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# MOMENTS IN TWO-WAY CONCRETE FLOOR SLABS

BY

CHESTER P. SIESS

AND

NATHAN M. NEWMARK

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# MOMENTS IN TWO-WAY CONCRETE FLOOR SLABS

A REPORT OF AN INVESTIGATION

CONDUCTED BY

THE ENGINEERING EXPERIMENT STATION
UNIVERSITY OF ILLINOIS

BY

CHESTER P. SIESS
Research Assistant Professor of Civil Engineering

AND

NATHAN M. NEWMARK
Research Professor of Structural Engineering

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ABSTRACT

The development of an approximate moment-distribution procedure for the analysis of uniformly loaded plates continuous in two directions over rigid beams is described in detail. The procedure is then used to study the effects of several important variables on the moments in two-way slabs.

The distribution procedure is analogous to the Cross method for continuous beams and frames except that it is not exact. Approximate numerical coefficients for fixed-edge moments, stiffnesses, and carry-over factors are derived and presented. Comparisons are made between moments computed by means of the approximate procedure and those obtained from more rigorous solutions.

Moments in a number of continuous slabs are computed by the distribution procedure. Variables studied include 1) the effect of discontinuous edges; 2) torsional restraint of supporting beams; 3) type of loading; 4) ratio of sides of panels; and 5) combinations of panels of various sizes and shapes. The moments obtained from these analyses are presented in full, in tabular form. Conclusions are drawn regarding the effects of the several variables, and recommendations are made regarding the types of loading and values of beam torsional stiffness which should be considered in the development of a design procedure for two-way concrete floor slabs.

Exact solutions for moments in rectangular plates with various edge conditions are presented in three appendixes.

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## I. Introduction

### 1. *Object and Scope*

The studies reported in this bulletin were undertaken in an attempt to obtain a more rational basis for the design of the type of building floor construction known as the two-way slab. This type of structure consists of a reinforced concrete slab supported either by reinforced concrete beams or by rolled steel beams arranged in a rectangular grid pattern. The beams are in turn supported on columns at the intersections of the grid.

The data presented herein are entirely analytical in nature and represent two steps toward the ultimate objective of a design procedure: 1) the development of a simple, rapid, numerical procedure for the calculation of moments in uniformly loaded continuous plates; 2) the application of this method of calculation to the study of moments in typical two-way slab structures.

The numerical method of analysis for continuous slabs is a moment distribution procedure, generally analogous to the Hardy Cross method for continuous beams and frames. It provides a means for computing the average or total moment on the edges of a uniformly loaded rectangular plate supported on all four sides and continuous in any directions with other rectangular plates which need not be of the same size or shape. The average positive moments on sections in the interior of the plate may also be obtained. Although it must be assumed that the beams supporting the edges of the plates do not deflect vertically, the distribution procedure may be used to consider the torsional restraint offered by monolithically cast reinforced concrete beams or by monolithically cast concrete fire-proofing encasement around steel beams.

The relative simplicity of the distribution procedure made it possible to undertake a large number of analyses and to investigate the effects of the following variables on the moments in two-way slabs: (a) effects of discontinuous edges, (b) various ratios of beam torsional stiffness to slab flexural stiffness, (c) three patterns of loading including all panels loaded and two types of partial loading, (d) ratios of sides of panels, and (e) combinations of panels of various span lengths and ratios of sides.

The actual development of a design procedure, based on the results obtained, is not presented in this bulletin. It is a separate problem and one that requires a number of assumptions regarding the behavior of reinforced concrete slabs. Moreover, many of these assumptions are associated with current concepts of design and with the values of moment coefficients and allowable stresses specified in current building codes and specifications.

The results presented in this bulletin, however, have been used in the development of a design procedure for two-way reinforced concrete slabs, which has been described elsewhere by the writers.[1]

2. *Outline of Bulletin*

The study of moments in two-way slabs has been approached in two distinct steps:

(a) The development of a distribution procedure for the calculations of moments in continuous plates.

(b) The use of this procedure to study the effects of several important variables on the moments in two-way slabs.

A brief outline of the remaining chapters of this bulletin is presented below:

Chapter II contains a complete description of the distribution procedure, tables and graphs of the various constants, and an example illustrating the use of the method.

Chapter III describes the manner in which the distribution procedure was developed and the methods used to obtain the numerical values of the various constants. Comparisons of the results obtained by the distribution procedure with those obtained by more exact methods are also included.

Chapter IV contains the results of the analyses of two-way slabs by means of the distribution procedure. Conclusions are drawn regarding the effects of several variables. Two types of slabs were studied: one type consisted of twenty-five panels of equal size and shape arranged in rows of five in each direction, while the other consisted of sixteen panels of two different sizes and shapes.

Chapter V is a summary.

Supplementary data in the form of exact solutions for moments in various types of slabs are presented in three appendixes.

Appendix A contains the results of a number of exact solutions for moments in uniformly loaded rectangular plates with various edge conditions. These moments, in all cases for a single isolated panel, were used extensively in the derivation of the distribution constants and in checking the distribution procedure.

Appendix B contains exact solutions for moments in uniformly loaded continuous slabs having several panels. These solutions furnished the principal basis for checking the accuracy of the distribution procedure.

[1] C. P. Siess and N. M. Newmark, "Rational Analysis and Design of Two-Way Concrete Slabs," *Jour. American Concrete Institute*, Vol. 20, No. 3, December 1948, pp. 273-315.

Appendix C contains the results of exact solutions for moments in rectangular plates with concentrated loads. These moments were used to determine the applicability of the distribution procedure to plates with concentrated loads.

The letter symbols used herein are defined where they first appear, and are assembled for convenience of reference in Appendix D.

3. *Basic Assumptions*

The formal distribution procedure for the analysis of continuous slabs is based upon the exact solutions for moments in rectangular plates presented in Appendixes A and B. These solutions were obtained by means of the ordinary theory of flexure for plates, and involve the following assumptions:

(a) The material in the slab is homogeneous, elastic, isotropic, and of constant thickness in each panel.

(b) The resultant of the normal stress acting on any cross-section of the slab is a pure couple.

(c) Flexural strains vary linearly through the depth of the slab.

(d) The beams exert only vertical forces on the slab; there is no shear between the top flange of the beam and the bottom of the slab.

(e) The reaction of the beam acts on the slab along a line and is not distributed over a finite width. This reaction, however, may include a moment or couple representing the torsional restraint offered by the beam.

(f) The supporting beams do not deflect.

In all calculations in this bulletin the value of Poisson's ratio is taken as zero. There is good reason for assuming such a value if the results are to be applied to reinforced concrete slabs, particularly if the conditions existing in the slab after cracking are to be considered. For other applications of the distribution procedure, however, it may be desirable to consider other values of Poisson's ratio. The moments on the edges of rectangular plates supported without deflection on all four sides are independent of the value of Poisson's ratio. The moments in the interior of a panel of such plates are not independent of Poisson's ratio, $\mu$, but may be determined from the moments for $\mu = 0$ by means of the following expressions.

$$M_x^{\mu} = M_x^{\mu=0} + \mu M_y^{\mu=0}$$

$$M_y^{\mu} = M_y^{\mu=0} + \mu M_x^{\mu=0}$$

wherein $M_x$ and $M_y$ are moments per unit of width in the direction of $x$ or $y$, respectively, and the superscripts indicate the value of $\mu$ considered.

The distribution procedure, though based on the exact moments for single panels, is nevertheless approximate. In addition to the assumptions listed above, it was necessary in the derivation of the distribution procedure to make additional assumptions and approximations regarding the relations between average rotations and average moments within a panel, and between average rotations on the common edge of adjoining panels.

None of the assumptions listed above, except possibly (c), are satisfied completely for a reinforced concrete slab supported on monolithically cast reinforced concrete beams. Consequently, the moments obtained by means of the distribution procedure require some modification before they may be used in the design of two-way slabs. Deflection of the beams, for example, changes both the magnitude and the distribution of moments in the slab from those computed by means of the distribution procedure. This problem, however, is outside the scope of this bulletin.

4. *Acknowledgment*

This bulletin is based upon a thesis by C. P. Siess submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Engineering in the Graduate College of the University of Illinois, 1948. The thesis was written under the direction of Professor N. M. Newmark, in the Department of Civil Engineering of which Professor W. C. Huntington is head.

## II. Distribution Procedure for the Computation of Moments in Plates Continuous in Two Directions

### 5. *General Description of Procedure*

The distribution procedure described in this chapter is applicable to the calculation of average moments in uniformly loaded rectangular elastic plates continuous in two directions over nondeflecting supports. The procedure is strictly analogous to the Cross[1] moment distribution method for the analysis of continuous beams and frames; that is, the procedure is one in which fixed-edge moments are calculated, unbalanced moments are distributed in proportion to the relative stiffness of the elements of the structure, and portions of the distributed moments are "carried over" to the other edges of the panel. The method as applied to plates, however, is not exact, since the values of the stiffness and carry-over factors may be determined only approximately for certain assumed conditions.

Since the moments on the edge of a continuous plate are distributed in some non-uniform manner across the width of the plate, it is necessary to decide what value of moment is to be dealt with in the distribution procedure. Several possibilities present themselves: moment at the middle of the edge, maximum moment, total moment, or average moment. From the standpoint of the physical interpretation of the distribution procedure, the use of total moment is probably to be desired. However, the relation between the shape of the panel and the carry-over factors is somewhat simpler if the average moment is used instead. Since the average moment is simply the total moment divided by the width of the panel, no particular difficulties of interpretation are introduced by this substitution.

The first step in the application of the distribution procedure is the determination of the average moments on the edges of each loaded panel, for all edges considered fixed. Coefficients for these moments, as well as numerical values of the other distribution constants, are given in the following section.

In general, the fixed-edge moments at the junction of two panels of the slab along the line of a support will be different for the two panels; that is, the moments at an edge will not be statically balanced. The unbalanced moment is equal to the algebraic difference in moments for the two panels. This unbalanced moment is distributed to the two panels in such a manner as to equalize the moments on the two sides of the edge, both in magnitude and in sign. If torsional

[1] See, for example, Hardy Cross, "Analysis of Continuous Frames by Distributing Fixed-End Moments," *Trans. ASCE*, Vol. 96 (1932), pp. 1-156; or Hardy Cross and Newlin D. Morgan, "Continuous Frames of Reinforced Concrete" (John Wiley and Sons, New York, 1932), Chapter IV.

rigidity of the supporting beams is considered, the moments in the beams may be handled in the same way as moments in columns of a continuous frame. The unbalanced moments at each edge are distributed to the adjacent panels in proportion to the relative values of the panel *stiffness factors*, $K$, as defined in the following section.

The distribution of unbalanced moments may be visualized as the releasing of restraints on an edge which was previously held fixed. Distribution to the various elements in proportion to their stiffness satisfies the condition that the slopes resulting from the distributed moment must be equal on both sides of a "joint." In this method, it is the "average" slopes which are made equal, instead of the slopes at every point along the edge, and therein lies one of the basic approximations of the distribution procedure.

The redistribution of moments occurring when one edge of a panel is released introduces additional unbalanced moments at the other edges of that panel. These moments are said to be "carried over" from the released edge, and their magnitude is determined as the product of the distributed moment and certain *carry-over factors*, $C$, which are functions of the ratio of sides, $b/a$, for the panel. Four carry-over factors are needed, two for the long edges and two for the short edges of a panel; of each pair, one determines the moment carried over to an opposite edge, the other the moment to an adjacent edge. Numerical values of these constants are given in the following section.

As in the Cross method, the two operations of distributing moments and carrying over are performed at each joint in turn, or at all joints simultaneously, as the analyst prefers, and are then repeated as many times as necessary until the moments on the two sides of an edge are balanced to the degree of accuracy desired. When this has been accomplished, the average moments on the edges of the continuous panels may be determined by adding algebraically the fixed-edge moments, the distributed moments, and the "carried-over" moments at each location.

The positive moments in the interior of a panel may be obtained as the summation of the following quantities:

(1) the moments due to the load acting on the panel simply supported on all edges,

(2) the moments produced in the interior of the panel by the moments acting on each edge.

The values of the moments in a simply-supported slab with uniform load are given in Section 6. The average interior moments produced in each direction by the edge moments on each edge are determined by means of *positive moment correction factors*, $F$, as described in the following section. The results given by this method of computing the positive moments in a panel are far from exact, but it is believed that the values obtained are conservative in all cases, without being too far on the side of safety. The principal reason for the relative crudeness of the calculation is that the location of the section on which the maximum positive moment occurs varies considerably with the edge conditions of a panel. Consequently it has been necessary to choose values of the positive moment correction factors, $F$, which will give a reasonably correct answer for any condition. This subject is discussed further in Section 17.

6. *Numerical Values of Constants*

The numerical values of the moment coefficients and distribution constants referred to in the preceding section are given herein. The manner in which these values were obtained is discussed in detail in Chapter III.

It should be pointed out that the identifying subscripts $a$ and $b$, which are used frequently in this section, refer at all times to an edge or to a section parallel to an edge having a length of $a$ or $b$, respectively, where $b$ is the shorter span. Thus the expression $M_b$ refers to the moment on the short edge of the panel, or on a section parallel to the short edge, and not to the moment in the short span of length $b$.

All the moment coefficients and distribution constants are tabulated and plotted as functions of the ratio of sides, $b/a$. Although in some cases values are given for the full range of $b/a$ from zero to one, the user is warned that the accuracy of the distribution procedure has not been verified for $b/a$ less than 0.5.

*Fixed-Edge Moments.*—The coefficients of average edge moments for panels fixed on all sides are given in Table 1 and plotted in Fig. 1. In both cases the values are coefficients of $wb^2$, wherein $w$ is the magnitude of the uniformly distributed load per unit of area and $b$ is the length of the short side. These moments are exact and were determined either directly or by interpolation from the values given in Section 31, Appendix A.

*Moments in Simply-Supported Slab.*—The coefficients of average positive moments in simply-supported slabs are given in Table 1 and in Fig. 2. As for the fixed-edge moments, the values given are coefficients of $wb^2$. The moments considered are the maximum in each direction. In the short span, the maximum occurs on the section at midspan. In the long span, the maximum occurs at midspan

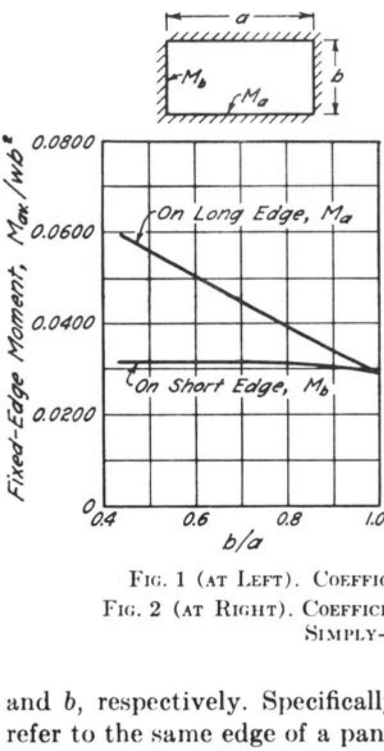
TABLE 1
NUMERICAL VALUES OF CONSTANTS FOR USE IN DISTRIBUTION PROCEDURE

| Constant | b/a | | | | | |
|---|---|---|---|---|---|---|
| | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| **Average Fixed-Edge Moment:** | | | | | | |
| On long edge, $M_a/wb^2$ | −0.0556 | −0.0503 | −0.0447 | −0.0389 | −0.0337 | −0.0290 |
| On short edge, $M_b/wb^2$ | −0.0315 | −0.0315 | −0.0315 | −0.0311 | −0.0302 | −0.0290 |
| **Average Simple-Span Moment:** | | | | | | |
| In short span, $M_a/wb^2$ | 0.0651 | 0.0543 | 0.0444 | 0.0362 | 0.0293 | 0.0236 |
| In long span, $M_b/wb^2$ | 0.0164 | 0.0175 | 0.0193 | 0.0213 | 0.0229 | 0.0236 |
| **Stiffness Factors:** | | | | | | |
| On long edge, $k_a$ | 5.00 | 5.44 | 5.96 | 6.56 | 7.24 | 8.00 |
| On short edge, $k_b$ | 7.50 | 7.60 | 7.70 | 7.80 | 7.90 | 8.00 |
| **Carry-over Factors:** | | | | | | |
| Long edge to long edge, $C_{aa}$ | −0.30 | −0.24 | −0.18 | −0.12 | −0.06 | 0 |
| Long edge to short edge, $C_{ab}$ | −0.30 | −0.30 | −0.30 | −0.30 | −0.30 | −0.30 |
| Short edge to short edge, $C_{bb}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| Short edge to long edge, $C_{ba}$ | −0.15 | −0.18 | −0.21 | −0.24 | −0.27 | −0.30 |
| **Positive Moment Correction Factors:** | | | | | | |
| Moment on long edge, $F_{aa}$ | 0.250 | 0.210 | 0.170 | 0.130 | 0.090 | 0.050 |
| $F_{ab}$ | 0.050 | 0.070 | 0.090 | 0.110 | 0.130 | 0.150 |
| Moment on short edge, $F_{bb}$ | 0.025 | 0.030 | 0.035 | 0.040 | 0.045 | 0.050 |
| $F_{ba}$ | 0.150 | 0.150 | 0.150 | 0.150 | 0.150 | 0.150 |

for values of $b/a$ greater than about 0.75; for smaller values the maximum moves towards the ends of the span, and for $b/a = 0.5$ or less it occurs at a distance of approximately $0.33b$ from the short edge.

The positive moments given in Fig. 2 and Table 1 are exact values determined from the data and sources described in Section 26, Appendix A.

*Stiffness Factors.*—The stiffnesses $K_a$ and $K_b$ represent the relation between the average moment and the average rotation on edges $a$



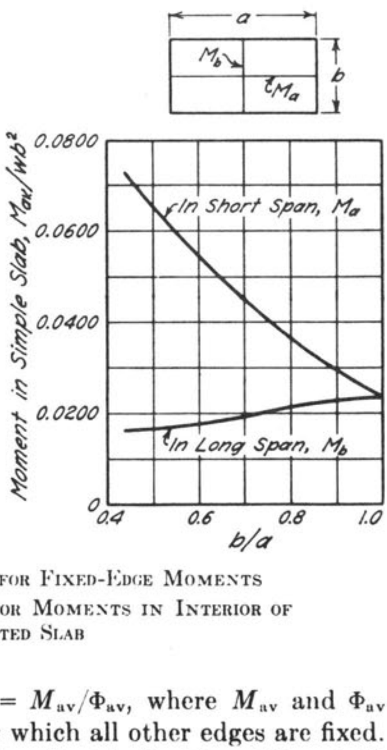




FIG. 1 (AT LEFT). COEFFICIENTS FOR FIXED-EDGE MOMENTS
FIG. 2 (AT RIGHT). COEFFICIENTS FOR MOMENTS IN INTERIOR OF SIMPLY-SUPPORTED SLAB

and $b$, respectively. Specifically, $K = M_{av}/\Phi_{av}$, where $M_{av}$ and $\Phi_{av}$ refer to the same edge of a panel, for which all other edges are fixed. The derivation of these factors is described in Sections 12 and 14 of the following chapter.

The stiffnesses for a given value of $b/a$ are determined from the expressions

$$K_a = k_a \frac{N}{b}$$

$$K_b = k_b \frac{N}{b}$$

in which $N = Et^3/12$, the stiffness of an element of the slab, and $k_a$ and $k_b$ are coefficients given in Table 1 and Fig. 3 as functions of $b/a$.

The coefficients, $k$, vary with the shape of the panel in such a manner that the stiffness factors, $K$, for adjacent panels having different span lengths do not vary as much as would the corresponding stiffnesses for a continuous beam. This is illustrated by Fig. 4, in which the stiffnesses, $K$, are plotted for panels of various dimensions. The portion of the curve on the right represents the stiffness, $K_b$, on

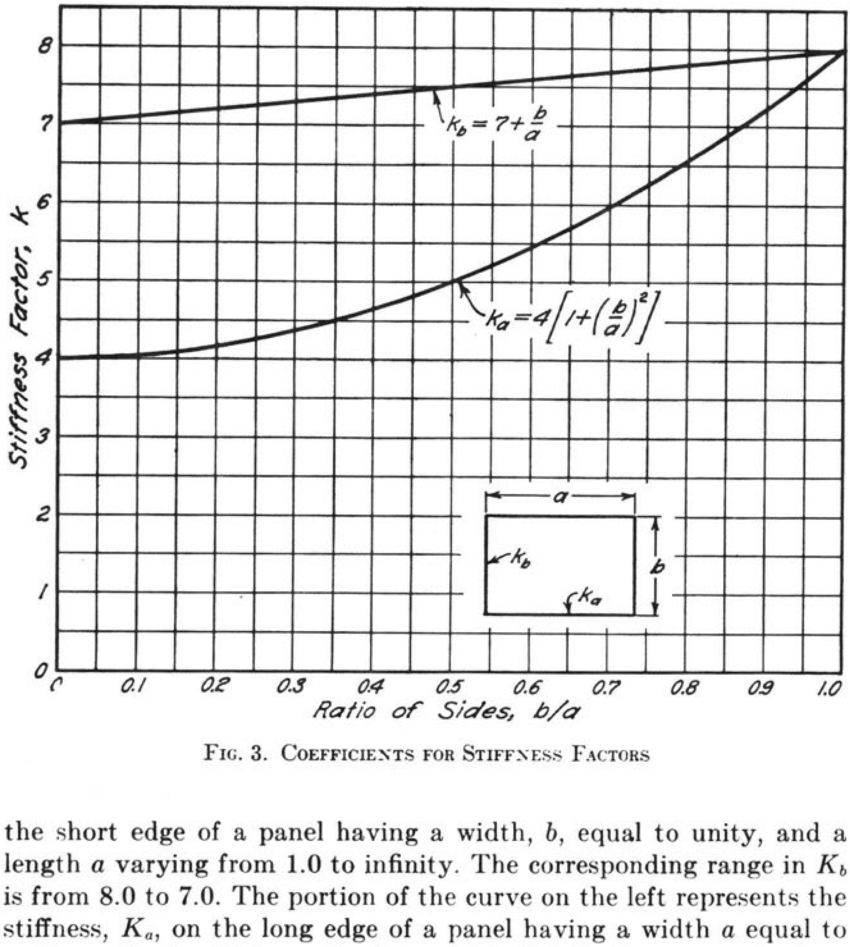



FIG. 3. COEFFICIENTS FOR STIFFNESS FACTORS

the short edge of a panel having a width, $b$, equal to unity, and a length $a$ varying from 1.0 to infinity. The corresponding range in $K_b$ is from 8.0 to 7.0. The portion of the curve on the left represents the stiffness, $K_a$, on the long edge of a panel having a width $a$ equal to unity, and a length $b$ varying from 1.0 to 0.4. In the range of values from $b = 0.5$ to 1.0, the value of $K_a$ ranges from 10.0 to 8.0, a relatively small variation. For smaller values of $b$, however, the stiffness approaches that for a beam and consequently varies nearly inversely as the span length.

*Carry-Over Factors.*—Consider a rectangular panel having three edges fixed and the other simply supported. If a moment is applied to the simply-supported edge, the moments induced on each of the fixed edges may be expressed as the product of the applied moment and the appropriate carry-over factor. Four such factors are required to define the behavior of the slab. For a moment applied on a short edge, $b$, the carry-over factors are $C_{bb}$ for the opposite short edge, and $C_{ba}$ for the adjacent long edges. Similarly, for a moment applied on a long edge, $a$, the factors are $C_{aa}$ for the opposite long edge, and $C_{ab}$ for the adjacent short edges. In each case the first letter of the subscript refers to the edge on which the moment is applied, while the second letter refers to the edge to which the moment is "carried over." The system of notation is further illustrated in Fig. 5.

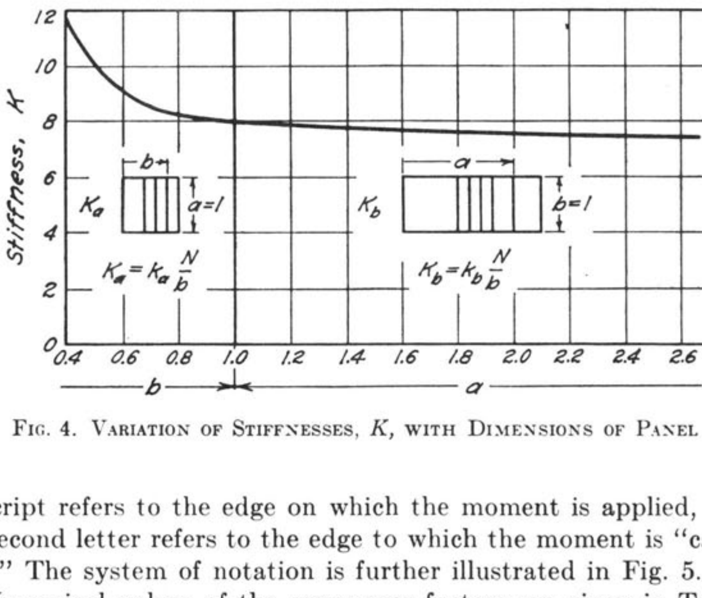



FIG. 4. VARIATION OF STIFFNESSES, $K$, WITH DIMENSIONS OF PANEL

Numerical values of the carry-over factors are given in Table 1 and Fig. 5. All the values are negative, in accordance with the sign convention of moment adopted for use with the distribution procedure: positive moment produces compression at the top of the slab. The carry-over factors are approximate in nature; their derivation is described in Sections 13 and 14.

*Positive Moment Correction Factors.*—The positive moment correction factors may be defined as the average moments produced at certain sections in the interior of a simply-supported rectangular panel by the application of a unit average moment to one edge of the panel. Numerical values are given in Table 1 and Fig. 6. The notation used is illustrated in the figure, and may be further explained as follows: If a moment is applied on a short edge, $b$, the positive moment correction factor applying to a section parallel to that edge is $F_{bb}$, and the factor for a section perpendicular to that edge is $F_{ba}$. Similarly, if the moment is applied on a long edge, $a$, the factors are designated $F_{aa}$ and $F_{ab}$ for the directions parallel and perpendicular to that edge, respectively.

The derivation of the positive moment correction factors is described in Section 17 of the following chapter. These constants are

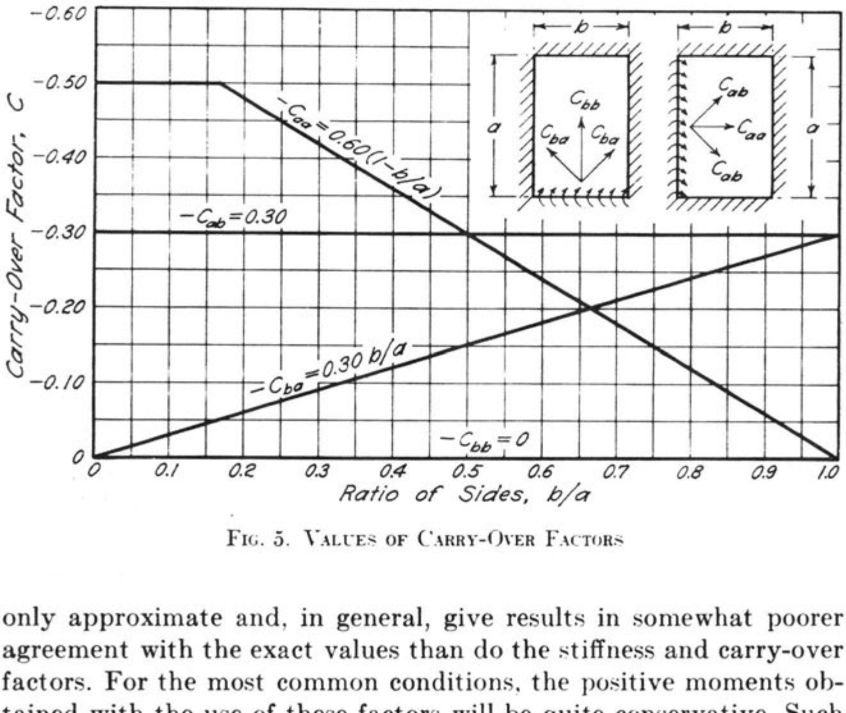



FIG. 5. VALUES OF CARRY-OVER FACTORS

only approximate and, in general, give results in somewhat poorer agreement with the exact values than do the stiffness and carry-over factors. For the most common conditions, the positive moments obtained with the use of these factors will be quite conservative. Such conservatism is necessary in order to take account of the fact that the location of the maximum positive moments in a continuous panel is dependent upon the edge conditions.

## 7. *Modified Stiffness and Carry-Over Factors*

The values given in Section 6 for the stiffness and carry-over factors are for a panel with all edges fixed except the one to which the constants apply. If one or more of these edges are simply supported, or if certain conditions of symmetry exist, modified values

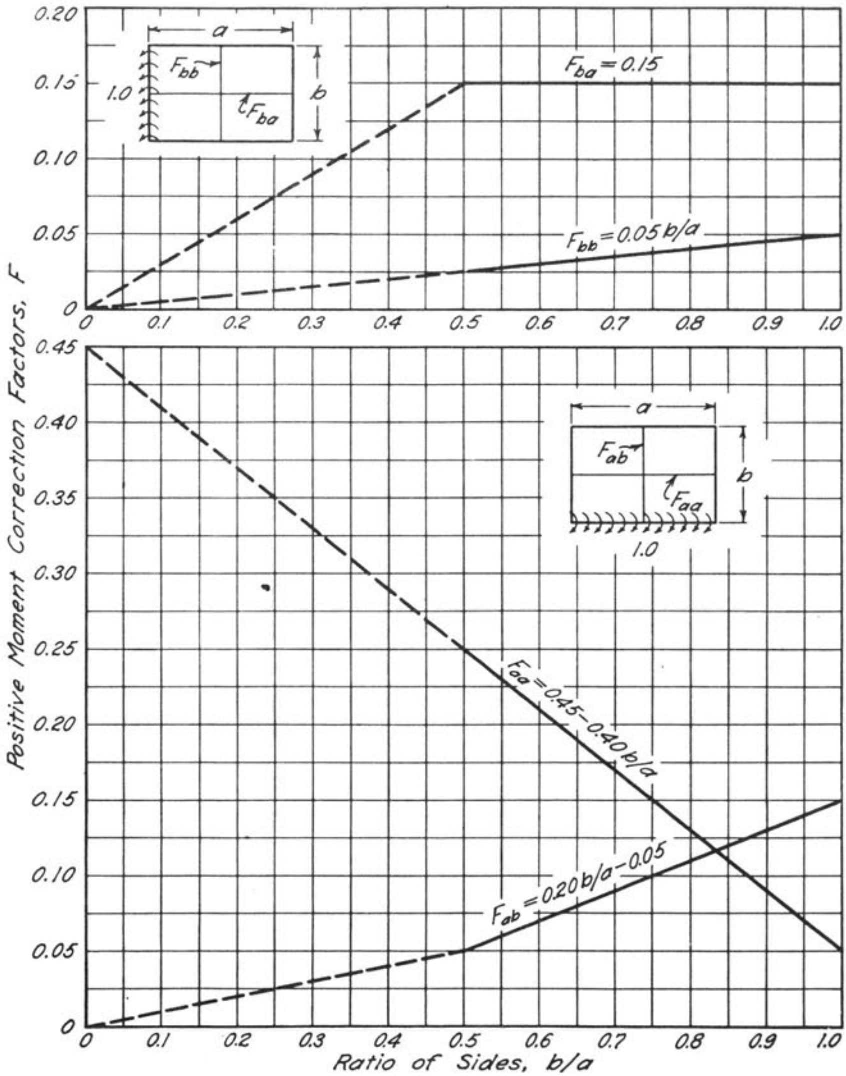



FIG. 6. VALUES OF POSITIVE MOMENT CORRECTION FACTORS

of the distribution constants may be used in order to simplify the numerical work involved.

Expressions for modified stiffness and carry-over factors for a number of common cases have been derived by the method described in Section 16. The cases considered, and the notation used, are illustrated in Fig. 7. The algebraic expressions are given below for the constants on a short edge, $b$. Corresponding expressions applying to a long edge may be obtained by simply interchanging the subscripts $a$ and $b$ in each case.

*Case 0. Basic panel:*

Stiffness factor: $k_b$

Carry-over factors: $C_{bb}$ and $C_{ba}$

*Case 1. Opposite edge simply-supported:*

$$k'_b = k_b(1 - C_{bb}^2)$$

$$C'_{ba} = \frac{C_{ba}}{1 + C_{bb}}$$

*Case 2. Adjacent edge simply-supported:*

$$k''_b = k_b(1 - C_{ab}C_{ba})$$

$$C''_{bb} = \frac{C_{bb} - C_{ab}C_{ba}}{1 - C_{ab}C_{ba}}$$

$$C''_{ba} = C_{ba} \cdot \frac{1 - C_{aa}}{1 - C_{ab}C_{ba}}$$

*Case 3. Opposite and adjacent edge simply-supported:*

$$k'''_b = k''_b(1 - C''^2_{bb})$$

$$C'''_{ba} = \frac{C''_{ba}}{1 + C''_{bb}}$$

*Case 4. Symmetrical conditions of loading and deformation:*

$$k^s_b = k_b(1 + C_{bb})$$

$$C^s_{ba} = \frac{2C_{ba}}{1 + C_{bb}} = 2C'_{ba}$$

*Case 5. Symmetry plus adjacent edge simply-supported:*

$$k''^s_b = k''_b(1 + C''_{bb})$$

$$C''^s_{ba} = \frac{2C''_{ba}}{1 + C''_{bb}} = 2C'''_{ba}$$

The moments on the fixed edges of panels with one or more edges simply supported may be obtained from the values of fixed-edge moments by means of the distribution procedure. This calculation is easily performed as the first step in the solution. (See the illustrative example in the following section.)

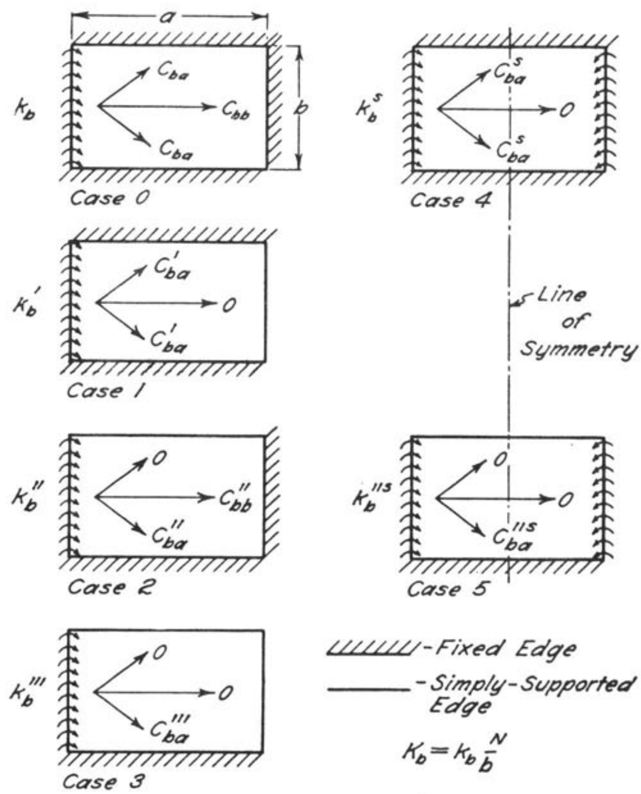



FIG. 7. NOTATION FOR MODIFIED STIFFNESS AND CARRY-OVER FACTORS

### 8. *Illustrative Example*

To illustrate the application of the distribution procedure, it will be used to compute moments in the continuous slab shown in Fig. 8. The loading considered will be $w$ = 100 lb per sq ft, uniformly distributed over the entire area of the slab. The thickness of the slab is assumed to be the same for all panels.

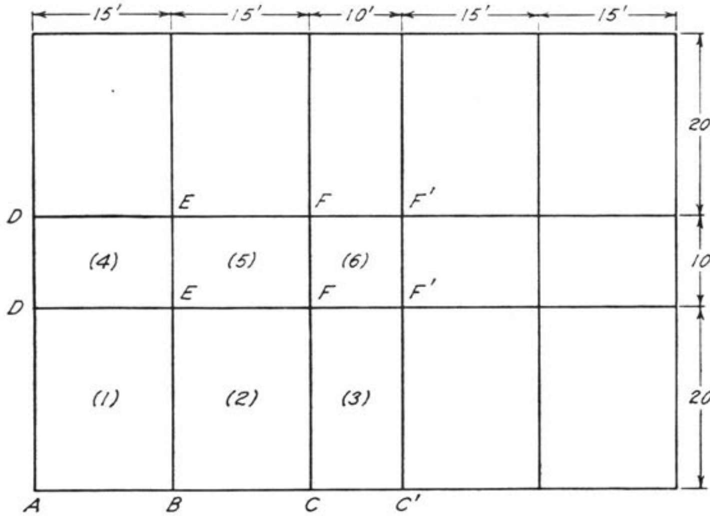



FIG. 8. PLAN OF SLAB FOR ILLUSTRATIVE EXAMPLE

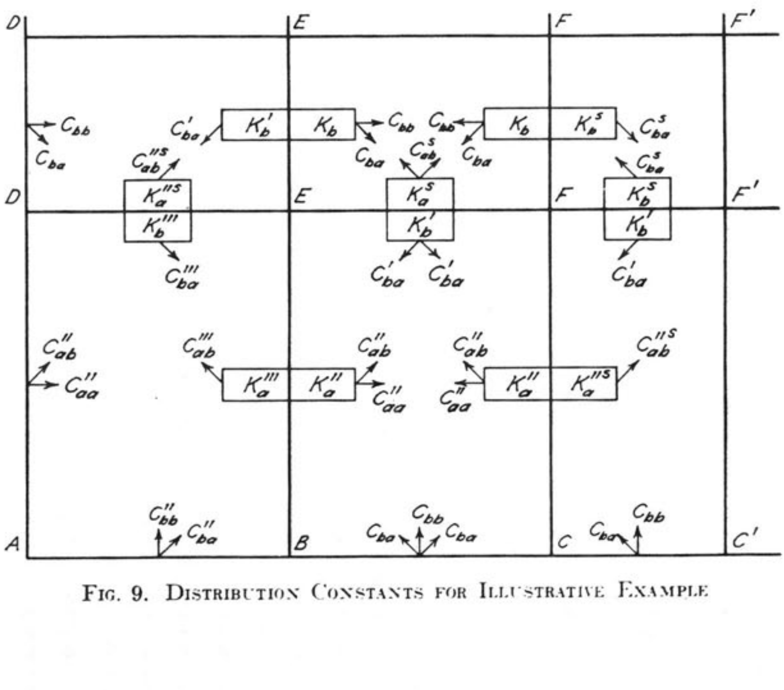



FIG. 9. DISTRIBUTION CONSTANTS FOR ILLUSTRATIVE EXAMPLE

Since both the structure to be analyzed and the loading are symmetrical about both centerlines, only six of the fifteen panels need be considered. Furthermore, in panels (3), (4), (5), and (6), modified stiffnesses and carry-over factors may be used to take account of the symmetrical conditions.

The beams supporting the edges of the panels are assumed to be nondeflecting and to offer no restraint against rotation of the slab; that is, they have no torsional rigidity. The exterior edges of the slab are therefore simply supported, and modified distribution constants may be used in the edge panels.

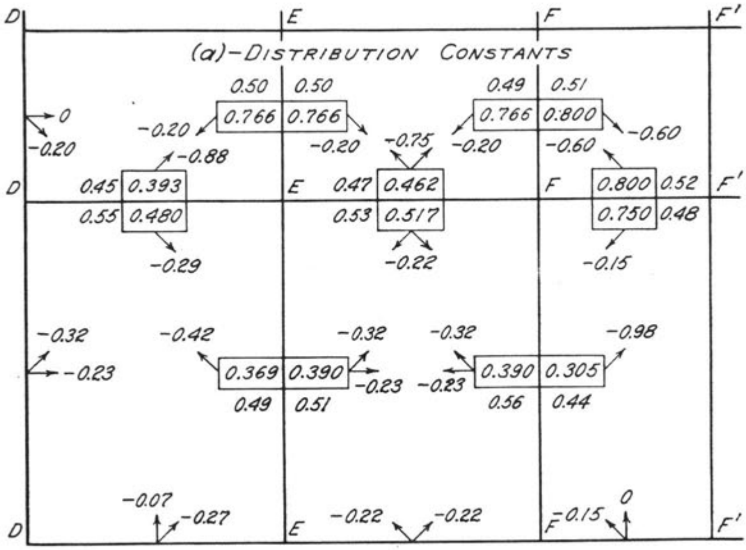
TABLE 2
CALCULATIONS FOR BASIC STIFFNESS FACTORS AND FIXED-EDGE MOMENTS FOR ILLUSTRATIVE EXAMPLE

| Panel | $\frac{b}{a}$ | Edge | $k$ | Span, $b$, ft | $K$* | F.E.M. $\frac{-M_{av}}{wb^2}$ | F.E.M.† $M_{av}$, lb |
|---|---|---|---|---|---|---|---|
| 1, 2 | 0.75 | Long | 6.25 | 15 | 0.417 | 0.0415 | −934 |
| | | Short | 7.75 | 15 | 0.517 | 0.0313 | −704 |
| 3 | 0.50 | Long | 5.00 | 10 | 0.500 | 0.0556 | −556 |
| | | Short | 7.50 | 10 | 0.750 | 0.0314 | −314 |
| 4, 5 | 0.67 | Long | 5.78 | 10 | 0.578 | 0.0464 | −464 |
| | | Short | 7.67 | 10 | 0.766 | 0.0315 | −315 |
| 6 | 1.00 | All | 8.00 | 10 | 0.800 | 0.0290 | −290 |

* $N = 1$.
† $w = 100$ lb per sq ft.

The particular distribution constants required in the solution of this problem are indicated by the appropriate symbols on the sketch in Fig. 9. Basic values of the carry-over factors were determined from the curves and formulas in Fig. 5, and values of the modified factors were computed by means of the equations in Section 7. The resulting numerical values are given in Fig. 10a.

The calculation of the basic stiffness factors is shown in Table 2. The values of $k$ were obtained from Fig. 3, and the quantity $K$ was computed for a value of $N = 1$, since the slab thickness is equal for all panels. The numerical values of $K$ thus obtained are also shown in the boxes in Fig. 10a. The distribution factors which determine the proportion of the unbalanced moment to be distributed to each panel are written next to the boxes. These factors are computed as in the Cross method of moment distribution for continuous



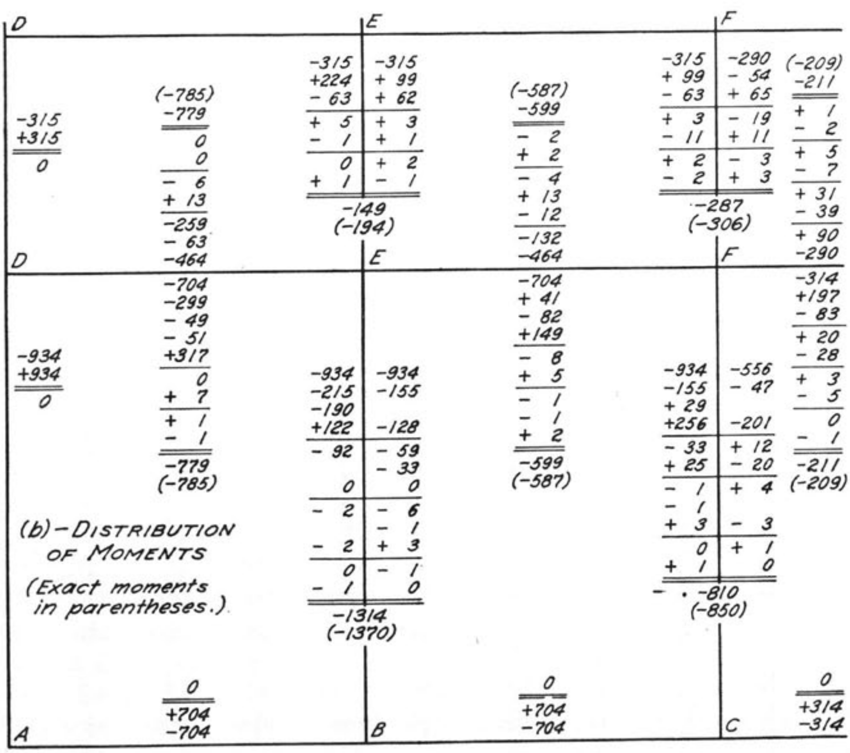




Fig. 10. Illustrative Computation Sheet

beams. For example: on edge BE the total stiffness is $0.369+0.390=0.759$, of which $0.369 \quad 0.759=0.49$ is contributed by panel (1); and $0.390 \quad 0.759=0.51$ by panel (2).

The calculation of the average fixed-edge moment is also illustrated in Table 2. Values of the coefficients of $M_{av}/wb^2$ were obtained from Table 1 and Fig. 1.

All necessary distribution constants are given on the sketch in Fig. 10a, and the complete calculations for edge moments are given in Fig. 10b. The operations carried out in these calculations may be described step by step as follows:

(a) Write in the average fixed-edge moment for each edge at a location adjacent to that edge as shown.

(b) Release the restraints on all exterior edges, which are simply supported in the actual structure, by balancing that moment on each edge to zero.

(c) Carry over the proper proportion of the balancing moment from step (b) to the interior edges of each panel. In the corner panel (1), the carry-over factors are modified so that moments need not be carried over to the adjacent simply-supported edge.

From this point forward, the use of modified distribution factors for the interior edges obviates the carrying over of any moments to the simply-supported edges.

(d) Release the restraints on edge BE and distribute the unbalanced moment of $-250$ to the two panels in proportion to the distribution factors, thus: $+122$, or 49 percent, to panel (1); $-128$, or 51 percent, to panel (2).

(e) Carry over the distributed moments to the remaining interior edges of panels (1) and (2). For example, in panel (2) carry over $-128 \times -0.32 = +41$ to edge EF, and $-128 \times -0.23 = +29$ to edge CF.

(f) Repeat the processes of steps (d) and (e) for edges CF, DE, EF, FF′, EE, and FF in turn, first balancing and then carrying over at each edge before proceeding to the next edge.

(g) Start again at edge BE, and repeat the process at successive edges until no unbalanced moment exists on any edge.

(h) Add all the moments in each panel at each edge. These are the desired average moment on the edges of the continuous slab.

In this problem the procedure has been followed of distributing moments at each edge successively. In other problems it may be more convenient to release all edges simultaneously. In either case the procedure has a direct parallel with that used for continuous beams or frames.

The calculation of average positive moment on sections in the interior of each panel is illustrated in Fig. 11. The positive moment correction factors, $F$, obtained from Fig. 6, are given on the small sketches for each group of panels having the same values of $b/a$. Average positive moments, computed from the coefficients in Table 1 and Fig. 2, are written on a sketch of each panel adjacent to the section to which they refer. Corrections due to the edge moments, obtained as the product of those moments and the appropriate factor, $F$, are written separately for the moments on the long and short edges. The net average positive moment in each direction is obtained as the sum of the positive moment for the simply-supported slab and the corrections due to the edge conditions.

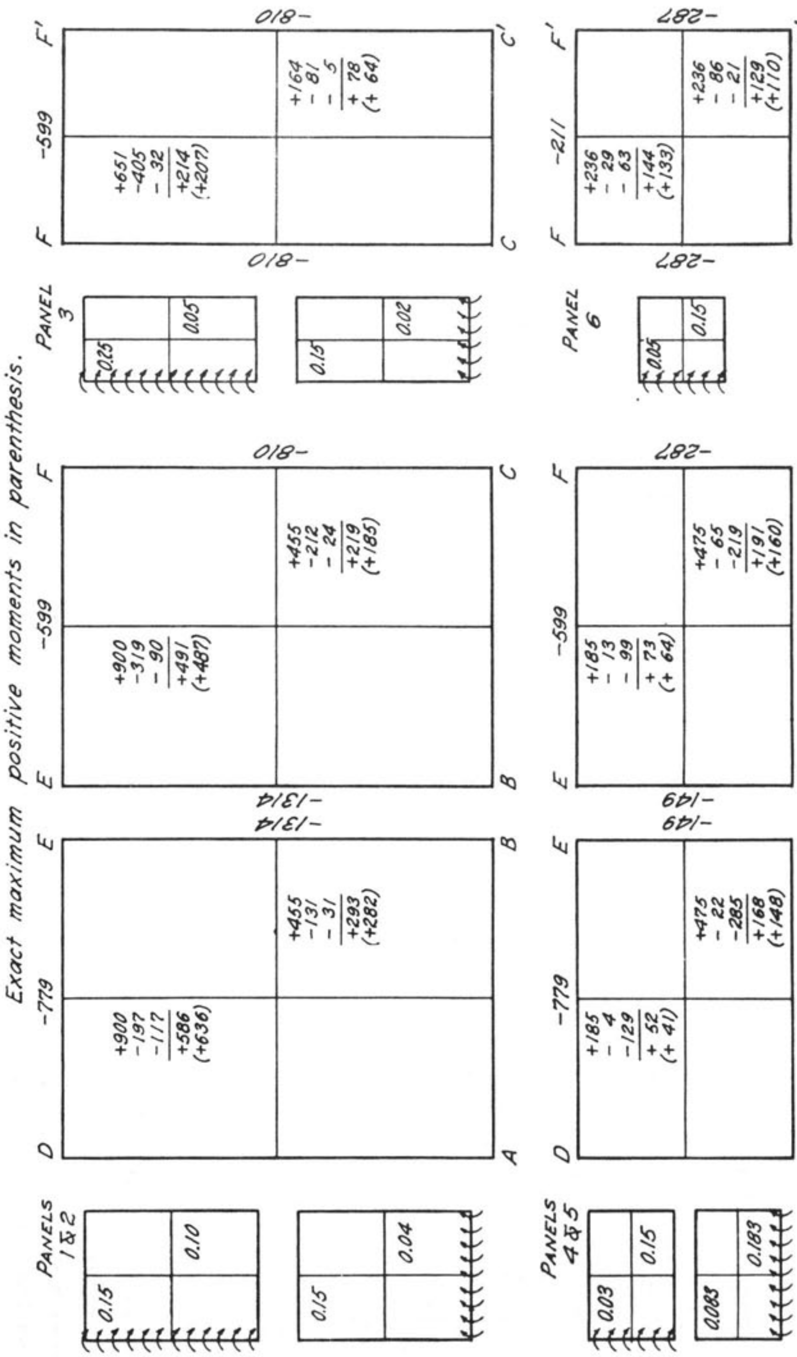



Fig. 11. Calculation of Positive Moments

## III. Development of the Distribution Procedure

### 9. *Introduction*

The distribution procedure is not exact, chiefly because of the approximate nature of the stiffnesses, carry-over factors, and positive moment correction factors, as applied to average moments. The other constants in the procedure, fixed-edge moments and moments in the simply-supported slab, are obtained from exact solutions and do not contribute to the approximate nature of the procedure, except insofar as average moments are used. The method of distributing moments is not in itself approximate, but is of course no more accurate than the constants involved in its use.

It is the purpose of this chapter to describe in detail the manner in which the final values of the various distribution constants were obtained, and to establish their validity and degree of approximation by applying the distribution procedure to the solution of certain problems for which the exact answers are known.

In the derivation of sets of elastic constants to be used in an approximate procedure for analyzing continuous plates, it is necessary first to make some decision regarding the manner in which moments and rotations are distributed along the edges of a panel. One problem in the development of the procedure, therefore, was to determine the extent to which the various constants were affected by changes in the assumed shapes of the distribution curves. This was done by first investigating two approximate procedures that had previously been developed for the analysis of continuous plates.[1] Each of these procedures is based on a different set of assumptions regarding the shapes of the curves of moment and rotation, and from the results of each set of assumptions it was possible to derive elastic constants corresponding to the stiffness and carry-over factors used in the moment distribution procedure.

The next step was the derivation of another set of stiffness and carry-over factors by means of a semi-rational procedure based on the moments in uniformly loaded isolated rectangular plates. These factors corresponded to still another set of assumptions regarding the manner in which the moments and slopes are distributed along an edge. The three sets of constants thus obtained were compared and the effects of variations in the assumptions were ascertained.

[1] After the completion of the thesis on which this bulletin is based, a third procedure came to the authors' attention. This method is similar to that described in Chapter II in that the moment distribution concept is used, but the underlying assumptions are different. In spite of these differences, the final results in the form of stiffnesses and carry-over factors are not greatly different from those proposed herein. The reference to this paper is "En Metode for Tilnaermet Beregning av Kontinuerlige Toveisplater," by Knud Engelbreth, *Betong*, 1945, Vol. 30, No. 2, pp. 99-115.

It was then necessary to determine the effect of variations in the constants on the moments computed by means of the distribution procedure. A study of this problem revealed that the moments were only slightly affected by relatively large differences in the constants obtained for the different assumptions. Consequently, a set of constants for use in the distribution procedure was selected, and was then checked by using them in the analysis of continuous slabs for which the correct moments were known.

The derivation and checking of the positive moment correction factors was an entirely separate step, described in Section 17.

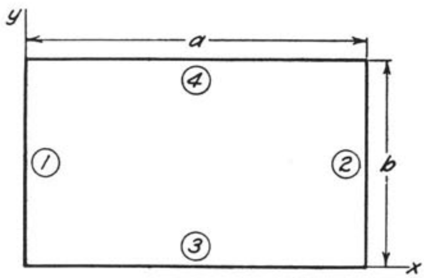



FIG. 12. NOTATION USED IN DERIVATION OF BITTNER AND MAUGH-PAN CONSTANTS

## 10. *Bittner's Method of Analysis*

This procedure was proposed by E. Bittner[1] in 1938 for the calculation of moments in continuous slabs. The basis of the method is the expression of the rotations on each edge of a panel in terms of the moments on all edges and the applied load. For continuous slabs the resultant slope in one panel must be equal to the resultant slope on the same edge in an adjacent panel. These conditions, plus the boundary conditions at the edges of the slab, permit the solution of the various equations for the unknown moments. This procedure is essentially algebraic, in contrast to the numerical procedure described in Chapter II of this bulletin.

The only part of Bittner's procedure that is of direct interest here relates to the elastic constants used to express the edge rotations in terms of edge moments. Both the assumptions made in the derivation of these factors and their numerical values are of interest. Bittner's approach to this problem is as follows: Consider a rectangular panel supported on all edges as shown in Fig. 12. If a single

[1] E. Bittner, "Momententafeln und Einflussflächen für kreuzweise bewehrte Eisenbetonplatten," Julius Springer, Vienna, 1938.

sine wave of moment, $M_1 \sin \frac{\pi y}{b}$, is applied on edge (1), the rotation of edge (1) is also a single sine wave, $\Phi_1 \sin \frac{\pi y}{b}$, in which

$$\Phi_1 = \beta_y M_1$$

and $\beta_y$ is the designation used by Bittner to represent the relation between $\Phi_1$ and $M_1$ on the short edge of a panel. The rotation on edge (2) is $\Phi_2 \sin \frac{\pi y}{b}$, wherein

$$\Phi_2 = \beta'_y M_1.$$

The rotations on edges (3) and (4) are not single sine waves and therefore cannot be expressed as simple functions of $M_1$ except in an approximate manner. This Bittner does by first expressing the actual rotations on edges (3) and (4) as a sine series, then considering only the first term of that series. The resulting rotations are thus $\Phi_3 \sin \frac{\pi x}{a}$ and $\Phi_4 \sin \frac{\pi x}{a}$, wherein

$$\Phi_3 = \Phi_4 = \beta_{xy} M_1.$$

The corresponding expressions for moment applied on a long edge such as (3) or (4) may be obtained from the above expressions by interchanging the subscripts $x$ and $y$, and $a$ and $b$.

General expressions for single sine waves of rotation on each edge due to single sine waves of moment on all four edges may be written as follows:

$$\begin{aligned}
\Phi_1 &= \beta_y M_1 + \beta'_y M_2 + \beta_{yx}(M_3 + M_4) \\
\Phi_2 &= \beta_y M_2 + \beta'_y M_1 + \beta_{yx}(M_3 + M_4) \\
\Phi_3 &= \beta_x M_3 + \beta'_x M_4 + \beta_{xy}(M_1 + M_2) \\
\Phi_4 &= \beta_x M_4 + \beta'_x M_3 + \beta_{xy}(M_1 + M_2).
\end{aligned} \tag{1}$$

The constants $\beta_y$ and $\beta_x$ may be considered as inverted stiffness factors relating single sine waves of moment and rotation on the same edge of a simply-supported panel. Thus

$$\frac{M_3}{\Phi_3} = \frac{1}{\beta_x} = \frac{2\pi\lambda M}{a}$$

wherein $\lambda$ is the stiffness factor for which numerical values are given by Bittner in his Table 1 for values of $a/b$ ($l/h$ in his notation) varying from 0 to 10.0. It may be noted that Bittner's stiffness factor, $\lambda$, is directly related to the modified stiffness, $S$, given by Newmark in Bulletin 304.[1] The relation between numerical values

[1] N. M. Newmark, "A Distribution Procedure for the Analysis of Slabs Continuous over Flexible Beams," Univ. of Ill. Eng. Exp. Sta. Bul. 304. 1938.

of these constants is as follows:

$$2\pi\lambda = C_s \frac{s}{b},$$

where $C_s$ is given in Table 1 of Bulletin 304, and $s$ and $b$ are the lengths of the sides of the panel under consideration.

Constants which may be considered as carry-over factors for rotations in a simply-supported slab may be derived from ratios of the $\beta$-constants in Eq. (1). This is done below, using Bittner's notation.

$$\frac{\beta'_x}{\beta_x} = \varphi_x \qquad \frac{\beta'_y}{\beta_y} = \varphi_y$$

$$\frac{\beta_{xy}}{\beta_x} = \varphi_{xy} \qquad \frac{\beta_{yx}}{\beta_y} = \varphi_{yx}.$$

The various $\varphi$-constants above are given by Bittner in Table 2, for values of $a/b$ ($l_y/l_x$ in his notation) ranging from 1.0 to 1.5. Again, it may be noted that the carry-overs to an opposite edge, $\varphi_x$ and $\varphi_y$, are identical with the carry-over factor, $k$, given by Newmark in Table 2 of Bulletin 304.

The stiffnesses and carry-over factors derived by Bittner apply to a simply-supported panel subjected to a single sine wave of moment on one edge and subject to certain assumptions regarding the distribution of rotation on the adjacent edges. It is possible, however, to derive from these constants another set, referring to a fixed-edge slab with a unit rotation applied to one edge. Such constants would be equivalent to those used in the distribution procedure developed herein.

The equivalent stiffness and carry-over factors may be obtained from Bittner's constants in the following manner: First, consider only the constants corresponding to a moment applied on the short edge of a panel. If in Eq. (1)

$$\Phi_1 = 1$$

$$\Phi_2 = \Phi_3 = \Phi_4 = 0,$$

then

$$\frac{M_3}{M_1} = C_{ba}$$

$$\frac{M_2}{M_1} = C_{bb}$$

$$\frac{M_1}{\Phi_1} = M_1 = K_b = k_b \frac{M}{b},$$

and the following expressions are obtained:

$$C_{ba} = \frac{1 - \varphi_y}{2\varphi_{yx} - \dfrac{1 + \varphi_x}{\varphi_{xy}}}$$

$$C_{bb} = -1 - \frac{1 + \varphi_x}{\varphi_{xy}} \cdot C_{ba}$$

$$k_b = \frac{2\pi\lambda}{1 + \varphi_y C_{bb} + 2\varphi_{yx} C_{ba}}.$$

The constants $C_{ab}$, $C_{aa}$, and $k_a$, referring to a long edge, may be obtained by a similar calculation.

TABLE 3
EQUIVALENT STIFFNESS AND CARRY-OVER FACTORS COMPUTED FROM BITTNER CONSTANTS

| Constant | $b/a$ | | | |
|---|---|---|---|---|
| | 0.50 | 0.67 | 0.75 | 1.00 |
| Stiffness Factors: $k_a$ | 4.92 | 5.72 | 6.21 | 7.79 |
| $k_b$ | 6.84 | 7.18 | 7.39 | 7.79 |
| Carry-over Factors: $C_{aa}$ | −0.317 | −0.199 | −0.140 | −0.017 |
| $C_{ab}$ | −0.274 | −0.304 | −0.307 | −0.268 |
| $C_{bb}$ | +0.062 | +0.062 | +0.053 | −0.017 |
| $C_{ba}$ | −0.099 | −0.161 | −0.193 | −0.268 |

Numerical values of the equivalent factors computed from Bittner's constants are given in Table 3. It will be noted that values are given for $b/a = 0.5$, although Bittner does not tabulate the $\varphi$-values for $b/a$ less than 0.67. For this case $\varphi_x$ and $\varphi_y$ were obtained from their equality with the constants in Bulletin 304, and values of $\varphi_{xy}$ and $\varphi_{yx}$ were obtained approximately by extrapolation from plotted data.

The values given in Table 3 are simply one of a number of possible sets of constants describing the behavior of a fixed-edge slab with a unit rotation applied on one edge. The values of these constants depend to some extent on the manner in which the moments and rotations are distributed along the various edges of the panels. In Bittner's solutions, both of these quantities are assumed to be distributed as a single sine wave on each edge.

11. *Method of Maugh and Pan*

This approximate method of analysis for continuous slabs was presented by L. C. Maugh and C. W. Pan[1] in 1941. It is similar in principle to Bittner's procedure and involves algebraic expressions for edge rotations in terms of edge moments. Somewhat different assumptions were made, however, in treating the rotations on the adjacent edges, and the carry-over factors to those edges are thus different from those obtained by Bittner. The carry-over factors to the opposite edge and the stiffness factors are the same as those given in the preceding section.

The chief assumptions made by Maugh and Pan are 1) that all edge moments are distributed as single sine waves and 2) that continuity between various panels will be maintained only at the middle of each edge. The first assumption is also made by Bittner, but the second is different, as may be seen from the following discussion.

Consider the panel in Fig. 12 with a single sine wave of moment $M_1 \sin \frac{\pi y}{b}$ acting on edge (1). The rotation on edge (1) is also a single sine wave, $\Phi_1 \sin \frac{\pi y}{b}$, and the rotation at the middle of the edge, $\Phi_1$, is given by the expression

$$\Phi_1 = \alpha_b \frac{b}{N} M_1,$$

wherein $\alpha_b$ is the authors' designation for the stiffness factor involved. Similarly, the rotation at the middle of edge (2) may be expressed as

$$\Phi_2 = \beta_b \frac{b}{N} M_1,$$

in which $\beta_b$ is Maugh and Pan's notation and should not be confused with the $\beta$-terms used by Bittner. In the above expressions both the moments and the rotations are distributed as single sine waves, and the results obtained are thus the same as those obtained by Bittner. The relations are as follows:

$$\text{(Maugh) } \alpha_b \frac{b}{N} = \text{(Bittner) } \beta_y$$

$$\text{(Maugh) } \beta_b \frac{b}{N} = \text{(Bittner) } \beta'_y.$$

[1] L. C. Maugh and C. W. Pan, "Moments in Continuous Rectangular Slabs on Rigid Supports," *Trans. ASCE*, Vol. 107 (1942), p. 1118.

The rotations on edges (3) and (4) are not distributed as a single sine wave. In this case, Bittner expanded the rotation into a sine series and dealt only with the first term. Maugh and Pan, however, retain the actual distribution but consider only the value of the rotation at the middle of the edge. These rotations are expressed by the following equation:

$$\Phi_3 = \Phi_4 = \gamma_b \frac{b}{N} M_1.$$

Maugh and Pan's constant $\gamma_b$ is thus different from Bittner's constant $\beta_{xy}$.

All the above expressions apply only for a moment on a short edge. Similar expressions for moment on a long edge may be obtained by interchanging $a$ and $b$.

The general expressions corresponding to Eq. (1) of Section 11 are as follows:

$$\begin{aligned}
\Phi_1 &= \alpha_b \frac{b}{N} M_1 + \beta_b \frac{b}{N} M_2 + \gamma_a \frac{a}{N} (M_3 + M_4) \\
\Phi_2 &= \alpha_b \frac{b}{N} M_2 + \beta_b \frac{b}{N} M_1 + \gamma_a \frac{a}{N} (M_3 + M_4) \\
\Phi_3 &= \alpha_a \frac{a}{N} M_3 + \beta_a \frac{a}{N} M_4 + \gamma_b \frac{b}{N} (M_1 + M_2) \\
\Phi_4 &= \alpha_a \frac{a}{N} M_4 + \beta_a \frac{a}{N} M_3 + \gamma_b \frac{b}{N} (M_1 + M_2).
\end{aligned} \tag{2}$$

Expressions for equivalent stiffness and carry-over factors were obtained from the above equations in a manner identical with that described in the preceding section in connection with Eq. (1). The expressions for constants referring to a short edge are:

$$C_{ba} = \frac{b}{a^2} \frac{\gamma_b(\alpha_b - \beta_b)}{\gamma_a\gamma_b - \alpha_b(\alpha_a + \beta_a)}$$

$$C_{bb} = -1 - \frac{a}{b} \frac{(\alpha_a + \beta_a)}{\gamma_b} \cdot C_{ba}$$

$$k_b = \frac{1}{\alpha_b + \beta_b C_{bb} + 2\gamma_a \dfrac{a}{b} C_{ba}}.$$

Constants for a long edge may be obtained in a similar manner.

Numerical values of the equivalent stiffness and carry-over factors computed from Maugh and Pan's constants are given in Table 4.

These values represent the results of a set of assumptions different from those made by Bittner, since in their solution for the simply-supported slab Maugh and Pan have assumed a single sine wave distribution of moment and rotation on only two edges of a panel and have considered an unsymmetrical distribution on the other two edges. The corresponding distributions of both moment and rotation for the fixed slab are not single-sine waves as was the case for Bittner's solution. Although only one pair of constants for the simply-supported slab was affected by the difference in the assumptions, all values in Tables 3 and 4 for the fixed-edge slab are changed, since all of the constants for the simply-supported slab enter into each expression for the fixed-edge slab.

TABLE 4
EQUIVALENT STIFFNESS AND CARRY-OVER FACTORS COMPUTED FROM MAUGH AND PAN CONSTANTS

| Constant | b/a | | | |
|---|---|---|---|---|
| | 0.50 | 0.67 | 0.75 | 1.00 |
| Stiffness Factors: $k_a$ | 4.80 | 5.59 | 6.03 | 7.46 |
| $k_b$ | 6.60 | 6.97 | 7.13 | 7.46 |
| Carry-over Factors: $C_{aa}$ | −0.346 | −0.231 | −0.178 | −0.062 |
| $C_{ab}$ | −0.247 | −0.278 | −0.275 | −0.227 |
| $C_{bb}$ | +0.031 | +0.032 | +0.020 | −0.062 |
| $C_{ba}$ | −0.063 | −0.127 | −0.159 | −0.227 |

## 12. *Derivation of Stiffness Factors*

The stiffness of a beam is defined as the moment corresponding to a unit rotation at one end of the beam, the other end being fixed. For the purposes of this bulletin the stiffness of a slab is defined in a similar manner, as the moment corresponding to a unit rotation on one edge of the slab, all other edges being fixed. The determination of this stiffness, however, is not as simple for a slab as for a beam, since a question immediately arises regarding the distribution of rotation and moment on the edge being considered. The procedure followed by Bittner is equivalent to assuming a single sine wave distribution for both moment and rotation; the method used by Maugh and Pan results in an unknown distribution of rotation when applied to a fixed-edge panel.

The method finally adopted herein for determining the stiffness factors for a slab is based on the moments and rotations of uniformly loaded plates with fixed edges. It may be described as follows: Consider the plate shown in Fig. 12, with edges (2), (3), and (4) fixed and edge (1) simply-supported. The rotation of edge (1) due to a uniform load $w$ on the plate may then be computed. (Such values are given in Section 30, Appendix A.) If a rotation equal and opposite to that for the simply-supported edge is applied to edge (1), the corresponding moment is equal to the moment on edge (1) of a uniformly loaded fixed-edge plate. This moment is given in Table 1 and in Section 31, Appendix A.

In effect, the above procedure consists of applying a known rotation to the simply-supported edge of a plate fixed on three edges, and determining the moment produced on that edge as a result of the applied rotation. In accordance with the assumptions of the distribution procedure, only average rotations and moments are considered. Specifically, the rotation applied is the average rotation on the simply-supported edge of a uniformly loaded plate fixed on three edges, and the moment produced is the average moment on the corresponding edge of a uniformly loaded plate fixed on all edges. For example, consider a plate having $b/a = 0.5$, and fixed on edges (2), (3), and (4) of Fig. 12. The average rotation on the simply-supported short edge (1) due to a uniform load $w$ is

$$\Phi_{av} = 0.0019 \frac{a}{N} wb^2 = 0.0038 \frac{b}{N} wb^2.$$

The average moment on the short edge of a uniformly loaded plate fixed on all edges is

$$M_{av} = 0.0314wb^2.$$

The stiffness on the short edge is thus

$$K_b = \frac{M_{av}}{\Phi_{av}} = 8.25 \frac{N}{b},$$

and the stiffness factor $k_b = 8.25$.

Data are available in Appendix A to make similar calculations for the long edge of a plate having $b/a = 0.5$ and for a plate having $b/a = 1.0$. The calculations for the stiffness factors $k$ are shown in Table 5. To distinguish this set of constants from the simplified constants discussed subsequently, these values will be referred to as the "derived" stiffness factors.

The stiffness factors from Table 5 are plotted versus $b/a$ in Fig. 13, together with the corresponding values based on Bittner's constants (Table 3) and Maugh's constants (Table 4). The stiffness factors obtained by the various procedures are not greatly different, the maximum difference being about 20 percent, for $k_b$ at $b/a = 0.5$. There is even less difference in the relative stiffnesses for adjacent panels as determined from the three sets of constants.

TABLE 5
CALCULATION OF DERIVED STIFFNESS FACTORS

| $b/a$ | Stiffness Factor | $\frac{-M_{av}}{wb^2}$ | $\frac{-\Phi_{av}}{wb^2}$ | $k$ |
|---|---|---|---|---|
| 1.0 | $k_b$ | 0.0290 | $0.0035\,\frac{b}{N}$ | 8.30 |
| 0.5 | $k_b$ | 0.0314 | $0.0038\,\frac{b}{N}$ | 8.25 |
| 0.5 | $k_a$ | 0.0556 | $0.0110\,\frac{b}{N}$ | 5.06 |

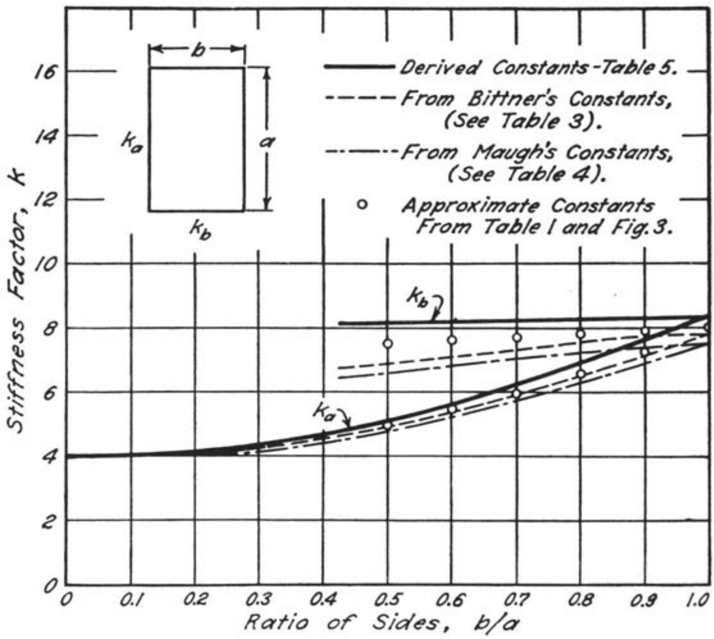



FIG. 13. COMPARISON OF STIFFNESS FACTORS

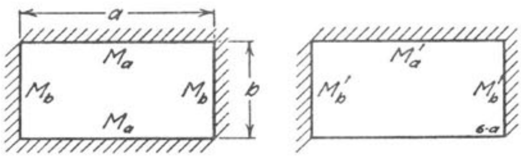
### 13. *Derivation of Carry-Over Factors*

The manner of obtaining the "derived" carry-over factors is described in this section. For the purpose of this bulletin the carry-over factors for average moments are defined in a manner analogous to that used for a beam. Consider a rectangular panel, fixed on three edges and simply-supported on the fourth. If a moment is applied to the simply-supported edge, corresponding moments are induced at each of the fixed edges. The ratios of these induced moments to the applied moment are the carry-over factors.

The problem of what distribution of moment should be considered on the various edges is treated here, as in the calculation of stiffness factors, by considering the moments in uniformly loaded plates. The panel with three edges fixed is assumed to be uniformly loaded, and the moments on the fixed edges are determined from Section 30, Appendix A. A moment of just sufficient magnitude to produce the condition of all edges fixed is then applied to the simply-supported

TABLE 6A

CALCULATION OF DERIVED CARRY-OVER FACTORS FOR MOMENT ON A LONG EDGE




$C_{aa} = \frac{M_a - M_a'}{M_a}$ $C_{ab} = \frac{M_b - M_b'}{M_a}$

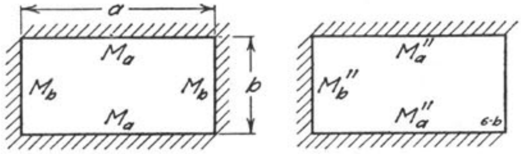
All values for moments are $M_{av}/wb^2$

| $b/a$ | $-M_a$ | $-M_a'$ | $-M_b$ | $-M_b'$ | $C_{aa}$ | $C_{ab}$ |
|---|---|---|---|---|---|---|
| 0.500 | 0.0556 | 0.0726 | 0.0315 | 0.0487 | −0.306 | −0.309 |
| 0.571 | 0.0518 | (0.0654) | 0.0315 | (0.0484) | −0.262 | −0.326 |
| 0.667 | 0.0464 | (0.0557) | 0.0315 | (0.0475) | −0.200 | −0.345 |
| 0.750 | (0.0417) | 0.0482 | (0.0314) | 0.0455 | −0.156 | −0.338 |
| 0.800 | 0.0389 | (0.0440) | 0.0311 | (0.0440) | −0.131 | −0.332 |
| 1.000 | 0.0290 | 0.0307 | 0.0290 | 0.0381 | −0.059 | −0.314 |

Values in parentheses obtained by interpolation from data in Appendix A.

TABLE 6B

CALCULATION OF DERIVED CARRY-OVER FACTORS FOR MOMENT ON A SHORT EDGE




$C_{bb} = \frac{M_b - M_b''}{M_b}$ $C_{ba} = \frac{M_a - M_a''}{M_b}$

All values for moments are $M_{av}/wb^2$

| $b/a$ | $-M_b$ | $-M_b''$ | $-M_a$ | $-M_a''$ | $C_{bb}$ | $C_{ba}$ |
|---|---|---|---|---|---|---|
| 0.500 | 0.0315 | 0.0314 | 0.0556 | 0.0610 | +0.003 | −0.172 |
| 0.571 | 0.0315 | (0.0314) | 0.0518 | (0.0580) | +0.003 | −0.197 |
| 0.667 | 0.0315 | (0.0314) | 0.0464 | (0.0532) | +0.003 | −0.224 |
| 0.750 | (0.0314) | 0.0314 | (0.0417) | 0.0495 | 0 | −0.248 |
| 0.800 | 0.0311 | (0.0312) | 0.0389 | (0.0470) | −0.003 | −0.262 |
| 1.000 | 0.0290 | 0.0307 | 0.0290 | 0.0381 | −0.059 | −0.314 |

Values in parentheses obtained by interpolation from data in Appendix A.

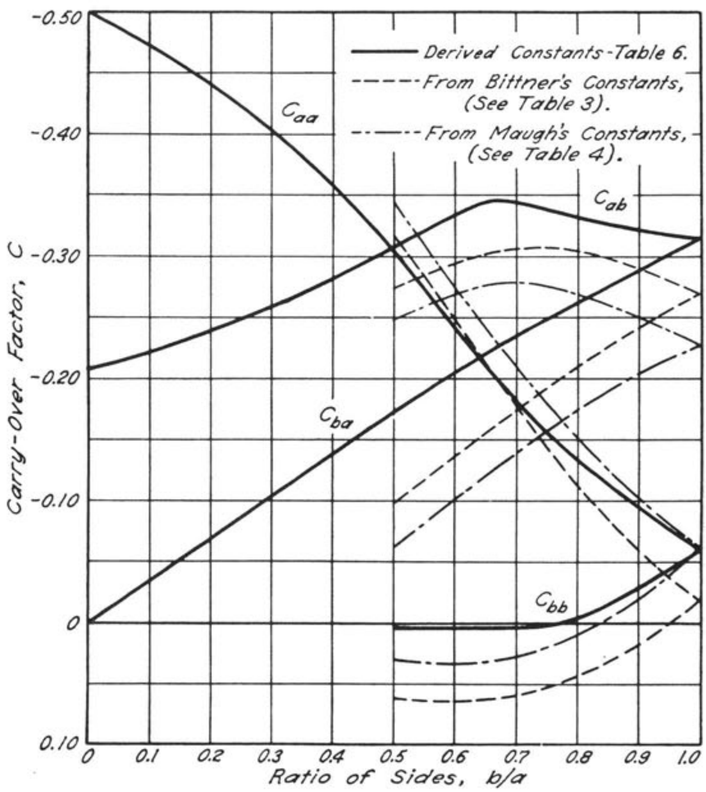



FIG. 14. COMPARISON OF CARRY-OVER FACTORS

edge. These moments are obtained from Section 31. The carry-over factors are then determined as the differences in average edge moments for three edges fixed and for all edges fixed, divided by the moment applied to the simply-supported edge (actually the moment for all edges fixed).

The calculation of the derived carry-over factors by the above procedure is illustrated in Tables 6a and 6b. The notation used is indicated on the sketches accompanying each table. Expressions for the various carry-over factors are indicated, and the values of the necessary moments are given in the tables.

Values of the carry-over factors given in Table 6 are plotted versus $b/a$ in Fig. 14, together with the corresponding factors based on Bittner's and Maugh's constants as given in Tables 3 and 4, respectively. Values of the derived constants for $b/a$ less than 0.5 were computed from the following approximate expressions for the average moments. The notation is that of Table 6.

$$-M_a/wb^2 \cong 0.0833 - 0.0554b/a$$

$$-M_b/wb^2 \cong 0.0315$$

$$-M_a'/wb^2 \cong 0.1250 - 0.1048b/a$$

$$-M_b'/wb^2 \cong 0.0487$$

$$-M_a''/wb^2 \cong 0.0833 - 0.0446b/a.$$

The agreement between the three sets of carry-over factors in Fig. 14 is not so good as that for the stiffness factors in Fig. 13. As might be expected from the nature of the assumptions, the greatest differences occur for the lateral carry-over factors, $C_{ba}$ and $C_{ab}$. The differences in values of $C_{bb}$ are not particularly significant, since this constant is small. The actual importance of the differences, however, can be assessed only by using the constants in the distribution procedure for the solution of problems to which the correct answers are known. Some of the results obtained from such solutions are discussed in the following sections.

## 14. *Simplified Distribution Constants*

In the preceding sections of this chapter the manner of obtaining three different sets of distribution constants, based on three different sets of assumptions, has been described. It may be seen from Figs. 13 and 14 that there are fairly large variations between the values thus obtained. The problem at this point is to determine the extent to which these variations affect the moments computed by means of the distribution procedure, and to decide which values of the constants shall be used. To this end, the distribution procedure using each set of constants in turn was applied to the analysis of three continuous slabs for which the exact moments were known. The three slabs analyzed were designated I, II, and III; the exact solutions are given in Sections 33, 34, and 35, respectively, of Appendix B. Plans of the slabs are given in Fig. 16.

The maximum variation between the average moments computed using the "Bittner," "Maugh," and "derived" constants was about 10 percent, plus or minus, with two exceptions. The exceptions were the moments on edge EE of slab II and on edge DD of slab III. In both these cases, use of the derived constants resulted in average moments considerably smaller than the correct values. This deficiency of the procedure was not considered unduly important, however, since the moments on the edges in question were small compared to the other moments in the structure.

The results of the study mentioned above indicated that relatively large variations in the distribution constants produced only small variations in the average moments. The three sets of constants were replaced by a single set of modified constants chosen to represent roughly the averages of the original three sets:

$$-C_{aa} = 0.560 - 0.535\, b/a, \qquad \text{for } b/a = 0.3\text{–}1.0$$

$$-C_{bb} = \frac{1}{16}(b/a - 0.6), \qquad \text{for } b/a = 0.6\text{–}1.0$$

$$= 0, \qquad \text{for } b/a = 0\text{–}0.6$$

$$-C_{ab} = 0.200 + 0.180\, b/a, \qquad \text{for } b/a = 0\text{–}0.7$$

$$= 0.433 - 0.153\, b/a, \qquad \text{for } b/a = 0.7\text{–}1.0$$

$$-C_{ba} = 0.280\, b/a, \qquad \text{for } b/a = 0\text{–}1.0$$

$$k_a = 4\,[1 + (b/a)^2]$$

$$k_b = 7 + \frac{b}{a}.$$

The above modified constants were then used for the solution of the three problems previously mentioned. The average moments thus obtained were little different from those computed with the derived constants or with either of the other sets. In general the modified constants gave results slightly better than did the derived factors. This first attempt at simplification was therefore successful in that it reduced the complexity of the relation between the distribution constants and $b/a$ without affecting the accuracy of the procedure.

It was finally decided to attempt an even greater modification of the constants in an attempt to secure even more simple relations with $b/a$. The equations for the stiffness factors were not changed; the expressions for the carry-over factors are as follows:

$$-C_{aa} = 0.60\,(1 - b/a), \text{ but not more than } 0.5$$

$$-C_{bb} = 0$$

$$-C_{ab} = 0.30$$

$$-C_{ba} = 0.30\, b/a.$$

Values of the constants determined from the above equations are given in Table 1 and Fig. 5. Values of the stiffness factors from the equations given previously are found in Table 1 and Fig. 3, and are also indicated by open circles in Fig. 13.

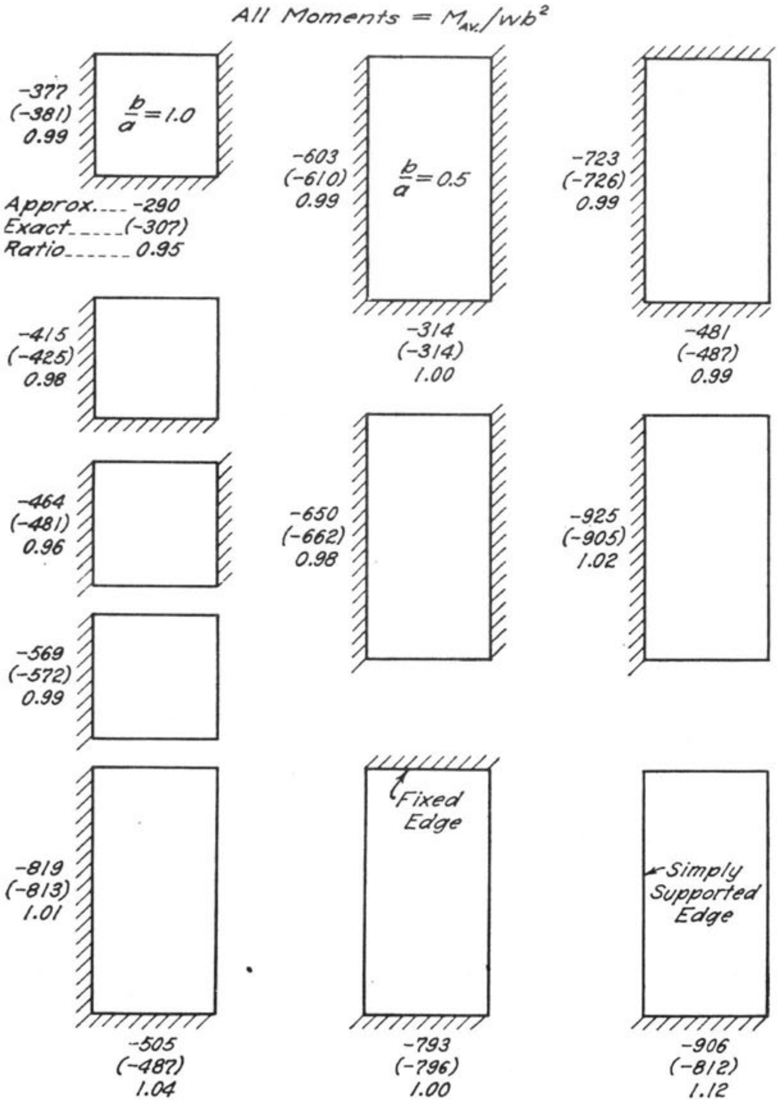



FIG. 15. COMPARISON OF APPROXIMATE AND EXACT MOMENTS FOR UNIFORMLY LOADED SINGLE PANELS

## 15. *Verification of Distribution Constants*

The simplified distribution constants derived in the preceding section were checked in two ways: 1) the carry-over factors alone were used to compute the average edge moments in uniformly loaded single

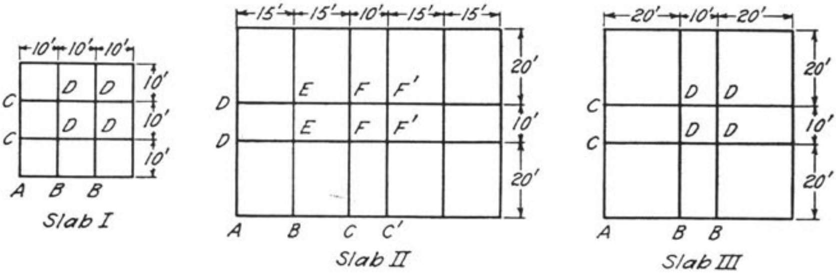



FIG. 16. PLANS OF CONTINUOUS SLABS USED IN CHECK OF DISTRIBUTION CONSTANTS

panels with one, two, or three edges fixed; 2) the entire distribution procedure, utilizing the simplified constants, was applied to the analysis of uniformly loaded continuous slabs for which the exact moments were known. These two procedures are described below.

The first check was made by using the simplified carry-over factors to determine the average edge moments in uniformly loaded single panels with one, two, or three edges fixed. Panels having $b/a$ ratios of 1.0 and 0.5 were used. The procedure was as follows: Average edge moments for a panel with all four edges fixed were obtained from Table 1. The various edges were then successively released and the appropriate moment was carried over to the other edges by means of the simplified carry-over factors of Section 14 and Table 1. The results are given in Fig. 15, in which the various panels and edge conditions considered are shown by sketches. Cross-hatching is used to indicate a fixed edge and all other edges are simply-supported. The approximate moments determined from the distribution procedure are given for each fixed edge, together with the exact average moments in parentheses and the ratio of approximate to exact moment. The exact moments were obtained from the data in Appendix A.

The agreement between approximate and exact moments in Fig. 15 is quite good, generally within 5 percent plus or minus. An exception occurs for the rectangular panel having one short edge fixed, for which the distribution procedure gives a moment 12 percent too high.

The second check involved the entire distribution procedure and consisted of the analysis of three continuous slabs, plans of which are given in Fig. 16. All slabs were considered to carry a uniform load, $W$, of 100 lb per sq ft over their entire area. The exact solutions for moments in these slabs are described in detail in Appendix B.

Slab II was used for the illustrative example in Section 8, and its solution by means of the distribution procedure is given in Figs. 9 and 10.

Comparisons of the approximate and exact average edge moments for the three slabs are given in Table 7. The agreement is quite good except at the two locations mentioned previously, edge EE of Slab II and edge DD of Slab III, for which the ratios of approximate to exact average moments are 0.77 and 0.71 respectively. It should be noted, however, that the moments on these edges are small compared to the other moments in the structures. A study of the behavior of the slabs suggests that discrepancies of this magnitude are the result of conditions which also result in small moments on the edges in question.

In general, the agreement between approximate and exact moments indicated in Fig. 15 and Table 7 was believed to be sufficiently good to warrant the use of the simplified constants and, in addition, to constitute adequate verification of the distribution procedure as a means of computing average moments in continuous slabs.

### 16. *Calculation of Modified Constants*

Expressions for modified distribution constants for use in cases of symmetry or in panels with simply-supported edges have been given in Section 7. The manner of obtaining these expressions is described in this section. Since a single example should be sufficient to explain the procedure, the constants $K_b''$, $C_{bb}''$, and $C_{ba}''$ will be derived for the short edge of a panel in which an adjacent long edge is simply-supported. This is Case 2 of Section 7.

Consider the rectangular panel of Fig. 12. The average moment on each edge may be expressed in terms of the average rotations on all edges and the distribution constants $K$ and $C$ as follows:

$$
\begin{aligned}
M_1 &= K_b\Phi_1 + C_{bb}K_b\Phi_2 + C_{ab}K_a(\Phi_3 + \Phi_4)\\
M_2 &= K_b\Phi_2 + C_{bb}K_b\Phi_1 + C_{ab}K_a(\Phi_3 + \Phi_4)\\
M_3 &= K_a\Phi_3 + C_{aa}K_a\Phi_4 + C_{ba}K_b(\Phi_1 + \Phi_2)\\
M_4 &= K_a\Phi_4 + C_{aa}K_a\Phi_3 + C_{ba}K_b(\Phi_1 + \Phi_2).
\end{aligned}
\qquad (3)
$$

To determine the modified constants for the case being considered apply a unit rotation to edge (1), edges (2) and (3) being considered fixed, and edge (4) simply-supported. These conditions are stated as follows:

$$
\begin{aligned}
\Phi_1 &= 1\\
\Phi_2 &= \Phi_3 = 0\\
M_4 &= 0.
\end{aligned}
$$

The values given above are then substituted into Equations (3) and $M_1$, $M_2$, and $M_3$ are solved for. The modified constants desired are then determined as follows:

$$K''_b = \frac{M_1}{\Phi_1} = M_1$$

$$C''_{bb} = \frac{M_2}{M_1}$$

$$C''_{ba} = \frac{M_3}{M_1}.$$

TABLE 7
COMPARISON OF APPROXIMATE AND EXACT EDGE MOMENTS FOR UNIFORMLY LOADED CONTINUOUS SLABS

See Fig. 16 for notation. All values are average moments in pounds for a uniform load $w$ over entire slab of 100 lb per sq ft.

| Slab | Edge | Approximate Average Moment* | Exact Average Moment† | Ratio (3)/(4) |
|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) |
| I | BD | − 398 | − 409 | 0.97 |
| | DD | − 285 | − 298 | 0.96 |
| II | BE | −1314 | −1370 | 0.96 |
| | CF | − 810 | − 850 | 0.95 |
| | EE | − 149 | − 194 | 0.77 |
| | FF | − 287 | − 306 | 0.94 |
| | DE | − 779 | − 785 | 0.99 |
| | EF | − 599 | − 587 | 1.02 |
| | FF′ | − 211 | − 209 | 1.01 |
| III | BD | −1182 | −1209 | 0.98 |
| | DD | − 99 | − 139 | 0.71 |

* Using distribution procedure with simplified constants (Table 1).
† For method of calculation see Appendix B.

The resulting algebraic expressions for these constants are given in Section 7.

A similar procedure was followed in calculating all the modified constants for which expressions are given in Section 7.

## 17. *Derivation of Positive Moment Factors*

The positive moment correction factors have been discussed briefly in Section 6, and the numerical values finally adopted for use with the distribution procedure are given in Table 1 and Fig. 6. These factors are used to correct the average positive moments in the interior of a simply-supported panel for the effect of edge moments resulting from continuity.

It was not considered feasible to derive the positive moment factors by direct calculation, because of the wide range of edge conditions for which they must be applicable and because of the manner in which the location of the maximum positive moment in a given span depends on the edge conditions. Consequently a cut-and-try procedure was adopted, based first on the exact moments given in Appendix A for single panels with various edge conditions, and finally on the exact solutions for continuous slabs given in Appendix B. Steps followed in the derivation of these factors are outlined below.

The first step involved the consideration of single rectangular panels having values of $b/a$ equal to 0.5 to 1.0 and having various numbers of edges fixed. Exact values of the average edge moments and the average positive moments in the interior of such panels are given in Appendix A. A panel simply-supported on all edges was considered first. Various edges were then fixed, and the relations between the resulting edge moments and the changes in the average positive moments were noted. These relations were used to obtain preliminary values of the desired factors for $b/a$ equal to 0.5 and 1.0.

The second step consisted of using the tentative factors obtained from the single panels to calculate the positive moments in panels of the continuous slabs having appropriate values of $b/a$. In all cases the value of average positive moment used was that for a section having the maximum moment. Except for nearly symmetrical conditions, the maximum positive moment does not occur at midspan. As a result of these checks with exact moments for continuous slabs, the tentative values were revised and modified. In making these modifications, greater weight was given to the results for the continuous slabs than to those for the single panels.

The two steps described above furnished values of the positive moment factors for $b/a$ equal to 0.5 and 1.0. The third step involved the determination of these factors for intermediate values of $b/a$. This was done first by rough interpolation from curves plotted for a range of $b/a$ from 0.5 to 2.0. Tentative values were then taken from such curves for $b/a$ equal to 0.67 and 0.75 and were applied to the calculation of positive moments in the panels of Slab II having the corresponding values of $b/a$. Revisions in the tentative values having been made as a result of these calculations, the revised values were used to correct the curves referred to above. The adjusted curves are shown by solid lines on Fig. 17.

Calculations made with several different sets of positive moment factors, some of which differed by large amounts, indicated that small variations in these factors had no serious effect on the positive moments obtained. With this fact in mind, it was decided to attempt a simplification of the relations between the positive moment factors and b a. The simplified values are shown by dash lines on Fig. 17. These values are the same as those given in Table 1 and Fig. 6. As a final step the simplified factors were used to compute moments in all panels of the three continuous slabs. The results are given in Table 8.

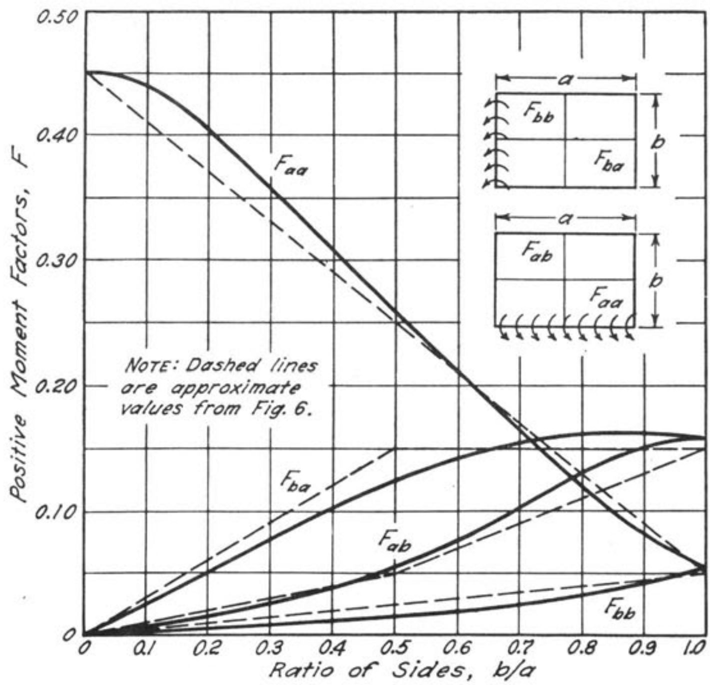



FIG. 17. PRELIMINARY VALUES OF POSITIVE MOMENT CORRECTION FACTORS

The approximate moments in Table 8 were obtained entirely by means of the distribution procedure, using the constants given in Table 1. The calculations for positive moments in the panels of Slab II are illustrated in Fig. 11, and described in Section 8 in connection with the illustrative example. The exact average moments were obtained from the solutions in Appendix B. It may be noted from the data in Table 8 that the approximate procedure almost always results in positive moments that are larger than the correct values. This is the result of a deliberate attempt to be conservative in this phase of the procedure, partly because of the greater importance of the positive moments and partly to allow for some uncertainty regarding the location of the section for maximum moment in slabs with different edge conditions. In general the moments obtained with the simplified constants are slightly greater than those obtained with the constants indicated by the solid curves in Fig. 17.

TABLE 8
COMPARISON OF APPROXIMATE AND EXACT POSITIVE MOMENTS FOR UNIFORMLY LOADED CONTINUOUS SLABS

See Fig. 16 for plans of slabs and notation. All values are average moments in pounds for a uniform load $w$ of 100 lb per sq ft over entire slab. Edge moments obtained by distribution procedure using approximate constants (see Table 7).

| Slab | Panel | $\frac{b}{a}$ | Span | Approximate Average Moment | Exact Average Moment* | Ratio $\frac{(5)}{(6)}$ |
|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) |
| I | DDDD | 1.0 | ..... | +122 | +104 | 1.17 |
| | BBDD | 1.0 | BB | +153 | +143 | 1.07 |
| | | | DD | +103 | + 89 | 1.16 |
| | ABCD | 1.0 | ..... | +156 | +153 | 1.02 |
| II | ABDE | 0.75 | Long | +293 | +282 | 1.04 |
| | | | Short | +586 | +636 | 0.92 |
| | BCEF | 0.75 | Long | +219 | +185 | 1.18 |
| | | | Short | +491 | +487 | 1.01 |
| | CC'FF' | 0.50 | Long | + 78 | + 64 | 1.22 |
| | | | Short | +214 | +207 | 1.03 |
| | DEDE | 0.67 | Long | + 52 | + 41 | 1.27 |
| | | | Short | +168 | +148 | 1.14 |
| | EFEF | 0.67 | Long | + 73 | + 64 | 1.14 |
| | | | Short | +191 | +160 | 1.19 |
| | FF'FF' | 1.0 | FF' | +144 | +133 | 1.08 |
| | | | FF | +129 | +110 | 1.17 |
| III | ABCD | 1.0 | ..... | +708 | +707 | 1.00 |
| | BBDD | 0.5 | Long | + 44 | + 25 | .... |
| | | | Short | + 45 | − 35 | .... |
| | DDDD | 1.0 | ..... | +196 | +164 | 1.20 |

* Based on data in Appendix B.

Moreover, it should be noted that the excess of approximate over exact moment is due in part to the fact that the edge moments obtained by the distribution procedure are usually a little on the low side, as may be verified by reference to Table 7.

On the basis of the data in Table 8 it was concluded that the simplified positive moment correction factors could be used with the distribution procedure to give conservatively high values of the positive moments. This conclusion was also borne out by calculations of positive moments in single panels with various edge conditions, the results of which are not given here.

## IV. Application of Distribution Procedure to Calculation of Moments in Two-Way Reinforced Concrete Slabs

### 18. *Introduction*

Although the distribution procedure described in the preceding chapters was developed primarily for the calculation of moments in two-way slabs of reinforced concrete, it is recognized that the usual assumptions of elasticity, isotropy, and homogeneity are not satisfied completely by a reinforced concrete slab at working stresses. There is some precedent, however, for the use of a theory based on these assumptions for the analysis of reinforced concrete structures.[1] Although the elastic theory is used primarily because nothing better is available, comparisons with test results have indicated that the computed moments and stresses are almost without exception on the safe side.

In applying the distribution procedure to the calculation of moments in reinforced concrete slabs supported on reinforced concrete beams cast monolithically with the slabs, two factors are introduced which have an important effect on the moments. The first of these is the restraint to rotation of the edges of the slab resulting from the torsional rigidity of the supporting beams. This effect, quite important for certain types of loading, is considered in some detail in this chapter. The second important factor is that the beams are not nondeflecting. Since the distribution procedure is valid only for the case of nondeflecting beams, this feature of the behavior of two-way slabs requires special treatment. Consequently, throughout this bulletin the beams are considered to be nondeflecting, and the effects of deflection are left for future consideration.

The calculations described in the following sections had as their object the determination of the effects of several variables on the moments in panels of continuous slabs. The variables studied are listed and discussed briefly below.

(a) *Location of Panel.*—The panels of a continuous slab may be classified on the basis of location in three categories: interior, edge, and corner panels. In order that the slabs to be analyzed should have panels typical of all three types, a structure consisting of twenty-five equal panels arranged in five rows of five panels each was chosen. In such a structure the center panel is two rows distant from an edge and should be fairly representative of a typical interior

[1] See for example H. M. Westergaard, "Formulas for the Design of Rectangular Floor Slabs and the Supporting Girders," *Proc. ACI*, Vol. XXII, 1926, pp. 26-43.

panel. Similarly, a panel at the middle of a side of the slab is two panels distant from a corner and may be considered as a typical edge panel. Data may also be obtained for panels only one row removed from an edge or corner.

(b) *Ratio of Sides, b/a.*—Under similar conditions of loading and restraint the moments in a given panel are dependent to a large extent on the ratio of sides, $b/a$. Three values of this variable were considered: 0.5, 0.8, and 1.0. For each value, the slab analyzed consisted of twenty-five panels of equal size. The analyses for the intermediate ratio, 0.8, were less extensive than for the other ratios.

(c) *Torsional Stiffness of Beams.*—The effect of torsional stiffness of the supporting beams is to increase the restraint offered to rotation of the edges of a panel. This in turn results in a decrease in both positive and negative moments for an interior panel, but in an increase in negative moment on the exterior edge of an edge or corner panel. Three values of the ratio of beam torsional stiffness to slab flexural stiffness were considered, one of them being zero. The basis for choosing the particular values used is discussed in Section 19.

(d) *Type of Loading.*—A floor slab is subjected to two types of loading: (1) dead load, which is a uniformly distributed load over the entire floor area, and (2) live load, which may or may not be uniformly distributed. Throughout this bulletin the live load is assumed to be uniformly distributed over each panel,[1] but not all panels are assumed to be loaded. Two types of partial loading were considered in addition to the dead-load condition of all panels loaded. They are described in Section 20.

(e) *Variation in Size of Panel.*—All of the preceding discussion has referred to slabs in which all panels are of the same size and shape. It is also necessary to investigate the slab in which the adjacent continuous panels have different spans and different values of $b/a$. One such slab was analyzed. It consisted of sixteen panels having two different $b/a$ ratios and two different span lengths. One value of beam torsional stiffness and one type of live load only were considered for this structure. Further details are given in Section 21.

### 19. *Torsional Rigidity of Beams*

Two problems are considered in this section: first, the means of computing torsional rigidity of a beam, and second, the determination of typical values of the ratio of beam torsional stiffness to slab flexural stiffness.

[1] A limited amount of data for slabs with concentrated loads is given in Appendix C.

The resistance a beam offers to rotation about its axis, its torsional stiffness, may be computed if the geometrical and physical properties of the section are known, and if some assumption is made regarding the distribution of twisting moment along the length of the beam. The distribution assumed herein is that of a sine-wave having zero ordinates at the ends of the beam. This distribution is not unreasonable and results in the following simple approximate expression for the torsional stiffness, $T_b$, of a beam having a span $b$:[1]

$$T_b = \frac{\pi^2 GJ}{b^2} \tag{4}$$

wherein

$G$ = modulus of elasticity in shear of the material in the beam,

$J$ = measure of torsional rigidity of the cross-section of the beam.

The quantity $J$ for a rectangular section having a width $v$ and depth $d$ is given by Timoshenko[2] as:

$$J = \frac{v^3 d}{3} f_1. \tag{5}$$

The notation of Equation (5) has been changed from that used by Timoshenko. The quantity $f_1$ is a function of $v/d$ and is tabulated by Timoshenko; for the usual range of dimensions of reinforced concrete beams, it may be expressed as:

$$f_1 = 1 - 0.63 \frac{v}{d}. \tag{6}$$

The stiffness of a T-beam section is given by Nylander[3] as

$$J' = J + t^3 (0.33u - 0.17v - 0.21t) \tag{7}$$

wherein

$J'$ = torsional stiffness of T-beam section

$J$ = torsional stiffness of rectangular beam having dimensions $v$ by $d$

$d$ = over-all depth of T-beam

$v$ = width of stem of T-beam

$u$ = width of flange of T-beam

$t$ = thickness of flange.

In the analysis of continuous slabs, it is the ratio of beam torsional stiffness to slab flexural stiffness that is of interest. Consider

[1] N. M. Newmark, "A Distribution Procedure for the Analysis of Slabs Continuous over Flexible Beams," Univ. of Ill. Eng. Exp. Sta. Bul. 304 (1938); see pp. 28-29.

[2] S. Timoshenko, "Theory of Elasticity." McGraw-Hill, New York, 1934. Equation (5) is taken from the author's Eq. (157) on p. 249. Values of $f_1$ are tabulated as $k_1$ on p. 248.

[3] Henrik Nylander. "Torsion and Torsional Restraint of Concrete Structures" (in Swedish). Meddelanden, Statens Kommitté for Byggnadsforskning, Nr. 3, 1945, Stockholm. See Table 1(c), p. 124.

a slab panel having dimensions $b$ by $a$. What is desired is the ratio $T_b/K_b$ or $T_a/K_a$, where $T_b$ and $T_a$ are the torsional stiffnesses for the beams of span $b$ and $a$ respectively, and $K_b$ and $K_a$ are the flexural stiffnesses for the slab on edges $b$ and $a$ respectively. Considering only the beam of span $b$, and assuming a rectangular beam section, the stiffnesses are

$$T_b = \frac{\pi^2 G}{b^2} \frac{v^3 d}{3} \left(1 - 0.63 \frac{v}{d}\right) \tag{8}$$

and

$$K_b = \frac{k_b N}{b} = \frac{k_b E t^3}{12b}. \tag{9}$$

The stiffness ratio may then be expressed as

$$\frac{T_b}{K_b} = \frac{4\pi^2}{k_b} \cdot \frac{G}{E} \cdot \frac{d}{b} \cdot \frac{v^3}{t^3} \left(1 - 0.63 \frac{v}{d}\right). \tag{10}$$

A similar expression may be obtained for $T_a/K_a$ by substituting $a$ for $b$ in Eq. (10), adding the term $b/a$ to account for the lack of reciprocity in the definition of $K_b$ and $K_a$, and substituting the values of $v$ and $d$ for the beam with span $a$.

In Eq. (10) the gross cross-section has been considered in computing the stiffness of both the slab and the beam in accordance with the usual practice for reinforced concrete. There may be some question, however, as to the reliability of such a procedure in the case of torsional stiffness. Another question may be raised: whether the T-beam section should be used rather than a rectangular section as is done in Eq. (10). Calculations made for a few of the typical panels mentioned subsequently, and based on a flange width, $u$, of one-fourth the span, indicated that the value of $J'$ for a T-beam section was only 20–40 percent greater than the value of $J$ for the corresponding rectangular section. The possible errors from using the gross section and from using the rectangular section are of opposite sign; consequently, it was decided to use Eq. (10) for the calculation of the stiffness ratios.

The next step was to determine, if possible, the values of $T/K$ for typical designs of two-way building slabs. The two sources of such designs were:

(a) "Proposed Manual of Standard Practice for Detailing Reinforced Concrete Structures," published by the American Concrete Institute, Detroit, 1946, and referred to herein as ACI Manual. See Drawing 22 for two-way slab, on page 41.

(b) "Cost Estimates of Reinforced Concrete Floors," published by Portland Cement Association, Chicago, 1940, and referred to herein as PCA Cost Estimates.

The ACI Manual contained typical drawings for a two-way slab floor of 21 panels, having values of $b/a$ ranging from 0.77 to 1.0, and span lengths of 17.50 to 22.75 ft. Values of $T/K$ were computed by means of Eq. (10) for 8 typical edge beams and 18 typical interior beams. The average value thus obtained was 1.51, the minimum 1.21, and the maximum 1.87. There were no significant differences noted for edge and interior beams, and only a slight effect of either span length or $b/a$ was observed for the range of values considered.

The data in PCA Cost Estimates consisted of typical designs for two-way slab floors having square panels with spans of 15, 20, or 25 ft and designed for uniform live loads of 100 or 150 lb per sq ft. Slab thicknesses varied from 4 to 7.5 in. Data were thus available for six different floors. Values of $T/K$ were computed as before by means of Eq. (10). The average value of the ratio was 1.77, the minimum 1.18, and the maximum 2.28. The average ratio for designs with 150-lb live load was about 30 percent greater than for designs with 100-lb live load. A definite tendency was noted for the value of $T/K$ to decrease with increasing span length. The average for the 25-ft spans was about 72 percent of that for the 15-ft spans.

The range of $T/K$ for all of the panels considered was from 1.21 to 2.29, with an average of about 1.60. The use of a T-beam section would have increased these values by 20 to 40 percent. On the other hand, the stiffness of the beam cross-section may not be as great as that assumed, due to the effects of cracking. On the basis of the above results it does not seem unreasonable to assume that the stiffness ratio $T/K$ will normally lie in the range 1.5–2.0, and that it will seldom if ever fall below 1.0.

## 20. *Types of Loading*

Three types of loading were considered in the analyses described in this chapter. They are listed and discussed briefly below.

(a) Uniform load over entire area of slab, referred to herein as "uniform loading." The dead load, or weight of the structure itself, is of this type; and in certain types of storage buildings the live load may correspond more closely to this type of loading than to either of the other types mentioned below. In the design procedures for flat slab floors specified in both the Joint Committee Report[1] and the ACI Building Code[2] the assumption of a uniform loading is

[1] "Recommended Practice and Standard Specifications for Concrete and Reinforced Concrete." Report of the Joint Committee on Standard Specifications for Concrete and Reinforced Concrete. Published by American Concrete Institute, and others, June 1940.

[2] "Building Regulations for Reinforced Concrete (ACI 318-47)." American Concrete Institute, Detroit, 1941. (See especially Section 1003.)

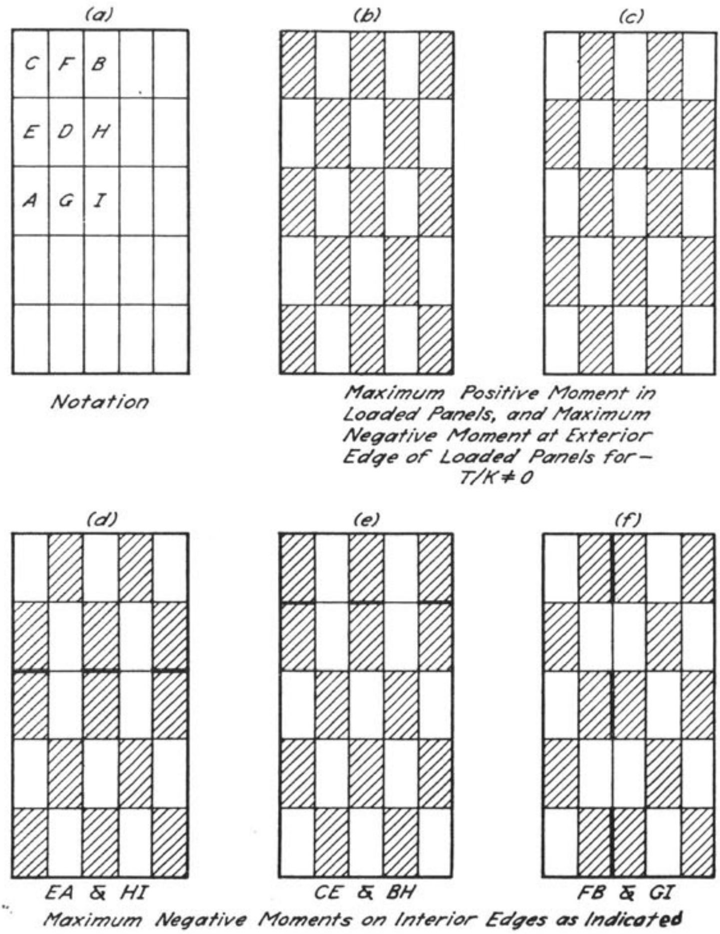



FIG. 18. TYPICAL CHECKERBOARD LOADINGS

implied by the nature of the expression used for total moment.

(b) Loading for maximum moments, referred to herein as "checkerboard loading." In this type, the load is considered as uniformly distributed over each panel, but only those panels are loaded which contribute positively to the moment being considered. Typical patterns of loading are illustrated in Fig. 18. For positive moment in the interior of a panel, alternate panels are loaded in checkerboard fashion. The same pattern is used to produce maximum negative moment at the exterior edge of an edge panel, when torsional restraint of the beams is present. For maximum negative moment over an interior beam, the checkerboard pattern is applied on either side of, and symmetrical about, a line passing along the edge in question. Such loadings are illustrated by the lower sketches in Fig. 18. Checkerboard loading has been used by Westergaard for the calculation of both positive and negative moments in two-way slabs.[1] It is recommended for positive moments in both the Joint Committee Report, Sec. 803(a)(1), and the ACI Building Code, Sec. 702(a)(2). It is also specified in the ACI Code, Sec. 1002(a)(6), for the design of flat slabs as continuous frames.

(c) The third type of loading is designated herein as "single-panel loading." For positive moment, or for negative moment at an exterior edge, only the panel in question is loaded. For negative moment over an interior beam, only the two adjacent panels are loaded. The only precedent for this type of loading for positive moment is the Joint Committee Report, Sec. 806(b)(2), with reference to moments in one-way slabs. Its use for negative moments, however, is recommended throughout both the Joint Committee Report, sections 803(a)(2) and 806(b)(1), and the ACI Code, sections 702(a)(2) and 1002(a)(6).

Three possible arrangements of live load which may be considered in the design of two-way slabs have been presented. In order of severity, as measured by the maximum moments produced, they rank as follows: 1) checkerboard, 2) single-panel, and 3) uniform. Some consideration also should be given to the relative probabilities of occurrence of each of these patterns. Obviously, the probability of obtaining a "single-panel loading" is very high, almost certain, since only one or two panels need be loaded. Second in order comes the uniform loading which is quite likely to occur in warehouses or similar structures if the aisle space is small or if the aisles are several panels apart. Even with wide or closely spaced aisles, it is possible to have several panels in the same row or in alternate rows fully loaded. The least probable of all the loading patterns considered is the checkerboard loading, which is necessary to produce absolute maximum moments. While a partial checkerboard loading might occur occasionally, the probability of exact duplication of the necessary pattern would seem to be small.

Data are given in Section 22 which permit a comparison of the moments produced by each of the three types of loading described in this section.

[1] H. M. Westergaard, "Formulas for the Design of Rectangular Floor Slabs and the Supporting Girders," *Proc.* ACI, Vol. XXII, 1926, pp. 26-43.

## 21. *Outline of Analyses*

Two types of structures are considered in this chapter: slabs with equal panels and slabs with unequal panels. The calculations for moments in each type are outlined and described briefly in the following paragraphs.

The major portion of this investigation was devoted to slabs having twenty-five equal panels arranged in five rows of five panels each. Three variables were considered: 1) the ratio of sides, $b/a$; 2) the ratio of beam torsional stiffness to slab flexural stiffness, $T/K$; and 3) the type of loading. Three $b/a$ ratios were considered: 0.5, 0.8, and 1.0. Plans of slabs having each of these ratios are given in Fig. 19a, b, and c respectively. The values 0.5 and 1.0 represent the limits of applicability of the distribution procedure. The intermediate value, 0.8, was used for a limited number of analyses in order to establish the nature of the variation of the most significant quantities between the limiting values of $b/a = 0.5$ and 1.0.

Three values of $T/K$ were considered: 0, 1, and 2. The zero value applies to slabs supported on bare steel beams and to other cases in which the beams provide no torsional restraint. The value of $T/K = 1$ was considered a minimum probable value for this ratio in an actual two-way slab supported on beams cast monolithically with the slab. Since all interior moments are decreased as the value of $T/K$ is increased a conservative design procedure should be based on a reasonable minimum for this ratio. The value of $T/K = 2$ was chosen as a maximum. This case must be considered, since the edge moment at a discontinuous edge exists only as a result of torsional restraint offered by the edge beam, and thus increases as $T/K$ is increased.

The three types of loading described in Section 20 were used in these analyses. The particular loadings used for each of the slabs analyzed are indicated in Table 9. It may be noted from this table that only eight different slabs were analyzed; a value of $T/K = 0$ was not considered for $b/a = 0.8$. All three types of loading were considered for the slabs with $b/a = 0.5$ and 1.0, while only the uniform and single-panel loadings were used for the structure with $b/a = 0.8$. Since the case of $T/K = 2$ was of interest only for moments at a discontinuous edge, the live-load types of loading were usually applied only for the calculation of such moments for this value of $T/K$.

In all the slabs having equal panels, the length, $b$, of the short span was taken as 10 ft, and the uniform load, $w$, for the loaded

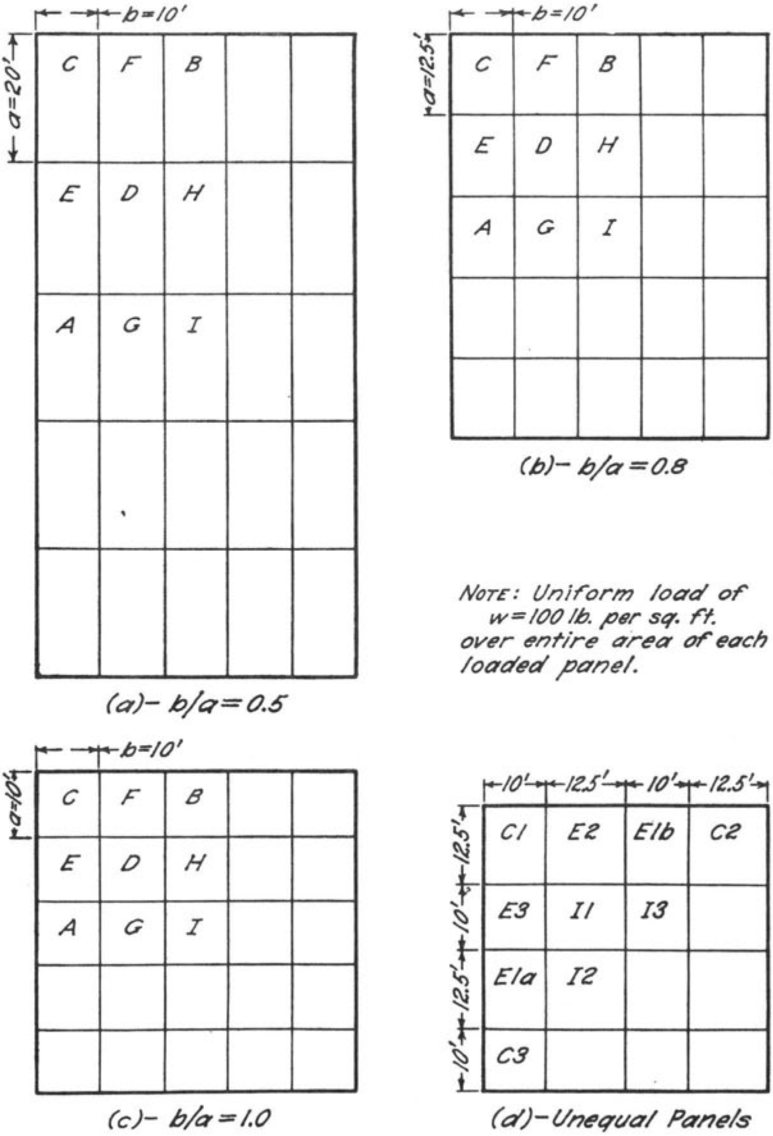



Fig. 19. Plans of Slabs Analyzed

panels was assumed to be 100 lb per sq ft. These values were so chosen as to result in convenient units for the average moments.

All moments were computed by means of the distribution procedure described in Chapter II, using the constants given in Table 1. The method of calculation finally adopted consisted of loading one panel at a time and computing the resulting moments on the edges of all panels. Edge moments for a particular combination of loaded panels were then obtained by superposition, and positive moments in the interior of the panels were computed from these values of the

TABLE 9
SUMMARY OF TYPES OF LOADING CONSIDERED IN ANALYSES OF SLABS HAVING TWENTY-FIVE EQUAL PANELS

| $\frac{T}{K}$ | Values of $b/a$ | | |
|---|---|---|---|
| | 0.5 | 0.8 | 1.0 |
| 0 | Uniform<br>Checkerboard<br>Single-panel | | Uniform<br>Checkerboard<br>Single-panel |
| 1 | Uniform<br>Checkerboard<br>Single-panel | Uniform<br><br>Single-panel | Uniform<br>Checkerboard<br>Single-panel |
| 2 | Uniform<br>Checkerboard*<br>Single-panel* | Uniform<br><br>Single-panel* | Uniform<br>Checkerboard<br>Single-panel |

* Loaded only for maximum edge moment at an exterior edge and maximum positive moments in edge panels.

edge moments. In practically all cases, however, the moments for all panels loaded were obtained by direct calculation rather than by superposition. The direct procedure was also followed for the checkerboard loadings on the slabs having $T/K = 0$. These results were later used as a check on the superposition procedure employed for the single-panel loadings on the same structures.

For the structures having $T/K = 0$, the exterior edges were simply-supported and use was made of the modified distribution constants in order to simplify the calculations. Modified constants were used likewise in those cases for which the loading was symmetrical. Advantage was also taken of symmetry in reducing the number of panels for which the moments had to be computed. For example, in the slabs having $b/a = 0.5$ or 0.8, moments were computed only for the panels designated by the letters in Fig. 19a and b. For the slab with $b/a = 1.0$, moments were computed only for the panels labeled C, F, B, D, H, and I in Fig. 19c. A total of 72 different analyses were made: 33 for $b/a=0.5$, 28 for $b/a=1.0$, and 11 for $b/a=0.8$.

The one slab having unequal panels is illustrated in Fig. 19d. It consists of 16 panels of three different sizes or shapes arranged in such a way that each type of panel occupies a corner, edge, and interior position. Two ratios of sides were used, $b/a=0.8$ and 1.0. The square slabs were of two sizes, 10 ft and 12.5 ft on a side respectively. The three shapes are designated as follows: 1) rectangular panel, 2) large square panel, and 3) small square panel. The letter designation indicates the position of the panel—corner, edge, or interior. In Fig. 19d it may be noted that there are two edge positions for the rectangular slab: one with the short side on the edge (E1b), and the other with the long side on the edge (E1a).

The object in analyzing the slab with unequal panels was to determine how the maximum moments in a given panel were affected by variations in the size and shape of the adjoining panels. It was deemed sufficient for this purpose to consider only $T/K=1$, and to use only the uniform and single-panel loadings. As in the case of the other slabs, a uniform load, $w$, of 100 lb per sq ft was used. All moment calculations were made by the distribution procedure.

Throughout all the analyses discussed in this section, the moments computed were the average moments on the section in question. The results for both types of slabs are presented and discussed in the following sections.

## 22. *Results of Analyses for Slabs with Equal Panels*

The moments obtained from the analyses of slabs having twenty-five panels are given in Tables 10, 11, and 12. The values tabulated are average moments in ft-lb per foot of width, or simply pounds. Plans of the slabs analyzed are shown in Fig. 19. The moments in each table are grouped by types as follows: negative moments over the interior beams, negative moments at an exterior edge, and positive moments. Moments in the short and long spans of the rectangular panels are listed separately. The edge of a panel is designated by two letters corresponding to the designations for the two adjoining panels. For example, CF refers to the edge between panels C and F. In each category the panels or edges are arranged with the corner panel first, then the edge panels, and finally the interior panels.

In this section the effects of three important variables are considered: 1) location of panel with respect to an edge, 2) type of loading, and 3) value of $T/K$, the relative torsional stiffness of the beams.

*Effect of Location of Panels.*—Several important conclusions may be drawn from the data in Tables 10, 11, and 12 concerning the variation of moment for panels at various locations relative to the edge of the slab. The general trend is for the moments in a panel to increase as the restraints at one or more edges are diminished. Thus the smallest moments are usually found for the interior panels; the next largest for an edge panel, with one discontinuous edge; the greatest for a corner panel, with two discontinuous edges.

The greatest difference in moments for the various panels is found for uniform loading on a slab having $T/K = 0$; that is, no

TABLE 10

MOMENTS IN CONTINUOUS SLABS HAVING TWENTY-FIVE EQUAL PANELS: $b/a = 0.5$

See Fig. 19a for plan of slab and system of designating panels. All values are average moments in panels for a uniform load $w$ of 100 lb per sq ft.

| LOADING* | | | $T/K=0$ | | | $T/K=1$ | | | $T/K=2$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | UL | SP | CB | UL | SP | CB | UL | SP | CB |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) |
| Negative Moment over Interior Beams | Short Span | CF | 716 | 815 | 846 | 644 | 687 | 692 | 617 | | |
| | | FB | 586 | 760 | 794 | 577 | 665 | 678 | 572 | | |
| | | ED | 639 | 771 | 806 | 612 | 677 | 685 | 597 | | |
| | | AG | 640 | 772 | 810 | 612 | 676 | 683 | 598 | | |
| | | DH | 545 | 720 | 781 | 554 | 655 | 674 | 554 | | |
| | | GI | 544 | 720 | 782 | 554 | 655 | 676 | 556 | | |
| | Long Span | CE | 536 | 683 | 725 | 412 | 483 | 494 | 376 | | |
| | | EA | 517 | 668 | 719 | 407 | 482 | 495 | 376 | | |
| | | FD | 277 | 549 | 594 | 305 | 447 | 456 | 311 | | |
| | | BH | 328 | 546 | 622 | 318 | 450 | 474 | 316 | | |
| | | DG | 283 | 539 | 599 | 306 | 445 | 460 | 310 | | |
| | | HI | 324 | 537 | 623 | 317 | 449 | 473 | 317 | | |
| Negative Moment at Exterior Edge | Short Span | C | 0 | 0 | 0 | 298 | 346 | 354 | 386 | 429 | 435 |
| | | E | 0 | 0 | 0 | 282 | 339 | 352 | 375 | 425 | 433 |
| | | A | 0 | 0 | 0 | 282 | 339 | 352 | 374 | 425 | 433 |
| | Long Span | C | 0 | 0 | 0 | 207 | 246 | 254 | 251 | 285 | 290 |
| | | F | 0 | 0 | 0 | 152 | 228 | 237 | 208 | 275 | 281 |
| | | B | 0 | 0 | 0 | 159 | 229 | 243 | 211 | 275 | 283 |
| Positive Moment | Short Span | C | 393 | 485 | 522 | 324 | 360 | 368 | 306 | 330 | 333 |
| | | F | 286 | 423 | 469 | 289 | 348 | 353 | 284 | | 329 |
| | | B | 310 | 420 | 480 | 292 | 347 | 362 | 286 | 324 | 330 |
| | | E | 333 | 445 | 492 | 304 | 353 | 361 | 296 | | 331 |
| | | A | 336 | 444 | 493 | 305 | 353 | 363 | 295 | 327 | 331 |
| | | D | 272 | 393 | 461 | 276 | 341 | 357 | 277 | | 330 |
| | | G | 271 | 393 | 462 | 277 | 341 | 358 | 277 | | 329 |
| | | H | 281 | 391 | 466 | 278 | 341 | 359 | 279 | | 331 |
| | | I | 282 | 391 | 467 | 278 | 341 | 360 | 279 | 322 | 330 |
| | Long Span | C | 115 | 132 | 139 | 102 | 108 | 110 | 98 | 103 | 103 |
| | | F | 92 | 120 | 129 | 94 | 106 | 107 | 93 | | 103 |
| | | B | 98 | 119 | 131 | 95 | 106 | 108 | 94 | 102 | 103 |
| | | E | 106 | 126 | 135 | 99 | 108 | 109 | 96 | | 103 |
| | | A | 106 | 126 | 135 | 99 | 108 | 110 | 96 | 102 | 103 |
| | | D | 91 | 116 | 127 | 92 | 105 | 108 | 93 | | 103 |
| | | G | 91 | 116 | 127 | 92 | 105 | 108 | 93 | | 103 |
| | | H | 94 | 115 | 129 | 93 | 105 | 108 | 93 | | 103 |
| | | I | 94 | 115 | 129 | 93 | 105 | 108 | 92 | 102 | 102 |

* UL = Uniform loading over all panels.
SP = Single-panel loading.
CB = Checkerboard loading.

torsional stiffness of the beams. Even for this case, however, there was no significant difference between the positive moments in any of the interior panels, whether they were one or two rows distant from the edge of the slab. In a square panel, or in the long span of a rectangular panel, the positive moments in an edge panel on a section parallel to a discontinuous edge were no different than those for an interior panel. However, the moments on a section perpen-

TABLE 11

MOMENTS AND COMPARISONS FOR CONTINUOUS SLABS HAVING TWENTY-FIVE EQUAL PANELS: $b/a = 0.8$

See Fig. 19b for plan of slab and system of designating panels. All moments are average moments in pounds for $w = 100$ lb per sq ft.

| LOADING* | | | MOMENTS | | | | COMPARISONS | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | $\frac{UL}{SP}$ % | | $\frac{T/K=2}{T/K=1}$ % | |
| | | | | | | $T/K=$ | 1 | 2 | UL | SP |
| | | | $T/K=1$ | | $T/K=2$ | | $\frac{(4)}{(5)}$ | $\frac{(6)}{(7)}$ | $\frac{(6)}{(4)}$ | $\frac{(7)}{(5)}$ |
| | | | UL | SP | UL | SP | | | | |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) |
| Negative Moment over Interior Beams | Short Span | CF | 449 | 487 | 429 | | 92 | | 95 | |
| | | FB | 426 | 478 | 414 | | 89 | | 97 | |
| | | ED | 404 | 473 | 401 | | 85 | | 99 | |
| | | AG | 404 | 473 | 401 | | 85 | | 99 | |
| | | DH | 389 | 465 | 389 | | 84 | | 100 | |
| | | GI | 389 | 465 | 389 | | 84 | | 100 | |
| | Long Span | CE | 376 | 426 | 353 | | 88 | | 94 | |
| | | EA | 371 | 422 | 351 | | 88 | | 95 | |
| | | FD | 309 | 401 | 310 | | 77 | | 100 | |
| | | BH | 311 | 401 | 311 | | 78 | | 100 | |
| | | DG | 309 | 399 | 310 | | 77 | | 100 | |
| | | HI | 311 | 399 | 311 | | 78 | | 100 | |
| Negative Moment at Exterior Edge | Short Span | C | 218 | 248 | 279 | 305 | 88 | 91 | 128 | 123 |
| | | E | 195 | 240 | 259 | 300 | 81 | 86 | 133 | 125 |
| | | A | 195 | 240 | 259 | 300 | 81 | 86 | 133 | 125 |
| | Long Span | C | 180 | 219 | 236 | 261 | 82 | 90 | 131 | 119 |
| | | F | 154 | 206 | 206 | 253 | 75 | 81 | 134 | 123 |
| | | B | 156 | 206 | 207 | 253 | 76 | 82 | 133 | 123 |
| Positive Moment | Short Span | C | 190 | 211 | 182 | 195 | 90 | 93 | 96 | 92 |
| | | F | 181 | 207 | 177 | | 87 | | 98 | |
| | | B | 181 | 207 | 177 | 192 | 87 | 92 | 98 | 93 |
| | | E | 172 | 204 | 171 | | 84 | | 99 | |
| | | A | 173 | 204 | 171 | 192 | 85 | 89 | 99 | 94 |
| | | D | 167 | 200 | 168 | | 83 | | 101 | |
| | | G | 167 | 200 | 168 | | 83 | | 101 | |
| | | H | 168 | 200 | 168 | | 84 | | 100 | |
| | | I | 168 | 200 | 168 | | 84 | | 100 | |
| | Long Span | C | 117 | 129 | 111 | 120 | 91 | 92 | 95 | 93 |
| | | F | 100 | 123 | 100 | | 81 | | 100 | |
| | | B | 100 | 123 | 101 | 117 | 81 | 86 | 101 | 95 |
| | | E | 117 | 128 | 112 | | 92 | | 96 | |
| | | A | 117 | 128 | 112 | 120 | 92 | 93 | 96 | 94 |
| | | D | 102 | 123 | 102 | | 83 | | 100 | |
| | | G | 102 | 123 | 102 | | 83 | | 100 | |
| | | H | 102 | 123 | 102 | | 83 | | 100 | |
| | | I | 102 | 123 | 102 | | 83 | | 100 | |

* See note at bottom of Table 10.

dicular to the discontinuous edge were greater than those for an interior panel. This may be explained by the fact that a span with restrained edges at both ends is stiffer than a span with one simply-supported end. Consequently, a greater proportion of the load is carried to the supports by the stiffer span, and the moments on sections normal to the simply-supported edge are thus increased. In the short span of a rectangular edge panel, the positive moments are not greatly different from those in an interior panel. However, the differences are just the opposite of those for the long span; that is, the moments on a section parallel to a discontinuous edge are increased more than those on a section perpendicular to such an edge.

Negative moments in a square panel or in the long span of a rectangular panel are practically the same over all interior beams, except over a beam perpendicular to a discontinuous edge where they are slightly higher. In the short span of a rectangular panel this effect is reversed; the moment over a beam opposite a discontinuous edge is appreciably higher than over an interior beam, while little difference is noted for the moment over a beam perpendicular to an edge.

TABLE 12

MOMENTS IN CONTINUOUS SLABS HAVING TWENTY-FIVE EQUAL PANELS: $b/a = 1.0$

See Fig. 19c for plan of slab and system of designating panels. All values are average moments in pounds for uniform load $w$ of 100 lb per sq ft.

| LOADING* | | | $T/K=0$ | | | $T/K=1$ | | | $T/K=2$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | UL | SP | CB | UL | SP | CB | UL | SP | CB |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) |
| Negative Moment over Interior Beams | CF | | 401 | 475 | 488 | 339 | 377 | 378 | 321 | 347 | 348 |
| | FB | | 376 | 455 | 479 | 337 | 372 | 379 | 319 | 346 | 348 |
| | FD | | 291 | 410 | 428 | 291 | 357 | 363 | 290 | 339 | 342 |
| | BH | | 290 | 409 | 431 | 290 | 357 | 364 | 291 | 339 | 342 |
| | DH | | 293 | 399 | 435 | 291 | 355 | 366 | 291 | 338 | 342 |
| | HI | | 290 | 398 | 434 | 290 | 356 | 371 | 292 | 338 | 342 |
| Negative Moment at Exterior Edge | C | | 0 | 0 | 0 | 170 | 194 | 197 | 217 | 235 | 239 |
| | F | | 0 | 0 | 0 | 145 | 184 | 190 | 192 | 229 | 231 |
| | B | | 0 | 0 | 0 | 145 | 184 | 191 | 194 | 229 | 233 |
| Positive Moment | C | | 156 | 186 | 196 | 134 | 146 | 148 | 128 | 136 | 137 |
| | F | ⊥† | 154 | 181 | 195 | 138 | 147 | 150 | 132 | 137 | 139 |
| | F | ‖ | 106 | 157 | 171 | 114 | 140 | 142 | 116 | 133 | 137 |
| | B | ⊥ | 154 | 181 | 195 | 137 | 147 | 150 | 131 | 137 | 138 |
| | B | ‖ | 109 | 157 | 173 | 113 | 140 | 142 | 115 | 133 | 135 |
| | D | | 119 | 156 | 177 | 120 | 140 | 145 | 120 | 134 | 137 |
| | H | | 118 | 156 | 177 | 120 | 140 | 146 | 120 | 134 | 137 |
| | I | | 120 | 156 | 178 | 120 | 140 | 146 | 119 | 134 | 136 |

* See note at bottom of Table 10.
† (⊥) indicates moment perpendicular to edge of slab.
(‖) indicates moment parallel to edge of slab.
Moments in panel *H* are same in both directions.

The effects described in the preceding paragraphs apply qualitatively to all the analyses made. Their magnitude, however, was affected by a number of factors. It has been mentioned that the greatest differences were obtained for uniform loading and $T/K=0$. The use of checkerboard loading, however, reduced the differences appreciably, as may be observed by comparing columns (4) and (6) in either Table 10 or Table 12. This effect is understandable, since the loading of alternate panels tends to decrease the effects of continuity. The differences for single-panel loadings were intermediate to those for checkerboard and uniform loadings, as would be expected.

If a value of $T/K=1$, representative of a minimum amount of torsional stiffness, is assumed, the differences between the panels at various locations are reduced to an almost negligible amount for both the checkerboard and single-panel loadings, although fairly large differences still exist for the uniform loading. For $T/K=1$, the maximum difference between the moments in a corner panel and in an interior panel is 5 percent for the checkerboard loading and 8 percent for the single-panel loading. For higher values of $T/K$ these differences are proportionately less; for $T/K=2$ they are about one-half as great.

In summary, it may be concluded that the maximum moments obtained by partial loading of either the checkerboard or single-panel type are practically independent of the location of the panel with respect to an edge if torsional beam stiffness corresponding to $T/K=1$ or more is considered.

*Effect of Type of Loading.*—The greatest moments were produced by the checkerboard loading, next greatest by the single-panel loading, and least by the uniform loading of all panels. In order to make quantitative comparisons of the moments from the three types of loading, the ratios in Tables 11, 13, and 14 have been computed. In preparing these tables the single-panel loading was used as a base.

A comparison of considerable interest is that between the moments for the two types of partial loadings. Ratios of checkerboard to single-panel moments are given in columns (4), (5), and (6) of Tables 13 and 14, for $b/a=0.5$ and 1.0 respectively. In all cases, the greatest moment is given by the checkerboard loading. For $T/K=0$, the difference is from 3 to 16 percent for negative moments and from 5 to 19 percent for positive moments. However, the assumption of torsional restraint by the beams reduces considerably the differences between the moments for the two types of loadings. For $T/K=1$, the maximum difference is 6 percent and the average only about 3 percent. For

$T/K = 2$, the maximum is 3 percent. Thus for a relatively small amount of beam torsional stiffness, the difference between moments for the two types of partial loading becomes negligible.

Comparisons between the moment due to single-panel and uniform loadings are of interest because the uniform loading represents dead load while the single-panel load is used for live load, and thus the relation between dead- and live-load moment coefficients may be studied. Data for the comparisons are given in columns (7), (8), and (9) of Tables 13 and 14, and in columns (8) and (9) of Table 11.

TABLE 13
COMPARISONS OF MOMENTS IN CONTINUOUS SLABS HAVING TWENTY-FIVE EQUAL PANELS: $b/a = 0.5$

See Fig. 19a for plan of slab and system of designating panels. All tabulated values are ratios of moments in Table 10, expressed in percent.

| | | | $\frac{CB^*}{SP}$ % | | | $\frac{UL}{SP}$ % | | | $\frac{T/K=0}{T/K=1}$ % | | | $\frac{T/K=2}{T/K=1}$ % | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $T/K =$ | | 0 | 1 | 2 | 0 | 1 | 2 | UL* | SP | CB | UL | SP | CB |
| Column Numbers from Table 10 | | | (6)/(5) | (9)/(8) | (12)/(11) | (4)/(5) | (7)/(8) | (10)/(11) | (4)/(7) | (5)/(8) | (6)/(9) | (10)/(7) | (11)/(8) | (12)/(9) |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | (14) | (15) |
| Negative Moment over Interior Beams | Short Span | CF | 104 | 101 | | 88 | 94 | | 111 | 119 | 122 | 96 | | |
| | | FB | 105 | 102 | | 77 | 87 | | 102 | 114 | 117 | 99 | | |
| | | ED | 105 | 101 | | 83 | 90 | | 104 | 114 | 118 | 98 | | |
| | | AG | 105 | 101 | | 83 | 90 | | 105 | 114 | 119 | 98 | | |
| | | DH | 109 | 103 | | 76 | 85 | | 99 | 110 | 116 | 100 | | |
| | | GI | 109 | 103 | | 76 | 85 | | 98 | 110 | 116 | 101 | | |
| | Long Span | CE | 106 | 102 | | 79 | 85 | | 130 | 141 | 147 | 91 | | |
| | | EA | 108 | 103 | | 77 | 82 | | 127 | 139 | 145 | 92 | | |
| | | FD | 108 | 102 | | 50 | 68 | | 91 | 123 | 130 | 102 | | |
| | | BH | 114 | 105 | | 60 | 71 | | 103 | 121 | 132 | 99 | | |
| | | DG | 111 | 103 | | 53 | 69 | | 92 | 121 | 130 | 101 | | |
| | | HI | 116 | 105 | | 60 | 71 | | 102 | 120 | 132 | 100 | | |
| Negative Moment at Exterior Edge | Short Span | C | | 102 | 102 | | 86 | 90 | | | | 130 | 124 | 123 |
| | | E | | 104 | 102 | | 83 | 88 | | | | 133 | 125 | 123 |
| | | A | | 104 | 102 | | 83 | 88 | | | | 133 | 125 | 123 |
| | Long Span | C | | 103 | 102 | | 84 | 88 | | | | 121 | 116 | 114 |
| | | F | | 104 | 102 | | 67 | 76 | | | | 137 | 120 | 118 |
| | | B | | 106 | 103 | | 69 | 77 | | | | 133 | 120 | 116 |
| Positive Moment | Short Span | C | 108 | 102 | 101 | 81 | 90 | 95 | 121 | 135 | 142 | 95 | 92 | 91 |
| | | F | 111 | 101 | | 68 | 83 | | 99 | 121 | 133 | 98 | | 93 |
| | | B | 114 | 104 | 102 | 74 | 84 | 88 | 106 | 121 | 133 | 98 | 98 | 91 |
| | | E | 111 | 102 | | 75 | 86 | | 110 | 126 | 136 | 98 | | 92 |
| | | A | 111 | 103 | 101 | 75 | 86 | 90 | 110 | 126 | 136 | 97 | 93 | 91 |
| | | D | 117 | 105 | | 69 | 81 | | 98 | 115 | 129 | 100 | | 92 |
| | | G | 118 | 105 | | 69 | 81 | | 98 | 115 | 129 | 100 | | 92 |
| | | H | 119 | 105 | | 72 | 82 | | 101 | 115 | 130 | 100 | | 92 |
| | | I | 119 | 106 | 103 | 72 | 82 | 87 | 101 | 115 | 130 | 100 | 98 | 92 |
| | Long Span | C | 105 | 102 | 100 | 87 | 94 | 95 | 113 | 122 | 126 | 96 | 95 | 94 |
| | | F | 108 | 101 | | 77 | 89 | | 98 | 113 | 121 | 99 | | 96 |
| | | B | 110 | 102 | 101 | 82 | 90 | 92 | 103 | 112 | 121 | 99 | 96 | 96 |
| | | E | 107 | 101 | | 84 | 92 | | 107 | 117 | 124 | 97 | | 95 |
| | | A | 107 | 102 | 101 | 84 | 92 | 94 | 107 | 117 | 123 | 97 | 94 | 94 |
| | | D | 110 | 103 | | 79 | 88 | | 99 | 110 | 118 | 101 | | 96 |
| | | G | 110 | 103 | | 79 | 88 | | 99 | 110 | 118 | 101 | | 96 |
| | | H | 112 | 103 | | 82 | 89 | | 101 | 109 | 119 | 100 | | 96 |
| | | I | 112 | 103 | 100 | 82 | 89 | 90 | 101 | 109 | 119 | 99 | 100 | 95 |

* See note at bottom of Table 10.

Tabulated values are the ratios in percent of the uniform-loading moments to the single-panel moments. In all cases the ratio is less than 100 percent. The ratio of moments for uniform loading to those for single-panel loading for $T/K = 1$ varies between 71 and 94 percent for negative moments over interior beams, and between 67 and 92 percent for negative moments at an exterior edge. Corresponding ratios for positive moments are 68 to 96 percent. In all cases the percentage values increase as $T/K$ is increased. In rectangular panels the ratio is greater for moments in the long span than for moments in the short span. Ratios for the short span are approximately the same as those for a square panel. In general, the differences between moments for the two types of loading were greatest for an interior panel and least for a corner panel. With uniform loading, considerable restraint is imposed at the interior edge of a panel as a result of continuity with adjacent loaded panels. This effect is of course greatest for interior panels and least for edge and corner panels. With single-panel loadings, however, the difference in degree of restraint at continuous and discontinuous edges is not so great since the adjacent panels are unloaded.

TABLE 14

COMPARISONS OF MOMENTS IN CONTINUOUS SLABS HAVING TWENTY-FIVE EQUAL PANELS: $b/a = 1.0$

See Fig. 19c for plan of slab and system of designating panels. All tabulated values are ratios of moments in Table 12 expressed in percent.

| | | | $\frac{CB^*}{SP}$ % | | | $\frac{UL}{SP}$ % | | | $\frac{T/K=0}{T/K=1}$ % | | | $\frac{T/K=2}{T/K=1}$ % | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $T/K =$ | | 0 | 1 | 2 | 0 | 1 | 2 | UL* | SP | CB | UL | SP | CB |
| Column Numbers from Table 12 | | | (6)/(5) | (9)/(8) | (12)/(11) | (4)/(5) | (7)/(8) | (10)/(11) | (4)/(7) | (5)/(8) | (6)/(9) | (10)/(7) | (11)/(8) | (12)/(9) |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) | (10) | (11) | (12) | (13) | (14) | (15) |
| Negative Moment over Interior Beams | CF | | 103 | 100 | 100 | 84 | 90 | 93 | 118 | 126 | 129 | 95 | 92 | 92 |
| | FB | | 105 | 102 | 101 | 83 | 91 | 92 | 112 | 122 | 126 | 95 | 93 | 92 |
| | FD | | 104 | 102 | 101 | 71 | 81 | 86 | 100 | 115 | 118 | 100 | 95 | 94 |
| | BH | | 106 | 102 | 101 | 71 | 81 | 86 | 100 | 115 | 119 | 100 | 95 | 94 |
| | DH | | 109 | 103 | 101 | 73 | 82 | 86 | 101 | 112 | 119 | 100 | 95 | 94 |
| | HI | | 109 | 104 | 101 | 73 | 82 | 86 | 100 | 112 | 117 | 101 | 95 | 92 |
| Negative Moment at Exterior Edge | C | | | 101 | 102 | | 88 | 92 | | | | 128 | 121 | 121 |
| | F | | | 103 | 101 | | 79 | 84 | | | | 132 | 125 | 121 |
| | B | | | 104 | 102 | | 79 | 85 | | | | 134 | 125 | 122 |
| Positive Moment | C | | 105 | 101 | 101 | 84 | 92 | 94 | 116 | 127 | 133 | 95 | 93 | 92 |
| | F | ⊥* | 108 | 102 | 102 | 85 | 94 | 96 | 112 | 123 | 130 | 96 | 93 | 93 |
| | F | ‖ | 109 | 102 | 103 | 68 | 82 | 87 | 93 | 112 | 120 | 102 | 95 | 96 |
| | B | ⊥ | 108 | 102 | 101 | 85 | 93 | 96 | 112 | 123 | 130 | 96 | 93 | 92 |
| | B | ‖ | 110 | 102 | 101 | 69 | 91 | 86 | 97 | 112 | 122 | 102 | 95 | 95 |
| | D | | 113 | 104 | 102 | 76 | 86 | 89 | 99 | 111 | 122 | 100 | 96 | 94 |
| | H | | 113 | 104 | 102 | 76 | 86 | 89 | 98 | 111 | 121 | 100 | 96 | 94 |
| | I | | 114 | 104 | 102 | 77 | 86 | 89 | 100 | 111 | 122 | 99 | 96 | 93 |

* See Tables 10 and 12 for notation.

*Effects of Beam Torsional Stiffness.*—Several effects of variations in $T/K$ have been noted in the preceding paragraphs. Further comparisons are made possible by the ratios given in columns (10) through (15) of Tables 13 and 14, and columns (10) and (11) of Table 11. In each case the value of $T/K=1$ has been taken as a base, and ratios of moment for other values of $T/K$ to those for this value are tabulated.

In general, all interior moments are greater for $T/K=0$ than for $T/K=1$. A few exceptions to this rule are noted for uniform loading where decreases were obtained at certain locations. The observed increases were greatest for checkerboard loadings and least for uniform loadings. As would be expected the differences were greater for corner or edge panels than for interior panels. In rectangular panels the increase in negative moments for $T/K=0$ was greatest for the long span, while for positive moments it was greatest for the short span. Percentage increases for the square panels lay between those for the long and short spans of the panels having $b/a=0.5$. The increases in moments from $T/K=1$ to $T/K=0$ varied considerably depending on the type of loading, $b/a$, etc. Average values, however, were as follows: 4 percent for uniform loading, 17 percent for single-panel loading, and 26 percent for checkerboard loading.

As the value of $T/K$ was increased from 1 to 2, all interior moments were decreased and all negative moments at an exterior edge were increased. The greatest decrease in the interior moments was 9 percent, and the average about 5 percent. This 5 percent decrease from $T/K=1$ to 2 may be compared with the decrease of about 18 percent from $T/K=0$ to 1 for interior moments and partial loadings. It is thus seen that by far the greatest proportion of the decrease in interior moments due to the assumption of beam torsional stiffness is obtained by assuming $T/K=1$.

The increase in edge negative moments corresponding to a change in $T/K$ from 1 to 2 was greatest for uniform loading and about equal for the other two loadings. For uniform loading the increase ranged from 21 to 37 percent with an average of 32 percent; for either checkerboard or single-panel loadings the range was 14 to 25 percent, with an average of 22 percent. Obviously, even greater increases would be obtained if higher values of $T/K$ were considered. For $T/K$ equal to infinity the edge moments are those for a fixed-edge panel. It is informative to note that for single-panel loadings the edge moments for $T/K=1$ averaged 65 percent of those for $T/K=$ infinity; for $T/K=2$ the corresponding figure was 80 percent. It was estimated that for this type of loading a moment equal to about 85 percent of the fixed-edge moment would be obtained for $T/K=3$. This constitutes an increase of only about 5 percent over the moments for $T/K=2$. Whereas a value of $T/K=1$ constituted a conservative assumption for the calculation of interior moments which are decreased by an increase in torsional stiffness, it is evident that a value of $T/K$ not less than 2 should be used when computing the moments at an exterior edge since they are increased as $T/K$ increases.

*Positive Moments in Interior Panels.*—The interior panels of a slab subjected to uniform loading are for all practical purposes fixed on all four edges. This is easily verified by comparing the edge moments for such panels with the fixed-edge moments in Table 1. It is the positive moments in these panels which are of interest here. Consider first the interior panel, I, of a slab having $b/a=1$. The positive moment due to uniform loading is obtained from column (4), (7), or (10) of Table 12 as 120. This is the value computed by means of the distribution procedure and the positive moment correction factors of Table 1. The correct value of the average positive moment in a fixed-edge slab is determined from Section 31, Appendix A, as 96. The approximate procedure thus gives a value for the positive moment in a square panel about 25 percent too high. Similarly, in an interior panel having $b/a=0.5$, the computed positive moment in the short span is 278 as compared to 255 for the exact average moment in a fixed-edge slab. In this case the error is only 9 percent. For the long span the approximate moment is 93 and the exact value is 51. The rather large discrepancy in this case is of little importance, however, since the moments in question are small.

The above differences between the approximate and exact positive moments for the few cases which permit comparisons to be made are probably typical of all the results for partial loadings with $T/K=1$ or more. The errors for the other panels, however, are not likely to exceed those given above for the interior panels. The source of these errors is of course the conservatism of the positive moment factors which has been discussed previously. Since a high degree of accuracy could not be obtained, these factors were chosen so that the larger errors would always be on the side of safety.

### 23. *Results of Analyses for Slab with Unequal Panels*

The slab analyzed consisted of sixteen panels of three different sizes or shapes, as illustrated in Fig. 19d. Torsional stiffness of the beams corresponding to a value of $T/K=1$ was assumed throughout; the value of $K$ used was the average for the two adjacent panels.

The types of loading considered included only uniform loading over all panels and single-panel loading. In the latter, one panel is loaded to obtain maximum positive moments or maximum negative moments at an exterior edge, while two adjacent panels are loaded to obtain maximum negative moments over an interior beam. Moments

TABLE 15

POSITIVE AND EDGE NEGATIVE MOMENTS FOR SLAB WITH SIXTEEN UNEQUAL PANELS

See Fig. 19d for plan of slab and notation. All values are average moments in pounds for a uniform load $w$ of 100 lb per sq ft. See text, Section 23, for description of calculations.

| | Direction of Moment* | Panel | Single-Panel Loading | | Uniform Loading | |
|---|---|---|---|---|---|---|
| | | | By Direct Distribution | From Equal Panels | By Direct Distribution | From Equal Panels |
| (1) | (2) | (3) | (4) | (5) | (6) | (7) |
| Positive Moment | $M_x$ | C1 | 209 | 211 | 190 | 190 |
| | | E2 | 230 | 230 | 212 | 216 |
| | | E1b | 206 | 207 | 178 | 181 |
| | | C2 | 229 | 228 | 213 | 209 |
| | | E3 | 140 | 140 | 112 | 114 |
| | | I1 | 123 | 123 | 99 | 102 |
| | | I3 | 140 | 140 | 120 | 120 |
| | | E1a | 203 | 204 | 174 | 173 |
| | | I2 | 219 | 219 | 191 | 187 |
| | | C3 | 146 | 146 | 133 | 134 |
| | $M_y$ | C1 | 128 | 129 | 114 | 117 |
| | | E2 | 218 | 218 | 185 | 178 |
| | | E1b | 123 | 123 | 95 | 100 |
| | | C2 | 229 | 228 | 213 | 209 |
| | | E3 | 147 | 147 | 138 | 138 |
| | | I1 | 200 | 200 | 166 | 167 |
| | | I3 | 140 | 140 | 120 | 120 |
| | | E1a | 128 | 128 | 115 | 117 |
| | | I2 | 219 | 219 | 191 | 187 |
| | | C3 | 146 | 146 | 133 | 134 |
| Negative Moment at Exterior Edge | $M_x$ | C1 | −250 | −248 | −217 | −218 |
| | | E3 | −185 | −184 | −142 | −145 |
| | | E1a | −240 | −240 | −197 | −195 |
| | | C3 | −194 | −194 | −169 | −170 |
| | $M_y$ | C1 | −221 | −219 | −185 | −180 |
| | | E2 | −286 | −288 | −236 | −226 |
| | | E1b | −206 | −206 | −144 | −156 |
| | | C2 | −302 | −303 | −274 | −266 |

* Axes indicated on Fig. 19d.

for each case were computed by means of the distribution procedure described in Chapter II. The results are given in Table 15 and Fig. 20.

Results are given in Table 15 only for positive moments and for negative moments at an exterior edge. The values in columns (4) and (6) are those determined by means of the distribution procedure for the slab with unequal panels. The values in columns (5) and (7) were obtained from the analyses of slabs having twenty-five equal panels. For each of the unequal panels in the slab of Fig. 19d, a

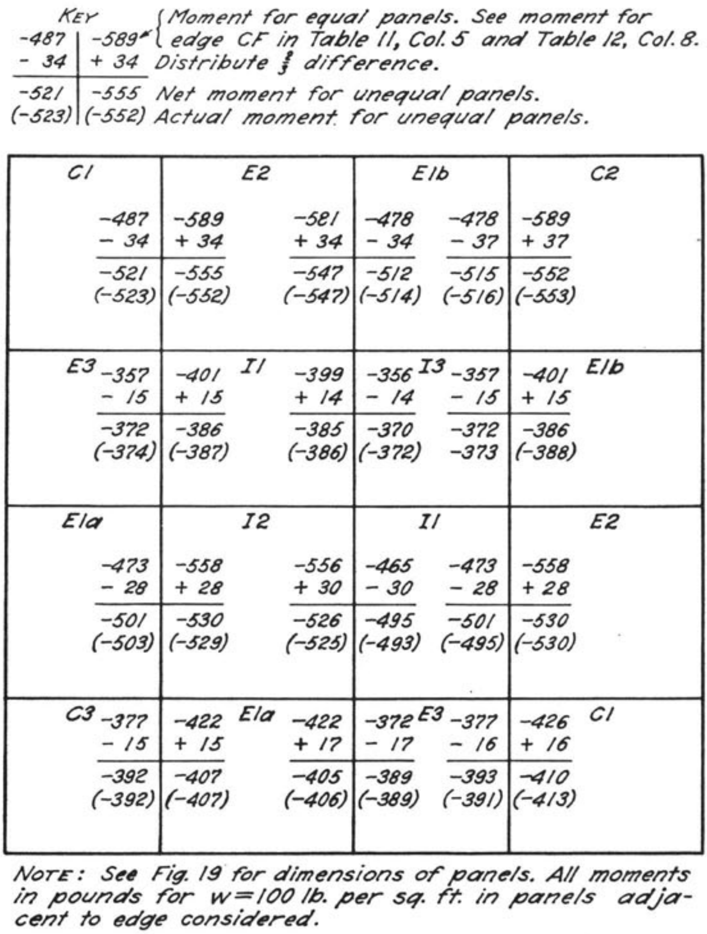



FIG. 20. NEGATIVE MOMENTS OVER INTERIOR BEAMS FOR SLAB WITH SIXTEEN UNEQUAL PANELS; SINGLE-PANEL LOADING, AND $T/K = 1$

corresponding panel having the same $b/a$ and $T/K$ ratios, occupying the same position relative to an edge or corner, and loaded in the same manner was chosen from the slabs of Fig. 19a, b, or c. The moments in this panel, corrected if necessary for differences in span length, are those recorded in columns (5) and (7). For example, panel C1 of Fig. 19d corresponds to panel C of Fig. 19b; thus the positive moment of 211 in the first line of column (5), Table 15, was obtained from column (5) of Table 11 on the line labeled "Positive Moment; Short Span; C." Similarly the positive moment of 230 in the second line of column (5), Table 15, is equal to $(12.5/10)^2$ times the moment of 147 in column (8) of Table 12 on the line labeled "Positive Moment; F; (⊥)." In this case a correction is necessary to take account of the difference in span length.

The results given in Table 15 for the single-panel loading indicate that the positive moments and the negative moments at an exterior edge of a given panel are practically independent of the size and shape of the adjacent panels. Although in this study the variation in span length of adjacent panels was only 25 percent, the very close agreement obtained suggests that much greater differences could exist without seriously affecting the accuracy of the results. For uniform loading, the agreement between moments for equal and unequal panels is not as good as for the single-panel loading. The maximum difference is about 8 percent, which is not excessive in view of the nature of the approximation involved. Some caution should be exercised, however, in extending this procedure to structures having differences in adjacent span lengths greater than about 25 percent if uniform loading is considered.

Negative moments over the interior beams for single-panel loadings are presented in Fig. 20. In this figure the moments applying to each edge are written in the appropriate panel adjacent to that edge. The values in the top line were obtained from the analyses for slabs with equal panels in the same manner as that described in connection with Table 15. For example, the moment of $-487$ at the left in the "Key" was obtained from column (5), Table 11, on the line labeled "Negative Moment; Short Span; CF." Similarly, the moment of $-589$ at the right is equal to $(12.5/10)^2$ times the moment of $-377$ on the first line of column (8), Table 12.

As may be seen from the figure, the moments obtained from the equal panel solutions are different on the two sides of an edge because of the difference in size and shape of the adjacent panels. These different moments may be "balanced," however, by using the distribution procedure in a manner similar to that used in balancing fixed-edge moments. Since the slab stiffness $K$ differs only slightly for adjacent unequal panels,[1] the value of $T$ used in the distribution procedure was assumed equal to the average of the $K$ values for the adjacent panels. For balancing the equal panel moments, a further simplification may be made by assuming the values of $K$ to be equal in adjacent panels. If $T/K = 1$, the correction moments are thus one-third of the difference between the moments in adjacent panels. This fraction would be different for other values of

[1] See Fig. 4 and discussion in Section 6.

$T/K$. Moreover, for greater variations in adjacent span lengths than those considered here or for slabs with panels having different thicknesses, more accurate assumptions regarding the relative magnitudes of $K$ for adjacent panels might have to be made. The correction moments, obtained as described above, are written on the second line of the calculations in Fig. 20.

The corrected equal-panel moments are written on the third line, and may be compared with the moments obtained by means of the distribution procedure for the slab with unequal panels, which are written on the bottom line in parentheses. The agreement between these two sets of moments is seen to be excellent. The maximum difference is only about 1 percent, and it seems likely that slabs with a considerably larger variation in adjacent span lengths could be handled by means of the procedure used in this case.

Comparisons similar to those in Fig. 20 were also made for uniform loading. The agreement for this case was not as good as for single-panel loading, but the maximum error did not exceed 6 percent. This relation between the results for uniform and single-panel loadings is substantially the same as that obtained for positive moments and for negative moments at an exterior edge.

Although the studies described in this section were not very extensive, it is believed that the following conclusions are justified. (1) For slabs consisting of dissimilar panels having the same thickness and not more than a 25 percent variation in adjacent span lengths, the positive moments and the negative moments at an exterior edge may be assumed equal to the corresponding moments in typical panels of a slab made up entirely of equal panels. (2) For the same conditions as those just stated, the negative moments over interior beams may be obtained from the corresponding moments in a group of equal panels, corrected as follows: For a given edge, determine the negative moment for each abutting panel from the equal-panel solutions; then, where these moments are different, distribute two-thirds of the difference equally to the two panels. If the slab thickness is different in adjacent panels, the first conclusion above regarding positive moments and negative moments at an exterior edge is still valid. However, for the calculation of interior negative moments, two-thirds of the difference should be distributed to the two panels in proportion to their respective stiffness factors, which in this case may differ appreciably because of the difference in thickness. The principles stated above have application in extending a design procedure based on the results of equal-panel analyses to the case of slabs consisting of dissimilar panels.

24. *Summary*

The distribution procedure for the analysis of continuous slabs has been applied to the study of moments in two-way reinforced concrete floors. The relative ease with which moments may be computed by this procedure made it feasible to undertake a large number of analyses and to investigate the effects of several variables. The following factors were studied: 1) location of panel with respect to an edge of the slab; 2) torsional restraint offered by the supporting beams; 3) various types of loading, including all panels loaded and two types of partial loadings; 4) ratio of sides of panels, $b/a$, for slabs in which all panels are similar; and 5) combinations of panels of various span lengths and values of $b/a$.

Eight slabs consisting of twenty-five equal panels and one slab consisting of sixteen unequal panels were analyzed for various loading conditions. A total of 83 different analyses were made. A summary of the calculations for the slab with equal panels is given in Table 9. The numerical values of the moments obtained from the analyses are presented in Tables 10–15 and in Fig. 20.

The following conclusions are believed to be justified on the basis of the data presented and discussed in this chapter:

1. The ratio, $T/K$, of the torsional stiffness of a beam to the flexural stiffness of the slab for typical two-way slabs supported on concrete beams was found to vary from 1.2 to 2.3, with an average of about 1.5. A reasonable minimum value to cover most cases would appear to be 1.0, neglecting both T-beam action and cracking of the concrete, two effects which act in opposite directions.

2. For $T/K$ equal to or greater than one, the moments produced in a continuous slab as a result of either single-panel or checkerboard loadings are practically the same for all panels, regardless of their location with respect to a discontinuous edge. Moments at a discontinuous edge are of course different. For these same conditions, the moments due to uniform loading are different in corner or edge panels but are the same for all interior panels. In an edge panel, the greatest effects were usually observed for moments on a section perpendicular to a discontinuous edge.

3. The greatest moments were produced by the checkerboard loading, and the least by the uniform loading. For $T/K$ equal to one or more, there was little difference between the moments for checkerboard and single-panel loadings. The moments for uniform loading, however, were somewhat smaller, ranging from 67 to 96 percent of those for partial loadings.

4. As the torsional stiffness of the beams is increased, both the positive moments and the negative moments over interior beams are decreased. An average decrease of 18 percent was produced by an increase in $T/K$ from zero to one, and an additional decrease of only 5 percent was produced by a further increase from one to two. It thus appears that a major portion of the effect of torsional stiffness on the interior moments is accounted for by the assumption of $T/K=1$.

5. As the torsional stiffness of the beams is increased, the negative moments at an exterior edge are likewise increased. For an increase in $T/K$ from one to two, this increase averaged 32 percent for uniform loading and 22 percent for the partial loadings. The resulting edge moments were on the average about 80 percent of those for a completely fixed edge. It was estimated that an increase of $T/K$ from two to three would increase the edge moments only about 5 percent more. Thus a value of $T/K=2$ would appear to be a reasonable upper limit for the beam torsional stiffness.

6. Average positive moments in interior panels and probably also in other panels were shown to be as much as 25 percent higher than the exact average moments. This difference was explained as the result of the conservative values of positive moment correction factors used with the distribution procedure.

7. For slabs consisting of a number of panels of unequal sizes and different shapes, but having not more than 25 percent variation in span length for adjacent panels, the positive moments and the negative moments at exterior edges may be taken as equal to the corresponding moments in panels of a slab consisting solely of equal panels. For the same conditions, negative moments over interior beams may also be obtained from the moments in slabs with equal panels, but a correction must be made as described in Section 23.

Four recommendations regarding the values of the several variables that should be used in the development of a design procedure for two-way concrete slabs supported on beams providing torsional restraint are based on the conclusions in the preceding paragraphs:

(a) The single-panel type of loading should be used for the calculation of moments due to live load.

(b) Uniform loading of all panels should be used for the calculation of moments due to dead load.

(c) Positive moments in all panels and negative moments over interior beams should be computed for an assumed torsional stiffness of the beams corresponding to $T/K=1$.

(d) Negative moments at an exterior discontinuous edge should be computed for $T/K=2$.

## V. Summary

The objective of the studies reported in this bulletin was to obtain a better understanding of the behavior of two-way slabs. The approach to this problem has been made in two steps: 1) the development of an approximate moment-distribution procedure for the calculation of moments in uniformly loaded plates continuous in two directions over rigid beams; 2) the application of this procedure to the study of the effects of several important variables on the moments in two-way slabs.

The distribution procedure which has been derived herein is analogous to the Cross moment-distribution method for the analysis of continuous beams and frames; that is, the procedure is one in which fixed-edge moments are calculated, unbalanced moments are distributed in proportion to the relative stiffnesses of the elements of the structure, and portions of the distributed moments on each edge of a panel are carried over to the other edges. The distribution procedure as applied herein to plates is approximate, however, and in this respect differs importantly from the Cross method. The moments considered are the average moments acting on the edge of a panel and are functions of the ratio of sides of the panel as well as the span length and loading. The stiffness factors are generally similar to those of the Cross method, but in addition to being functions of the span length and depth of the slab, they too are dependent on the ratio of sides, as are also the carry-over factors. The effect of edge moments on the positive moments in the interior of a simply-supported slab is determined approximately by means of correction factors which are functions only of the ratio of sides of the panel. The distribution procedure is described in Chapter II, and its development is explained in detail in Chapter III.

The distribution procedure was applied to the calculation of moments in a number of continuous slabs. Several variables were studied, including 1) the location of a panel with respect to a discontinuous edge; 2) torsional restraint offered by the supporting beams; 3) various types of loading, including all panels loaded, and two types of partial loading; 4) ratio of sides of panels for slabs in which all panels were similar; and 5) combination of panels of various span lengths and ratios of sides. Eight slabs consisting of twenty-five equal panels arranged in five rows of five panels each and one slab consisting of sixteen unequal panels were analyzed for various loading conditions. A total of 83 different analyses were made. From these studies, certain conclusions were reached regarding the effects of the several variables, and the types of loading and the values of torsional beam stiffness which should be considered in the development of a design procedure. The studies mentioned above are described in Chapter IV, and the results obtained and conclusions drawn are summarized in Section 24.

Supplementary data in the form of exact solutions for moments in various types of slabs are presented in the three appendixes following. The solutions in Appendix A for uniformly loaded single rectangular plates, and those in Appendix B for uniformly loaded continuous plates, were used extensively in the development of the distribution procedure. Rectangular plates with concentrated loads are considered briefly in Appendix C.

# Appendix A

## Moments in Uniformly Loaded Rectangular Plates with Various Edge Conditions

### 25. *Introduction*

Solutions are given in this Appendix for moments in uniformly loaded rectangular plates supported on all four sides and having various combinations of fixed and simply-supported edges. The moments were computed by means of the so-called ordinary theory of flexure for slabs, and are referred to herein as "exact" values. Actually, all the solutions involved the use of infinite series, and the correctness of the results is consequently dependent upon the number of terms used in the calculations. In general, the accuracy of the analyses was such that the moments given herein are believed to be correct to within three or four units in the last significant figure. Moments on the edges of a panel, which are computed directly from the terms in the series, are likely to be more accurate than this; moments in the interior, which are obtained by one or more corrections to the simply-supported slab moments, are probably the least accurate.

The various types of edge conditions considered include 1) simply-supported plates, 2) plates with one edge fixed, 3) plates with two opposite edges fixed, 4) plates with two adjacent edges fixed, 5) plates with three edges fixed, and 6) plates with all four edges fixed. The solutions for one or two edges fixed, and for three edges fixed with the short edge simply-supported, were obtained by the writers. However, in several cases, the solutions obtained by other investigators were extended, and moments were computed at additional points in the structure. This computation usually consisted of using the terms in the series for edge moments to determine the distribution of those moments across an edge, and to obtain the correction moments for the calculation of positive moments at points in the interior of a panel.

So far as possible a uniform notation has been used throughout this Appendix, and wherever necessary the notation used in solutions from other sources has been changed to conform. The system of coordinates is shown in Fig. 12. The designations $M_x$ and $M_y$ refer to moments per unit of width in the direction of $x$ and $y$ respectively, acting on a section normal to the $x$ or $y$ axis respectively. The spans $a$ and $b$ are in the direction of $x$ and $y$ respectively, and except where otherwise noted, the dimension $b$ refers to the shorter side.

All the solutions given herein are for a value of Poisson's ratio equal to zero. The moments at an edge are independent of this ratio; the moments in the interior are not, but may be determined for other values of Poisson's ratio by means of the equations given in Section 3 of this bulletin.

Average moments are computed in almost every case considered. Where the moment in question is expressed as a sine series, the average was obtained directly by computing the total moment from the terms in the series. Thus, if

$$M = \sum A_n \sin \frac{n\pi x}{a}$$

then

$$M_{\text{tot}} = \sum \frac{2}{n\pi} A_n, \quad \text{for } n \text{ odd only,}$$

and $M_{\text{av}}$ is equal to $M_{\text{tot}}$ divided by the width of the section being considered. Where series terms were not available but where values of the moment were known for points spaced at regular intervals across the section, as was frequently the case for positive moments in the interior of a panel, the average moment was computed from these values by means of Simpson's one-third rule for obtaining areas.

A brief summary of the moments computed and tabulated for the various cases considered is given in Table 16.

## 26. *Simply-Supported Plates*

The moments in uniformly loaded simply-supported plates were obtained from the results given by Leitz[1] and Bittner.[2] These are the only sources known to the writers in which moments are given for a large number of points in the interior of the plate.

Leitz has tabulated moments in plates having $b/a = 0.5$, 0.8, and 1.0 for points at the intersections of lines at $x/a$ and $y/b = 0$, 0.1, 0.2, 0.3, 0.4, and 0.5; and moments for $b/a = 0$ at the intersections of lines at $x/b = 0$, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.8, 1.0, and 2.0, and $y/b = 0$, 0.1, 0.2, 0.3, 0.4, 0.5. Coefficients are given for both $M_x$ and $M_y$ at each point, for the twisting moment $M_{xy}$, and for the principal moments and their directions. The tabulated moments are for a value of Poisson's ratio equal to zero. In presenting the moments herein, the notation used by Leitz has been converted to that of Fig. 12. Values are also given by Leitz for moments in both directions at the middle of the plate, for $b/a = 0$, 0.200, 0.250, 0.333, 0.400, 0.500, 0.571, 0.667, 0.800, and 1.0.

[1] H. Leitz, "Die Beruchnung der frei aufliegenden, rechteckigen Platten." *Forsch. auf dem Gebiete des Eisenbetons*, Heft XXIII. Wilhelm Ernst u. Sohn, Berlin, 1914.

[2] E. Bittner, "Momententafeln und Einflussflächen für kreuzweise bewehrte Eisenbetonplatten." Julius Springer, Vienna, 1938.

The values tabulated by Bittner are coefficients for $M_x$ and $M_y$ at the middle of a simply-supported plate loaded with a load, $P$, distributed over a centrally loaded rectangle having a width $t_x$ in the direction of the short span, $l_x$, and a length $t_y$ in the direction of the long span, $l_y$. Values of $M_x/P$ and $M_y/P$ are tabulated for $l_y/l_x = 1.0$, 1.1, 1.2, 1.3, 1.4, and 1.5, for various values of $t_y/l_x$ and $t_x/l_x$. Because of certain reciprocal relations, the tabulated value of $M/P$ for given values of $t_x$ and $t_y$ is numerically equal to the quantity $M'/4xyw$, where $M'$ is the moment at point $x$, $y$ in a simply-supported

TABLE 16
SUMMARY OF MOMENTS GIVEN IN APPENDIX A FOR UNIFORMLY LOADED RECTANGULAR PLATES

| Section | Edge Condition | Values of $b/a$ | Moments Given |
|---|---|---|---|
| 26 | Simply-Supported Plates | 0.5<br>0.5<br>1.0 | Moments in interior of panel at 1/10-points in both directions |
| | | 0.667<br>0.714<br>0.770<br>0.800<br>0.833<br>0.909 | Moments in interior of panel at 1/10-points in short span and at various intervals in long span |
| 27 | One Short Edge Fixed | 0.500<br>0.625<br>0.714<br>0.833<br>0.909<br>1.000 | Maximum and average moments on fixed edge and at midspan of both spans for all $b/a$. Average moments on certain sections in interior for $b/a = 0.5$ and 1.0, and maximum value of average positive moment in both spans |
| | One Long Edge Fixed | 0.5<br>0.6<br>0.7<br>0.8<br>0.9<br>1.0 | Same as above |
| 28 | Same as 27 except for two opposite edges fixed | | |
| 29 | Two Adjacent Edges Fixed | 0.5<br>1.0 | Moments at 1/12-points on fixed edges. Average moments on various sections in interior, and maximum value of average positive moment in both spans |
| 30 | One Short Edge Simply-Supported | 0.50<br>0.75 | Moments at 1/12-points on fixed edges. Average moments on various sections in interior, and maximum value of average positive moment in both spans |
| | One Long Edge Simply-Supported | 0.33<br>0.50<br>0.75<br>1.00 | Moments at 1/4-points on short edges and at 1/12-points on long edges. Average moments on various sections in interior, and maximum value of average positive moment in short span, for $b/a = 0.5$ and 1.0 |
| 31 | Four Edges Fixed | 0.500<br>1.000 | Moments in both directions at 1/10-points on edges and in interior of panel |
| | | 0.667 | Moments at 1/10-points on edges only |
| | | 0.571<br>0.800 | Maximum and average moments on edges only |

plate loaded with a uniformly distributed load, $w$, when $x = \frac{t_x}{2}$ and $y = \frac{t_y}{2}$. Using this relation, it is possible to compute the moments at a number of points in the interior of the plates, limited only by the range of tabulated values of $t_y/l_x$ and $t_x/l_x$. In the notation of this bulletin, values of $M_x$ and $M_y$ may be determined at $y/b = 0.1$, 0.2, 0.25, 0.3, 0.4, and 0.5, and at various values of $x/a$ depending on the particular value of $b/a$ considered. In general, data were available at sufficient values of $x/a$ to permit easy interpolation for moments at the one-tenth points of span $a$.

The most significant moments obtained from the solutions by Leitz and Bittner are given in Tables 17 and 18. Moments are given in Table 17 for the one-tenth points across the width of the plate at midspan. For the moment in the long span, $M_x$, values are also given for a section at or near that for which this moment is a maximum. For $b/a$ greater than 0.75, the maximum moment in the long span occurs at midspan; for smaller values of $b/a$ the location of the section for maximum moment is indicated in Table 18. The maximum value of the moment in the short span, $M_y$, occurs at midspan for all values of $b/a$. The average moments on the sections considered are given in the last column of Table 17.

Maximum and average moments for the various value of $b/a$ are summarized in Table 18. All values for $M_y$ are for a section at midspan. For the moment, $M_x$, in the long span, the average moment is given both for the section at midspan and for the section on which this moment is greatest.

The moments given by Leitz and Bittner for a plate having $b/a = 1.0$ were never different by more than one in the last place.

## 27. *Plates with One Edge Fixed*

Moments are given in this section for uniformly loaded rectangular plates having one edge fixed and the other three edges simply supported. Moments were computed at the fixed edge and in the interior of plates having either a short or a long edge fixed. For a long edge fixed, solutions were obtained for values of $b/a = 0.5$, 0.6, 0.7, 0.8, 0.9, and 1.0; for a short edge fixed, the calculations were made for $b/a = 0.500$, 0.555, 0.625, 0.714, 0.833, 0.909, and 1.000.

All solutions were obtained using the procedure developed by N. M. Newmark and described in Bulletin 304.[1] This procedure involves the use of infinite series, and the degree of accuracy of the

[1] N. M. Newmark, "A Distribution Procedure for the Analysis of Slabs Continuous over Flexible Beams," Univ. of Ill. Eng. Exp. Sta. Bul. 304. 1938.

results is dependent on the number of terms in the series considered. The number of terms used herein for each of the plates studied were as follows: For a long edge fixed, for $b/a = 0.5$, 0.6, and 0.7—eleven terms; for a long edge fixed for $b/a = 0.8$, 0.9, and 1.0—nine terms; for a short edge fixed for all values of $b/a$—seven terms. By eleven terms is meant that the terms in the series through $n = 11$ were used, even though in this case the even-numbered terms are equal to zero. The number of terms used in each case was so chosen as to obtain moments which would not be in error by more than about two in the fourth decimal place. This degree of accuracy applies to the moments on a fixed edge and also to the moments in the interior produced by the edge moments. These latter will be referred

TABLE 17
MOMENTS IN UNIFORMLY LOADED SIMPLY-SUPPORTED RECTANGULAR PLATES
Poisson's Ratio = 0. See Fig. 12 for notation.

| $\frac{b}{a}$ | $\frac{y}{b}$ | $M_y/wb^2$, in short span, at $x/a =$ | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.2 | 0.25 | 0.3 | 0.4 | 0.5 | Av |
| 0.500 | 0.5 | 0.0350 | 0.0631 | ...... | 0.0825 | 0.0934 | 0.0964 | 0.0651 |
| 0.667 | 0.5 | 0.0241 | 0.0449 | 0.0532 | 0.0605 | 0.0698 | 0.0727 | 0.0475 |
| 0.714 | 0.5 | 0.0218 | 0.0409 | 0.0485 | 0.0549 | 0.0637 | 0.0665 | 0.0433 |
| 0.770 | 0.5 | 0.0193 | 0.0362 | 0.0432 | 0.0490 | 0.0569 | 0.0595 | 0.0385 |
| 0.800 | 0.5 | 0.0180 | 0.0341 | ...... | 0.0460 | 0.0535 | 0.0560 | 0.0362 |
| 0.833 | 0.5 | 0.0167 | 0.0315 | 0.0377 | 0.0430 | 0.0500 | 0.0523 | 0.0338 |
| 0.909 | 0.5 | 0.0142 | 0.0267 | 0.0321 | 0.0367 | 0.0427 | 0.0447 | 0.0288 |
| 1.000 | 0.5 | 0.0115 | 0.0220 | 0.0263 | 0.0299 | 0.0351 | 0.0368 | 0.0236 |

| $\frac{b}{a}$ | $\frac{x}{a}$ | $M_x/wb^2$, in long span, at $y/b =$ | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.2 | 0.25 | 0.3 | 0.4 | 0.5 | Av |
| 0 | ($x/b =$ 0.3) | 0.0077 | 0.0143 | ...... | 0.0193 | 0.0227 | 0.0232 | 0.0152 |
| 0.500 | 0.5 | 0.0054 | 0.0103 | ...... | 0.0141 | 0.0166 | 0.0174 | 0.0111 |
| | 0.2 | 0.0079 | 0.0149 | ...... | 0.0203 | 0.0236 | 0.0248 | 0.0160 |
| 0.667 | 0.5 | 0.0087 | 0.0165 | 0.0199 | 0.0227 | 0.0266 | 0.0280 | 0.0179 |
| | 0.3 | 0.0091 | 0.0172 | 0.0206 | 0.0236 | 0.0275 | 0.0289 | 0.0185 |
| 0.714 | 0.5 | 0.0094 | 0.0179 | 0.0215 | 0.0246 | 0.0288 | 0.0302 | 0.0193 |
| | 0.357 | 0.0095 | 0.0181 | 0.0217 | 0.0248 | 0.0291 | 0.0305 | 0.0195 |
| 0.770 | 0.5 | 0.0101 | 0.0192 | 0.0230 | 0.0263 | 0.0309 | 0.0325 | 0.0207 |
| 0.800 | 0.5 | 0.0104 | 0.0198 | ...... | 0.0271 | 0.0318 | 0.0334 | 0.0213 |
| 0.833 | 0.5 | 0.0107 | 0.0203 | 0.0244 | 0.0279 | 0.0327 | 0.0343 | 0.0219 |
| 0.909 | 0.5 | 0.0113 | 0.0214 | 0.0256 | 0.0292 | 0.0342 | 0.0359 | 0.0230 |
| 1.000 | 0.5 | 0.0115 | 0.0220 | 0.0263 | 0.0299 | 0.0351 | 0.0368 | 0.0236 |

Moments for $b/a = 0$, 0.5, 0.8, and 1.0 from Leitz.
All others, and $M_x$ at $y/b = 0.25$ for $b/a = 1.0$, from Bittner.
All values of $M_y$ from Bittner except for $x/a = 0.5$ were obtained by interpolation from moments for odd values of $x/a$, and may be in error by three or four in the last place.
Averages by Simpson's One-third Rule.

TABLE 18
SUMMARY OF MAXIMUM AND AVERAGE MOMENTS IN UNIFORMLY LOADED SIMPLY-SUPPORTED RECTANGULAR PLATES
Poisson's Ratio = 0. See Fig. 12 for notation.

| $\frac{b}{a}$ | $M_y$: $\frac{M_{max}}{wb^2}$ at $x/a=0.5$ $y/b=0.5$ | $M_y$: $\frac{M_{av}}{wb^2}$ at $y/b=0.5$ | $M_x$: $\frac{M_{max}}{wb^2}$ at $x/a=0.5$ $y/b=0.5$ | $M_x$: $\frac{M_{av}}{wb^2}$ at $x/a=0.5$ | $M_x$: Maximum $M_{av}$: $\frac{M_{av}}{wb^2}$ | $M_x$: Maximum $M_{av}$: at $x/a=$ | Source |
|---|---|---|---|---|---|---|---|
| 0 | 0.1250 | 0.1250 | 0 | 0 | 0.0154 | $\left(\frac{x}{b}=0.32\right)$ | Leitz |
| 0.250 | 0.1231 | ...... | ...... | ...... | ...... | .... | Leitz |
| 0.333 | 0.1172 | ...... | ...... | ...... | ...... | .... | Leitz |
| 0.500 | 0.0964 | 0.0651 | 0.0174 | 0.0111 | 0.0164 | 0.16 | Leitz |
| 0.667 | 0.0727 | 0.0475 | 0.0280 | 0.0179 | 0.0185 | 0.30 | Bittner |
| 0.714 | 0.0665 | 0.0433 | 0.0302 | 0.0193 | 0.0195 | 0.36 | Bittner |
| 0.770 | 0.0595 | 0.0385 | 0.0325 | 0.0207 | 0.0207 | 0.50 | Bittner |
| 0.800 | 0.0560 | 0.0362 | 0.0334 | 0.0213 | 0.0213 | 0.50 | Leitz |
| 0.833 | 0.0523 | 0.0338 | 0.0343 | 0.0219 | 0.0219 | 0.50 | Bittner |
| 0.909 | 0.0447 | 0.0288 | 0.0359 | 0.0230 | 0.0230 | 0.50 | Bittner |
| 1.000 | 0.0368 | 0.0236 | 0.0368 | 0.0236 | 0.0236 | 0.50 | Leitz, Bittner |

to as the correction moments, since the actual moments in the interior of a plate are equal to the moments in a uniformly loaded simply-supported plate plus these correction moments resulting from the edge conditions. Final moments in the interior are probably not as accurate as those on an edge, since in some cases the simply-supported plate moments had to be obtained from the data given in Section 26 by interpolation. Consequently, these moments may be in error by as much as three or four in the last place given.

Moments in the fixed span are given in Table 19 for the several plates analyzed, and moments in the simply-supported span are given in Table 20. It should be noted that in both tables the moments for $b/a=0$ are given for values of $x/b$ rather than $x/a$, since the length $a$ is infinite. An exception occurs in Table 19 in the column giving $M_y$ at $x/a=0.5$. In this case the moments for $b/a=0$ are for an arbitrary location near the middle of the infinitely long span $a$.

Average moments are given in every case considered. In Table 19 the average moments at a fixed edge, and the average correction moments in the interior, were obtained directly from the terms in the series for the respective moments by means of the expression given in Section 25. The average positive moments given in Table 19 are equal to the sum of the average positive moments that are obtained from the data in Section 26, by interpolation if necessary, and the average correction moments obtained as described above.

In Table 20, the average moments were computed from the moments at the one-tenth points by the application of Simpson's one-third rule. Where the moments at the one-tenth points are not given in this table, the average moments were obtained as the sum of the average simply-supported plate moments and the average correction moments, which were computed by means of Simpson's one-third rule from values at the one-tenth points.

TABLE 19

MOMENTS IN FIXED SPAN OF UNIFORMLY LOADED RECTANGULAR PLATES FIXED ON ONE EDGE AND SIMPLY-SUPPORTED ON OTHERS

Poisson's Ratio = 0. See Fig. 12 for notation. Fixed edge is at $x=0$ or $y=0$ as noted in table headings. Average moments from terms in series.

| $\frac{b}{a}$ | $\frac{x}{a}$ | $\frac{M_x}{wb^2}$ in fixed span for short edge at $x=0$ fixed | | $\frac{b}{a}$ | $\frac{y}{b}$ | $\frac{M_y}{wb^2}$ in fixed span for long edge at $y=0$ fixed | |
|---|---|---|---|---|---|---|---|
| | | $y/b=0.5$ | $M_{av}$ | | | $x/a=0.5$ | $M_{av}$ |
| 0 | $x/b=0$ | −0.1248 | −0.0832 | 0 | 0 | −0.1250 | −0.1250 |
| | 0.1 | −0.0644 | −0.0410 | | 0.500 | 0.0625 | 0.0625 |
| | 0.2 | −0.0644 | −0.0161 | | 0.625* | 0.0702 | 0.0702 |
| | 0.3 | −0.0035 | −0.0017 | | | | |
| | 0.4 | 0.0092 | 0.0060 | | | | |
| | 0.5 | 0.0151 | 0.0098 | | | | |
| | 0.6 | 0.0174 | 0.0111 | | | | |
| | 0.68* | 0.0181 | 0.0115 | | | | |
| | 0.8 | 0.0158 | 0.0100 | | | | |
| | 1.0 | 0.0120 | 0.0076 | | | | |
| 0.500 | 0 | −0.1216 | −0.0812 | 0.5 | 0 | −0.1210 | −0.0905 |
| | 0.1 | −0.0239 | −0.0145 | | 0.5 | 0.0582 | 0.0410 |
| | 0.2 | 0.0115 | 0.0075 | | 0.6 | | 0.0452 |
| | 0.3 | 0.0217 | 0.0132 | | 0.62* | | 0.0453 |
| | 0.4 | 0.0213 | 0.0135 | | 0.7 | | 0.0435 |
| | 0.5 | 0.0206 | 0.0130 | | 0.8 | | 0.0359 |
| | 0.6 | 0.0212 | 0.0134 | | | | |
| | 0.7 | 0.0236 | 0.0150 | 0.6 | 0 | −0.1156 | −0.0836 |
| | 0.8 | 0.0259 | 0.0167 | | 0.5 | 0.0539 | 0.0367 |
| | 0.85* | ...... | 0.0169 | | | | |
| | 0.9 | 0.0225 | 0.0151 | | | | |
| 0.555 | 0 | −0.1185 | −0.0794 | 0.7 | 0 | −0.1086 | −0.0767 |
| | 0.5 | 0.0243 | 0.0153 | | 0.5 | 0.0487 | 0.0321 |
| 0.625 | 0 | −0.1147 | −0.0771 | 0.8 | 0 | −0.1009 | −0.0700 |
| | 0.5 | 0.0280 | 0.0178 | | 0.5 | 0.0428 | 0.0279 |
| 0.714 | 0 | −0.1084 | −0.0727 | | | | |
| | 0.5 | 0.0312 | 0.0199 | 0.9 | 0 | −0.0922 | −0.0633 |
| 0.833 | 0 | −0.0981 | −0.0664 | | 0.5 | 0.0371 | 0.0242 |
| | 0.5 | 0.0327 | 0.0210 | | | | |
| 0.909 | 0 | −0.0916 | −0.0623 | | | | |
| | 0.5 | 0.0328 | 0.0210 | | | | |
| 1.000 | 0 | −0.0840 | −0.0572 | 1.0 | Same as $M_x$ at left | | |
| | 0.3 | 0.0146 | 0.0098 | | | | |
| | 0.4 | 0.0259 | 0.0168 | | | | |
| | 0.5 | 0.0318 | 0.0205 | | | | |
| | 0.6* | 0.0341 | 0.0219 | | | | |
| | 0.7 | 0.0329 | 0.0213 | | | | |

* Maximum moment obtained by algebraic interpolation using parabola fitted through three points.

### TABLE 20

### MOMENTS IN SIMPLY-SUPPORTED SPAN OF UNIFORMLY LOADED RECTANGULAR PLATES FIXED ON ONE EDGE AND SIMPLY-SUPPORTED ON OTHERS

Poisson's Ratio = 0. See Fig. 12 for notation. Fixed edge is at $x=0$ or $y=0$ as noted in table headings. Average moments by Simpson's One-third rule.

| $\frac{b}{a}$ | $\frac{y}{b}$ | $M_y/wb^2$ in short span for short edge at $x=0$ fixed, for $x/a=$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | $M_{av}$ |
| 0 | 0.5 | $x/b=0.1$ 0.0042 | $x/b=0.2$ 0.0149 | $x/b=0.3$ 0.0286 | $x/b=0.4$ 0.0427 | $x/b=0.5$ 0.0562 | $x/b=0.6$ 0.0686 | $x/b=0.8$ 0.0884 | $x/b=1.0$ 0.1019 | $x/b=2.0$ 0.1234 | |
| 0.500 | 0.5 | 0.0146 | 0.0407 | 0.0645 | 0.0806 | 0.0880 | 0.0878 | 0.0793 | 0.0611 | 0.0342 | 0.0555 |
| 0.555 | 0.5 | | | | | | | | | | 0.0483 |
| 0.625 | 0.5 | | | | | | | | | | 0.0411 |
| 0.714 | 0.5 | 0.0066 | 0.0210 | 0.0356 | 0.0468 | 0.0532 | 0.0537 | 0.0481 | 0.0367 | 0.0195 | 0.0323 |
| 0.833 | 0.5 | 0.0042 | 0.0143 | 0.0250 | 0.0337 | 0.0386 | 0.0393 | 0.0352 | 0.0266 | 0.0142 | 0.0233 |
| 1.000 | 0.5 | 0.0025 | 0.0084 | 0.0151 | 0.0208 | 0.0243 | 0.0248 | 0.0222 | 0.0168 | 0.0090 | 0.0145 |

| $\frac{b}{a}$ | $\frac{x}{a}$ | $M_x/wb^2$ in long span for long edge at $y=0$ fixed, for $y/b=$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | $M_{av}$ |
| 0.5 | 0.1 | | | | | | | | | | 0.0105 |
| | 0.15 | | | | | | | | | | 0.0106 |
| | 0.2 | | | | | | | | | | 0.0092 |
| | 0.3 | | | | | | | | | | 0.0063 |
| | 0.4 | | | | | | | | | | 0.0041 |
| | 0.5 | 0.0006 | 0.0016 | 0.0033 | 0.0048 | 0.0059 | 0.0064 | 0.0058 | 0.0045 | 0.0024 | 0.0035 |
| 0.6 | 0.5 | | | | | | | | | | 0.0061 |
| 0.7 | 0.5 | | | | | | | | | | 0.0089 |
| 0.8 | 0.5 | 0.0017 | 0.0060 | 0.0112 | 0.0157 | 0.0186 | 0.0193 | 0.0173 | 0.0131 | 0.0070 | 0.0110 |
| 0.9 | 0.5 | | | | | | | | | | 0.0132 |
| 1.0 | 0.5 | 0.0025 | 0.0084 | 0.0151 | 0.0208 | 0.0243 | 0.0248 | 0.0222 | 0.0168 | 0.0090 | 0.0145 |

## 28. *Plates with Two Opposite Edges Fixed*

Moments are given in this section for uniformly loaded rectangular plates having two opposite edges fixed and the other two simply-supported. These solutions are generally parallel to those described in Section 27, and differ only in that an additional edge is considered fixed, opposite to the single fixed edge of that section.

TABLE 21

MOMENTS IN FIXED SPAN OF UNIFORMLY LOADED RECTANGULAR PLATES FIXED ON TWO OPPOSITE EDGES AND SIMPLY-SUPPORTED ON OTHERS

Poisson's Ratio = 0. See Fig. 12 for notation. All moments symmetrical about both centerlines of plate. Long or short edges fixed as indicated in table headings.

| $\frac{b}{a}$ | $\frac{x}{a}$ | $\frac{M_x}{wb^2}$ in fixed span for short edges fixed | |
|---|---|---|---|
| | | $y/b = 0.5$ | $M_{av}$ |
| 0.100 | 0 | −0.1242 | ...... |
| 0.167 | 0 | −0.1244 | ...... |
| 0.200 | 0 | −0.1250 | ...... |
| 0.500 | 0 | −0.1192 | −0.0796 |
| | 0.1 | −0.0227 | −0.0139 |
| | 0.2 | 0.0128 | 0.0084 |
| | 0.3 | 0.0224 | 0.0142 |
| | 0.4 | 0.0237 | 0.0150 |
| | 0.5 | 0.0234 | 0.0148 |
| 0.555 | 0 | −0.1150 | −0.0771 |
| | 0.5 | 0.0272 | 0.0171 |
| 0.625 | 0 | −0.1089 | −0.0732 |
| | 0.5 | 0.0300 | 0.0192 |
| 0.714 | 0 | −0.0997 | −0.0672 |
| | 0.5 | 0.0320 | 0.0204 |
| 0.833 | 0 | −0.0866 | −0.0589 |
| | 0.5 | 0.0317 | 0.0201 |
| 0.909 | 0 | −0.0785 | −0.0538 |
| | 0.5 | 0.0306 | 0.0197 |
| 1.000 | 0 | −0.0697 | −0.0481 |
| | 0.5 | 0.0284 | 0.0184 |

| $\frac{b}{a}$ | $\frac{y}{b}$ | $\frac{M_y}{wb^2}$ in fixed span for long edges fixed | |
|---|---|---|---|
| | | $x/a = 0.5$ | $M_{av}$ |
| 0.3 | 0 | −0.0836 | ...... |
| 0.4 | 0 | −0.0841 | ...... |
| 0.5 | 0 | −0.0841 | −0.0662 |
| | 0.5 | 0.0416 | 0.0305 |
| 0.6 | 0 | −0.0833 | −0.0627 |
| | 0.5 | 0.0402 | 0.0283 |
| 0.7 | 0 | −0.0814 | −0.0592 |
| | 0.5 | 0.0361 | 0.0255 |
| 0.8 | 0 | −0.0783 | −0.0556 |
| | 0.5 | 0.0352 | 0.0231 |
| 0.9 | 0 | −0.0743 | −0.0519 |
| | 0.5 | 0.0317 | 0.0210 |
| 1.0 | 0 | −0.0697 | −0.0481 |
| | 0.5 | 0.0284 | 0.0184 |

NOTE: Values of $M_x$ at various $x/b$ for $b/a = 0$ are same as those given in Table 19.

The method of solution, the values of $b/a$, and the number of terms in the series are the same for both sets of solutions. Consequently the discussion in Section 27 regarding the accuracy of the results applies also herein. Average moments and positive moments were, in general, computed by the same procedures for the two groups of plates.

The moments are given in Tables 21 and 22, which are comparable in content to Tables 19 and 20 respectively. In Table 21, the values of $M_x$ for $b/a = 0$ have been omitted, as they are identical with those given in Table 19. Also, because of symmetry, the maximum positive moments nearly always occurred at midspan, and moments

TABLE 22
MOMENTS IN SIMPLY-SUPPORTED SPAN OF UNIFORMLY LOADED RECTANGULAR PLATES FIXED ON TWO OPPOSITE EDGES AND SIMPLY-SUPPORTED ON OTHERS

Poisson's Ratio = 0. See Fig. 12 for notation. All moments symmetrical about both centerlines of plate. Long or short edges fixed as indicated in table headings.

| $\frac{b}{a}$ | $\frac{y}{b}$ | $\frac{M_y}{wb^2}$ for short edges fixed, for $x/a =$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | $M_{av}$ |
| 0.500 | 0.5 | 0.0142 | 0.0395 | 0.0617 | 0.0758 | 0.0796 | 0.0463 |
| 0.555 | 0.5 | | | | | 0.0692* | 0.0395 |
| 0.625 | 0.5 | | | | | 0.0562* | 0.0313 |
| 0.714 | 0.5 | 0.0058 | 0.0185 | 0.0305 | 0.0388 | 0.0422 | 0.0230 |
| 0.833 | 0.5 | 0.0035 | 0.0119 | 0.0202 | 0.0260 | 0.0281 | 0.0152 |
| 1.000 | 0.5 | 0.0019 | 0.0064 | 0.0111 | 0.0146 | 0.0158 | 0.0084 |

| $\frac{b}{a}$ | $\frac{x}{a}$ | $\frac{M_x}{wb^2}$ for long edges fixed, for $y/b =$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | $M_{av}$ |
| 0.5 | 0.1 | | | | | | 0.0078 |
| | 0.15 | | | | | | 0.0071 |
| | 0.2 | | | | | | 0.0055 |
| | 0.3 | | | | | | 0.0029 |
| | 0.4 | | | | | | 9.0015 |
| | 0.5 | 0.0005 | 0.0006 | 0.0012 | 0.0015 | 0.0016 | 0.0009 |
| 0.6 | 0.5 | | | | | 0.0041 | 0.0021 |
| 0.7 | 0.5 | | | | | 0.0071* | 0.0038 |
| 0.8 | 0.5 | 0.0010 | 0.0035 | 0.0070 | 0.0095 | 0.0102 | 0.0052 |
| 0.9 | 0.5 | | | | | 0.0130* | 0.0070 |
| 1.0 | 0.5 | 0.0019 | 0.0064 | 0.0111 | 0.0146 | 0.0158 | 0.0084 |

* Obtained by graphical interpolation from moments for other values of $b/a$.
NOTE: Values of $M_y$ at various $x/b$ for $b/a = 0$ are same as those given in Table 20.

at other values of $x/a$ or $y/b$ were not computed, except for $M_x$ in the plate having the short sides fixed and $b/a = 0.5$.

Table 22 differs from Table 20 principally in the omission of the moments at $x/a$ or $y/b$ between 0.5 and 1.0. Because of symmetry, these moments need not be tabulated. The moments $M_x$ for $b/a = 0$ are also omitted from this table, since they are identical with those given in Table 20.

### 29. *Plates with Two Adjacent Edges Fixed*

Moments are given in this section for uniformly loaded rectangular plates having two adjacent edges fixed and the other two simply-supported. Moments were computed at the fixed edges and in the interior, for plates having ratios of sides equal to 0.5 and 1.0.

The moments given herein were computed by the writers, using a modification of the method developed by Timoshenko.[1] The procedure described by Timoshenko involves the expression of moments and rotations on the edges of the plate in terms of infinite series. Equations are derived for the rotation at each edge in terms of the moments on that edge and on the other edges. At a fixed edge the net rotation must equal zero; that is, the rotation produced by the bending moments on the various edges must be equal and opposite to the rotation produced by the given loading on a simply-supported plate. To satisfy this condition each term in the series for rotation at a fixed edge is set equal to zero, and a number of equations are obtained, one for each term in the series for each fixed edge. By considering only a finite number of terms in the series, the resulting finite number of equations may be solved for the coefficients of the terms in the series for edge moments.

The equations given by Timoshenko have been modified herein so as to utilize the constants tabulated in Bulletin 304. Consider a rectangular plate supported on all four sides and having the dimensions and coordinate system indicated in Fig. 12. Let the moments be expressed in the following terms:

$$\begin{aligned} M_1 &= \sum_n A_n \sin \frac{n\pi y}{b} \\ M_2 &= \sum_n B_n \sin \frac{n\pi y}{b} \\ M_3 &= \sum_m C_m \sin \frac{m\pi x}{a} \\ M_4 &= \sum_m D_m \sin \frac{m\pi x}{a}, \end{aligned} \qquad (11)$$

wherein $A_n$, $B_n$, $C_m$, and $D_m$ are the coefficients whose values are to be determined. The rotations on each edge may be expressed in a similar manner as follows:

[1] S. P. Timoshenko, "Bending of Rectangular Plates with Clamped Edges," *Proc. Fifth International Congress for Applied Mechanics*. Wiley, New York, 1940; pp. 40-43. See also "Theory of Plates and Shells" by the same author (McGraw-Hill, New York, 1940), pp. 222 ff.

$$
\begin{aligned}
\Phi_1 &= \sum_n P_n \sin \frac{n\pi y}{b} \\
\Phi_2 &= \sum_n Q_n \sin \frac{n\pi y}{b} \\
\Phi_3 &= \sum_m R_m \sin \frac{m\pi x}{a} \\
\Phi_4 &= \sum_m U_m \sin \frac{m\pi x}{a}.
\end{aligned}
\tag{12}
$$

The rotations on each edge may be expressed in terms of the moments on all edges by means of the following equations relating the series coefficients for rotations and moments.

$$
\begin{aligned}
P_n &= \frac{A_n}{\frac{N}{a}(C_s)_n} + \frac{B_n(k)_n}{\frac{N}{a}(C_s)_n} \\
&\quad + \frac{a}{N}\frac{2n}{\pi^2}\frac{b^2}{a^2}\sum_m \frac{m}{\left(n^2 + m^2\frac{b^2}{a^2}\right)^2}(C_m - \cos n\pi D_m) \\
Q_n &= \frac{B_n}{\frac{N}{a}(C_s)_n} + \frac{A_n(k)_n}{\frac{N}{a}(C_s)_n} \\
&\quad + \frac{a}{N}\frac{2n}{\pi^2}\frac{b^2}{a^2}\sum_m \frac{m(-\cos m\pi)}{\left(n^2 + m^2\frac{b^2}{a^2}\right)^2}(C_m - \cos n\pi D_m) \\
R_m &= \frac{C_m}{\frac{N}{b}(C_s)_m} + \frac{D_m(k)_m}{\frac{N}{b}(C_s)_m} \\
&\quad + \frac{b}{N}\frac{2m}{\pi^2}\frac{a^2}{b^2}\sum_n \frac{n}{\left(m^2 + n^2\frac{a^2}{b^2}\right)^2}(A_n - \cos m\pi B_n) \\
U_m &= \frac{D_m}{\frac{N}{b}(C_s)_m} + \frac{C_m(k)_m}{\frac{N}{b}(C_s)_m}
\end{aligned}
\tag{13}
$$

$$+\frac{b}{N}\frac{2m}{\pi^2}\frac{a^2}{b^2}\sum_n \frac{n(-\cos n\pi)}{\left(m^2+n^2\frac{a^2}{b^2}\right)^2}(A_n-\cos m\pi B_n).$$

In the above equations the terms $(C_s)_n$ and $(C_s)_m$ are modified stiffness factors as defined in Bulletin 304. Numerical values may be obtained from Table 1 of that bulletin for values of $(b/s)=na/b$ of $mb/a$ as indicated by the subscripts $n$ or $m$, respectively. The terms $(k)_n$ and $(k)_m$ are carry-over factors in the long and short spans respectively, as defined in Bulletin 304; numerical values may be obtained from Table 2 of that bulletin for $(b/s)=na/b$ or $mb/a$, as indicated by the subscripts $n$ or $m$ respectively.

The rotations of an edge of a simply-supported slab are given by the following expressions for the coefficients of the terms in the series:

$$P_n = Q_n = \frac{-(M^F)_n(1+k_n)}{\frac{N}{a}(C_s)_n}$$
$$R_m = U_m = \frac{-(M^F)_m(1+k_m)}{\frac{N}{b}(C_s)_m}. \tag{14}$$

The term $(M^F)$ is the fixed-edge moment as defined in Bulletin 304. Numerical values may be obtained from the tables of that Bulletin for various types of loading. For a uniformly distributed load of magnitude, $w$, the fixed-edge moments are

$$(M^F)_n = -(c_m)_n\,wa^2\frac{4}{n\pi}, \text{ for } n \text{ odd only}$$
$$(M^F)_m = -(c_m)_m\,wb^2\frac{4}{m\pi}, \text{ for } m \text{ odd only.} \tag{15}$$

The quantity $(c_m)$ may be obtained from Table 5 of Bulletin 304 for $(b/s)=na/b$ or $mb/a$ as indicated by the subscripts $n$ or $m$ respectively.

It is more convenient when solving for moments in plates with one or more fixed edges to use the relations stated above in somewhat different form. If the relations in Equations (13) and (14) are combined to express the rotation of each edge as a function of the loading and of the edge moments, and if these expressions are each in turn set equal to zero, the resulting equations may be used to express the relation between the moment on a fixed edge and the moments on the other edges and the loading. These relations are:

$$
\begin{aligned}
A_n &= (M^F)_n\,(1 + k_n) - B_n k_n \\
&\qquad - n(C_s)_n \frac{2}{\pi^2}\frac{b^2}{a^2}\sum_m \frac{m}{\left(n^2 + m^2\dfrac{b^2}{a^2}\right)^2}\,(C_m - \cos n\pi D_m) \\
B_n &= (M^F)_n\,(1 + k_n) - A_n k_n \\
&\qquad - n(C_s)_n \frac{2}{\pi^2}\frac{b^2}{a^2}\sum_m \frac{m(-\cos m\pi)}{\left(n^2 + m^2\dfrac{b^2}{a^2}\right)^2}\,(C_m - \cos n\pi D_m) \\
C_n &= (M^F)_m\,(1 + k_m) - D_m k_m \\
&\qquad - m(C_s)_m \frac{2}{\pi^2}\frac{b^2}{a^2}\sum_n \frac{n}{\left(n^2 + m^2\dfrac{b^2}{a^2}\right)^2}\,(A_n - \cos m\pi B_n) \\
D_n &= (M^F)_m\,(1 + k_m) - C_m k_m \\
&\qquad - m(C_s)_m \frac{2}{\pi^2}\frac{b^2}{a^2}\sum_n \frac{n(-\cos n\pi)}{\left(n^2 + m^2\dfrac{b^2}{a^2}\right)^2}\,(A_n - \cos m\pi B_n).
\end{aligned}
\tag{16}
$$

The plates considered in this section have the edges at $x = 0$ and $y = 0$ fixed, and the edges at $x = a$ and $y = b$ simply-supported. For these conditions, the moments $M_2$ and $M_4$ are equal to zero, and thus $B_n = D_m = 0$. Equations to be solved for values of $A_n$ and $C_m$ may then be written using only the first and third of Equations (16). These equations are given below in the form used in these calculations.

$$
\begin{aligned}
\frac{-A_n}{wb^2} &= (c_m)_n\,(1 + k_n)\,\frac{4}{n\pi}\frac{a^2}{b^2} \\
&\qquad - n(C_s)_n \frac{2}{\pi^2}\frac{b^2}{a^2}\sum_m \frac{m}{\left(n^2 + m^2\dfrac{b^2}{a^2}\right)^2}\cdot\frac{-C_m}{wb^2} \\
\frac{-C_m}{wb^2} &= (c_m)_m\,(1 + k_m)\,\frac{4}{m\pi} \\
&\qquad - m(C_s)_m \frac{2}{\pi^2}\frac{b^2}{a^2}\sum_n \frac{n}{\left(n^2 + m^2\dfrac{b^2}{a^2}\right)^2}\cdot\frac{-A_n}{wb^2}.
\end{aligned}
$$

In the numerical solutions, eleven terms were used for both $n$ and $m$. For $b/a = 0.5$, this resulted in 22 equations involving the 22 unknown coefficients for moment, $A_1, A_2, A_3, \ldots A_{11}$ and $C_1, C_2, C_3, \ldots C_{11}$. For $b/a = 1.0$, the moments $M_1$ and $M_3$ are equal and only eleven equations in eleven unknowns were obtained. In each case, the equations were solved by a process of successive approximations. The values of the coefficients thus determined are given below for the two plates analyzed. The results are accurate to the last place given.

| $n$ or $m$ | $b/a=0.5$ | | $b/a=1.0$ |
|---|---|---|---|
| | $A_n$ | $C_m$ | $A_n=C_m$ |
| 1 | −0.077422 | −0.125769 | −0.067323 |
| 2 | +0.015362 | +0.011695 | +0.011572 |
| 3 | +0.001853 | −0.007965 | +0.000945 |
| 4 | +0.003452 | +0.006493 | +0.003091 |
| 5 | +0.000996 | +0.000341 | +0.000794 |
| 6 | +0.001300 | +0.003106 | +0.001154 |
| 7 | +0.000514 | +0.000722 | +0.000393 |
| 8 | +0.000641 | +0.001634 | +0.000535 |
| 9 | +0.000304 | +0.000519 | +0.000207 |
| 10 | +0.000374 | +0.000938 | +0.000285 |
| 11 | +0.000201 | +0.000343 | +0.000119 |

The various moments computed from the above coefficients are given in Table 23. Negative moments on the fixed edges were obtained at intervals of one-twelfth the width of the plate. Positive moments in the interior were obtained only on certain sections, and with one exception, only the average moment was computed. The calculation for positive moments consisted first of determining the moments in a simply-supported slab from the data given in Section 26, or from the sources mentioned therein. The corrections to these moments due to the moments acting on the edges were then obtained by means of the procedure in Bulletin 304.

The computations described above for $b/a = 0.5$ and 1.0 were also made for only one term in the series expression for moment; that is, for a single sine wave of moment on each edge. For $b/a = 0.5$ the following edge moments were obtained:

$$\text{at } x = 0, \quad \frac{M_x}{wb^2} = -0.0743 \sin \frac{\pi y}{b}$$

$$\text{at } y = 0, \quad \frac{M_y}{wb^2} = -0.1258 \sin \frac{\pi x}{a}.$$

## TABLE 23

### MOMENTS IN UNIFORMLY LOADED RECTANGULAR PLATES FIXED ON TWO ADJACENT EDGES AND SIMPLY-SUPPORTED ON OTHERS

Poisson's Ratio = 0. See Fig. 12 for notation. Fixed edges are at $x = 0$ and $y = 0$. Average moments on edges from terms in series, all others based on calculations by Simpson's one-third rule.

| $\frac{b}{a}$ | $\frac{x}{a}$ | $\frac{y}{b}$ | $\frac{M_x}{wb^2}$ | $\frac{y}{b}$ | $\frac{x}{a}$ | $\frac{M_y}{wb^2}$ |
|---|---|---|---|---|---|---|
| 0.5 | 0 | 1/12 | −0.0043 | 0 | 1/12 | −0.0202 |
| | | 2/12 | −0.0216 | | 2/12 | −0.0582 |
| | | 3/12 | −0.0394 | | 3/12 | −0.0852 |
| | | 4/12 | −0.0571 | | 4/12 | −0.1038 |
| | | 5/12 | −0.0701 | | 5/12 | −0.1132 |
| | | 6/12 | −0.0787 | | 6/12 | −0.1180 |
| | | 7/12 | −0.0813 | | 7/12 | −0.1180 |
| | | 8/12 | −0.0782 | | 8/12 | −0.1140 |
| | | 9/12 | −0.0683 | | 9/12 | −0.1042 |
| | | 10/12 | −0.0524 | | 10/12 | −0.0853 |
| | | 11/12 | −0.0297 | | 11/12 | −0.0532 |
| | | Av | −0.0487 | | Av | −0.0813 |
| | 0.5 | Av | 0.0368 | 0.5 | Av | 0.0048 |
| | 0.6 | Av | 0.0404 | 0.6 | Av | 0.0051 |
| | 0.62* | Av | 0.0405 | 0.7 | Av | 0.0070 |
| | 0.7 | Av | 0.0390 | 0.8 | Av | 0.0094 |
| | 0.8 | Av | 0.0323 | 0.85 | Av | 0.0107 |
| | | | | 0.87* | Av | 0.0108 |
| | | | | 0.9 | Av | 0.0105 |
| 1.0 | 0 | 1/12 | −0.0052 | | | |
| | | 2/12 | −0.0208 | | | |
| | | 3/12 | −0.0368 | | | |
| | | 4/12 | −0.0512 | | | |
| | | 5/12 | −0.0615 | | | |
| | | 6/12 | −0.0678 | | | |
| | | 7/12 | −0.0694 | | | |
| | | 8/12 | −0.0663 | | | |
| | | 9/12 | −0.0582 | | | |
| | | 10/12 | −0.0448 | | | |
| | | 11/12 | −0.0257 | | | |
| | | Av | −0.0425 | | | |
| | 0.4 | Av | 0.0118 | | | |
| | 0.5 | Av | 0.0140 | | | |
| | 0.6 | 0.1 | 0.0030 | | | |
| | | 0.2 | 0.0090 | | | |
| | | 0.3 | 0.0160 | | | |
| | | 0.4 | 0.0215 | | | |
| | | 0.5 | 0.0249 | | | |
| | | 0.6 | 0.0252 | | | |
| | | 0.7 | 0.0225 | | | |
| | | 0.8 | 0.0169 | | | |
| | | 0.9 | 0.0089 | | | |
| | | Av | 0.0149 | | | |
| | 0.7 | Av | 0.0147 | | | |

* Maximum moment obtained by algebraic interpolation using parabola fitted through three points.

The moment, $M_x$, at the middle of the short edge is thus $-0.0743$ from the one-term solution, as compared with $-0.0787$ from the eleven-term solution in Table 23. At the middle of the long edge the corresponding moments, $M_y$, are $-0.1258$ from the one-term solution and $-0.1180$ from Table 23. The accuracy of the one-term solution is somewhat better if the average moments are considered. On the short edge, the average moment, $M_x$, is $-0.0472$ for one-term, and $-0.0487$ for eleven terms; while on the long edge the corresponding values of average moment, $M_y$, are $-0.0800$ and $-0.0813$.

For $b/a = 1.0$ the edge moment for only one term in the series was as follows:

$$\text{at } x = 0, \quad \frac{M_x}{wb^2} = -0.0663 \sin \frac{\pi y}{b}.$$

The moment at the middle of the edge is $-0.0663$ for one term, as compared to $-0.0678$ from Table 23, while the respective average moments are $-0.0422$ and $-0.0425$.

It is evident from the above comparisons that the one-term solution is reasonably accurate for average moments, and since it is these moments that are of particular interest in this Bulletin, an additional solution was made for a plate having $b/a = 0.7$. The edge moments thus obtained were as follows:

$$\text{at } x = 0, \quad \frac{M_x}{wb^2} = -0.0729 \sin \frac{\pi y}{b}$$

$$\frac{M_{av}}{wb^2} = -0.0464$$

$$\text{at } y = 0, \quad \frac{M_y}{wb^2} = -0.1010 \sin \frac{\pi x}{a}$$

$$\frac{M_{av}}{wb^2} = -0.0643.$$

## 30. *Plates with Three Edges Fixed*

Moments are given in this section for uniformly loaded rectangular plates having three edges fixed and the other edge simply-supported. The moments in plates having a long edge simply-supported are based on the solutions obtained by Young[1] for values of $b/a = 0.33$, 0.50, 0.75, and 1.0. The moments for plates having a short edge

[1] Dana Young, "Analysis of Clamped Rectangular Plates," *Jour. Appl. Mech.*, December 1940, Vol. 7, No. 4, pp. A-139–142.

simply-supported were obtained from solutions by the writers for $b/a = 0.50$ and 0.75.

Coefficients for the terms in the series for moments on the edges of plates having a long edge simply-supported are given by Young in Table 3 of the paper referred to. Moments at the one-twelfth points and average moments on the edges were computed directly from these coefficients. The values thus obtained at the middle of an edge differed by not more than 0.0002 from the values given by Young. In order to compute positive moments in the interior for the plates having $b/a = 0.5$ and 1.0, the moments given by Young in the form of a cosine series were converted into a sine series, and the correction moments in the interior were obtained by means of the

TABLE 24

MOMENTS IN UNIFORMLY LOADED RECTANGULAR PLATES FIXED ON THREE EDGES AND SIMPLY-SUPPORTED ON ONE LONG EDGE

Poisson's Ratio = 0. See Fig. 12 for notation. Simply-supported edge is at $y = b$. See Table 25 for $b/a = 1.0$.

| $\frac{b}{a}$ | $\frac{x}{a}$ | $\frac{y}{b}$ | $\frac{M_x}{wb^2}$ | $\frac{y}{b}$ | $\frac{x}{a}$ | $\frac{M_y}{wb^2}$ |
|---|---|---|---|---|---|---|
| 0.33 | 0 | 1/4 | −0.0403 | 0 | 1/12 | −0.0393 |
| | | 1/2 | −0.0788 | | 2/12 | −0.0852 |
| | | 3/4 | −0.0686 | | 3/12 | −0.1098 |
| | | Av | −0.0487 | | 4/12 | −0.1200 |
| | | | | | 5/12 | −0.1235− |
| | | | | | 6/12 | −0.1244 |
| | | | | | Av | −0.0902 |
| 0.50 | 0 | 1/12 | −0.0052 | 0 | 1/12 | −0.0207 |
| | | 2/12 | −0.0210 | | 2/12 | −0.0574 |
| | | 3/12 | −0.0403 | | 3/12 | −0.0856 |
| | | 4/12 | −0.0566 | | 4/12 | −0.1026 |
| | | 5/12 | −0.0701 | | 5/12 | −0.1117 |
| | | 6/12 | −0.0786 | | 6/12 | −0.1148 |
| | | 7/12 | −0.0810 | | Av | −0.0726 |
| | | 8/12 | −0.0781 | | | |
| | | 9/12 | −0.0683 | | | |
| | | 10/12 | −0.0522 | | | |
| | | 11/12 | −0.0297 | | | |
| | | Av | −0.0487 | | | |
| | 0.5 | Av | 0.0055 | 0.5 | Av | 0.0284 |
| | 0.6 | Av | 0.0060 | 0.6 | Av | 0.0327 |
| | 0.66* | Av | 0.0063 | 0.64* | Av | 0.0331 |
| | 0.7 | Av | 0.0062 | 0.7 | Av | 0.0324 |
| | 0.8 | Av | 0.0046 | 0.8 | Av | 0.0274 |
| | 0.9 | Av | −0.0067 | | | |
| 0.75 | 0 | 1/12 | −0.0055− | 0 | 1/12 | −0.0097 |
| | | 2/12 | −0.0207 | | 2/12 | −0.0327 |
| | | 3/12 | −0.0388 | | 3/12 | −0.0545 |
| | | 4/12 | −0.0537 | | 4/12 | −0.0705 |
| | | 5/12 | −0.0658 | | 5/12 | −0.0804 |
| | | 6/12 | −0.0730 | | 6/12 | −0.0838 |
| | | 7/12 | −0.0749 | | Av | −0.0482 |
| | | 8/12 | −0.0720 | | | |
| | | 9/12 | −0.0630 | | | |
| | | 10/12 | −0.0483 | | | |
| | | 11/12 | −0.0276 | | | |
| | | Av | −0.0455− | | | |

* Maximum moment obtained by algebraic interpolation using parabola fitted through three points.

procedure of Bulletin 304. The simply-supported slab moments were obtained from the sources given in Section 26.

The moments computed from Young's coefficients are given in Table 24 for $b/a = 0.33$, 0.50, and 0.75, and in Table 25 for $b/a = 1.0$.

Solutions for plates having a short side simply-supported were obtained by the writers for $b/a = 0.50$ and 0.75 by means of the procedure described in Section 29. In each case there are only two unknown moments, and terms in the series through $n = 11$ were used for each. Since the moment on one edge is symmetrical and the even values of $n$ are thus equal to zero for that moment, only 17 equations involving 17 unknown coefficients were obtained. These

TABLE 25

MOMENTS IN UNIFORMLY LOADED RECTANGULAR PLATES FIXED ON THREE EDGES AND SIMPLY-SUPPORTED ON ONE SHORT EDGE

Poisson's Ratio = 0. See Fig. 12 for notation. Simply-supported edge is at $x = a$.

| $\frac{b}{a}$ | $\frac{x}{a}$ | $\frac{y}{b}$ | $\frac{M_x}{wb^2}$ | $\frac{y}{b}$ | $\frac{x}{a}$ | $\frac{M_y}{wb^2}$ |
|---|---|---|---|---|---|---|
| 1.0 | 0 | 1/12 | −0.0053 | 0 | 1/12 | −0.0056 |
| | | 2/12 | −0.0196 | | 2/12 | −0.0193 |
| | | 3/12 | −0.0341 | | 3/12 | −0.0345 |
| | | 4/12 | −0.0454 | | 4/12 | −0.0461 |
| | | 5/12 | −0.0526 | | 5/12 | −0.0551 |
| | | 6/12 | −0.0551 | | 6/12 | −0.0601 |
| | | Av | −0.0307 | | 7/12 | −0.0610 |
| | | | | | 8/12 | −0.0586 |
| | | | | | 9/12 | −0.0515 |
| | | | | | 10/12 | −0.0399 |
| | | | | | 11/12 | −0.0231 |
| | | | | | Av | −0.0381 |
| | 0.4 | Av | 0.0080 | 0.5 | Av | 0.0137 |
| | 0.5 | Av | 0.0090 | | | |
| | 0.6 | Av | 0.0095 | | | |
| | 0.7 | Av | 0.0097 | | | |
| | 0.8 | Av | 0.0092 | | | |
| 0.75 | 0 | 6/12 | −0.0571 | 0 | 5/12 | −0.0740 |
| | | Av | −0.0314 | | 6/12 | −0.0750 |
| | | | | | 7/12 | −0.0757 |
| | | | | | 8/12 | −0.0726 |
| | | | | | Av | −0.0495 |
| 0.50 | 0 | 1/12 | −0.0045− | 0 | 5/12 | −0.0816 |
| | | 2/12 | −0.0200 | | 6/12 | −0.0837 |
| | | 3/12 | −0.0343 | | 7/12 | −0.0836 |
| | | 4/12 | −0.0468 | | 8/12 | −0.0819 |
| | | 5/12 | −0.0541 | | Av | −0.0610 |
| | | 6/12 | −0.0571 | | | |
| | | Av | −0.0314 | | | |
| | 0.1 | Av | −0.0004 | 0.5 | 0.1 | 0.0089 |
| | 0.2 | Av | 0.0053 | | 0.2 | 0.0219 |
| | 0.3 | Av | 0.0038 | | 0.3 | 0.0319 |
| | 0.4 | Av | 0.0023 | | 0.4 | 0.0381 |
| | 0.5 | Av | 0.0013 | | 0.5 | 0.0412 |
| | 0.6 | Av | 0.0015 | | 0.6 | 0.0426 |
| | 0.7 | Av | 0.0032 | | 0.7 | 0.0399 |
| | 0.8 | Av | 0.0057 | | 0.8 | 0.0321 |
| | 0.85 | Av | 0.0071 | | 0.9 | 0.0187 |
| | 0.88* | Av | 0.0078 | | Av | 0.0279 |
| | 0.9 | Av | 0.0075 | | | |

* Maximum moment obtained by algebraic interpolation using parabola fitted through three points.

equations were solved, as in the previous case, by a process of successive approximations. The terms in the series are given below for a plate having the simply-supported edge at $x=a$. If the moments are expressed, in the notation of Fig. 12, as:

$$\frac{M_x}{wb^2} \text{ at } x = 0 = \sum_n X_n \sin \frac{n\pi y}{b}$$

$$\frac{M_y}{wb^2} \text{ at } y = 0 \text{ or } b = \sum_n Y_n \sin \frac{n\pi x}{a}$$

then the coefficients $X_n$ and $Y_n$ are as follows:

| $n$ | $b/a=0.5$ | | $b/a=0.7$ | |
|---|---|---|---|---|
| | $X_n$ | $Y_n$ | $X_n$ | $Y_n$ |
| 1 | −0.052266 | −0.092564 | −0.052310 | −0.077353 |
| 2 | 0 | +0.007011 | 0 | +0.009637 |
| 3 | +0.006479 | −0.009712 | +0.006428 | −0.002046 |
| 4 | 0 | +0.004908 | 0 | +0.003854 |
| 5 | +0.002635 | −0.000544 | +0.002570 | +0.000623 |
| 6 | 0 | +0.002560 | 0 | +0.001635 |
| 7 | +0.001301 | +0.000380 | +0.001175 | +0.000462 |
| 8 | 0 | +0.001413 | 0 | +0.000808 |
| 9 | +0.000757 | +0.000372 | +0.000634 | +0.000279 |
| 10 | 0 | +0.000839 | 0 | +0.000446 |
| 11 | +0.000493 | +0.000274 | +0.000384 | +0.000170 |

The moments computed from the above coefficients are given in Table 25.

Values of the average moments on the fixed edges for other values of $b/a$ were needed for use in the development of the distribution procedure described in Chapter III. These moments were obtained by graphical interpolation from a curve of average moment vs. $b/a$. For convenience, all of the average edge moments are summarized in Table 26. The accuracy of the interpolated moments for $b/a=0.571$, 0.667, and 0.800 is obviously less than that indicated by the number of significant figures given. It is believed, however, that these moments are not in error by more than five in the last place.

*Calculation of Rotations.*—Rotations of the simply-supported edge were computed for three of the cases considered above: $b/a=1.0$, $b/a=0.5$ with a long edge simply-supported, and $b/a=0.5$ with a short edge simply-supported. The rotation on each simply-supported edge was expressed as a sine series, and the coefficient for each

TABLE 26

SUMMARY OF AVERAGE MOMENTS ON FIXED EDGES OF UNIFORMLY LOADED RECTANGULAR PLATES FIXED ON THREE EDGES

See Fig. 12 for notation. Asterisk (*) indicates moments obtained graphically from curves of moment vs. $b/a$.

| $\frac{b}{a}$ | Average Moments | | $\frac{b}{a}$ | Average Moments | |
|---|---|---|---|---|---|
| | $\frac{M_x}{wb^2}$ at $x = 0$ | $\frac{M_y}{wb^2}$ at $y = 0$ | | $\frac{M_x}{wb^2}$ at $x = 0$ | $\frac{M_y}{wb^2}$ at $y = 0$ |
| Long edge at $y=b$ simply-supported | | | Short edge at $x=a$ simply-supported | | |
| 0 | −0.0487 | −0.1250 | 0 | −0.0314 | −0.0833 |
| 0.333 | −0.0487 | −0.0902 | 0.500 | −0.0314 | −0.0610 |
| 0.500 | −0.0487 | −0.0726 | 0.571* | −0.0314 | −0.0580 |
| 0.571* | −0.0484 | −0.0654 | 0.667* | −0.0314 | −0.0532 |
| 0.667* | −0.0475 | −0.0557 | 0.750 | −0.0314 | −0.0495 |
| 0.750 | −0.0455− | −0.0482 | 0.800 | −0.0312 | −0.0470 |
| 0.800* | −0.0440 | −0.0440 | 1.000 | −0.0307 | −0.0381 |
| 1.000 | −0.0381 | −0.0307 | | | |

term was computed by means of Equations (13) and (14) of Section 29. The net rotation is the sum of that due to the loading, as determined from the appropriate Equations (14), and that due to the edge moments, from Equations (13). The series expressions are given below.

For the long edge at $y=b$ simply-supported:

$$\frac{N}{b}\frac{\Phi_y}{wb^2} = \sum_m \Phi_{om} \sin \frac{m\pi x}{a}.$$

For the short edge at $x=a$ simply-supported:

$$\frac{N}{a}\frac{\Phi_x}{wb^2} = \sum_n \Phi_{on} \sin \frac{n\pi y}{b}.$$

The terms in the series and the average rotation for the cases considered are given in the following table:

| $n$ or $m$ | $\Phi_{om}$ | | $\Phi_{on}$ |
|---|---|---|---|
| | $b/a=0.5$ | $b/a=1.0$ | $b/a=0.5$ |
| 1 | +0.017555 | +0.005780 | +0.003166 |
| 3 | −0.000628 | −0.000635− | −0.000372 |
| 5 | −0.000631 | −0.000207 | −0.000110 |
| 7 | −0.000329 | −0.000087 | −0.000045 |
| 9 | −0.000182 | −0.000044 | −0.000023 |
| 11 | −0.000109 | −0.000025 | −0.000013 |
| Av | +0.0110 | +0.0035 | +0.0019 |

TABLE 27
MOMENTS IN UNIFORMLY LOADED RECTANGULAR PLATES FIXED ON ALL FOUR EDGES
Poisson's Ratio = 0. See Fig. 12 for notation.

| $\frac{b}{a}$ | $\frac{y}{b}$ | $\frac{M_y}{wb^2}$, in short span, for $x/a$ = | | | | | |
|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | $M_{av}$ |
| 0.500 | 0 | −0.0247 | −0.0572 | −0.0743 | −0.0808 | −0.0828 | −0.0556 |
| | 0.1 | −0.0090 | −0.0220 | −0.0311 | −0.0355 | −0.0369 | −0.0233 |
| | 0.2 | 0.0006 | 0.0002 | −0.0007 | −0.0018 | −0.0020 | −0.0005 |
| | 0.3 | 0.0063 | 0.0144 | 0.0187 | 0.0212 | 0.0218 | 0.0143 |
| | 0.4 | 0.0094 | 0.0216 | 0.0301 | 0.0346 | 0.0357 | 0.0228 |
| | 0.5 | 0.0100 | 0.0237 | 0.0336 | 0.0392 | 0.0404 | 0.0255 |
| 0.571 | 0 | | | | | −0.0805 | −0.0518 |
| 0.667 | 0 | −0.0167 | −0.0434 | −0.0620 | −0.0723 | −0.0756 | −0.0464 |
| 0.800 | 0 | | | | | −0.0664 | −0.0389 |
| 1.000 | 0 | −0.0077 | −0.0247 | −0.0389 | −0.0482 | −0.0513 | −0.0290 |
| | 0.1 | −0.0027 | −0.0073 | −0.0125 | −0.0158 | −0.0171 | −0.0094 |
| | 0.2 | 0.0006 | 0.0013 | 0.0018 | 0.0020 | 0.0020 | 0.0014 |
| | 0.3 | 0.0019 | 0.0054 | 0.0089 | 0.0111 | 0.0118 | 0.0067 |
| | 0.4 | 0.0022 | 0.0071 | 0.0120 | 0.0153 | 0.0164 | 0.0090 |
| | 0.5 | 0.0024 | 0.0077 | 0.0126 | 0.0164 | 0.0175 | 0.0096 |

| $\frac{b}{a}$ | $\frac{x}{a}$ | $\frac{M_x}{wb^2}$, in long span, for $y/b$ = | | | | | |
|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | $M_{av}$ |
| 0.500 | 0 | −0.0077 | −0.0257 | −0.0422 | −0.0531 | −0.0573 | −0.0314 |
| | 0.1 | 0.0010 | −0.0001 | −0.0017 | −0.0023 | −0.0024 | −0.0008 |
| | 0.2 | 0.0011 | 0.0041 | 0.0068 | 0.0084 | 0.0091 | 0.0050 |
| | 0.3 | 0.0012 | 0.0034 | 0.0055 | 0.0071 | 0.0078 | 0.0042 |
| | 0.4 | 0.0004 | 0.0019 | 0.0033 | 0.0045 | 0.0050 | 0.0025 |
| | 0.5 | 0.0006 | 0.0015 | 0.0025 | 0.0036 | 0.0039 | 0.0020 |
| 0.571 | 0 | | | | | −0.0571 | −0.0315 |
| 0.667 | 0 | −0.0079 | −0.0257 | −0.0422 | −0.0532 | −0.0570 | −0.0315 |
| 0.800 | 0 | | | | | −0.0559 | −0.0311 |
| 1.000 | | Same as $M_y$ above | | | | | |

The average rotation on each edge is equal to the coefficient from the above table multiplied by $wb^2 \cdot b/N$ or $wb^2 \cdot a/N$ for the long and short edges respectively.

## 31. *Plates with Four Edges Fixed*

Moments are given in this section for uniformly loaded rectangular plates having all four edges fixed.

The moments given in Table 27 for $b/a = 0.500$, 0.571, 0.667, 0.800, and 1.000 are based on the results given by Wojtaszak,[1]

[1] I. A. Wojtaszak, "The Calculation of Maximum Deflection, Moment, and Shear for Uniformly Loaded Rectangular Plate with Clamped Edges," *Jour. Appl. Mech.*, December 1937, Vol. 4, pp. A-173–176.

modified slightly in certain cases in order to simplify the calculation of moments in the interior of the plate. The solutions given by Wojtaszak consist of a parabolic term plus a correction term in the form of a cosine series. Values of the coefficients for terms in this series are tabulated for $n=1$ through 27. The expressions given by Wojtaszak were used without change to obtain all the moments on an edge given in Table 27, except certain values for $b/a=0.5$ and 1.0 which are mentioned later. For the calculation of moments in the interior of the plates having $b/a=0.5$ and 1.0, the expressions for edge moments obtained from Wojtaszak were modified as follows: First, the origin of coordinates was shifted from the middle of the plate to a corner as in Fig. 12, and the cosine series was converted to a sine series referred to the new axes. The parabolic term was then expanded into a sine series which was added term by term to the other. The result was an expression for edge moments involving only a sine series whose terms were easily computed from Wojtaszak's tabulated coefficients and the expansion of the parabolic term. Correction moments in the interior of the plate were then computed by means of the procedure in Bulletin 304. Positive moments in the simply-supported plate at the one-tenth points were obtained from Leitz. (See Section 26.) Moments at the one-tenth points on the edges for $b/a=0.5$ and 1.0 were computed from these modified expressions.

Average moments on an edge were obtained by the integration of Wojtaszak's equations for moment. Average moments on sections in the interior were obtained by the application of Simpson's one-third rule to the values at the one-tenth points.

Certain of the moments given in Table 27, especially those at the middle of an edge or at the center of the plate, have also been computed by other investigators. Wojtaszak gives moments for the middle of the long edge which differ from those given herein by not more than 0.0001. Young[1] has computed moments for $b/a=0.500$, 0.667, and 1.000, based on Wojtaszak's solutions. His values for moments at the middle of an edge are not more than 0.0001 different from those in Table 27. The difference for moments at the center of the plate is usually not more than 0.0001, but for $b/a=0.5$ is as high as 0.0004. Some errors are introduced, however, in making the comparisons since the values given by Young are for Poisson's ratio equal to 0.3, and are expressed in terms of the square of the long side instead of the short side as used herein. In general the agreement is satisfactory, but where differences exist, the interior

[1] Dana Young, "Analyses of Clamped Rectangular Plates," *Jour. Appl. Mech.*, December 1940, Vol. 7, No. 4, pp. A-139–142.

moments given by Young are probably more nearly correct since they were obtained by a more direct calculation.

Moments at the one-tenth points on the edges and in the interior of a square plate are given by Leitz.[1] The values given, however, differ from those in Table 27 by as much as 0.0009. At the center of the plate the coefficient given by Leitz is 0.0184, as compared to 0.0175 herein and 0.0176 from Young's results. In view of this comparison it is believed that the values given in Table 27 are more nearly correct than those obtained by Leitz. However, the solution by Leitz is more complete, and in some ways more useful, if a lower degree of accuracy is acceptable. He has tabulated, in addition to $M_x$ and $M_y$, the twisting moments, $M_{xy}$, and the principal moments and their directions, all for Poisson's ratio equal to zero.

The results given by Hencky[2] for $b/a = 1.0$ are in good agreement with those in Table 27, and with those computed by Wojtaszak and Young. His values for $b/a = 0.5$, however, differ considerably and are believed to be in error. The reason for this is that only the terms through $n = 11$ were used in his calculations for $b/a = 0.5$. For $b/a = 1.0$, the terms through $n = 25$ were used, and good agreement would be expected with Wojtaszak's solution for terms through $n = 27$.

Maximum and average moments on the edges are summarized in Table 28 for a number of values of $b/a$. The additional moments given were taken from the results obtained by Evans,[3] and are believed to be as accurate as the values from Table 27. Evans computed the terms through $n = 11$, and obtained a number of additional terms by extrapolation based on the observed trends. He states that the values are not in error by as much as 0.1 percent, which corresponds to less than 0.0001 for the moments given.

[1] N. Leitz, "Berechnung der eingespannten rechteckigen Platte," *Zeits. für Math. u. Phys.*, Bd. 64, 1916.
[2] H. Hencky, "Der Spannungszustand in rechteckigen Platten." Oldenbourg, Munich, 1913.
[3] T. H. Evans, "Tables of Moments and Deflections for a Rectangular Plate Fixed on All Edges and Carrying a Uniformly Distributed Load," *Jour. Appl. Mech.*, March 1939, Vol. 6, No. 1, pp. A-7–11.

TABLE 28

SUMMARY OF MAXIMUM AND AVERAGE MOMENTS ON EDGES OF UNIFORMLY LOADED RECTANGULAR PLATES FIXED ON ALL EDGES

Coordinate system and notation is that of Fig. 12.

| $\frac{b}{a}$ | $\frac{M_y}{wb^2}$, on long edge | | $\frac{M_x}{wb^2}$, on short edge | | Basis for Moments |
|---|---|---|---|---|---|
| | At Middle of Edge | Average Moment | At Middle of Edge | Average Moment | |
| 0 | −0.0833 | −0.0833 | | | |
| 0.500 | −0.0828 | −0.0556 | −0.0573 | −0.0314 | Wojtaszak* |
| 0.526 | −0.0822 | | −0.0571 | | Evans |
| 0.556 | −0.0812 | | −0.0571 | | Evans |
| 0.571 | −0.0805 | −0.0518 | −0.0571 | −0.0315 | Wojtaszak* |
| 0.588 | −0.0799 | | −0.0571 | | Evans |
| 0.625 | −0.0780 | | −0.0571 | | Evans |
| 0.667 | −0.0756 | −0.0464 | −0.0570 | −0.0315 | Wojtaszak* |
| 0.714 | −0.0726 | | −0.0568 | | Evans |
| 0.750 | | −0.0417 | | −0.0314 | Interpolated† |
| 0.769 | −0.0687 | | −0.0563 | | Evans |
| 0.800 | −0.0664 | −0.0389 | −0.0559 | −0.0311 | Wojtaszak* |
| 0.833 | −0.0639 | | −0.0554 | | Evans |
| 0.909 | −0.0581 | | −0.0538 | | Evans |
| 1.000 | −0.0513 | −0.0290 | −0.0513 | −0.0290 | Wojtaszak* |

* Moments given are those computed by authors from Wojtaszak's solution. In some cases they differ by 0.0001 from values given by Wojtaszak.

† Obtained by graphical interpolation from plot of moment vs. $b/a$.

# Appendix B

## Moments in Uniformly Loaded Continuous Rectangular Plates

### 32. *Introduction*

Solutions are given in this Appendix for moments in uniformly loaded rectangular plates continuous in two directions over rigid supporting beams. The results set forth in the following sections were used extensively in the derivation of the constants for the distribution procedure and in the verification of the procedure as a whole.

Moments are given for three continuous slabs, designated as Slabs I, II, and III, plans of which are given in Fig. 16. In each case the structure has been assumed to carry a uniform load, $w$, distributed over the entire area of all panels. The beams are assumed to be nondeflecting but to have no torsional stiffness. The exterior edges of the slabs are thus simply-supported.

The solutions for edge moments for Slabs I and II were obtained by C. W. Pan,[1] and the writers are indebted to Professor L. C. Maugh of the Civil Engineering Department of the University of Michigan for his kindness in permitting their presentation and use herein. The calculations for Slab III were made by the writers, using the equations and procedure described in Section 29 of Appendix A. In all slabs, the positive moments in the interior of the panels were computed by means of the procedure described in Bulletin 304. Moments due to the uniform load on a simply-supported slab were obtained from the sources mentioned in Section 26 of Appendix A.

The notation used throughout this Appendix is generally the same as that of Figs. 12 and 16. Panels in the various slabs are identified by the letter designations indicated on Fig. 16, while points within the panels are referred to the coordinate system of Fig. 12. The origin of coordinates is always taken at the lower left-hand corner of the panel, with the $x$-axis extending horizontally and the $y$-axis vertically. Irrespective of their relative lengths, the span lengths are designated as $b$ in the $y$-direction and $a$ in the $x$-direction, and points in the panel will therefore always be located by the coordinates $x/a$ and $y/b$.

All the numerical results presented in the following sections are for a value of Poisson's ratio equal to zero.

[1] C. W. Pan. "Analysis of Continuous Slabs," a dissertation submitted in partial fulfilment of the requirements for the degree of Doctor of Science in the University of Michigan, Ann Arbor, Michigan, 1939.

TABLE 29

MOMENTS IN UNIFORMLY LOADED CONTINUOUS SLAB HAVING NINE SQUARE PANELS: SLAB I

Poisson's Ratio = 0. For notation see Figs. 12 and 16. Values in table are average moments in pounds for uniform load $w = 100$ lb per sq ft and dimensions of panels given in Fig. 16.

| Panel | $\frac{x}{a}$ | $\frac{y}{b}$ | $M_x$ | $\frac{y}{b}$ | $\frac{x}{a}$ | $M_y$ |
|---|---|---|---|---|---|---|
| ABCD | 0.2 | Av | +145 | 0.2 | Av | +145 |
| | 0.3 | Av | +153 | 0.3 | Av | +153 |
| | 0.4 | Av | +150 | 0.4 | Av | +150 |
| | 0.5 | Av | +134 | 0.5 | Av | +134 |
| | 1.0 | Av | −409 | 1.0 | Av | −409 |
| | | 0.5 | −658 | | 0.5 | −658 |
| BBDD | 0 | 0.5 | −658 | 0.1 | Av | + 66 |
| | | Av | −409 | 0.2 | Av | + 86 |
| | | | | 0.3 | Av | + 89 |
| | | | | 0.4 | Av | + 87 |
| | 0.5 | Av | +143 | 0.5 | Av | + 81 |
| | | | | 1.0 | Av | −298 |
| | | | | | 0.5 | −573 |
| DDDD | 0 | 0.5 | −573 | 0 | 0.5 | −573 |
| | | Av | −298 | | Av | −298 |
| | 0.5 | Av | +104 | 0.5 | Av | +104 |

## 33. *Analysis of Slab with Nine Square Panels*

Moments are given in this section for a uniformly loaded continuous slab having nine square panels arranged in three rows of three panels each. The beams are assumed to be nondeflecting, and to have zero torsional stiffness. The notation used and the dimensions of the slab are those of Slab I in Fig. 16. A uniform load $w$ of 100 lb per sq ft over the entire area of the slab is considered. The structure analyzed is symmetrical about both centerlines and moments are therefore given for three panels only.

The moments given in Table 29 are based on the solutions obtained by C. W. Pan and reported in the thesis referred to in the preceding section. The method of analysis used has been described in a paper by L. C. Maugh and C. W. Pan.[1] Expressions in terms of infinite series are written for the slopes on all edges of each panel, in terms of the restraining moments on all edges and the applied loading. At each support, the expressions for edge slopes in the two adjacent panels are equated term by term; the resulting equations are solved for the desired restraining moments, which are also expressed as a trigonometric series. In Pan's solution, terms in the series for moment through $n = 3$ were obtained as described above, and additional terms through $n = 5$ were computed by an approximate procedure.

[1] L. C. Maugh and C. W. Pan, "Moments in Continuous Rectangular Slabs on Rigid Supports," *Trans. ASCE*, Vol. 107 (1942), p. 1118.

All moments given in Table 29 for the edge of a panel were obtained directly from the five terms in the series given by Pan. Average moments on sections in the interior were computed from the edge moments and the moments in a uniformly loaded simply-supported slab. In this latter calculation, average moments in the simply-supported slab were obtained by means of Simpson's one-third rule from the solutions by Leitz, referred to in Section 26. Correction moments in the interior were then obtained from the terms in the series for edge moments using the procedure described in Bulletin 304, and average values were obtained either by Simpson's one-third rule or by direct summation, depending on the nature of the expression.

34. *Analysis of Slab with Fifteen Unequal Panels*

Moments are given in this section for a uniformly loaded continuous slab consisting of fifteen unequal panels having the dimensions given in Fig. 16 for Slab II. The supporting beams are assumed to be nondeflecting, and to have zero torsional stiffness. The uniform load $w$ on the entire area of the slab is taken as 100 lb per sq ft.

The moments given in Table 30 are based on the solutions obtained by Pan, and were computed from the terms in the series for edge moments in exactly the same way as those in the preceding section. The analysis of this slab is similar in every respect to that for the slab having nine square panels as described in Section 33.

The analysis of the slab considered in this section is described in the ASCE paper by Maugh and Pan, referred to in the preceding section, and values are given therein for moments at the middle of each edge and at the middle of each panel. The edge moments agree with those in Table 30; no moments are given herein corresponding to those at the centers of the panels.

35. *Analysis of Slab with Nine Unequal Panels*

Moments are given in this section for a uniformly loaded continuous slab consisting of nine unequal panels having the dimensions given in Fig. 16 for Slab III. The supporting beams are assumed to be nondeflecting and to have zero torsional stiffness. The uniform load $w$ over the entire area of the slab is taken as 100 lb per sq ft.

The analysis of this slab was made by means of a procedure similar to that used by Pan and described briefly in Section 33. Slopes on the continuous edges of each panel were expressed in terms of the restraining moments and the applied loading by means of Equations (13) and (14) of Section 30. Since both the slab and loading are symmetrical, only four equations for edge slopes were needed, as follows:

For slope at edge BD of panel ABCD,
For slope at edge BD of panel BBDD,
For slope at edge DD of panel BBDD,
For slope at edge DD of panel DDDD.

### Table 30
### Moments in Uniformly Loaded Continuous Slab Having Fifteen Unequal Panels: Slab II

Poisson's Ratio = 0. For notation see Figs. 12 and 16. Values in table are moments in pounds for uniform load $w = 100$ lb per sq ft and dimensions of panels given in Fig. 16.

| Panel | $\frac{x}{a}$ | $\frac{y}{b}$ | $M_x$ | $\frac{y}{b}$ | $\frac{x}{a}$ | $M_y$ |
|---|---|---|---|---|---|---|
| ABDE | 0.1 | Av | + 321 | 0.1 | Av | + 219 |
| | 0.2 | Av | + 517 | 0.2 | Av | + 278 |
| | 0.3 | Av | + 606 | 0.27* | Av | + 282 |
| | 0.38* | Av | + 636 | 0.3 | Av | + 281 |
| | 0.4 | Av | + 635 | 0.4 | Av | + 267 |
| | 0.5 | Av | + 579 | 0.5 | Av | + 256 |
| | 1.0 | 0.5 | −2056 | 1.0 | 0.5 | −1217 |
| | | Av | −1370 | | Av | − 785 |
| BCEF | 0 | 0.5 | −2056 | 0.1 | Av | + 166 |
| | | Av | −1370 | 0.2* | Av | + 185 |
| | 0.4 | Av | + 416 | 0.3 | Av | + 166 |
| | 0.5 | Av | + 481 | 0.4 | Av | + 144 |
| | 0.54* | Av | + 487 | 0.5 | Av | + 138 |
| | 0.6 | Av | + 474 | 1.0 | 0.5 | −1005 |
| | 0.7 | Av | + 361 | | Av | − 587 |
| | 1.0 | 0.5 | −1214 | | | |
| | | Av | − 850 | | | |
| CC′FF′ | 0 | 0.5 | −1214 | 0.1* | Av | + 64 |
| | | Av | − 850 | 0.15 | Av | + 47 |
| | | | | 0.2 | Av | + 18 |
| | 0.5 | Av | + 207 | 0.3 | Av | − 25 |
| | | | | 0.4 | Av | − 42 |
| | | | | 0.5 | Av | − 38 |
| | | | | 1.0 | 1/12 | − 3 |
| | | | | | 2/12 | − 78 |
| | | | | | 3/12 | − 215 |
| | | | | | 4/12 | − 351 |
| | | | | | 5/12 | − 409 |
| | | | | | 6/12 | − 426 |
| | | | | | Av | − 209 |
| DEDE | 0.1* | Av | + 41 | 0 | 0.5 | −1217 |
| | 0.2 | Av | + 12 | | Av | − 785 |
| | 0.3 | Av | − 22 | | | |
| | 0.4 | Av | − 36 | 0.5 | Av | + 148 |
| | 0.5 | Av | − 34 | | | |
| | 1.0 | 1/12 | + 18 | | | |
| | | 2/12 | − 58 | | | |
| | | 3/12 | − 208 | | | |
| | | 4/12 | − 336 | | | |
| | | 5/12 | − 388 | | | |
| | | 6/12 | − 395 | | | |
| | | Av | − 194 | | | |
| EFEF | 0 | Same as DEDE at $x/a = 1.0$ | | 0 | 0.5 | −1005 |
| | | | | | Av | − 587 |
| | 0.1 | Av | + 5 | | | |
| | 0.2 | Av | + 61 | 0.5 | Av | + 160 |
| | 0.23* | Av | + 64 | | | |
| | 0.3 | Av | + 45 | | | |
| | 0.4 | Av | + 20 | | | |
| | 0.5 | Av | + 7 | | | |
| | 1.0 | 0.5 | − 540 | | | |
| | | Av | − 306 | | | |
| FF′FF′ | 0 | 0.5 | − 540 | 0 | Same as CC′FF′ at $y/b = 1.0$ | |
| | | Av | − 306 | | | |
| | 0.5 | Av | + 133 | 0.5 | Av | + 110 |

* Maximum moment obtained by algebraic interpolation using parabola fitted through three points.

The expressions for edge slopes in the two adjacent panels at edges BD and DD were then equated term by term, and two sets of equations were obtained, involving the two sets of coefficients for terms in the series for moments on those edges. In the numerical solution, terms in the series through $n = 11$ were used for both moments. Since the even-numbered terms in the series for moment on edge DD are equal to zero because of symmetry, only 17 equations involving 17 unknown coefficients were obtained. These were solved by a process of successive approximations identical with that used for the solutions described in Sections 29 and 30 of Appendix A. Four cycles of substitution (requiring four and one-half hours' work with a computing machine) were required to obtain terms in the series accurate to the second decimal place, corresponding to three to six significant figures. Because of the limited number of terms considered, the edge moments in Table 31, other than the average

TABLE 31

MOMENTS IN UNIFORMLY LOADED CONTINUOUS SLAB HAVING NINE UNEQUAL PANELS: SLAB III

Poisson's Ratio = 0. For notation see Figs. 12 and 16. Values in table are moments in pounds for uniform load $w = 100$ lb per sq ft and dimensions of panels given in Fig. 16.

| Panel | $\frac{x}{a}$ | $\frac{y}{b}$ | $M_x$ | $\frac{y}{b}$ | $\frac{x}{a}$ | $M_y$ |
|---|---|---|---|---|---|---|
| ABCD | 0.2 | Av | + 594 | Same as $M_x$ because of symmetry about diagonal | | |
| | 0.3 | Av | + 683 | | | |
| | 0.4* | Av | + 707 | | | |
| | 0.5 | Av | + 681 | | | |
| | 1.0 | 1/12 | − 718 | | | |
| | | 2/12 | −1209 | | | |
| | | 3/12 | −1538 | | | |
| | | 4/12 | −1732 | | | |
| | | 5/12 | −1818 | | | |
| | | 6/12 | −1813 | | | |
| | | 7/12 | −1710 | | | |
| | | 8/12 | −1521 | | | |
| | | 9/12 | −1216 | | | |
| | | 10/12 | − 821 | | | |
| | | 11/12 | − 326 | | | |
| | | Av | −1209 | | | |
| BBDD | 0 | Same as ABCD at $x/a = 1.0$ | | 0.1* | Av | + 25 |
| | | | | 0.15 | Av | − 1 |
| | | | | 0.2 | Av | − 37 |
| | 0.5* | Av | − 35 | 0.3 | Av | − 94 |
| | | | | 0.4 | Av | −123 |
| | | | | 0.5 | Av | −122 |
| | | | | 1.0 | 1/12 | + 26 |
| | | | | | 2/12 | − 55 |
| | | | | | 3/12 | −146 |
| | | | | | 4/12 | −232 |
| | | | | | 5/12 | −283 |
| | | | | | 6/12 | −304 |
| | | | | | Av | −139 |
| DDDD | Same as $M_y$ because of symmetry about diagonal | | | 0 | Same as BBDD at $y/b = 1.0$ | |
| | | | | 0.5* | Av | +164 |

* Maximum positive moment.

moments, are not equally accurate, and may be in error by as much as 2 or 3 in the last place. The average moments on an edge are believed to be correct to the number of places given. However, because of the several corrections involved and because of uncertainties regarding the accuracy of average moments for the simply-supported slab, the average moments given for points in the interior of the panels may be in error by as much as 5 in the last place.

Average moments and moments at the one-twelfth points on an edge were computed directly from the terms in the series. Moments in the interior of a panel were obtained from the edge moments and the simply-supported slab moments, using the procedure of Bulletin 304 in the manner described in Section 33.

A second analysis of this slab was made using only the terms in the series through $n = 3$. Such an analysis is relatively simple to carry out, and it was desired to compare the results obtained with those from the more refined calculations using eleven terms. Moments at the one-twelfth points of edges BD and DD obtained from the two solutions are compared in Table 32. The agreement is seen to be excellent for average moments; and, in general, the three-term solution yielded results which would be satisfactory for many purposes. With the exception of the one-twelfth and eleven-twelfths points, the maximum difference for edge BD is about 4 percent plus or minus. Larger differences, however, were obtained for edge DD.

TABLE 32

COMPARISON OF EDGE MOMENTS IN SLAB III COMPUTED FOR ELEVEN TERMS AND FOR THREE TERMS IN SERIES

Poisson's Ratio = 0. See notes in Table 31. Moments for eleven terms are from Table 31.

| $\frac{y}{b}$ | $M_x$ on edge BD | | $\frac{x}{a}$ | $M_y$ on edge DD | |
|---|---|---|---|---|---|
| | 11 Terms | 3 Terms | | 11 Terms | 3 Terms |
| 1/12 | − 718 | − 624 | 1/12 | + 26 | − 3 |
| 2/12 | −1209 | −1163 | 2/12 | − 55 | − 37 |
| 3/12 | −1538 | −1557 | 3/12 | −146 | −113 |
| 4/12 | −1732 | −1785 | 4/12 | −232 | −212 |
| 5/12 | −1818 | −1860 | 5/12 | −283 | −296 |
| 6/12 | −1813 | −1813 | 6/12 | −304 | −329 |
| 7/12 | −1710 | −1676 | | | |
| 8/12 | −1521 | −1468 | | | |
| 9/12 | −1216 | −1191 | | | |
| 10/12 | − 821 | − 846 | | | |
| 11/12 | − 326 | − 441 | | | |
| Av | −1209 | −1210 | Av | −139 | −138 |

# Appendix C

## Moments in Rectangular Plates with Concentrated Loads

### 36. *Introduction*

A limited number of solutions for moments in rectangular plates subjected to concentrated loads are given in this Appendix. This type of loading is not considered in the main body of this bulletin, and is not usually considered in the design of two-way concrete floor slabs for buildings. Nevertheless it was deemed worthwhile to investigate, at least partially, the applicability of the moment distribution procedure to plates carrying concentrated loads.

Table 33

Edge Moments for Concentrated Load at Center of Rectangular Plates Having All Edges Fixed

Moments from solution by Young (*Jour. Appl. Mech.*, September, 1939). See Fig. 12 for coordinate system and notation. Values in table are moments in terms of concentrated load $P$ and length of short span $b$.

| $\frac{b}{a}$ | $M_y$ at $y=0$ or $b$ | | | $M_x$ at $x=0$ or $a$ | | |
|---|---|---|---|---|---|---|
| | $\frac{M_y}{P}$ at $x/a=0.5$ | Av $\frac{M_y}{P}$ | Total $\frac{M_y}{Pb}$ | $\frac{M_x}{P}$ at $y/b=0.5$ | Av $\frac{M_x}{P}$ | Total $\frac{M_x}{Pb}$ |
| 0 | −0.1677* | 0 | −0.1250* | 0 | 0 | Same as Average |
| 0.500 | −0.1674 | −0.0611 | −0.1222 | −0.0162 | −0.0066 | |
| 0.555 | −0.1667 | −0.0656 | −0.1181 | −0.0263 | −0.0116 | |
| 0.625 | −0.1651 | −0.0695 | −0.1112 | −0.0425 | −0.0192 | |
| 0.715 | −0.1604 | −0.0713 | −0.0998 | −0.0648 | −0.0299 | |
| 0.833 | −0.1490 | −0.0688 | −0.0826 | −0.0935 | −0.0438 | |
| 1.000 | −0.1257 | −0.0590 | −0.0590 | −0.1257 | −0.0590 | |

* Computed from Bulletin 304.

Moments were computed in rectangular plates having three or four edges fixed, and loaded with a single concentrated load at various locations. Two values of the ratio of sides, $b/a$, were considered: 0.5 and 1.0. Three load positions in the central portion of the plate were considered for the square plates ($b/a=1.0$). Analyses were made for a load at each of these positions for the condition of four edges fixed, but for a load at only two of them for the case of three edges fixed. Only a single load at the middle of the plate was considered for the rectangular plates having $b/a=0.5$ and either four edges fixed or three edges fixed with the long edge simply-supported.

The moments given in Sections 38 and 39 were obtained by the writers by means of the procedure described in Section 29, Appendix A. Solutions obtained by other investigators, some of which are for the same cases as those considered herein, are discussed in Section 37. The last section contains a brief discussion of the applicability of the distribution procedure described in Chapter II to slabs with concentrated loads.

37. *Previous Work*

The most important previous solutions for concentrated loads in fixed rectangular plates are those reported by Young[1] for a central load on plates fixed on all edges. The method of calculation was that proposed by S. Timoshenko and referred to in Section 29, Appendix A. The results are given as coefficients for the terms in the series for edge moments for plates having values of $b/a = 0.500$, 0.555, 0.625, 0.715, 0.833, and 1.000. For $b/a = 0.500$ and 1.000, terms in the series through $n = 13$ are given; for the other values of $b/a$ only the terms through $n = 9$ were obtained.

Maximum moments at the middle of the edges, average moments, and total moments are given in Table 33. The moment $M_y$ at the middle of the long edge was given by Young; the other moments were computed by the writers from the terms in the series.

Moments in the plate with $b/a = 1.0$ have also been obtained by Hencky,[2] using terms in the series through $n = 11$. His results are in reasonably good agreement with those obtained by Young.

Analyses of plates having $b/a = 0.5$ and 1.0 were also made by the writers, using only the terms through $n = 5$. The moments obtained differ somewhat from those obtained by Young for a greater number of terms. The two sets of values may be compared by reference to Tables 35 and 38.

Other solutions for concentrated loads on plates with fixed edges have been reported by Young in another paper.[3] Two of the cases considered in that paper are of interest here. In the first, moments are given at the middle of an edge for the following conditions: $b/a = 0.5$, three edges fixed with a long edge simply-supported, and a concentrated load at the middle of the plate. Terms in the series are not given, however, and it was necessary to repeat this solution in Section 39 in order to obtain data for the calculation of average moments and moments at other points on the edges. The values obtained by Young and by the writers may be compared by reference to Table 38.

The other solution of interest here is for a square plate having

[1] Dana Young, "Clamped Rectangular Plates with a Central Concentrated Load," *Jour. Appl. Mech.*, September 1939, Vol. 6, No. 3, pp. A-114–116.
[2] H. Hencky, "Der Spannungszustand in rechteckigen Platten." Oldenbourg, Munich, 1913.
[3] Dana Young, "Analysis of Clamped Rectangular Plates," *Jour. Appl. Mech.*, December 1940, Vol. 7, No. 14, pp. A-139–142.

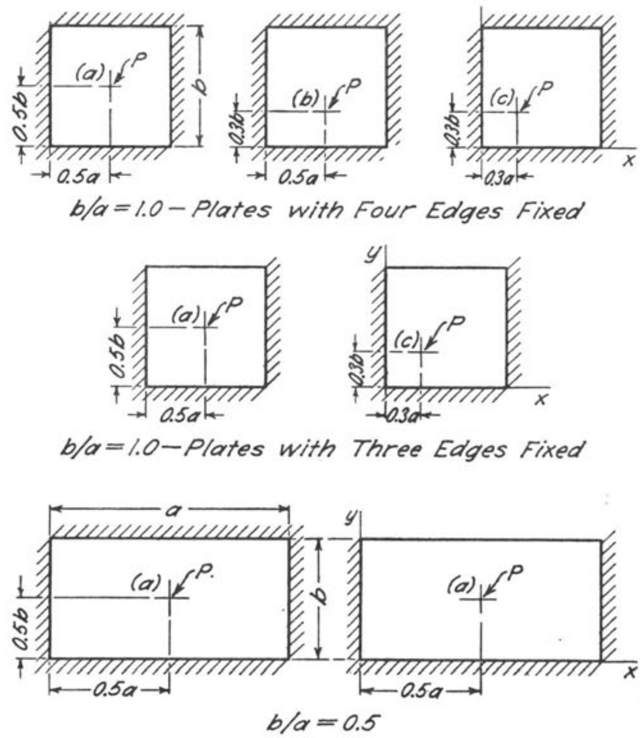



FIG. 21. EDGE CONDITIONS AND LOCATIONS OF LOADS FOR PLATES ANALYZED IN APPENDIX C

four edges fixed and loaded at the quarter-point of one centerline. Again, terms in the series are not given but moments are tabulated for points at the middle and quarter-points of the edges equidistant from the load, and at the middle of the other edges. These moments are not presented herein.

### 38. *Moments in Square Plates*

Moments are given in this section for a concentrated load at various locations on square plates supported on all sides and having either three or four edges fixed. Moments were computed only at the edges of the plates, but terms in the sine series for edge moment are given so that moments in the interior may be obtained, if desired, by means of the procedure described in Bulletin 304. Five plates were analyzed, having the various combinations of edge conditions and locations of load indicated in Fig. 21. Loads at positions (a) and

(c) were considered for both three and four edges fixed, while a load at position (b) was considered only for the condition of four edges fixed.

All results given herein were obtained by the procedure described in Section 29 of Appendix A, using Equations (16). For the case of a concentrated load $P$ on the plate at a point $x = u$, $y = v$, the quantity $(M^F)$ in Equations (16) is given by the following expressions:

$$(M^F)_n = -2P\frac{a}{b}\sin\frac{n\pi y}{b}(C_M)_n$$

$$(M^F)_m = -2P\frac{b}{a}\sin\frac{m\pi u}{a}(C_M)_m,$$

where $(C_M)$ is obtained from Table 3 of Bulletin 304 for $(b/s)$ in the table equal to $na/b$ or $mb/a$ as indicated by the subscripts $n$ and $m$ respectively, and $(v/b)$ in the table equal to $v/b$, $1-v/b$, $u/a$, or $1-u/a$, depending on the edge being considered. It should be noted that the term $(M^F)$ $(1+k)$ in Equation (16) must be separated into its component parts, $(M^F)$ and $(kM^F)$, if the load is not symmetrical, since the value of $(M^F)$ for one edge is obtained from Bulletin 304 for $(v/b) = u/a$ or $v/b$, and for the opposite edge for $(v/b) = 1-u/a$ or $1-v/b$. That is, the fixed-edge moments on two opposite edges will not be equal except for symmetrically placed loads.

Only the terms in the series through $n = 5$ were considered, and from 3 to 15 equations were obtained for the various plates analyzed. These were solved, as before, by a process of successive approximations, continued until values of the coefficients were obtained correct to the fifth decimal place.[1] Terms in the series are given in Table 34.

Average moments and moments at the twelfth points on the edges, computed from the terms in the series, are given in Table 35 for four edges fixed and in Table 36 for three edges fixed. Although these moment coefficients are given to four decimal places, they are not that accurate because of the small number of terms in the series used. Some idea of the probable error may be obtained by comparing the values for a central load computed herein using 5 terms with those computed by Young using 13 terms, as given in parentheses in Table 35. The difference is only about 1 percent for the maximum moment and considerably less for the average moment. However, for unsymmetrical cases the values given herein may be more in error.

[1] The time required for solution of the equations, using a fully automatic electric computing machine, varied from 35 to 213 min, depending on the number of equations. The time per equation varied from 10 to 20 min.

TABLE 34

TERMS IN SERIES FOR EDGE MOMENTS FOR SQUARE PLATES WITH CONCENTRATED LOADS

See Fig. 21 for notation. Values in table are coefficients of sin $n\pi x/a$ or sin $n\pi y/b$.

| Edge Conditions | Location of Load | $n$ | Coefficients for Edge Moment $M/P$ | | | |
|---|---|---|---|---|---|---|
| | | | $M_x$ at $x=0$ | $M_x$ at $x=a$ | $M_y$ at $y=0$ | $M_y$ at $y=b$ |
| Four Edges Fixed | (a) | 1<br>2<br>3<br>4<br>5 | −0.1025<br>0<br>+0.0264<br>0<br>+0.0043 | Same | Same | Same |
| | (b) | 1<br>2<br>3<br>4<br>5 | −0.0742<br>−0.0156<br>+0.0124<br>+0.0076<br>+0.0036 | Same as $M_x$ at $x=0$ | −0.1690<br>0<br>+0.0526<br>0<br>+0.0003 | −0.0324<br>0<br>+0.0189<br>0<br>+0.0056 |
| | (c) | 1<br>2<br>3<br>4<br>5 | −0.1227<br>−0.0598<br>+0.0109<br>+0.0181<br>+0.0134 | −0.0277<br>−0.0039<br>+0.0068<br>+0.0013<br>+0.0014 | Same as $M_x$ at $x=0$ | Same as $M_x$ at $x=a$ |
| Three Edges Fixed | (a) | 1<br>2<br>3<br>4<br>5 | −0.1279<br>+0.0191<br>+0.0182<br>+0.0046<br>+0.0021 | Same as $M_x$ at $x=0$ | −0.1090<br>0<br>+0.0284<br>0<br>+0.0048 | 0<br>0<br>0<br>0<br>0 |
| | (b) | 1<br>2<br>3<br>4<br>5 | −0.1300<br>−0.0541<br>+0.0082<br>+0.0196<br>+0.0126 | −0.0341<br>+0.0007<br>+0.0050−<br>+0.0023<br>+0.0009 | −0.1244<br>−0.0578<br>−0.0114<br>+0.0181<br>+0.0135 | 0<br>0<br>0<br>0<br>0 |

## 39. *Moments in Rectangular Plates with $b/a=0.5$*

Moments are given in this section for a concentrated load at the center of a rectangular plate having $b/a=0.5$ and supported on all four sides. Two cases are considered: a plate having all four edges fixed, and one having three edges fixed and the long edge at $y=b$ simply-supported. All calculations for moments in this section were made by exactly the same procedure as that described in Section 38 for the square plates. The same equations from Section 29 were used, terms in the series through $n=5$ were considered, and the accuracy of the results is the same as that discussed in the preceding section.[1]

Terms in the sine series for edge moments are given in Table 37. No moments were computed at points in the interior of the plate, but the coefficients in this table may be used with the method of Bulletin 304 to compute correction moments in the interior if desired.

[1] In the solution for the plate with three edges fixed, eight equations were obtained and the time required for their solution by successive approximations was 52 min.

Average moments and moments at the twelfth points on the edges are given in Table 38. The probable accuracy of these moments may be estimated by comparing them with the more precise values obtained by Young using 13 terms in the series. Young's values are given in parentheses in the table.

## 40. *Applicability of Distribution Procedure to Plates with Concentrated Loads*

Although the data given in the preceding sections are too limited to support any definite conclusions, certain of the moments tabulated may be used to make a rough check on the applicability of the carry-

TABLE 35

EDGE MOMENTS FOR CONCENTRATED LOAD AT VARIOUS LOCATIONS ON SQUARE PLATE WITH FOUR EDGES FIXED

See Fig. 21 for notation. Values in parentheses are moments from solution by Young (see Section 37).

| Location of Load | Edge Moment $\frac{M_x}{P}$ | | | Edge Moment $\frac{M_y}{P}$ | | |
|---|---|---|---|---|---|---|
| | $\frac{x}{a}$ | $\frac{y}{b}$ | $\frac{M_x}{P}$ | $\frac{y}{b}$ | $\frac{x}{a}$ | $\frac{M_y}{P}$ |
| (a) | 0 or 1.0 | 1/12 | −0.0037 | 0 or 1.0 | Same as $M_x$ | |
| | | 2/12 | −0.0227 | | | |
| | | 3/12 | −0.0569 | | | |
| | | 4/12 | −0.0925 | | | |
| | | 5/12 | −0.1166 | | | |
| | | 6/12 | −0.1246 (−0.1257) | | | |
| | | Av | −0.0591 (−0.0590) | | | |
| (b) | 0 or 1.0 | 1/12 | −0.0081 | 0 | 1/12 | −0.0063 |
| | | 2/12 | −0.0297 | | 2/12 | −0.0318 |
| | | 3/12 | −0.0619 | | 3/12 | −0.0825 |
| | | 4/12 | −0.0876 | | 4/12 | −0.1466 |
| | | 5/12 | −0.0940 | | 5/12 | −0.2000 |
| | | 6/12 | −0.0830 | | 6/12 | −0.2213 |
| | | 7/12 | −0.0650− | | Av | −0.0964 |
| | | 8/12 | −0.0471 | | | |
| | | 9/12 | −0.0307 | 1.0 | 1/12 | +0.0105− |
| | | 10/12 | −0.0161 | | 2/12 | +0.0056 |
| | | 11/12 | −0.0059 | | 3/12 | −0.0135 |
| | | Av | −0.0441 | | 4/12 | −0.0330 |
| | | | | | 5/12 | −0.0432 |
| | | | | | 6/12 | −0.0457 |
| | | | | | Av | −0.0159 |
| (c) | 0 | 1/12 | −0.0254 | 1.0 | 1/12 | −0.0019 |
| | | 2/12 | −0.0799 | | 2/12 | −0.0086 |
| | | 3/12 | −0.1484 | | 3/12 | −0.0196 |
| | | 4/12 | −0.1854 | | 4/12 | −0.0297 |
| | | 5/12 | −0.1683 | | 5/12 | −0.0343 |
| | | 6/12 | −0.1201 | | 6/12 | −0.0332 |
| | | 7/12 | −0.0772 | | 7/12 | −0.0281 |
| | | 8/12 | −0.0504 | | 8/12 | −0.0206 |
| | | 9/12 | −0.0287 | | 9/12 | −0.0118 |
| | | 10/12 | −0.0076 | | 10/12 | −0.0041 |
| | | 11/12 | +0.0032 | | 11/12 | −0.0002 |
| | | Av | −0.0741 | | Av | −0.0160 |
| | 1.0 | Same as $M_y$ at $y/b = 1.0$ | | 0 | Same as $M_x$ at $x/a = 0$ | |

## Table 36

### Edge Moments for Concentrated Load at Various Locations on Square Plate with Three Edges Fixed

See Fig. 21 for notation. Edge at $y = b$ is simply-supported.

| Location of Load | Edge Moment $\frac{M_x}{P}$ | | | Edge Moment $\frac{M_y}{P}$ | | |
|---|---|---|---|---|---|---|
| | $\frac{x}{a}$ | $\frac{y}{b}$ | $\frac{M_x}{P}$ | $\frac{y}{b}$ | $\frac{x}{a}$ | $\frac{M_y}{P}$ |
| (a) | 0 or 1.0 | 1/12 | −0.0045 | 0 | 1/12 | −0.0035 |
| | | 2/12 | −0.0241 | | 2/12 | −0.0237 |
| | | 3/12 | −0.0599 | | 3/12 | −0.0604 |
| | | 4/12 | −0.1000 | | 4/12 | −0.0985 |
| | | 5/12 | −0.1303 | | 5/12 | −0.1241 |
| | | 6/12 | −0.1440 | | 6/12 | −0.1326 |
| | | 7/12 | −0.1414 | | Av | −0.0628 |
| | | 8/12 | −0.1251 | | | |
| | | 9/12 | −0.0982 | | | |
| | | 10/12 | −0.0652 | | | |
| | | 11/12 | −0.0318 | | | |
| | | Av | −0.0773 | | | |
| (e) | 0 | 1/12 | −0.0257 | 0 | 1/12 | −0.0253 |
| | | 2/12 | −0.0804 | | 2/12 | −0.0802 |
| | | 3/12 | −0.1492 | | 3/12 | −0.1494 |
| | | 4/12 | −0.1874 | | 4/12 | −0.1870 |
| | | 5/12 | −0.1722 | | 5/12 | −0.1703 |
| | | 6/12 | −0.1256 | | 6/12 | −0.1223 |
| | | 7/12 | −0.0841 | | 7/12 | −0.0792 |
| | | 8/12 | −0.0597 | | 8/12 | −0.0520 |
| | | 9/12 | −0.0410 | | 9/12 | −0.0297 |
| | | 10/12 | −0.0207 | | 10/12 | −0.0079 |
| | | 11/12 | −0.0056 | | 11/12 | +0.0032 |
| | | Av | −0.0794 | | Av | −0.0751 |
| | 1.0 | 1/12 | −0.0021 | | | |
| | | 2/12 | −0.0090 | | | |
| | | 3/12 | −0.0205 | | | |
| | | 4/12 | −0.0317 | | | |
| | | 5/12 | −0.0379 | | | |
| | | 6/12 | −0.0382 | | | |
| | | 7/12 | −0.0346 | | | |
| | | 8/12 | −0.0289 | | | |
| | | 9/12 | −0.0219 | | | |
| | | 10/12 | −0.0142 | | | |
| | | 11/12 | −0.0067 | | | |
| | | Av | −0.0205 | | | |

## Table 37

### Terms in Series for Edge Moments for Rectangular Plates with Concentrated Load: $b/a = 0.5$

See Fig. 21 for notation. Values in table are coefficients of $\sin n\pi x/a$ or $\sin n\pi y/b$. Concentrated load $P$ at center of plate.

| Edge Conditions | Location of Load | $n$ | Coefficients for Edge Moment $M/P$ | | | |
|---|---|---|---|---|---|---|
| | | | $M_x$ at $x=0$ | $M_x$ at $x=a$ | $M_y$ at $y=0$ | $M_y$ at $y=b$ |
| Four Edges Fixed | (a) | 1 | −0.0116 | Same as $M_x$ at $x=0$ | −0.1101 | Same as $M_y$ at $y=0$ |
| | | 2 | 0 | | 0 | |
| | | 3 | +0.0045 | | +0.0460 | |
| | | 4 | 0 | | 0 | |
| | | 5 | +0.0010 | | −0.0085 | |
| Three Edges Fixed | (a) | 1 | −0.0344 | Same as $M_x$ at $x=0$ | −0.1477 | 0 |
| | | 2 | +0.0112 | | 0 | 0 |
| | | 3 | +0.0036 | | +0.0536 | 0 |
| | | 4 | +0.0015− | | 0 | 0 |
| | | 5 | +0.0008 | | −0.0068 | 0 |

TABLE 38

EDGE MOMENTS FOR CONCENTRATED LOAD AT CENTER OF RECTANGULAR PLATE WITH THREE OR FOUR EDGES FIXED: $b/a = 0.5$

See Fig. 21 for notation. Edge at $y = b$ is simply-supported for plate with three edges fixed. Values in parentheses are moments from solution by Young (see Section 37).

| Edge Conditions | Edge Moment $\frac{M_x}{P}$ | | | Edge Moment $\frac{M_y}{P}$ | | |
|---|---|---|---|---|---|---|
| | $\frac{x}{a}$ | $\frac{y}{b}$ | $\frac{M_x}{P}$ | $\frac{y}{b}$ | $\frac{x}{a}$ | $\frac{M_y}{P}$ |
| Four Edges Fixed | 0 or 1.0 | 1/12 | +0.0012 | 0 or 1.0 | 1/12 | −0.0043 |
| | | 2/12 | −0.0008 | | 2/12 | −0.0134 |
| | | 3/12 | −0.0057 | | 3/12 | −0.0393 |
| | | 4/12 | −0.0109 | | 4/12 | −0.0879 |
| | | 5/12 | −0.0142 | | 5/12 | −0.1410 |
| | | 6/12 | −0.0152 | | 6/12 | −0.1646 |
| | | | (−0.0162) | | | (−0.1674) |
| | | Av | −0.0063 | | Av | −0.0614 |
| | | | (−0.0066) | | | (−0.0611) |
| Three Edges Fixed | 0 or 1.0 | 1/12 | +0.0013 | 0 | 1/12 | −0.0069 |
| | | 2/12 | −0.0022 | | 2/12 | −0.0237 |
| | | 3/12 | −0.0112 | | 3/12 | −0.0617 |
| | | 4/12 | −0.0221 | | 4/12 | −0.1220 |
| | | 5/12 | −0.0313 | | 5/12 | −0.1823 |
| | | 6/12 | −0.0372 | | 6/12 | −0.2081 |
| | | | (−0.0389) | | | (−0.2115) |
| | | 7/12 | −0.0399 | | Av | −0.0835 |
| | | 8/12 | −0.0389 | | | |
| | | 9/12 | −0.0335− | | | |
| | | 10/12 | −0.0241 | | | |
| | | 11/12 | −0.0124 | | | |
| | | Av | −0.0210 | | | |

over factors from Section 6 to the edge moments resulting from concentrated loads located in the middle third of the plate. Moreover, the coefficients given in Tables 35 and 38 may be used for the calculation of average fixed-edge moments in plates loaded with concentrated loads. No data are available for use in checking the applicability of the stiffness factors proposed for uniformly loaded plates, but is probable that they will be satisfactory if the carry-over factors are.

To check the carry-over factors it is necessary that average moments on all edges be known for plates with four edges fixed and with three edges fixed, for the same condition of loading. Three such sets of moments are available from the tables in this Appendix: (1) average moments in a square plate with a concentrated load at the center, location (a), are given in Table 35 for four edges fixed and in Table 36 for three edges fixed; (2) similar data for a load on the diagonal at location (c) are also given in Tables 35 and 36; (3) average moments are found in Table 38 for a centrally loaded rectangular plate having $b/a = 0.5$ and either four or three edges fixed. The check of the carry-over factors is made as follows: First, consider a plate having four edges fixed. Then release the edge which is simply-supported in the case with three edges fixed, and carry over the average moments from that edge using the carry-over factors from Table 1. Add these "carried-over" moments to those already existing on the edges, and compare the total with the correct average moments computed in this appendix for plates with only three edges fixed.

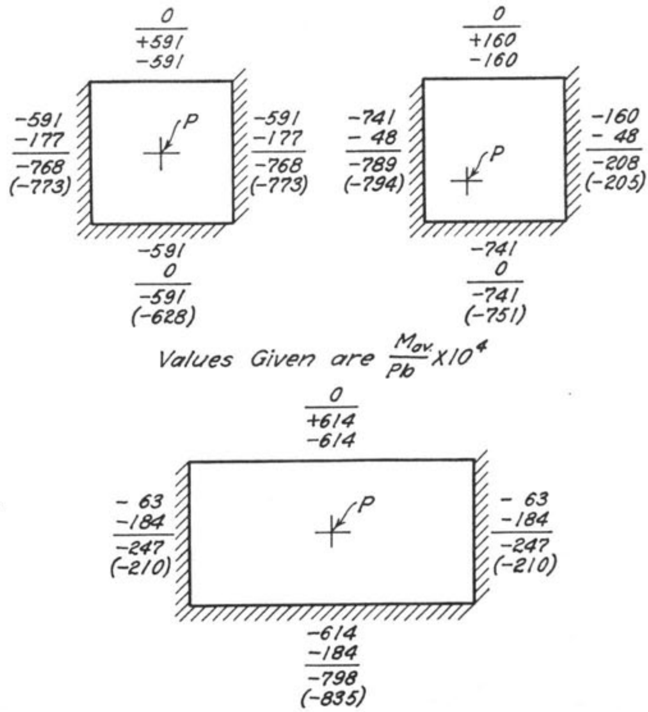



FIG. 22. CHECK OF CARRY-OVER FACTORS FOR PLATES WITH CONCENTRATED LOADS

The calculations described above are illustrated in Fig. 22. The quantity written below the line is the approximate average moment in a plate having three edges fixed, as determined by means of the distribution procedure using the carry-over factors derived for uniformly loaded plates. The moment in parentheses is the correct value from the analyses described in the preceding sections. For a square plate, the approximate procedure results in moments from 1 to 6 percent too low; while for the rectangular plate the range is from

4 percent low to 18 percent high. Although the agreement is not exceptionally good, it is close enough to suggest that the distribution procedure developed for uniform loads might also be applicable to plates carrying concentrated loads in the middle third of their area.

If concentrated loads are applied close to an edge, the distribution procedure proposed herein is not suitable, because of the relatively large edge moments which are produced in the neighborhood of the load. The concept of design for average moments, which underlies the distribution procedure, is unsatisfactory if local effects may be important as is the case for concentrated loads near an edge. It should also be kept in mind that for a concentrated load at any point on the plate, the local positive moments under the load are usually much more critical than the moments at an edge.

## Appendix D

### List of Symbols

The letter symbols and other notation used throughout this bulletin are listed below:

$x, y$ = horizontal rectangular coordinates having their origin at the corner of a panel.

$a$ = length of long span of slab panel.

$b$ = length of short span of slab panel.

$b/a$ = ratio of short span to long span.

$t$ = total thickness of slab.

$I$ = $t^3/12$ = moment of inertia per unit width of the slab in a particular panel.

$I_b$ = moment of inertia of the cross-section of a beam.

$E$ = modulus of elasticity of the material in the slab.

$E_b$ = modulus of elasticity of the material in a beam.

$\mu$ = Poisson's ratio of the material in the slab, taken equal to zero throughout this bulletin.

$N$ = $EI(1 - \mu^2)$ = measure of stiffness of an element of the slab in a particular panel.

$H$ = $E_bI_b/bN$ or $E_bI_b/aN$ = a dimensionless coefficient which is a measure of the stiffness of a beam relative to that of the slab. The term $b$ or $a$ in the denominator corresponds to the span length of the beam being considered.

$w$ = load per unit of area uniformly distributed over a panel of the slab.

$P$ = concentrated load applied to the slab.

$M_x, M_y$ = bending moments per unit of width in the direction of $x$ or $y$, respectively, acting on a section normal to the $x$ or $y$ axis, respectively, positive when producing compression at the top of the slab.

$M_b$ = $M_x$ = bending moment acting on a section parallel to the side of a panel having a length equal to $b$.

$M_a$ = $M_y$ = bending moment acting on a section parallel to the side of a panel having a length equal to $a$.

$M_{tot}$ = total bending moment acting on a section extending the full width of a panel in the direction normal to that of the moment, expressed in ft-lb or equivalent units.

$M_{av}$ = average bending moment per unit of width, acting on a section extending the full width of a panel, expressed in ft-lb per foot of width, or simply in pounds. $M_{av} = M_{tot}/b$ or $M_{tot}/a$, depending on the direction of the moment.

$K_a$, $K_b$ = flexural stiffness of a panel of the slab at an edge having a length of $a$ or $b$, respectively.

$k_a$, $k_b$ = dimensionless coefficients used to determine the stiffness of a slab panel; $K_a = k_a N/b$ and $K_b = k_b N/b$.

$C_{bb}$, $C_{ba}$, $C_{aa}$, $C_{ab}$ = carry-over factors for average moments acting on the edges of a panel. The significance of the subscripts is discussed in Section 6.

$K'_a$, $C'_{bb}$, etc. = modified stiffness and carry-over factors defined in Section 7.

$F_{bb}$, $F_{ba}$, $F_{aa}$, $F_{ab}$ = positive moment correction factors employed to determine the moments produced in the interior of a panel by moments acting on the edges, defined in Section 6.

$\Phi_{av}$ = the average rotation of an edge of a panel.

$G = E_b/[2(1+\mu)]$ = modulus of elasticity in shear of the material in a beam.

$J$ = measure of the torsional rigidity of the cross-section of a beam.

$T_b$, $T_a$ = torsional stiffness of a beam having a span of $b$ or $a$, respectively, defined in Section 19.

$T/K$ = a measure of the torsional stiffness of a beam relative to that of the slab along the edge adjacent to the beam.

Symbols and expressions taken from the works of others and referred to in this bulletin are not included in the foregoing list, but are defined in the text where first introduced.

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History of the Development of Building Construction in Chicago, by Frank A. Randall. 1949. A publication of the University of Illinois Press; sponsored jointly by the Engineering Experiment Station and the Graduate College. 388 pages. Price: cloth, *five dollars*; paper, *four dollars*.

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