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Classical inequality indices, welfare and illfare functions, and the dual decomposition

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Abstract

In the traditional framework, social welfare functions depend on the mean income and on the income inequality. An alternative illfare framework has been developed to take into account the disutility of unfavorable variables. The illfare level is assumed to increase with the inequality of the distribution. In some social and economic fields, such as those related to employment, health, education, or deprivation, the characteristics of the individuals in the population are represented by bounded variables, which encode either achievements or shortfalls. Accordingly, both the social welfare and the social illfare levels may be assessed depending on the framework we focus on. In this paper we propose a unified dual framework in which welfare and illfare levels can both be investigated and analyzed in a natural way. The dual framework leads to the consistent measurement of achievements and shortfalls, thereby overcoming one important difficulty of the traditional approach, in which the focus on achievements or shortfalls often leads to different inequality rankings.

A number of welfare functions associated with inequality indices are OWA operators. Specifically this paper considers the welfare functions associated with the classical inequality measures due to Gini, Bonferroni, and De Vergottini. These three indices incorporate different value judgments in the measurement of inequality, leading to different behavior under income transfers between individuals in the population. In the bounded variables representation, we examine the dual decomposition and the orness degree of the three classical welfare/illfare functions in the standard framework of aggregation functions on the $[0,1]^n$ domain. The dual decomposition of each welfare/illfare function into a self-dual central index and an anti-self-dual inequality index leads to the consistent measurement of achievements and shortfalls.

Keywords: Income inequality and social welfare; Classical Gini, Bonferroni, and De Vergottini inequality indices; Welfare functions; Illfare functions; Aggregation functions; WA and OWA operators; Dual decomposition; Orness

1 Introduction

Income inequality plays a crucial role in Economics and Social Welfare. It has been proved that income inequality has an important impact in terms of development, poverty, and public finance. Typical issues that arise in these contexts are the evolution of inequality over time in some particular region, the differences in the inequality level across different countries, and the effect of different policies in the evolution of inequality. In order to address these and related questions the choice of inequality measure is a central issue.

Basically, an inequality measure is a summary statistic of the income dispersion. Several inequality indices have been proposed in the literature, for comprehensive surveys on inequality measures see Silber [46] and Chakravarty [12]. One of the most widely used is the Gini index (Gini [27]), based on the absolute values of all pairwise income differences. This index has a very intuitive appeal for its geometrical interpretation in terms of the Lorenz curve and, unlike other inequality measures, it easily accommodates negative incomes. One drawback of the Gini index is that it is insensitive to the position of income transfers within the ordered income profile. In order to overcome this difficulty, a single-parameter class of inequality measures that generalizes the

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Gini index referred to as the S-Gini family, has been introduced and characterized (Donaldson and Weymark [16,17], Weymark [49] and Bossert [9]). In this family, different value judgments can be considered by means of a weighting function of incomes.

The Bonferroni and De Vergottini indices are two other classical inequality indices that are recently receiving growing attention, see for instance Nygard and Sandstrom [41], Giorgi [28,29], Tarsitano [48], Giorgi and Mondani [31], Giorgi and Crescenzi [30], Chakravarty and Muliere [13], Piesch [42], Chakravarty [11] and Bárcena and Imedio [3]. Similarly to the Gini index, they also permit negative incomes.

The Bonferroni index ([8]) measures inequality comparing the overall income mean with the income means of the poorest individuals in the population. The De Vergottini index ([15]) complements the information provided by the Bonferroni index since inequality is captured by comparing the overall income mean with the income means of the richest individuals in the population. The three classical inequality indices –Gini, Bonferroni, and De Vergottini– are formally similar but introduce distinct and complementary information in the study of income inequality. Moreover, in contrast with the Gini index, the Bonferroni and De Vergottini indices are sensitive to the specific position of income transfers within the ordered income profile.

An inequality index is relative if it is invariant when an additional amount of income is proportionally distributed among the whole population. This corresponds to the rightist viewpoint, according to Kolm's designation [35]. In turn, the leftist view requires that inequality remains unchanged when each individual in the population receives the same amount of the extra income. This invariance condition is fulfilled by the absolute inequality indices, which are obtained by multiplying the corresponding relative indices by the mean income.

Choosing a particular index to measure inequality involves a value judgment, because different choices can lead to different results. One criterion is to select ethical indices, that are indices with a normative interpretation. This means that there is an explicit relationship between the inequality measure and a social welfare ordering defined on incomes. In other words, for these indices it is possible to construct a social welfare function whose contours specify the tradeoffs between inequality and efficiency, as measured by the mean income.

More recently, the interest in unfavorable variables, such as unemployment, illiteracy, morbidity or poverty gaps, has led to the development of an alternative illfare social framework (see for instance Riese and Brunner [44] and Chakravarty [12]). Basically, the social illfare functions make use of the disutility of the individuals and depend also on the mean of the variable and on the inequality level: the higher the inequality of the distribution is, the greater the social illfare is.

In a bounded setting, each individual's characteristics may be represented in terms of achievements or shortfalls. In this paper we propose a unified framework in which the illfare functions associated with the shortfall distributions may be derived as the counter-part of the welfare achievement functions. However a problem arises when the decomposition into the mean and the inequality terms needs to be identified. The difficulty stems from the difference between the achievement and the shortfall inequality levels. Recent papers (among them Clarke *et al.* [14], Erreygers [20], Lambert and Zheng [36], and Lasso de la Vega and Aristondo [37]) deal with this issue in health measurement.

An interesting feature of several inequality indices is that the associated welfare functions are of the OWA type. Accordingly, when bounded variables are involved, they can be analyzed in the framework of the dual decomposition of aggregation functions proposed by García-Lapresta and Marques Pereira [25], where each aggregation operator is decomposed into a self-dual core and an associated anti-self-dual remainder [1]. In this paper we show that the two terms of the dual decomposition of the welfare function can be interpreted as a weighted mean and an inequality term. The inequality indices generated from the anti-self-dual remainder component are consistent in the sense that the levels of the achievement inequality and the shortfall inequality coincide. This fact allows us to introduce pairs of welfare and illfare functions associated with the same inequality indices and the highlighted difficulty is overcome.

Moreover, the dual decomposition offers interesting insight on the distinct and complementary nature of the three classical inequality indices. In the Gini index case, the dual decomposition reproduces in a natural way the construction of the associated welfare function. As for the Bonferroni and the De Vergottini indices, the corresponding self-dual cores and anti-self-dual remainders express the underlying relationship between the two indices.

The paper is organized as follows. In Section 2, we introduce the basic notation and properties of aggregation functions and we describe the general framework of the dual decomposition of an aggregation function into a self-dual core and an associated anti-self-dual remainder. Moreover, we briefly review the dual decomposition of OWA operators. Section 3 is devoted to inequality indices and the associated welfare and illfare functions, focusing on the classical Gini, Bonferroni, and De Vergottini indices. In Section 4 we examine the dual decomposition of the welfare and illfare functions associated to these indices and show the possibilities of this decomposition in the unified achievement-shortfall framework. In Section 5, an illustrative example is provided. Finally,

[1] Other applications of the dual decomposition to the field of Welfare Economics can be found in García-Lapresta *et al.* [23] and Aristondo *et al.* [1].

Section 6 contains some concluding remarks.

2 Aggregation functions

In this section we present notation and basic definitions regarding aggregation functions on $[0,1]^n$ and functions on $[0,\infty)^n$, with $n \in \mathbb{N}$ and $n \geq 2$ throughout the text.

Notation. Vectors in $[0,\infty)^n$ are denoted as $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\mathbf{0} = (0, \ldots, 0)$, $\mathbf{1} = (1, \ldots, 1)$. Accordingly, for every $x \in [0,\infty)$, we have $x \cdot \mathbf{1} = (x, \ldots, x)$. Given $\boldsymbol{x}, \boldsymbol{y} \in [0,\infty)^n$, by $\boldsymbol{x} \geq \boldsymbol{y}$ we mean $x_i \geq y_i$ for every $i \in \{1, \ldots, n\}$, and by $\boldsymbol{x} > \boldsymbol{y}$ we mean $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Given $\boldsymbol{x} \in [0,\infty)^n$, the increasing and decreasing reorderings of the coordinates of $\boldsymbol{x}$ are indicated as $x_{(1)} \leq \cdots \leq x_{(n)}$ and $x_{[1]} \geq \cdots \geq x_{[n]}$, respectively. In particular, $x_{(1)} = \min\{x_1, \ldots, x_n\} = x_{[n]}$ and $x_{(n)} = \max\{x_1, \ldots, x_n\} = x_{[1]}$. Clearly, $x_{[k]} = x_{(n-k+1)}$ for every $k \in \{1, \ldots, n\}$. In general, given a permutation σ on $\{1, \ldots, n\}$, we denote $\boldsymbol{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Finally, the arithmetic mean is denoted $\mu(\boldsymbol{x}) = (x_1 + \cdots + x_n)/n$. Occasionally, we refer to $\boldsymbol{x}$ as achievements and to $\boldsymbol{y} = \mathbf{1} - \boldsymbol{x}$ as shortfalls.

We begin by defining standard properties of real functions on $\mathbb{R}^n$. For further details the interested reader is referred to Fodor and Roubens [22], Calvo *et al.* [10], Beliakov *et al.* [4], García-Lapresta and Marques Pereira [25] and Grabisch *et al.* [32].

Definition 1 *Let $A : D^n \longrightarrow \mathbb{R}$ be a function with $D = [0,1]$ or $D = [0,\infty)$.*

(1) A is idempotent *if for every $x \in D$:*

$$A(x \cdot \mathbf{1}) = x.$$

(2) A is symmetric *if for every permutation σ on $\{1, \ldots, n\}$ and every $\boldsymbol{x} \in D^n$:*

$$A(\boldsymbol{x}_\sigma) = A(\boldsymbol{x}).$$

(3) A is monotonic *if for all $\boldsymbol{x}, \boldsymbol{y} \in D^n$:*

$$\boldsymbol{x} \geq \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) \geq A(\boldsymbol{y}).$$

(4) A is strictly monotonic *if for all $\boldsymbol{x}, \boldsymbol{y} \in D^n$:*

$$\boldsymbol{x} > \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) > A(\boldsymbol{y}).$$

(5) A is compensative *(or* internal*) if for every $\boldsymbol{x} \in D^n$:*

$$x_{(1)} \leq A(\boldsymbol{x}) \leq x_{(n)}.$$

(6) A is self-dual[2] *if* $D = [0,1]$ *and for every* $\boldsymbol{x} \in [0,1]^n$*:*

$$A(\mathbf{1} - \boldsymbol{x}) = 1 - A(\boldsymbol{x}).$$

(7) A is anti-self-dual *if* $D = [0,1]$ *and for every* $\boldsymbol{x} \in [0,1]^n$*:*

$$A(\mathbf{1} - \boldsymbol{x}) = A(\boldsymbol{x}).$$

(8) A is invariant for translations *if for every* $\boldsymbol{x} \in D^n$*:*

$$A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x})$$

for all $t \in \mathbb{R}$ *such that* $\boldsymbol{x} + t \cdot \mathbf{1} \in D^n$.

(9) A is stable for translations *if for every* $\boldsymbol{x} \in D^n$*:*

$$A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x}) + t$$

for all $t \in \mathbb{R}$ *such that* $\boldsymbol{x} + t \cdot \mathbf{1} \in D^n$.

(10) A is ratio scale invariant *(or* positively homogeneous of degree 0*) if for every* $\boldsymbol{x} \in D^n$*:*

$$A(\lambda \cdot \boldsymbol{x}) = A(\boldsymbol{x})$$

for all $\lambda > 0$ *such that* $\lambda \cdot \boldsymbol{x} \in D^n$.

(11) A is positively homogeneous of degree 1 *if for every* $\boldsymbol{x} \in D^n$*:*

$$A(\lambda \cdot \boldsymbol{x}) = \lambda \cdot A(\boldsymbol{x})$$

for all $\lambda > 0$ *such that* $\lambda \cdot \boldsymbol{x} \in D^n$.

Definition 2 *Let* $\{A^{(k)}\}_{k \in \mathbb{N}}$ *be a sequence of functions, with* $A^{(k)} : D^k \longrightarrow \mathbb{R}$ *and* $A^{(1)}(x) = x$ *for every* $x \in D$*, where* $D = [0,1]$ *or* $D = [0,\infty)$*.* $\{A^{(k)}\}_{k \in \mathbb{N}}$ *is* invariant for replications *(or* strongly idempotent*) if for all* $\boldsymbol{x} \in D^n$ *and any number of replications* $m \in \mathbb{N}$ *of* $\boldsymbol{x}$*:*

$$A^{(mn)}(\overbrace{\boldsymbol{x}, \ldots, \boldsymbol{x}}^{m}) = A^{(n)}(\boldsymbol{x}).$$

Definition 3 *Consider the binary relation* $\succcurlyeq$ *on* D^n*, with* $D = [0,1]$ *or* $D = [0,\infty)$*, defined as*

$$\boldsymbol{x} \succcurlyeq \boldsymbol{y} \;\Leftrightarrow\; \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \;\; \text{and} \;\; \sum_{i=1}^{k} x_{(i)} \leq \sum_{i=1}^{k} y_{(i)},$$

for every $k \in \{1, \ldots, n-1\}$*. With respect to the binary relation* $\succcurlyeq$*, the notions of* Schur-convexity *(S-convexity) and* Schur-concavity *(S-concavity) of a function A are defined as follows.*

[2] A more general notion of self-duality is $A(\mathbf{1} - \boldsymbol{x}) = A(\mathbf{0}) + A(\mathbf{1}) - A(\boldsymbol{x})$.

(1) $A : D^n \longrightarrow D$ *is* S-convex *if for all* $\boldsymbol{x}, \boldsymbol{y} \in D^n$*:*

$$\boldsymbol{x} \succcurlyeq \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) \geq A(\boldsymbol{y}).$$

(2) $A : D^n \longrightarrow D$ *is* S-concave *if for all* $\boldsymbol{x}, \boldsymbol{y} \in D^n$*:*

$$\boldsymbol{x} \succcurlyeq \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) \leq A(\boldsymbol{y}).$$

Moreover, in each case, the S-convexity *(resp.* S-concavity*) of a function* A *is said to be strict if* $A(\boldsymbol{x}) > A(\boldsymbol{y})$ *(resp.* $A(\boldsymbol{x}) < A(\boldsymbol{y})$*) whenever* $\boldsymbol{x} \neq \boldsymbol{y}$*.*

Definition 4 *Given* $\boldsymbol{x}, \boldsymbol{y} \in D^n$*, we say that* $\boldsymbol{y}$ *is obtained from* $\boldsymbol{x}$ *by a* progressive transfer *if there exist two individuals* $i, j \in \{1, \ldots, n\}$ *and* $h > 0$ *such that* $x_i < x_j$*,* $y_i = x_i + h \leq x_j - h = y_j$ *and* $y_k = x_k$ *for every* $k \in \{1, \ldots, n\} \setminus \{i, j\}$*.*

A classical result (see Marshall and Olkin [39, Ch. 4, Prop. A.1]) establishes that $\boldsymbol{x} \succcurlyeq \boldsymbol{y}$ if and only if $\boldsymbol{y}$ can be derived from $\boldsymbol{x}$ by means of a finite sequence of permutations and/or progressive transfers.

Definition 5 *A function* $A : [0,1]^n \longrightarrow [0,1]$ *is called an* n-ary aggregation function *if it is monotonic and satisfies* $A(\mathbf{1}) = 1$ *and* $A(\mathbf{0}) = 0$*. An aggregation function is said to be* strict *if it is strictly monotonic.*

For the sake of simplicity, the n-arity is omitted whenever it is clear from the context.

It is easy to see that every idempotent aggregation function is compensative, and viceversa. Self-duality and stability for translations are important properties of aggregation functions. In turn, anti-self-duality and invariance for translations are incompatible with idempotency. Nevertheless, anti-self-duality and invariance for translations play an important role in this paper as far as they are properties of important functions associated with aggregation functions, such as we shall discuss later.

2.1 Dual decomposition of aggregation functions

In this section we briefly recall the so-called *dual decomposition* of an aggregation function into its self-dual core and associated anti-self-dual remainder, due to García-Lapresta and Marques Pereira [25]. First we introduce the concepts of self-dual core and anti-self-dual remainder of an aggregation function, establishing which properties are inherited in each case from the original aggregation function. Particular emphasis is given to the properties of stability for translations (self-dual core) and invariance for translations (anti-self-dual remainder).

Definition 6 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation function. The aggregation function* $A^* : [0,1]^n \longrightarrow [0,1]$ *defined as*

$$A^*(\boldsymbol{x}) = 1 - A(\mathbf{1} - \boldsymbol{x})$$

is known as the dual *of the aggregation function* A.

Clearly, $(A^*)^* = A$, which means that dualization is an *involution.* An aggregation function A is self-dual if and only if $A^* = A$.

The dual of an arbitrary function A is given by $A^*(\boldsymbol{x}) = A(\mathbf{0})+A(\mathbf{1})-A(\mathbf{1}-\boldsymbol{x})$.

Remark 1 The dual A^* inherits from the aggregation function A the properties of continuity, idempotency (hence, compensativeness), symmetry, strict monotonicity, self-duality, stability for translations and invariance for replications, whenever A has these properties. In addition, A^* is S-convex (resp. S-concave) whenever A is S-concave (resp. S-convex) (see García-Lapresta *et al.* [23]). It is easy to see that A^* is strictly S-convex (resp. strictly S-concave) whenever A is strictly S-concave (resp. strictly S-convex)

2.1.1 The self-dual core of an aggregation function

Aggregation functions are not in general self-dual. However, a self-dual aggregation function can be associated with any aggregation function in a simple manner. The construction of the so-called *self-dual core* of an aggregation function A is as follows.

Definition 7 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation function. The function* $\widehat{A} : [0,1]^n \longrightarrow [0,1]$ *defined as*

$$\widehat{A}(\boldsymbol{x}) = \frac{A(\boldsymbol{x}) + A^*(\boldsymbol{x})}{2} = \frac{A(\boldsymbol{x}) - A(\mathbf{1} - \boldsymbol{x})}{2} + \frac{1}{2}$$

is called the core *of the aggregation function* A.

Since $\widehat{A}$ is self-dual, we say that $\widehat{A}$ is the *self-dual core* of the aggregation function A. Notice that $\widehat{A}$ is clearly an aggregation function.

It is interesting to note that the self-dual core reduces to the arithmetic mean in the simple case of $n = 2$, but not in higher dimensions.

The following results [3] can be found in García-Lapresta and Marques Pereira [25].

[3] Excepting that invariance for replications is inherited by the core (the proof is immediate).

Proposition 1 *An aggregation function* $A : [0,1]^n \longrightarrow [0,1]$ *is self-dual if and only if* $\widehat{A}(\boldsymbol{x}) = A(\boldsymbol{x})$ *for every* $\boldsymbol{x} \in [0,1]^n$.

Proposition 2 *The self-dual core* $\widehat{A}$ *inherits from the aggregation function* A *the properties of continuity, idempotency (hence, compensativeness), symmetry, strict monotonicity, stability for translations, and invariance for replications, whenever* A *has these properties.*

2.1.2 The anti-self-dual remainder of an aggregation function

We now introduce the *anti-self-dual remainder* $\widetilde{A}$, which is simply the difference between the original aggregation function A and its self-dual core $\widehat{A}$.

Definition 8 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation function. The function* $\widetilde{A} : [0,1]^n \longrightarrow \mathbb{R}$ *defined as* $\widetilde{A}(\boldsymbol{x}) = A(\boldsymbol{x}) - \widehat{A}(\boldsymbol{x})$ *, that is,*

$$\widetilde{A}(\boldsymbol{x}) = \frac{A(\boldsymbol{x}) - A^*(\boldsymbol{x})}{2} = \frac{A(\boldsymbol{x}) + A(\mathbf{1} - \boldsymbol{x})}{2} - \frac{1}{2},$$

is called the remainder *of the aggregation function* A.

Since $\widetilde{A}$ is anti-self-dual, we say that $\widetilde{A}$ is the *anti-self-dual remainder* of the aggregation function A. Clearly, $\widetilde{A}$ is not an aggregation function. In particular, $\widetilde{A}(\mathbf{0}) = \widetilde{A}(\mathbf{1}) = 0$ violates idempotency and implies that $\widetilde{A}$ is either non monotonic or everywhere null. Moreover, $-0.5 \leq \widetilde{A}(\boldsymbol{x}) \leq 0.5$ for every $\boldsymbol{x} \in [0,1]^n$.

The following results [4] can be found in García-Lapresta and Marques Pereira [25].

Proposition 3 *An aggregation function* $A : [0,1]^n \longrightarrow [0,1]$ *is self-dual if and only if* $\widetilde{A}(\boldsymbol{x}) = 0$ *for every* $\boldsymbol{x} \in [0,1]^n$.

Proposition 4 *The anti-self-dual remainder* $\widetilde{A}$ *inherits from the aggregation function* A *the properties of continuity, symmetry, invariance for replications, plus also strict S-convexity and S-concavity, whenever* A *has these properties.*

Summarizing, every aggregation function A decomposes additively $A = \widehat{A} + \widetilde{A}$ in two components: the self-dual core $\widehat{A}$ and the anti-self-dual remainder $\widetilde{A}$, where only $\widehat{A}$ is an aggregation function. The so-called *dual decomposition* $A = \widehat{A} + \widetilde{A}$ clearly shows some analogy with other algebraic decompositions,

[4] Excepting that invariance for replications is inherited by the remainder (the proof is immediate) and that strict S-convexity and S-concavity are also inherited by the remainder.

such as that of square matrices and bilinear tensors into their symmetric and skew-symmetric components.

The following result concerns two more properties of the anti-self-dual remainder based directly on the definition $\widetilde{A} = A - \widehat{A}$ and the corresponding properties of the self-dual core (see García-Lapresta and Marques Pereira [25]).

Proposition 5 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation function.*

(1) If A is idempotent, then $\widetilde{A}(x \cdot \mathbf{1}) = 0$ for every $x \in [0,1]$.
(2) If A is stable for translations, then $\widetilde{A}$ is invariant for translations.

These properties of the anti-self-dual remainder are suggestive. The first statement establishes that anti-self-dual remainders of idempotent aggregation functions are null on the main diagonal. The second statement applies to the subclass of stable aggregation functions. In such case, self-dual cores are stable and therefore anti-self-dual remainders are invariant for translations. In other words, if the aggregation function A is stable for translations, the value $\widetilde{A}(\boldsymbol{x})$ does not depend on the average value of the $\boldsymbol{x}$ coordinates, but only on their numerical deviations from that average value. These properties of the anti-self-dual remainder $\widetilde{A}$ suggest that it may give some indication on the dispersion of the $\boldsymbol{x}$ coordinates.

In Maes *et al.* [38], the authors propose a generalization of the dual decomposition framework introduced in García-Lapresta and Marques Pereira [25], based on a family of binary aggregation functions satisfying a form of twisted self-duality condition. Each binary aggregation function in that family corresponds to a particular way of combining an aggregation function A with its dual A^* for the construction of the self-dual core $\widehat{A}$. As particular cases of the general framework proposed in Maes *et al.* [38], one obtains García-Lapresta and Marques Pereira's construction, based on the arithmetic mean, and Silvert's construction, based on the symmetric sums formula (see Silvert [47]). However, the dual decomposition framework introduced in García-Lapresta and Marques Pereira [25] remains the only one which preserves stability under translations, a crucial requirement in the present analysis of welfare and illfare functions.

2.2 OWA operators

In 1988 Yager [52] introduced OWA operators as a tool for aggregating numerical values in multi-criteria decision making. An OWA operator is similar to a weighted mean, but with the values of the variables previously ordered in a decreasing way. Thus, contrary to the weighted means, the weights are not associated with concrete variables and, therefore, they are symmetric. Because of these properties, OWA operators have been widely used in the literature (see, for instance, Yager and Kacprzyk [53] and Yager *et al.* [54]).

Definition 9 *Given a weighting vector* $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$ *satisfying* $\sum_{i=1}^n w_i = 1$*, the* OWA operator associated with $\boldsymbol{w}$ *is the aggregation function* $A_{\boldsymbol{w}} : [0,1]^n \longrightarrow [0,1]$ *defined as*

$$A_{\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_{[i]}.$$

Simple examples of OWA operators are

$$A_{\boldsymbol{w}}(\boldsymbol{x}) = \begin{cases} \max\{x_1, \ldots, x_n\}\,, & \text{when } \boldsymbol{w} = (1, 0, \ldots, 0)\,, \\ \min\{x_1, \ldots, x_n\}\,, & \text{when } \boldsymbol{w} = (0, \ldots, 0, 1\,)\,, \\ \dfrac{x_1 + \cdots + x_n}{n}\,, & \text{when } \boldsymbol{w} = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})\,. \end{cases}$$

OWA operators are continuous, idempotent (hence, compensative), symmetric, and stable for translations. Moreover, an OWA operator $A_{\boldsymbol{w}}$ is self-dual if and only if $w_{n-i+1} = w_i$ for every $i \in \{1, \ldots, n\}$ (see García-Lapresta and Llamazares [24, Proposition 5]).

In general, the dual $A^*_{\boldsymbol{w}}$, the self-dual core $\widehat{A}_{\boldsymbol{w}}$ and the anti-self-dual remainder $\widetilde{A}_{\boldsymbol{w}}$ of an OWA operator $A_{\boldsymbol{w}}$ can be written as

$$\begin{aligned} A^*_{\boldsymbol{w}}(\boldsymbol{x}) &= \sum_{i=1}^{n} w_{n-i+1} \, x_{[i]} = \sum_{i=1}^{n} w_i \, x_{(i)} \\ \widehat{A}_{\boldsymbol{w}}(\boldsymbol{x}) &= \sum_{i=1}^{n} \frac{w_i + w_{n-i+1}}{2} \, x_{[i]} \\ \widetilde{A}_{\boldsymbol{w}}(\boldsymbol{x}) &= \sum_{i=1}^{n} \frac{w_i - w_{n-i+1}}{2} \, x_{[i]}\,. \end{aligned} \tag{1}$$

As we know, the self-dual core $\widehat{A}_{\boldsymbol{w}}$ is an aggregation function. Moreover, since

$$\sum_{i=1}^{n} \frac{w_i + w_{n-i+1}}{2} = 1,$$

the self-dual core $\widehat{A}_{\boldsymbol{w}}$ is again an OWA operator, that is $\widehat{A}_{\boldsymbol{w}} = A_{\widehat{\boldsymbol{w}}}$ with

$$\widehat{w}_i = \frac{w_i + w_{n-i+1}}{2}$$

for every $i \in \{1, \ldots, n\}$. Notice that $\widehat{A}_{\boldsymbol{w}}$ reduces to the arithmetic mean in the simple case $n = 2$, but not in higher dimensions.

The self-dual core and the anti-self-dual remainder can be equivalently written as follows

$$\widehat{A}_{\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, \frac{x_{[i]} + x_{[n-i+1]}}{2} \quad \text{and} \quad \widetilde{A}_{\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, \frac{x_{[i]} - x_{[n-i+1]}}{2} \, .$$

These expressions show clearly that the self-dual core is a weighted average of pairwise averages of $\boldsymbol{x}$ coordinates (*quasi-midranges*), whereas the anti-self-dual remainder is a weighted average of pairwise differences of $\boldsymbol{x}$ coordinates (*quasi-ranges*). The anti-self-dual remainder is therefore independent of the overall average of the coordinates of $\boldsymbol{x}$ and constitutes a form of dispersion measure. Moreover, it is straightforward to prove that $w_1 \geq \cdots \geq w_n$ implies $\widetilde{A}_{\boldsymbol{w}}(\boldsymbol{x}) \geq 0$ and $w_1 \leq \cdots \leq w_n$ implies $\widetilde{A}_{\boldsymbol{w}}(\boldsymbol{x}) \leq 0$.

3 Inequality indices and welfare/illfare functions

In this paper we assume the following definitions of inequality index and welfare and illfare functions.

Definition 10

(1) A relative inequality index *is a function* $I : [0, \infty)^n \longrightarrow [0, \infty)$ *that is continuous in* $[0, \infty)^n \setminus \{\mathbf{0}\}$*, ratio scale invariant, strictly S-convex, and satisfies* $I(x \cdot \mathbf{1}) = 0$ *for every* $x \in [0, \infty)$*.*

(2) An absolute inequality index *is a function* $I : [0, \infty)^n \longrightarrow [0, \infty)$ *that is continuous, invariant for translations, strictly S-convex, and satisfies* $I(x \cdot \mathbf{1}) = 0$ *for every* $x \in [0, \infty)$*.*

Certain properties which are generally considered to be inherent to the concept of inequality have come to be accepted as basic properties for an inequality measure. The crucial axiom in this field is the *Pigou-Dalton transfer principle.* This axiom establishes that a progressive transfer, that is, a transfer from a richer to a poorer person that does not change the relative positions of the donor and the recipient, should decrease inequality. Marshall and Olkin [39] show that strict S-convexity implies symmetry and inequality reduction under progressive transfers. Conversely, symmetry and inequality reduction under progressive transfers imply strict S-convexity.

Definition 11 *A* welfare function *is a function* $W : [0, \infty)^n \longrightarrow [0, \infty)$ *that is continuous, monotonic, and strictly S-concave.*

In analogy with the inequality case, strict S-concavity is equivalent to symmetry and the increment of the welfare level under progressive transfers. Any welfare function allows the definition of the "equally distributed equivalent income", as the income level that if equally distributed among the population would generate the same value of the W function.

An inequality index is called *ethical* if it implies, and is implied, by a welfare function. If the welfare function W is positively homogeneous of degree 1, there is a one-to-one relationship between W and a relative inequality index (see Blackorby and Donaldson [6]). Following the Atkinson [2], Kolm [34], and Sen [45] approaches, every relative inequality index I may be associated with a positively homogeneous of degree 1 welfare function $W : [0, \infty)^n \longrightarrow [0, \infty)$ according to the following expression

$$W(\boldsymbol{x}) = \mu(\boldsymbol{x}) \left(1 - I(\boldsymbol{x})\right). \tag{2}$$

Conversely, given a positively homogeneous of degree 1 welfare function, we can recover the relative index associated using the above relation as follows

$$I(\boldsymbol{x}) = \begin{cases} 1 - \dfrac{W(\boldsymbol{x})}{\mu(\boldsymbol{x})}, & \text{if } \boldsymbol{x} \neq \boldsymbol{0}\,, \\ 0\,, & \text{if } \boldsymbol{x} = \boldsymbol{0}. \end{cases}$$

The index $I(\boldsymbol{x})$ gives the fraction of total income that could be saved if society distributed the remaining amount equally without any welfare loss. In other words, it can be interpreted as the proportion of welfare loss due to inequality.

In turn, Kolm [35] and Blackorby and Donaldson [7] approaches allow the derivation of a translatable welfare function from an absolute inequality index according to the following expression,

$$W(\boldsymbol{x}) = \mu(\boldsymbol{x}) - I_A(\boldsymbol{x}). \tag{3}$$

This absolute index, I_A, represents the per capita income that could be saved if society distributed incomes equally without any loss of welfare.

A class of welfare functions that will play an important role in this paper is that of the generalized Gini welfare functions (see Mehran [40], Donaldson and Weymark [16,17], Weymark [49], Yaari [50,51], Ebert [19], Quiggin [43], and Ben-Porath and Gilboa [5]).

Definition 12 *Given a weighting vector $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$, with $0 < w_1 < \cdots < w_n$ and $\sum_{i=1}^n w_i = 1$, the* generalized Gini welfare function *(or* rank dependent general welfare function*) associated with $\boldsymbol{w}$ is the function*

$W_{\boldsymbol{w}} : [0,\infty)^n \longrightarrow [0,\infty)$ *defined as*

$$W_{\boldsymbol{w}}(\boldsymbol{x}) = \sum_{i=1}^{n} w_i\, x_{[i]}.$$

Positivity of w_i guarantees that $W_{\boldsymbol{w}}$ satisfies the Pareto Principle, that is, it is increasing in x_i. Increasingness of the sequence of coefficients is necessary and sufficient for S-concavity of $W_{\boldsymbol{w}}$. On the other hand, all the functions $W_{\boldsymbol{w}}$ are stable for translations and positively homogeneous of degree 1.

Sometimes the variable under consideration is unfavorable. Think for instance of the duration of unemployment that each member in a society suffers. Then, an alternative illfare framework has been developed (see Riese and Brunner [44]). The illfare is assumed to be an increasing S-convex function of the (bad) variable since an addition increment is the more severe the higher the variable (the more severe an additional unemployment week, the longer the unemployment has already lasted).

Definition 13 *An* illfare function *is a function* $L : [0,\infty)^n \longrightarrow [0,\infty)$ *that is continuous, monotonic, and strictly S-convex.*

The strict S-convexity is equivalent to symmetry and the decrement of the illfare level under progressive transfers.

Remark 2 The generalized Gini welfare functions are OWA aggregation functions as long as the variables are restricted to the domain $[0,1]^n$. However, it could be possible to extend to $[0,\infty)^n$ the definitions of the self-dual-core and the anti-self-dual remainder associated with a generalized Gini welfare function in the following way. Consider $\lambda > 0$ an upper bound of the distribution $\boldsymbol{x} \in [0,\infty)^n$. Then, define

$$\widehat{A}_{\boldsymbol{w}}(\boldsymbol{x}) = \lambda \cdot \widehat{A}_{\boldsymbol{w}}\left(\frac{1}{\lambda} \cdot \boldsymbol{x}\right) \quad \text{and} \quad \widetilde{A}_{\boldsymbol{w}}(\boldsymbol{x}) = \lambda \cdot \widetilde{A}_{\boldsymbol{w}}\left(\frac{1}{\lambda} \cdot \boldsymbol{x}\right).$$

Since $\widehat{A}_{\boldsymbol{w}}$ and $\widetilde{A}_{\boldsymbol{w}}$ are linear for every generalized Gini welfare function $A_{\boldsymbol{w}}$, the previous extension does not depend on the chosen upper bound λ.

Remark 3 From now on, we will assume that all the variables are restricted to $[0,1]^n$ (see Remark 2). Then, the generalized Gini welfare functions correspond to OWA aggregation functions. In this setting, a welfare function is an aggregation function $W : [0,1]^n \longrightarrow [0,1]$ that is continuous and strictly S-concave. Analogously, an illfare function is an aggregation function $L : [0,1]^n \longrightarrow [0,1]$ that is continuous and strictly S-convex.

We will denote by $\boldsymbol{x}$ the achievement distribution and by $\boldsymbol{y}$ the shortfall distribution. Moreover, we assume that the welfare function is applied to the achievements whereas the illfare function to the shortfalls.

In the following proposition we establish that the dual of a welfare (illfare) function is always an illfare (welfare) function. This result allows us to introduce the generalized Gini, Bonferroni, and De Vergottini illfare functions as the duals of the corresponding welfare functions.

Proposition 6 *If* $W : [0,1]^n \longrightarrow [0,1]$ *is a welfare function, then* $L : [0,1]^n \longrightarrow [0,1]$ *defined as* $L(\boldsymbol{y}) = W^*(\boldsymbol{y}) = 1 - W(\mathbf{1} - \boldsymbol{y})$ *is an illfare function. Conversely, if* $L : [0,1]^n \longrightarrow [0,1]$ *is an illfare function, then* $W : [0,1]^n \longrightarrow [0,1]$ *defined as* $W(\boldsymbol{x}) = L^*(\boldsymbol{x}) = 1 - L(\mathbf{1} - \boldsymbol{x})$ *is a welfare function.*

PROOF: By Definition 6 and Remark 1. ■

Remark 4 Given a weighting vector $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$, with $0 < w_1 < \cdots < w_n$ and $\sum_{i=1}^n w_i = 1$, by Proposition 6 and equation (1), the *generalized Gini illfare function* is defined as $L_{\boldsymbol{w}}(\boldsymbol{y}) = W^*_{\boldsymbol{w}}(\boldsymbol{y}) = \sum\limits_{i=1}^{n} w_i \, y_{(i)}$.

3.1 The Gini index

The Gini index (Gini [27]), the most popular measure of inequality, was introduced by Corrado Gini in 1912. It is based on the average of the absolute differences between all possible pairs of observations. The Gini index is defined as half of the ratio of that average to the mean of the distribution (hence proposing a relative measure of variability). Specifically, for any unordered income distribution the formula given by Gini [27] is

$$G(\boldsymbol{x}) = \frac{1}{2n^2\mu(\boldsymbol{x})} \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j| \,, \quad \text{with } \boldsymbol{x} \neq \mathbf{0}. \tag{4}$$

This index varies between 0, which reflects complete equality, and 1. It is relative and invariant under replications of the population, which allows inequality comparisons between societies with different incomes and different populations. Moreover, inequality as measured by this index depends on the significance of the income gaps in society.

Graphically, the Gini index can be computed as twice the area between the line of equality and the Lorenz curve (Gastwirth [26], Kendall and Stuart [33], Dorfman [18]). This curve plots the cumulative income share, ranked in increasing order, on the vertical axis against the distribution of the population on the horizontal axis.

Mehran [40] highlights the linear structure of the index and the implicit weighting scheme involved in (4), which assigns a particular weight to an individual according to his ranking in the income distribution (Sen [45]). In particular, it can be shown that an alternative formula for $G(\boldsymbol{x})$ is

$$G(\boldsymbol{x}) = 1 - \frac{1}{n^2\mu(\boldsymbol{x})}\sum_{i=1}^{n}(2i-1)\,x_{[i]}\,, \quad \text{with } \boldsymbol{x} \neq \mathbf{0}.$$

See Yitzhaki [55] for alternative formulations of the Gini index.

The decrease of $G(\boldsymbol{x})$ under a progressive transfer does not depend where the transfer takes place as long as it occurs between two persons with a fixed rank difference. In other words, this index is insensitive to the incomes of the individuals involved in the transfers.

When the Gini coefficient is multiplied by the mean income an absolute index is obtained.

Definition 14 *The* absolute Gini inequality index *is defined as*

$$G_A(\boldsymbol{x}) = \mu(\boldsymbol{x}) - \frac{1}{n}\sum_{i=1}^{n}\frac{2i-1}{n}\,x_{[i]}.$$

Remark 5 From equations (2) and (3), the *Gini welfare function* is equivalently obtained as

$$W_G(\boldsymbol{x}) = \mu(\boldsymbol{x})(1-G(\boldsymbol{x})) = \mu(\boldsymbol{x}) - G_A(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n}\frac{2i-1}{n}\,x_{[i]}.$$

Remark 6 By Proposition 6 and equation (1), the *Gini illfare function* is defined as

$$L_G(\boldsymbol{y}) = W_G^*(\boldsymbol{y}) = \frac{1}{n}\sum_{i=1}^{n}\frac{2i-1}{n}\,y_{(i)}.$$

Notice that by Proposition 6 and equation (3), we also have

$$L_G(\boldsymbol{y}) = 1 - W_G(\mathbf{1}-\boldsymbol{y}) = 1 - (\mu(\mathbf{1}-\boldsymbol{y}) - G_A(\mathbf{1}-\boldsymbol{y})) = \mu(\boldsymbol{y}) + G_A(\mathbf{1}-\boldsymbol{y}).$$

Given that $G_A(\mathbf{1}-\boldsymbol{y}) = G_A(\boldsymbol{y})$, see Proposition 12 below, we obtain

$$L_G(\boldsymbol{y}) = \mu(\boldsymbol{y}) + G_A(\mathbf{1}-\boldsymbol{y}) = \mu(\boldsymbol{y}) + G_A(\boldsymbol{y}).$$

Thus, the Gini illfare function may be decomposed into the mean of the shortfalls and the absolute Gini inequality index either of the achievements, $G_A(\mathbf{1}-\boldsymbol{y})$, or the shortfalls, $G_A(\boldsymbol{y})$. Since, by Remark 5, $W_G(\boldsymbol{x}) = \mu(\boldsymbol{x}) - G_A(\boldsymbol{x})$, the welfare and illfare Gini functions share a similar decomposition.

The Bonferroni index is another example of relative index that has a natural upper bound 1. It is based on the comparison of the partial means and the general mean of an income distribution.

Let us denote by $m_i(\boldsymbol{x})$ the mean income of the $n-i+1$ persons with lowest income, that is

$$m_i(\boldsymbol{x}) = \frac{1}{n-i+1}\sum_{j=i}^{n} x_{[j]}.$$

The Bonferroni index is defined by

$$B(\boldsymbol{x}) = \frac{1}{n\mu(\boldsymbol{x})}\sum_{i=1}^{n}(\mu(\boldsymbol{x}) - m_i(\boldsymbol{x}))\,, \quad \text{with} \;\; \boldsymbol{x} \neq \mathbf{0}.$$

and it depends on how much the $m_i(\boldsymbol{x})/\mu(\boldsymbol{x})$ ratios fall short of unity.

The Bonferroni index B is not invariant for replications. However it fulfils a stronger redistributive criterion than the Pigou-Dalton condition. The decrement in the B index due to a progressive transfer is larger the poorer are the two participants. This property is referred to as the *principle of positional transfer sensitivity* (Mehran [40] and Zoli [56]).

When multiplied by the mean income it becomes an absolute index.

Definition 15 *The* absolute Bonferroni inequality index *is defined as*

$$B_A(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n}(\mu(\boldsymbol{x}) - m_i(\boldsymbol{x})) = \mu(\boldsymbol{x}) - \frac{1}{n}\sum_{i=1}^{n} m_i(\boldsymbol{x}).$$

Remark 7 From equations (2) and (3), the *Bonferroni welfare function* is simultaneously obtained as

$$W_B(\boldsymbol{x}) = \mu(\boldsymbol{x})(1 - B(\boldsymbol{x})) = \mu(\boldsymbol{x}) - B_A(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n} m_i(\boldsymbol{x}).$$

Proposition 7 *The Bonferroni welfare function is expressed by*

$$W_B(\boldsymbol{x}) = \sum_{i=1}^{n} u_i\, x_{[i]},$$

where $u_i = \displaystyle\sum_{j=n-i+1}^{n} \frac{1}{jn}$, *for* $i = 1, \ldots, n$.

Proof: The derivation is as follows

$$\sum_{i=1}^{n} m_i(\boldsymbol{x}) = \frac{x_{[n]}}{1} + \frac{x_{[n-1]} + x_{[n]}}{2} + \cdots + \frac{x_{[1]} + \cdots + x_{[n]}}{n} =$$

$$= \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right) x_{[n]} + \left(\frac{1}{2} + \cdots + \frac{1}{n}\right) x_{[n-1]} + \cdots + \frac{1}{n}\, x_{[1]} =$$

$$= n \sum_{i=1}^{n} u_i\, x_{[i]}. \blacksquare$$

Remark 8 The weights introduced in the previous proposition satisfy the following conditions

(1) $0 < u_1 < u_2 < \cdots < u_{n-1} < u_n < 1$.

(2) $u_1 = \dfrac{1}{n^2}$ and $u_{i+1} = u_i + \dfrac{1}{(n-i)\,n}$, for $i = 1, \ldots, n-1$.

(3) $\sum_{i=1}^{n} u_i = 1$, since

$$\sum_{i=1}^{n} \left(\sum_{j=n-i+1}^{n} \frac{1}{j} \right) = \frac{1}{n} + \left(\frac{1}{n-1} + \frac{1}{n}\right) + \left(\frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}\right) +$$

$$+ \cdots + \left(\frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n}\right) + \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n}\right) =$$

$$= 1\,\frac{1}{1} + 2\,\frac{1}{2} + \cdots + (n-2)\,\frac{1}{n-2} + (n-1)\,\frac{1}{n-1} + n\,\frac{1}{n} = n\,.$$

Remark 9 From Propositions 6 and 7 and equation (1), the *Bonferroni illfare function* is defined as

$$L_B(\boldsymbol{y}) = W_B^*(\boldsymbol{y}) = \sum_{i=1}^{n} u_i\, y_{(i)}.$$

Notice that by Proposition 6 and equation (3), we also have

$$L_B(\boldsymbol{y}) = 1 - W_B(\mathbf{1} - \boldsymbol{y}) = 1 - (\mu(\mathbf{1} - \boldsymbol{y}) - B_A(\mathbf{1} - \boldsymbol{y})) = \mu(\boldsymbol{y}) + B_A(\mathbf{1} - \boldsymbol{y}).$$

Nevertheless, $B_A(\mathbf{1} - \boldsymbol{y})$ is different from $B_A(\boldsymbol{y})$ and an inconsistency in the inequality indices appears. To show that $B_A(\boldsymbol{y})$ and $B_A(\mathbf{1} - \boldsymbol{y})$ are often different, let us see the following example. Consider $\boldsymbol{y} = (0.65, 0.30, 0.10, 0.09)$ the shortfalls of a distribution, and $\mathbf{1} - \boldsymbol{y} = (0.35, 0.70, 0.90, 0.91)$ as the corresponding achievements distribution. The Bonferroni index for shortfalls and achievements are $B_A(\boldsymbol{y}) = 0.127$ and $B_A(\mathbf{1} - \boldsymbol{y}) = 0.155$, respectively, obtaining different results for achievements and shortfalls. This kind on inconsistency should be avoided by using the dual decomposition (Subsection 4.2).

3.3 The De Vergottini index

The De Vergottini index ([15]) captures another aspect of the inequality. It compares the total mean income with the mean of the i-richest person group. If $M_i(\boldsymbol{x})$ denotes the mean income of the i-persons with highest incomes, that is

$$M_i(\boldsymbol{x}) = \frac{1}{i}\sum_{j=1}^{i} x_{[j]},$$

then the De Vergottini index is

$$V(\boldsymbol{x}) = \frac{1}{n\mu(\boldsymbol{x})}\sum_{i=1}^{n}(M_i(\boldsymbol{x}) - \mu(\boldsymbol{x})), \quad \text{with} \quad \boldsymbol{x} \neq \mathbf{0}.$$

With respect to other redistributive criteria, the reduction in the V index due to a progressive transfer is larger the richer are the two participants.

V is also a compromise index in the sense that if multiplied by the mean, then the counterpart absolute index is obtained.

Definition 16 *The* absolute De Vergottini inequality index *is defined as*

$$V_A(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^{n}(M_i(\boldsymbol{x}) - \mu(\boldsymbol{x})) = \frac{1}{n}\sum_{i=1}^{n} M_i(\boldsymbol{x}) - \mu(\boldsymbol{x}).$$

In contrast with the relative Bonferroni index, whose maximum value is

$$B_{max} = \frac{n-1}{n}$$

in correspondence with the income profile in which only one individual accumulates all the income, the De Vergottini index does not have a unit upper bound. The maximum inequality value corresponds to the same income profile as for the Bonferroni index, $x_{[1]} = n\mu(\boldsymbol{x})$, $x_{[2]} = \cdots = x_{[n]} = 0$, but the value is now

$$V_{\max} = \sum_{j=2}^{n} \frac{1}{j}.$$

This value only depends on the population size and may be used to normalize the index. Our proposal is to use the normalization factor

$$c = \frac{n}{n-1}\, V_{\max},$$

because it ensures that the maximum value of the normalized De Vergottini index, $\overline{V}(\boldsymbol{x}) = V(\boldsymbol{x})/c$, is the same as that of $B(\boldsymbol{x})$, i.e. $(n-1)/n$. Similarly, we denote the absolute normalized De Vergottini index by $\overline{V}_A = V_A/c$.

Remark 10 From equations (2) and (3), the *normalized De Vergottini welfare function* is equivalently obtained as

$$W_{\overline{V}}(\boldsymbol{x}) = \mu(\boldsymbol{x})\left(1 - \overline{V}(\boldsymbol{x})\right) = \mu(\boldsymbol{x}) - \overline{V}_A(\boldsymbol{x}) = \frac{c+1}{c}\,\mu(\boldsymbol{x}) - \frac{1}{c\,n}\sum_{i=1}^{n} M_i(\boldsymbol{x}).$$

Remark 11 For $n = 2$, the Gini, Bonferroni and normalized De Vergottini welfare functions coincide:

$$W_G(x_1, x_2) = W_B(x_1, x_2) = W_{\overline{V}}(x_1, x_2) = \frac{x_{[1]} + 3\,x_{[2]}}{4}.$$

However, this fact is not true in higher dimensions. For instance, for $n = 3$ we have

$$W_G(x_1, x_2, x_3) = \frac{10\,x_{[1]} + 30\,x_{[2]} + 50\,x_{[3]}}{90}$$

$$W_B(x_1, x_2, x_3) = \frac{10\,x_{[1]} + 25\,x_{[2]} + 55\,x_{[3]}}{90}$$

$$W_{\overline{V}}(x_1, x_2, x_3) = \frac{10\,x_{[1]} + 34\,x_{[2]} + 46\,x_{[3]}}{90}.$$

Proposition 8 *The weighting scheme implicit in the normalized De Vergottini welfare function* $W_{\overline{V}}$ *is expressed by*

$$\frac{1}{n}\sum_{i=1}^{n} M_i(\boldsymbol{x}) = \sum_{i=1}^{n} v_i\, x_{[i]},$$

where $v_i = \sum_{j=i}^{n} \frac{1}{j\,n}$, *for* $i = 1, \ldots, n$.

PROOF: The derivation is as follows

$$\sum_{i=1}^{n} M_i(\boldsymbol{x}) = \frac{x_{[1]}}{1} + \frac{x_{[1]}+x_{[2]}}{2} + \cdots + \frac{x_{[1]}+\cdots+x_{[n]}}{n} =$$

$$= \left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}\right) x_{[1]} + \left(\frac{1}{2}+\cdots+\frac{1}{n}\right) x_{[2]} + \cdots + \frac{1}{n}\, x_{[n]} =$$

$$= n \sum_{i=1}^{n} v_i\, x_{[i]}. \blacksquare$$

Remark 12 The weights introduced in the previous proposition satisfy the following conditions

(1) $0 < v_n < v_{n-1} < \cdots < v_2 < v_1 < 1$.

(2) $v_n = \dfrac{1}{n^2}$ and $v_{i-1} = v_i + \dfrac{1}{(i-1)\,n}$, for $i = 2, \ldots, n$.

(3) $\sum_{i=1}^{n} v_i = 1$, since

$$\sum_{i=1}^{n} \left(\sum_{j=i}^{n} \frac{1}{j} \right) = \sum_{i=1}^{n} \left(\sum_{j=n-i+1}^{n} \frac{1}{j} \right) = n\,.$$

Remark 13 The normalized De Vergottini welfare function can be written as

$$W_{\overline{V}}(\boldsymbol{x}) = \sum_{i=1}^{n} w_i^V x_{[i]}, \quad \text{where} \quad w_i^V = \frac{c+1-n\,v_i}{c\,n} \quad \text{for} \quad i = 1, \ldots, n.$$

Notice that $\sum_{i=1}^{n} w_i^B = \sum_{i=1}^{n} w_i^V = 1$ and the lowest Bonferroni and De Vergottini weights are $w_1^B = w_1^V = 1/n^2$, since $w_1^B = u_1 = 1/n^2$ and

$$w_1^V = \frac{c+1-n\,v_1}{c\,n} = \frac{n(n\,v_1-1)+(1-n\,v_1)(n-1)}{n(n\,v_1-1)n} = \frac{1}{n^2},$$

where we have used that $c = \dfrac{n(n\,v_1-1)}{n-1}$.

Remark 14 By Proposition 6 and equation (1), the *normalized De Vergottini illfare function* can be written as

$$L_{\overline{V}}(\boldsymbol{y}) = W^*_{\overline{V}}(\boldsymbol{y}) = \sum_{i=1}^{n} w_i^V\, y_{(i)}\,,$$

with w_i^V as in Remark 12.

Notice that by Proposition 6 and equation (3), we also have

$$L_{\overline{V}}(\boldsymbol{y}) = 1 - W_{\overline{V}}(\mathbf{1}-\boldsymbol{y}) = 1 - (\mu(\mathbf{1}-\boldsymbol{y}) - \overline{V}_A(\mathbf{1}-\boldsymbol{y})) = \mu(\boldsymbol{y}) + \overline{V}_A(\mathbf{1}-\boldsymbol{y}).$$

The same example given in Remark 9 allows us to conclude that $\overline{V}_A(\boldsymbol{y})$ and $\overline{V}_A(\mathbf{1}-\boldsymbol{y})$ are in general different. Again this kind on inconsistency should be avoided by using the dual decomposition (Subsection 4.2).

3.4 Orness of the Gini, Bonferroni and normalized De Vergottini welfare functions

The notion of *orness* (or *attitudinal character*) of OWA operators was introduced by Yager [52] for reflecting the andlike or orlike aggregation behavior of OWA operators.

Definition 17 *Let $A_{\boldsymbol{w}}$ the OWA operator associated with the weighting vector $\boldsymbol{w}=(w_1,\ldots,w_n)\in[0,1]^n$. The* orness *of $A_{\boldsymbol{w}}$ is defined as*

$$A^o_{\boldsymbol{w}}=\frac{1}{n-1}\sum_{i=1}^{n}(n-i)\,w_i.$$

Remark 15 The orness of $A_{\boldsymbol{w}}$ coincides with the value $A_{\boldsymbol{w}}(\boldsymbol{x}^o)$, where $x^o_i=\dfrac{n-i}{n-1}$, i.e.,

$$A^o_{\boldsymbol{w}}=w_1+w_2\,\frac{n-2}{n-1}+\cdots+w_{n-1}\,\frac{1}{n-1}.$$

The orness of the extreme OWA operators maximum, arithmetic mean and minimum are 1, 0.5 and 0, respectively:

(1) $A_{\boldsymbol{w}}(\boldsymbol{x})=\max\{x_1,\ldots,x_n\}$, where $\boldsymbol{w}=(1,0,\ldots,0)$: $A^o_w=1$.

(2) $A_{\boldsymbol{w}}(\boldsymbol{x})=\dfrac{x_1+\cdots+x_n}{n}$, where $\boldsymbol{w}=\left(\dfrac{1}{n},\dfrac{1}{n},\ldots,\dfrac{1}{n}\right)$: $A^o_w=\dfrac{1}{2}$.

(3) $A_{\boldsymbol{w}}(\boldsymbol{x})=\min\{x_1,\ldots,x_n\}$, where $\boldsymbol{w}=(0,\ldots,0,1)$: $A^o_{\boldsymbol{w}}=0$.

The orness of the OWA operator $A_{\boldsymbol{w}}$ can also be written in the following way (see Filev and Yager [21]):

$$A^o_{\boldsymbol{w}}=\frac{1}{2}+\sum_{i=1}^{n}\frac{n-2i+1}{2(n-1)}\,w_i=\frac{1}{2}+\sum_{i=1}^{n}\frac{n-2i+1}{4(n-1)}\,(w_i-w_{n-i+1}),$$

which implies $0\le A^o_{\boldsymbol{w}}\le 0.5$ when $0\le w_1\le\cdots\le w_n\le 1$, and $0.5\le A^o_{\boldsymbol{w}}\le 1$ when $0\le w_n\le\cdots\le w_1\le 1$. In either case, $A^o_{\boldsymbol{w}}=0.5$ only if $w_1=\cdots=w_n=1/n$.

Accordingly, in the case of generalized Gini welfare functions, with $0<w_1<\cdots<w_n<1$, we always have $0<A^o_{\boldsymbol{w}}<0.5$, which reflects the greater importance given to the poorer incomes in the population. In what follows we explicitly compute the orness of the Gini, Bonferroni, and normalized De Vergottini welfare functions.

Proposition 9 *The orness of the Gini welfare function is* $W_G^o = \dfrac{1}{3} - \dfrac{1}{6n}$.

Proof: From the definition

$$W_G(\boldsymbol{x}) = \frac{1}{n} \sum_{i=1}^{n} \frac{2i-1}{n} \, x_{[i]}$$

and since $W_G^o = W_G(\boldsymbol{x}^o)$ with $x_{[i]}^o = \dfrac{n-i}{n-1}$, we obtain

$$W_G^o = W_G(\boldsymbol{x}^o) = \sum_{i=1}^{n} w_i^G \, x_{[i]}^o = \frac{1}{n} \sum_{i=1}^{n} \frac{2i-1}{n} \, \frac{n-i}{n-1} =$$

$$= \frac{1}{(n-1)\,n^2} \left(-n^2 \; + \; (2n+1) \sum_{i=1}^{n} i \; - 2 \sum_{i=1}^{n} i^2 \right) =$$

$$= \frac{1}{(n-1)\,n^2} \left(-n^2 \; + \; (2n+1) \, \frac{n(n+1)}{2} \; - 2 \, \frac{n(n+1)(2n+1)}{6} \right) =$$

$$= \frac{2n-1}{6n} = \frac{1}{3} - \frac{1}{6n},$$

where we have used $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. ∎

Proposition 10 *The orness of the Bonferroni welfare function is* $W_B^o = \dfrac{1}{4}$.

Proof: From the definition

$$W_B(\boldsymbol{x}) = \frac{1}{n} \sum_{i=1}^{n} m_i(\boldsymbol{x}) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{n-i+1} \sum_{j=i}^{n} x_{[j]}$$

and since $W_B^o = W_B(\boldsymbol{x}^o)$ with $x_{[i]}^o = \dfrac{n-i}{n-1}$, we obtain

$$
\begin{aligned}
W_B^o = W_B(\boldsymbol{x}^o) &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{n-i+1}\sum_{j=i}^{n}\frac{n-j}{n-1} = \\
&= \frac{1}{n(n-1)}\sum_{i=1}^{n}\frac{1}{n-i+1}\left(n(n-i+1)-\sum_{j=i}^{n} j\right) = \\
&= \frac{1}{n(n-1)}\sum_{i=1}^{n}\frac{1}{n-i+1}\left(n(n-i+1)-\frac{(n-i+1)(n+i)}{2}\right) = \\
&= \frac{1}{n(n-1)}\sum_{i=1}^{n}\left(\frac{n}{2}-\frac{i}{2}\right) = \frac{1}{n(n-1)}\left(\frac{n^2}{2}-\frac{n(n+1)}{4}\right) = \\
&= \frac{1}{n(n-1)}\,\frac{n(n-1)}{4} = \frac{1}{4},
\end{aligned}
$$

where we have used that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Proposition 11 *The orness of the normalized De Vergottini welfare function is* $W_{\overline{V}}^o = \dfrac{1}{2} - \dfrac{1}{4\,c}$.

Proof: From the definition

$$
W_{\overline{V}}(\boldsymbol{x}) = \frac{c+1}{c}\,\mu(\boldsymbol{x}) - \frac{1}{c\,n}\sum_{i=1}^{n} M_i(\boldsymbol{x}) = \frac{c+1}{c}\,\mu(\boldsymbol{x}) - \frac{1}{c\,n}\sum_{i=1}^{n}\left(\frac{1}{i}\sum_{j=1}^{i} x_{[j]}\right)
$$

and since $W^o_{\overline{V}} = W_{\overline{V}}(\boldsymbol{x}^o)$ with $x^o_{[i]} = \dfrac{n-i}{n-1}$, we obtain

$$W^o_{\overline{V}} = W_{\overline{V}}(\boldsymbol{x}^o) = \frac{c+1}{c}\,\mu(\boldsymbol{x}^o) - \frac{1}{c\,n}\sum_{i=1}^{n}\left(\frac{1}{i}\sum_{j=1}^{i}\frac{n-j}{n-1}\right) =$$

$$= \frac{c+1}{c}\,\frac{1}{2} - \frac{1}{c\,n(n-1)}\sum_{i=1}^{n}\left(n - \frac{1}{i}\sum_{j=1}^{i} j\right) =$$

$$= \frac{c+1}{2\,c} - \frac{1}{c\,n(n-1)}\sum_{i=1}^{n}\left(n - \frac{i+1}{2}\right) =$$

$$= \frac{c+1}{2\,c} - \frac{1}{c\,n(n-1)}\sum_{i=1}^{n}\left(\frac{2n-1}{2} - \frac{i}{2}\right) =$$

$$= \frac{c+1}{2\,c} - \frac{1}{c\,n(n-1)}\left(\frac{n(2n-1)}{2} - \frac{n(n+1)}{4}\right) =$$

$$= \frac{c+1}{2\,c} - \frac{1}{c\,n(n-1)}\,\frac{3n(n-1)}{4} = \frac{c+1}{2\,c} - \frac{3}{4\,c} = \frac{1}{2} - \frac{1}{4\,c},$$

where we have used that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\mu(\boldsymbol{x}^o) = \dfrac{1}{2}$.

Summarizing, the orness values of the welfare functions associated with the three classical inequality indices are all in the range $(0, 1/2)$, as expected. We see that the focus on the poorer individuals in the population slightly decreases with n in the cases of Gini and normalized De Vergottini (given the orness slightly increases), not so in the Bonferroni case. Moreover, in the large population asymptotic limit, the orness values of the Gini $(1/3)$, Bonferroni $(1/4)$, and normalized De Vergottini $(1/2)$ welfare functions express the respective emphasis on poorer incomes and establish the Gini inequality index and welfare function as an intermediate case between the Bonferroni and the normalized De Vergottini frameworks.

4 Dual decomposition of Gini, Bonferroni and normalized De Vergottini welfare and illfare functions

This section analyzes the dual decomposition in the self-dual core and the anti-self-dual remainder for any welfare and illfare function. Specifically, the dual decomposition of the Gini, Bonferroni and normalized De Vergottini welfare and illfare functions are identified and highlighted the relationship among them.

As shown in Proposition 2, the self-dual core $\widehat{A}$ inherits from the OWA operator A the properties of continuity, idempotency, symmetry and stability for translations. The positivity of the normalized weights in A implies that $\widehat{A}$ is increasing with normalized weights. Hence, the self-dual core component can be interpreted as a weighted mean.

In turn, the remainder $\widetilde{A}$ is symmetric, fulfils $\widetilde{A}(x_1, \ldots, x_q) = 0$ if and only if $x_1 = \cdots = x_q$, and Proposition 4 ensures that $\widetilde{A}$ inherits from A the strict S-concavity (respectively, the strict S-convexity). Consequently, the Pigou-Dalton transfer principle is satisfied for $-\widetilde{A}$ (respectively $\widetilde{A}$). Hence, we can obtain a direct interpretation of this component as a measure of inequality. In addition, the anti–self-duality ensures that $\widetilde{A}(\mathbf{1} - \boldsymbol{x}) = \widetilde{A}(\boldsymbol{x})$. Thus, the anti-self-dual remainder measures equally the inequality of achievements and shortfalls.

Welfare and illfare functions can be decomposed in the following way

$$
\begin{aligned}
W(\boldsymbol{x}) &= \widehat{W}(\boldsymbol{x}) + \widetilde{W}(\boldsymbol{x}) \\
L(\boldsymbol{y}) &= \widehat{L}(\boldsymbol{y}) + \widetilde{L}(\boldsymbol{y}) = \widehat{W^*}(\boldsymbol{y}) + \widetilde{W^*}(\boldsymbol{y}) = \widehat{W}(\boldsymbol{y}) - \widetilde{W}(\boldsymbol{y}).
\end{aligned}
$$

4.1 The dual decomposition of the Gini welfare and illfare functions

Proposition 12 *The absolute Gini inequality index is anti-self-dual, i.e., it satisfies* $G_A(\mathbf{1} - \boldsymbol{x}) = G_A(\boldsymbol{x})$, *for every* $\boldsymbol{x} \in [0, 1]^n$.

PROOF: The derivation is as follows:

$$G_A(\mathbf{1}-\boldsymbol{x}) = \mu(\mathbf{1}-\boldsymbol{x}) - \frac{1}{n}\sum_{i=1}^{n}\frac{2i-1}{n}\,(1-x)_{[i]} =$$

$$= 1-\mu(\boldsymbol{x}) - \frac{1}{n}\sum_{i=1}^{n}\frac{2(n-i+1)-1}{n}\,\left(1-x_{[i]}\right) =$$

$$= 1-\mu(\boldsymbol{x}) - \frac{1}{n}\sum_{i=1}^{n}\left(2-\frac{2i-1}{n}\right)\left(1-x_{[i]}\right) =$$

$$= 1-\mu(\boldsymbol{x}) - \frac{1}{n}\sum_{i=1}^{n}2\left(1-x_{[i]}\right) + \frac{1}{n}\sum_{i=1}^{n}\frac{2i-1}{n}\,\left(1-x_{[i]}\right) =$$

$$= 1-\mu(\boldsymbol{x}) - 2 + 2\,\mu(\boldsymbol{x}) + 1 - \frac{1}{n}\sum_{i=1}^{n}\frac{2i-1}{n}\,x_{[i]} =$$

$$= \mu(\boldsymbol{x}) - \frac{1}{n}\sum_{i=1}^{n}\frac{2i-1}{n}\,x_{[i]} = G_A(\boldsymbol{x}),$$

where we have used that $\displaystyle\sum_{i=1}^{n}\frac{2i-1}{n} = n.$ ■

Remark 16 Since $W_G(\mathbf{1}-\boldsymbol{x}) = \mu(\mathbf{1}-\boldsymbol{x}) - G_A(\mathbf{1}-\boldsymbol{x}) = 1-\mu(\boldsymbol{x}) - G_A(\boldsymbol{x})$, the dual Gini welfare function can be written as $W_G^*(\boldsymbol{x}) = 1 - W_G(\mathbf{1}-\boldsymbol{x}) = \mu(\boldsymbol{x}) + G_A(\boldsymbol{x})$.

Proposition 13 *The self-dual core of the Gini welfare function is the arithmetic mean.*

PROOF: Taking into account Remark 16, we have

$$\widehat{W}_G(\boldsymbol{x}) = \frac{W_G(\boldsymbol{x}) + W_G^*(\boldsymbol{x})}{2} = \frac{\mu(\boldsymbol{x}) - G_A(\boldsymbol{x}) + \mu(\boldsymbol{x}) + G_A(\boldsymbol{x})}{2} = \mu(\boldsymbol{x}).\blacksquare$$

Proposition 14 *The anti-self-dual remainder of the Gini welfare function is minus the absolute Gini index.*

PROOF: Taking into account Remark 16, we have

$$\widetilde{W}_G(\boldsymbol{x}) = \frac{W_G(\boldsymbol{x}) - W_G^*(\boldsymbol{x})}{2} = \frac{\mu(\boldsymbol{x}) - G_A(\boldsymbol{x}) - \mu(\boldsymbol{x}) - G_A(\boldsymbol{x})}{2} = -G_A(\boldsymbol{x}).\blacksquare$$

Remark 17 The Gini welfare function can be decomposed as

$$W_G(\boldsymbol{x}) = \mu(\boldsymbol{x}) - G_A(\boldsymbol{x}),$$

whereas the decomposition of the Gini illfare function is

$$L_G(\boldsymbol{y}) = \mu(\boldsymbol{y}) + G_A(\boldsymbol{y}).$$

In this case the absolute Gini index takes part as the inequality term in the two sides. In fact the absolute Gini index measures equally the inequality of achievements and shortfalls.

Notice that the above decompositions were also obtained in Remarks 5 and 6.

4.2 The dual decomposition of the Bonferroni and De Vergottini welfare and illfare functions

Proposition 15 *The duality relation between the absolute Bonferroni and the absolute De Vergottini inequality indices is expressed by*

$$B_A(\mathbf{1} - \boldsymbol{x}) = V_A(\boldsymbol{x}) \quad \text{and} \quad V_A(\mathbf{1} - \boldsymbol{x}) = B_A(\boldsymbol{x}),$$

for every $\boldsymbol{x} \in [0, 1]^n$

Proof: The derivation is as follows:

$$B_A(\mathbf{1} - \boldsymbol{x}) = \mu(\mathbf{1} - \boldsymbol{x}) - \frac{1}{n}\sum_{i=1}^{n} m_i(\mathbf{1} - \boldsymbol{x}) = (1 - \mu(\boldsymbol{x})) - \frac{1}{n}\sum_{i=1}^{n}(1 - M_i(\boldsymbol{x})) =$$
$$= 1 - \mu(\boldsymbol{x}) - 1 + \frac{1}{n}\sum_{i=1}^{n} M_i(\boldsymbol{x}) = V_A(\boldsymbol{x}).$$

On the other hand, $V_A(\mathbf{1} - \boldsymbol{x}) = B_A(\mathbf{1} - (\mathbf{1} - \boldsymbol{x})) = B_A(\boldsymbol{x})$. ■

Remark 18 Since $W_B(\mathbf{1}-\boldsymbol{x}) = \mu(\mathbf{1}-\boldsymbol{x}) - B_A(\mathbf{1}-\boldsymbol{x}) = 1-\mu(\boldsymbol{x}) - V_A(\boldsymbol{x})$, the dual Bonferroni welfare function can be written as $W_B^*(\boldsymbol{x}) = 1 - W_B(\mathbf{1}-\boldsymbol{x}) = \mu(\boldsymbol{x}) + V_A(\boldsymbol{x})$.

Proposition 16 *The self-dual core and the anti-self-dual remainder of the Bonferroni welfare function are given by*

$$\widehat{W}_B(\boldsymbol{x}) = \mu(\boldsymbol{x}) - \frac{B_A(\boldsymbol{x}) - V_A(\boldsymbol{x})}{2} \quad \text{and} \quad \widetilde{W}_B(\boldsymbol{x}) = -\frac{B_A(\boldsymbol{x}) + V_A(\boldsymbol{x})}{2}.$$

PROOF: Taking into account Remark 18, the derivations are as follows:

$$\widehat{W}_B(\boldsymbol{x}) = \frac{W_B(\boldsymbol{x}) + W_B^*(\boldsymbol{x})}{2} = \frac{\mu(\boldsymbol{x}) - B_A(\boldsymbol{x}) + \mu(\boldsymbol{x}) + V_A(\boldsymbol{x})}{2} =$$

$$= \mu(\boldsymbol{x}) - \frac{B_A(\boldsymbol{x}) - V_A(\boldsymbol{x})}{2}.$$

$$\widetilde{W}_B(\boldsymbol{x}) = \frac{W_B(\boldsymbol{x}) - W_B^*(\boldsymbol{x})}{2} = \frac{\mu(\boldsymbol{x}) - B_A(\boldsymbol{x}) - \mu(\boldsymbol{x}) - V_A(\boldsymbol{x})}{2} =$$

$$= -\frac{B_A(\boldsymbol{x}) + V_A(\boldsymbol{x})}{2}. \blacksquare$$

Remark 19 The Bonferroni welfare function can be written as follows

$$W_B(\boldsymbol{x}) = \left(\mu(\boldsymbol{x}) - \frac{B_A(\boldsymbol{x}) - V_A(\boldsymbol{x})}{2}\right) + \left(-\frac{B_A(\boldsymbol{x}) + V_A(\boldsymbol{x})}{2}\right)$$

and the Bonferroni illfare function as

$$L_B(\boldsymbol{y}) = \left(\mu(\boldsymbol{y}) - \frac{B_A(\boldsymbol{y}) - V_A(\boldsymbol{y})}{2}\right) + \left(\frac{B_A(\boldsymbol{y}) + V_A(\boldsymbol{y})}{2}\right).$$

Consequently, the Bonferroni welfare and illfare functions can be decomposed in terms of a central index and a consistent inequality index.

Remark 20 It is easy to see that the self-dual core and the anti-self-dual remainder of the Bonferroni welfare function can be also expressed as

$$\widehat{W}_B(\boldsymbol{x}) = \sum_{i=1}^{n} \frac{v_i + u_i}{2} x_{[i]} \quad \text{and} \quad \widetilde{W}_B(\boldsymbol{x}) = \sum_{i=1}^{n} \frac{v_i - u_i}{2} x_{[i]}.$$

On the one hand, $\widehat{W}_B$ is a self-dual OWA operator with larger outer weights and smaller inner weights. It is therefore not S-concave, which means that it is not a welfare function. However, it can be considered as a *self-dual central index*. On the other hand, $\widetilde{W}_B$ is strictly S-convex (the coefficients of the $x_{[i]}$ are decreasing) and invariant for translations (the coefficients of the $x_{[i]}$ have zero sum). Then, it is an *anti-self-dual* (thus consistent) *inequality index*.

Remark 21 Since

$$W_{\overline{V}}(\mathbf{1}-\boldsymbol{x}) = \mu(\mathbf{1}-\boldsymbol{x}) - \frac{V_A(\mathbf{1}-\boldsymbol{x})}{c} = 1 - \mu(\boldsymbol{x}) - \frac{B_A(\boldsymbol{x})}{c},$$

the dual normalized De Vergottini welfare function can be written as

$$W^*_{\overline{V}}(\boldsymbol{x}) = 1 - W_{\overline{V}}(\mathbf{1}-\boldsymbol{x}) = \mu(\boldsymbol{x}) + \frac{B_A(\boldsymbol{x})}{c}.$$

Proposition 17 *The self-dual core and the anti-self-dual remainder of the normalized De Vergottini welfare function are given by*

$$\widehat{W}_{\overline{V}}(\boldsymbol{x}) = \mu(\boldsymbol{x}) + \frac{B_A(\boldsymbol{x}) - V_A(\boldsymbol{x})}{2\,c} \qquad \textit{and} \qquad \widetilde{W}_{\overline{V}}(\boldsymbol{x}) = -\frac{B_A(\boldsymbol{x}) + V_A(\boldsymbol{x})}{2\,c}.$$

PROOF: Taking into account Remark 21, the derivations are as follows:

$$\widehat{W}_{\overline{V}}(\boldsymbol{x}) = \frac{W_{\overline{V}}(\boldsymbol{x}) + W^*_{\overline{V}}(\boldsymbol{x})}{2} = \frac{\mu(\boldsymbol{x}) - \frac{1}{c}V_A(\boldsymbol{x}) + \mu(\boldsymbol{x}) + \frac{1}{c}B_A(\boldsymbol{x})}{2} =$$

$$= \mu(\boldsymbol{x}) + \frac{B_A(\boldsymbol{x}) - V_A(\boldsymbol{x})}{2\,c}.$$

$$\widetilde{W}_{\overline{V}}(\boldsymbol{x}) = \frac{W_{\overline{V}}(\boldsymbol{x}) - W^*_{\overline{V}}(\boldsymbol{x})}{2} = \frac{\mu(\boldsymbol{x}) - \frac{1}{c}V_A(\boldsymbol{x}) - \mu(\boldsymbol{x}) - \frac{1}{c}B_A(\boldsymbol{x})}{2} =$$

$$= -\frac{B_A(\boldsymbol{x}) + V_A(\boldsymbol{x})}{2\,c}. \blacksquare$$

Remark 22 The De Vergottini welfare function can be written as follows

$$W_{\overline{V}}(\boldsymbol{x}) = \left(\mu(\boldsymbol{x}) - \frac{B_A(\boldsymbol{x}) - V_A(\boldsymbol{x})}{2c}\right) + \left(-\frac{B_A(\boldsymbol{x}) + V_A(\boldsymbol{x})}{2c}\right)$$

and the Vergottini illfare function

$$L_{\overline{V}}(\boldsymbol{y}) = \left(\mu(\boldsymbol{y}) - \frac{B_A(\boldsymbol{y}) - V_A(\boldsymbol{y})}{2c}\right) + \left(\frac{B_A(\boldsymbol{y}) + V_A(\boldsymbol{y})}{2c}\right).$$

Consequently, the De Vergottini welfare and illfare functions can be decomposed in terms of a central index and a consistent inequality index.

Remark 23 Analogous statements to those provided in Remark 20 apply for the self-dual core and the anti-self-dual remainder of the normalized De Vergottini welfare function.

It is worth noting the dual behavior of the decomposition components for the Bonferroni and the normalized De Vergottini welfare functions. On the one hand, the anti-self-dual remainders are equal but for the normalization constant. The role played by the absolute Gini index in the anti-self-dual remainder of the Gini welfare function is replaced now by an average of the respective absolute indices. On the other hand, the components of the self-dual cores are completely symmetric but, once again, the normalization constant is present.

Finally, the dual decomposition of the Bonferroni and De Vergottini welfare functions suggests the possibility of constructing an extended family of welfare functions, with their associated self-dual central indices and anti-self-dual (thus consistent) inequality indices. The idea is that of generalizing the concepts of upper and lower averages involved in the construction of B_A and V_A by means of generalized coefficients u_i (positive increasing, unit sum) and v_i (positive decreasing, unit sum), such that $u_{n-i+1} = v_i$ as usual for $i = 1, \ldots, n$. The corresponding expressions for the generalized Bonferroni and De Vergottini inequality indices would be again (the unchanged notation is meant to emphasize the analogy with the classical construction)

$$B_A(\boldsymbol{x}) = \sum_{i=1}^{n}(1/n - u_i)x_{[i]}\,, \quad V_A(\boldsymbol{x}) = \sum_{i=1}^{n}(v_i - 1/n)x_{[i]}$$

and therefore the generalized self-dual central index and anti-self-dual (thus consistent) inequality index would be as before (Remarks 19 and 20), respectively,

$$\mu(\boldsymbol{x}) - \frac{1}{2}(B_A(\boldsymbol{x}) - V_A(\boldsymbol{x})) = \sum_{i=1}^{n}\frac{v_i + u_i}{2}x_{[i]}\,, \ \frac{1}{2}(B_A(\boldsymbol{x}) + V_A(\boldsymbol{x})) = \sum_{i=1}^{n}\frac{v_i - u_i}{2}x_{[i]},$$

with corresponding expressions for the generalized welfare functions (choosing the normalization in an appropriate way). This new families of central and (consistent) inequality indices, which includes the classical Gini (linear coefficients), Bonferroni, and De Vergottini classical constructions, is now being investigated and will be the subject of a future publication.

5 An illustrative example

Imagine that we want to analyze the welfare due to employment in a society of four individuals. Consider $\boldsymbol{x} = (0.35, 0.70, 0.90, 0.91)$ the distribution of the employment duration rates. On the other hand, we could analyze the illfare due to unemployment in the same society. The unemployment duration rate distribution is $\boldsymbol{y} = \boldsymbol{1} - \boldsymbol{x} = (0.65, 0.30, 0.10, 0.09)$, that represents the shortfall distribution. The corresponding means are $\mu(\boldsymbol{x}) = 0.715$ and $\mu(\boldsymbol{y}) = 0.285$, respectively.

In this example we measure the social welfare/illfare according to the the Gini, Bonferroni and normalized De Vergottini functions. Table 1 shows these values and also the corresponding inequality values of employment and unemployment.

Table 1. Gini, Bonferroni and normalized De Vergottini welfare/illfare functions and inequality measures

$W_G(\boldsymbol{x})$	$W_B(\boldsymbol{x})$	$W_{\overline{V}}(\boldsymbol{x})$	$L_G(\boldsymbol{y})$	$L_B(\boldsymbol{y})$	$L_{\overline{V}}(\boldsymbol{y})$
0.598	0.560	0.627	0.403	0.440	0.373
$G_A(\boldsymbol{x})$	$B_A(\boldsymbol{x})$	$\overline{V}_A(\boldsymbol{x})$	$G_A(\boldsymbol{y})$	$B_A(\boldsymbol{y})$	$\overline{V}_A(\boldsymbol{y})$
0.118	0.155	0.088	0.118	0.127	0.107

As shown in this table, all the employment welfare functions can be decomposed as the subtraction of the mean and the inequality of employment rates. Nevertheless, only the Gini unemployment illfare function allows a decomposition as the sum of the mean and the inequality of unemployment. This is not true for the Bonferroni and the normalized De Vergottini unemployment illfare functions. In these cases, the inequality of employment and unemployment do not coincide.

Table 2 shows the dual decomposition of the welfare and ilffare functions associated with Gini, Bonferroni and normalized De Vergottini indices proposed in this paper. As we have proved, the two terms can be interpreted as a weighted mean and an inequality term. The particularity of this proposal is that the inequality terms measure equally the inequality of achievements and shortfalls. In this case, the employment welfare (illfare) functions can be decomposed as the subtraction (sum) of a weighted mean of the rate of employment (unemployment) and an inequality measure of employment (unemployment).

Table 2. The dual decomposition of the Gini, Bonferroni and normalized De Vergottini welfare and illfare functions

$\widehat{W}_G(\boldsymbol{x})$	$\widetilde{W}_G(\boldsymbol{x})$	$\widehat{W}_B(\boldsymbol{x})$	$\widetilde{W}_B(\boldsymbol{x})$	$\widehat{W}_{\overline{V}}(\boldsymbol{x})$	$\widetilde{W}_{\overline{V}}(\boldsymbol{x})$
0.715	−0.118	0.701	−0.141	0.725	−0.098
$\widehat{L}_G(\boldsymbol{y})$	$\widetilde{L}_G(\boldsymbol{y})$	$\widehat{L}_B(\boldsymbol{y})$	$\widetilde{L}_B(\boldsymbol{y})$	$\widehat{L}_{\overline{V}}(\boldsymbol{y})$	$\widetilde{L}_{\overline{V}}(\boldsymbol{y})$
0.285	0.118	0.299	0.141	0.275	0.098

6 Concluding remarks

We have examined the dual decomposition of the OWA welfare functions associated with the Gini, Bonferroni, and De Vergottini indices in the standard framework of aggregation functions on the $[0,1]^n$ domain. Variables bounded on the $[0,1]^n$ domain can encode achievements or shortfalls. In this way a complementary illfare framework can be introduced. We have shown that the complementary illfare functions are also OWA operators and the dual decomposition has allowed us to identify and establish the relationship between the respective inequality and mean terms.

In addition, the dual decomposition highlights the distinct and complementary nature of the three classical inequality indices. In the Gini index case, the central result is $G_A(\mathbf{1} - \boldsymbol{x}) = G_A(\boldsymbol{x})$ and the dual decomposition reproduces in a natural way the canonical construction of the associated welfare function. In the Bonferroni and De Vergottini cases, the central result is $B_A(\mathbf{1} - \boldsymbol{x}) = V_A(\boldsymbol{x})$ (and vice-versa) and the natural dual relationship between the two indices emerges very clearly in the way the self-dual cores and anti-self-dual remainders of the two welfare functions combine the two inequality indices. An appropriate normalization of the De Vergottini index is considered.

Finally, the orness of the welfare functions associated with three classical inequality indices has been computed, obtaining values in the $(0, 1/2)$ interval due to the common emphasis on poorer incomes. In the large population asymptotic limit, the orness values of the Gini (1/3), Bonferroni (1/4), and De Vergottini (1/2) welfare functions recall the character of the associated classical inequality indices and constitute further evidence of the duality pattern illustrated by the dual decomposition.

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