - 1)} \\
S_1 &= -\frac{n}{a} u_1 - \frac{w'_1}{n} + \frac{(n - 1/n)^3}{4a^2}\left[2 - \frac{1}{4n^2 - 1}\right] w_1^2 w'_1 \\
S_2 &= -\frac{2n}{a} u_2 - \frac{(n - 1/n)(2n^2 + 1)}{2(4n^2 - 1)} \frac{w_1 w'_1}{a} \\
S_3 &= -\frac{3n}{a} u_3 + \frac{(n - 1/n)^3}{4(4n^2 - 1)} \frac{w_1^2 w'_1}{a^2}
\end{aligned}\right\} \quad (20)
$$

5. Membrane Energy

In Sections 3 and 4, expressions for the membrane strains were derived in the form of Fourier series; and the coefficients of the series for the displacement components v and w were expressed in terms of the single parameter w_1. We now proceed to develop an expression for the increment of membrane energy due to buckling in terms of the parameter w_1 and the coefficients u_0, u_1, u_2, u_3 of the Fourier series for the displacement component u.

The membrane energy is (16)

$$U_m = \frac{Eha}{1 - \nu^2}\int_0^{L/2} dx \int_0^{2\pi}\left[\epsilon_x^2 + \epsilon_\theta^2 + 2\nu\epsilon_x\epsilon_\theta + \frac{1}{2}(1 - \nu)\gamma_{x\theta}^2\right] d_\theta \quad (21)$$

in which E is Young's modulus, ν is Poisson's ratio, and h is the thickness of the shell. Eqs. (19) and (21) yield

$$U_m = \frac{\pi Eha}{1 - \nu^2}\int_0^{L/2}\Big[2(\epsilon_x^{(0)} + C_0)^2 + C_1^2 + C_2^2 + C_3^2 + C_4^2 + 2(\epsilon_\theta^{(0)})^2 + 4\nu(\epsilon_x^{(0)} + C_0)\epsilon_\theta^{(0)}$$

$$+ \frac{1}{2}(1-\nu)(S_1^2 + S_2^2 + S_3^2)\Big] dx$$

in which L is the length of the shell.

The membrane energy just before buckling is

$$U_m^{(0)} = \frac{\pi Eha}{1-\nu^2}\int_0^{L/2}[2(\epsilon_x^{(0)})^2 + 2(\epsilon_\theta^{(0)})^2 + 4\nu\epsilon_x^{(0)}\epsilon_\theta^{(0)}]\, dx$$

This result is obtained by discarding the C's and S's from the preceding equation.

The increment of membrane energy due to buckling is $\Delta U_m = U_m - U_m^{(0)}$. Consequently, by the two preceding equations,

$$\Delta U_m = \frac{\pi Eha}{1-\nu^2}\int_0^{L/2}\Big[4(\epsilon_x^{(0)} + \nu\epsilon_\theta^{(0)})C_0 + 2C_0^2 + C_1^2 + C_2^2 + C_3^2 + C_4^2 + \frac{1}{2}(1-\nu)(S_1^2 + S_2^2 + S_3^2)\Big] dx \quad (22)$$

The initial axial stress is

$$\sigma_x^{(0)} = \frac{E}{1-\nu^2}(\epsilon_x^{(0)} + \nu\epsilon_\theta^{(0)})$$

By statics,

$$\sigma_x^{(0)} = \frac{-F}{2\pi ah} = \frac{-\lambda pa}{2\pi h}$$

where F is the axial compressive force. We set $F = \lambda pa^2$ where p is the external pressure on the lateral surface. Consequently,

$$\frac{E}{1-\nu^2}(\epsilon_x^{(0)} + \nu\epsilon_\theta^{(0)}) = \frac{-\lambda pa}{2\pi h}$$

With this relation, the initial strains may be eliminated from Eq. (22). Then Eqs. (20) and (22) yield

$$\left.\begin{aligned}\Delta U_m = \frac{\pi Eha}{1-\nu^2}\int_0^{L/2}\Big\{&2u_0'^2 + k_4u_0'w_1'^2 + k_5u_0'(w_1/a)^2w_1'^2 + k_1w_1'^4 \\ &+ k_2(w_1/a)^4w_1'^4 + k_3(w_1/a)^2w_1'^4 \\ &+ \frac{1}{2}(1-\nu)\Big[k_6 - \frac{1+\nu}{\pi}k_4\frac{\lambda ap}{Eh}\Big]w_1'^2 \\ &- \frac{1}{2}(1-\nu)\Big[k_7 + \frac{1+\nu}{\pi}k_5\frac{\lambda ap}{Eh}\Big](w_1/a)^2w_1'^2 \\ &+ \frac{1}{2}(1-\nu)k_8(w_1/a)^4w_1'^2 - \frac{2\lambda ap(1-\nu)^2}{\pi Eh}u_0'\Big\}\, dx \\ &+ \frac{\pi Eha}{1-\nu^2}(\Psi + \mathrm{X} + \Upsilon)\end{aligned}\right\} \quad (23)$$

where $k_1, k_2, \ldots$ are functions of n only, and Ψ, X, Υ respectively represent the integrals that contain u_1, u_2, and u_3. These integrals are

$$\Psi = \int_0^{L/2}\Big[u_1'^2 - k_9u_1'\frac{w_1}{a}w_1'^2 + \frac{1}{2}(1-\nu)\frac{n^2}{a^2}u_1^2 + (1-\nu)\frac{u_1}{a}w_1' - \frac{1}{2}(1-\nu)k_{10}\frac{u_1}{a}\Big(\frac{w_1}{a}\Big)^2w_1'\Big] dx \quad (24)$$

$$\mathrm{X} = \int_0^{L/2}\Big[u_2'^2 + 2(1-\nu)\Big(\frac{n}{a}\Big)^2u_2^2 + k_{11}u_2'w_1'^2 + k_{12}u_2'\Big(\frac{w_1}{a}\Big)^2w_1'^2 + (1-\nu)k_{13}\frac{u_2}{a}\frac{w_1}{a}w_1'\Big] dx \quad (25)$$

$$\Upsilon = \int_0^{L/2}\Big[u_3'^2 + \frac{9}{2}(1-\nu)\Big(\frac{n}{a}\Big)^2u_3^2 + k_{14}u_3'\frac{w_1}{a}w_1'^2 - (1-\nu)k_{15}\frac{u_3}{a}\Big(\frac{w_1}{a}\Big)^2w_1'\Big] dx \quad (26)$$

The constants k_i are defined by

$$\left.\begin{aligned}k_1 &= \frac{3n^4 + 2n^2 + 3}{16n^4} \\ k_2 &= \frac{(n-1/n)^8}{256(4n^2-1)^4}[2(32n^4 - 12n^2 + 3)^2 + 17(4n^2-1)^2] \\ k_3 &= \frac{(n-1/n)^4}{16n^2(4n^2-1)^2}[96n^6 + 44n^4 - 17n^2 + 5] \\ k_4 &= 1 + \frac{1}{n^2} \\ k_5 &= \frac{1}{4}(n-1/n)^4\Big[2 + \frac{4n^2+1}{(4n^2-1)^2}\Big] \\ k_6 &= 1/n^2 \\ k_7 &= \frac{(n-1/n)^2}{4n^2(4n^2-1)^2}(60n^6 - 108n^4 + 45n^2 - 6) \\ k_8 &= \frac{(n-1/n)^6}{8(4n^2-1)^2}(32n^4 - 24n^2 + 5)\end{aligned}\right\} \quad (27)$$

$$\left.\begin{aligned}
k_9 &= \frac{(n-1/n)^2\,(8n^2+1)}{2\,(4n^2-1)} \\
k_{10} &= \frac{n\,(n-1/n)^3\,(8n^2-3)}{2\,(4n^2-1)} \\
k_{11} &= \frac{1}{2}\,(1-1/n^2) \\
k_{12} &= \frac{(n-1/n)^4}{2\,(4n^2-1)} \\
k_{13} &= \frac{(n^2-1)\,(2n^2+1)}{4n^2-1} \\
k_{14} &= \frac{(n-1/n)^2}{2\,(4n^2-1)} \\
k_{15} &= \frac{3n\,(n-1/n)^3}{4\,(4n^2-1)}
\end{aligned}\right\} \tag{27}$$

The notations K_1, K_2, . . . , are reserved for certain combinations of the quantities k_1, k_2, etc.

6. Potential Energy of External Forces

The potential energy of the external forces consists of two parts, the potential energy of the axial force F and the potential energy of the lateral pressure p. If the ends of the shell are rigid, the Fourier coefficients u_1, u_2, u_3 vanish at the ends. Then the increment of potential energy of the axial force is simply $2Fu_0(L/2)$.

If the ends of the shell are not rigid, the potential energy of the force F depends on the way in which the axial load is distributed. We assume that it is distributed so that the axial stress σ_x in the cylindrical wall is constant at the ends, $x = \pm L/2$. Then the increment of potential energy of the force F is

$$\Delta\Omega_F = -2\int_0^{2\pi} ah\sigma_x(L/2)u(L/2)d\theta$$

Since $\sigma_x(L/2)$ is constant, $F = -2\pi ah\sigma_x(L/2)$. Consequently

$$\Delta\Omega_F = \frac{F}{\pi}\int_0^{2\pi} u(L/2)d\theta$$

With Eq. (11), this yields $\Delta\Omega_F = 2Fu_0(L/2)$. This is the same result that was obtained for a shell with rigid ends.

Since $F = \lambda a^2 p$, the preceding formula yields

$$\Delta\Omega_F = 2\lambda a^2 p u_0(L/2)$$

Since, by symmetry, u vanishes at the center section, $x = 0$, this equation may be written in an integral form, as follows:

$$\Delta\Omega_F = 2\lambda a^2 p\int_0^{L/2} u'_0 dx \tag{28}$$

To calculate the potential energy of the lateral pressure p, we must determine the area A^* of a cross section of the buckled shell. The intersection of the plane $x =$ constant with the middle surface of the buckled cylindrical wall is represented parametrically by $y^* = y^*(\theta)$, $z^* = z^*(\theta)$. These functions are given explicitly by Eq. (1).

The area enclosed by the curve $y^* = y^*(\theta)$, $z^* = z^*(\theta)$ is

$$A^* = -\int_0^{2\pi} y^*\frac{\partial z^*}{\partial\theta}\,d\theta \tag{29}$$

The sign on the right side of this equation is negative, since the positive sense of θ runs clockwise. Eq. (29) is a special consequence of Green's theorem.

By Eq. (1),

$$\begin{aligned}
y^* &= a\sin\theta + v\cos\theta + w\sin\theta \\
\frac{\partial z^*}{\partial\theta} &= -a\sin\theta - v_\theta\sin\,\theta - v\cos\theta \\
&\quad + w_\theta\cos\theta - w\sin\theta
\end{aligned}$$

With Eqs. (11) and(15), these equations yield

$$\begin{aligned}
y^* &= (a+w_0)\sin\theta - \frac{w_1}{n}\cos\theta\sin n\theta \\
&\quad + w_1\sin\theta\cos n\theta - \frac{2nw_0}{4n^2-1}\cos\theta\sin 2n\theta \\
&\quad + \frac{w_0}{4n^2-1}\sin\theta\cos 2n\theta \\
\frac{-\partial z^*}{\partial\theta} &= (a+w_0)\sin\theta - w_0\sin\theta\cos 2n\theta \\
&\quad + (n-1/n)\,w_1\cos\theta\sin n\theta
\end{aligned}$$

Consequently, if $n > 2$, Eq. (29) yields

$$A^* = \pi\,(a+w_0)^2 - \frac{1}{2}\,\pi(1-1/n^2)w_1^2 - \frac{\pi w_0^2}{2(4n^2-1)}$$

By means of Eq. (16), w_0 may be eliminated from this equation.

The effect of the deformation before buckling on the incremental cross-sectional area will be neglected. Then the increment of cross-sectional area due to buckling is $\Delta A = A^* - \pi a^2$. Consequently,

$$\Delta A = \pi\left[-k_{16}w_1^2 + k_{17}\frac{w_1^4}{a^2}\right] \tag{30}$$

where

$$\begin{aligned}
k_{16} &= \frac{1}{2}\,(n^2-1) \\
k_{17} &= \frac{1}{16}\,(n-1/n)^4\left[1 - \frac{1}{2(4n^2-1)}\right]
\end{aligned} \tag{31}$$

Although Eq. (30) has been derived for $n > 2$, it remains valid for $n=2$, as we see by carrying out the integration specifically for $n=2$.

The increment of potential energy of the lateral pressure due to buckling is approximately

$$\Delta\Omega_p = 2p\int_0^{L/2} \Delta A\,dx \tag{32}$$

Eq. (32) implies the approximation that the axial displacement u does not influence the work of the lateral pressure p when the shell buckles.

The total increment of potential energy of the external forces due to buckling is $\Delta\Omega = \Delta\Omega_F + \Delta\Omega_p$. Consequently, Eqs. (28), (30), and (32) yield

$$\Delta\Omega = 2\lambda a^2 p\int_0^{L/2} u'_0 dx + 2\pi a^2 p\int_0^{L/2}\left[-k_{16}\left(\frac{w_1}{a}\right)^2 + k_{17}\left(\frac{w_1}{a}\right)^4\right]dx \tag{33}$$

7. Elimination of u_0 from the Increment of Total Potential Energy

The increment of total potential energy due to buckling is $\Delta V = \Delta U_m + \Delta\Omega + \Delta U_b$, in which ΔU_m is the increment of membrane energy, $\Delta\Omega$ is the incremental potential energy of the external forces, and ΔU_b is the incremental strain energy of bending. The slight axial bending that exists before buckling will be neglected. Accordingly, ΔU_b is approximated by the total strain energy of bending U_b. A detailed analysis of the term U_b is developed in Section 12.

Since the term U_b does not depend on the axial displacement u, Eqs. (23) and (33) yield

$$\Delta V = \frac{\pi Eha}{1-\nu^2}\int_0^{L/2}\left[2u_0'^2 + k_4 u'_0 w_1'^2 + k_5 u'_0 (w_1/a)^2 w_1'^2 - \frac{2\lambda ap(1-\nu)^2}{\pi Eh}u'_0\right]dx + 2\lambda a^2 p\int_0^{L/2} u'_0 dx + \text{terms that do not contain } u'_0.$$

By the principle of minimum potential energy, the axial displacement u provides a minimum to ΔV. Consequently, u'_0 minimizes the integrand in the preceding equation. A necessary condition for the value of the integrand ϕ to be a minimum is $\partial\phi/\partial u'_0 = 0$. Furthermore, this condition is sufficient to insure a relative minimum, since

$$\frac{\partial^2\phi}{\partial u_0'^2} = \frac{4\pi Eha}{1-\nu^2} > 0$$

Consequently,

$$u'_0 = -\frac{1}{4}k_4 w_1'^2 - \frac{1}{4}k_5 (w_1/a)^2 w_1'^2 \tag{34}$$

Eliminating u'_0 from ΔV by means of Eq. (34), we obtain

$$\left.\begin{aligned}\Delta V = U_b + \frac{\pi Eha}{1-\nu^2}\int_0^{L/2}&\left\{\left(k_1 - \frac{1}{8}k_4^2\right)w_1'^4 + \left(k_2 - \frac{1}{8}k_5^2\right)\left(\frac{w_1}{a}\right)^4 w_1'^4\right.\\ &+ \left(k_3 - \frac{1}{4}k_4k_5\right)\left(\frac{w_1}{a}\right)^2 w_1'^4\\ &+ \frac{1}{2}(1-\nu)\left[k_6 - \frac{1+\nu}{\pi}k_4\frac{\lambda ap}{Eh}\right]w_1'^2\\ &- \frac{1}{2}(1-\nu)\left[k_7 + \frac{1+\nu}{\pi}k_5\frac{\lambda ap}{Eh}\right]\left(\frac{w_1}{a}\right)^2 w_1'^2\\ &+ \frac{1}{2}(1-\nu)k_8\left(\frac{w_1}{a}\right)^4 w_1'^2\\ &- 2(1-\nu^2)k_{16}\frac{ap}{Eh}\left(\frac{w_1}{a}\right)^2\\ &\left. + 2(1-\nu^2)k_{17}\frac{ap}{Eh}\left(\frac{w_1}{a}\right)^4\right\}dx\\ &+ \frac{\pi Eha}{1-\nu^2}(\Psi + \mathrm{X} + \Upsilon)\end{aligned}\right\} \tag{35}$$

It will be assumed that

$$\frac{w_1}{a} = \frac{W_0\cos\frac{\pi x}{L}}{n - 1/n} = \omega\cos\frac{\pi x}{L} \tag{36}$$

in which W_0 is a constant. Observations of buckled cylindrical shells suggest that this is a reasonable assumption. Since this assumption implies an artificial constraint, it possibly raises the computed buckling pressure, but it cannot lower it. Eq. (36) yields

$$\left.\begin{aligned}\int_0^{L/2} w_1'^4 dx &= \frac{3\pi^4 a^4 W_0^4}{16(n-1/n)^4 L^3}\\ \int_0^{L/2} w_1^4 w_1'^4 dx &= \frac{3\pi^4 a^8 W_0^8}{256(n-1/n)^8 L^3}\\ \int_0^{L/2} w_1^2 w_1'^4 dx &= \frac{\pi^4 a^6 W_0^6}{32(n-1/n)^6 L^3}\\ \int_0^{L/2} w_1'^2 dx &= \frac{\pi^2 a^2 W_0^2}{4(n-1/n)^2 L}\\ \int_0^{L/2} w_1^2 w_1'^2 dx &= \frac{\pi^2 a^4 W_0^4}{16(n-1/n)^4 L}\end{aligned}\right\} \tag{37}$$

$$\left.\begin{aligned}\int_0^{L/2} w_1^4 w_1'^2 dx &= \frac{\pi^2 a^6 W_0^6}{32\,(n - 1/n)^6\,L}\\ \int_0^{L/2} w_1^2 dx &= \frac{a^2 L W_0^2}{4\,(n - 1/n)^2}\\ \int_0^{L/2} w_1^4 dx &= \frac{3\,a^4 W_0^4\,L}{16\,(n - 1/n)^4}\end{aligned}\right\} \quad (37)$$

Consequently, by Eq. (35),

$$\left.\begin{aligned}\Delta V = U_b &+ \left[K_6 \frac{Ea^3h}{L} - K_4 \frac{a^4\lambda p}{L} - K_{16}a^2pL\right] W_0^2 + \left[K_1 \frac{Ea^5h}{L^3} - K_7 \frac{Ea^3h}{L} - K_5 \frac{a^4\lambda p}{L} + K_{17}a^2pL\right] W_0^4\\ &+ \left[K_3 \frac{Ea^5h}{L^3} + K_8 \frac{Ea^3h}{L}\right] W_0^6 + K_2 \frac{Ea^5h}{L^3} W_0^8 + \frac{\pi Eha}{1 - \nu^2}(\Psi + \mathrm{X} + \Upsilon)\end{aligned}\right\} \quad (38)$$

in which

$$\left.\begin{aligned}K_1 &= \frac{3\pi^5\left(k_1 - \frac{1}{8}k_4^2\right)}{16\,(1 - \nu^2)\,(n - 1/n)^4} = \frac{3\pi^5}{256\,(1 - \nu^2)\,(n^2 - 1)^2}\\ K_2 &= \frac{3\pi^5\left(k_2 - \frac{1}{8}k_5^2\right)}{256\,(1 - \nu^2)\,(n - 1/n)^8} = \frac{51\pi^5}{65536\,(1 - \nu^2)\,(4n^2 - 1)^2}\\ K_3 &= \frac{\pi^5\left(k_3 - \frac{1}{4}k_4k_5\right)}{32\,(1 - \nu^2)\,(n - 1/n)^6} = \frac{\pi^5\left[\frac{5n^2}{(4n^2 - 1)^2} + \frac{7n^2 - 1}{4n^2 - 1} + 2n^2\right]}{256\,(1 - \nu^2)\,(n^2 - 1)^2}\\ K_4 &= \frac{\pi^2 k_4}{8\,(n - 1/n)^2} = \frac{\pi^2\,(n^2 + 1)}{8\,(n^2 - 1)^2}\\ K_5 &= \frac{\pi^2 k_5}{32\,(n - 1/n)^4} = \frac{\pi^2}{128}\left[2 + \frac{4n^2 + 1}{(4n^2 - 1)^2}\right]\\ K_6 &= \frac{\pi^3 k_6}{8\,(1 + \nu)\,(n - 1/n)^2} = \frac{\pi^3}{8\,(1 + \nu)\,(n^2 - 1)^2}\\ K_7 &= \frac{\pi^3 k_7}{32\,(1 + \nu)\,(n - 1/n)^4} = \frac{\pi^3}{32\,(1 + \nu)}\left[\frac{-n^2}{(4n^2 - 1)^2} + \frac{2n^2 - 1}{2\,(n^2 - 1)\,(4n^2 - 1)} + \frac{3n^2 - 4}{4\,(n^2 - 1)^2}\right]\\ K_8 &= \frac{\pi^3 k_8}{64\,(1 + \nu)\,(n - 1/n)^6} = \frac{\pi^3}{512\,(1 + \nu)}\left[\frac{1}{(4n^2 - 1)^2} - \frac{2}{4n^2 - 1} + 2\right]\\ K_{16} &= \frac{\pi k_{16}}{2\,(n - 1/n)^2} = \frac{\pi n^2}{4\,(n^2 - 1)}\\ K_{17} &= \frac{3\pi k_{17}}{8\,(n - 1/n)^4} = \frac{3\pi}{128}\left[1 - \frac{1}{2\,(4n^2 - 1)}\right]\end{aligned}\right\} \quad (39)$$

These constants have been tabulated (Table 1).

8. Elimination of u_1 from the Increment of Total Potential Energy

Since the origin of x is the mid-section of the shell, the displacement component u is an odd function of x. Therefore, u_1 is an odd function of x. By the principle of minimum potential energy, this function provides a minimum to Ψ. When w_1 is eliminated by means of Eq. (36) the equation for Ψ becomes

$$\Psi = \int_0^{L/2}\left[u_1'^2 - k_9 \frac{\pi^2 a^2 \omega^3}{L^2} u_1' \cos\frac{\pi x}{L} \sin^2\left(\frac{\pi x}{L}\right) - (1 - \nu)\frac{\pi\omega}{L} u_1 \sin\frac{\pi x}{L} + \frac{1}{2}(1 - \nu)\frac{\pi\omega^3}{L} k_{10} u_1 \sin\frac{\pi x}{L}\cos^2\left(\frac{\pi x}{L}\right) + \frac{1}{2}(1 - \nu)\frac{n^2}{a^2} u_1^2\right] dx$$

where, for brevity, $\omega = \dfrac{W_0}{n - 1/n}$

With the trigonometric identities,

$$\cos\frac{\pi x}{L}\sin^2\left(\frac{\pi x}{L}\right) = \frac{1}{4}\left(\cos\frac{\pi x}{L} - \cos\frac{3\pi x}{L}\right)$$
$$\sin\frac{\pi x}{L}\cos^2\left(\frac{\pi x}{L}\right) = \frac{1}{4}\left(\sin\frac{\pi x}{L} + \sin\frac{3\pi x}{L}\right),$$

this yields

$$\Psi = \int_0^{L/2}\left[u_1'^2 - k_9 \frac{\pi^2 a^2 \omega^3}{4L^2} u_1' \left(\cos\frac{\pi x}{L} - \cos\frac{3\pi x}{L}\right)\right.$$

$$+(1-\nu)\frac{\pi\omega}{L}u_1\sin\frac{\pi x}{L}+(1-\nu)\frac{\pi\omega^3}{8L}\left(\sin\frac{\pi x}{L}+\sin\frac{3\pi x}{L}\right)k_{10}u_1+\frac{1}{2}(1-\nu)\frac{n^2}{a^2}u_1^2\Big]dx$$

Since $u_1(0)=0$, integration by parts now yields

$$\Psi=\int_0^{L/2}\left[u_1'^2+k^2u_1^2+\frac{2A}{L}u_1\sin\frac{\pi x}{L}+\frac{2B}{L}u_1\sin\frac{3\pi x}{L}\right]dx \tag{40}$$

in which

$$\left.\begin{aligned}A&=\frac{1}{2}\left[\frac{\pi}{8}(1-\nu)k_{10}-\frac{\pi^3}{4}k_9\frac{a^2}{L^2}\right]\omega^3-\frac{1}{2}(1-\nu)\pi\omega\\B&=\frac{1}{2}\left[\frac{\pi}{8}(1-\nu)k_{10}+\frac{3\pi^3}{4}k_9\frac{a^2}{L^2}\right]\omega^3\\k&=\frac{n}{a}\sqrt{\frac{1-\nu}{2}}\end{aligned}\right\} \tag{41}$$

For integrals of this particular type, Euler's equation of the calculus of variations is both necessary and sufficient for a relative minimum.(17) Euler's equation for Ψ is

$$u''_1-k^2u_1=\frac{A}{L}\sin\frac{\pi x}{L}+\frac{B}{L}\sin\frac{3\pi x}{L} \tag{42}$$

The general odd solution of Eq. (42) is

$$u_1=CL\sinh kx-\frac{AL}{\alpha}\sin\frac{\pi x}{L}-\frac{BL}{\beta}\sin\frac{3\pi x}{L} \tag{43}$$

in which C is an arbitrary constant of integration, and

$$\alpha=\pi^2+k^2L^2,\qquad \beta=9\pi^2+k^2L^2 \tag{44}$$

Substituting Eq. (43) into Eq. (40), we obtain

$$\Psi=\frac{1}{2}kC^2L^2\sinh kL-\frac{A^2L}{4\alpha}-\frac{B^2L}{4\beta} \tag{45}$$

If there are no constraints at the ends that affect the axial displacement u, the constant C must be chosen to minimize Ψ. Obviously, this condition implies that $C=0$. Then,

$$\Psi=-\frac{A^2L}{4\alpha}-\frac{B^2L}{4\beta} \tag{46}$$

Evidently, Ψ is negative and it reduces ΔV. This circumstance might have been anticipated, since the introduction of u_1 effectively gives the shell additional degrees of freedom.

Let us set

$$\left.\begin{aligned}K_9&=\frac{\pi^3k_9}{8(n-1/n)^3}=\frac{\pi^3n(8n^2+1)}{16(n^2-1)(4n^2-1)}\\K_{10}&=\frac{\pi(1-\nu)k_{10}}{16(n-1/n)^3}=\frac{\pi(1-\nu)n(8n^2-3)}{32(4n^2-1)}\\K_{11}&=\frac{(1-\nu)\pi n}{2(n^2-1)},\qquad r=a/L\end{aligned}\right\} \tag{47}$$

Then, by Eq. (41),

$$\begin{aligned}A&=(K_{10}-K_9r^2)W_0^3-K_{11}W_0\\B&=(K_{10}+3K_9r^2)W_0^3\\\frac{1}{2}kL&=\xi=\frac{n}{2r}\sqrt{\frac{1-\nu}{2}}\end{aligned} \tag{48}$$

Accordingly, Eq. (46) yields

$$\frac{\pi}{1-\nu^2}\frac{\Psi}{L}=W_0^2f_1(\xi)+[f_2(\xi)+r^2f_3(\xi)]W_0^4+[f_4(\xi)-r^2f_5(\xi)+r^4f_6(\xi)]W_0^6 \tag{49}$$

where

$$\left.\begin{aligned}f_1(\xi)&=\frac{-\pi K_{11}^2}{4(1-\nu^2)\alpha} & f_4(\xi)&=\frac{-\pi K_{10}^2}{4(1-\nu^2)}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\\f_2(\xi)&=\frac{\pi K_{10}K_{11}}{2(1-\nu^2)\alpha} & f_5(\xi)&=\frac{\pi K_9K_{10}}{2(1-\nu^2)}\left(\frac{-1}{\alpha}+\frac{3}{\beta}\right)\\f_3(\xi)&=\frac{-\pi K_9K_{11}}{2(1-\nu^2)\alpha} & f_6(\xi)&=\frac{-\pi K_9^2}{4(1-\nu^2)}\left(\frac{1}{\alpha}+\frac{9}{\beta}\right)\end{aligned}\right\} \tag{50}$$

9. Elimination of u_2 from the Increment of Total Potential Energy

Like u_1, u_2 is an odd function of x. This function must be chosen to minimize X. When w_1 is eliminated by means of Eq. (36), the equation for X becomes

$$\begin{aligned}X=\int_0^{L/2}\Big\{&u_2'^2+2(1-\nu)n^2\left(\frac{u_2}{a}\right)^2\\&+\frac{\pi^2a^2}{16L^2}\left[4(1-1/n^2)\omega^2\left(1-\cos\frac{2\pi x}{L}\right)+\frac{(n-1/n)^4\omega^4}{4n^2-1}\left(1-\cos\frac{4\pi x}{L}\right)\right]u_2'\\&-\frac{(1-\nu)\pi(n^2-1)(2n^2+1)}{2L(4n^2-1)}\omega^2u_2\sin\frac{2\pi x}{L}\Big\}dx\end{aligned} \tag{51}$$

Integration by parts yields

$$X=\int_0^{L/2}\left[u_2'^2+4k^2u_2^2-\frac{2A}{L}u_2\sin\frac{2\pi x}{L}-\frac{2B}{L}u_2\sin\frac{4\pi x}{L}\right]dx+\frac{1}{2}\pi^2r^2(1-1/n^2)\omega^2u_2(L/2) \tag{52}$$

where

$$\left.\begin{aligned} A &= \frac{\pi}{4}\left[\frac{(1-\nu)(n^2-1)(2n^2+1)}{4n^2-1} + \pi^2 r^2 (1 - 1/n^2)\right]\omega^2 \\ B &= \frac{\pi^3 r^2}{8}\,\frac{(n-1/n)^4}{4n^2-1}\,\omega^4 \\ k &= \frac{n}{a}\sqrt{\frac{1-\nu}{2}}, \qquad r = a/L \end{aligned}\right\} \tag{53}$$

Euler's equation for the integral X is

$$u''_2 - 4k^2 u_2 = -\frac{A}{L}\sin\frac{2\pi x}{L} - \frac{B}{L}\sin\frac{4\pi x}{L}$$

The general odd solution of this equation is

$$u_2 = CL\sinh 2kx + \frac{AL}{4\alpha}\sin\frac{2\pi x}{L} + \frac{BL}{4\gamma}\sin\frac{4\pi x}{L} \tag{54}$$

where

$$\alpha = \pi^2 + k^2L^2, \qquad \gamma = 4\pi^2 + k^2L^2 \tag{55}$$

Substituting Eq. (54) into Eq. (52), we obtain

$$\begin{aligned} \mathrm{X} = {} & kL^2C^2\sinh 2kL - \frac{A^2L}{16\alpha} - \frac{B^2L}{16\gamma} \\ & - \frac{\pi ACL}{\alpha}\sinh kL + \frac{2\pi BCL}{\gamma}\sinh kL \\ & + \frac{\pi^2a^2}{2L}(1 - 1/n^2)\,\omega^2 C\sinh kL \end{aligned} \tag{56}$$

If the ends are not constrained against warping, C must be chosen to minimize X. Then Eq. (56) yields

$$\begin{aligned} \mathrm{X} = {} & -\frac{A^2L}{16\alpha} - \frac{B^2L}{16\gamma} - \frac{1}{8k}\left[\frac{\pi^2a^2}{2L^2}(1-1/n^2)\,\omega^2 \right. \\ & \left. - \frac{\pi A}{\alpha} + \frac{2\pi B}{\gamma}\right]^2 \tanh kL \end{aligned} \tag{57}$$

Set

$$\left.\begin{aligned} K_{12} &= \frac{\pi(1-\nu)\,n^2(2n^2+1)}{4(n^2-1)(4n^2-1)} \\ K_{13} &= \frac{\pi^3}{4(n^2-1)}, \qquad K_{18} = \frac{\pi^3}{8(4n^2-1)} \end{aligned}\right\} \tag{58}$$

Then, by Eq. (53),

$$\left.\begin{aligned} & A = (K_{12} + K_{13}r^2)\,W_0^2 \qquad B = K_{18}r^2W_0^4 \\ & \frac{1}{2}kL = \xi = \frac{n}{2r}\sqrt{\frac{1-\nu}{2}} \end{aligned}\right\} \tag{59}$$

Consequently, to sixth degree terms in W_0,

$$\frac{\pi}{1-\nu^2}\,\frac{\mathrm{X}}{L} = -W_0^4\phi_1(\xi) - W_0^6\phi_2(\xi) \tag{60}$$

where

$$\left.\begin{aligned} \phi_1(\xi) = {} & \frac{\pi(K_{12}+K_{13}r^2)^2}{16(1-\nu^2)\,\alpha} \\ & + \frac{\pi^3\tanh 2\xi}{16(1-\nu^2)\,\xi}\left[\frac{\pi r^2}{2(n^2-1)} - \frac{K_{12}+K_{13}r^2}{\alpha}\right]^2 \\ \phi_2(\xi) = {} & \frac{\pi^3K_{18}r^2\tanh 2\xi}{4(1-\nu^2)\,\gamma\xi}\left[\frac{\pi r^2}{2(n^2-1)} - \frac{K_{12}+K_{13}r^2}{\alpha}\right] \end{aligned}\right\} \tag{61}$$

Numerical calculations show that the eighth degree terms in W_0 are entirely negligible.

10. Elimination of u_3 from the Increment of Total Potential Energy

The Fourier coefficient u_3 is an odd function of x that must be chosen to minimize ϒ. When w_1 is eliminated by means of Eq. (36), the equation for ϒ becomes

$$\Upsilon = \int_0^{L/2}\left[u'^2_3 + 9k^2u_3^2 + \frac{\pi^2k_{14}a^2\omega^3}{L^2}u'_3\sin^2\frac{\pi x}{L}\cos\frac{\pi x}{L} + (1-\nu)\frac{\pi k_{15}\omega^3}{L}u_3\cos^2\frac{\pi x}{L}\sin\frac{\pi x}{L}\right]dx$$

With the trigonometric identities,

$$\sin^2\left(\frac{\pi x}{L}\right)\cos\frac{\pi x}{L} = \frac{1}{4}\left(\cos\frac{\pi x}{L} - \cos\frac{3\pi x}{L}\right)$$

$$\cos^2\left(\frac{\pi x}{L}\right)\sin\frac{\pi x}{L} = \frac{1}{4}\left(\sin\frac{\pi x}{L} + \sin\frac{3\pi x}{L}\right)$$

this equation yields

$$\begin{aligned} \Upsilon = \int_0^{L/2} & \left[u'^2_3 + 9k^2u_3 + \frac{\pi^2k_{14}a^2\omega^3}{4L^2}u'_3\left(\cos\frac{\pi x}{L} \right.\right. \\ & \left. - \cos\frac{3\pi x}{L}\right) + (1-\nu)\frac{\pi k_{15}\omega^3}{4L}u_3\left(\sin\frac{\pi x}{L} \right. \\ & \left.\left. + \sin\frac{3\pi x}{L}\right)\right]dx \end{aligned}$$

Integration by parts now yields

$$\Upsilon = \int_0^{L/2}\left[u'^2_3 + 9k^2u_3^2 + \frac{2A}{L}u_3\sin\frac{\pi x}{L} - \frac{2B}{L}u_3\sin\frac{3\pi x}{L}\right]dx \tag{62}$$

where

$$\left.\begin{aligned} A &= \frac{\pi\omega^3}{8}\left[\pi^2k_{14}r^2 + (1-\nu)\,k_{15}\right] \\ B &= \frac{\pi\omega^3}{8}\left[3\pi^2k_{14}r^2 - (1-\nu)\,k_{15}\right], \quad r = a/L \end{aligned}\right\} \tag{63}$$

Euler's equation for the integral ϒ is

$$u_3'' - 9\,k^2u_3 = \frac{A}{L}\sin\frac{\pi x}{L} - \frac{B}{L}\sin\frac{3\pi x}{L}$$

The general odd solution of this equation is

$$u_3 = CL \sinh 3kx - \frac{AL}{\delta} \sin \frac{\pi x}{L} + \frac{BL}{9\alpha} \sin \frac{3\pi x}{L} \qquad (64)$$

where C is a constant of integration, and

$$\alpha = \pi^2 + k^2L^2, \qquad \delta = \pi^2 + 9k^2L^2 \qquad (65)$$

Substituting Eq. (64) into Eq. (62), we obtain

$$\Upsilon = \frac{3}{2} C^2kL^2 \sinh 3kL - \frac{A^2L}{4\delta} - \frac{B^2L}{36\alpha}$$

To minimize Υ, we set $C = 0$. Then,

$$\Upsilon = -\frac{A^2L}{4\delta} - \frac{B^2L}{36\alpha} \qquad (66)$$

The following notations are introduced:

$$\left. \begin{aligned} K_{14} &= \frac{\pi^3 k_{14}}{8\,(n - 1/n)^3} = \frac{\pi^3 n}{16\,(n^2 - 1)\,(4n^2 - 1)} \\ K_{15} &= \frac{\pi\,(1 - \nu)\,k_{15}}{8\,(n - 1/n)^3} = \frac{3\pi\,(1 - \nu)\,n}{32\,(4n^2 - 1)} \end{aligned} \right\} \qquad (67)$$

Then, by Eq. (66),

$$\frac{\pi}{1 - \nu^2} \frac{\Upsilon}{L} = -W_0^6 g(\xi) \qquad (68)$$

where

$$g(\xi) = \frac{\pi}{4\,(1 - \nu^2)} \left[\frac{(K_{14}r^2 + K_{15})^2}{\delta} + \frac{(3K_{14}r^2 - K_{15})^2}{9\alpha} \right] \qquad (69)$$

$$\xi = \frac{kL}{2} = \frac{n}{2r} \sqrt{\frac{1 - \nu}{2}}$$

11. Further Simplification of ΔV

In Sections 5 to 10, we have developed expressions for the various components of the increment of the total potential energy due to buckling. In this article, we proceed to further simplify the expression for ΔV (Eq. 38).

Set

$$p = KE\frac{h}{a}, \quad r = a/L, \quad \xi = \frac{n}{2r}\sqrt{\frac{1 - \nu}{2}} \qquad (70)$$

Then Eqs. (38), (49), (60), and (68) yield

$$\frac{\Delta V}{EahL} = \frac{U_b}{EahL} + (b_1 - Ka_1)\, W_0^2 + (b_2 + Ka_2)\, W_0^4 + b_3 W_0^6 \qquad (71)$$

in which

$$\left. \begin{aligned} a_1 &= K_4\lambda r^2 + K_{16} \\ a_2 &= -K_5\lambda r^2 + K_{17} \\ b_1 &= f_1 + K_6 r^2 \\ b_2 &= f_2 - (K_7 - f_3)\, r^2 + K_1 r^4 - \phi_1 \\ b_3 &= f_4 + (K_8 - f_5)\, r^2 + (K_3 + f_6)\, r^4 - \phi_2 - g \end{aligned} \right\} \qquad (72)$$

An eighth degree term in W_0 has not been included in Eq. (71), since numerical calculations show that it is very small. The constants (a_1, a_2) come from the potential energy of the external forces, and the constants (b_1, b_2, b_3) come from the membrane energy.

The constants $K_1, K_2, \ldots$ have been tabulated for $\nu = 0.30$. (Table 1). With these data, the constants a_1, a_2, b_1, b_2, b_3 have been tabulated for $\lambda = \pi$. (Tables 2 to 20). It is to be noted that λ affects only a_1 and a_2; it does not enter into b_1, b_2, b_3. In the construction of the tables, it was convenient to treat ξ as an independent variable. The corresponding values of L/a have been included in the tables.

12. Strain Energy of Bending of an Elastic Cylinder

In Eq. (71), ΔV is expressed in terms of the single parameter W_0 and the strain energy U_b due to bending. By the theory of this article, U_b is also expressed in terms of W_0. Also, the strain energy of reinforcing rings is expressed in terms of W_0. Thus, the increment of the total potential energy is reduced to a function of the single parameter W_0.

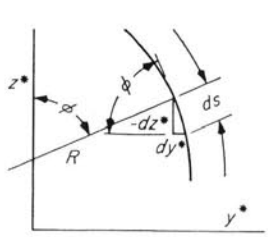


Fig. 3. Arc of Buckled Cylindrical Shell

For an elastic cylindrical shell, the strain energy of bending, per unit area of the middle surface is [18]

$$\frac{1}{2} D \left[\kappa_x^2 + \kappa_\theta^2 + 2\nu\kappa_x\kappa_\theta + 2\,(1 - \nu)\,\tau^2\right] \qquad (73)$$

where κ_x is the change of curvature in the longitudinal direction, κ_θ is the change of curvature in the circumferential direction, and τ is the local twist. The flexural rigidity D is defined by

$$D = \frac{Eh^3}{12\,(1 - \nu^2)} \qquad (74)$$

To express κ_θ in terms of the displacement components (v,w) of the middle surface, we refer to Fig. 3. The arc in the figure represents a part of the cross section of the middle surface of a buckled cylindrical shell. Since, by assumption, there is no incremental hoop strain due to buckling, $ds^* = ds = a d\theta$. Also, by Fig. 3, $R d\phi = ds$, where

R is the radius of curvature of the arc. Furthermore, we see by Fig. 3,

$$\sin\phi = -dz^*/ds, \qquad \cos\phi = dy^*/ds \tag{75}$$

Differentiating these equations, we obtain

$$\frac{d\phi}{ds}\cos\phi = -\frac{d^2z^*}{ds^2}, \qquad \frac{d\phi}{ds}\sin\phi = -\frac{d^2y^*}{ds^2} \tag{76}$$

By geometry, $1/R = d\phi/ds$. Consequently,

$$\frac{1}{R} = \frac{-d^2z^*/ds^2}{\cos\phi} = \frac{-d^2y^*/ds^2}{\sin\phi} \tag{77}$$

Introducing the parameter θ we obtain by Eq. (77)

$$\frac{1}{R} = -\frac{1}{a}\frac{z_{\theta\theta}^*}{y_\theta^*} = \frac{1}{a}\frac{y_{\theta\theta}^*}{z_\theta^*}$$

Since $(y_\theta^*)^2 + (z_\theta^*)^2 = a^2$, this equation yields

$$\frac{1}{R} = \frac{z_\theta^* y_{\theta\theta}^* - y_\theta^* z_{\theta\theta}^*}{a^3} \tag{78}$$

By Eq. (1),

$$\left.\begin{aligned}
y_\theta^* &= a\cos\theta + v_\theta\cos\theta - v\sin\theta \\
&\quad + w_\theta\sin\theta + w\cos\theta \\
z_\theta^* &= -a\sin\theta - v_\theta\sin\theta - v\cos\theta \\
&\quad + w_\theta\cos\theta - w\sin\theta \\
y_{\theta\theta}^* &= -a\sin\theta - 2v_\theta\sin\theta + v_{\theta\theta}\cos\theta - v\cos\theta \\
&\quad + 2w_\theta\cos\theta + w_{\theta\theta}\sin\theta - w\sin\theta \\
z_{\theta\theta}^* &= -a\cos\theta - 2v_\theta\cos\theta - v_{\theta\theta}\sin\theta + v\sin\theta \\
&\quad - 2w_\theta\sin\theta + w_{\theta\theta}\cos\theta - w\cos\theta
\end{aligned}\right\} \tag{79}$$

Eqs. (78) and (79) yield

$$\begin{aligned}
\frac{1}{R} = \frac{1}{a^3}\Big\{&[a + (v_\theta + w)]\,[a + 2(v_\theta + w) \\
&- (w_{\theta\theta} + w)] - (v - w_\theta)\,[(v_{\theta\theta} + w_\theta) \\
&- (v - w_\theta)]\Big\}
\end{aligned} \tag{80}$$

By Eqs. (12) and (16),

$$\left.\begin{aligned}
v_\theta + w &= w_0 - w_0\cos 2n\theta \\
v - w_\theta &= (n - 1/n)\, w_1 \sin n\theta \\
w_{\theta\theta} + w &= w_0 - (n^2 - 1)\, w_1\cos n\theta \\
&\quad - w_0\cos 2n\theta \\
v_{\theta\theta} + w_\theta &= 2nw_0\sin 2n\theta
\end{aligned}\right\} \tag{81}$$

Since $\kappa_\theta = \dfrac{1}{R} - \dfrac{1}{a}$, Eqs. (80) and (81) yield

$$\begin{aligned}
\kappa_\theta = \frac{1}{a^3}\Big\{&\Big[2aw_0 + \frac{3}{2}w_0^2 + \frac{1}{2}(n - 1/n)^2 w_1^2\Big] \\
&+ n(n - 1/n)\Big(a - \frac{1}{2}w_0\Big) w_1\cos n\theta \\
&- \Big[2(a + w_0)\, w_0 + \frac{1}{2}(n - 1/n)^2 w_1^2\Big] \\
&\cos 2n\theta + \frac{1}{2}n(n - 1/n)\, w_0 w_1\cos 3n\theta \\
&+ \frac{1}{2}w_0^2\cos 4n\theta\Big\}
\end{aligned}$$

With Eq. (16), this yields

$$\begin{aligned}
a\,\kappa_\theta = nW\cos n\theta &+ \frac{1}{2}nW^3\sin^2 n\theta\cos n\theta \\
&+ \frac{1}{4}W^4\sin^4 n\theta
\end{aligned} \tag{82}$$

where $W = (n - 1/n)\, w_1/a$

Eq. (82) yields

$$\begin{aligned}
\frac{1}{2}Da\int_0^{2\pi}\kappa_\theta^2 d\theta = \frac{\pi Eh^3}{24(1-\nu^2)\,a}\Big[n^2W^2 + \frac{1}{4}n^2W^4 \\
+ \frac{1}{32}n^2W^6 + \frac{35}{1024}W^8\Big]
\end{aligned} \tag{83}$$

Subsequent calculations show that rarely, if ever, does W exceed the value $\frac{1}{2}$. Consequently, the eighth degree term in W is quite negligible, and it will not be included in the subsequent analysis.

Eq. (36) may be written

$$W = W_0\cos\frac{\pi x}{L} \tag{84}$$

Eqs. (83) and (84) yield

$$\begin{aligned}
\frac{1}{2}Da\int_{-L/2}^{L/2}dx\int_0^{2\pi}\kappa_\theta^2 d\theta = \frac{\pi En^2h^3L}{48a(1-\nu^2)}\Big[W_0^2 \\
+ \frac{3}{16}W_0^4 + \frac{5}{256}W_0^6\Big]
\end{aligned} \tag{85}$$

Eq. (85) represents the principal part of the strain energy of bending. Since the longitudinal curvature is a minor effect, we shall use the approximation,

$$\kappa_x = -w_{xx} \tag{86}$$

Then, by Eq. (11),

$$\kappa_x = -(w_0'' + w_1''\cos n\theta + w_2''\cos 2n\theta) \tag{87}$$

Consequently,

$$\frac{1}{2}Da\int_0^{2\pi}\kappa_x^2 d\theta = \frac{\pi Eh^3a}{24(1-\nu^2)}(2w''^2_0 + w''^2_1 + w''^2_2)$$

With Eq. (16), this yields

$$\begin{aligned}
\frac{1}{2}Da\int_0^{2\pi}\kappa_x^2 d\theta = \frac{\pi Eh^3a^3}{24(1-\nu^2)}\Big\{\frac{W''^2}{(n-1/n)^2} \\
+ \frac{1}{4}\Big[2 + \frac{1}{(4n^2-1)^2}\Big] \\
[W'^4 + 2WW'^2W'' + W^2W''^2]\Big\}
\end{aligned} \tag{88}$$

Eqs. (84) and (88) yield

$$\begin{aligned}
\frac{1}{2}Da\int_{-L/2}^{L/2}dx\int_0^{2\pi}\kappa_x^2 d\theta = \frac{\pi^5Eh^3a^3}{48(1-\nu^2)L^3}\Big\{\frac{W_0^2}{(n-1/n)^2} \\
+ \frac{1}{8}\Big[2
\end{aligned}$$

$$+ \frac{1}{(4n^2-1)^2}\Big] W_0^4 \Big\} \quad (89)$$

Eqs. (84) and (87) yield

$$\int_0^{2\pi} 2\nu\kappa_x\kappa_\theta d\theta = -\frac{2\pi\nu}{a}\Big[\frac{3}{16} w''_0 W^4 + w''_1\Big(nW + \frac{1}{8} nW^3\Big) - \frac{1}{8} w''_2 W^4\Big]$$

With Eq. (16), this yields

$$\int_0^{2\pi} 2\nu\kappa_x\kappa_\theta d\theta = -2\pi\nu\Big[\frac{3}{32}(W'^2 + WW'')W^4 + \frac{W''}{n-1/n}\Big(nW + \frac{1}{8} nW^3\Big) + \frac{W^4}{16(4n^2-1)}(W'^2 + WW'')\Big] \quad (90)$$

Eqs. (84) and (90) yield

$$\frac{1}{2} Da\int_{-L/2}^{L/2} dx\int_0^{2\pi} 2\nu\kappa_x\kappa_\theta d\theta = \frac{\pi^3\nu Eh^3 a}{24(1-\nu^2)L}\Big\{\frac{n^2}{n^2-1} W_0^2 + \frac{3n^2}{32(n^2-1)} W_0^4 - \frac{1}{64}\Big[3 - \frac{2}{4n^2-1}\Big] W_0^6\Big\} \quad (91)$$

The twist τ is defined by $\tau = \partial\phi/\partial x$ (Fig. 3). Since $\sin\phi = -\partial z^*/\partial s$ and $\cos\phi = \partial y^*/\partial s$, (see Eq. 75),

$$\tau\cos\phi = -\frac{\partial^2 z^*}{\partial x\partial s} = -\frac{1}{a} z_{x\theta}^*,$$

$$\tau\sin\phi = -\frac{\partial^2 y^*}{\partial x\partial s} = -\frac{1}{a} y_{x\theta}^*$$

Hence,

$$\tau\cos^2\phi = -\frac{1}{a^2} y_\theta^* z_{x\theta}^*, \quad \tau\sin^2\phi = \frac{1}{a^2} z_\theta^* y_{x\theta}^*$$

Adding these equations, we get

$$a^2\tau = z_\theta^* y_{x\theta}^* - y_\theta^* z_{x\theta}^* \quad (92)$$

Eqs. (1) and (92) yield

$$a^2\tau = a(v' - w_\theta') + (v_\theta + w)(v' - w_\theta') - (v_\theta' + w')(v - w_\theta) \quad (93)$$

Consequently, by Eqs. (16) and (82),

$$a\tau = \Big(aW' - \frac{3}{2} Ww'_0 + \frac{3}{2} W'w_0\Big)\sin n\theta + \frac{1}{2}(Ww'_0 - w_0W')\sin 3n\theta \quad (94)$$

Hence,

$$2a(1-\nu)\int_0^{2\pi}\tau^2 d\theta = \frac{2\pi(1-\nu)}{a}\Big[\Big(aW' - \frac{3}{2} Ww'_0 + \frac{3}{2} W'w_0\Big)^2 + \frac{1}{4}(Ww'_0 - w_0W')^2\Big] \quad (95)$$

By Eq. (16), $w_0 = -\frac{1}{4} aW^2$. Consequently,

$$2a(1-\nu)\int_0^{2\pi}\tau^2 d\theta = 2\pi(1-\nu)a\Big(W'^2 + \frac{3}{4} W^2W'^2 + \frac{5}{32} W^4W'^2\Big)$$

Hence, by Eq. (84),

$$2a(1-\nu)\int_{-L/2}^{L/2} dx\int_0^{2\pi}\tau^2 d\theta = \pi^3(1-\nu)\frac{a}{L}\Big[W_0^2 + \frac{3}{32} W_0^4 + \frac{5}{256} W_0^6\Big] \quad (96)$$

By Eqs. (73), (85), (89), (91), and (96), the total strain energy of bending of an elastic cylindrical shell, excluding the strain energy of reinforcing rings, is determined by

$$\frac{U_b}{EahL} = \frac{h^2}{a^2} W_0^2(c_1 + c_2W_0^2 + c_3W_0^4) \quad (97)$$

where

$$r = a/L$$

$$\left.\begin{aligned} c_1 &= \frac{\pi}{48(1-\nu^2)}\Big[n^2 + 2\pi^2r^2\Big(1 + \frac{\nu}{n^2-1}\Big) + \frac{\pi^4n^2r^4}{(n^2-1)^2}\Big] \\ c_2 &= \frac{\pi}{256(1-\nu^2)}\Big[n^2 + \pi^2r^2\Big(1 + \frac{\nu}{n^2-1}\Big) + \frac{2}{3}\pi^4r^4\Big(2 + \frac{1}{(4n^2-1)^2}\Big)\Big] \\ c_3 &= \frac{\pi}{12288(1-\nu^2)}\Big[5n^2 + 2\pi^2r^2\Big(5 - 17\nu + \frac{8\nu}{4n^2-1}\Big)\Big] \end{aligned}\right\} \quad (98)$$

The constants c_1, c_2, c_3 have been tabulated (Tables 2 to 20).

The strain energy of a reinforcing ring is

$$U_r = \frac{1}{2} a\int_0^{2\pi} EI\kappa_\theta^2 d\theta$$

in which I is the moment of inertia of the cross section of the ring. Consequently, if I is constant for a ring, Eq. (83) yields

$$U_r = \frac{\pi EI}{2a}\Big[n^2W^2 + \frac{1}{4} n^2W^4 + \frac{1}{32} n^2W^6\Big] \quad (99)$$

The eighth degree term in W has been discarded from Eq. (83).

Eq. (84) determines the value of W for any ring. The strain energy of all reinforcing rings is represented by $\sum U_r$, where the sum extends over all rings.

It is assumed in the derivation of Eq. (99) that the centroidal axis of the ring coincides with the middle surface of the shell. Actually, this condition rarely occurs in practice. Possibly Eq. (99) can be retained for off-center rings if EI is modified to take account of an effective width of shell that acts with the ring. However, this problem is not examined in the present investigation.

Eqs. (71) and (97) yield

$$\frac{\Delta V}{EahL} = (B_1 - Ka_1)\, W_0^2 + (B_2 + Ka_2)\, W_0^4 + B_3 W_0^6 + \frac{\Sigma U_r}{EahL} \qquad (100)$$

in which

$$B_1 = b_1 + c_1 \frac{h^2}{a^2},\ B_2 = b_2 + c_2 \frac{h^2}{a^2},$$

$$B_3 = b_3 + c_3 \frac{h^2}{a^2} \qquad (101)$$

The constants a_1, a_2, b_1, b_2, b_3, c_1, c_2, c_3 have been tabulated (Tables 2 to 20). The last term in Eq. (100) representing the effect of reinforcing rings, is a sixth degree polynomial in W_0, in which only even powers occur (see Eq. 99).

III. POTENTIAL ENERGY OF A SHELL WITH RIGID ENDS

13. Shell with Rigid Ends

In the preceding analysis, it has been assumed that the ends of the shell are free to warp out of their planes; that is, that the ends of the shell impose no restrictions on the axial displacement u. If the ends of the shell are rigid plates, the Fourier coefficients u_1, u_2, u_3 (See Eq. 11) vanish at the ends. Accordingly, the functions Ψ, X, and Υ must be modified. Numerical computations show that the effect of u_3 is quite small, and consequently the function Υ will be discarded.

Eqs. (40), (41), (42), (43), (44), and (45) remain valid. However, the constant C must be chosen so that u_1 vanishes for $x = L/2$. Consequently,

$$C = \frac{A}{\alpha} - \frac{B}{\beta} \operatorname{csch} \frac{kL}{2} \tag{102}$$

Eqs. (45) and (102) yield

$$\Psi = -\frac{LA^2}{4\alpha} - \frac{LB^2}{4\beta} + 2\left(\frac{A}{\alpha} - \frac{B}{\beta}\right)^2 L\xi \coth \xi \tag{103}$$

in which ξ is defined by Eq. (48). Consequently,

$$\frac{\Psi}{L} = A^2F_1(\xi) - 2ABF_2(\xi) + B^2F_3(\xi) \tag{104}$$

where

$$\left.\begin{aligned} F_1(\xi) &= \frac{2\xi \coth \xi - \left(\xi^2 + \frac{1}{4}\pi^2\right)}{(\pi^2 + 4\xi^2)^2} \\ F_2(\xi) &\doteq \frac{2\xi \coth \xi}{(\pi^2 + 4\xi^2)(9\pi^2 + 4\xi^2)} \\ F_3(\xi) &= \frac{2\xi \coth \xi - (\xi^2 + 9\pi^2/4)}{(9\pi^2 + 4\xi^2)^2} \end{aligned}\right\} \tag{105}$$

Hence,

$$\frac{\pi}{1-\nu^2}\frac{\Psi}{L} = W_0^2\psi_1(\xi) + [\psi_2(\xi) + r^2\psi_3(\xi)]\, W_0^4 + [\psi_4(\xi) - r^2\psi_5(\xi) + r^4\psi_6(\xi)]\, W_0^6 \tag{106}$$

where

$$\left.\begin{aligned} \psi_1(\xi) &= \frac{\pi K_{11}^2 F_1(\xi)}{1-\nu^2} \\ \psi_2(\xi) &= \frac{2\pi K_{10}K_{11}(F_2 - F_1)}{1-\nu^2} \\ \psi_3(\xi) &= \frac{2\pi K_9K_{11}(F_1 + 3F_2)}{1-\nu^2} \\ \psi_4(\xi) &= \frac{\pi K_{10}^2(F_1 - 2F_2 + F_3)}{1-\nu^2} \\ \psi_5(\xi) &= \frac{2\pi K_9K_{10}(F_1 + 2F_2 - 3F_3)}{1-\nu^2} \\ \psi_6(\xi) &= \frac{\pi K_9^2(F_1 + 6F_2 + 9F_3)}{1-\nu^2} \end{aligned}\right\} \tag{107}$$

It may be shown that the expression on the right side of Eq. (104) is a negative definite quadratic form in A and B. Consequently, Ψ always reduces the membrane energy. For a shell with rigid ends, the functions $\psi_1, \ldots \psi_6$ replace the functions $f_1, \ldots f_6$ of Eqs. (50) and (72).

Turning attention to the function X, we observe that Eqs. (54) and (56) remain valid. The end condition, $u_2 = 0$, obviously requires that $C = 0$. Consequently, Eq. (56) yields

$$\mathrm{X} = -\frac{A^2L}{16\alpha} - \frac{B^2L}{16\gamma} \tag{108}$$

Hence,

$$\frac{\pi}{1-\nu^2}\frac{\mathrm{X}}{L} = -W_0^4\mathrm{X}_1(\xi) \tag{109}$$

where

$$\mathrm{X}_1(\xi) = \frac{\pi(K_{12} + K_{13}r^2)^2}{16(1-\nu^2)(\pi^2 + 4\xi^2)^2} \tag{110}$$

Accordingly, the function X_1 replaces the function ϕ_1 of Eqs. (60) and (72). The functions ϕ_2 and g are discarded from Eq. (72). The second term in Eq. (108) has been neglected, since B^3 is of eighth degree W_0.

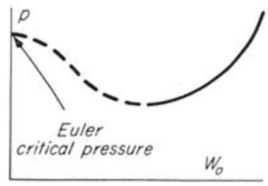
With the above modifications, the preceding theory applies for a shell whose ends are rigid plates.

IV. PRESSURE-DEFLECTION RELATIONS

14. Load-Deflection Curves

For a given shell and a given value of the pressure p, we may plot a graph of $\Delta V/(EahL)$ versus W_0 by means of Eq. (100). The forms of the graphs, corresponding to several values of p, are illustrated by Fig. 4. The pressures indicated on the curves are such that $p_1 < p_2 < p_3 < p_4$. The minima on the curves represent configurations of stable equilibrium, and the maxima represent configurations of unstable equilibrium. If $p < p_4$, the unbuckled state is stable, since the configuration $W_0 = 0$ then provides a relative minimum to the potential energy. However, if $p \geqq p_4$, the unbuckled state becomes a configuration of maximum potential energy; hence, it is unstable. Accordingly, p_4 is the Euler critical pressure.

We may pick the maximum and minimum points from the curves of Fig. 4, and thus plot p versus W_0. The resulting curve, illustrated by Fig. 5, represents all equilibrium configurations. Fig. 5 is effectively a load-deflection curve for the buckled cylinder, since W_0 is roughly proportional to the incremental deflection at the center of a lobe due to buckling. The intercept of the curve with the p-axis is the Euler critical pressure. The falling part of the curve (dotted in Fig. 5) represents unstable configurations, since the points on this part of the curve correspond to maxima on the curves of Fig. 4. The rising part of the curve represents stable configurations, as the points on this part of the curve correspond to minima on the curves of Fig. 4. The minimum ordinate on the curve of Fig. 5 is the lowest pressure for which the shell will not snap back if it is forced into the buckled form. It is equal to p_1, if the curve corresponding to p_1 in Fig. 4 is considered to have an inflection point with a horizontal tangent.



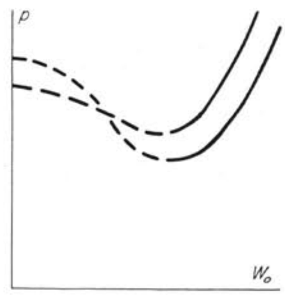


Fig. 5 (above). Pressure-Deflection Curve for Ideal Shell

Fig. 6 (right). Intersecting Pressure-Deflection Curves

We may plot Fig. 5 directly by means of Eq. (100). The points on Fig. 5 are solutions of the equation $dy/dx = 0$, where, for brevity, $y = \Delta V/(EahL)$ and $x = W_0^2$. If there are no reinforcing rings, this condition yields

$$K = \frac{B_1 + 2B_2W_0^2 + 3B_3W_0^4}{a_1 - 2a_2W_0^2}, \quad p = KEh/a \qquad (111)$$

In view of Eq. (99), the effect of reinforcing rings is merely to modify the coefficients b_1, b_2, b_3. Consequently, the form of Eq. (111) remains valid for a cylinder that is reinforced by elastic rings. Fig. 5 is a graph of Eq. (111). It is irrelevant whether the ordinate is p or K.

A curve of the type illustrated in Fig. 5 corresponds to each value of the integer n. It is necessary to choose n by trial to provide the minimum buckling pressure. In some cases, the curves corresponding to two consecutive values of n intersect, as illustrated by Fig. 6. Then the number of lobes in the final buckled form may possibly be different from the number of lobes in the infinitesimal pattern that precipitates buckling. However, since the

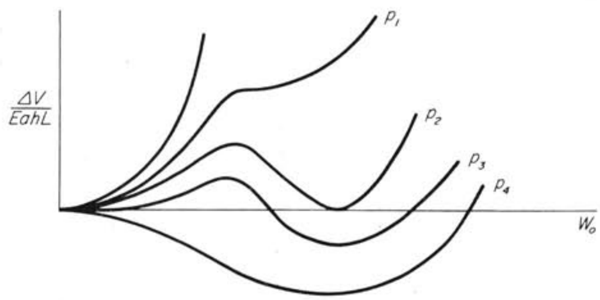


Fig. 4. Increment of Potential Energy versus Deflection Parameter

collapse of an ideal shell is a sudden phenomenon that carries the shell over the unstable region onto the rising part of an equilibrium curve (Fig. 6), dynamical processes undoubtedly play an important part in determining the final pattern.

15. Tsien Critical Pressure

The curve corresponding to p_1 (Fig. 4) is considered to have an inflection point with a horizontal tangent. Then the curve corresponding to any value of p in the range $p_1 < p < p_4$ possesses two relative minima, one representing the unbuckled state, and the other representing a buckled form. Consequently, if $p > p_1$, the shell will maintain a buckled form if it is initially forced into that condition by external disturbances. Since initial disturbances and imperfections always exist, von Kármán and Tsien[15] originally conjectured that p_1 is the maximum safe pressure.

Tsien[19] later concluded that, although p_1 is the greatest lower bound for the pressures at which buckled configurations can persist, there is little danger of a shell passing into a buckled configuration unless, in doing so, it loses potential energy. In Tsien's words, "The most probable equilibrium state is the state with the lowest potential energy. — This principle of lowest energy level is verified by comparing experimental data with theoretical predictions. However, in view of the prerequisite that arbitrary disturbances of finite magnitude have to exist, the buckling load determined by this principle may be called the 'lower buckling load.' The classic buckling load that assumes only the existence of disturbances of infinitesimal magnitude may be called the 'upper buckling load.' Of course, by extreme care in avoiding all disturbances during a test, the upper buckling load can be approached. The lower buckling load, however, has to be used as a correct basis for design."

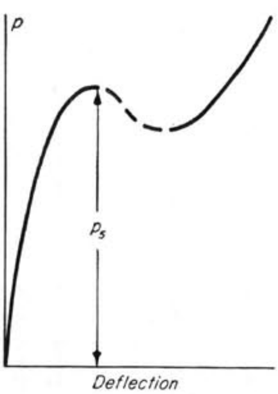


Fig. 7. Pressure-Deflection Curve for Imperfect Shell

According to Tsien's reasoning, the pressure p_2 (Fig. 4) is the maximum safe pressure. This is the pressure at which the potential energy of the unbuckled form ceases to be an *absolute* minimum.

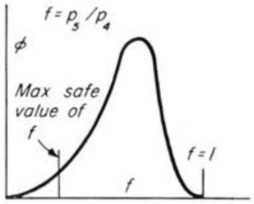


Fig. 8. Statistical Distribution Curve for Imperfect Shell

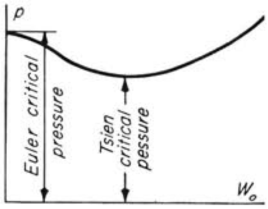


Fig. 9. Determination of Tsien Critical Pressure

It will be designated as the "Tsien critical pressure." The curve in Fig. 4 that corresponds to p_2 is tangent to the axis of W_0 at a point to the right of the origin.

The buckling pressure of an imperfect shell poses a statistical problem. Load-deflection curves for imperfect shells have the general form shown by Fig. 7. This figure is to be contrasted with the load-deflection curve for an ideal shell (Fig. 5). Donnell has emphasized that the designer is concerned principally with the maximum pressure on the load-deflection curve (denoted by p_5 on Fig. 7). Since the falling part of the curve (dotted in Fig. 7) represents unstable equilibrium configurations, the maximum point lies at the boundary of the stable range. Therefore, p_5 is the Euler critical pressure for the imperfect shell. This pressure may be expressed conveniently as a fraction f of the Euler critical pressure p_4 for a perfect shell; that is, $f = p_5/p_4$. Since p_5 depends on initial imperfections in the shell, tests of a large number of shells with the same dimensions would lead to a statistical distribution curve of the general form shown in Fig. 8. The ordinate ϕ of this curve is defined by the condition that ϕdf is the probability that a random shell will fall in the interval $(f, f + df)$.

The specification of a safe pressure is somewhat arbitrary. Under some circumstances, an operating pressure would be considered safe if 95% of all specimens would fail above that pressure. In other cases, the safety limit might be raised to 99%, or some other value. Tsien implied that his definition of the lower critical pressure provides a value that lies near the maximum safe pressure (Fig. 8). At present, this conclusion is largely conjectural, but since Tsien's critical pressure affords a ready empirical criterion for safe design of shells, it has been charted by Euler critical pressure (Fig. 13).

The Tsien critical pressure is determined by the equation $\Delta V = 0$. If we select the intercepts of the curves of Fig. 4 with the w_0-axis, we can plot the resulting relation between p and W_0. The graph has the general form shown in Fig. 9. The minimum ordinate of the curve is the Tsien critical pressure, and the corresponding value of W_0 determines the deformation of the buckled shell, if the applied pressure equals the Tsien critical pressure. The intercept of the curve with the p-axis is the Euler critical pressure. Although Fig. 9 looks like Fig. 5, the two curves are distinct, since they are derived by different formulas. Fig. 9 is not a graph of equilibrium configurations; it merely serves to show how the Tsien critical pressure may be computed.

If there are no reinforcing rings, and if the shell is elastic, the equation $\triangle V = 0$ yields

$$K = \frac{B_1 + B_2 W_0^2 + B_3 W_0^4}{a_1 - a_2 W_0^2} \tag{112}$$

The form of Eq. (112) remains valid for a cylinder that is reinforced by elastic rings, if the coefficients b_1, b_2, b_3 are modified suitably (See Eq. 99).

Plotting K versus W_0 by means of Eq. (112), we obtain a curve that is essentially equivalent to Fig. 9, although the ordinate is K rather than p. The intercept of the curve with the K-axis is determined by setting $W_0 = 0$ in Eq. (112). Consequently, the value of K corresponding to the Euler critical pressure is

$$K_i = B_1/a_1 \tag{113}$$

The notation K_i denotes the value of K that is obtained by the infinitesimal theory of buckling.

The value of K that corresponds to the Tsien critical pressure, denoted by K_{st} (the subscripts "st" denote snap-through) is the minimum value of the function defined by Eq. (112). To minimize K, W_0^2 must be a root of the equation,

$$W_0^2 = -\frac{1}{2}\left[\frac{B_2}{B_3} + \frac{a_2 B_1}{a_1 B_3}\right] + \frac{a_2 W_0^4}{2a_1} \tag{114}$$

Although Eq. (114) may be solved by the quadratic formula, it is usually poorly conditioned for this type of solution, and it is most easily solved by iteration. The procedure is to obtain an approximation of W_0^2 by neglecting the fourth degree term on the right side of Eq. (112), and to use this approximation to refine the first approximation. The process of refinement may be iterated, and it converges quite rapidly. The value of W_0^2, determined by Eq. (114), must be substituted into Eq. (112). Thus, the Tsien critical pressure is determined. It is represented by $p = K_{st}Eh/a$, where the subscripts "st" denote "snap-through." The

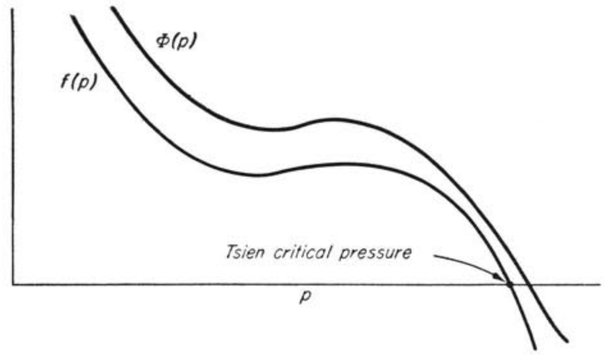


Fig. 10. Graphs of f(p) and φ(p)

buckling coefficient K_{st} is plotted versus W_0 in Fig. 13 for values of a/h of 100 and 1000 and $\lambda = \pi$.

16. Effect of Assumptions on the Tsien Critical Pressure

It is well known that arbitrary assumptions about the deformation of a structure cause the computed value of the Euler critical load to be too high. The same conclusion applies for the Tsien critical pressure. To verify this assertion, we observe that the potential energy V is a functional of the displacement components (u, v, w) and the pressure p. Let Class I be the set of all continuous differentiable functions (u, v, w) that satisfy the forced boundary conditions.

It has been found that there exists a pressure p' for a given shell, such that a buckled configuration will persist if $p > p'$. The buckled configuration, being stable, provides a relative minimum to V among functions of Class I. This *minimum* of V depends only on p; hence, it will be denoted by $f(p)$. (See Fig. 4.)

It has been found that there exists a pressure p'' (the Tsien critical pressure), such that $f(p) > 0$ in the range $p' < p < p''$, $f(p'') = 0$, and $f(p) < 0$ in the range $p > p''$.

Let Class II be a given subset of Class I, as determined, for example, by assumptions about the nature of the deformation pattern. We have employed two assumptions of this type: (1) the shell buckles without incremental hoop strain. (2) The function w_1 is represented by a single term of a Fourier series in x (see Eq. 36). When the functions (u, v, w) are restricted to Class II, the minimum value of V is $\phi(p)$. If our assumptions are good, $\phi(p)$ differs but slightly from $f(p)$.

Since Class II is a subset of Class I, $\phi(p) \geqq f(p)$. Consequently, the graphs of $f(p)$ and $\phi(p)$ have the general features shown by Fig. 10. The essential characteristics of these functions are that ϕ and f are positive for small values of p and negative for large values of p, and that the curve representing $\phi(p)$ lies above or in contact with the curve representing $f(p)$.

The Tsien critical pressure is the intercept of the graph of $f(p)$ with the p-axis (Fig. 10). Evidently if $\phi(p)$ is used as an approximation for $f(p)$, the computed value of the Tsien critical pressure is too high.

17. Potential Energy Barriers

The maximum on the curve corresponding to p_2 (Fig. 4) represents a potential energy barrier that the shell must cross to arrive at the buckled form, if the pressure is exactly equal to the Tsien critical pressure. Therefore, it serves as a rough indication of the imminence of snap-through. The value of this maximum may be derived from Eq. (100). For brevity, Eq. (100) is written as follows:

$$\left.\begin{aligned} &y = ax - bx^2 + cx^3 \\ &y = \frac{\Delta V}{EahL}, \quad x = W_0^2 \\ &a = B_1 - Ka_1, \quad b = -(B_2 + Ka_2), \quad c = B_3 \end{aligned}\right\} \quad (a)$$

The graph of y is as shown in Fig. 11.

The maximum value of y is determined by differentiation with respect to x. Thus,

$$a - 2bx + 3cx^2 = 0 \qquad (b)$$

The roots of Eq. (b) are

$$\left.\begin{aligned} x_1 &= \frac{b}{3c}\left[1 - \sqrt{1 - \frac{3ac}{b^2}}\right] \\ x_2 &= \frac{b}{3c}\left[1 + \sqrt{1 - \frac{3ac}{b^2}}\right] \end{aligned}\right\} \quad (c)$$

Fig. 11. Potential Energy Barrier

The root x_1 provides the maximum, and the root x_2 provides the minimum (Fig. 11). Since the value of the minimum is zero,

$$a - bx_2 + cx_2^2 = 0 \qquad (d)$$

Eqs. (c) and (d) yield

$$b = 2\sqrt{ac} \qquad (e)$$

Consequently, Eq. (c) yields

$$x_1 = \frac{1}{3}\sqrt{\frac{a}{c}}, \quad x_2 = \sqrt{\frac{a}{c}} \qquad (f)$$

Eqs. (a) and (f) yield

$$y_{\max} = \frac{4c}{27} x_2^3 \qquad (g)$$

In terms of our previous notations, this equation yields,

$$\left(\frac{\Delta V}{Ea_{\max}^3}\right) = \frac{4}{27}\frac{hL}{a^2} B_3 \overline{W}_0^6 \qquad (115)$$

where $\overline{W}_0$ is the root of Eq. (114) that corresponds to the point of tangency with the x-axis (Fig. 11). Eq. (115) is plotted in Fig. 13 for $a/h = 1000$, $\lambda = \pi$ and for $n = 2$ to 20.

18. Numerical Example

Consider a shell with the following proportions: $a/h = 100$, $L/a = 0.6010$. The value $L/a = 0.6010$ is selected to coincide with a tabulated value. This condition is unimportant. If a selected value of L/a does not appear in the tables, interpolation must be used.

(a). *Euler Critical Pressure for Shell with Simply-Supported Ends and No Axial End Constraint.*

The equilibrium pressure corresponding to any given state of deformation is represented in the following form: $p = KEh/a$. The constant K is evidently equal to the compression hoop strain that exists in the unbuckled shell at pressure p. The value of K corresponding to the Euler critical pressure is denoted by K_i. By Eq. (113), $K_i = B_1/a_1$, where $B_1 = b_1 + c_1h^2/a^2$. Accordingly, in this example, $B_1 = b_1 + 0.0001\ c_1$. The constants a_1, b_1, c_1 for a shell with simply supported ends have been tabulated (Tables 2 to 20). The number of lobes in the buckled form must be determined by trial to minimize K_i. For very long shells, $n = 2$. In general, n increases with decreasing L or h. In the present example, L/a is small, but h/a is large. Therefore, a moderate value of n — for example, $n = 10$ — might be estimated. It is found by several trials that the value $n = 9$ actually provides a minimum to K_i. For $n = 9$, Table 9 yields (with $\lambda = \pi$) $a_1 = 0.9327$, $b_1 = 0.0006329$, $c_1 = 10.45$. Consequently, $B_1 = 0.001678$. Accordingly, Eq. (113) yields $K_i = 0.001678/0.9327 = 0.001799$.

The condition $\lambda = \pi$ indicates that a uniform hydrostatic pressure acts on the ends of the shell. If $\lambda = 0$, the axial force due to the pressure on the ends is removed. Then $a_1 = 0.7952$, as noted at the bottom of Table 9. Since b_1 and c_1 are independent of λ, B_1 has the same value as before. Thus, if

$\lambda = 0$, then $K_i = B_1/a_1 = 0.001678/0.7952 = 0.002110$. Accordingly, in this example, the hydrostatic pressure on the ends reduces the Euler critical pressure about 15%.

For the case $\lambda = 0$, von Mises[1] derived a formula that may be put in the following form:

$$K_i = \frac{1}{(n^2-1)\left(1+\frac{n^2L^2}{\pi^2a^2}\right)^2} + \frac{h^2}{12\,(1-\nu^2)\,a^2} \cdot \left[n^2 - 1 + \frac{2n^2-1-\nu}{1+\frac{n^2L^2}{\pi^2a^2}}\right] \tag{116}$$

In the present numerical example, von Mises' formula yields $K_i = 0.001900$. This result is about 10% lower than that computed by the present theory. There is seemingly a systematic deviation between the present infinitesimal theory and von Mises' theory, for thick shells that are short compared to their radii. However, in all cases, the Tsien criterion yields lower values than the von Mises' theory.

(b) *Pressure-Deflection Curves for Elastic Shell with Simply-Supported Ends and No Axial End Constraint.*

The pressure-deflection curve is essentially a graph of K versus W_0, where K is defined as above. This curve may be plotted by means of Eq. (111). Setting $n = 9$, we obtain from Table 9 (with $\lambda = \pi$),

$a_1 = 0.9327, \quad a_2 = -1.270,$
$b_1 = 0.0006329, \quad b_2 = -0.02334, \quad b_3 = 0.3077,$
$c_1 = 10.45, \quad c_2 = 14.89, \quad c_3 = 0.1124.$

The b's and c's are independent of λ. The quantities B_1, B_2, B_3 are determined by $B_1 = b_1 + c_1h^2/a^2$, $B_2 = b_2 + c_2\,h^2/a^2$, $B_3 = b_3 + c_3\,h^2/a^2$. In the present example, $h^2/a^2 = 0.0001$. Hence,

$$B_1 = 0.001678,\ B_2 = -\,0.02185,\ B_3 = 0.3077.$$

Substituting these values of a_1, a_2, B_1, B_2, B_3 into Eq. (111), we obtain an equation whose graph is shown in Fig. 12. Since the curves corresponding to $n = 8$ and $n = 9$ intersect each other, the curve for $n = 8$ is also plotted.

The intercept of the curve for $n = 9$ with the vertical axis is the Euler critical hoop strain, $K_i = 0.001799$. The minimum value on the curve for $n = 9$ is $K_{min} = 0.001170$. The minimum value on the curve for $n = 8$ is $K_{min} = 0.001080$. This is the lowest minimum that occurs for any value of n. Therefore, it determines the lowest pressure at which the shell will maintain a buckled form, if it is perfectly elastic.

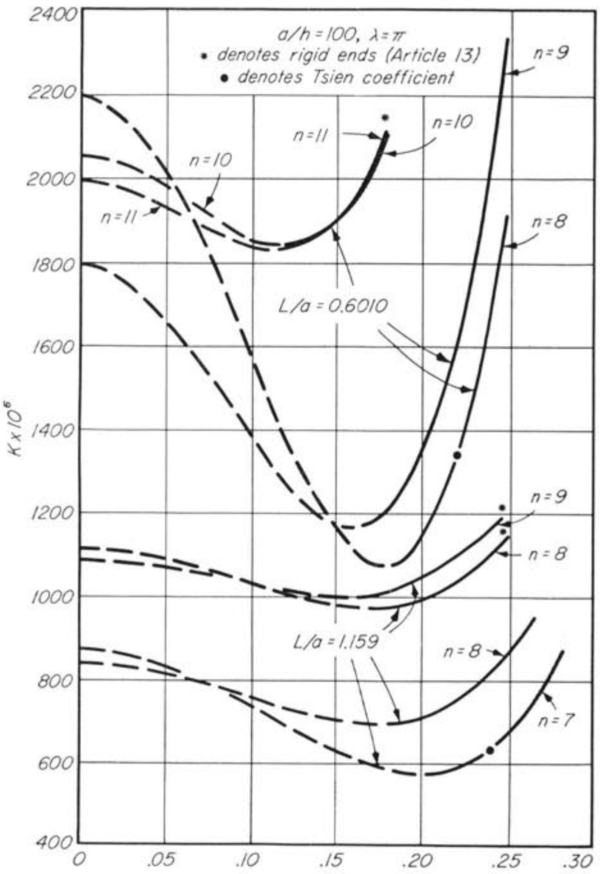


Fig. 12. Buckling Coefficient K versus Deflection Parameter W_0

Hence, in this example, the lowest pressure at which the elastic shell will maintain a buckled form is 60% of the Euler critical pressure.

The value of K corresponding to the Tsien critical pressure has been denoted by K_{st}. For this case, $K_{st} = 0.00133$. This result may be obtained from Eq. (112). The Tsien critical pressures have been marked on the curves of Fig. 15. The Euler critical pressure for this case (see above) is $K_i = 0.001799$ which is 35% higher than K_{st}.

(c) *Effect of Rigid Ends.*

If the ends of the shell are hinged, but the axial displacements are constrained by the action of rigid end plates, the buckling pressure is increased significantly. Eqs. (111) and (113) remain valid, but the coefficients b_1, b_2, b_3 are changed. The constants a_1, a_2, c_1, c_2, c_3 are not altered.

The constants b_1, b_2, b_3 have not been tabulated for a shell with rigid ends. Consequently, their values must be computed by means of Eqs. (105), (107, (110), and (72).

The constraint imposed by rigid end plates generally increases the number of lobes in the buckled form. Trying $n = 10$, we obtain by Eq. (48), $\xi = 1.778$. Hence, by Eq. (105),

$$F_1(\xi) = -0.003675, \quad F_2(\xi) = 0.001648,$$
$$F_3(\xi) = -0.002098.$$

Using the values of the K's from Table 1, we obtain by Eq. (107),

$$\psi_1 = -0.0001565, \quad \psi_2 = 0.005605,$$
$$\psi_3 = 0.000382, \quad \psi_4 = -0.05900,$$
$$\psi_5 = 0.02203, \quad \psi_6 = -0.006755.$$

Eq. (110) yields $X_1 = 0.002364$.

In Eq. (72), the functions $f_1, f_2, \ldots$ are to be replaced by $\psi_1, \psi_2, \ldots$, respectively. Also, X_1 replaces ϕ_1. The functions ϕ_2 and g are discarded since they are negligible.

Hence, $b_1 = 0.0006858$, $b_2 = -0.01210$, $b_3 = 0.2929$. Interpolating values from Table 10, we obtain

$$a_1 = 0.904,\ a_2 = -1.27,\ c_1 = 11.68,\ c_2 = 15.14,$$
$$c_3 = 0.139.$$

Since $B_1 = b_1 + c_1h^2/a^2$, $B_2 = b_2 + c_2h^2/a^2$, and $B_3 = b_3 + c_3h^2/a^2$, $B_1 = 0.00185$, $B_2 = -0.0106$, $B_3 = 0.293$. Eq. (111) now provides a graph of K versus W_0 (Fig. 12).

It is necessary to repeat the calculations for several other values of n. It is found by trial that the value $n = 11$ provides the lowest buckling pressure. The curves corresponding to $n = 10$ and $n = 11$ are plotted in Fig. 12. It is seen that these curves are significantly higher than the curves obtained for a shell without axial end constraints. Similar calculations have been performed for $L/a = 1.159$ and the results have been plotted in Fig. 12. All the curves for $L/a = 1.159$ are lower than the corresponding curves for $L/a = 0.6010$.

V. SUMMARY

A theory, based on an energy analysis, has been developed for the snap-through and post-buckling behavior of simply-supported ideal shells under the action of external pressure. The principal results of the theory are given: (a) by Eqs. (71), (72), (97), (100), (101), (111), (112), and (113) for elastic shells whose ends are free to warp out of their planes, and (b) by Eqs. (72), (105), (107), (110), (111), (112), and (113), and the modifications indicated in Article 13 for elastic shells whose ends are rigid plates.

The main results of the computations are presented in the form of tables and graphs. Tables 1 to 20 list the parameters needed for calculation of the buckling coefficient K given by $p_{cr} = KEh/a$. The use of Tables 1 to 20 is illustrated by a numerical example (Article 18). Table 21 gives values of K, for elastic shells whose ends are free to warp out of their planes, as determined by the infinitesimal theory and the Tsien snap-through theory for various values of L/a and λ, and for $a/h = 1000$. For no axial pressure ($\lambda = 0$), some values of K

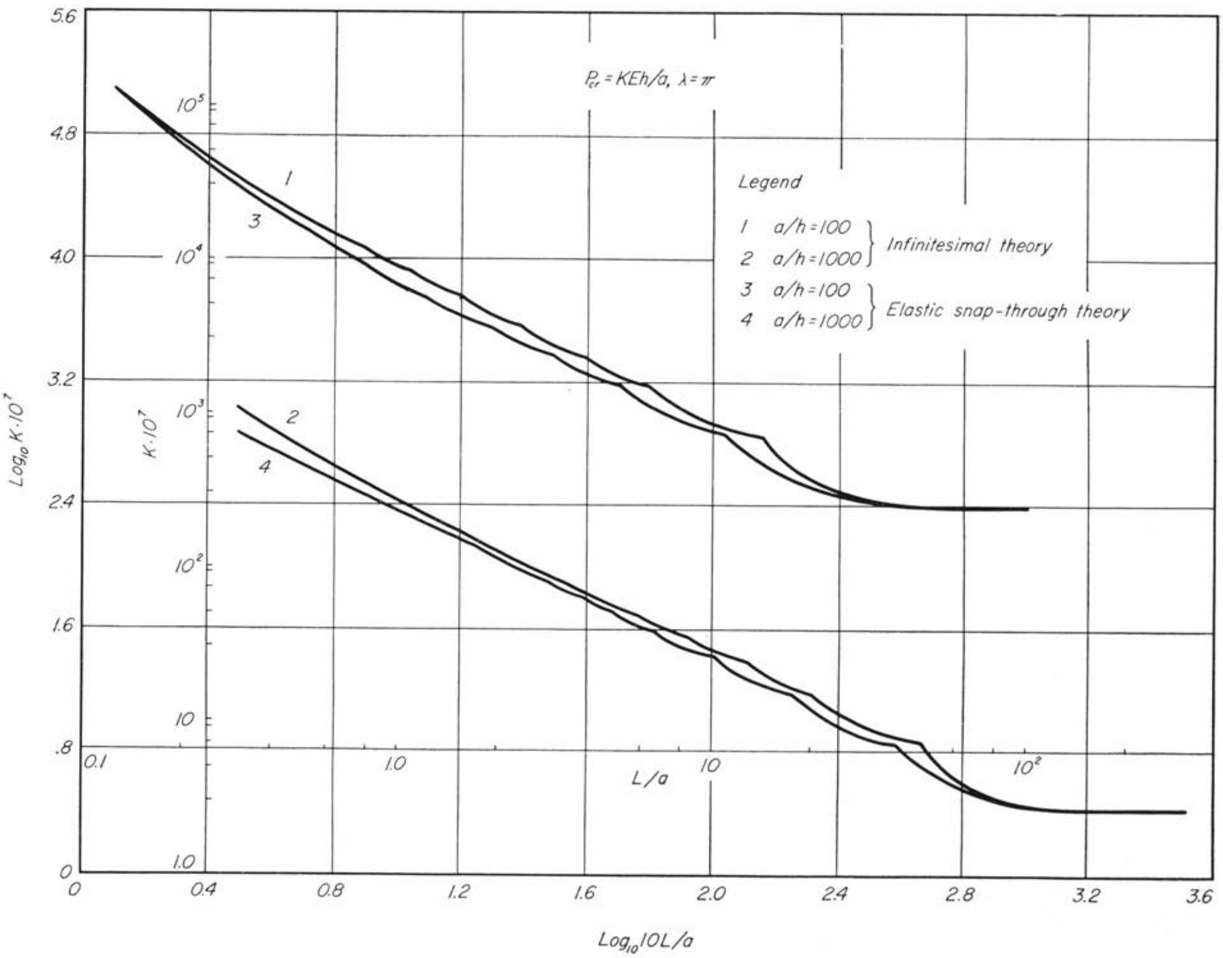


Fig. 13. Buckling Coefficients for Cylindrical Shells Subjected to Hydrostatic Pressure

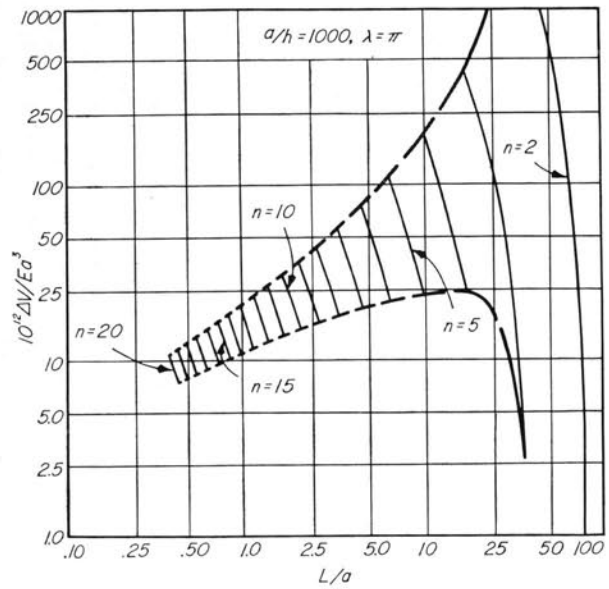


Fig. 14. Potential Energy Barriers Separating Buckled and Unbuckled Forms

as calculated by von Mises' theory are given for comparison. Table 22 lists similar values of K for $a/h = 100$.

Discrepancies between von Mises' theory and the present infinitesimal theory are greatest for short thick shells. Apparently, the trouble lies in the assumption that the shell buckles without incremental hoop strain. Von Mises did not make this assumption.

For elastic cylinders whose ends are free to warp out of their planes, the Euler buckling coefficient (Infinitesimal Theory) and the Tsien buckling coefficient (Elastic Snap-Through Theory) are plotted versus L/a in Fig. 13 for $a/h = 100$ and $a/h = 1000$ with $\lambda = \pi$. Some of the data of Fig. 13 are reproduced in Tables 21 and 22. For long slender cylinders (see Tables 21 and 22), the Euler buckling coefficient is only slightly higher than the Tsien coefficient. However, for relatively small values of L/a (say, $L/a = 0.6010$), the Euler coefficient may be 30 to 35% higher than the Tsien coefficient. In the numerical example of Article 18, the Tsien coefficient is approximately 14% higher than the minimum pressure under which the elastic shell will maintain a buckled form. Prevention of end warping raises the critical pressure (Fig. 12).

For $\lambda = 0$ (no end pressure), all the critical pressures as determined by the different theories are raised. The effect of axial compression is greatest for small values of L/a. It becomes insignificant for very large values of L/a (Tables 21 and 22).

The negative slopes of the load-deflection curves (Fig. 12) denote a condition favorable to snap-through. The potential energy barrier that the shell must overcome to snap-through is discussed in Article 17. Fig. 14 is a chart that shows these barriers for $a/h = 1000$ and $\lambda = \pi$. The curve is discontinuous because of sudden changes in n. The dashed curves have no significance; they merely outline the region in which the discontinuous curve lies. The points of discontinuity correspond to the cusps on curve 4 of Fig. 13. For example, if $L/a = 0.6$ and $a/h = 1000$, Fig. 14 shows that $n = 17$ and $10^{12} \Delta V/Ea^3 = 11.5$. Hence, if $a = 20$ in. and $E = 30{,}000{,}000$ psi, $\Delta V = 2.76$ in.-lb $= 0.23$ ft-lb. This result means that only 0.23 ft-lb of work must be supplied from the outside to cause snap-through. Accidental disturbances might easily supply this much energy. Imperfections are perhaps a more frequent cause of snap-through than accidental disturbances, although submarine hulls may be subjected to damaging shocks.

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VII. APPENDIX

Table 1

Values of K's for $\nu = 0.30$

Subscripts denote number of zeros preceding first significant figure.

n	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9
2	0.43787	$0.0_2 11631$	1.44334	0.68539	0.16004	0.33126	0.21035	0.087164	2.8422
3	0.061575	$0.0_3 21363$	0.40657	0.19277	0.15654	0.046584	0.084115	0.090544	1.5157
4	0.017515	$0.0_4 65935$	0.19723	0.093213	0.15548	0.013250	0.045660	0.091700	1.0582
5	$0.0_2 68417$	$0.0_4 26701$	0.11807	0.055688	0.15501	$0.0_2 51760$	0.028753	0.092232	0.81969
6	$0.0_2 32170$	$0.0_4 12798$	0.079100	0.037263	0.15476	$0.0_2 24338$	0.019794	0.092518	0.67139
7	$0.0_2 17104$	$0.0_5 68822$	0.056878	0.026773	0.15461	$0.0_2 12940$	0.014467	0.092691	0.56956
8	$0.0_3 99290$	$0.0_5 40245$	0.042946	0.020204	0.15452	$0.0_3 75116$	0.011039	0.092804	0.49506
9	$0.0_3 61575$	$0.0_5 25084$	0.033611	0.015807	0.15445	$0.0_3 46584$	$0.0_2 87018$	0.092880	0.43805
10	$0.0_3 40208$	$0.0_5 16438$	0.027041	0.012713	0.15441	$0.0_3 30419$	$0.0_2 70368$	0.092935	0.39296
11	$0.0_3 27367$	$0.0_5 11218$	0.022236	0.010452	0.15437	$0.0_3 20704$	$0.0_2 58084$	0.092975	0.35638
12	$0.0_3 19271$	$0.0_6 79152$	0.018613	$0.0_2 87479$	0.15435	$0.0_3 14580$	$0.0_2 48762$	0.093006	0.32609
13	$0.0_3 13963$	$0.0_6 57437$	0.015813	$0.0_2 74309$	0.15433	$0.0_3 10563$	$0.0_2 41519$	0.093030	0.30058
14	$0.0_3 10364$	$0.0_6 42685$	0.013603	$0.0_2 63916$	0.15431	$0.0_4 78405$	$0.0_2 35779$	0.093049	0.27879
15	$0.0_4 78540$	$0.0_6 32380$	0.011827	$0.0_2 55568$	0.15430	$0.0_4 59418$	$0.0_2 31153$	0.093064	0.25997
16	$0.0_4 60605$	$0.0_6 25006$	0.010379	$0.0_2 48760$	0.15429	$0.0_4 45849$	$0.0_2 27366$	0.093076	0.24354
17	$0.0_4 47512$	$0.0_6 19617$	$0.0_2 91817$	$0.0_2 43134$	0.15428	$0.0_4 35944$	$0.0_2 24237$	0.093087	0.22908
18	$0.0_4 37773$	$0.0_6 15605$	$0.0_2 81810$	$0.0_2 38432$	0.15427	$0.0_4 28576$	$0.0_2 21613$	0.093096	0.21624
19	$0.0_4 30408$	$0.0_6 12568$	$0.0_2 73359$	$0.0_2 34460$	0.15427	$0.0_4 23004$	$0.0_2 19394$	0.093103	0.20477
20	$0.0_4 24754$	$0.0_6 10235$	$0.0_2 66155$	$0.0_2 31075$	0.15426	$0.0_4 18727$	$0.0_2 17500$	0.093110	0.19446

Table 1 (Concluded)

n	K_{10}	K_{11}	K_{12}	K_{13}	K_{14}	K_{15}	K_{16}	K_{17}	K_{18}
2	0.26572	0.73304	0.43982	2.5839	0.086128	0.027489	1.04720	0.071177	0.25839
3	0.40644	0.41234	0.33576	0.09895	0.020763	0.017671	0.88357	0.072579	0.11074
4	0.54541	0.29322	0.30718	0.51677	$0.0_2 82027$	0.013090	0.83776	0.073047	0.061520
5	0.68375	0.22907	0.29502	0.32298	$0.0_2 40780$	0.010412	0.81812	0.073259	0.039149
6	0.82178	0.18850	0.28868	0.22147	$0.0_2 23231$	$0.0_2 86504$	0.80784	0.073374	0.027103
7	0.95964	0.16035	0.28493	0.16149	$0.0_2 14493$	$0.0_2 74009$	0.80176	0.073442	0.019876
8	1.09740	0.13963	0.28254	0.12304	$0.0_3 96502$	$0.0_2 64680$	0.79786	0.073487	0.015199
9	1.2351	0.12370	0.28091	0.096894	$0.0_3 67496$	$0.0_2 57446$	0.79522	0.073517	0.011999
10	1.3727	0.11107	0.27975	0.078299	$0.0_3 49059$	$0.0_2 51671$	0.79333	0.073539	$0.0_2 97137$
11	1.5103	0.10079	0.27890	0.064596	$0.0_3 36778$	$0.0_2 46953$	0.79194	0.073555	$0.0_2 80244$
12	1.6479	0.092271	0.27826	0.054207	$0.0_3 28282$	$0.0_2 43026$	0.79089	0.073567	$0.0_2 67405$
13	1.7854	0.085085	0.27776	0.046140	$0.0_3 22216$	$0.0_2 39706$	0.79007	0.073576	$0.0_2 57419$
14	1.9230	0.078943	0.27736	0.039752	$0.0_3 17769$	$0.0_2 36862$	0.78943	0.073584	$0.0_2 49499$
15	2.0605	0.073631	0.27704	0.034605	$0.0_3 14435$	$0.0_2 34399$	0.78890	0.073590	$0.0_2 43112$
16	2.1980	0.068992	0.27678	0.030398	$0.0_3 11886$	$0.0_2 32245$	0.78848	0.073595	$0.0_2 37886$
17	2.3355	0.064905	0.27656	0.026915	$0.0_4 99038$	$0.0_2 30345$	0.78812	0.073599	$0.0_2 33556$
18	2.4730	0.061276	0.27638	0.023999	$0.0_4 83393$	$0.0_2 28656$	0.78783	0.073603	$0.0_2 29929$
19	2.6105	0.058032	0.27623	0.021532	$0.0_4 70878$	$0.0_2 27146$	0.78758	0.073606	$0.0_2 26859$
20	2.7480	0.055116	0.27610	0.019427	$0.0_4 60749$	$0.0_2 25787$	0.78736	0.073608	$0.0_2 24239$

Table 2

Values of Coefficients for Computing Buckling Loads — $n = 2$

$n = 2$ $\lambda = \pi$ $\nu = 0.30$

ξ	L/a	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3
4	6.7612	1.0943	0.06018	$0.0_3 9680$		$0.0_3 9558$	0.3233	0.05798	$0.0_2 5626$
5	8.4516	1.0773	0.06414	$0.0_3 4165$	$-0.0_3 7060$	$0.0_3 4230$	0.3102	0.05634	$0.0_2 5624$
6	10.142	1.0681	0.06629	$0.0_3 2065$	$-0.0_3 3504$	$0.0_3 2144$	0.3032	0.05553	$0.0_2 5622$
7	11.832	1.0625	0.06759	$0.0_3 1134$	$-0.0_3 1926$	$0.0_3 1197$	0.2990	0.05508	$0.0_2 5621$
8	13.522	1.0589	0.06843	$0.0_4 6720$	$-0.0_3 1146$	$0.0_4 7193$	0.2963	0.05480	$0.0_2 5621$
10	16.903	1.0547	0.06942	$0.0_4 2788$	$-0.0_4 4744$	$0.0_4 3041$	0.2932	0.05448	$0.0_2 5620$
12	20.284	1.0524	0.06996	$0.0_4 1354$	$-0.0_4 2327$	$0.0_4 1494$	0.2915	0.05431	$0.0_2 5620$
15	25.355	1.0506	0.07040	$0.0_5 5574$	$-0.0_5 9597$	$0.0_5 6222$	0.2901	0.05417	$0.0_2 5620$
20	33.806	1.0491	0.07074	$0.0_5 1769$	$-0.0_5 3054$	$0.0_5 1996$	0.2890	0.05407	$0.0_2 5619$
25	42.258	1.0484	0.07090	$0.0_6 7238$	$-0.0_5 1254$	$0.0_6 8235$	0.2886	0.05402	$0.0_2 5619$
30	50.709	1.0480	0.07098	$0.0_6 3483$	$-0.0_6 6066$	$0.0_6 3989$	0.2883	0.05400	$0.0_2 5619$
40	67.612	1.0477	0.07107	$0.0_6 1094$	$-0.0_6 1921$	$0.0_6 1270$	0.2880	0.05397	$0.0_2 5619$
50	84.516	1.0475	0.07111	$0.0_7 4439$	$-0.0_7 7986$	$0.0_7 5228$	0.2879	0.05396	$0.0_2 5619$
60	101.42	1.0474	0.07113	$0.0_7 2111$	$-0.0_7 3838$	$0.0_7 2535$	0.2878	0.05396	$0.0_2 5619$
75	126.77	1.0473	0.07115	$0.0_8 8454$	$-0.0_7 1617$	$0.0_7 1046$	0.2878	0.05395	$0.0_2 5619$
100	169.03	1.0473	0.07116	$0.0_8 2501$	$-0.0_8 5054$	$0.0_8 3360$	0.2877	0.05395	$0.0_2 5619$
125	211.29	1.0472	0.07117	$0.0_9 9686$	$-0.0_8 2122$	$0.0_8 1408$	0.2877	0.05394	$0.0_2 5619$
150	253.55	1.0472	0.07117	$0.0_9 4119$	$-0.0_8 1011$	$0.0_9 6960$	0.2877	0.05394	$0.0_2 5619$
200	338.06	1.0472	0.07117	$0.0_{10} 9321$	$-0.0_9 3034$	$0.0_{10} 198$	0.2877	0.05394	$0.0_2 5619$

If $\lambda = 0$, $a_1 = 1.0472$ and $a_2 = 0.07118$. The b's and c's are independent of λ.

Table 3

Values of Coefficients for Computing Buckling Loads — $n = 3$

$n = 3$ $\lambda = \pi$ $\nu = 0.30$

ξ	L/a	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3
2.4	2.704	0.9664	$0.0_{2}5343$	$0.0_{2}1910$	$-0.0_{2}7619$	0.01077	0.8671	0.1730	0.01262
2.8	3.155	0.9444	0.02318	$0.0_{2}1120$	$-0.0_{2}4485$	$0.0_{2}6490$	0.8052	0.1519	0.01262
3.2	3.606	0.9301	0.03476	$0.0_{3}6956$	$-0.0_{2}2793$	$0.0_{2}4137$	0.7664	0.1424	0.01263
4	4.508	0.9134	0.04837	$0.0_{3}3063$	$-0.0_{2}1234$	$0.0_{2}1911$	0.7222	0.1324	0.01263
5	5.634	0.9026	0.05709	$0.0_{3}1318$	$-0.0_{3}5394$	$0.0_{3}8630$	0.6947	0.1275	0.01264
6	6.761	0.8968	0.06182	$0.0_{4}6536$	$-0.0_{3}2639$	$0.0_{3}4443$	0.6800	0.1252	0.01264
7	7.888	0.8933	0.06468	$0.0_{4}3589$	$-0.0_{3}1450$	$0.0_{3}2511$	0.6712	0.1240	0.01264
8	9.015	0.8910	0.06653	$0.0_{4}2128$	$-0.0_{4}8596$	$0.0_{3}1522$	0.6656	0.1233	0.01264
10	11.269	0.8883	0.06871	$0.0_{5}8834$	$-0.0_{4}3569$	$0.0_{4}6506$	0.6590	0.1226	0.01264
12	13.523	0.8869	0.06989	$0.0_{5}4292$	$-0.0_{4}1754$	$0.0_{4}3220$	0.6554	0.1222	0.01264
15	16.904	0.8857	0.07086	$0.0_{5}1768$	$-0.0_{5}7145$	$0.0_{4}1350$	0.6525	0.1219	0.01264
20	22.538	0.8848	0.07161	$0.0_{6}5622$	$-0.0_{5}2271$	$0.0_{5}4352$	0.6502	0.1216	0.01264
25	28.172	0.8843	0.07196	$0.0_{6}2307$	$-0.0_{6}9335$	$0.0_{5}1799$	0.6492	0.1216	0.01264
30	33.81	0.8841	0.07215	$0.0_{6}1113$	$-0.0_{6}4494$	$0.0_{6}8725$	0.6486	0.1215	0.01264
40	45.08	0.8839	0.07234	$0.0_{7}3527$	$-0.0_{6}1421$	$0.0_{6}2780$	0.6480	0.1214	0.01264
50	56.34	0.8838	0.07242	$0.0_{7}1454$	$-0.0_{7}583$	$0.0_{6}1145$	0.6478	0.1214	0.01264
60	67.61	0.8837	0.07247	$0.0_{8}6938$	$-0.0_{7}279$	$0.0_{7}5550$	0.6476	0.1214	0.01264
75	84.52	0.8836	0.07251	$0.0_{8}2879$	$-0.0_{7}117$	$0.0_{7}2294$	0.6475	0.1214	0.01264
100	112.69	0.8836	0.07254	$0.0_{9}9986$	$-0.0_{8}34$	$0.0_{8}729$	0.6474	0.1214	0.01264

If $\lambda = 0$, $a_1 = 0.8836$ and $a_2 = 0.07258$. The b's and c's are independent of λ.

Table 4

Values of Coefficients for Computing Buckling Loads — $n = 4$

$n = 4$ $\lambda = \pi$ $\nu = 0.30$

ξ	L/a	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3
2.0	1.690	0.9402	−0.09791	$0.0_{2}1769$	−0.01273	0.03261	1.719	0.4779	0.02260
2.4	2.028	0.9089	−0.04568	$0.0_{3}9656$	$-0.0_{2}6993$	0.01837	1.532	0.3522	0.02256
2.8	2.366	0.8900	−0.01418	$0.0_{3}5662$	$-0.0_{2}4116$	0.01112	1.425	0.2959	0.02254
3.2	2.704	0.8778	$0.0_{2}6266$	$0.0_{3}3516$	$-0.0_{2}2562$	$0.0_{2}7116$	1.358	0.2671	0.02252
4	3.381	0.8634	0.03031	$0.0_{3}1548$	$-0.0_{2}1132$	$0.0_{2}3311$	1.281	0.2410	0.02251
5	4.226	0.8542	0.04569	$0.0_{4}6660$	$-0.0_{3}4877$	$0.0_{2}1507$	1.233	0.2289	0.02250
6	5.071	0.8492	0.05405	$0.0_{4}3302$	$-0.0_{3}2421$	$0.0_{3}7797$	1.208	0.2237	0.02249
7	5.916	0.8461	0.05909	$0.0_{4}1812$	$-0.0_{3}1330$	$0.0_{3}4422$	1.192	0.2211	0.02248
8	6.761	0.8442	0.06236	$0.0_{4}1074$	$-0.0_{4}7886$	$0.0_{3}2687$	1.183	0.2196	0.02248
10	8.452	0.8419	0.06621	$0.0_{5}4454$	$-0.0_{4}3275$	$0.0_{3}1152$	1.171	0.2180	0.02248
12	10.14	0.8406	0.06830	$0.0_{5}2161$	$-0.0_{4}1592$	$0.0_{4}5715$	1.165	0.2172	0.02248
15	12.68	0.8396	0.07001	$0.0_{6}8881$	$-0.0_{5}6562$	$0.0_{4}2399$	1.160	0.2167	0.02248
20	16.90	0.8388	0.07134	$0.0_{6}2809$	$-0.0_{5}2088$	$0.0_{5}7749$	1.156	0.2163	0.02248
25	21.13	0.8384	0.07195	$0.0_{6}1145$	$-0.0_{6}8585$	$0.0_{5}3205$	1.154	0.2161	0.02248
30	25.35	0.8382	0.07229	$0.0_{7}5478$	$-0.0_{6}4152$	$0.0_{5}1555$	1.153	0.2160	0.02248
40	33.81	0.8380	0.07262	$0.0_{7}1698$	$-0.0_{6}1322$	$0.0_{6}495$	1.152	0.2159	0.02248
50	42.26	0.8379	0.07277	$0.0_{8}6805$	$-0.0_{7}545$	$0.0_{6}205$	1.152	0.2158	0.02248

If $\lambda = 0$, $a_1 = 0.8378$ and $a_2 = 0.07305$. The b's and c's are independent of λ.

Table 5

Values of Coefficients for Computing Buckling Loads — $n = 5$

$n = 5$ $\lambda = \pi$ $\nu = 0.30$

ξ	L/a	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3
1.6	1.082	0.9676	−0.3429	$0.0_{2}2170$	−0.02441	0.09734	3.248	1.731	0.03476
2.0	1.352	0.9138	−0.1931	$0.0_{2}1080$	−0.01227	0.04999	2.675	0.9347	0.03489
2.4	1.623	0.8846	−0.1117	$0.0_{3}5895$	$-0.0_{2}6733$	0.02822	2.388	0.6409	0.03496
2.8	1.893	0.8669	−0.06262	$0.0_{3}3457$	$-0.0_{2}3963$	0.01712	2.223	0.5111	0.03500
3.2	2.164	0.8555	−0.03077	$0.0_{3}2147$	$-0.0_{2}2467$	0.01096	2.119	0.4459	0.03503
4	2.704	0.8420	$0.0_{2}6678$	$0.0_{4}9454$	$-0.0_{2}1089$	$0.0_{2}5124$	2.000	0.3883	0.03506
5	3.381	0.8334	0.03065	$0.0_{4}4068$	$-0.0_{3}4693$	$0.0_{2}2338$	1.926	0.3623	0.03508
6	4.057	0.8288	0.04367	$0.0_{4}2017$	$-0.0_{3}2329$	$0.0_{2}1213$	1.886	0.3518	0.03509
7	4.733	0.8259	0.05151	$0.0_{4}1108$	$-0.0_{3}1280$	$0.0_{3}6890$	1.863	0.3466	0.03510
8	5.409	0.8241	0.05661	$0.0_{5}6567$	$-0.0_{4}7586$	$0.0_{3}4191$	1.848	0.3438	0.03510
10	6.761	0.8219	0.06261	$0.0_{5}2726$	$-0.0_{4}3140$	$0.0_{3}1802$	1.830	0.3409	0.03511
12	8.113	0.8208	0.06686	$0.0_{5}1324$	$-0.0_{4}1530$	$0.0_{4}8942$	1.820	0.3396	0.03511
15	10.142	0.8198	0.06852	$0.0_{6}5456$	$-0.0_{5}6305$	$0.0_{4}3757$	1.812	0.3386	0.03511
20	13.52	0.8191	0.07060	$0.0_{6}1734$	$-0.0_{5}2005$	$0.0_{4}1215$	1.806	0.3379	0.03511
25	16.90	0.8187	0.07155	$0.0_{7}7115$	$-0.0_{6}8229$	$0.0_{5}5027$	1.803	0.3376	0.03512
30	20.28	0.8185	0.07208	$0.0_{7}3432$	$-0.0_{6}4012$	$0.0_{5}2439$	1.802	0.3375	0.03512
40	27.04	0.8184	0.07259	$0.0_{7}1086$	$-0.0_{6}1257$	$0.0_{6}7783$	1.800	0.3373	0.03512
50	33.81	0.8183	0.07283	$0.0_{8}4452$	$-0.0_{7}5173$	$0.0_{6}3205$	1.799	0.3372	0.03512

If $\lambda = 0$, $a_1 = 0.8181$ and $a_2 = 0.07326$. The b's and c's are independent of λ.

Table 6

Values of Coefficients for Computing Buckling Loads — $n = 6$

$n = 6$ $\lambda = \pi$ $\nu = 0.30$

ξ	L/a	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3
1.2	0.6761	1.0639	−0.9902	$0.0_{2}3362$	−0.05376	0.3080	6.707	9.160	0.04956
1.6	0.9015	0.9519	−0.5249	$0.0_{2}1470$	−0.02393	0.1386	4.663	3.303	0.05000
2.0	1.127	0.9000	−0.3095	$0.0_{3}7312$	−0.01202	0.07128	3.844	1.677	0.05021
2.4	1.352	0.8719	−0.1925	$0.0_{3}3991$	$-0.0_{2}6542$	0.04028	3.434	1.083	0.05032
2.8	1.578	0.8549	−0.1220	$0.0_{3}2341$	$-0.0_{2}3888$	0.02445	3.198	0.8222	0.05038
3.2	1.803	0.8438	−0.07619	$0.0_{3}1454$	$-0.0_{2}2418$	0.01570	3.049	0.6925	0.05043
4	2.254	0.8309	−0.02234	$0.0_{4}6401$	$-0.0_{2}1067$	$0.0_{2}7341$	2.879	0.5798	0.05048
5	2.817	0.8226	0.01111	$0.0_{4}2754$	$-0.0_{3}4600$	$0.0_{2}3356$	2.773	0.5302	0.05051
6	3.381	0.8181	0.03083	$0.0_{4}1365$	$-0.0_{3}2282$	$0.0_{2}1743$	2.716	0.5106	0.05053
7	3.944	0.8154	0.04212	$0.0_{5}7497$	$-0.0_{3}1254$	$0.0_{3}9908$	2.682	0.5014	0.05054
8	4.507	0.8136	0.04944	$0.0_{5}4444$	$-0.0_{4}7385$	$0.0_{3}6030$	2.660	0.4963	0.05055
10	5.634	0.8115	0.05806	$0.0_{5}1844$	$-0.0_{4}3087$	$0.0_{3}2594$	2.634	0.4914	0.05056
12	6.761	0.8104	0.06274	$0.0_{6}8956$	$-0.0_{4}1500$	$0.0_{3}1288$	2.621	0.4892	0.05056
15	8.452	0.8095	0.06657	$0.0_{6}3687$	$-0.0_{5}6181$	$0.0_{4}5412$	2.609	0.4877	0.05056
20	11.27	0.8088	0.06954	$0.0_{6}1170$	$-0.0_{5}1966$	$0.0_{4}1749$	2.600	0.4866	0.05057
25	14.08	0.8084	0.07092	$0.0_{7}4790$	$-0.0_{6}8035$	$0.0_{5}7237$	2.596	0.4862	0.05057

If $\lambda = 0$, $a_1 = 0.8078$ and $a_2 = 0.07337$. The b's and c's are independent of λ.

Table 7

Values of Coefficients for Computing Buckling Loads — $n = 7$

$n = 7$ $\lambda = \pi$ $\nu = 0.30$

ξ	L/a	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3
1.0	0.4829	1.1624	−2.009	$0.0_{2}3948$	−0.08504	0.6668	12.388	33.43	0.06675
1.2	0.5795	1.0522	−1.373	$0.0_{2}2433$	−0.05314	0.4162	9.098	16.59	0.06738
1.6	0.7727	0.9426	−0.7400	$0.0_{2}1064$	−0.02364	0.1875	6.335	5.798	0.06802
2.0	0.9659	0.8919	−0.4472	$0.0_{3}5292$	−0.01188	0.09646	5.227	2.817	0.06831
2.4	1.159	0.8644	−0.2881	$0.0_{3}2889$	$-0.0_{2}6580$	0.05455	4.670	1.731	0.06847
2.8	1.352	0.8478	−0.1922	$0.0_{3}1694$	$-0.0_{2}3838$	0.03314	4.350	1.258	0.06857
3.2	1.545	0.8370	−0.1299	$0.0_{3}1052$	$-0.0_{2}2388$	0.02129	4.148	1.024	0.06863
4	1.932	0.8243	−0.05671	$0.0_{4}4634$	$-0.0_{2}1054$	$0.0_{2}9965$	3.918	0.8224	0.06870
5	2.415	0.8162	$-0.0_{2}9858$	$0.0_{4}1994$	$-0.0_{3}4544$	$0.0_{2}4560$	3.774	0.7353	0.06875
6	2.898	0.8118	0.01559	$0.0_{5}9889$	$-0.0_{3}2255$	$0.0_{2}2370$	3.696	0.7016	0.06877
7	3.381	0.8091	0.03094	$0.0_{5}5431$	$-0.0_{3}1239$	$0.0_{2}1348$	3.650	0.6859	0.06879
8	3.864	0.8074	0.04090	$0.0_{5}3220$	$-0.0_{4}7345$	$0.0_{3}8206$	3.621	0.6776	0.06880
10	4.829	0.8054	0.05262	$0.0_{5}1330$	$-0.0_{4}3050$	$0.0_{3}3532$	3.586	0.6698	0.06881

If $\lambda = 0$, $a_1 = 0.8018$ and $a_2 = 0.07344$. The b's and c's are independent of λ.

Table 8

Values of Coefficients for Computing Buckling Loads — $n = 8$

$n = 8$ $\lambda = \pi$ $\nu = 0.30$

ξ	L/a	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3
1.0	0.4226	1.153	−2.645	$0.0_{2}2993$	−0.08441	0.8667	16.13	56.54	0.08709
1.2	0.5071	1.045	−1.814	$0.0_{2}1844$	−0.05273	0.5411	11.86	27.87	0.08795
1.6	0.6761	0.9367	−0.9884	$0.0_{3}8064$	−0.02346	0.2438	8.264	9.537	0.08880
2.0	0.8452	0.8867	−0.6061	$0.0_{3}4012$	−0.01178	0.1255	6.822	4.483	0.08920
2.4	1.014	0.8596	−0.3985	$0.0_{3}2190$	$-0.0_{2}6470$	0.07103	6.097	2.649	0.08942
2.8	1.183	0.8432	−0.2732	$0.0_{3}1284$	$-0.0_{2}3808$	0.04316	5.680	1.852	0.08954
3.2	1.352	0.8326	−0.1920	$0.0_{4}7975$	$-0.0_{2}2370$	0.02774	5.417	1.460	0.08963
4	1.690	0.8201	−0.09641	$0.0_{4}3512$	$-0.0_{2}1046$	0.01299	5.116	1.124	0.08973
5	2.113	0.8121	−0.03525	$0.0_{4}1511$	$-0.0_{3}4509$	$0.0_{2}5950$	4.928	0.9809	0.08979
6	2.535	0.8077	$-0.0_{2}2024$	$0.0_{5}7492$	$-0.0_{3}2237$	$0.0_{2}3093$	4.828	0.9263	0.08982
7	2.958	0.8051	0.01801	$0.0_{5}4113$	$-0.0_{3}1213$	$0.0_{2}1760$	4.768	0.9012	0.08985
8	3.381	0.8034	0.03101	$0.0_{5}2438$	$-0.0_{4}7285$	$0.0_{2}1072$	4.729	0.8882	0.08986
10	4.226	0.8014	0.04630	$0.0_{5}1012$	$-0.0_{4}3024$	$0.0_{3}4614$	4.683	0.8761	0.08988

If $\lambda = 0$, $a_1 = 0.7979$ and $a_2 = 0.07349$. The b's and c's are independent of λ.

Table 9

Values of Coefficients for Computing Buckling Loads — $n = 9$

$n = 9$ $\lambda = \pi$ $\nu = 0.30$

ξ	L/a	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3
1.0	0.3756	1.147	−3.366	$0.0_{2}2349$	−0.08398	1.0934	20.38	90.02	0.1101
1.2	0.4507	1.040	−2.315	$0.0_{2}1448$	−0.05292	0.6827	14.99	44.18	0.1112
1.6	0.6010	0.9327	−1.270	$0.0_{3}6329$	−0.02334	0.3077	10.45	14.89	0.1124
2.0	0.7512	0.8832	−0.7862	$0.0_{3}3149$	−0.01172	0.1585	8.629	6.828	0.1129
2.4	0.9015	0.8563	−0.5235	$0.0_{3}1719$	$-0.0_{2}6437$	0.08970	7.714	3.908	0.1131
2.8	1.052	0.8401	−0.3651	$0.0_{3}1008$	$-0.0_{2}3788$	0.05453	7.186	2.644	0.1133
3.2	1.202	0.8296	−0.2623	$0.0_{4}6260$	$-0.0_{2}2357$	0.03505	6.855	2.024	0.1134
4	1.502	0.8172	−0.1414	$0.0_{4}2756$	$-0.0_{2}1041$	0.01642	6.474	1.495	0.1136
5	1.878	0.8093	−0.06404	$0.0_{4}1186$	$-0.0_{3}4484$	$0.0_{2}7525$	6.237	1.271	0.1136
6	2.254	0.8050	−0.02201	$0.0_{5}5880$	$-0.0_{3}2225$	$0.0_{2}3913$	6.110	1.186	0.1137
7	2.629	0.8024	$0.0_{2}3333$	$0.0_{5}3228$	$-0.0_{3}1222$	$0.0_{2}2223$	6.034	1.148	0.1137
8	3.005	0.8007	0.01978	$0.0_{5}1914$	$-0.0_{4}7247$	$0.0_{2}1356$	5.985	1.129	0.1137
10	3.756	0.7987	0.03913	$0.0_{6}7941$	$-0.0_{4}3008$	$0.0_{3}5839$	5.927	1.111	0.1137
12	4.507	0.7977	0.04964	$0.0_{6}3856$	$-0.0_{4}1462$	$0.0_{3}2901$	5.896	1.103	0.1137
15	5.634	0.7968	0.05823	$0.0_{6}1587$	$-0.0_{5}6022$	$0.0_{3}1068$	5.871	1.098	0.1137
20	7.512	0.7961	0.06492	$0.0_{7}5037$	$-0.0_{5}1914$	$0.0_{4}3941$	5.851	1.095	0.1137

If $\lambda = 0$, $a_1 = 0.7952$ and $a_2 = 0.07352$. The b's and c's are independent of λ.

Table 10

Values of Coefficients for Computing Buckling Loads — $n = 10$

$n = 10$ $\lambda = \pi$ $\nu = 0.30$

ξ	L/a	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3
0.8	0.2704	1.339	−6.559	$0.0_{2}3302$	−0.1431	2.3217	40.02	330.6	0.1334
1.0	0.3381	1.143	−4.171	$0.0_{2}1894$	−0.08367	1.3467	25.12	136.6	0.1359
1.2	0.4057	1.036	−2.874	$0.0_{2}1167$	−0.05227	0.8410	18.48	66.83	0.1373
1.6	0.5409	0.9298	−1.584	$0.0_{3}5103$	−0.02326	0.3792	12.89	22.27	0.1387
2.0	0.6761	0.8807	−0.9876	$0.0_{3}2539$	−0.01168	0.1953	10.65	10.02	0.1393
2.4	0.8113	0.8540	−0.6634	$0.0_{3}1386$	$-0.0_{2}6412$	0.1106	9.520	5.593	0.1397
2.8	0.9466	0.8379	−0.4677	$0.0_{4}8127$	$-0.0_{2}3794$	0.06724	8.871	3.679	0.1399
3.2	1.082	0.8275	−0.3410	$0.0_{4}5047$	$-0.0_{2}2348$	0.04323	8.461	2.742	0.1400
4	1.352	0.8152	−0.1918	$0.0_{4}2222$	$-0.0_{2}1037$	0.02026	7.992	1.945	0.1402
5	1.690	0.8073	−0.09624	$0.0_{5}9563$	$-0.0_{3}4468$	$0.0_{2}9287$	7.699	1.610	0.1403
6	2.028	0.8030	−0.04437	$0.0_{5}4742$	$-0.0_{3}2217$	$0.0_{2}4831$	7.543	1.484	0.1404
7	2.366	0.8005	−0.01309	$0.0_{5}2604$	$-0.0_{3}1218$	$0.0_{2}2745$	7.449	1.428	0.1404
8	2.704	0.7988	$0.0_{2}7217$	$0.0_{5}1543$	$-0.0_{4}7221$	$0.0_{2}1675$	7.388	1.400	0.1404
10	3.381	0.7968	0.03109	$0.0_{6}6407$	$-0.0_{4}3000$	$0.0_{3}7215$	7.317	1.373	0.1404
12	4.057	0.7958	0.04406	$0.0_{6}3112$	$-0.0_{4}1454$	$0.0_{3}3586$	7.279	1.363	0.1404
15	5.071	0.7949	0.05467	$0.0_{6}1282$	$-0.0_{5}6006$	$0.0_{3}1508$	7.248	1.356	0.1404
20	6.761	0.7942	0.06293	$0.0_{7}4074$	$-0.0_{5}1911$	$0.0_{4}4882$	7.223	1.352	0.1404

If $\lambda = 0$, $a_1 = 0.7933$ and $a_2 = 0.07354$. The b's and c's are independent of λ.

Table 11

Values of Coefficients for Computing Buckling Loads — $n = 11$

$n = 11$ $\lambda = \pi$ $\nu = 0.30$

ξ	L/a	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3
0.7	0.2151	1.501	−10.40	$0.0_{2}3732$	−0.1937	3.830	66.94	822.2	0.1585
0.8	0.2459	1.335	−7.949	$0.0_{2}2720$	−0.1428	2.805	48.36	483.2	0.1612
1.0	0.3073	1.140	−5.061	$0.0_{2}1560$	−0.08234	1.627	30.37	199.4	0.1643
1.2	0.3688	1.033	−3.492	$0.0_{3}9612$	−0.05213	1.016	22.35	97.30	0.1661
1.6	0.4917	0.9277	−1.932	$0.0_{3}4202$	−0.02319	0.4581	15.60	32.14	0.1678
2.0	0.6147	0.8788	−1.210	$0.0_{3}2091$	−0.01165	0.2360	12.88	14.26	0.1686
2.4	0.7376	0.8523	−0.8179	$0.0_{3}1141$	$-0.0_{2}6394$	0.1336	11.52	7.795	0.1690
2.8	0.8605	0.8363	−0.5814	$0.0_{4}6929$	$-0.0_{2}3763$	0.08127	10.73	5.006	0.1693
3.2	0.9834	0.8259	−0.4279	$0.0_{4}4156$	$-0.0_{2}2342$	0.05226	10.24	3.642	0.1694
4	1.229	0.8137	−0.2474	$0.0_{4}1830$	$-0.0_{2}1034$	0.02450	9.670	2.487	0.1696
5	1.537	0.8058	−0.1318	$0.0_{5}7876$	$-0.0_{3}4456$	0.01123	9.316	2.002	0.1698
6	1.844	0.8016	−0.06907	$0.0_{5}3906$	$-0.0_{3}2211$	$0.0_{2}5844$	9.126	1.822	0.1698
7	2.151	0.7990	−0.03123	$0.0_{5}2145$	$-0.0_{3}1215$	$0.0_{2}3327$	9.013	1.742	0.1699
8	2.459	0.7974	−0.01667	$0.0_{5}1271$	$-0.0_{4}7204$	$0.0_{2}2027$	8.940	1.702	0.1699
10	3.073	0.7954	0.02221	$0.0_{6}5279$	$-0.0_{4}2993$	$0.0_{3}8731$	8.854	1.666	0.1699
12	3.688	0.7944	0.03790	$0.0_{6}2565$	$-0.0_{4}1454$	$0.0_{3}4340$	8.808	1.651	0.1699
15	4.610	0.7935	0.05074	$0.0_{6}1057$	$-0.0_{5}5994$	$0.0_{3}1825$	8.770	1.642	0.1700
20	6.147	0.7928	0.06072	$0.0_{7}3361$	$-0.0_{5}1907$	$0.0_{4}5906$	8.740	1.636	0.1700

If $\lambda=0$, $a_1=0.7919$ and $a_2=0.07356$. The b's and c's are independent of λ.

Table 12

Values of Coefficients for Computing Buckling Loads — $n = 12$

$n = 12$ $\lambda = \pi$ $\nu = 0.30$

ξ	L/a	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3
0.7	0.1972	1.4976	−12.395	$0.0_{2}3128$	−0.1933	4.552	79.56	1163	0.1886
0.8	0.2254	1.3320	−9.4730	$0.0_{2}2279$	−0.1425	3.333	57.49	683.4	0.1918
1.0	0.2817	1.1372	−6.0362	$0.0_{2}1307$	−0.08328	1.933	36.12	281.7	0.1956
1.2	0.3381	1.0314	−4.1693	$0.0_{3}8056$	−0.05204	1.208	26.58	137.2	0.1976
1.6	0.4508	0.9262	−2.3131	$0.0_{3}3522$	−0.02314	0.5446	18.55	45.03	0.1997
2.0	0.5634	0.8775	−1.4539	$0.0_{3}1752$	−0.01162	0.2806	15.33	19.74	0.2006
2.4	0.6761	0.8510	−0.9872	$0.0_{4}9566$	$-0.0_{2}6381$	0.1589	13.70	10.61	0.2011
2.8	0.7888	0.8350	−0.7057	$0.0_{4}5610$	$-0.0_{2}3756$	0.09665	12.77	6.680	0.2014
3.2	0.9015	0.8247	−0.5231	$0.0_{4}3484$	$-0.0_{2}2336$	0.06216	12.18	4.758	0.2016
4.0	1.1269	0.8125	−0.3083	$0.0_{4}1534$	$-0.0_{2}1032$	0.02915	11.51	3.133	0.2019
5.0	1.4086	0.8047	−0.1708	$0.0_{5}6603$	$-0.0_{3}4446$	0.01336	11.09	2.454	0.2020
6.0	1.690	0.8005	−0.09615	$0.0_{5}3275$	$-0.0_{3}2207$	$0.0_{2}6953$	10.86	2.203	0.2021
7.0	1.972	0.7980	−0.05112	$0.0_{5}1799$	$-0.0_{3}1212$	$0.0_{2}3959$	10.73	2.092	0.2021
8.0	2.254	0.7963	−0.02190	$0.0_{5}1066$	$-0.0_{4}7187$	$0.0_{2}2412$	10.64	2.036	0.2022

If $\lambda=0$, $a_1=0.79089$ and $a_2=0.073567$. The b's and c's are independent of λ.

Table 13

Values of Coefficients for Computing Buckling Loads — $n = 13$

$n = 13$ $\lambda = \pi$ $\nu = 0.30$

ξ	L/a	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3
0.7	0.1820	1.4946	−14.558	$0.0_{2}2660$	−0.1934	5.336	93.28	1601	0.2212
0.8	0.2080	1.3295	−11.1289	$0.0_{2}1938$	−0.1423	3.907	67.41	940.4	0.2250
1.0	0.2600	1.1353	−7.0960	$0.0_{2}1112$	−0.08443	2.267	42.36	387.2	0.2295
1.2	0.3121	1.0298	−4.9053	$0.0_{3}6850$	−0.05195	1.416	31.18	188.4	0.2319
1.6	0.4161	0.9249	−2.7270	$0.0_{3}2994$	−0.02310	0.6386	21.77	61.49	0.2343
2.0	0.5201	0.8764	−1.7188	$0.0_{3}1490$	−0.01160	0.3291	17.99	26.71	0.2354
2.4	0.6241	0.8500	−1.1711	$0.0_{4}8132$	$-0.0_{2}6381$	0.1864	16.08	14.16	0.2360
2.8	0.7281	0.8341	−0.8409	$0.0_{4}4769$	$-0.0_{2}3749$	0.1134	14.99	8.762	0.2364
3.2	0.8322	0.8238	−0.6266	$0.0_{4}2962$	$-0.0_{2}2333$	0.07292	14.30	6.124	0.2366
4.0	1.040	0.8116	−0.3745	$0.0_{4}1304$	$-0.0_{2}1030$	0.03420	13.50	3.898	0.2369
5.0	1.300	0.8039	−0.2132	$0.0_{5}5611$	$-0.0_{3}4439$	0.01568	13.01	2.971	0.2371
6.0	1.560	0.7996	−0.1256	$0.0_{5}2782$	$-0.0_{3}2203$	$0.0_{2}8162$	12.75	2.629	0.2372
7.0	1.820	0.7971	−0.07274	$0.0_{5}1528$	$-0.0_{3}1210$	$0.0_{2}4648$	12.59	2.479	0.2372
8.0	2.080	0.7955	−0.03845	$0.0_{6}9055$	$-0.0_{4}7177$	$0.0_{2}2832$	12.48	2.403	0.2373

If $\lambda=0$, $a_1=0.79007$ and $a_2=0.073576$. The b's and c's are independent of λ.

Table 14

Values of Coefficients for Computing Buckling Loads — $n = 14$

$n = 14$ $\lambda = \pi$ $\nu = 0.30$

ξ	L/a	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3
0.7	0.1690	1.4922	−16.8938	$0.0_{2}2289$	−0.19283	6.184	108.10	2152.9	0.2565
0.8	0.1932	1.3275	−12.9170	$0.0_{2}1668$	−0.14213	4.528	78.13	1263.9	0.2609
1.0	0.2415	1.1338	−8.2404	$0.0_{3}9568$	−0.08305	2.627	49.10	520.09	0.2661
1.2	0.2898	1.0286	−5.7000	$0.0_{3}5896$	−0.05188	1.641	36.15	252.67	0.2689
1.6	0.3864	0.9239	−3.1741	$0.0_{3}2578$	−0.02307	0.7402	25.24	82.14	0.2717
2.0	0.4829	0.8755	−2.0049	$0.0_{3}1282$	−0.01159	0.3815	20.86	35.41	0.2730
2.4	0.5795	0.8492	−1.3698	$0.0_{4}7000$	$-0.0_{2}6363$	0.2160	18.65	18.57	0.2737
2.8	0.6761	0.8334	−0.9869	$0.0_{4}4105$	$-0.0_{2}3744$	0.1314	17.38	11.32	0.2742
3.2	0.7727	0.8230	−0.7383	$0.0_{4}2549$	$-0.0_{2}2330$	0.08453	16.58	7.779	0.2744
4.0	0.9659	0.8110	−0.4460	$0.0_{4}1122$	$-0.0_{2}1029$	0.03965	15.66	4.798	0.2748
5.0	1.207	0.8032	−0.2590	$0.0_{5}4829$	$-0.0_{3}4432$	0.01818	15.09	3.559	0.2750
6.0	1.449	0.7990	−0.1574	$0.0_{5}2394$	$-0.0_{3}2218$	$0.0_{2}9462$	14.78	3.104	0.2751
7.0	1.690	0.7965	−0.09609	$0.0_{5}1314$	$-0.0_{3}1208$	$0.0_{2}5378$	14.60	2.904	0.2751
8.0	1.932	0.7948	−0.05632	$0.0_{6}7788$	$-0.0_{4}7162$	$0.0_{2}3283$	14.48	2.805	0.2752

If $\lambda=0$, $a_1=0.78943$ and $a_2=0.073584$. The b's and c's are independent of λ.

Table 15

Values of Coefficients for Computing Buckling Loads — $n = 15$

$n = 15$ $\lambda = \pi$ $\nu = 0.30$

ξ	L/a	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3
0.7	0.1578	1.49031	−19.4030	0.0_219918	−0.1926	7.0940	124.016	2835.839	0.294380
0.8	0.1803	1.32592	−14.8381	0.0_214514	−0.1420	5.1947	89.6418	1664.536	0.299463
1.0	0.2254	1.13259	−9.4699	0.0_383243	−0.08296	3.0135	56.3471	684.529	0.305441
1.2	0.2704	1.02757	−6.5538	0.0_351298	−0.05183	1.8823	41.4908	332.244	0.308688
1.6	0.3606	0.92315	−3.6543	0.0_322427	−0.02305	0.8492	28.9734	107.647	0.311916
2.0	0.4507	0.87482	−2.3123	0.0_311157	−0.01158	0.4376	23.9406	46.1196	0.313411
2.4	0.5409	0.84857	−1.5833	0.0_460908	-0.0_26356	0.2479	21.4086	23.9515	0.314223
2.8	0.6310	0.83274	−1.1437	0.0_435719	-0.0_23740	0.1508	19.9506	14.4137	0.314712
3.2	0.7212	0.82246	−0.85839	0.0_422182	-0.0_22328	0.09700	19.0320	9.76469	0.315030
4.0	0.9015	0.81038	−0.52288	0.0_59769	-0.0_21028	0.04551	17.9794	5.85008	0.315403
5.0	1.127	0.80265	−0.30813	0.0_54204	-0.0_34428	0.02087	17.3216	4.22540	0.315642
6.0	1.352	0.79845	−0.19151	0.0_520846	-0.0_32197	0.01086	16.9695	3.63096	0.315772
7.0	1.578	0.79591	−0.12118	0.0_511447	-0.0_31207	0.0_26185	16.7589	3.37054	0.315851
8.0	1.803	0.79427	−0.075527	0.0_667869	-0.0_47156	0.0_23768	16.6229	3.24099	0.315901
10.0	2.254	0.79234	−0.021845	0.0_628178	-0.0_42971	0.0_21624	16.4637	3.12838	0.315961
12.0	2.704	0.79129	$+0.0_27316$	0.0_613692	-0.0_41443	0.0_38071	16.3776	3.08521	0.315994

If $\lambda = 0$, $a_1 = 0.7889$ and $a_2 = 0.07359$. The b's and c's are independent of λ.

Table 16

Values of Coefficients for Computing Buckling Loads — $n = 16$

$n = 16$ $\lambda = \pi$ $\nu = 0.30$

ξ	L/a	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3
0.7	0.1479	1.4887	−22.0856	0.0_21749	−0.1924	8.067	141.0	3670	0.3349
0.8	0.1690	1.3246	−16.8920	0.0_21274	−0.1419	5.907	101.9	2154	0.3407
1.0	0.2113	1.1316	−10.7844	0.0_37308	−0.08288	3.427	64.09	885.2	0.3475
1.2	0.2535	1.0268	−7.4667	0.0_34503	−0.05178	2.141	47.20	429.3	0.3512
1.6	0.3381	0.9225	−4.1678	0.0_31969	−0.02303	0.9658	32.96	138.7	0.3549
2.0	0.4226	0.8743	−2.6409	0.0_49794	−0.01156	0.4978	27.24	59.12	0.3566
2.4	0.5071	0.8480	−1.8115	0.0_45346	-0.0_26349	0.2820	24.36	30.46	0.3575
2.8	0.5916	0.8322	−1.3114	0.0_43135	-0.0_23754	0.1715	22.70	18.13	0.3581
3.2	0.6761	0.8220	−0.9868	0.0_41947	-0.0_22340	0.1103	21.65	12.12	0.3584
4.0	0.8452	0.8099	−0.6050	0.0_58574	-0.0_21035	0.05177	20.46	7.072	0.3589
5.0	1.056	0.8022	−0.3607	0.0_53689	-0.0_34466	0.02374	19.71	4.978	0.3591
6.0	1.268	0.7980	−0.2280	0.0_51829	-0.0_32194	0.01236	19.31	4.213	0.3593
7.0	1.479	0.7955	−0.1480	0.0_51004	-0.0_31205	0.0_27037	19.07	3.879	0.3594
8.0	1.690	0.7938	−0.09606	0.0_65951	-0.0_47140	0.0_24288	18.94	3.713	0.3594

If $\lambda = 0$, $a_1 = 0.7885$ and $a_2 = 0.07360$. The b's and c's are independent of λ.

Table 17

Values of Coefficients for Computing Buckling Loads — $n = 17$

$n = 17$ $\lambda = \pi$ $\nu = 0.30$

ξ	L/a	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3
0.7	0.1392	1.4874	−24.9387	0.0_21548	−0.1923	9.100	159.14	4675.5	0.3779
0.8	0.1591	1.3235	−19.0764	0.0_21128	−0.1418	6.664	115.05	2743.6	0.3845
1.0	0.1989	1.1308	−12.1824	0.0_36468	−0.08283	3.867	72.34	1127.3	0.3922
1.2	0.2386	1.0261	−8.4375	0.0_33986	−0.05175	2.416	53.27	546.36	0.3964
1.6	0.3182	0.9220	−4.7139	0.0_31742	−0.02302	1.090	37.21	176.11	0.4006
2.0	0.3977	0.8738	−2.9904	0.0_48669	−0.01156	0.5618	30.75	74.74	0.4025
2.4	0.4773	0.8476	−2.0542	0.0_44732	-0.0_26347	0.3182	27.50	38.24	0.4036
2.8	0.5568	0.8318	−1.4897	0.0_42775	-0.0_23734	0.1936	25.62	22.55	0.4042
3.2	0.6364	0.8216	−1.1233	0.0_41723	-0.0_22324	0.1245	24.44	14.91	0.4046
4.0	0.7954	0.8095	−0.6924	0.0_57589	-0.0_21026	0.05843	23.09	8.483	0.4051
5.0	0.9943	0.8018	−0.4166	0.0_53265	-0.0_34422	0.02680	22.25	5.824	0.4054
6.0	1.193	0.7976	−0.2668	0.0_51619	-0.0_32194	0.01395	21.80	4.855	0.4056
7.0	1.392	0.7951	−0.1765	0.0_68890	-0.0_31205	0.0_27944	21.52	4.432	0.4057
8.0	1.591	0.7935	−0.1179	0.0_65269	-0.0_47148	0.0_24840	21.35	4.223	0.4058

If $\lambda = 0$, $a_1 = 0.7881$ and $a_2 = 0.07360$. The b's and c's are independent of λ.

Table 18

Values of Coefficients for Computing Buckling Loads — $n = 18$

$n = 18$ $\lambda = \pi$ $\nu = 0.30$

ξ	L/a	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3
0.7	0.1315	1.4864	−27.967	0.0_21379	−0.1922	10.199	178.35	5875.0	0.4236
0.8	0.1502	1.3227	−21.395	0.0_21005	−0.1417	7.469	128.94	3447.0	0.4310
1.0	0.1878	1.1301	−13.666	0.0_35765	−0.08279	4.334	81.08	1415.8	0.4397
1.2	0.2254	1.0255	−9.468	0.0_33552	−0.05171	2.707	59.71	685.86	0.4444
1.6	0.3005	0.9215	−5.294	0.0_31553	−0.02300	1.222	41.71	220.64	0.4491
2.0	0.3756	0.8734	−3.361	0.0_47727	−0.01155	0.6296	34.47	93.29	0.4513
2.4	0.4507	0.8473	−2.312	0.0_44218	-0.0_26342	0.3567	30.82	47.45	0.4525
2.8	0.5259	0.8315	−1.679	0.0_42473	-0.0_23732	0.2170	28.73	27.75	0.4532
3.2	0.6010	0.8212	−1.2682	0.0_41536	-0.0_22325	0.1396	27.40	18.16	0.4536
4.0	0.7512	0.8092	−0.7851	0.0_56764	-0.0_21026	0.06551	25.89	10.10	0.4542
5.0	0.9391	0.8015	−0.4760	0.0_52910	-0.0_34417	0.03005	24.94	6.773	0.4545
6.0	1.127	0.7973	−0.3081	0.0_51443	-0.0_32192	0.01564	24.44	5.560	0.4547
7.0	1.315	0.7948	−0.2068	0.0_67923	-0.0_31204	0.0_28891	24.13	5.033	0.4548
8.0	1.502	0.7932	−0.1411	0.0_64089	-0.0_47137	0.0_25428	23.94	4.772	0.4549

If $\lambda = 0$, $a_1 = 0.7878$ and $a_2 = 0.07360$. The b's and c's are independent of λ.

Table 19

Values of Coefficients for Computing Buckling Loads — $n = 19$

$n = 19$ $\lambda = \pi$ $\nu = 0.30$

ξ	L/a	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3
0.7	0.1246	1.4854	−31.1693	$0.0_{2}1237$	−0.1921	11.360	198.65	7292.1	0.4719
0.8	0.1423	1.3219	−23.8467	$0.0_{3}9015$	−0.1416	8.319	143.63	4278.0	0.4802
1.0	0.1779	1.1295	−15.235	$0.0_{3}5171$	−0.08275	4.827	90.32	1756.7	0.4899
1.2	0.2135	1.0250	−10.5576	$0.0_{3}3186$	−0.05169	3.016	66.52	850.6	0.4951
1.6	0.2847	0.9212	−5.9065	$0.0_{3}1393$	−0.02299	1.361	46.47	273.2	0.5004
2.0	0.3559	0.8731	−4.1669	$0.0_{4}6930$	−0.01155	0.7014	38.40	115.0	0.5028
2.4	0.4270	0.8470	−3.7536	$0.0_{4}3783$	$-0.0_{2}6339$	0.3973	34.34	58.27	0.5041
2.8	0.4982	0.8312	−2.5842	$0.0_{4}2219$	$-0.0_{2}3731$	0.2417	32.00	33.84	0.5049
3.2	0.5694	0.8210	−1.4214	$0.0_{4}1378$	$-0.0_{2}2321$	0.1555	30.53	21.94	0.5054
4.0	0.7117	0.8087	−0.8832	$0.0_{5}6067$	$-0.0_{2}1025$	0.07298	28.84	11.96	0.5060
5.0	0.8896	0.8010	−0.5388	$0.0_{5}2611$	$-0.0_{3}4417$	0.03348	27.79	7.833	0.5064
6.0	1.068	0.7968	−0.3516	$0.0_{5}1294$	$-0.0_{3}2192$	0.01742	27.22	6.334	0.5066
7.0	1.246	0.7943	−0.2388	$0.0_{6}7109$	$-0.0_{3}1204$	$0.0_{2}9905$	26.89	5.682	0.5067
8.0	1.423	0.7927	−0.1556	$0.0_{6}4214$	$-0.0_{4}7141$	$0.0_{2}6047$	26.67	5.361	0.5068

If $\lambda = 0$, $a_1 = 0.7876$ and $a_2 = 0.07361$. The b's and c's are independent of λ.

Table 20

Values of Coefficients for Computing Buckling Loads — $n = 20$

$n = 20$ $\lambda = \pi$ $\nu = 0.30$

ξ	L/a	a_1	a_2	b_1	b_2	b_3	c_1	c_2	c_3
0.7	0.1183	1.4847	−34.5428	$0.0_{2}1116$	−0.1920	12.584	220.06	8950.8	0.52288
0.8	0.1352	1.3213	−26.4295	$0.0_{3}8132$	−0.1416	9.215	195.11	5250.7	0.53203
1.0	0.1690	1.1291	−16.8884	$0.0_{3}4664$	−0.08272	5.347	100.06	2155.6	0.54278
1.2	0.2028	1.02464	−11.7056	$0.0_{3}2874$	−0.05167	3.340	73.70	1043.3	0.54862
1.6	0.2704	0.9208	−6.5522	$0.0_{3}1256$	−0.02298	1.507	51.48	334.6	0.55443
2.0	0.3381	0.8728	−4.1669	$0.0_{4}6251$	−0.01154	0.7770	42.55	140.6	0.55712
2.4	0.4057	0.8467	−2.8712	$0.0_{4}3412$	$-0.0_{2}6336$	0.4402	38.05	70.87	0.55858
2.8	0.4733	0.8309	−2.0899	$0.0_{4}2001$	$-0.0_{2}3692$	0.2678	35.46	40.90	0.55946
3.2	0.5409	0.8207	−1.5828	$0.0_{4}1243$	$-0.0_{2}2321$	0.1723	33.83	26.31	0.56003
4.0	0.6761	0.8087	−0.9865	$0.0_{5}5472$	$-0.0_{2}1025$	0.08085	31.96	14.07	0.56070
5.0	0.8452	0.8010	−0.6049	$0.0_{5}2354$	$-0.0_{3}4415$	0.03709	30.79	9.014	0.56114
6.0	1.014	0.7969	−0.3976	$0.0_{5}1167$	$-0.0_{3}2191$	0.01930	30.17	7.179	0.56137
7.0	1.183	0.7943	−0.2726	$0.0_{6}6409$	$-0.0_{3}1203$	0.01097	29.79	6.383	0.56151
8.0	1.352	0.7927	−0.1914	$0.0_{6}3798$	$-0.0_{4}7135$	$0.0_{2}6700$	29.55	5.991	0.56160
10.0	1.690	0.7908	−0.09601	$0.0_{6}1576$	$-0.0_{4}2963$	$0.0_{2}2887$	29.27	5.655	0.56171
12.0	2.028	0.7897	−0.04418	$0.0_{7}7653$	$-0.0_{4}1439$	$0.0_{2}1435$	29.12	5.530	0.56177

If $\lambda = 0$, $a_1 = 0.7874$ and $a_2 = 0.07361$. The b's and c's are independent of λ.

Table 21

Coefficients K_{st}, K_i, and K_{vm}, ($a/h = 1000$)

		$\lambda = 0$		$\lambda = \pi$		$\lambda = 0$
L/a	n	$K_{st} \times 10^6$	$K_i \times 10^6$	$K_{st} \times 10^6$	$K_i \times 10^6$	$K_{vm} \times 10^6$ (Von Mises Formula)
0.4057	20	63.38	91.66	57.50	85.24	88.66
0.4982	19	50.89	68.80	47.05	65.19	
0.5259	18	47.83	67.86	44.51	64.29	65.93
0.6364	17	39.33	52.87	37.21	50.05	
0.6761	16	36.61	52.15	34.64	50.02	50.70
0.9659	14	25.71	34.05	24.82	33.14	
1.04	13	23.88	33.59	23.06	32.70	
1.127	12	22.71	33.95	21.92	33.05	
1.409	12	17.77	22.37	17.38	21.98	21.77
1.537	11	16.20	21.71	15.85	21.33	
1.690	10	15.07	21.76	14.73	21.38	
2.028	10	12.34	15.48	12.15	15.29	
2.254	9	11.17	15.08	11.00	14.89	
2.629	9	9.59	11.65	9.49	11.54	11.33
2.958	8	8.576	11.130	8.480	11.03	
3.381	8	7.493	8.983	7.427	8.920	
3.864	7	6.547	8.532	6.489	8.472	
5.634	6	4.466	5.543	4.442	5.517	5.356
6.761	5	3.963	5.569	3.941	5.543	
8.113	5	3.102	3.843	3.089	3.830	
10.142	5	2.611	2.882	2.605	2.876	
12.68	4	1.967	2.445	1.962	2.439	1.973
16.90	4	1.597	1.715	1.595	1.713	
22.538	3	1.094	1.372	1.092	1.370	
28.172	3	0.9090	0.9958	0.9080	0.9950	0.9728
33.81	3	0.8372	0.8600	0.8366	0.8595	
50.71	2	0.4479	0.6079	0.4475	0.6074	
67.61	2	0.3566	0.3795	0.3564	0.3793	0.3722
84.52	2	0.3172	0.3173	0.3172	0.3172	

Table 22

Coefficients K_{st}, K_i, and K_{vm}, (a/h = 100)

	$\lambda = 0$				$\lambda = \pi$				$\lambda = 0$
L/a	n	$K_{st} \times 10^6$	n	$K_i \times 10^6$	n	$K_{st} \times 10^6$	n	$K_i \times 10^6$	$K_{vm} \times 10^6$
0.1690							20	9273	
0.1779			19	12120			19	8454	
0.1803					15	7577.6			
0.1878			18	11020			18	7684	
0.1932	14	12000			14	6707			
0.1989			17	10000			17	6969	
0.2113			16	9055			16	6310	5344
0.2415			14	7432			14	5175	4544
0.2600	13	6484			13	4161			
0.2817	12	5731			12	3651			
0.3121			13	4813			13	3693	3530
0.3688	11	3484			11	2473			
0.4917			11	2500			11	2134	2185
0.5409	10	1905	10	2268	10	1544	10	1935	
0.6761	8	1402	10	1663	8	1132	10	1498	1548
0.7512			9	1481			9	1334	
0.8452	8	1041			8	904.4			
0.9659	7	882.0			7	765.1			
1.014			8	1039			8	964.0	
1.159	7	712.0			7	644.1			
1.352	6	605.5	7	753.8	6	547.2	7	712.9	
1.545			7	648.5			7	621.3	634.7
1.578	6	506.0			6	469.5			
1.803	6	451.7	6	557.4			6	533.6	
1.893	5	425.5			5	394.0			
2.164	5	361.5			5	340.9			
2.254			6	435.6			6	423.5	
2.704	5	297.0	5	360.0	5	286.1	5	349.8	
3.381	4	230.4	5	285.2	4	221.6	5	279.9	281.1
4.226	4	186.4	4	226.7	4	181.9	4	222.3	
5.071	4	167.1	4	183.6	4	164.4	4	181.1	180.6
5.634	3	140.5			3	136.8			
5.916			4	163.9			4	162.3	
6.761	3	113.4	3	151.0	3	111.4	3	148.8	
7.888	3	98.93	3	116.6	3	97.62	3	115.3	
9.015	3	91.13	3	99.41	3	90.23	3	98.59	97.46
11.269	3	83.28	3	84.57			3	84.13	
11.832	2	71.08			2	69.81			
13.522	2	56.08			2	55.32			
16.903	2	42.96	2	54.62	2	42.58	2	54.23	
20.284	2	37.22	2	40.76	2	37.00	2	40.56	
25.355	2	32.89	2	33.02	2	32.79	2	32.91	32.59
33.806			2	29.28			2	29.22	

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