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I. INTRODUCTION

During the past few years, in this country and abroad, considerable work has been done in the theoretical and experimental study of stress and strength conditions arising when two cylinders are rolled together under pressure. This increased interest has been a result of the higher demands placed upon machine members which are constrained to work under similar conditions. Prime examples of such machine elements are gears and roller bearings.

Up to the present time the methods for determining the stress distribution and strength conditions in the contact zone of machine parts have been based, in general, on static rather than dynamic loading conditions. It is apparent that in modern high-speed machines the results based on such an approximation only approach actual values.

1. Purpose and Scope of Investigation

The purpose of the following investigation was twofold:

(1) To extend knowledge of the stress conditions in the contact zone of two compressed cylinders when rolled together with their axes parallel, by studying (a) the stresses outside the plane of symmetry and (b) their variation during the loading cycle by rolling of the cylinders.

(2) To investigate the strength condition of materials so studied and to suggest a method for its calculation in machine parts working under similar conditions.

The purposes mentioned above were carried out as follows: A review was made of the literature on character of failure of machine members as well as on experimental and theoretical investigations of the contact problem. For determination of stress variation in the contact zone of two rolling cylinders two methods were suggested — one in which the influence lines of stresses caused by the concentrated force are used, and the other an analytical method.

The first method does not require complicated mathematical calculations and has the further advantage that the loading curve of any form can be investigated. But the results obtained by using this method are only approximate. It does give satisfactory results for points which are located sufficiently far from the contact surface.

The analytical method was used for determinating stress components which are used for strength calculation. This method is based on the Belayev's three-dimensional solution of the problem of two cylinders of infinite length pressed together. Using this solution, the author investigated the variation of normal and tangential stress components in the contact zone when the cylinders are rolling one upon the other. On the basis of these investigations, a number of diagrams were plotted representing the variation of the normal and tangential components for the points located at various distances from the contact surface. One of the significant results from these investigations was that the stress component τ_{xy} has a completely reversed range.

The data on the absolute values and range ratios of stress at various layers in the contact zone were used to develop an equation for expressing the dynamic criterion of failure for rolling cylinders, or members operating under similar conditions. This equation is based on the Huber-Mises energy hypothesis. A table and a diagram showing the relationship between the critical pressure on the contact surface and the mechanical properties of different steels were prepared from data obtained by using this equation. The table also shows the location of the critical layer in the contact zone and its dependence upon the mechanical properties of materials. It should be noted that the distance of the critical layer from the contact surface for the conditions of dynamic loading does not coincide with the location of the point where the maximum shearing stress exists. In the author's opinion, the results obtained give a basis for calculating the strength of machine members working under similar conditions of dynamic loading.

The investigation is limited to the case of two cylinders rolling together without sliding. The case of rolling with sliding is likewise of great interest. At present, however, the actual distribution of the tangential forces acting on the lubricated surfaces of cylinders rolling with relative sliding is not known. It seems probable that in this case the friction between dry, half-dry, and lubricated surfaces takes place simultaneously and that the tangential loading on the contact surface is not proportional to the normal loading at the corresponding points. In the author's opinion, a study of the actual distribution of the tangential loading on the surface for this case must be made before the problem of the stress distribution and strength condition can be solved.

2. Acknowledgments

This work is based partly upon previous investigations conducted by the author at the Institute of Applied Mechanics of Academy of Sciences

of the Ukrainian SSR, Kiev, before coming to the United States. World War II prevented completion of this work until the present time.

The author wishes to thank Professors N. A. Parker, head of the Department of Mechanical Engineering, and D. G. Ryan, who is in charge of the Machine Design Group, for the opportunity to complete this investigation as part of the research program of the Machine Design Group. The assistance of D. H. Offner, instructor in the Department of Mechanical Engineering, is acknowledged for his help in the initial presentation of the written material.

II. PREVIOUS DATA ON CHARACTER OF FAILURE OF VARIOUS MACHINE MEMBERS

3. Character of Fatigue Failure of Gear Teeth and of Ball and Roller Bearings

Gear teeth failures caused by repeated contact stresses usually occur on the surface of the tooth along the intersection of the tooth face and the pitch cylinder. The failure appears as a crumbling of fine metal particles, causing the surface to be pitted. Such a condition results in severe wear.

A similar phenomenon can be observed in ball and roller bearings, where the fatigue failure causes the surface metal of the bearing races to come off in the form of thin flakes. The surface soon becomes rough and the bearing noisy. Failure progresses rapidly and the member must be replaced soon after flaking begins. According to Turkish*, the phenomena of pitting and flaking can likewise be observed on the working faces of cams and tappets of valve gears of internal combusion engines. The flaking phenomenon also develops whenever excessive variable contact stresses are present in machine parts operating under similar conditions, such as rails or on the wheels of cranes.

Various theories have been proposed to explain this phenomenon of pitting and flaking. At present it is fairly well established that pitting and flaking are caused by a great many cycles of repeated stresses which cause a fatigue failure of the metal in a relatively thin layer near the working surface of the members.

4. Experimental Investigations with Roller Machines

In the past fifteen years several experimental investigations have been conducted to determine the factors that cause pitting and flaking. The experiments made by S. Way†, T. Nishihara and T. Kobayashi‡, and A. Meldahl¶ are especially significant.

The experiments were made with roller machines so constructed that two metal cylinders could roll while being pressed together with any desired force. In Way's experiments the cylinders were made to roll

* M. C. Turkish, "Valve Gear Design," Detroit, 1946.

† S. Way, Journ. of Applied Mechanics, Vol. 2, Nos. 2 and 3, 1935; Machine Design, March 1939; Journ. of Applied Mechanics, Vol. 7, No. 4, 1940; Trans. A.S.M.E., Vol. 57, 1935.

‡ T. Nishihara and T. Kobayashi, Trans. Soc. Mech. Engrs. Japan, Vol. 3, 1937. (Japanese)

¶ A. Meldahl, Engineering, Vol. 148, July 21, 1939.

against each other without sliding. However, in the experiments of Nishihara and Kobayashi, and Meldahl the relative surface speeds of two cylinders were adjustable, permitting sliding to occur.

The results of these investigations showed clearly that pitting is caused by fatigue failure of the surface layer of metal and that it depended upon the magnitude and number of repeated load applications. The relationship between the number of stress applications and the pressure causing pitting on the surface was found to be very similar to the ordinary fatigue curves.

These experiments indicated also how the critical pressure depended on the relative sliding speed. It was found that the "surface endurance limit" decreased slightly when the relative sliding speed was increased from 0 to 20 percent, but increased again as the relative sliding speed went above 20 percent. This result was probably due to increased friction which caused the surface layer to wear off before fatigue failure could take place.

III. DISTRIBUTION OF STRESSES IN ZONE OF CONTACT

5. H. Hertz' Investigation

The general method for determining the distribution of stresses in the zone of contact of two elastic bodies was indicated by H. Hertz* in 1881. Of most interest are the following basic types of contact between two bodies: (1) bodies initially having point contact before the deformation, such as two spheres, a sphere and a plane, or a sphere and a cylinder, and (2) bodies having straight line contact before deformation, such as two cylinders with parallel axes or a cylinder and a plane. Practical illustrations of the first type of contact are ball bearings and their supports and wheels rolling on convex rails; illustrations of the second type are roller bearings and gears.

Hertz derived mathematical expressions for the distribution of pressure on the contact area of bodies having an initial point contact. He also was able to determine the limiting size of the axes of the elliptical contact area and the relative displacements of the bodies. For circular areas of contact, Hertz was able to determine the stress in the center of the area. For bodies having initial line contact, he obtained expressions defining the width of the contact strip and the pressure distribution across this strip.

According to Hertz, the pressure distribution on the surface of two compressed bodies is defined by the expression (see Fig. 1)

$$p = p_{\max} \sqrt{1 - \frac{y^2}{b^2}} \qquad (1)$$

where $p_{\max}$ = maximum pressure at center of contact area for the initial point contact and at the middle of strip for the initial line contact

y = the distance from the x-axis

b = semi-axis of elliptical contact area or one-half the width of contact strip

Figure 2 shows the pressure distribution across the surface of the contact strip of two cylinders. It also shows the position of the axes

* H. Hertz, Journ. Math., Vol. 92, 1881. See also H. Hertz' "Gesammelte Werke," Vol. 1, Leipzig, 1885.

in the coordinate system. The value of the maximum pressure on the middle of the contact strip is

$$p_{\max} = \sqrt{\frac{1}{\pi(1-\mu^2)} \cdot \frac{P}{l} \cdot \frac{\frac{1}{R_1}+\frac{1}{R_2}}{\frac{1}{E_1}+\frac{1}{E_2}}} \tag{2}$$

where μ = Poisson's ratio for the two materials
P = force pressing the two cylinders together
E_1, E_2 = modulus of elasticity in compression for cylinders 1 and 2
R_1, R_2 = radii of the two cylinders
l = length of the cylinders

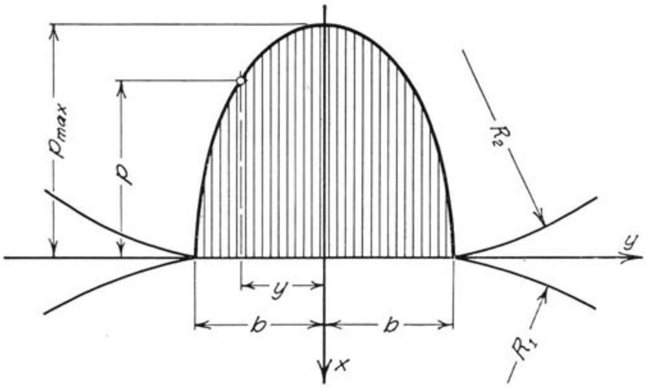

Fig. 1. Pressure Distribution on Contact Surface

The value of b for two cylinders with parallel axes can be obtained from the expression:

$$b = 2\sqrt{\frac{1-\mu^2}{\pi} \frac{P\left(\frac{1}{E_1}+\frac{1}{E_2}\right)}{l\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}} \tag{3}$$

For steel, using $\mu = 0.3$ and $E_1 = E_2 = E$, the magnitude of one-half the width of the contact strip is:

$$b = 1.52\sqrt{\frac{P}{El} \frac{R_1 R_2}{R_1 + R_2}} \tag{4}$$

and the maximum pressure on the middle of the contact strip is:

$$p_{\max} = 0.418 \sqrt{\frac{PE}{l}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)} \tag{5}$$

In making this analysis Hertz used the following assumptions:

1. The bodies in contact are isotropic
2. The proportional limit of materials is not exceeded
3. The loading acts perpendicular to the surface
4. The dimensions of the compressed area are small when compared with the whole surface of the bodies pressed together
5. The radii of curvature of the contact areas are very large compared with the dimensions of these areas

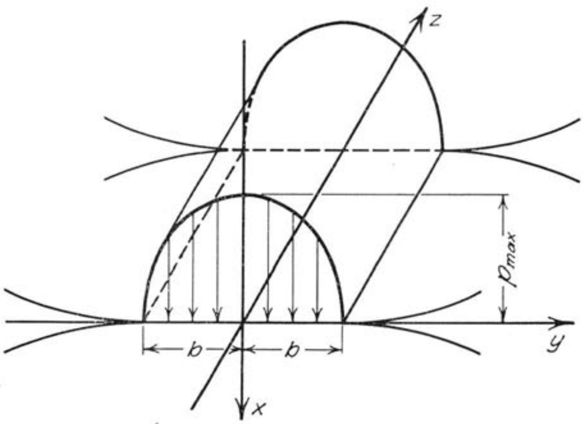

Fig. 2. Pressure Distribution Across the Strip of Contact Surface of Two Cylinders

6. Later Investigations of the Hertz Problem

Using the same assumptions listed above, later investigators extended Hertz' work. In 1924 Belayev* in Russia, and independently Thomas and Hoersch† in this country and Föppl‡ in Germany determined the law of distribution for the stress components along the x-axis in the plane of symmetry of the loading curve. Figure 3 shows the distribution of stress components σ_x, σ_y, and σ_z, and the stress τ_{45° along the line of symmetry for two cylinders pressed together with parallel axes. The ordinate representing the distance of the points from the contact surface is given in terms of b. Except for the values of normal stress components, the diagram also represents the variation in value of τ_{45°. The stress τ_{45° is the

* N. M. Belayev, Mag. Eng. Buildings and Applied Mech., St. Petersburg, 1924. (Russian)
† H. R. Thomas and V. A. Hoersch, "Stresses Due to the Pressure of one Elastic Solid upon Another," Univ. of Ill. Eng. Exp. Sta. Bul. No. 212, 1930.
‡ L. Föppl, Forschung auf d. Gebiete d. Ingenieurwesens, No. 5, 1936.

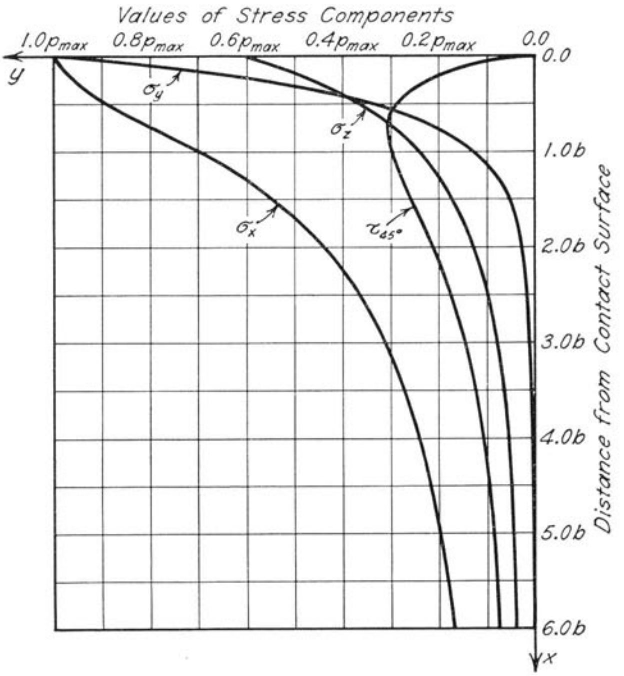

Fig. 3. Distribution of Stresses in Contact Zone Along Line of Symmetry

shearing stress on a plane which makes an angle of 45 deg with the y-axis. The value is

$$\tau_{45^\circ} = \frac{\sigma_x - \sigma_y}{2} \tag{6}$$

at the plane of symmetry.

Belayev, Thomas and Hoersch, and Föppl all found that the maximum shearing stress in the zone of contact of two cylinders having parallel axes acts on a plane which bisects the angle formed by the y- and x-axes ($\varphi = 45°$). This stress is located at the point where $x \cong 0.78\, b$ from the contact surface. The value of the shearing stress τ_{xy} in the plane of symmetry is zero. According to Belayev, the value of maximum shearing stress when $\mu = 0.3$ is

$$\tau_{\max} = 0.304 p_{\max} = 0.304 \frac{2P}{l\pi b} = 0.304 \frac{2q_0}{\pi b}$$

where q_0 is the loading per unit length of cylinder.

The more recent works of Poritsky* and Liu† should also be mentioned. Poritsky expressed the equation for two-dimensional stresses and deflections of two cylindrical bodies in contact under normal and tangential reactions in terms of Airy's function. He has taken the tangential load

* H. Poritsky, Journal of Applied Mechanics, Vol. 17, No. 2, June 1950.
† C. K. Liu, Thesis for the degree of Doctor of Philosophy, University of Illinois, 1950.

as being proportional to the normal Hertz' load over the same area. He considered that the ratio between the tangential and normal load was equal to the constant coefficient of friction, 0.3.

Liu independently solved the same problem by the method of real variables both for cases of plane stress and plane strain. He pointed out that the stresses which appear in the region of contact caused by the presence of tangential forces deserve attention. He also assumed that the intensity of the tangential force at every point in the contact area is proportional to the normal force at the same point. His factor of proportionality is equal to $\frac{1}{3}$.

7. Factors of the Hertz Problem Not Considered

There are some indications that tensile stresses occur on the surface layer of compressed bodies at points which lie outside of, but very close to, the area of pressure contact. When two bodies are rolled together, the surface areas are stressed in tension and then in compression. For brittle materials it is possible that these tensile stresses, which occur near the border of the contact area, contribute to the fatigue failure of the surface. It seems possible that this phenomenon would be important in the case of two cylinders having a large difference between their radii. The condition could be especially critical on the surface of the larger cylinder because it would have a concave contact surface.

This phenomenon was not taken into account by Hertz' analysis and those expanding his work.

Lubrication is also a factor which can have some influence on the stress distribution on the contact zone, and which was not taken into account by Hertz' analysis. S. Way believed that initial cracks were formed very near the surface due to the repeated contact stresses. When these cracks progress to the surface, the oil is pressed into the openings, causing further cracking, and resulting in pitting as the metal is crumbled and torn from the surface. Way suggests that by using oil of high viscosity, the time at which pitting appears may be increased. However, the experiments of Nishihara and Kobayashi do not confirm this opinion. Apparently, the use of high viscosity lubricants decreases the duration of ball bearing life.

Very little reliable data are available regarding the manner in which the use of lubricants affects the fatigue failure of compressed bodies. The oil layer between the rolling cylinders also affects the pressure distribution on the surface of compressed bodies, and thus also changes the stress conditions in the contact zone. The results of some investigations indicate that these changes in pressure distribution due to lubrication are relatively small and affect the stress very little. Figure 4 shows how Eichinger suggested that the pressure distribution may be effected by lubrication.

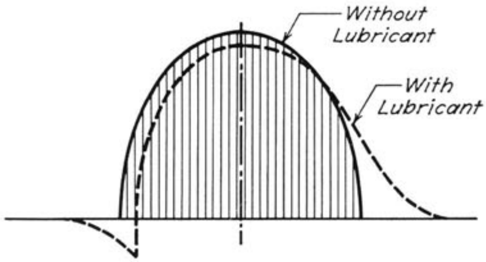

Fig. 4. Pressure Distribution With and Without Lubricant, According to Eichinger

8. Strength Calculation of Materials Based on Data Obtained from Stress Distribution Along the Plane of Symmetry

Several authors have suggested using the maximum shear stress for gear calculations. Among them are Schmitter*, Petrusyevich†, Dollezhal‡. Buckingham¶ and Wissman§ recommended using the maximum pressure on the contact area as an initial value in gear calculation. T. Barish** also used the maximum Hertz' pressure as a basic stress for calculations on ball bearings.

Since the maximum shearing stress in elastic bodies which are pressed together is directly proportional to the maximum pressure on the contact area, there is no difference of principle between the methods of determining the permissible stress as proposed by Schmitter and Petrusyevich or Buckingham, Wissman and Barish.

Eichinger†† suggested using the reduced stress σ_r for calculating the strength in the contact zone of two cylinders with parallel axes. This

* W. P. Schmitter, Machine Design, July 1934.

† Petrusyevich, The works of "Central Bureau Reducter Building Orgametall," Moscow, 1939. (Russian)

‡ V. Dollezhal, Techn. of Air Fleet, No. 5, May 1939. (Russian)

¶ E. Buckingham, Manual of Gears Design, Sect. No. 2, 1935; Manual of Gears Design, Sect. No. 3, 1937.

§ K. Wissman, Berechnung und Konstruktion von Zahnrädern für Kräne und ähnliche Maschinen, Dissertation, Duisburg, 1930.

** Thomas Barish, "A Theoretical Derivation of Ball Bearing Ratings," Paper No. 46-A-75 prepared by Engineering and Research Corporation, Riverdale, Maryland, for presentation at the annual ASME meeting of 1946 (Advance copy).

†† E. Eichinger, "Reibung und Verschliess," Stuttgart, 1938.

reduced stress σ_r should be compared to the mechanical properties of metal by direct pressure, for example to the compressive yield point σ_s.

The value of the reduced stress according to Eichinger (using the Mises-Hencky's hypothesis) may be found from the equation:

$$\sigma_r = \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_x\sigma_z} \quad (7)$$

Figure 5, shows the variation in the value of σ_r along the axis of symmetry. On the axis of symmetry, the maximum value of σ_r is

$$(\sigma_r)_{\max} = 0.55\sigma_{ox}$$

where $\sigma_{ox} = p_{\max}$. The maximum reduced stress occurs at a point $x \cong 2/3\, b$ from the contact surface.

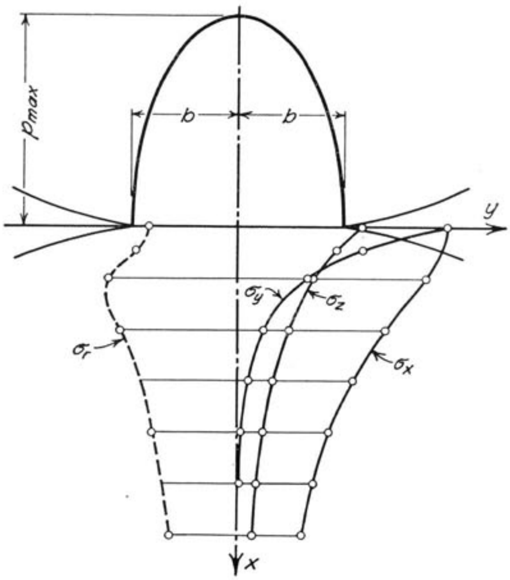

Fig. 5. Variation in Value of Reduced Stress Along Axis of Symmetry, According to Eichinger

On the surface at the middle of the contact strip

$$\sigma_r = 0.4p_{\max}$$

Mundt* and Treyer† used the reduced normal stresses at the middle of the contact area as the basis for calculations on ball and roller bearings.

* R. Mundt, Forschung auf d. Gebiete d. Ingenieurwesens, Bd. 3, 1932, S. 127.
† Treyer, The Theory and Calculation of Ball and Roller Bearings, 2d ed., Moscow, 1936. (Russian)

IV. ANALYSIS OF STRESS VARIATION IN CONTACT ZONE OF TWO COMPRESSED CYLINDERS ROLLING TOGETHER

The fatigue strength of materials subjected to repeated loading is known to depend not only on the maximum stress but also upon the manner in which the stresses vary during a loading cycle. It was the purpose of this investigation to determine this stress variation at every desired point in the contact zone of two rolling cylinders. Two methods of analysis were used in this investigation*.

9. Method of Influence Lines of Stresses

According to Flamant† the normal components of stress σ_x and σ_y and the shearing stress τ_{xy}, which occur in an elastic medium limited by a plane due to a normal force P concentrated at a point, can be determined as follows: (See Fig. 6a)

$$\left.\begin{aligned} \sigma_x &= -\frac{2P}{\pi H}\cos^4\theta \\ \sigma_y &= -\frac{2P}{\pi H}\sin^2\theta\cos^2\theta \\ \tau_{xy} &= -\frac{2P}{\pi H}\sin\theta\cos^3\theta \end{aligned}\right\} \qquad (8)$$

These equations were derived by Flamant using the Boussinesq‡ three-dimensional solution. Defining φ as the angle between the y-axis and plane on which the stresses to be found are acting (see Fig. 6b), the equation for the normal and shearing stresses on this plane can be determined as follows:

$$\sigma = \sigma_x\cos^2\varphi + \sigma_y\sin^2\varphi + 2\tau_{xy}\sin\varphi\cos\varphi \qquad (9)$$

$$\tau = \tau_{xy}(\cos^2\varphi - \sin^2\varphi) + (\sigma_y - \sigma_x)\sin\varphi\cos\varphi \qquad (10)$$

* These methods are briefly discussed in the author's article, "About the Strength by Repeated Contact Stresses," Inst. of Mech. Knowledge, Academy of Sciences USSR, 1941. (Russian)

† Flamant, "Comptes rendus," Vol. 114. p. 1465, Paris, 1892.

‡ J. Boussinesq, "Application des potentiels à l'étude de l'équilibre et du mouvement des solides elastiques," Paris, 1885. See also S. Timoshenko, "Theory of Elasticity," New York, 1934.

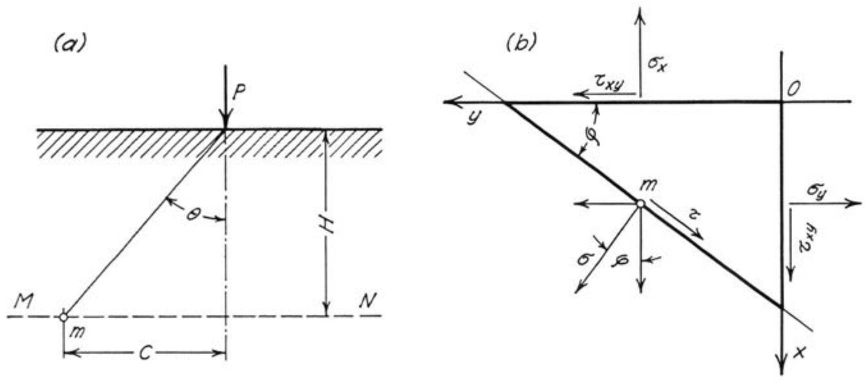

Fig. 6. Single Concentrated Force Acting Perpendicular to a Plane

By substituting Eqs. 8 into Eqs. 9 and 10, it is possible to find the values of σ and τ at any desired point in the stressed zone caused by the concentrated force P. In Fig. 7 values of stresses σ_x and τ_{xy} at a set of points a, b, c, ... n, located a distance H below the surface, have been plotted and connected by the smooth lines. These lines show how the stresses vary along the line p–q due to the concentrated load P. At the same time they can be considered as the influence lines for the stresses (in this case stresses σ_x and τ_{xy}), at the point B due to the concentrated force P.

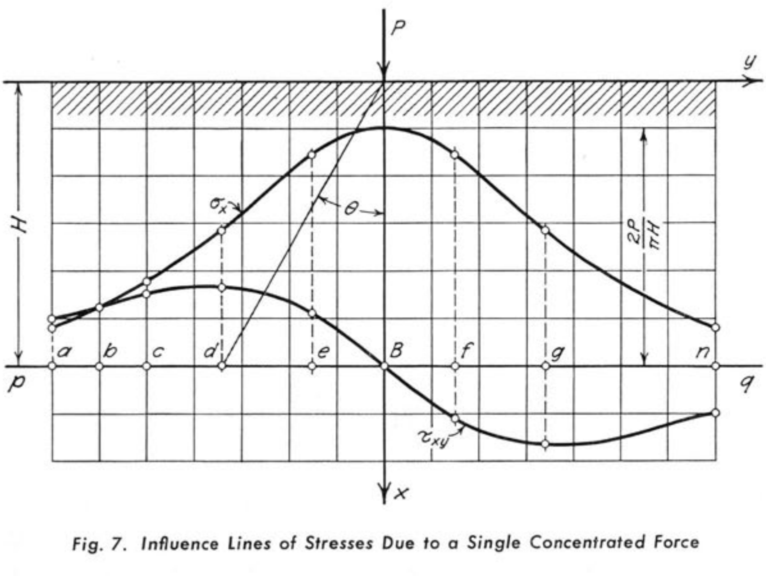

Fig. 7. Influence Lines of Stresses Due to a Single Concentrated Force

This method was applied to the stress analysis of two rolling cylinders as follows:

1. Referring to Fig. 8a, a curve showing the pressure distribution across the contact strip of a cylinder of unit length was constructed from the base line A–B. For the case of two cylinders with parallel axes, the curve is a semi-ellipse. This area was divided into a number of parts by vertical lines. Concentrated forces $P_1, P_2, \ldots P_n$ were assumed to be applied at the center of gravity of each area and their magnitudes were assumed equal to the areas they replaced.

2. Influence lines of the stresses were then constructed for the single concentrated force P_1 having an assumed magnitude of unity. In the example of Fig. 8, the stresses $\tau_{45°}$ and τ_{xy} are shown at the point where $x = 0.78\ b$. According to Belayev, the shearing stress $\tau_{45°}$ reaches its maximum value at this point. In Fig. 8b, curve 1 represents the influence line of stress $\tau_{45°}$ and curve 2 represents the influence line of stress τ_{xy} for the unit force, P_1.

3. Using the influence lines 1 and 2, the influence lines of stresses $\tau_{45°}$ and τ_{xy} for the whole system of forces $P_1, P_2, \ldots P_n$ can be constructed (Fig. 8c). These curves now show the variation $\tau_{45°}$ and τ_{xy} at the point lying in a layer where $x = 0.78\ b$, by moving the system of forces along the line A–B.

The results thus obtained are an approximate analysis of the stress variations existing in two rolling cylinders. The degree of accuracy can be

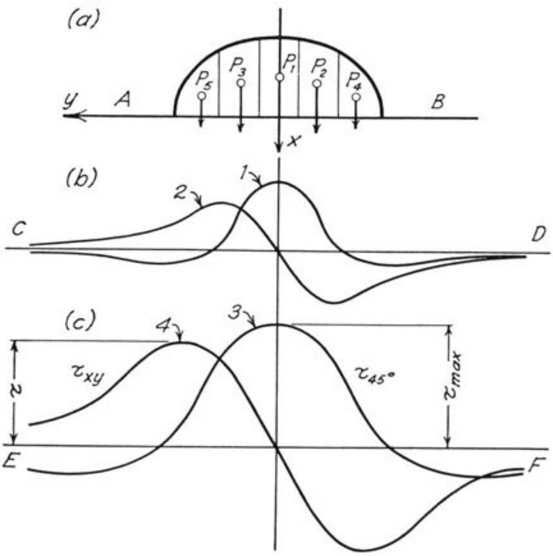

Fig. 8. Method Using Influence Lines of Stresses, Caused by a Series of Concentrated Forces

increased by dividing the loading into a greater number of parts and by increasing the distance H between the points and the surface of the body.

By comparing the different results it was found that when the area of the semi-ellipse was divided into seven parts, satisfactory values of stress were obtained at points about 0.5 b or farther from the surface.

10. Analytical Method of Analysis

The second method of analysis for determining the stress distribution during a loading cycle is entirely analytical. This analysis is based on work done by Belayev*. He found that the stresses on planes perpendicular to the coordinate axes can be calculated by the following equations expressed in elliptic coordinates:

$$\left.\begin{aligned}
\sigma_y &= -\frac{2q}{\pi b} e^{-\alpha} \sin\beta + \frac{2q}{\pi b} \sin\beta \sinh\alpha \left(1 - \frac{\sinh 2\alpha}{\cosh 2\alpha - \cos 2\beta}\right) \\
\sigma_x &= -\frac{2q}{\pi b} e^{-\alpha} \sin\beta - \frac{2q}{\pi b} \sin\beta \sinh\alpha \left(1 - \frac{\sinh 2\alpha}{\cosh 2\alpha - \cos 2\beta}\right) \\
\sigma_z &= -\frac{2q}{\pi b} \frac{\lambda}{\lambda + \mu} e^{-\alpha} \sin\beta \\
\tau_{xy} &= -\frac{2q}{\pi b} \sinh\alpha \sin\beta \frac{\sin 2\beta}{\cosh 2\alpha - \cos 2\beta} \\
\tau_{yz} &= 0 \\
\tau_{xz} &= 0
\end{aligned}\right\} \quad (11)$$

where q is the loading per unit length of contact strip, λ and μ are Lame's constants and b is one-half the width of the contact strip. The equations of transformation are

$$\left.\begin{aligned}
y &= b \cosh\alpha \cos\beta \\
x &= b \sinh\alpha \sin\beta
\end{aligned}\right\} \quad (12)$$

where y and x are the rectangular coordinates. The locations of the points in elliptical coordinates used in Eqs. 11 are the intersection of the ellipse

$$\frac{y^2}{b^2 \cosh^2\alpha} + \frac{x^2}{b^2 \sinh^2\alpha} = 1 \quad (13)$$

with the hyperbola

$$\frac{y^2}{b^2 \cos^2\beta} - \frac{x^2}{b^2 \sin^2\beta} = 1 \quad (14)$$

* N. M. Belayev, loc. cit., page 12. Belayev's equations are also quoted in Stewart Way, Journ. of Applied Mechanics, ASME, 1935; and Fr. Karas. Forschung auf d. Gebiete d. Ingenieurwesens, VDI, Bd. 11, No. 6, 1940.

We can define the $\sin \beta$ and $\sinh \alpha$ functions in terms of x and y by solving the Eqs. 13 and 14 as follows:

$$\left.\begin{aligned} \sin \beta &= \pm \sqrt{\frac{(b^2 - y^2 - x^2) + \sqrt{(b^2 - y^2 - x^2)^2 + 4b^2x^2}}{2b^2}} \\ \text{and} \\ \sinh \alpha &= \pm \sqrt{\frac{-(b^2 - y^2 - x^2) + \sqrt{(b^2 - y^2 - x^2)^2 + 4b^2x^2}}{2b^2}} \end{aligned}\right\} \quad (15)$$

The other functions in Eqs. 11 can be expressed in terms of α and β as

$$\begin{aligned} \cos \beta &= \sqrt{1 - \sin^2 \beta} & \sinh 2\alpha &= 2 \sinh \alpha \cosh \alpha \\ \cosh \alpha &= \sqrt{1 + \sinh^2 \alpha} & \cosh 2\alpha &= \cosh^2 \alpha + \sinh^2 \alpha \\ \sin 2\beta &= 2 \sin \beta \cos \beta & e^{-\alpha} &= \cosh \alpha - \sinh \alpha \\ \cos 2\beta &= 1 - 2 \sin^2 \beta \end{aligned} \quad (16)$$

By using Eqs. 15 and 11 the values of the stress components σ_x, σ_y, σ_z and τ_{xy} can be evaluated for any desired point having coordinates (x,y), and the normal and the shearing stress on any plane at these points of contact zone can now be found by selecting the proper angle φ and solving Eqs. 9 and 10. (See Fig. 6b)

11. Stress Variation in Contact Zone Caused by Rolling

Using Eqs. 11, 15, and 10 the values of the stress components and $\tau_{45°}$ were evaluated for a number of points located along lines parallel to the y-axis and at various distances from the contact surface. The calculations are summarized in Fig. 9 which shows the magnitude of the stress components, σ_x, σ_y, σ_z and τ_{xy}, in terms of $p_{\max}$, the maximum pressure at the middle of the contact surface, plotted against the distance y from the plane of symmetry in terms of b. The four parts of Fig. 9 show these quantities for the four distances $x = 1.0\ b$, $0.78\ b$, $0.50\ b$ and $0.10\ b$ from the contact surface, respectively. Figures 9b and 9c also show curves of $\tau_{45°}$.

The curves in Fig. 9 also represent the influence lines of stresses at points in these layers. Thus, the curves not only give an idea about the maximum stresses at the desired point but also indicate the manner in which they vary during the loading cycle.

In Fig. 9a are plotted curves representing the stresses in the layer $x = 1.0\ b$ from the contact surface. Because the normal stresses σ_x and σ_z reach their maximum values in the plane of symmetry, they must occur when the point under consideration crosses the center line of cylinders. As this point moves away from the x-axis, these normal stresses gradually diminish and asymptotically approach zero. The curve of the normal stress σ_y has a saddle form symmetrical with the x-axis. The maximum values of this stress are outside of the plane of symmetry. Therefore, the stress passes twice through its maximum value during the loading cycle. The shearing stress component τ_{xy} is equal to zero at the plane of symmetry, increasing at first and then approaching zero asymptotically as the contact point moves away from the x-axis. During the loading cycle these stresses change their absolute value as well as their direction. Thus, as the two cylinders are rolled together, the stress τ_{xy} changes from $-\tau_{xy}$ to $+\tau_{xy}$. This reversal of stress determines to a great extent the fatigue limit of the materials.

In Fig. 9b are shown the stresses σ_x, σ_y, σ_z, τ_{xy} and $\tau_{45°}$ in the layer $x = 0.78\ b$, where, according to Belayev and Thomas and Hoersch, the shearing stress reaches its maximum value.

The diagram shows that, even though the absolute value of the stress reaches its maximum in this layer, this stress apparently is not the determining factor in causing the material to fail under the existing conditions. It seems probable that the stress τ_{xy} has more to do with the fatigue failure. This stress has an absolute value somewhat smaller than the maximum value of $\tau_{45°}$, but changes its direction during the loading cycle. From this curve it is seen that the ordinates of the influence line τ_{xy} reach their maximum absolute values at points located between $y = \pm\ 0.75\ b$ and $y = \pm\ 1.0\ b$ from the line of symmetry.

In Fig. 9c are shown the influence lines of stress components and the shearing stress $\tau_{45°}$ in the layer which is a distance $x = 0.5\ b$ from the surface. Here the shearing stress τ_{xy} reaches about its maximum value. This maximum absolute value of these ordinates is located about $y = \pm 0.85\ b$ from the line of symmetry and has a magnitude of

$$(\tau_{xy})_{\max} \cong 0.256 p_{\max}$$

Finally Fig. 9d shows the stresses σ_x, σ_y, σ_z and τ_{xy} in a layer $x = 0.1\ b$ from the contact surface.

The results of the investigation of the stress condition on the surface ($x = 0$) are shown in Fig. 10. Here, the normal stresses σ_x and σ_y are equal to the pressure on the contact area, and their maximum value is equal to the maximum pressure on the middle of contact strip

$$(\sigma_x)_{\max} = (\sigma_y)_{\max} = 1.0 p_{\max} = \frac{2q}{\pi b}$$

Using the value for the Poisson's ratio as $\mu = 0.3$*, the maximum value of stress component σ_z at the surface is equal to

$$(\sigma_z)_{\max} = 0.6 p_{\max}$$

The shearing stress τ_{xy} on the surface is equal to zero.

* This value for the Poisson's ratio is also used in the other evaluations.

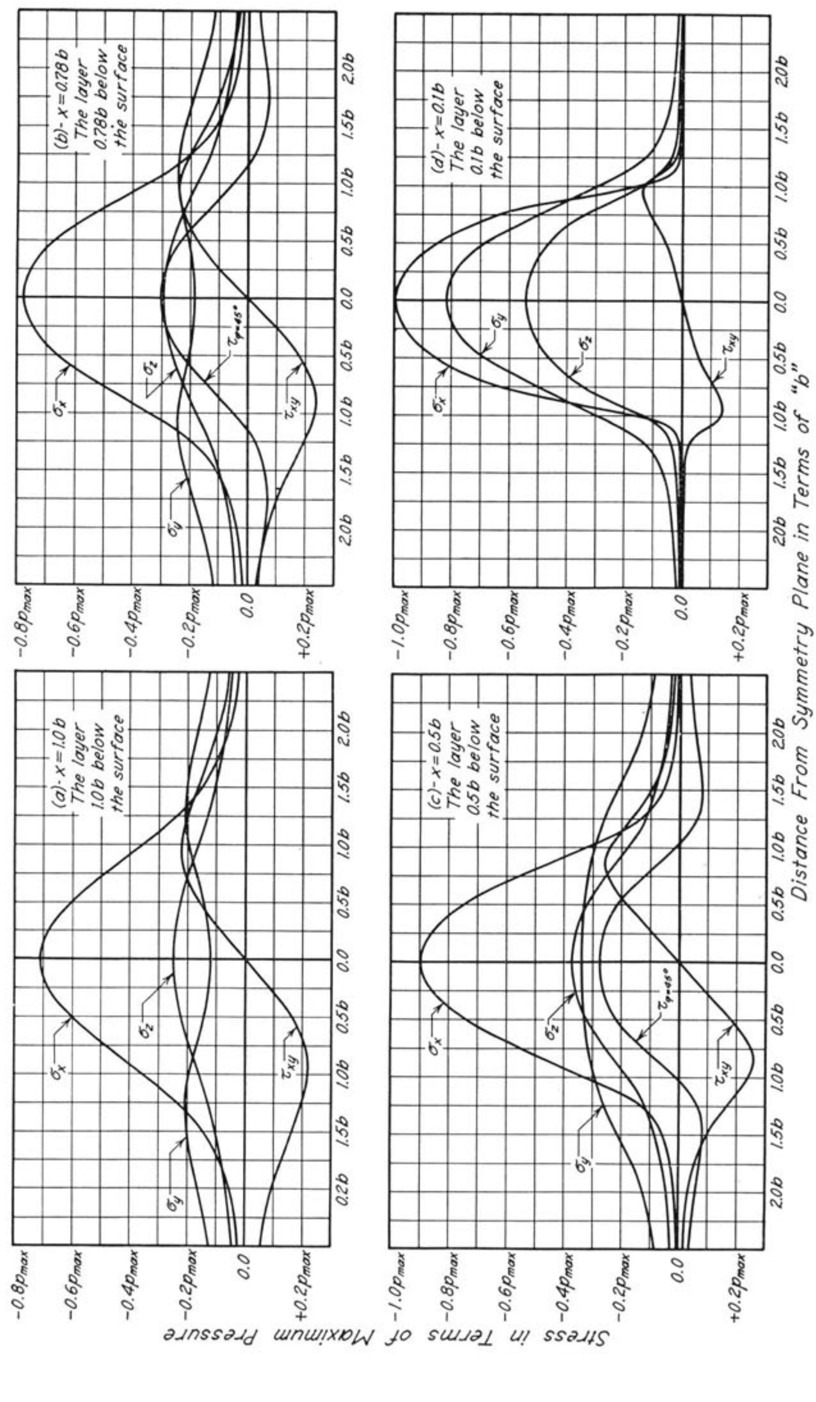

Fig. 9. Variation of Stress Components in Contact Zone During Loading Cycle

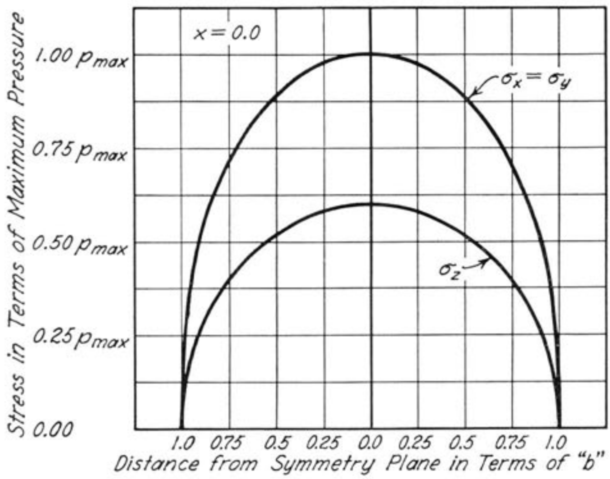

Fig. 10. Variation of Normal Stresses on Contact Surface During Loading Cycle

The data in Figs. 9 and 10 are incorporated with those given in Table 1. The values tabulated for σ_x, σ_y, σ_z and $\tau_{45°}$ refer to these stresses which occur at the axis of symmetry in the x–z plane. These are maximum values except for σ_y in layers farther away than $x = 0.5\ b$. Also listed are the maximum values of the shearing stress τ_{xy}, which occur outside the axis of symmetry. The range ratios of $\tau_{45°}$ during a stress cycle are listed in the last column in Table 1. These range ratios are the ratio of algebraical values of minimum stress to maximum stress during the cycle. That is

$$r = \frac{\sigma_{\min}}{\sigma_{\max}}; \qquad r = \frac{\tau_{\min}}{\tau_{\max}}$$

The normal stresses σ_x, σ_y and σ_z all vary from zero to negative maximum; therefore $\sigma_{\min} = 0$ and the value of the range ratio is always

$$r = \frac{0}{\sigma_{\max}} = 0$$

The shearing stress τ_{xy}, on the other hand, varies from a positive maximum to a negative maximum value; hence the range ratio is always

$$r = \frac{-\tau_{\max}}{+\tau_{\max}} = -1$$

The stress $\tau_{45°}$ changes in an asymmetrical cycle which gives rise to a range ratio whose value depends on the distance x from the contact surface. At the point where it reaches its maximum value the range ratio for $\tau_{45°}$ is approximately

$$r \cong -0.25$$

Several authors, including Belayev, Thomas and Hoersch, Schmitter, Petrusievich and others, believe that the critical point of the contact zone should be taken as the point where the maximum shearing

Table 1

Maximum Values of Stress in Contact Zone in Terms of p_{max}

Distance from Contact Surface, x	Maximum Stresses* in Terms of					Range Ratio of Stress $\tau_{45°}$
	σ_y	σ_x	σ_z	τ_{xy}	$\tau_{45°}$	
0.0	−1.000	−1.000	−0.600	±0.000	0.000	......
0.1 b	−0.815	−0.995	−0.543	±0.146	0.090	......
0.2 b	−0.659	−0.980	−0.492	±0.195	0.160	......
0.5 b	−0.342	−0.894	−0.371	±0.256	0.276	−0.29
0.78 b	−0.188	−0.788	−0.293	±0.233	0.300	−0.25
1.0 b	−0.121	−0.707	−0.248	±0.218	0.293	−0.21
1.2 b	−0.084	−0.640	−0.217	±0.205	0.267	......
1.5 b	−0.054	−0.552	−0.182	±0.178	0.248	......

* The values tabulated for σ_x, σ_y, σ_z and $\tau_{45°}$ refer to the stresses which occur at the axis of symmetry in the x–z plane. The stresses τ_{xy} occur outside of the plane of symmetry, while the stresses σ_y given for x greater than about 0.5 b are those on the axis of symmetry and are somewhat less than $(\sigma_y)_{max}$.

stress occurs. If this is true, then the critical point lies in the layer at a distance $x = 0.78\ b$ from the contact surface where maximum shearing stress reaches its highest value

$$(\tau_{45°})_{max} \cong 0.3 p_{max}$$

However, this way of determining the critical point would seem to be restricted to the conditions of constant loading and would not be valid for variable stress conditions existing when two compressed cylinders are rolling. If we consider the shearing stress critical, for the case of rolling cylinders, it would seem more reasonable to take the critical point nearer to the surface where the maximum stress τ_{xy} occurs. Although the absolute value of $(\tau_{xy})_{max}$ is lower, its range ratio is always equal to minus one, constituting a complete stress reversal.

V. STRENGTH CONDITIONS OF MATERIALS SUBJECTED TO VARIABLE CONTACT STRESSES

It has already been mentioned that some authors have based their strength calculations on the maximum pressure at the contact surface. Others base the strength on the maximum shearing stresses. There is no real difference between the two methods since the maximum shearing stresses are directly proportional to the maximum pressure on the contact surface. However, neither one of these methods for determining the critical and permissible loads takes into consideration the complex stress conditions due to the variable contact loading.

By knowing the absolute values and range ratios of stresses at various layers in the contact zone, it is now possible to develop an equation for expressing the conditions of strength based on these variable stresses. Normal and shearing stress data, the manner in which they change, and the properties of materials important in fatigue strength determinations are used in arriving at this relationship.

12. Huber-Mises' Condition on Static Loading

According to the opinion of some investigators, there is good correlation between theoretical calculations based on the Huber-Mises' hypothesis and experimental results for the case of plane stress conditions. By this hypothesis, the strength condition for constant loading can be expressed by the equation:

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_s^2 \tag{17}$$

where σ_1 and σ_2 are the principal stresses and σ_s is the yield point of the material. The same condition expressed in terms of the components (see Appendix A) can be written in the form:

$$(\sigma_x + \sigma_y)^2 - 3\sigma_x\sigma_y + 3\tau_{xy}^2 = \sigma_s^2 \tag{18}$$

In order to satisfy the correlation existing between the torsional yield point and the tensile or compressive yield point for different materials, a correction factor should be introduced in the plastic condition (see Sorensen* and Lahmann†). With this correction, Eq. 17 takes the form:

$$(\sigma_1 + \sigma_2)^2 - \left(\frac{\sigma_s}{\tau_s}\right)^2 \sigma_1\sigma_2 = \sigma_s^2 \tag{19}$$

* S. V. Sorensen, "Theory of Strength in Variable Loading," Inst. of Appl. Mech. N. 33, Academy of Sciences of the Ukrainian SSR, Kiev, 1938. (Ukrainian)
† K. Lahmann, Ingenieur, Archiv. H. 3, 1930

or using the components σ_x, σ_y and τ_{xy} (see Appendix B)

$$(\sigma_x + \sigma_y)^2 - \left(\frac{\sigma_s}{\tau_s}\right)^2 (\sigma_x\sigma_y - \tau_{xy}{}^2) = \sigma_s{}^2 \tag{20}$$

13. General Case of Plane Stress Condition Due to Variable Loading

Equation 18 can be applied to the general case of plane stress condition due to variable loading as was shown by Sorensen. In this case, Eq. 18 becomes

$$\begin{aligned}&\left[\left(\frac{\sigma_{xm}}{\sigma_s} + \frac{\sigma_{xv}}{\sigma_{-1}}\right) + \left(\frac{\sigma_{ym}}{\sigma_s} + \frac{\sigma_{yv}}{\sigma_{-1}}\right)\right]^2 \\ &\quad -3\left[\left(\frac{\sigma_{xm}}{\sigma_s} + \frac{\sigma_{xv}}{\sigma_{-1}}\right)\left(\frac{\sigma_{ym}}{\sigma_s} + \frac{\sigma_{yv}}{\sigma_{-1}}\right)\right] + 3\left[\frac{\tau_m}{\sigma_s} + \frac{\tau_v}{\sigma_{-1}}\right]^2 = 1\end{aligned} \tag{21}$$

In this equation the stresses are reduced to a statical form according to Goodman's linear relation as shown in Appendix C. The reduced stresses are

$$(\sigma_x)_r = \sigma_{xm} + \frac{\sigma_s}{\sigma_{-1}}\sigma_{xv}; \quad (\sigma_y)_r = \sigma_{ym} + \frac{\sigma_s}{\sigma_{-1}}\sigma_{yv}; \quad \tau_r = \tau_m + \frac{\tau_s}{\tau_{-1}}\tau_v$$

where

σ_{xv}, σ_{yv} and τ_v are the amplitudes of the alternating parts of the stress components

σ_{xm}, σ_{ym} and τ_m are the steady parts

σ_s and τ_s the yield point and

σ_{-1} and τ_{-1} the endurance limits due to a symmetrical stress cycle ($r = -1$)

By using the reduced stresses, Eq. 20 can be written:

$$\begin{aligned}&\left[\left(\frac{\sigma_{xv}}{\sigma_{-1}} + \frac{\sigma_{xm}}{\sigma_s}\right) + \left(\frac{\sigma_{yv}}{\sigma_{-1}} + \frac{\sigma_{ym}}{\sigma_s}\right)\right]^2 \\ &\quad -\left(\frac{\sigma_s}{\tau_s}\right)^2\left[\left(\frac{\sigma_{xv}}{\sigma_{-1}} + \frac{\sigma_{xm}}{\sigma_s}\right)\left(\frac{\sigma_{yv}}{\sigma_{-1}} + \frac{\sigma_{ym}}{\sigma_s}\right)\right] + \left[\frac{\tau_v}{\tau_{-1}} + \frac{\tau_m}{\tau_s}\right]^2 = 1\end{aligned} \tag{22}$$

For the case of variable stress conditions the extreme value of normal stress components must be put into this equation. Therefore, the normal stress σ_x and σ_z must be put into Eq. 22 to determine the strength of the material in the layer very near the surface. To determine the strength of layer of material further from the surface, where $x \geqq 0.5\, b$, the normal stress components σ_x and σ_y must be used (see Table 1 and Figs. 9–10).

14. Endurance Limit of Steels Subjected to Variable Compressive Stress

There are sufficient data about the endurance limits of steels due to symmetrical alternate loading but not about the endurance limit for the case of the asymmetrical loading cycle. The endurance limit of steels due to variable compressive stress with a zero range ratio is considerably higher than the same limit for tensile stresses. According to R. Thomas' and I. Lowther's data*, steel having a tensile yield point about 37,000 lb per sq in. and tensile strength about 60,000 lb per sq in. has a compressive endurance limit $\sigma_{op} = 48{,}000$ lb per sq in. These results are somewhat conditional because the specimens which had an asymmetrical cross section were tested by repeated bending and the maximum compressive stresses were determined by the expression $\sigma_{op} = M/Z$ where M is the bending moment and Z is the section modulus.

This method would be correct to use if the stresses at the points under consideration were not higher than the yield point. Since this is not the case, it is reasonable to expect a plastic deformation at the points of maximum stresses, and consequently a change in the distribution of the stresses at the cross section of the specimen. Because of this phenomenon, the maximum value of the actual stress in the zone of compression will be smaller than the value calculated from the expression above. Nevertheless, these experiments show that the value of σ_{op} is close to the yield point.

G. Seeger†, H. F. Moore‡, J. O. Smith¶ and some other investigators also considered the problem of fatigue failure under repeated compression. Investigations on the fatigue strength of a carbon steel and a high-strength heated alloy steel by asymmetrical loading cycles were also made by the author§ and were found to have the following properties:

	Carbon Steel	Ni-Cr-Mo Steel
Ultimate tensile strength, σ_u	78 700 psi	257 500 psi
Yield point, $\sigma_{s_{0.2}}$	45 500 psi	172 000 psi
Elastic limit, $\sigma_{0.003}$		152 000 psi
Brinell hardness number, H_B .	155–157	477
Percent elongation, δ_{10}	25.5	5.5
Percentage area reduction, ψ	53.7	14.0
Fatigue strength for symmetrical tensile-compressive cycle, σ_{-1}	25 000 psi	64 000 psi
Fatigue strength for tensile loading from zero to maximum, σ_{ot}	37 700 psi	89 600 psi
Fatigue strength for compressive loading from zero to maximum, σ_{op}	49 200 psi	160 800 psi
Fatigue strength for compression with cycle ratio, $r = +0.2$; $\sigma_{0.2p}$	54 000 psi	

* R. Thomas and I. Lowther, ASTM Proceedings of the Thirty-fifth Annual Meeting, Vol. 32, Part II, June 1932.

† G. Seeger, Wirkung von Drückvorspannung auf die Dauerfestigkeit metallischer Werkstoffe, VDI, Berlin, 1935.

‡ H. F. Moore, Report of the Rails Investigation for the Quarter Ending December 31, 1932. Univ. of Ill. Eng. Exp. Sta.

¶ J. O. Smith, "The Effect of Range of Stress on the Fatigue Strength of Metals," Univ. of Ill. Eng. Exp. Sta. Bul. No. 334, 1942.

§ E. I. Radzimovsky, Report of Institute of Appl. Mech. of Academy of Sciences of Ukrainian SSR, 1940 (Unpublished).

The results of these experiments are shown on the Goodman diagrams in Fig. 11. The steady parts of the stresses are scaled along the abscissa and the limiting stresses along the ordinate.

The above mentioned experiments, and others in the same field, indicate that the fatigue limit of steel due to variable compressive stresses with a zero range ratio is near to the tensile yield point, or somewhat higher. Thus, it is possible to use

$$\sigma_{op} = \sigma_s \tag{23}$$

for calculation purposes.

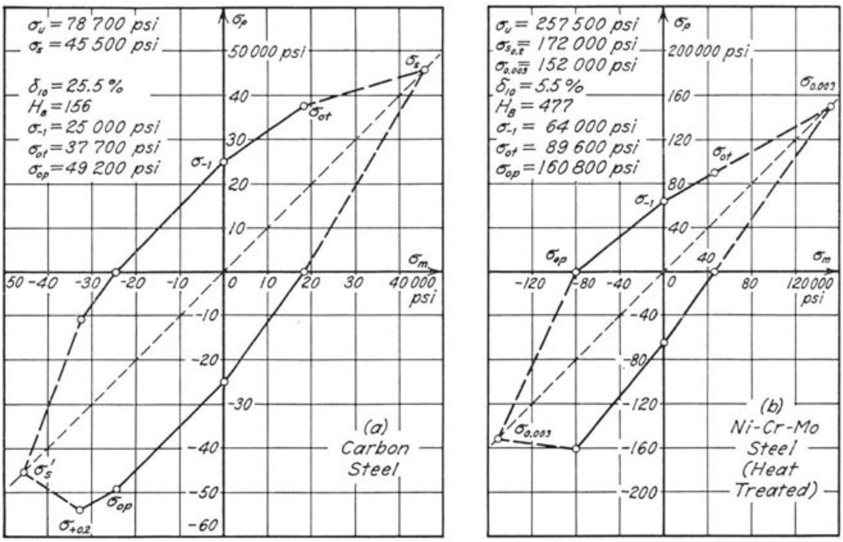

Fig. 11. Strength Limits of Steels Subjected to Asymmetrical Loading Cycles

15. Strength Condition of Materials Subjected to Contact Stresses Produced by Two Cylinders Rolling

As shown in Section 11, the range ratio of the normal stresses σ_x, σ_y and σ_z is $r = 0$, and the range ratio of the stress τ_{xy} is $r = -1$ for the case of two rolling cylinders pressed together. Therefore,

$$\frac{\sigma_{xm}}{\sigma_s} + \frac{\sigma_{xv}}{\sigma_{-1}} = \frac{\sigma_{xm} + \sigma_{xv}}{\sigma_{op}}; \quad \frac{\sigma_{ym}}{\sigma_s} + \frac{\sigma_{yv}}{\sigma_{-1}} = \frac{\sigma_{ym} + \sigma_{yv}}{\sigma_{op}}$$
$$\frac{\sigma_{zm}}{\sigma_s} + \frac{\sigma_{zv}}{\sigma_{-1}} = \frac{\sigma_{zm} + \sigma_{zv}}{\sigma_{op}}; \quad \tau_m = 0; \quad \tau_v = \tau_{xy} \tag{24}$$

In addition

$$\sigma_{xm} + \sigma_{xv} = \sigma_x; \quad \sigma_{ym} + \sigma_{yv} = \sigma_y; \quad \sigma_{zm} + \sigma_{zv} = \sigma_z; \quad \sigma_{op} = \sigma_s \quad (24a)$$

where σ_{op} is the endurance limit on pressure by the range ratio $r = 0$, and σ_s is the tensile yield point. By substituting Eqs. 24 and 24a into Eq. 22, it is simplified considerably to

$$\left(\frac{\sigma_x + \sigma_y}{\sigma_s}\right)^2 - \left(\frac{\sigma_s}{\tau_s}\right)^2 \left(\frac{\sigma_x \sigma_y}{\sigma_s^2}\right) + \left(\frac{\tau_{xy}}{\tau_{-1}}\right)^2 = 1 \quad (25)$$

Equation 25 shows the strength condition of steel in the contact zone of two rolling cylinders. From Table 1, it is evident that values for σ_x and σ_y must be used in the strength calculations for points which are in layers a distance $x \geqq 0.5\ b$ from the surface. For points nearer the surface than 0.5 b, σ_x and σ_z, which are now the extreme stresses, must be used in the calculations. In this case, Eq. 25 must be modified to

$$\left(\frac{\sigma_x + \sigma_z}{\sigma_s}\right)^2 - \left(\frac{\sigma_s}{\tau_s}\right)^2 \left(\frac{\sigma_x \sigma_z}{\sigma_s^2}\right) + \left(\frac{\tau_{xy}}{\tau_{-1}}\right)^2 = 1 \quad (25a)$$

As was shown previously, the various stress components are not in phase. The maximum of the stresses σ_x and σ_z both occur at the plane of symmetry, at which points the stress τ_{xy} is equal to zero. The stress τ_{xy}, on the other hand, reaches its maximum value outside of the plane of symmetry. It would seem that this circumstance could influence the conditions for failure. However, the results of experiments conducted by Sorensen* show that such is not the case. Sorensen tested a steel specimen by a simultaneous repeated loading in bending and torsion. Two series of experiments were conducted. In the first the cycles for bending and torsion loading were in phase. In the second series of tests the cycles were out of phase by 90 deg. The fatigue curves plotted from the data of these tests were the same for both loading conditions. It was concluded that the endurance limit is not affected by the difference in phase relationships of the stress components.

The values of σ_x, σ_y, σ_z and τ_{xy} can be evaluated in terms of $p_{\max}$, which is the maximum pressure on the contact strip, for any point in the contact zone. The value $p_{\max}$ depends upon the magnitude of loading and upon the size, form, and mechanical properties of the materials of the two compressed bodies. It can be calculated using Eqs. 2 and 5.

The limiting value of the maximum pressure at which the initial fatigue failure in the contact zone is expected is the critical value of $p_{\max}$

* S. Sorensen, "About the Strength Conditions in Variable Loadings for the Plane and Volume Stress Condition," Eng. Mag. Inst. Mechan. Acad. of Sciences of USSR, Vol. I, ser. 1, 1941. (Russian)

and is denoted by the symbol σ_g. Since the stresses σ_x, σ_y, σ_z and τ_{xy} at any point in the contact zone are directly dependent upon the value of $p_{\max}$, they can also be expressed in terms of σ_g, as follows:

$$\sigma_x = \alpha\sigma_g; \quad \sigma_y = \beta\sigma_g; \quad \sigma_z = \gamma\sigma_g; \quad \tau_{xy} = \delta\sigma_g \tag{26}$$

Using these relationships, Eq. 25 now can be expressed as

$$\frac{\sigma_g^2}{\sigma_s^2}(\alpha + \beta)^2 - \left(\frac{\sigma_s}{\tau_s}\right)^2 \alpha\beta \frac{\sigma_g^2}{\sigma_s^2} + \delta^2 \frac{\sigma_g^2}{\tau_{-1}^2} = 1$$

from which

$$\sigma_g = \sqrt{\frac{\sigma_s^2}{(\alpha + \beta)^2 - \alpha\beta\left(\frac{\sigma_s}{\tau_s}\right)^2 + \delta^2\left(\frac{\sigma_s}{\tau_{-1}}\right)^2}} \tag{27}$$

Letting $A = (\alpha + \beta)^2$, $B = \alpha\beta$ and $C = \delta^2$, Eq. 27 can be written

$$\sigma_g = \frac{\sigma_s}{\sqrt{A - B\left(\frac{\sigma_s}{\tau_s}\right)^2 + C\left(\frac{\sigma_s}{\tau_{-1}}\right)^2}} \tag{28}$$

or more compactly as

$$\sigma_g = K\sigma_s \tag{29}$$

where

$$K = \frac{1}{\sqrt{A - B\left(\frac{\sigma_s}{\tau_s}\right)^2 + C\left(\frac{\sigma_s}{\tau_{-1}}\right)^2}} \tag{30}$$

The coefficient K in Eqs. 29 and 30 shows the relationship between the yield point, σ_s, of the material and the critical pressure, σ_g, on the surface. The values α, β, and δ, and thus A, B, and C, depend only on the position of the point under consideration, while K and hence σ_g, the critical pressure on the contact surface, depend upon properties of the materials making up the bodies as well as the distance of the layer from the surface.

The critical layer in the contact zone is that one for which the value of σ_g is a minimum. This minimum value of σ_g is called the "surface endurance limit" of two rolling cylinders.

Since the extreme values of normal stress components are required in Eqs. 27-30, the coefficient A is equal to $(\alpha+\beta)^2$ and coefficient B to $\alpha\beta$ for points at a distance of $x \geqq 0.5\ b$; the coefficient A is equal to $(\alpha+\gamma)^2$ and coefficient B to $\alpha\gamma$ for points at a distance of $x < 0.5\ b$.

VI. RESULTS OF INVESTIGATIONS USING THE SUGGESTED STRENGTH CONDITION

Knowing the mechanical properties of the materials, it is possible now to find the critical value σ_g. The steels considered were the soft carbon steels up to the high-strength alloy steels. The mechanical properties of the steels used for the calculations are given in Table 2.

Table 2
Mechanical Properties of Steels

Sample No.	Type of Steel	Mechanical Properties, psi			
		Tensile Yield Point σ_s	Tensile Strength σ_u	Torsional Yield Point τ_s	Endurance Limit in Torsion τ_{-1}
1	Carbon	35 600	64 200	24 200	15 700
2	Carbon	44 200	78 300	27 000	19 900
3	Carbon	59 800	98 300	37 000	27 000
4	Chrome-Nickel	88 300	118 000	54 100	34 000
5	Chrome-Nickel-Vanadium	135 000	156 000	85 300	42 200
6	Chrome-Nickel-Molybdenum	142 200	170 500	86 800	42 700

In Table 3 are tabulated the values of coefficients *A, B,* and *C* for points located at various distances from the surface. The coefficients *K* for the various steels presented in Table 2 are also listed. The data in bold face type in Table 3 are for the points below the surface where σ_g has the smallest value. It is reasonable to expect that the fatigue crack starts in a corresponding layer in the contact zone.

The distance of the critical layer from the surface depends upon the mechanical properties of the steels. Table 3 shows that the critical layer for soft steels is closer to the surface than for the hard steels, but for all steels, these critical layers are closer to the surface than the point of maximum shearing stress.

Table 3
Coefficients A, B, and C and the Ratio, K, Between Critical Pressure on the Surface and Yield Point of Different Steels, for Various Layers Below Contact Surface

Distance from Surface, x	Coefficients			Ratio $K=\sigma_g/\sigma_s$ for Steel No.					
	A	B	C	1	2	3	4	5	6
0.0	2.560	0.600	0.0000	0.889	1.025	1.003	1.018	0.972	1.026
0.1 b	2.365	0.540	0.0213	**0.873**	0.989	0.971	0.965	0.901	0.931
0.2 b	2.161	0.482	0.0380	**0.872**	**0.972**	**0.961**	**0.938**	0.862	0.881
0.5 b	1.528	0.306	0.0655	0.909	0.985	0.976	**0.930**	**0.827**	**0.8205**
0.78 b	0.953	0.1482	0.0543	1.046	1.102	1.095	1.039	0.935	0.929
1.0 b	0.6856	0.0855	0.0475	1.157	1.125	1.199	1.134	1.020	1.008
1.2 b	0.5242	0.0538	0.042	1.275	1.304	1.302	1.227	1.102	1.087
1.5 b	0.367	0.0298	0.0317	1.465	1.502	1.499	1.413	1.388	1.251

These results possibly explain some of the discrepancies among the data of previous investigators who used the roller testing machines in order to study the character of failure and fatigue limit due to the repeated contact stresses arising from two rolling cylinders. In connection with this, the observation of M. Turkish* about failure on the faces of tappets of valve gears should be mentioned. This author stated that the failure in overstressed steel tappet faces originated in the form of microscopic cracks on the surface, but in the case of a chilled cast-iron tappet, a fatigue crack starts beneath the surface.

The values obtained by using Eq. 28 are shown in the diagram of Fig. 12. The abscissa represents the yield points of steels and the ordinate represents the surface endurance limit.

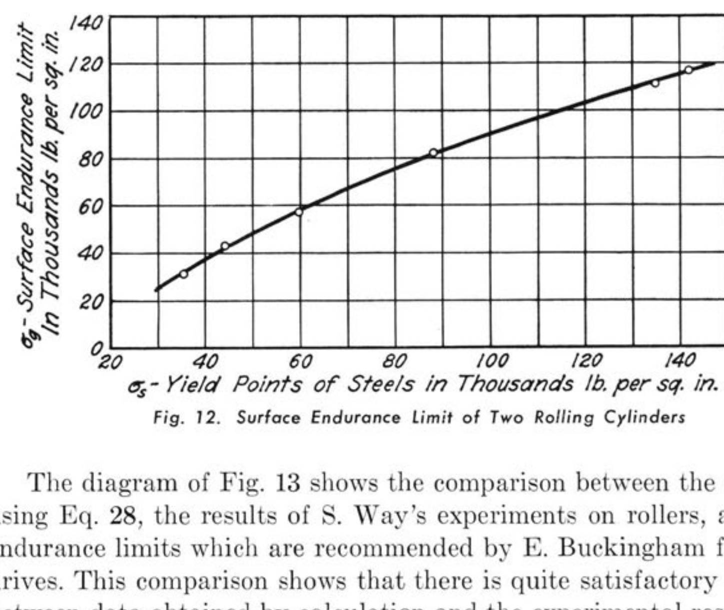

Fig. 12. Surface Endurance Limit of Two Rolling Cylinders

The diagram of Fig. 13 shows the comparison between the data found using Eq. 28, the results of S. Way's experiments on rollers, and surface endurance limits which are recommended by E. Buckingham for the gear drives. This comparison shows that there is quite satisfactory correlation between data obtained by calculation and the experimental results. Buckingham's tables gave relationships between surface endurance limit and the hardness of materials. Because of this, the abscissas in Fig. 13 represent the Brinell hardness. Dëhmer's† relation

$$H_B = \frac{\sigma_u - 4.8}{0.343}$$

where H_B = Brinell hardness number and σ_u = tensile strength of steel in kg per sq mm, was used for reducing the strength of materials to their hardness number.

* M. C. Turkish, loc. cit., page 8.
† P. Dëhmer, Die Brinellsche Kugeldruckprobe, Berlin, 1935.

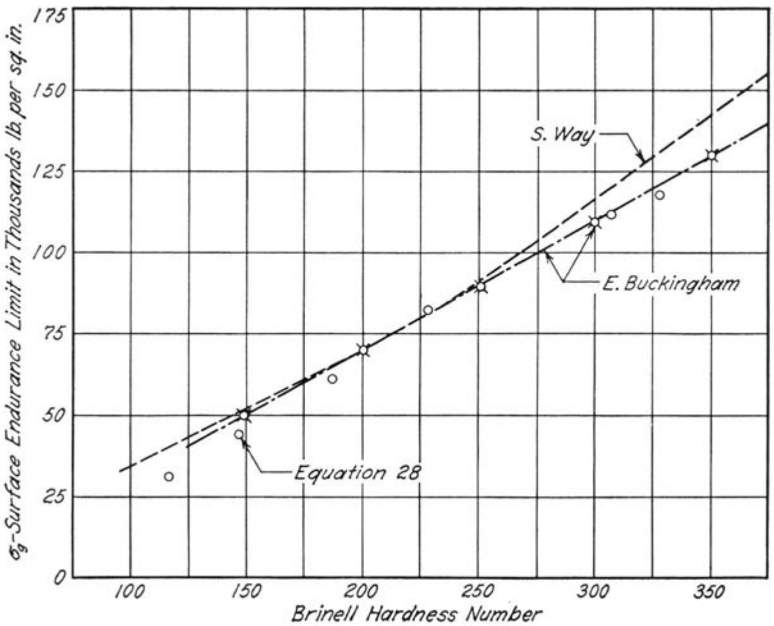

Fig. 13. Comparison Between Data from Eq. 28 and Experimental Results

The method presented in this bulletin for strength calculations does not consider the number of factors which could influence the strength. For example, the presence of a lubricating layer between the two bodies in contact affects, to some degree, the load distribution at the contact surface. The stress gradient in the contact zone which affects the strength of material and the effect of the cold working are also not considered. However, since the other methods currently in use do not consider these factors either, no great importance is attached to their omission. On the other hand the method suggested here does seem to consider more completely the stresses occurring in the contact zone and the manner in which these stresses vary, as well as the properties of the materials upon which the strength of machine members in the contact zone depends.

VII. SUMMARY AND CONCLUSIONS

The strength determination in the contact zone of two compressed cylinders rolling together should take into account the manner in which the stresses vary during a loading cycle, as well as the properties of the materials upon which the strength depends.

Two methods are suggested for determining the manner in which the stresses vary during the loading cycle.

a) The first method consists of substituting a system of concentrated forces for the given load distribution across the contact strip and using the influence lines due to these forces.

b) The second method consists of using Belayev's equations expressed in elliptic coordinates and transforming coordinates; introducing into these equations the locations of the points under consideration determined in Cartesian coordinates; and then determining the values of the stress components at any point in the contact zone, including points outside the plane of symmetry.

The manner in which the stress components change during the loading cycle at the various layers in the contact zone were investigated. The principal results were:

a) The normal stresses σ_x and σ_z change during the loading cycle as follows: $0 \rightarrow \sigma_{\max} \rightarrow 0$, with the maximum absolute value occurring at the plane of symmetry. These stresses approach zero asymptotically.

b) The curves of the normal stress component σ_y has a saddle form symmetrical about the x-axis at distances approximately greater than 0.5 b from the contact surface. In this case, the maximum values of the stress are outside of the plane of symmetry. Therefore, the stress component in these layers passes twice through its maximum value during the loading cycle.

c) The stress $\tau_{45°}$ reaches maximum values in the plane of symmetry at a distance about $x = 0.78\ b$ from the contact surface. This stress has various range ratios, which depend upon the distance of the points from the surface. At the points where $x = 0.78\ b$, the stress $\tau_{45°}$ changes with the range ratio $r \cong -0.25$; the maximum range ratio is about $r = -0.29$ at the layer where $x \cong 0.5\ b$.

d) Based on the strength of materials subjected to repeated loading, the shearing stress τ_{xy} is the critical stress. This stress is equal to zero at the plane of symmetry and changes in a symmetrical cycle with a range ratio $r = -1$. The maximum value of $\tau_{xy} \cong -0.256p_{max}$ occurs in the layer where $x \cong 0.5\ b$. The maximum τ_{xy} does not coincide in phase with the maximum normal stresses and maximum shearing stress.

Calculations based only on maximum normal stresses or on the maximum shearing stress cannot be considered as satisfactory, because of the complicated stress conditions which are present in the case of two compressed cylinders rolling together. The expression for the strength condition based on the Huber-Mises' hypothesis is suggested. The data on stress variations during the loading cycles, as well as the fatigue properties of materials, are used.

A table was made showing the values of critical pressure on the surface in terms of the yield point of different steels for various layers below the contact surfaces of two rolling cylinders. The data in this table which are based on the calculations of dynamic strength of materials differ from the results of calculation in which the contact stress condition was considered as a statical problem. The table shows that the relation between the yield point of steels and the critical pressure on the contact surface is not constant but depends upon the mechanical properties of materials. It shows that the distance of the critical layer from the contact surface also depends upon the mechanical properties of the steels. For the soft steels this layer is closer to the surface than for the hard steels. For all steels this layer is closer to the surface than the point of maximum shearing stress.

Diagrams were made showing the relations between the critical pressure on the surface and the yield points and hardness of the materials. The last diagram shows good correlation with experimental data.

Further experimental verification of the results is desirable. It would be particularly desirable to find experimentally the actual location where the initial fatigue crack begins in the contact zone. Results from such an investigation would help to evaluate the validity of the existing theories of failure for the strength conditions under consideration.

APPENDIX A. TRANSFORMATION OF EQ. 17 IN TERMS OF THE STRESS COMPONENTS

According to the Huber-Mises' hypothesis, the conditions of plastic deformation by statical loading are expressed by Eq. 17 in the text:

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_s^2 \tag{17}$$

where σ_1 and σ_2 are the principal stresses

σ_s is the yield point of the material

For expressing this condition in terms of the stress components we can use the known relationships:

$$\sigma_1 = \frac{1}{2}(\sigma_y + \sigma_x) + \frac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2} \tag{31}$$

and

$$\sigma_2 = \frac{1}{2}(\sigma_y + \sigma_x) - \frac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2} \tag{32}$$

By substituting Eqs. 31 and 32 into Eq. 17, it takes the form:

$$\begin{aligned}\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = &\left[\frac{1}{2}(\sigma_y + \sigma_x) + \frac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2}\right]^2 \\ &+ \left[\frac{1}{2}(\sigma_y + \sigma_x) - \frac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2}\right]^2 \\ &- \left[\frac{1}{2}(\sigma_y + \sigma_x) + \frac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2}\right] \\ &\times \left[\frac{1}{2}(\sigma_y + \sigma_x) - \frac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2}\right] = \sigma_s\end{aligned} \tag{33}$$

Letting

$$\frac{1}{2}(\sigma_y + \sigma_x) = a \tag{34}$$

and

$$\frac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2} = b \tag{35}$$

then by substitution

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = (a + b)^2 + (a - b)^2 - (a + b)(a - b) = a^2 + 3b^2 = \sigma_s^2$$

or

$$\left[\frac{1}{2}(\sigma_y + \sigma_x)\right]^2 + 3\left[\frac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2}\right]^2 = \sigma_s^2$$

Simplifying and collecting terms

$$\sigma_y^2 + \sigma_x^2 - \sigma_x\sigma_y + 3\tau_{xy}^2 = \sigma_s^2$$

or

$$(\sigma_x + \sigma_y)^2 - 3\sigma_x\sigma_y + 3\tau_{xy}^2 = \sigma_s^2 \qquad (18)$$

APPENDIX B. TRANSFORMATION OF EQ. 19 IN TERMS OF THE STRESS COMPONENTS

Equation 19 can be transformed by the same procedure used in Appendix A. It has the form

$$(\sigma_1 + \sigma_2)^2 - \left(\frac{\sigma_s}{\tau_s}\right)^2 \sigma_1\sigma_2 = \sigma_s^2 \tag{19}$$

Using Eqs. 31 and 32, 19 can be written

$$\begin{aligned}\sigma_s^2 = &\left[\frac{1}{2}(\sigma_y + \sigma_x) + \frac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2}\right.\\ &\left.+ \frac{1}{2}(\sigma_y + \sigma_x) - \frac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2}\right]^2\\ &- \left(\frac{\sigma_s}{\tau_s}\right)^2 \left[\frac{1}{2}(\sigma_y + \sigma_x) + \frac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2}\right]\\ &\times \left[\frac{1}{2}(\sigma_y + \sigma_x) - \frac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau_{xy}^2}\right]\end{aligned} \tag{36}$$

By putting Eqs. 34 and 35 into Eq. 36,

$$\sigma_s^2 = [a+b+a-b]^2 - \left(\frac{\sigma_s}{\tau_s}\right)^2 (a+b)\,(a-b) = (2a)^2 - \left(\frac{\sigma_s}{\tau_s}\right)^2 (a^2 - b^2)$$

then by substituting 34 and 35 for a and b again and simplifying,

$$(\sigma_x + \sigma_y)^2 - \left(\frac{\sigma_s}{\tau_s}\right)^2 (\sigma_x\sigma_y - \tau_{xy}^2) = \sigma_s^2 \tag{20}$$

APPENDIX C. REDUCTION OF VARIABLE STRESSES TO STATICAL CONDITIONS

In Eqs. 21 and 22 the stresses are reduced to a statical form according to Goodman's linear relation. The Goodman's linear strength relationship is represented in Fig. 14.

If the stress varies in an asymmetrical cycle with some range ratio r, its maximum value, σ_r, consists of a steady part, σ_m, and an alternating part σ_v. For reducing this variable stress, σ_r, to a statical form, we have to take its steady part, σ_m, and add it to the alternating part σ_v, reduced to the statical condition. For statical loading the strength of the material will be greater than that for the alternate loading by a factor σ_s/σ_{-1}, where σ_s is the yield point and σ_{-1} the endurance limit due to a symmetrical stress cycle. Thus to reduce the part σ_v to the statical condition, we have to multiply the alternating part σ_v by σ_s/σ_{-1}. The reduced stress σ_r therefore takes the form:

$$\sigma_r = \sigma_m + (\sigma_v)_{\text{red}} = \sigma_m + \frac{\sigma_s}{\sigma_{-1}} \sigma_v$$

and

$$\tau_r = \tau_m + \frac{\tau_s}{\tau_{-1}} \tau_v$$

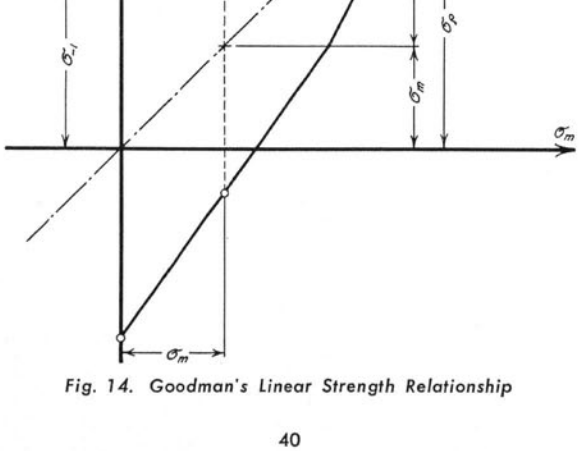

Fig. 14. Goodman's Linear Strength Relationship

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