TESE DE DOUTORAMENTO

QUALITATIVE ANALYSIS OF SOME MODELS OF DELAY DIFFERENTIAL EQUATIONS

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Qualitative analysis of some models of delay differential equations

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Preface/*Prefacio*/*Prefacio*

[EN] This document is the Ph.D. thesis of Sebastián Buedo Fernández. It is based on a major part of the work that the author has done during the period ranging from October 2016 to June 2021 as a student of the Doctoral Programme in Mathematics, at the University of Santiago de Compostela (USC), Spain; under the supervision of Eduardo Liz Marzán (University of Vigo) and Rosana Rodríguez López (USC).

This thesis, presented as a monograph, is mainly written in English and gathers the contents of the articles [17, 18, 19, 20, 21, 83] of the section 'Bibliography'. In compliance with the rules of USC, brief summaries in English (pages xiii–xxii), Galician (pages xxiii–xxxiii) and Spanish (pages xxxv–xlv) are provided and further details of the journals that published the above-mentioned articles are given at the end of the document, in the part named 'Further information'. Finally, also in such part, there is a list of the institutions that have provided funds to the author in order to develop his project, do internships and attend conferences.

[GL] *Este documento é a tese de doutoramento de Sebastián Buedo Fernández. Está baseada nunha gran parte do traballo que o autor realizou durante o período que vai desde outubro do 2016 ata xuño do 2021 como estudante do Programa de Doutoramento en Matemáticas, na Universidade de Santiago de Compostela (USC), España; baixo a dirección de Eduardo Liz Marzán (Universidade de Vigo) e Rosana Rodríguez López (USC).*

Esta tese, en formato monografía, está principalmente escrita en inglés e compila os contidos dos artigos [17, 18, 19, 20, 21, 83] da sección "Bibliography". En cumprimento das normativas da USC, proporciónanse breves resumos en inglés (páxinas xiii–xxii), galego (páxinas xxiii–xxxiii) e castelán (páxinas xxxv–xlv) e danse máis detalles das revistas que publicaron os devanditos artigos ao final do documento, na parte titulada "Further information". Por último, tamén nesa derradeira parte, recóllense as institucións que financiaron ao autor para desenvolver o seu proxecto, realizar estadías e asistir a congresos.

[ES] *Este documento es la tesis doctoral de Sebastián Buedo Fernández. Está basada en una gran parte del trabajo que el autor ha realizado durante el periodo que va desde octubre del 2016 hasta junio del 2021 como estudiante del Programa de Doctorado en Matemáticas, en la Universidad de Santiago de Compostela (USC), España; bajo la dirección de Eduardo Liz Marzán (Universidad de Vigo) y Rosana Rodríguez López (USC).*

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Con todo mi cariño, dedicado a mi padre, a mi madre y a mi hermana

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Una tesis doctoral solamente tiene un autor. Sin embargo, en primer lugar, debo reconocer que habría sido imposible finalizarla sin el apoyo de muchas personas, no solo desde el punto de vista científico, sino también, y más aún, si cabe, desde el ámbito personal. De hecho, me veo incapaz de reflejar en poco espacio todos aquellos gestos, pequeños y grandes, que tantos compañeros, amigos y familiares han hecho por mí durante estos últimos años.

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Cinco cursos de tesis suponen un largo camino, un viaje que se hizo posible gracias a tener una madre que no duda en entregarse plenamente siempre que sus hijos lo necesitan. Además, esta travesía puede ser tan intensa que todas las experiencias vividas ayudan a ir forjándose como persona y a unirse a aquellos que las comparten contigo: me siento muy feliz de que Ana y yo hayamos hecho juntos este peregrinaje doctoral. En muchas de sus etapas me han acompañado muy buenos amigos, pero en casi todas ha estado Daniel, con el que he compartido no solo un despacho, sino muchas vivencias, inquietudes, carcajadas y, cómo no, matemáticas. Quiero destacar también a José María y sus frecuentes palabras de ánimo y fe, que tanto me han sostenido a lo largo de estos años.

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Finalmente, gracias a ti, lector/a, por haberte interesado por mi trabajo.

Sebastián Buedo Fernández
Junio 2021

Summary

This thesis concerns the study of some qualitative properties of delay differential equations that are included within the family

$$x'(t) = -a(t)x(t) + \hat{f}(t, x_t), \tag{1}$$

where $x(t)$ stands for the state of a magnitude x (which may be scalar or vector) at time t and whose evolution, given by $x'(t)$, we are modelling. The notation x_t formally represents the history of x, that is, $x(s)$, for certain times $s \leq t$. Moreover, a and $\hat{f}$ are specific functions whose meaning we comment below.

Equation (1) has been widely considered to model several phenomena whose state affects the evolution of the system in a delayed way. For instance, it has been applied to models concerning population growth [49, 114], cell production [7, 96, 135], economic growth [102, 103], etc. We recommend the books by Kolmanovskii and Myshkis [68], Kuang [72] and Smith [124] to find more details and references for the above-mentioned and further applications.

In many contexts where (1) arises, only nonnegative (or even positive) solutions $x(t)$ of (1) have a real meaning. Therefore, if one assumes that both a is a nonnegative function and $\hat{f}$ is nonnegative for nonnegative solutions, then the terms appearing in the right-hand side of (1) are the *destruction* term $-a(t)x(t)$, which in this case is linear, and the *production* term $\hat{f}(t, x_t)$, which may be nonlinear and gathers the aforementioned delay effects; indeed, the latter term is also commonly known as the *delayed feedback*.

The basic theory on the existence and uniqueness of solutions to delay differential equations, such as (1), may be found in many references. For instance, we highlight the book by Smith [124] for an introduction to this topic and the one by Hale and Verduyn Lunel [58] for a deeper look into such equations. These references, among others, serve as a starting point of the core of this thesis (Section 1.1).

One of the main issues with equations like (1) is that one does not get a complete picture of the long-term behaviour of its solutions by considering them only as a problem defined in $\mathbb{R}^s$. In fact, one of the key ideas towards a full understanding of the dynamics of (1) is to bear in mind the nature of the so-called *segment* x_t. In many models, x_t is

just a vector function defined on a compact real interval $[-\tau, 0]$, that is, a function that gathers the information of the solution in the last time interval of length τ; this is the case of *finite delay.* Clearly, in order to endow equation (1) with sense, one has to consider initial conditions written in terms of the segments. Hence, instead of just considering the problem in the Euclidean space of dimension $s \in \mathbb{N}$, we shall also bear in mind how the segments x_t evolve as time goes by, i.e., the evolution of the system interpreted in a certain functional space. Then, we can recover many relevant facts from the theory of dynamical systems (see Section 1.2). Nonetheless, that requires working with an infinite-dimensional space.

In the first part of the thesis (Chapters 2 and 3), we mainly deal with a family of scalar delay-differential models as (1) having a constant decay rate $a(t) = a$ and a single discrete delay, i.e., with $\hat{f}(t, x_t) = f(x(t - \tilde{\tau}(t)))$, for certain real functions $f, \tilde{\tau}$ of a real variable. Under such assumptions, equation (1) may be written as

$$x'(t) = -ax(t) + f(x(t - \tilde{\tau}(t))). \tag{2}$$

One particular case of (2) that has become relevant in terms of the literature [71, 79] is the one of constant delay, i.e., the one satisfying $\tilde{\tau}(t) = \tau$. Such assumption yields that (2) takes the form

$$x'(t) = -ax(t) + f(x(t - \tau)). \tag{3}$$

The family we are interested in for the first part of this thesis is the one of the *gamma-models* [81, 114], which are a family of models that include equation (2) with

$$f(x) = x^{\gamma} h(x), \tag{4}$$

for $\gamma \geq 0$ and a nonincreasing function $h : [0, \infty) \to (0, \infty)$. Several well-known models can be included under the framework of (2) with (4) as its feedback [49, 96, 135]. The meaning of γ depends on the model. For instance, it modules the level of cooperation [81] in population dynamical models, the role of the capital on the commodity production in the neoclassical model of economic growth [103]; etc.

Our goal is to obtain sufficient conditions to ensure that equation (2) with (4) has a globally attracting equilibrium $p \in (0, \infty)$, which means that, roughly speaking, all its solutions must be defined for future times and converge to p as $t \to \infty$. To do so, we devote several pages (Section 1.3) to recall some insights on how equation (2), without necessarily imposing the particular case of the production function (4), may be approached. However, for the sake of simplicity regarding our explanations, we firstly do the latter for its particular case (3), and, finally, we explain what results can directly be extended to (2). Furthermore, taking into account this objective, it is natural to impose conditions of uniqueness of an equilibrium for (3), as we also do.

The following three items are relevant to study (3): the destruction rate a, the feedback f and the delay τ. The relation existing among those three objects determines the long-term behaviour of the solutions of (3). For instance, the relation f/a plays a fundamental role. Indeed, the asymptotic behaviour of the solutions of the corresponding difference equation

$$x_{n+1} = \frac{f(x_n)}{a} =: F(x_n), \tag{5}$$

that is, the long-term behaviour of the sequences

$$x_0,\ F(x_0),\ F(F(x_0)),\ F(F(F(x_0))),\ \ldots, \tag{6}$$

has direct implications in the corresponding behaviour of the solutions of (3). This fact is derived from the works by Mallet-Paret and Nussbaum [98] and by Ivanov and Sharkovsky [65]. Roughly speaking, they proved that, under desirable conditions (see Theorem 1.33), if (5) has a globally attracting equilibrium p, then p is a globally attracting equilibrium for (3) too. In other words, the existence of p such that any sequence as (6) tends to p implies that any solution of (3) tends to p as $t \to \infty$. This type of result is actually crucial to the study of the dynamics of (3): if one is able to obtain global asymptotic stability conditions for (5), then they are directly transferred to (3).

Although analysing the global attractivity of an equilibrium for (5) seems easier than doing the same for (3), it is not as trivial as, for instance, the study of its local asymptotic stability. Nevertheless, there exist families of maps F that satisfy the so-called property *'LAS implies GAS'* (acronym for Locally/Globally Asymptotically Stable). As its name suggests, it refers to a situation for which, if an equilibrium is locally attracting, then it is also globally attracting [43]. One of those families of maps, which is widely found in many applications [79], are those having at most one critical point and negative Schwarzian derivative (we refer to [67] for a historical introduction to this concept). Particularly, for a real map F of class $\mathcal{C}^3$ defined on a real interval I, the *Schwarzian derivative* of F is defined by

$$SF(x) = \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left(\frac{F''(x)}{F'(x)}\right)^2,$$

for all $x \in I$ such that $F'(x) \neq 0$. The results of Allwright [2] and Singer [122] originally highlighted the role of the negative Schwarzian derivative on the link between local and global dynamics, while a generalization was later provided by El-Morshedy and Jiménez López [33] (we recall it in Theorem 1.55), in which they consider such negativity hypothesis only on a certain subset of the domain of SF. In such cases, if a regular map F that has at most one critical point, also has a unique fixed point p such that $F'(p) \in [-1, 1)$, then p is globally attracting for (5). Besides, if we bear in mind the notation in (3) and (5), the information of the item f in the triple (a, f, τ) would be 'summarised' in the value $f'(p)$ and such a global asymptotic stability condition would read as

$$f'(p) \in [-a, a). \tag{7}$$

It is worth highlighting that the delay τ does not appear in condition (7), since it is obtained via (5), which is independent from the value τ. Thus, condition (7) is called a *delay-independent stability condition* or an *absolute stability condition* [124].

Afterwards, Győri and Trofimchuk [54] provided, under a negativity-type assumption on Sf, an extension of (7) by using a modified version of (5) that incorporates τ as part of a coefficient, and they obtained the *delay-dependent global stability condition*

$$f'(p) \in [-a, a) \quad \text{or} \quad \left[f'(p) < -a \text{ and } \tau \leq \frac{1}{a} \ln \left(\frac{f'(p)}{f'(p) + a} \right) \right]. \tag{8}$$

Within the quest of better global stability conditions for the case of feedbacks having a negative Schwarzian derivative, Liz et al. [86] provided an improvement of (8), which turned out to be the best possible for (2), but, in the particular case of (3), still leaves a tiny region before reaching the border of the set of parameters for which the equilibrium is locally attracting, which is given by

$$f'(p) \in [-a, a) \quad \text{or} \quad \left[f'(p) < -a \text{ and } \tau < \frac{\arccos\left(\frac{a}{f'(p)}\right)}{\sqrt{f'(p)^2 - a^2}} \right]. \tag{9}$$

We devote some pages to summarise those efforts and to visually explain the regions of parameters $(a, f'(p), \tau)$ that satisfy the previously mentioned stability conditions (see Remark 1.64 and Figure 1.8).

Having recalled the main properties of (2) (with a particular interest on (3)), we focus on the part of our project [17, 20, 83] that is devoted to the gamma-models, which, as said before, include equation (2) together with (4). In particular, we justify the use of $\gamma \in [0, 1]$ as a parameter that provides flexibility to many models and links several couples of different well-known equations. For instance, the two delay differential equations proposed by Mackey and Glass [96] to study a model concerning the hematopoiesis are connected under our approach.

In order to study the global dynamics of the gamma-models via the tools recalled above, we provide results showing which features a general feedback of the type of (4) with $\gamma \in [0, 1]$ provides to (2). For instance, the existence of a unique positive equilibrium p (or a unique constant positive solution), the way in which γ influences the value of p and crucial information about the dynamics of the delay differential equation in a neighbourhood of

p can be obtained through the analysis of its corresponding difference equation (Theorem 2.1). Later, we study whether the known stability conditions (7), (8) and (9) can be applied to several gamma-models that are of interest due to their applications. In the following paragraphs, we describe our main findings.

The first example is based on a version of the *Solow's neoclassical model of economic growth* [126], where x represents the *capital-labour rate* x, which is a quantity that relates the capital stock (e.g., machines) and the labour's force (workers, number of hours worked, etc.). The utilisation of the delay is explained as the time required to produce a commodity, as highlighted by Matsumoto and Szidarovszky [102]. If the production of such commodity is ruled by the Cobb-Douglas function [10, 126], we have $h(x) = \beta > 0$ for the feedback (4), so equation (2) takes the form

$$x'(t) = -ax(t) + \beta x^{\gamma}(t - \tilde{\tau}(t)). \tag{10}$$

In [20], we find that the unique equilibrium for (10) is globally asymptotically stable regardless of the value of the parameters $a, \beta > 0$, $\gamma \in (0, 1)$, and the bounded continuous delay $\tilde{\tau}(t)$ (see Theorem 2.7). Bearing in mind the context of the previous model, the time-dependent delay is harmless with respect to the eventual achievement of the steady-state capital-labour rate p.

The remaining cases also find applications when it comes to the Solow's model. They are based on the incorporation of a pollution factor in the feedback f, produced by large concentrations of capital [27, 102, 103], that is mathematically represented as a decreasing function. Hence, in the second case, we assume that $h(x) = \beta e^{-\delta x}$, with $\beta, \delta > 0$, in (4), so equation (2) is written as

$$x'(t) = -ax(t) + \beta x^{\gamma}(t - \tilde{\tau}(t))e^{-\delta x(t - \tilde{\tau}(t))}. \tag{11}$$

This is referred to as the *Lasota equation* due to its use for modelling the production of a certain type of blood cells in a paper by Lasota [74], yet it can also be interpreted as a *gamma-version of the Nicholson's blowflies equation*, utilised by Gurney et al. [49] to model the amount of individuals grown in lab colonies.

Regarding (11), the classical approach by Allwright and Singer does not work since the feedback f does not have negative Schwarzian derivative. Nevertheless, we can still apply the generalised version by El-Morshedy and Jiménez López to derive (Theorem 2.12) that the unique equilibrium of the difference equation (5) related to (11) is GAS provided it is LAS, that is, if

$$\frac{\beta}{a} \leq e^{\gamma+1} \left(\frac{\gamma + 1}{\delta} \right)^{1-\gamma}.$$

The latter condition is the sharpest delay-independent global stability condition for (11) and it is obtained from the analysis of its corresponding difference equation, which was developed by Liz [81]. Moreover, if the delay is constant, such condition can be refined by using the estimates provided by Győri and Trofimchuk to obtain (see also Theorem 2.12) the delay-dependent global stability condition

$$\frac{\beta}{a} \leq e^{\gamma+\frac{1}{1-e^{-a\tau}}} \left(\frac{\gamma + \frac{1}{1-e^{-a\tau}}}{\delta} \right)^{1-\gamma}.$$

Thus, we extend the study of the local dynamics of (11) by Matsumoto and Szidarovszky [103] and provide a link between the global dynamics for the cases $\gamma = 0$, provided by Győri [50], and $\gamma = 1$, described by Győri and Trofimchuk [54]. Additionally, with the support of [81], we also show that γ plays a subtle role in the stability of the unique equilibrium of (11). For example, there exist cases for which increasing γ may generate a couple of stability switches (see Figure 2.1).

The third one is the γ*-logistic delay differential equation*, related with the choice $h(x) = \beta(1-x)$, with $\beta > 0$, in (4). Hence, equation (2) is now written as

$$x'(t) = -ax(t) + \beta x^{\gamma}(t - \tilde{\tau}(t))(1 - x(t - \tilde{\tau}(t))). \tag{12}$$

This gamma-version of an equation with a logistic production function is a particular case of (2) that requires some preliminary remarks in order to become manageable by the techniques commented above. For instance, it turned out that one has to restrict the study to solutions with values in $(0,1)$ and impose a consistency condition on the parameters (see (2.39)). Under such constraints, we obtain (Theorem 2.22) that the unique equilibrium is GAS provided

$$\frac{\beta}{a} \leq (\gamma+1) \left(\frac{\gamma+2}{\gamma+1} \right)^{\gamma},$$

which, in an analogous way to the situation for the Lasota equation, constitutes the sharpest delay-independent global stability condition for (12). Furthermore, if the delay is constant, then the estimates by Győri and Trofimchuk are valid and we provide (also in Theorem 2.22) a delay-dependent global stability condition that improves the last one, and it is written as

$$\frac{\beta}{a} \leq \left(\gamma + \frac{1}{1-e^{-a\tau}} \right) \left(\frac{\gamma + 1 + \frac{1}{1-e^{-a\tau}}}{\gamma + \frac{1}{1-e^{-a\tau}}} \right)^{\gamma}.$$

Our global stability results for $\gamma \in [0,1]$ extend the study of the local dynamics of (12) by Matsumoto and Szidarovszky for $\gamma = 1$ [103].

In the fourth example, we assume that $h(x) = \frac{\beta}{1+\delta x^m}$, with $\beta, \delta, m > 0$, in (4). Then, one can rewrite (2) as

$$x'(t) = -ax(t) + \frac{\beta x^{\gamma}(t - \tilde{\tau}(t))}{1 + \delta x^m(t - \tilde{\tau}(t))}, \tag{13}$$

and we refer to the former equation as the γ-*Mackey-Glass delay differential equation*, in analogy with the couple of equations proposed by Mackey and Glass [96] to represent a model of blood cell generation. The analysis of (13) seems trickier than the former cases, since we could not find easy computations that lead to an application of the result by El-Morshedy and Jiménez López. However, in the particular case of $m = 1$, from the work by Liz [82] on the corresponding difference equation, where the Coppel's criterion [24] is used, the global stability of (13) is a straightforward consequence (Theorem 2.28). Nonetheless, no proper way to adapt such reasoning to the case $m \neq 1$ was found. Thus, a new Schwarzian-type formula was proposed with the aim of handling the property 'LAS implies GAS'. This is done in Chapter 3, which arises as a branch from Chapter 2 at this point. Namely, if in the corresponding difference equation (5) we denote $F(x) = f(x)/a = x^\gamma H(x)$, then our condition (see (3.3)), written in the form suggested by Jiménez López, is

$$SH(x) < \frac{H(x) - xH'(x)}{2(xH(x))^2}(xH(x))', \quad \text{for all } x > 0. \tag{14}$$

One of the principal strengths of formula (14) is that it does not take into account the Schwarzian derivative of the map F, but only of one of its factors, namely H, which helps simplifying the computations for several models. In fact, this formula comes from a logarithmic change of variables [78] that transforms the product $x^\gamma H(x)$ into a sum $\gamma x + H^*(x)$, where H^* is written in terms of H. Additionally, whenever $SH^* < 0$, it is possible to apply an enveloping-type result proved by Liz et al. [85] to H^* and derive, after undoing the change of variables, that the difference equation (5) related to (13) satisfies the property 'LAS implies GAS'. In other words, we obtain (Theorem 3.1) that, under condition (14), p is GAS for (5) provided it is LAS. In particular, if $F(x) = \frac{\beta x^\gamma}{a(1+\delta x^m)}$ in (5), the former conclusion is obtained provided

$$m \leq 1 + \gamma \quad \text{or} \quad \left[m > 1 + \gamma, \quad \frac{\beta}{a} \leq \frac{m}{m - 1 - \gamma} \left(\frac{\gamma + 1}{\delta(m - 1 - \gamma)} \right)^{\frac{1-\gamma}{m}} \right]. \tag{15}$$

Indeed, condition (15) turns out to be the sharpest delay-independent global stability condition for the equation (13) (Theorem 3.13). Furthermore, we are able to link some known estimates concerning the global dynamics of equation (13) for the particular cases of $\gamma = 0$ and $\gamma = 1$ by Gopalsamy et al. [44].

Besides, the reasoning involved in the main result of Chapter 3 (Theorem 3.1) can be slightly modified in order to provide an extension for the case $\gamma = 0$ and unimodal maps H (recall that they were originally assumed to be nonincreasing). Such result (see Theorem 3.5) is analogous to the classical ones given by Allwright and Singer, but in this case based on formula (14) instead of the inequality $SH < 0$.

We conclude the part related to globally attracting equilibria in gamma-models by studying the equation

$$x'(t) = -ax(t) + s(x(t))f(x(t - \tilde{\tau}(t))), \tag{16}$$

where $f(x) = Bx^{\gamma}$, with $B > 0$, which cannot be included under the framework of (2). Such an equation also appears in the context of the Solow's neoclassical model of economic growth by considering more general assumptions. Indeed, if the function s has certain appropriate properties coherent with its meaning (s is the so-called saving rate [10, 126]), then we are able to show (Theorem 2.34) that (16) has a globally attracting equilibrium by relating its dynamics to another difference equation, more general than the one in (5). Its proof is based on a generalisation (Theorem 2.32) of a result by Ivanov et al. [64] and, indeed, we use a simpler reasoning than that of [64] since we rely on a powerful feature of global attractivity in one-dimensional maps, namely the *strong attractivity* [90].

Later, in Chapter 4, we shift our focus to equation (1) in a multidimensional context. In particular, we propose the higher dimensional extension of (3) given by

$$x_i'(t) = -x_i(t) + f_i(x_1(t - \tau_{i1}), \dots, x_s(t - \tau_{is})), \quad i = 1, \dots, s, \tag{17}$$

and study whether there exists a multidimensional difference equation from which we can derive information about the long-term behaviour of the solutions of (17), as it happened with the relation between (3) and (5) in the scalar case. In fact, if we denote by $x_{\cdot,n} = (x_{1,n}, \dots, x_{s,n})$ the general term of a sequence in $\mathbb{R}^s$, then the natural candidate to be the difference equation corresponding to (17) is

$$x_{i,n+1} = f_i(x_{1,n}, \dots, x_{s,n}), \quad i = 1, \dots, s. \tag{18}$$

Nonetheless, the problem is certainly subtle in higher dimensions, as shown by Liz and Ruiz-Herrera [90, 91]. In fact, if the map $f = (f_1, \dots, f_s)$ in (18) has a unique fixed point p, then, assuming that p is globally attracting for f is not sufficient to ensure the analogous property for the system of delay differential equations (17) [91]. In other words, some stronger condition needs to be imposed to the multidimensional difference equation (18). The authors of [90, 91] proposed a concept, namely the *strong attractor* (it is recalled in Definition 4.1), which reveals the main features that turned out to be essential to prove the relation between those two equations in the scalar case $s = 1$. Notice that, as we remarked, this concept is also useful in the study of the scalar equation (16). The key idea is to assume the existence of certain families of nested Cartesian products of compact intervals that behave well under the application of the map f in (18). Thus, if an equilibrium of (18) is a strong attractor, which is a more restrictive notion than the global attractor if the dimension is $s \geq 2$, then the equilibrium p is globally attracting for (17) (we recall their result in Theorem 4.2).

Nevertheless, a profound look into the main reasoning shown in [90] allows us (see [18]) to extend it to a more general context. We introduce a new concept, specifically a *CC-strong attractor* (see Definition 4.3), which also reveals the features needed to relate the dynamics of (18) and (17) in the scalar case. Indeed, this concept turns out to be an intermediate notion between a global attractor and a strong attractor in the sense of [90], provided $s \geq 2$ (see our Remark 4.7). The main difference from the concept proposed by Liz and Ruiz-Herrera is that we 'weaken' the geometry of the sets involved: we do not need nested families of Cartesian product of compact intervals, it is sufficient to consider certain nested families of compact and convex sets. Therefore, the weaker assumption of (18) having a CC-strong attractor is still sufficient to prove that p is also globally attracting for (17). The latter constitutes the main result of this part of the thesis (Theorem 4.5).

Furthermore, the next part corresponds to Chapter 5. Here, we consider a scalar version of (1) with $\hat{f}$ being linear in the second variable, but now allowing the decay coefficient a and the coefficients of $\hat{f}$ to be non-constant, but depend on t. For instance, under such assumption, one of the simplest cases is the one of a single discrete variable delay, represented by

$$x'(t) = -a(t)x(t) + \beta(t)x(t - \tilde{\tau}(t)), \tag{19}$$

with β being a nonnegative function. The asymptotic behaviour of equation (19) is well-known if $a, \beta, \tilde{\tau}$ are constant, as it is deduced from the analysis of the *characteristic equation* [124]. In such a case, sufficient conditions to ensure that the origin is globally attracting are available in, e.g., [58]. Nevertheless, in the general case of (19), the techniques may change, and we could even deal with a model satisfying $\tilde{\tau}(t) \to \infty$ as $t \to \infty$, which is a case that represents a 'long-term unboundedness' of the delay. For example, the works by Győri and Horváth [52, 53] handle such kind of equations and provide sharp estimates for their solutions by working with an inequality of the type

$$x(t) \leq c + \int_{\sigma}^{t} b(s)x(s - \tilde{\tau}(s))\, ds.$$

The main ingredients that they use are the so-called *characteristic inequality* and an estimate of Gronwall-Bellman-type. The characteristic inequality is a generalisation of the widely-known characteristic equation for the case of constant functions $a, \beta, \tilde{\tau}$ (see, e.g., Theorems 2.7 and 5.1 in [53]). We show (Section 5.4) that their ideas can be adapted to equations more general than (19), such as

$$x'(t) = -a(t)x(t) + \beta(t)x(\nu(t)) + \int_{\sigma}^{\nu(t)} \eta(t,s)x(\nu(s))\, ds, \tag{20}$$

where $\nu(t) := t - \tilde{\tau}(t)$ and η is a nonnegative function that is nondecreasing in the first variable. By working with the equation (20), one may arrive to an integral inequality of the type

$$x(t) \leq c + \int_{\sigma}^{t} k(t,s)x(s-\tilde{\tau}(s))\,ds. \tag{21}$$

Similarly, the characteristic inequality needs to be generalised, as we show in [21], and the main result of this part of our work (Theorem 5.6), which also deals with Gronwall-Bellman estimates, generalises parts of the main conclusions in [52]. Additionally, it serves to prove that the origin is a globally attracting solution for some equations of the form (20). Moreover, we also extend a result by Norbury and Stuart [110] since, in the case of (21), they assume $\tilde{\tau} \equiv 0$.

In Chapter 6, we consider once more the scalar version of (1), but now with periodic coefficients and also allowing the solutions to have discontinuities produced by impulses at given certain times t_k, $k \in \mathbb{N}$. In other words, we consider the equation

$$\begin{cases} x'(t) = -a(t)x(t) + \hat{f}(t,x_t), \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), \end{cases} \tag{22}$$

where a and $\hat{f}$ are periodic functions in t, with the same period $\omega > 0$, the impulse functions $I_k : [0,\infty) \to \mathbb{R}$ follow a pattern of time length ω, and for which some other technical conditions are imposed in order to ensure that the solutions relative to nonnegative segments are nonnegative (see Section 6.2). Faria and Oliveira [40] studied the existence of positive ω-periodic solutions of (22) with finite delay and linear impulse functions. Our main result (Theorem 6.8), whose basic ingredients are a version of the Krasnoselskii Fixed Point Theorem [48, 70] and the imposition of a certain behaviour on $\hat{f}$ for 'close-to-zero' and 'sufficiently large' solutions, yields conclusions analogous to those in [40]. In [19], we generalise their assumptions in several ways, e.g., by including more general impulse functions I_k (whose signs may change) or via the consideration of *infinite delay* or a wider class of feedbacks. We also provide several corollaries that are useful for applications. In the last part of Chapter 6, we apply our results to integro-differential equations with infinite or periodic distributed delay. We also establish links with several results available in the literature.

As a conclusion, we provide a general overview of the results of the thesis by highlighting several lines of future work that may be of interest.

Resumo

Esta tese céntrase no estudo dalgunhas propiedades cualitativas das ecuacións diferenciais con retardo incluídas na familia

$$x'(t) = -a(t)x(t) + \hat{f}(t, x_t), \tag{23}$$

onde $x(t)$ se refire ao estado dunha magnitude x (que pode ser escalar ou vectorial) no tempo t e cuxa evolución, determinada mediante $x'(t)$, procuramos modelar. A notación x_t representa formalmente a historia de x, isto é, os valores $x(s)$, para certos tempos $s \leq t$. Ademais, a e $\hat{f}$ son certas funcións cuxo significado comentaremos deseguido.

A ecuación (23) foi amplamente considerada para a modelaxe de varios fenómenos cuxo estado presente afecta á evolución do sistema de xeito retardado. Por exemplo, foi usada en modelos de crecemento poboacionais [49, 114], de produción celular [7, 96, 135], de crecemento económico [102, 103], etc. Para atopar máis detalles e referencias sobre estas e outras aplicacións, recoméndanse os libros de Kolmanovskii e Myshkis [68], Kuang [72] e Smith [124].

En moitos contextos nos que a ecuación (23) é de interese, só as solucións $x(t)$ de (23) que sexan non negativas teñen un significado real. Así, se asumimos que a é unha función non negativa e $\hat{f}$ é non negativa para solucións non negativas, entón os termos do membro dereito de (23) son coñecidos como o termo de *destrución* $-a(t)x(t)$, que neste caso é linear, e o termo de *produción* $\hat{f}(t, x_t)$, que pode ser non linear e reúne aqueles efectos que inflúen na evolución do sistema de maneira retardada, como mencionamos anteriormente. Concretamente, o último termo é tamén coñecido como o *feedback retardado*.

A teoría básica de existencia e unicidade de solucións para as ecuacións diferenciais con retardo, como a que expresamos en (23), pódese consultar en multitude de referencias. A modo de exemplo, destacamos o libro de Smith [124] para unha introdución a este tema e o de Hale e Verduyn Lunel [58] para unha ollada máis profunda a estas ecuacións. Estas referencias, entre outras, serven para dar comezo ao núcleo desta tese (Sección 1.1).

Unha das principais problemáticas das ecuacións do tipo (23) é que non se pode obter unha descrición completa do comportamento a longo prazo das súas solucións se só as consideramos como un problema en $\mathbb{R}^s$. En particular, unha das ideas chave cara a unha comprensión total da dinámica relativa a (23) é ter en conta a natureza do *segmento* x_t. En moitos modelos, x_t é simplemente unha función vectorial definida nun intervalo real compacto $[-\tau, 0]$, isto é, unha función que compila a información da solución de (23) no último intervalo temporal de lonxitude τ; este é o caso das ecuacións con *retardo finito*. Claramente, para dotar de sentido á ecuación (23), cómpre considerar condicións iniciais que incorporen aos segmentos. Por isto, en vez de considerar o problema só no espazo euclidiano de dimensión $s \in \mathbb{N}$, tamén se debe ter en conta a evolución dos segmentos x_t, é dicir, a evolución do sistema nun certo espacio funcional. Deste modo, poderemos recuperar moitos feitos relevantes da teoría dos sistemas dinámicos (ver Sección 1.2). No entanto, iso provoca que teñamos que traballar cun espazo de dimensión infinita.

Na primeira parte da tese (Capítulos 2 e 3), traballamos principalmente cunha familia de ecuacións escalares do tipo (23) asociada a unha función de decaída linear constante $a(t) = a$ e un único retardo discreto, isto é, con $\hat{f}(t, x_t) = f(x(t - \tilde{\tau}(t)))$, para certas funcións reais $f, \tilde{\tau}$ de variable real. Baixo tales hipóteses, a ecuación (23) pode ser escrita como

$$x'(t) = -ax(t) + f(x(t - \tilde{\tau}(t))). \tag{24}$$

Un caso particular de (24) que adquiriu relevancia na literatura [71, 79] é o de retardo constante, é dicir, aquel no que $\tilde{\tau}(t) = \tau$. Asumir o anterior implica que (24) toma a forma

$$x'(t) = -ax(t) + f(x(t - \tau)). \tag{25}$$

A familia de ecuacións sobre a que recae o noso interese nesta primeira parte da tese é a dos chamados *modelos gamma* [81, 114], que son unha familia de modelos que incorporan á ecuación (24) coa escolla

$$f(x) = x^{\gamma} h(x), \tag{26}$$

sendo $\gamma \geq 0$ e $h : [0, \infty) \to (0, \infty)$ unha función monótona decrecente. Algúns modelos ben coñecidos pódense incluír no contexto de (24) co *feedback* (26) [49, 96, 135]. O significado de γ depende do modelo. Por exemplo, modula o nivel de cooperación [81] no eido de crecemento de poboacións, o rol que xoga o capital dispoñible na produción dunha mercadoría [103] no modelo neoclásico de crecemento económico, etc.

O noso obxectivo é obter condicións suficientes que garantan que a ecuación (24) co *feedback* (26) teña un equilibrio $p \in (0, \infty)$ que sexa globalmente atractor, o que vén a significar, grosso modo, que todas as súas solucións estean definidas para tempos futuros e converxan a p cando $t \to \infty$. Para cumprir dita meta, dedicamos algunhas páxinas (Sección 1.3) a lembrar algunhas claves sobre como se pode abordar o estudo da ecuación (24) sen necesariamente impoñer o caso particular da función de produción (26). Porén, por cuestións de simplicidade nas explicacións, facémolo, en primeiro lugar, para o caso de retardo constante (25) e, finalmente, explicamos que resultados se extenden directamente ao caso de (24). Ademais, tendo en conta esta meta, é natural impoñer condicións de existencia e unicidade de equilibrio para (25), como tamén facemos.

Os tres seguintes obxectos tórnanse relevantes para estudar a ecuación (25): o parámetro de destrución a, o *feedback* f e o retardo τ. A relación existente entre os anteriores elementos determina o comportamento a longo prazo das solucións de (25). De xeito particular, a relación f/a ten un papel fundamental. É máis, o comportamento asintótico das solucións da correspondente ecuación en diferenzas

$$x_{n+1} = \frac{f(x_n)}{a} =: F(x_n), \tag{27}$$

isto é, o comportamento a longo prazo das sucesións

$$x_0,\ \ F(x_0),\ \ F(F(x_0)),\ \ F(F(F(x_0))),\ \ \ldots, \tag{28}$$

ten implicacións directas no comportamento correspondente das solucións de (25). Este feito despréndese dos traballos de Mallet-Paret e Nussbaum [98] e de Ivanov e Sharkovsky [65]. A grandes rasgos, eles probaron que, baixo condicións apropiadas (ver Teorema 1.33), se (27) ten un equilibrio p globalmente atractor, entón p é tamén un equilibrio globalmente atractor para (25). Noutras palabras, a existencia dun p tal que toda sucesión do tipo (28) tende a p implica que calquera solución de (25) tende a p cando $t \to \infty$. Este tipo de resultados son realmente cruciais para o estudo da dinámica de (25): se un é quen de obter condicións de estabilidade asintótica global para (27), entón estas son tamén válidas para (25).

Aínda que analizar se un equilibrio é globalmente atractor para a ecuación (27) semella máis doado que facelo para (25), non é tan trivial como, por exemplo, si sería o estudo da súa estabilidade local. Secasí, existen familias de funcións F que satisfacen a propiedade chamada *"LAS implica GAS"* (polas súas siglas en inglés "Locally/Globally Asymptotically Stable", é dicir, localmente/globalmente asintóticamente estable). Como o seu nome indica, reflexa unha situación na que, se un equilibrio é localmente atractor, entón tamén o é globalmente [43]. Unha das devanditas familias, presente en moitas aplicacións [79], é a das funcións que teñen como moito un punto crítico e derivada de Schwarz negativa (remitimos ao artigo de Jiménez López e Parreño [67] para unha introdución histórica deste concepto). En particular, para unha función real F de clase $\mathcal{C}^3$ definida nun intervalo real I, a *derivada de Schwarz* de F defínese como

$$SF(x) = \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left(\frac{F''(x)}{F'(x)}\right)^2,$$

para todo $x \in I$ tal que $F'(x) \neq 0$. Os resultados de Allwright [2] e Singer [122] destacaron por vez primeira o papel da derivada de Schwarz negativa na ligazón entre as dinámicas local e global, mentres que El-Morshedy e Jiménez López [33] proporcionaron unha posterior xeneralización (lembrámola no Teorema 1.55), onde eles asumen tal negatividade de SF só nun certo subconxunto do seu dominio. Neses contextos, se unha función regular F que ten como moito un punto crítico ten tamén un único punto fixo p tal que $F'(p) \in [-1,1)$, entón p é un atractor global para (27). Ademais, se temos en conta a notación de (25) e (27), a información do obxecto f na terna (a, f, τ) estaría "resumida" no valor $f'(p)$ e a anterior condición de estabilidade asintótica global tería a forma

$$f'(p) \in [-a, a). \tag{29}$$

Convén remarcar que o retardo τ non aparece na condición (29), xa que se obtén vía (27), que é independente do valor de τ. A condición (29) é, polo tanto, unha *condición de estabilidade independente do retardo* ou unha *condición de estabilidade absoluta* [124].

Uns anos máis tarde, Győri e Trofimchuk [54] publicaron, para condicións de negatividade de Sf, unha extensión de (29) a partir do uso dunha modificación da ecuación en diferenzas (27) que incluía a τ como parte dun coeficiente e obtiveron a *condición de estabilidade dependente do retardo*

$$f'(p) \in [-a, a) \quad \text{ou} \quad \left[f'(p) < -a \ \text{e} \ \tau \leq \frac{1}{a} \ln \left(\frac{f'(p)}{f'(p) + a} \right) \right]. \tag{30}$$

Na procura de mellores condicións de estabilidade global para o caso de *feedbacks* con derivada de Schwarz negativa, Liz et al. [86] proporcionaron unha mellora de (30), que resultou ser a óptima para (24), mais, no caso particular de (25), aínda deixaba unha pequena marxe antes de chegar á fronteira da rexión de parámetros para os que o único equilibrio de (25) é localmente atractor, sendo esta última

$$f'(p) \in [-a, a) \quad \text{ou} \quad \left[f'(p) < -a \ \text{e} \ \tau < \frac{\arccos\left(\frac{a}{f'(p)}\right)}{\sqrt{f'(p)^2 - a^2}} \right]. \tag{31}$$

Dedicamos algunhas páxinas a resumir estes esforzos e a explicar dun modo visual as rexións de parámetros $(a, f'(p), \tau)$ que satisfán as condicións de estabilidade previamente mencionadas (ver a Observación 1.64 e a Figura 1.8).

Despois de lembrar as principais propiedades da ecuación (24) (cun interese particular en (25)), centrámonos na parte do proxecto [17, 20, 83] que está dedicada aos modelos gamma, que, como diximos antes, inclúen á ecuación (24) con (26). En particular, xustificamos o uso de $\gamma \in [0, 1]$ como un parámetro que proporciona flexibilidade a moitos modelos e conecta parellas de ecuacións que foron amplamente estudadas. Por exemplo, as dúas ecuacións diferenciais con retardo propostas por Mackey e Glass [96] para estudar un modelo de hematopoese están conectadas no noso enfoque.

Para estudar a dinámica global dos modelos gamma coas ferramentas que destacamos anteriormente, proporcionamos resultados que amosan que propiedades lle aporta á ecuación (24) un *feedback* xeral do tipo (26) con $\gamma \in [0, 1]$. A modo ilustrativo, a existencia dun único equilibrio positivo p (ou unha única solución positiva constante), o xeito no que γ inflúe no valor de p e información crucial sobre a dinámica da ecuación diferencial con retardo nas proximidades de p poden ser obtidas a través da análise da súa correspondente ecuación en diferenzas (Teorema 2.1). A continuación, estudamos se as condicións de estabilidade (29), (30) e (31) poden ser aplicadas a varios modelos gamma que son de interese de cara ás aplicacións. Nos seguintes parágrafos, describimos as nosas achegas.

O primeiro exemplo está baseado nunha versión do *modelo neoclásico de crecemento económico de Solow* [126], onde x representa o *capital-labour rate* x que, grosso modo, é unha taxa que relaciona os bens materiais (como máquinas) e a forza traballadora (traballadores, número de horas traballadas, etc.). O uso do retardo explícase polo tempo requirido na produción dun ben, como destacaron Matsumoto e Szidarovszky [102]. Se a produción dese ben vén dada pola función de Cobb-Douglas [10, 126], temos que $h(x) = \beta > 0$ no *feedback* (26), co que a ecuación (24) adquire a forma

$$x'(t) = -ax(t) + \beta x^{\gamma}(t - \tilde{\tau}(t)). \tag{32}$$

En [20], atopamos que o único equilibrio de (32) é globalmente atractor para calquera valor dos parámetros $a, \beta > 0$, $\gamma \in (0, 1)$ e do retardo continuo e limitado $\tilde{\tau}(t)$ (ver Teorema 2.7). Se temos en mente o contexto do modelo anterior, o noso resultado di que considerar un retardo como o anterior na produción dunha mercadoría [102] non evita que finalmente se alcance un equilibrio p para a devandita taxa.

O resto de casos que tratamos tamén atopan aplicacións en relación co modelo de Solow. Baséanse na incorporación dun factor de contaminación no *feedback* f, provocado por grandes concentracións de capital [27, 102, 103] e que matematicamente está representado por unha función estritamente decrecente. Así, no segundo caso asumimos que $h(x) = \beta e^{-\delta x}$, sendo $\beta, \delta > 0$, en (26), de modo que a ecuación (24) se escribe como

$$x'(t) = -ax(t) + \beta x^{\gamma}(t - \tilde{\tau}(t))e^{-\delta x(t-\tilde{\tau}(t))}. \tag{33}$$

Esta é coñecida como a *ecuación de Lasota* [74], polo seu uso na modelaxe da produción dun tipo fixado de células sanguíneas (por exemplo, glóbulos vermellos), aínda que tamén pode verse como unha *versión gamma da ecuación das moscas de Nicholson*, utilizada para determinar a cantidade de individuos presentes en colonias de insectos criados en laboratorio, segundo o traballo de Gurney et al. [49].

Para a ecuación (33), o enfoque clásico de Allwright e Singer non funciona, xa que f non ten derivada de Schwarz negativa. No entanto, si se pode aplicar a xeneralización de El-Morshedy e Jiménez López para concluír (Teorema 2.12) que o único equilibrio da ecuación en diferenzas (27) relativa a (33) é GAS sempre que este sexa LAS, isto é, se

$$\frac{\beta}{a} \leq e^{\gamma+1}\left(\frac{\gamma+1}{\delta}\right)^{1-\gamma}.$$

Esta é a condición de estabilidade global independente do retardo máis fina que se pode dar para (33) e pódese obter da análise da súa correspondente ecuación en diferenzas, proporcionada por Liz [81]. Adicionalmente, se o retardo é constante, a condición anterior pode ser mellorada utilizando as estimacións de Győri e Trofimchuk para obter (ver tamén o Teorema 2.12) a condición de estabilidade global dependente do retardo

$$\frac{\beta}{a} \leq e^{\gamma+\frac{1}{1-e^{-a\tau}}}\left(\frac{\gamma+\frac{1}{1-e^{-a\tau}}}{\delta}\right)^{1-\gamma}.$$

Con isto extendemos o estudo de Matsumoto e Szidarovszky [103] sobre a dinámica local para a ecuación (33) e conectamos o estudo da dinámica global dos casos $\gamma = 0$, analizado por Győri [50], e $\gamma = 1$, descrito por Győri e Trofimchuk [54]. Ademais, apoiándonos en [81], mostramos que o papel que xoga γ na estabilidade do único equilibrio de (33) é sutil. Por exemplo, existen casos para os que incrementar γ pode xerar un par de cambios na estabilidade (ver Figura 2.1).

O terceiro caso é a ecuación diferencial gamma-loxística con retardo, relacionada coa escolla $h(x) = \beta(1-x)$, con $\beta > 0$, en (26). Con isto, agora a ecuación (24) escribímola como

$$x'(t) = -ax(t) + \beta x^{\gamma}(t-\tilde{\tau}(t))(1-x(t-\tilde{\tau}(t))). \tag{34}$$

Esta versión gamma dunha ecuación con función de produción loxística é un caso particular de (24) que require dalgunhas observacións previas para poder ser tratada coas técnicas mencionadas máis arriba. En particular, precisamos restrinxir o estudo ás solucións con valores en $(0,1)$ e impoñer unha condición de consistencia nos parámetros (ver (2.39)). Baixo tales restricións, obtemos (Teorema 2.22) que o único equilibrio é GAS se

$$\frac{\beta}{a} \leq (\gamma+1)\left(\frac{\gamma+2}{\gamma+1}\right)^{\gamma},$$

o que, de xeito análogo á situación da ecuación de Lasota, constitúe a condición de estabilidade independente do retardo máis fina que se pode dar para (34). É máis, se o retardo é constante, as estimacións de Győri e Trofimchuk tamén son válidas e proporcionamos (tamén no Teorema 2.22) unha condición de estabilidade dependente do retardo que mellora á anterior e se escribe como

$$\frac{\beta}{a} \leq \left(\gamma+\frac{1}{1-e^{-a\tau}}\right)\left(\frac{\gamma+1+\frac{1}{1-e^{-a\tau}}}{\gamma+\frac{1}{1-e^{-a\tau}}}\right)^{\gamma}.$$

Os nosos resultados de estabilidade global para $\gamma \in [0,1]$ extenden o estudo de Matsumoto e Szidarovszky [103] sobre a dinámica local da ecuación (34) para o caso $\gamma = 1$.

No cuarto exemplo, supoñemos que $h(x) = \frac{\beta}{1+\delta x^m}$, con $\beta, \delta, m > 0$, en (26). Entón, reescribimos (24) como

$$x'(t) = -ax(t) + \frac{\beta x^{\gamma}(t-\tilde{\tau}(t))}{1+\delta x^m(t-\tilde{\tau}(t))} \tag{35}$$

e nos referimos a esta ecuación como a *versión gamma da ecuación diferencial con retardo de Mackey-Glass*, en analoxía coas dúas ecuacións propostas por Mackey e Glass [96], relacionadas cos casos $\gamma = 0$ e $\gamma = 1$, e utilizadas para representar un modelo de xeración de células sanguíneas. A análise de (35) é máis delicada ca nos casos anteriores, xa que non conseguimos atopar cálculos sinxelos que nos levasen a unha aplicación do resultado de El-Morshedy e Jiménez López. De todos os xeitos, no caso particular de $m = 1$ e a partir do traballo de Liz [82] sobre a correspondente ecuación en diferenzas, onde se usa o criterio de Coppel [24], si que se poder obter de maneira directa, como mostramos en [20], a estabilidade global do único equilibrio de (35) (Teorema 2.28). Porén, non atopamos ningunha maneira de adaptar os razoamentos para o caso $m \neq 1$. Con todo isto, proponse unha nova fórmula que involucra, unha vez máis, á derivada de Schwarz para poder dispoñer da propiedade "LAS implica GAS". Isto é feito no Capítulo 3, que xorde como unha ramificación do Capítulo 2 neste punto. Concretamente, se na correspondente ecuación en diferenzas (27) denotamos $F(x) = f(x)/a = x^{\gamma}H(x)$, entón a condición que damos (ver (3.3)) escrita na forma suxerida por Jiménez López, é

$$SH(x) < \frac{H(x) - xH'(x)}{2(xH(x))^2}(xH(x))', \quad \text{para todo } x > 0. \tag{36}$$

Unha das principais fortalezas da fórmula (36) é que non ten en conta a derivada de Schwarz da aplicación F, senón só a dun dos seus factores, H, o que axuda a simplificar os cálculos en moitos modelos. De feito, esta fórmula vén dun cambio de variables logarítmico [78] que transforma o produto $x^{\gamma}H(x)$ nunha suma $\gamma x + H^*(x)$, onde H^* se escribe en termos de H. Naqueles casos nos que $SH^* < 0$, pódese aplicar un resultado de tipo envoltura de Liz et al. [85] a H^* e concluír, despois de desfacer o cambio de variables, que a ecuación (27) satisfai a propiedade "LAS implica GAS". Noutras palabras, obtense (Teorema 3.1) que, baixo a condición (36), p é GAS para (27) sempre que sexa LAS. En particular, se $F(x) = \frac{\beta x^{\gamma}}{a(1+\delta x^m)}$ en (27), a conclusión anterior obtense se

$$m \leq 1+\gamma \quad \text{ou} \quad \left[m > 1+\gamma, \quad \frac{\beta}{a} \leq \frac{m}{m-1-\gamma}\left(\frac{\gamma+1}{\delta(m-1-\gamma)}\right)^{\frac{1-\gamma}{m}} \right]. \tag{37}$$

De feito, a condición (37) convértese na mellor condición de estabilidade global independente do retardo que se pode proporcionar para a ecuación (35) (Teorema 3.13). Ademais, somos quen de dar unha ligazón entre estimacións relativas á dinámica global da ecuación (35) para os casos particulares de $\gamma = 0$ e $\gamma = 1$, estudados por Gopalsamy et al. [44].

É máis, o razoamento no resultado principal do Capítulo 3 (Teorema 3.1) pode ser lixeiramente modificado para dar unha extensión no caso $\gamma = 0$ e unha aplicación unimodal H (recordemos que orixinariamente asumiamos que era monótona decrecente). Así, pódese dar un resultado (ver Teorema 3.5) que é análogo ao clásico proporcionado por Allwright e Singer, mais neste caso coa fórmula (36) en vez de coa desigualdade $SH < 0$.

Rematamos a parte relativa a equilibrios globalmente atractores nos modelos gamma estudando a ecuación

$$x'(t) = -ax(t) + s(x(t))f(x(t - \tilde{\tau}(t))), \tag{38}$$

onde $f(x) = Bx^{\gamma}$, con $B > 0$, que non se axusta ao marco de (24) con (26) pola introdución do factor $s(x(t))$. Tal ecuación tamén aparece no contexto do modelo neoclásico de crecemento económico de Solow se se consideran hipóteses máis xerais. En particular, se a función s ten certas propiedades convenientes e coherentes co seu significado (s é a chamada taxa de aforro [10, 126]), mostramos (Teorema 2.34) que a ecuación (38) ten un equilibrio globalmente atractor relacionando a súa dinámica con outra ecuación en diferenzas máis xeral que a de (27). A súa proba baséase na xeneralización (Teorema 2.32) dun resultado de Ivanov et al. [64] e para a que, de feito, damos un razoamento máis simple que o de dita referencia [64] xa que nos apoiamos nunha propiedade especial das funcións unidimensionais cun equilibrio globalmente atractor, concretamente, a *atracción forte* [90].

Máis adiante, no Capítulo 4, desprazamos a nosa atención á ecuación (23) nun contexto multidimensional. Particularmente, propoñemos a extensión da ecuación (25) a dimensións superiores a través do sistema

$$x_i'(t) = -x_i(t) + f_i(x_1(t - \tau_{i1}), \dots, x_s(t - \tau_{is})), \quad i = 1, \dots, s, \tag{39}$$

e cuestionámonos se existe unha ecuación en diferencias multidimensional da que se poida extraer información sobre o comportamento a longo prazo das solucións de (39), como acontecía coa relación entre (25) e (27) no caso escalar. En concreto, se denotamos por $x_{\cdot,n} = (x_{1,n}, \dots, x_{s,n})$ ao termo xeral dunha sucesión en $\mathbb{R}^s$, entón a candidata natural a ser a ecuación en diferenzas correspondente a (39) é

$$x_{i,n+1} = f_i(x_{1,n}, \dots, x_{s,n}), \quad i = 1, \dots, s. \tag{40}$$

Secasí, o problema é certamente sutil en dimensións máis altas, como foi mostrado por Liz e Ruiz-Herrera [90, 91]. De feito, se a función $f = (f_1, \dots, f_s)$ en (40) ten un único punto fixo p, entón asumir que p é globalmente atractor para f non é suficiente para asegurar a mesma propiedade para o sistema de ecuacións diferenciais (39) [91]. Dito doutro xeito, precisamos impoñer unha condición máis forte á ecuación en diferenzas (40). Os autores de [90, 91] propuxeron un concepto, o *atractor forte* (lémbrase na Definición 4.1), que recupera as principais propiedades que resultaron esenciais para relacionar esas dúas ecuacións no caso escalar $s = 1$. Teñamos en conta que, como destacamos, este concepto é útil tamén no estudo da ecuación escalar (38). A idea chave é asumir a existencia de certas familias de produtos cartesianos de intervalos compactos que son aniñados con respecto á aplicación da función f en (40). Así, se un equilibrio de (40) é un atractor forte, que é máis específico que ser globalmente atractor se a dimensión é $s \geq 2$, entón o equilibrio p é globalmente atractor para a ecuación (39) (como recordamos no Teorema 4.2).

No entanto, unha ollada profunda ao razoamento principal amosado en [90] permítenos (ver [18]) extendelo a un contexto máis xeral. Introducimos un novo concepto, especificamente, o *atractor CC-forte* (ver Definición 4.3), que tamén recupera as propiedades necesarias para relacionar a dinámica de (40) coa de (39) no caso escalar. Concretamente, este concepto convértese nunha noción intermedia entre un atractor global e un atractor forte no sentido de [90], sempre que $s \geq 2$ (ver a nosa Observación 4.7). A diferencia principal co concepto proposto por Liz e Ruiz-Herrera é que “debilitamos” a xeometría dos conxuntos involucrados: non necesitamos familias aniñadas de produtos cartesianos de intervalos compactos, chega con considerar certas familias aniñadas de conxuntos compactos e convexos. Polo tanto, a hipótese más débil de que (40) teña un atractor CC-forte é aínda unha condición suficiente para probar que p é tamén globalmente atractor para (39). O anterior constitúe o resultado principal deste capítulo da tese (Teorema 4.5).

Deseguido, describimos os contidos do Capítulo 5. Nel, consideramos a versión escalar de (23) con $\hat{f}$ linear na segunda variable, pero agora permitindo un coeficiente de decaída linear a non constante e que o *feedback* $\hat{f}$ teña coeficientes que tamén poidan depender de t. Por exemplo, nese marco, un dos casos máis simples é o de retardo discreto variable, representado por

$$x'(t) = -a(t)x(t) + \beta(t)x(t - \tilde{\tau}(t)), \tag{41}$$

onde β é unha función non negativa. O comportamento asintótico da ecuación (41) é ben coñecido se $a, \beta, \tilde{\tau}$ son constantes, xa que se deduce da análise da *ecuación característica* [124]. Nese caso, coñécense condicións suficientes sinxelas que garantan que a orixe sexa globalmente atractora. Porén, no caso xeral de (41), as técnicas deben ser outras, e incluso poderiamos ter que lidiar cun modelo que satisfaga $\tilde{\tau}(t) \to \infty$ cando $t \to \infty$, que representa un caso de retardo “non limitado a longo prazo”. Por exemplo, os traballos de Győri e Horváth [52, 53] manexan ese tipo de ecuacións e proporcionan estimacións óptimas para as súas solucións traballando cunha desigualdade do tipo

$$x(t) \leq c + \int_\sigma^t b(s)x(s - \tilde{\tau}(s))\, ds.$$

Os ingredientes principais que eles usan son a chamada *desigualdade característica* e unha cota de tipo Gronwall-Bellman. A desigualdade característica é unha xeneralización da ecuación característica amplamente coñecida para o caso de funcións constantes $a, \beta, \tilde{\tau}$ (ver, por exemplo, Teoremas 2.7 e 5.1 en [53]). Mostramos (Sección 5.4) que as súas ideas poden ser adaptadas a ecuacións máis xerais que (41), como

$$x'(t) = -a(t)x(t) + \beta(t)x(\nu(t)) + \int_\sigma^{\nu(t)} \eta(t,s)x(\nu(s))\, ds, \tag{42}$$

onde $\nu(t) := t - \tilde{\tau}(t)$ e η é unha función non negativa que é monótona crecente na primeira variable. Traballando coa ecuación (42), podemos chegar a unha desigualdade integral do tipo

$$x(t) \leq c + \int_\sigma^t k(t,s)x(s - \tilde{\tau}(s))\, ds. \tag{43}$$

De xeito paralelo, a desigualdade característica tamén necesita ser xeneralizada, como mostramos en [21], e o resultado principal desta parte do noso traballo (Teorema 5.6), que tamén está relacionado con cotas de tipo Gronwall-Bellman, xeneraliza parte das principais conclusións de [52] e sérvenos para probar que algunhas ecuacións do tipo (42) teñen á orixe como solución globalmente atractora. Ademais, extendemos un resultado de Norbury e Stuart [110] xa que, en particular, en (43), asumen $\tilde{\tau} \equiv 0$.

No Capítulo 6, consideramos unha vez máis a versión escalar de (23), pero agora con coeficientes periódicos e tamén permitindo ás solucións ter discontinuidades producidas por impulsos nuns certos instantes t_k, $k \in \mathbb{N}$. Noutras palabras, consideramos a ecuación

$$\begin{cases} x'(t) = -a(t)x(t) + \hat{f}(t, x_t), \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), \end{cases} \tag{44}$$

onde a e $\hat{f}$ son funcións periódicas na variable t co mesmo período $\omega > 0$, as funcións de impulso $I_k : [0, \infty) \to \mathbb{R}$ seguen un patrón de lonxitude temporal ω, e para a que se impoñen outras condicións técnicas para asegurar que as solucións relativas a segmentos non negativos sexan non negativas (ver Sección 6.2). Faria e Oliveira [40] estudaron a existencia de solucións ω-periódicas positivas para a ecuación (44) con retardo finito e funcións de impulso lineares. O noso principal resultado (Teorema 6.8), cuxos ingredientes básicos son unha versión do Teorema de Punto Fixo de Krasnoselskii [48, 70] e a imposición dun certo comportamento de $\hat{f}$ para solucións cercanas a cero ou suficientemente grandes, proporciona conclusións análogas ás de [40]. Porén, xeneralizamos (ver [19]) as súas hipóteses de varias maneiras como, por exemplo, incluíndo funcións de impulso I_k máis xerais (permitimos que os seus signos poidan variar) ou considerando *retardo infinito* ou unha clase máis ampla de *feedbacks* retardados. Tamén proporcionamos algúns corolarios que son útiles para algunhas aplicacións. Na parte final do Capítulo 6, aplicamos os nosos resultados a ecuacións integro-diferenciais con retardo distribuído infinito ou periódico. Tamén establecemos ligazóns con algúns dos resultados dispoñibles na literatura.

A modo de conclusión deste traballo, ofrecemos unha visión xeral dos resultados da tese destacando algunhas liñas de traballo futuro que poden ser de interese.

Resumen

Esta tesis se centra en el estudio de algunas propiedades cualitativas de las ecuaciones diferenciales con retardo incluidas en la familia

$$x'(t) = -a(t)x(t) + \hat{f}(t, x_t), \tag{45}$$

donde $x(t)$ denota el estado de una magnitud x (que puede ser escalar o vectorial) en el instante t y cuya evolución, determinada mediante $x'(t)$, buscamos modelizar. La notación x_t representa formalmente la historia de x, esto es, los valores $x(s)$, para ciertos tiempos $s \leq t$. Además, a y $\hat{f}$ son ciertas funciones cuyo significado comentaremos a continuación.

La ecuación (45) ha sido ampliamente utilizada para la modelización de varios fenómenos cuyo estado afecta a la evolución del sistema de manera retardada. Por ejemplo, ha sido empleada en modelos de crecimiento poblacional [49, 114], de producción celular [7, 96, 135], de crecimiento económico [102, 103], etc. Para encontrar más detalles y referencias sobre estas y otras aplicaciones, se recomiendan también los libros de Kolmanovskii y Myshkis [68], Kuang [72] y Smith [124].

En muchos contextos en los que la ecuación (45) es de interés, solo las soluciones $x(t)$ de (45) que sean no negativas poseen un significado real. Así, asumiendo que a es una función no negativa y $\hat{f}$ es no negativa para soluciones no negativas, entonces los términos del miembro derecho de (45) son conocidos como el término de *destrucción* $-a(t)x(t)$, que en este caso es lineal, y el término de *producción* $\hat{f}(t, x_t)$, que puede ser no lineal y reúne aquellos efectos que influyen en la evolución del sistema de manera retardada, como mencionamos anteriormente. Concretamente, el último término es también conocido como el *feedback retardado*.

La teoría básica de existencia y unicidad de soluciones para las ecuaciones diferenciales con retardo, como la que expresamos en (45), puede consultarse en multitud de referencias. A modo de ejemplo, destacamos el libro de Smith [124] para una introducción a este tema y el de Hale y Verduyn Lunel [58] para una mirada más profunda a estas ecuaciones. Estas referencias, entre otras, sirven para dar comienzo al núcleo de esta tesis (Sección 1.1).

Una de las principales problemáticas que presentan las ecuaciones del tipo (45) es que no se puede obtener una descripción completa del comportamiento a largo plazo de

sus soluciones si solo las consideramos como un problema en $\mathbb{R}^s$. En particular, una de las ideas clave hacia una comprensión total de la dinámica relativa a (45) es tener en cuenta la naturaleza del *segmento* x_t. En muchos modelos, x_t es simplemente una función vectorial definida en un intervalo real compacto $[-\tau, 0]$, esto es, una función que compila la información de la solución de (45) en el último intervalo temporal de longitud τ; este es el caso de *retardo finito*. Claramente, para dotar de sentido a la ecuación (45), uno tiene que considerar condiciones iniciales que incorporen a los segmentos. De ahí que, en vez de considerar el problema solamente en el espacio euclídeo de dimensión $s \in \mathbb{N}$, también se debe tener en cuenta la evolución de los segmentos x_t, es decir, la evolución del sistema en un cierto espacio funcional. De este modo, podremos recuperar muchos hechos relevantes de la teoría de los sistemas dinámicos (ver Sección 1.2). Sin embargo, eso provoca que tengamos que trabajar con un espacio de dimensión infinita.

En la primera parte de la tesis (Capítulos 2 y 3), trabajamos principalmente con una familia de ecuaciones escalares del tipo (45) con una función de decaimiento lineal constante $a(t) = a$ y un único retardo discreto, esto es, con $\hat{f}(t, x_t) = f(x(t - \tilde{\tau}(t)))$ para ciertas funciones reales $f, \tilde{\tau}$ de variable real. Bajo tales hipótesis, la ecuación (45) puede ser escrita como

$$x'(t) = -ax(t) + f(x(t - \tilde{\tau}(t))). \tag{46}$$

Un caso particular de (46) que ha adquirido relevancia en la literatura [71, 79] es el de retardo constante, es decir, aquel en el que $\tilde{\tau}(t) = \tau$. Asumir lo anterior implica que (46) toma la forma

$$x'(t) = -ax(t) + f(x(t - \tau)). \tag{47}$$

La familia de ecuaciones sobre la que recae nuestro interés en esta primera parte de la tesis es la de los llamados *modelos gamma* [81, 114], que son una familia de modelos que incorporan a la ecuación (46) con la elección

$$f(x) = x^\gamma h(x), \tag{48}$$

siendo $\gamma \geq 0$ y $h : [0, \infty) \to (0, \infty)$ una función monótona decreciente. Algunos modelos bastante conocidos pueden ser incluidos en el contexto de (46) con el *feedback* (48) [49, 96, 135]. El significado de γ depende del modelo. Por ejemplo, modula el nivel de cooperación [81] en el campo de crecimiento de poblaciones, el rol que juega el capital disponible en la producción de una mercancía [103] en el modelo neoclásico de crecimiento económico, etc.

Nuestro objetivo es obtener condiciones suficientes que aseguren que la ecuación (46) con el *feedback* (48) tenga un equilibrio $p \in (0, \infty)$ que sea un atractor global, lo que significa que, grosso modo, todas sus soluciones están definidas para tiempos futuros y convergen a p cuando $t \to \infty$. Para cumplir dicho objetivo, dedicamos algunas páginas (Sección 1.3) a recordar algunas claves sobre cómo se puede abordar el estudio de la ecuación (46) sin imponer necesariamente el caso particular de la función de producción (48). No obstante, por cuestiones de simplicidad en las explicaciones, lo hacemos en un primer lugar para el caso de retardo constante (47) y, finalmente, explicamos qué resultados se extienden directamente al caso de (46). Además, teniendo en cuenta esta meta, es natural imponer condiciones de existencia y unicidad de equilibrio para (47), como también hacemos.

Los tres siguientes objetos se convierten en elementos relevantes para estudiar la ecuación (47): el parámetro de destrucción a, el *feedback* f y el retardo τ. La relación existente entre los anteriores determina el comportamiento a largo plazo de las soluciones de (47). De modo particular, la relación f/a tiene un papel fundamental. Es más, el comportamiento asintótico de las soluciones de la correspondiente ecuación en diferencias

$$x_{n+1} = \frac{f(x_n)}{a} =: F(x_n), \tag{49}$$

esto es, el comportamiento a largo plazo de las sucesiones

$$x_0,\ \ F(x_0),\ \ F(F(x_0)),\ \ F(F(F(x_0))),\ \ \ldots, \tag{50}$$

tiene implicaciones directas en el correspondiente comportamiento de las soluciones de (47). Este hecho se desprende de los trabajos de Mallet-Paret y Nussbaum [98] y de Ivanov y Sharkovsky [65]. A grandes rasgos, ellos probaron que, bajo condiciones apropiadas (ver Teorema 1.33), si (49) tiene un equilibrio p globalmente atractor, entonces p es también un equilibrio globalmente atractor para (47). En otras palabras, la existencia de un p tal que toda sucesión del tipo (50) tiende a p implica que cualquier solución de (47) tiende a p cuando $t \to \infty$. Este tipo de resultados es realmente crucial para el estudio de la dinámica de (47): si uno es capaz de obtener condiciones de estabilidad asintótica global para (49), entonces estas son también válidas para (47).

Aunque analizar si un equilibrio es globalmente atractor para la ecuación (49) parece más fácil que hacerlo para (47), no es tan trivial como, por ejemplo, sí sería el estudio de su estabilidad local. Aún así, existen familias de funciones F que satisfacen la propiedad llamada *"LAS implica GAS"* (por sus siglas en inglés "Locally/Globally Asymptotically Stable", es decir, localmente/globalmente asintóticamente estable). Como su nombre indica, refleja la situación en la que se cumple que, si un equilibrio es localmente atractor, entonces también lo es globalmente [43]. Una de dichas familias, presente en muchas aplicaciones [79], es la de las funciones que tienen a lo sumo un punto crítico y derivada de Schwarz negativa (remitimos al artículo de Jiménez López y Parreño [67] para una introducción histórica de este concepto). En particular, para una función real F de clase $\mathcal{C}^3$ definida en un intervalo real I, la *derivada de Schwarz* de F se define como

$$SF(x) = \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left(\frac{F''(x)}{F'(x)}\right)^2,$$

para todo $x \in I$ tal que $F'(x) \neq 0$. Los resultados de Allwright [2] y Singer [122] destacaron en un primer momento el papel de la derivada de Schwarz en la conexión entre las dinámicas local y global, mientras que El-Morshedy y Jiménez López [33] proporcionaron una posterior generalización (la recordamos en el Teorema 1.55), donde ellos asumen la negatividad de SF solo en un cierto subconjunto de su dominio. En dichos contextos, si una función regular F que tiene como mucho un punto crítico tiene también un único punto fijo p tal que $F'(p) \in [-1, 1)$, entonces p es un atractor global para (49). Además, si tenemos en cuenta la notación de (47) y (49), la información del objeto f en la terna (a, f, τ) estaría "resumida" en el valor $f'(p)$ y la anterior condición de estabilidad asintótica global tomaría la forma

$$f'(p) \in [-a, a). \tag{51}$$

Conviene remarcar que el retardo τ no aparece en la condición (51) ya que se obtiene vía (49), que es independiente del valor de τ. La condición (51) es, por tanto, una *condición de estabilidad independiente del retardo* o una *condición de estabilidad absoluta* [124].

Unos años más tarde, Győri y Trofimchuk [54] publicaron, para condiciones de negatividad de Sf, una extensión de (51) a partir de una modificación de la ecuación en diferencias (49) que incluía a τ como parte de un coeficiente y obtuvieron la *condición de estabilidad dependiente del retardo*

$$f'(p) \in [-a, a) \quad \text{o} \quad \left[f'(p) < -a \ \text{ y } \ \tau \leq \frac{1}{a} \ln\left(\frac{f'(p)}{f'(p) + a} \right) \right]. \tag{52}$$

En la búsqueda de condiciones más generales de estabilidad global para el caso de *feedbacks* con derivada de Schwarz negativa, Liz et al. [86] proporcionaron una mejora de (52), que resultó ser la óptima para (46), pero que, en el caso particular de (47), aún dejaba un pequeño margen antes de llegar a la frontera de la región de parámetros para los que el único equilibrio de (47) es localmente atractor, siendo esta última

$$f'(p) \in [-a, a) \quad \text{o} \quad \left[f'(p) < -a \ \text{ y } \ \tau < \frac{\arccos\left(\frac{a}{f'(p)}\right)}{\sqrt{f'(p)^2 - a^2}} \right]. \tag{53}$$

Dedicamos algunas páginas a resumir estos esfuerzos y a explicar de un modo visual las regiones de parámetros $(a, f'(p), \tau)$ que satisfacen las condiciones de estabilidad previamente mencionadas (ver la Observación 1.64 y la Figura 1.8).

Después de recordar las principales propiedades de la ecuación (46) (con un interés particular en (47)), nos centramos en la parte del proyecto [17, 20, 83] que está dedicada a los modelos gamma, que, como dijimos antes, incluyen a la ecuación (46) con (48). En particular, justificamos el uso de $\gamma \in [0, 1]$ como un parámetro que proporciona flexibilidad a muchos modelos y conecta parejas de ecuaciones que han sido ampliamente estudiadas. Por ejemplo, las dos ecuaciones diferenciales con retardo propuestas por Mackey y Glass [96] para estudiar un modelo de hematopoyesis están conectadas en nuestro enfoque.

Para estudiar la dinámica global de los modelos gamma con las herramientas que destacamos anteriormente, proporcionamos resultados que muestran qué propiedades le aporta a la ecuación (46) un *feedback* general del tipo (48) con $\gamma \in [0, 1]$. A modo ilustrativo, la existencia de un único equilibrio positivo p (o una única solución positiva constante), el modo en el que γ influye en el valor de p e información crucial de la dinámica de la ecuación diferencial con retardo en las proximidades de p pueden ser obtenidas a través del análisis de su correspondiente ecuación en diferencias (Teorema 2.1). A continuación, estudiamos si las condiciones de estabilidad (51), (52) y (53) pueden ser aplicadas a varios modelos gamma que son de interés en las aplicaciones que mostramos. En los siguientes párrafos, describimos nuestras aportaciones.

El primer ejemplo está basado en una versión del *modelo neoclásico de crecimiento económico de Solow* [126], donde x representa el *cociente entre el capital disponible y la mano de obra* que, grosso modo, es una tasa que relaciona los bienes materiales (como máquinas) y la fuerza trabajadora (trabajadores, número de horas trabajadas, etc.). El uso del retardo se explica por el tiempo requerido en la producción de un cierto bien, como destacaron Matsumoto y Szidarovszky [102]. Si la producción de ese bien viene dada por la función de Cobb-Douglas [10, 126] tenemos que $h(x) = \beta > 0$ en el *feedback* (48), con lo que la ecuación (46) adquiere la forma

$$x'(t) = -ax(t) + \beta x^{\gamma}(t - \tilde{\tau}(t)). \tag{54}$$

En [20], probamos que el único equilibrio de (54) es globalmente atractor para cualquier valor de los parámetros $a, \beta > 0$, $\gamma \in (0, 1)$ y del retardo continuo y acotado $\tilde{\tau}(t)$ (ver Teorema 2.7). Si tenemos en mente el contexto del modelo anterior, esto significa que considerar un retardo como el anterior en la producción de una mercancía [102] no evita que finalmente se alcance un equilibrio p para la tasa mencionada.

El resto de casos que tratamos también tienen aplicaciones en relación con el modelo de Solow. Se basan en la incorporación de un factor de contaminación en el *feedback* f, provocado por grandes concentraciones de capital [27, 102, 103] y que matemáticamente está representado por una función estrictamente decreciente. Así, en el segundo caso, asumimos [20] que $h(x) = \beta e^{-\delta x}$, siendo $\beta, \delta > 0$, en (48), con lo que la ecuación (46) se escribe como

$$x'(t) = -ax(t) + \beta x^{\gamma}(t - \tilde{\tau}(t))e^{-\delta x(t-\tilde{\tau}(t))}. \tag{55}$$

Esta se conoce como la *ecuación de Lasota* [74] por su uso para modelar la producción de un tipo fijado de células sanguíneas (por ejemplo, glóbulos rojos), aunque también puede verse como una *versión gamma de la ecuación de las moscas de Nicholson*, utilizada para determinar la cantidad de individuos en colonias de insectos criados en laboratorio, según el trabajo de Gurney et al. [49].

Para la ecuación (55), el enfoque clásico de Allwright y Singer no funciona, ya que f no tiene derivada de Schwarz negativa. Sin embargo, sí se puede aplicar la generalización proporcionada por El-Morshedy y Jiménez López para concluir (Teorema 2.12) que el único equilibrio de la ecuación en diferencias (49) relativa a (55) es GAS siempre que este sea LAS, es decir, si

$$\frac{\beta}{a} \leq e^{\gamma+1}\left(\frac{\gamma+1}{\delta}\right)^{1-\gamma}.$$

Esta es la condición de estabilidad global independiente del retardo más fina que se puede dar para (55) y se puede obtener a partir del análisis de su correspondiente ecuación en diferencias, el cual fue desarrollado por Liz [81]. Adicionalmente, si el retardo es constante, la condición anterior se puede mejorar utilizando las estimaciones de Győri y Trofimchuk para obtener (ver también Teorema 2.12) la condición de estabilidad global dependiente del retardo

$$\frac{\beta}{a} \leq e^{\gamma+\frac{1}{1-e^{-a\tau}}}\left(\frac{\gamma+\frac{1}{1-e^{-a\tau}}}{\delta}\right)^{1-\gamma}.$$

Con esto extendemos el estudio de Matsumoto y Szidarovszky [103] sobre la dinámica local para la ecuación (55) y conectamos el estudio de la dinámica global de los casos $\gamma = 0$, analizado por Győri [50], y $\gamma = 1$, descrito por Győri y Trofimchuk [54]. Además, apoyándonos en [81], mostramos que el papel que juega γ en la estabilidad del único equilibrio de (55) es sutil. Por ejemplo, existen casos para los que incrementar γ puede generar un par de cambios en la estabilidad (ver Figura 2.1).

El tercer caso es la ecuación diferencial γ-logística con retardo, relacionada con la elección $h(x) = \beta(1-x)$, con $\beta > 0$, en (48) [17]. Con esta información, la ecuación (46) puede ser escrita como

$$x'(t) = -ax(t) + \beta x^{\gamma}(t-\tilde{\tau}(t))(1-x(t-\tilde{\tau}(t))). \tag{56}$$

Esta versión gamma de una ecuación con función de producción logística es un caso particular de (46) que requiere de algunas observaciones previas para poder ser tratada con las técnicas mencionadas más arriba. En particular, necesitamos restringir el estudio a las soluciones con valores en $(0,1)$ e imponer una condición de consistencia en los parámetros (ver (2.39)). Bajo tales restricciones, obtenemos (Teorema 2.22) que el único equilibrio es GAS si

$$\frac{\beta}{a} \leq (\gamma+1)\left(\frac{\gamma+2}{\gamma+1}\right)^{\gamma},$$

lo cual, de manera análoga a la situación de la ecuación de Lasota, constituye la condición de estabilidad independiente del retardo más fina que se puede dar para (56). Es más, si el retardo es constante, las estimaciones de Győri y Trofimchuk también son válidas y proporcionamos (también en el Teorema 2.22) una condición de estabilidad dependiente del retardo que mejora a la anterior y se escribe como

$$\frac{\beta}{a} \leq \left(\gamma + \frac{1}{1-e^{-a\tau}}\right)\left(\frac{\gamma + 1 + \frac{1}{1-e^{-a\tau}}}{\gamma + \frac{1}{1-e^{-a\tau}}}\right)^{\gamma}.$$

Nuestros resultados de estabilidad global para $\gamma \in [0,1]$ extienden el estudio de Matsumoto y Szidarovszky [103] sobre la dinámica local de la ecuación (56) para el caso $\gamma = 1$.

En el cuarto ejemplo, suponemos que $h(x) = \frac{\beta}{1+\delta x^m}$, con $\beta, \delta, m > 0$, en (48). Entonces, reescribimos (46) como

$$x'(t) = -ax(t) + \frac{\beta x^{\gamma}(t-\tilde{\tau}(t))}{1+\delta x^m(t-\tilde{\tau}(t))} \tag{57}$$

y nos referimos a esta ecuación como la *versión gamma de la ecuación diferencial con retardo de Mackey-Glass*, en analogía con las dos ecuaciones propuestas por Mackey y Glass [96], relacionadas con los casos $\gamma = 0$ y $\gamma = 1$ y utilizadas para representar un modelo de generación de células sanguíneas.

El análisis de (57) es más delicado que el de los casos anteriores, ya que no conseguimos encontrar cálculos sencillos que nos llevasen a una aplicación del resultado de El-Morshedy y Jiménez López. De todos modos, en el caso particular $m = 1$ y a partir del trabajo de Liz [82] sobre la correspondiente ecuación en diferencias, donde se usa el criterio de Coppel [24], sí que se puede obtener de forma directa, como mostramos en [20], la estabilidad global del único equilibrio de (57) (Teorema 2.28). No obstante, no encontramos ningún modo sencillo de adaptar los razonamientos para el caso $m \neq 1$. Con todo esto, se propone una nueva fórmula que involucra, una vez más, a la derivada de Schwarz para poder disponer de la propiedad “LAS implica GAS”. Esto se hace en el Capítulo 3, que surge como una ramificación del Capítulo 2 en este punto. Concretamente, si en la correspondiente ecuación en diferencias (49) denotamos $F(x) = f(x)/a = x^{\gamma}H(x)$, entonces la condición que proporcionamos (ver (3.3)), escrita en la forma sugerida por Jiménez López, es

$$SH(x) < \frac{H(x) - xH'(x)}{2(xH(x))^2}(xH(x))', \quad \text{para todo } x > 0. \tag{58}$$

Una de las principales fortalezas de la expresión (58) es que no tiene en cuenta la derivada de Schwarz de la aplicación F, sino solamente la de uno de sus factores, H, lo que ayuda a simplificar los cálculos en muchos modelos. De hecho, esta fórmula procede de un cambio de variables logarítmico [78] que transforma el producto $x^{\gamma}H(x)$ en una suma $\gamma x + H^*(x)$, donde H^* se escribe en función de H. En aquellos casos en los que $SH^* < 0$, se puede aplicar un resultado de tipo envoltura de Liz et al. [85] a H^* y concluir, después de deshacer el cambio de variables, que la ecuación (49) satisface la propiedad "LAS implica GAS". En otras palabras, se obtiene (Teorema 3.1) que, bajo la condición (58), p es GAS para (49) siempre que sea LAS. En particular, si $F(x) = \frac{\beta x^\gamma}{a(1+\delta x^m)}$ en (49), la conclusión anterior se obtiene si

$$m \leq 1+\gamma \quad \text{o} \quad \left[m > 1+\gamma, \quad \frac{\beta}{a} \leq \frac{m}{m-1-\gamma}\left(\frac{\gamma+1}{\delta(m-1-\gamma)}\right)^{\frac{1-\gamma}{m}} \right]. \tag{59}$$

De hecho, la condición (59) se convierte en la mejor condición de estabilidad global independiente del retardo que se pode proporcionar para la ecuación (57) (Teorema 3.13). Además, somos capaces de dar una conexión entre estimaciones relativas a la dinámica global de la ecuación (57) para los casos particulares de $\gamma = 0$ y $\gamma = 1$, estudiados por Gopalsamy et al. [44].

A mayores, el resultado principal que se proporciona en el Capítulo 3 (Teorema 3.1) puede ser ligeramente modificado para dar una extensión con $\gamma = 0$ y una aplicación unimodal H (recordemos que originariamente asumíamos que era monótona decreciente). Dicho resultado (ver el Teorema 3.5) es análogo al clásico dado por Allwright y Singer, pero en este caso basado en la condición (58) en vez de en la desigualdad $SH < 0$.

Terminamos la parte relativa a equilibrios globalmente atractores en los modelos gamma estudiando la ecuación

$$x'(t) = -ax(t) + s(x(t))f(x(t-\tilde{\tau}(t))), \tag{60}$$

donde $f(x) = Bx^\gamma$, con $B > 0$, que no se ajusta al marco de (46) con (48) por la introducción del factor $s(x(t))$ [20]. Tal ecuación también aparece en el contexto del modelo neoclásico de crecimiento económico de Solow si se consideran hipótesis más generales. En particular, si la función s posee ciertas propiedades apropiadas y coherentes con su significado (s es la llamada tasa de ahorro [10, 126]), mostramos (Teorema 2.34) que la ecuación (60) tiene un equilibrio globalmente atractor relacionando su dinámica con otra ecuación en diferencias, más general que la de (49). Su prueba se basa en la generalización (Teorema 2.32) de un resultado de Ivanov et al. [64] para la que, de hecho, damos un razonamiento más simple que el incluido en dicha referencia [64] ya que nos apoyamos en una propiedad especial de las funciones unidimensionales con un equilibrio globalmente atractor, concretamente, la *atracción fuerte* [90].

Más adelante, en el Capítulo 4, centramos nuestra atención en la ecuación (45) en un contexto multidimensional. Particularmente, proponemos la extensión de la ecuación (47) a dimensiones superiores a través del sistema

$$x_i'(t) = -x_i(t) + f_i(x_1(t-\tau_{i1}), \ldots, x_s(t-\tau_{is})), \quad i = 1, \ldots, s, \tag{61}$$

y nos cuestionamos si existe una ecuación en diferencias multidimensional de la que se pueda extraer información sobre el comportamiento a largo plazo de las soluciones de (61), como sucedía con la relación entre (47) y (49) en el caso escalar. En concreto, si denotamos por $x_{\cdot,n} = (x_{1,n}, \dots, x_{s,n})$ el término general de una sucesión en $\mathbb{R}^s$, entonces la candidata natural a ser la ecuación en diferencias correspondiente a (61) es

$$x_{i,n+1} = f_i(x_{1,n}, \dots, x_{s,n}), \quad i = 1, \dots, s. \tag{62}$$

De todos modos, el problema es ciertamente sutil en dimensiones superiores, como fue mostrado por Liz y Ruiz-Herrera [90, 91]. De hecho, si la función $f = (f_1, \dots, f_s)$ en (62) tiene un único punto fijo p, entonces asumir que p es globalmente atractor para f no es suficiente para asegurar la propiedad análoga para el sistema de ecuaciones diferenciales (61) [91]. Dicho de otro modo, necesitamos imponer una condición más fuerte a la ecuación en diferencias (62). Los autores de [90, 91] propusieron un concepto, el *atractor fuerte* (se recuerda en la Definición 4.1), que recupera las principales propiedades que resultaron esenciales para relacionar esas dos ecuaciones en el caso escalar $s = 1$. Ha de tenerse en cuenta que, como destacamos, este concepto es útil también en el estudio de la ecuación escalar (60). La idea clave es asumir la existencia de ciertas familias de productos cartesianos de intervalos compactos que son anidados con respecto a la aplicación de la función f en (62). Así, si un equilibrio de (62) es un atractor fuerte, que es más específico que ser globalmente atractor si la dimensión es $s \geq 2$, entonces el equilibrio p es globalmente atractor para la ecuación (61) (recordamos su resultado en el Teorema 4.2).

No obstante, una mirada profunda al razonamiento principal mostrado en [90] nos permite (ver [18]) extenderlo a un contexto más general. Introducimos un nuevo concepto, específicamente, el *atractor CC-fuerte* (ver Definición 4.3), que también recupera las propiedades necesarias para relacionar la dinámica de (62) con la de (61) en el caso escalar. Concretamente, este concepto se convierte en una noción intermedia entre un atractor global y un atractor fuerte en el sentido de [90], siempre que $s \geq 2$ (ver nuestra Observación 4.7). La diferencia principal con el concepto propuesto por Liz y Ruiz-Herrera es que "debilitamos" la geometría de los conjuntos involucrados: no necesitamos familias anidadas de productos cartesianos de intervalos compactos, pues es suficiente considerar ciertas familias anidadas de conjuntos compactos y convexos. Por lo tanto, la hipótesis más débil de que (62) tenga un atractor CC-fuerte es aún una condición suficiente para probar que p es también globalmente atractor para (61). Lo anterior constituye el resultado principal de este capítulo de la tesis (Teorema 4.5).

A continuación, describimos los contenidos del Capítulo 5. En él, consideramos la versión escalar de (45) con $\hat{f}$ lineal en la segunda variable, pero ahora permitiendo un coeficiente de decaimiento lineal a no constante y que el *feedback* $\hat{f}$ tenga coeficientes que también puedan depender de t. Por ejemplo, en ese marco, uno de los casos más simples es el de retardo discreto variable, representado por

$$x'(t) = -a(t)x(t) + \beta(t)x(t - \tilde{\tau}(t)), \tag{63}$$

donde β es una función no negativa. El comportamiento asintótico de la ecuación (63) es bien conocido si $a, \beta, \tilde{\tau}$ son constantes, ya que se deduce del análisis de la *ecuación característica* [124]. En dicho caso, se conocen condiciones suficientes sencillas que garanticen que el origen es globalmente atractor [58]. De todos modos, en el caso general de (63), las técnicas han de ser otras, e incluso podríamos tener que lidiar con un modelo que satisfaga $\tilde{\tau}(t) \to \infty$ cuando $t \to \infty$, que representa un caso de retardo "no acotado a largo plazo". Por ejemplo, los trabajos de Győri y Horváth [52, 53] manejan ese tipo de ecuaciones y proporcionan estimaciones óptimas para sus soluciones trabajando con una desigualdad del tipo

$$x(t) \leq c + \int_{\sigma}^{t} b(s)x(s - \tilde{\tau}(s))\, ds.$$

Los ingredientes principales que usan son la llamada *desigualdad característica* y una cota de tipo Gronwall-Bellman. La desigualdad característica es una generalización de la ecuación característica para el caso de funciones constantes $a, \beta, \tilde{\tau}$ (ver, por ejemplo, Teoremas 2.7 y 5.1 en [53]). Aquí mostramos (Sección 5.4) que sus ideas pueden ser adaptadas a ecuaciones más generales que (63), como

$$x'(t) = -a(t)x(t) + \beta(t)x(\nu(t)) + \int_{\sigma}^{\nu(t)} \eta(t,s)x(\nu(s))\, ds, \tag{64}$$

donde $\nu(t) := t - \tilde{\tau}(t)$ y η es una función no negativa que es monótona creciente en la primera variable. Trabajando con la ecuación (64), podemos llegar a una desigualdad integral del tipo

$$x(t) \leq c + \int_{\sigma}^{t} k(t,s)x(s - \tilde{\tau}(s))\, ds. \tag{65}$$

De este modo, la desigualdad característica también necesita ser generalizada, como mostramos en [21], y el resultado principal de esta parte de nuestro trabajo (Teorema 5.6), que también está relacionado con cotas de tipo Gronwall-Bellman, generaliza parte de las principales conclusiones de [52] y nos sirve para probar que algunas ecuaciones del tipo (64) tienen al origen como solución globalmente atractora. Además, extendemos un resultado de Norbury y Stuart [110] ya que, en particular, en (65), asumen $\tilde{\tau} \equiv 0$.

En el Capítulo 6, consideramos una vez más la versión escalar de (45), pero ahora con coeficientes periódicos y también permitiendo que las soluciones presenten discontinuidades producidas por impulsos en unos ciertos instantes t_k, $k \in \mathbb{N}$. En otras palabras, consideramos la ecuación

$$\begin{cases} x'(t) = -a(t)x(t) + \hat{f}(t, x_t), \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), \end{cases} \tag{66}$$

donde a y $\hat{f}$ son funciones periódicas en la variable t con el mismo periodo $\omega > 0$, las funciones de impulso $I_k : [0, \infty) \to \mathbb{R}$ siguen un patrón de longitud temporal ω, y para la que se imponen otras condiciones técnicas para asegurar que las soluciones relativas a segmentos no negativos sean no negativas (ver Sección 6.2). Faria y Oliveira [40] estudiaron la existencia de soluciones ω-periódicas positivas para la ecuación (66) con retardo finito y funciones de impulso lineales. Nuestro resultado principal (Teorema 6.8), cuyos ingredientes básicos son una versión del Teorema de Punto Fijo de Krasnoselskii [48, 70] y la imposición de un cierto comportamiento de $\hat{f}$ para soluciones cercanas a cero o suficientemente grandes, proporciona conclusiones análogas a las de [40]. Sin embargo, generalizamos (ver [19]) sus hipótesis de varias maneras como, por ejemplo, incluyendo funciones de impulso I_k más generales (permitimos que sus signos puedan variar) o considerando *retardo infinito* o una clase más amplia de *feedbacks* retardados. También proporcionamos algunos corolarios que son útiles para algunas aplicaciones. En la parte final de este capítulo, aplicamos nuestros resultados a ecuaciones integro-diferenciales con retardo distribuido infinito o periódico. También establecemos conexiones con algunos de los resultados disponibles en la literatura.

A modo de conclusión de este trabajo, damos una visión general de los resultados de la tesis destacando algunas líneas de trabajo futuro que pueden resultar de interés.

Contents

Introduction

The issue of mathematical modelling has acquired a huge relevance in the scientific community in the recent history. Mathematics as a whole has served as a great source for the understanding of many natural and social phenomena. In particular, differential equations have become a powerful tool to analyse how some of those phenomena evolve as time goes by.

For instance, ordinary differential equations have been used to explain classical problems in physical motion, and some types of partial differential equations are used in several phenomena with spatial-diffusive properties and can be interpreted as ordinary differential equations in Banach spaces. The latter equations are written in terms of a certain rule that yields the current evolution of the phenomenon in terms of its current state. The form of such rule depends on the deep nature of the phenomena they are attempting to model.

Nonetheless, there exist cases when the evolution of a certain phenomenon is not fully explained by its current state. For instance, imagine that we are interested in modelling how many adult individuals of a certain animal species live simultaneously. As the generation of new mature individuals from the adult ones usually requires a certain amount of time, the evolution of such group does not depend on the current number of adult individuals: there is a natural 'lag' in the effect they cause on the evolution of their group. Something similar happens with some processes inside, e.g., the human body. For instance, the generation of specific cells, which may take a relevant amount of time, can be a natural response to the current lack of such type of cells. Nevertheless, this does not restrict to biological or ecological situations. For example, one can think about a machine as an instrument to control physical variables, e.g., the temperature, if such magnitude has exceeded a certain value in the last hour. Even within social contexts one may find this kind of examples. For instance, making decisions that attempt to affect the wealth of a country, region, community, etc. does not produce instantaneous outputs.

Following this line, if $x(t)$ stands for the value of a certain magnitude at time t, then we are interested in studying the evolution of x as time goes by if the model fits within the expression of the differential equation

$$x'(t) = f(t, x_t), \tag{67}$$

where x_t is a common notation to denote the history of x (roughly speaking, x_t gathers the information relative to $x(s)$, $s \leq t$), and f represents the rule, which may be time-varying, that tells how the magnitude evolves according to its history. Equation (67) is a *delay differential equation*. The term *delay* represents the length of the time interval for which the history of the function x needs to be known in order for the equation to make sense.

A nice book for anyone interested in starting studying delay differential equations is the one by Smith [124]. For readers that would like to delve into the theory of this type of equations, the classical book by Hale and Verduyn Lunel [58] develops it in detail, and it contains plenty of supplementary remarks and references for the different particular topics that had been arising in the past decades. Furthermore, the book by Diekmann et al. [31] is also a great source with a thorough theoretical basis, where many specific underlying ideas of these equations are developed. Moreover, the book by Kolmanovskii and Myshkis [68] is another possible option to consult results and applications to modelling phenomena. In the latter line, the book by Kuang [72] is also well-known for its emphasis in the analysis of equations that are used in population models. Regarding the current research on this field, we refer to the recent survey by Walther [133], where many active topics are shown.

It is quite a natural issue to aim to obtain information about the long-term behaviour of the magnitude being modelled via the equation (67). Obviously, such task may be simplified if one is able to compute the explicit solutions of the differential equation. Nevertheless, in general, it is not possible to obtain explicit expressions for its solutions, that situation is, clearly, an excepcional case. This is an issue that also arises when working with ordinary differential equations in $\mathbb{R}^n$.

Nonetheless, it is here where the qualitative theory of differential equations naturally comes into consideration. In fact, when those explicit expressions are not available, one can still try to derive some relevant qualitative information with respect to the asymptotic behaviour of the solutions of the equation. For example, one might be interested to know whether the phenomenon modelled by such equation evolves towards a single state as time goes by, or even if a periodic pattern arises. This is done just by studying the properties of the rule f in (67), in an analogous way to what is done in the case of n-dimensional ordinary differential equations.

However, one of the main difficulties of the qualitative analysis of delay differential equations is that the deep nature of their dynamics is, generally speaking, infinite-dimensional, that is, the phase space X, the set where a complete picture of how the magnitude modelled via (67) evolves, lies in a functional Banach space. This is in fact related to the necessity of having a continuum of data about the history of the solution, in other words, the values of the solution at an infinite number of past times. For instance, classical choices for this phase space are certain sets of continuous functions with values in $\mathbb{R}^n$. Moreover, several scenarios that do not hold for ordinary differential equations in $\mathbb{R}^n$ might take place for delay differential equations. For instance, even in the context of existence and uniqueness of solutions, two different solutions can 'merge' at a certain 'future' time [58, Chapter 2], which means that the semiflow may not be injective. Thus, there may not be a unique backward solution, as in the case of difference equations.

Hence, if one is interested in a topological description of the long-term behaviour of the solution of a delay differential equation, an appropriate framework is to consider dynamics in metric spaces [115, 125]. Sometimes, they are also assumed to be complete [56], and some powerful particular results are obtained. When it comes to autonomous dynamics, i.e., when the rule of evolution f in (67) is not time-varying, then the behaviour may be explained via the well-known concept of the semiflow map. Otherwise, other more general concepts, like processes, need to be considered for the non-autonomous case [58].

Within this context, describing the so-called global attractor of bounded sets provides a complete picture of the long-term dynamics. This concept has been widely considered in the literature regarding the qualitative analysis of delay differential equations (see, e.g., [56, 71, 115, 125] and the references therein). However, we will not dwell into further details concerning this notion, since it exceeds by far the scope of the current work. We refer to [125, Chapter 2] for its theoretical background and to [7] for one of its applications.

In spite of the former consideration, we can still say that the long-term dynamics of an equation like (67) are 'simple' if we ensure the existence of $p \in \mathbb{R}^n$ such that any solution $x(t)$ of (67) satisfies

$$\lim_{t\to\infty} x(t) = p, \tag{68}$$

for a subset of admissible initial conditions that is as large as possible. Condition (68) can be rewritten, in terms of the announced underlying functional phase space X, as the existence of a global attractor of points, as we will see later. This will be our main aim.

Moreover, there exist phenomena that are modelled under equation (67) which depend on external factors that are themselves periodic; for instance, bear in mind the seasonal effects on population growth. This would mean that the rule f in (67) purely depends on the first variable, and, in fact, the rule is repeated in consecutive time intervals of a certain fixed length. Thus, it is also natural to wonder whether the magnitude whose evolution is modelled via equation (67) inherits such periodic pattern; in other words, whether there exists a periodic solution of (67). The quest for the existence of such solutions will also be one of our goals.

Nevertheless, we will not seek sufficient conditions to ensure the existence of globally attracting equilibria or periodic solutions for the general equation (67). Roughly speaking, this manuscript is focused on the particular family of delay differential equations of the type of (67) that are written in the form

$$x'(t) = -\mathsf{d}(t, x_t) + \mathsf{p}(t, x_t), \tag{69}$$

where p and d are nonnegative functions (on each coordinate, in case it is a multidimensional equation), respectively representing what are known as the production and destruction terms. The consideration of equation (69) comes from the work by an der Heiden and Mackey [5] and represents an appropriate framework to study many biological or economic models. In particular, positive solutions are usually the center of attention.

In the case of (69), the qualitative study is related to the analysis of the interplay between the destruction and production terms, since the rule f in (67) is written as $f = \mathsf{p} - \mathsf{d}$. Since the destruction term will be generally linear and 'non-delayed', it is the expression of the production term that acquires a major relevance, as we will see.

Aims

In particular, we summarise our goals below:

- To find sufficient conditions for the existence of globally attracting equilibria for equations of the form (69) by:
  - analysing to what extent the results available in the literature concerning global attractivity for difference equations are useful to study the role of the production function p in the attractivity properties of equation (69);
  - improving some known theoretical results regarding difference equations and their relation to (69), with the aim of studying the qualitative properties of broader classes of equations of the form (69);
  - extending the techniques concerning linear integral inequalities to obtain global attractivity results for (69) in case p and d are linear but the delay is not bounded.
- To provide sufficient conditions for the existence of positive periodic solutions when the functions p and d in (69) are periodic in the first variable:
  - in cases when the whole history of x is considered;
  - in cases when the solutions are allowed to have discontinuities at certain given times, corresponding to abrupt changes in the phenomena being modelled.

Organisation

To write this manuscript, as highlighted in the preface, the works [17, 18, 19, 20, 21, 83] have been considered and adapted to share a common notation and framework. Some remarks and results have been added too. The organisation of this manuscript is as follows.

In Chapter 1, the basics of delay differential equations are presented. The author has decided to put a particular emphasis on its development, with the help of his two supervisors, aiming to bring together the main concepts and several basic results of this theory (Section 1.1) and to provide some detailed explanations that could potentially facilitate the reading in subsequent chapters. Although proofs are generally omitted in this chapter, some guidelines, intuitive ideas and references to consult are given to help following the main path of this work. Nevertheless, no portion of this chapter is intended to be something new nor considered as a result of the research of the author; it is just his particular view.

Despite of the fact that the main goal of the present work is to obtain results concerning delay differential equations like the one in (69), one of the main mathematical tools that we will use are difference equations. Thus, there is also a relevant part of this work devoted to such equations. In order to include them in our study, in Section 1.2, a general exposition on the basics of semiflows is given (see the references cited therein) in order to save some pages due to the analogy between the theories with discrete and continuous-time.

Afterwards, in Section 1.3, special attention is given to scalar delay differential equations with a linear decay and a delayed feedback function (which might be nonlinear) terms. Those equations find applications in many models [5, Table 2.1]. In particular, the equation

$$x'(t) = -ax(t) + f^*(x(t-\tau)), \tag{70}$$

where $a, \tau > 0$, deserves a particular attention. Such equation has been called the Mackey-Glass-type differential equation [88] due to the particular equations that appeared in the very influential paper by Mackey and Glass [96] in the context of blood cells production. Many results regarding the latter equation are available if it has a unique equilibrium. For instance, the surveys by Liz [79] and Krisztin [71] serve as a great source that sum up the efforts that have been made with respect to that equation for the case when the equilibrium is stable or unstable, respectively. We are maily interested in global asymptotic stability conditions, so we are especially focusing on the cases related to [79]. In fact, some of those results are still true if one replaces τ by a certain nonnegative function $\tilde{\tau}(t)$, as we will see.

In fact, the role of f^* as a map turns out to be highly relevant. The information that comes from the iterates of f^* somehow dominates the asymptotic behaviour of the solutions of the delay differential equation (70), as it will be shown in Subsection 1.3.1.

In fact, we will recall the known conditions to guarantee the local stability of the unique equilibrium of the differential equation in Subsection 1.3.2.

In some cases, global asymptotic stability type results can be obtained through local information on the equilibrium. This is the case when the map f^* satisfies certain geometrical hypotheses. For instance, the Schwarzian derivative of a function will be one of our allies. It is defined by

$$Sf^*(x) = \frac{(f^*)'''(x)}{(f^*)'(x)} - \frac{3}{2}\left(\frac{(f^*)''(x)}{(f^*)'(x)}\right)^2,$$

if x is such that $(f^*)'(x) \neq 0$. When f^* is monotone or unimodal and $Sf^* < 0$ on a certain domain, then that local-global link becomes a realm and the study is facilitated. Such an expression comes from the work by H. Schwarz on conformal maps and Möbius transformations and some decades ago was found to have deep dynamical implications in the works by Allwright [2] and Singer [122]. We refer to [67, Section 1] for a great and thorough introduction of the Schwarzian derivative in terms of its historical background and applications to dynamical systems.

One may wonder whether imposing that f^* has at most one point of extremum or $Sf^* < 0$ are actually strong constraints for equation (70). However, the consideration of such 'humped' production functions is quite common [5, 133] and, as highlighted in [79], many models that have been widely considered also satisfy that hypothesis on the Schwarzian derivative. Thus, we have devoted a great part of Section 1.3.3 to show the main dynamical features of those maps. We also recall some results that have appeared in the literature in the last decades and recall some interesting open problems.

In Chapter 2, we take advantage of a significant part of the machinery that has been introduced in Chapter 1 to obtain results regarding a family of scalar models that are called gamma-models. They have been applied to certain economic or population growth models (see the references cited therein) and their name gives great importance to a certain parameter γ appearing in the production term, since it serves to connect several existing models. We provide a unified treatment for their study and give some results concerning the existence of globally attracting equilibria by using scalar difference equations as the main tool. Since some difficulties arise when trying to apply known conditions to certain particular cases, Chapter 3 is devoted to provide new theoretical results that help fixing such issues. The development of the latter couple of chapters corresponds to the articles [17, 20, 83].

Chapter 4 is devoted to analysing to what extent difference equations are still useful to study global attractivity of delay differential equations like (70), but in $\mathbb{R}^n$ with $n \geq 2$. This is the unique chapter in which we consider systems of delay differential equations and it is developed from the contents in [18].

The four articles corresponding to Chapters 2 to 4 are related with cases of bounded finite delay and correspond to the work that the author has done under the supervision of Prof. Eduardo Liz: a couple of them being co-authored and a couple of them being single-authored.

In Chapter 5, we present several results regarding inequalities of the Gronwall-Bellman-type that aim to find sharper bounds for the solutions of certain types of equations belonging to the family in (69) with a non-delayed linear destruction term. In this chapter, the production term is assumed to be a linear function too, but with the difference that we get rid of the long-term boundedness of the delay and consider what may be called a distributed dependence on past states. The contents of this chapter are based on that of [21] and it is the work that the author has developed with his other supervisor, Prof. Rosana Rodríguez-López.

The last part of the core of this work, Chapter 6, represents the most general branch, at least compared with the rest of the thesis. Although we deal with scalar delay differential equations with a linear term and a delayed feedback, three major differences are present. Firstly, we allow the equations to have infinite delay (even more general than the situation proposed in Chapter 5), which yields the consideration of different phase spaces. Moreover, as a second comment, we assume that such spaces are constituted by certain functions that are allowed to have 'jumps' to include, if necessary, the so-called impulses; thus, the continuity assumption of the elements of the phase space is no longer imposed. Thirdly, unlike the previous part of the thesis, which was related with globally attracting equilibria, the main aim of this chapter is to find periodic solutions. The development of this chapter comes from the one in [19], which is the work initiated during the internships that the author enjoyed at the University of Lisbon, Portugal, under the supervision of Prof. Teresa Faria.

Finally, in Chapter 7, we give a brief critical discussion of the results that we have obtained in these years. Besides, we highlight several future lines of work, some of which are currently being considered by the author of this manuscript. They are mostly related with questions that arise from the previous chapters, but we also give some comments regarding the other internship that the author has done, in this case, at the University of Szeged, Hungary, under the supervision of Prof. Gergely Röst.

We also remark that it might be recommendable taking a look to the chosen notation (see the ending part of the work) before starting the reading of the core of the thesis, although those choices are somehow standard in the fields of Analysis and Topology. Moreover, as natural, the basic notions and results from those areas are assumed to be known.

Methodology

The methodology of this project is based on the study of several well-known references in the field, e.g., [58, 68, 124], which explain the principal features concerning delay differential equations. Moreover, other works as [54, 65, 98] gather the main results regarding the study of the scalar delay differential equations with production and destruction terms [5] by the use of difference equations, as we highlighted above. Hence, acquiring knowledge regarding difference equations turns out to be a crucial step, which can be accomplished through the reading of references such as [29, 33, 119]. Furthermore, the works in [78, 81, 82, 102] serve as a starting point for the part of the project that concerns the global attractivity for the family of scalar gamma-models, while [90] is the basis for the analysis of multidimensional equations in terms of the link with difference equations that has been widely studied in the scalar case.

Additionally, handling the tecnhiques in [52] is relevant in order to provide sharp estimates for the solutions of Gronwall-Bellman inequalities with the aim of applying them to several non-autonomous delay differential equations that are of interest in this thesis.

When it comes to the incorporation of infinite delay and nonlinear impulses in the type of equations that we are considering, the question about the existence of a positive periodic solution provided that the coefficients are periodic may be tackled via the approach in [40].

Besides, the whole above-mentioned study has been complemented with the attendance to many seminars, courses and conferences in this and other related fields, providing knowledge that has become profitable for a better exposition of the contents of this thesis. In fact, the author has tried to balance taking care about the readability of the text and giving detailed explanations. He hopes that the reader finds some enlarged discussions both useful and finally time-saving rather than time-wasting. Indeed, the reasoning is usually supported by some key figures. Since this thesis has been typed via the wonderful tool of LaTeX, almost all of them have been depicted by the use of its package Ti*k*Z [130], while a few of them (whenever it is explicitly stated) have been obtained via numerical simulations based on the MATLAB program[1] *dde23* [101].

[1] On May 21, 2021, the link https://es.mathworks.com/help/matlab/ref/dde23.html is available to check further details.

Chapter 1

Preliminaries

All along the next pages, the reader can find the notation, the concepts and the results that constitute the basis of this manuscript.

Since the main aim of this thesis is to study certain models of delay differential equations, we will devote the first section of this chapter to introducing the basics of such type of equations, with special focus on what is needed in the subsequent chapters. Moreover, the second section deals with a dynamical approach to those equations, providing a convenient framework to study the asymptotic behaviour of the solutions in certain problems. Besides, the contents of that section are presented in a sufficiently general way to let us include difference equations too. With the aim of setting such a general framework, both sections are partially based on references [56, 58, 68, 124, 125].

Finally, taking advantage of the latter, the basic facts of a type of scalar delay differential equations with both linear decay and delayed feedback terms are presented in the third section. In particular, we show some well-known results on the relation between the solutions of such a type of delay differential equations and the solutions of a certain one-dimensional difference equation when it comes to their asymptotic behaviour. We refer to the surveys [71, 79, 133], which have helped to build up this section, and the references therein for further information on this topic.

1.1 Brief introduction to delay differential equations

In this section, we provide the most basic concepts and results to set the bases of the present work. Delay differential equations are a type of differential equations which take into account past states in the evolution of certain variables. The definition of what does a solution of these equations mean, how does the Cauchy problem look like, and when do they show existence, uniqueness and continuous dependence of solutions are important topics to work with, specially when we are interested in studying the limit properties of the solutions as time goes by.

Mathematically, an event that has occurred a certain time ago, namely τ, can be considered to have happened at time $-\tau$, provided that the present time is our reference time and therefore associated with the origin. Thus, past times correspond to elements of the interval $(-\infty, 0]$.

It may happen that the entire history or arbitrarily far time-distant past events of the phenomenon are relevant for their current evolution or it might be that only states regarding the last time interval of a certain finite length $\tau \geq 0$, i.e., $[-\tau, 0]$, actually affect the present changes. The first case will be known as *infinite delay*, while the second one will obviously be referred to as *finite delay*. The second case can be included in the first one, just by omitting what happened before time $-\tau$. Although we can recall the theory in the most general way by choosing the first case, that approach would require a careful treatment which will only be needed for the last part of the thesis and even in a more general context. Therefore, we will focus on the case of finite delay for the sake of introducing the text.

Hereafter, let $\tau \in \mathbb{R}_+$ and $n \in \mathbb{N}$. We define

$$C := \mathcal{C}([-\tau, 0], \mathbb{R}^n) = \{\varphi : [-\tau, 0] \to \mathbb{R}^n : \varphi \text{ is continuous}\}.$$

Since $[-\tau, 0]$ is compact, C is a real Banach space endowed with the norm $\|\cdot\| : C \to \mathbb{R}$ given by

$$\|\phi\| = \sup_{\theta \in [-\tau, 0]} \|\phi(\theta)\|_{\mathbb{R}^n},$$

where $\|\cdot\|_{\mathbb{R}^n}$ is a certain given norm in $\mathbb{R}^n$. Notice that the definition of C actually involves the value of the delay τ and the dimension of the equation n, but we have decided to simplify the notation since the value of that couple of parameters will be clear from the context. Moreover, we remark that recalling the theory of delay differential equations with infinite delay would have required further assumptions, as $\mathcal{C}((-\infty, 0], \mathbb{R}^n)$ is not a Banach space with such a norm (see Chapter 6 for more details).

Let $\sigma \in \mathbb{R}$, $b \in \mathbb{R}^+ \cup \{\infty\}$ and let $x : [\sigma - \tau, \sigma + b] \to \mathbb{R}^n$ be a function. Then, for any $t \in [\sigma, \sigma + b]$, the map $x_t : [-\tau, 0] \to \mathbb{R}^n$, which is defined by

$$x_t(\theta) := x(t + \theta), \quad \theta \in [-\tau, 0],$$

is called a *segment*. The segment related to a certain time t_* sums up the behaviour of the original function in the last time interval of length τ up to t_*. In fact, if x is continuous, x_t belongs to C and, additionally, the segments vary continuously in t [58, Lemma 2.1].

Let D be a subset of $\mathbb{R} \times C$ and $f : D \to \mathbb{R}^n$. Then

$$\dot{x}(t) = f(t, x_t), \tag{1.1}$$

where the dot stands for the right-hand derivative, is said to be a *delay differential equation (DDE)* [58].

If the function f in (1.1) does not depend on t, then we say that the DDE in (1.1) is *autonomous*. Otherwise, we say that the DDE in (1.1) is *non-autonomous*. Besides, the expression in (1.1) can be seen as a generalisation of an ordinary differential equation (ODE) in $\mathbb{R}^n$ by simply considering $\tau = 0$. Furthermore, in applications, it is common to find the next types of DDEs regarding the nature of the function f.

Example 1.1. If the function f in (1.1) is such that there exist $m \in \mathbb{N}$, nonzero bounded functions $\tilde{\tau}_i : \mathbb{R} \to \mathbb{R}_+$, $i \in \{1, \ldots, m\}$, and a certain function $f^* : \operatorname{Dom} f^* \subset \mathbb{R}^{m+2} \to \mathbb{R}^n$ such that

$$f(t, x_t) = f^*(t, x(t), x(t - \tilde{\tau}_1(t)), \ldots, x(t - \tilde{\tau}_m(t))), \tag{1.2}$$

then we say that equation (1.2) is a DDE with *discrete delays* or a *differential-difference equation* [58]. If, for a certain $i \in \{1, \ldots, m\}$, $\tilde{\tau}_i(t)$ is constant, we say that the i-th delay is a *constant delay*. Otherwise, we say that the i-th delay is a *variable delay*. In the case of equation (1.2), we will denote

$$\tau_i := \sup_{t \in \mathbb{R}} \tilde{\tau}_i(t) > 0, \; i \in \{1, \ldots, m\}, \quad \tau := \max_{i \in \{1, \ldots, m\}} \tau_i > 0.$$

Bear in mind that the notation of τ is consistent with what we said about the meaning of finite delays at the beginning of the current section. Besides, if $\tilde{\tau}_i$ is a constant positive function, we have $\tilde{\tau}_i(t) = \tau_i$, for all $t \in \mathbb{R}$, so we will directly use the notation τ_i when it comes to a constant delay. If $m = 1$, we will also avoid using subindexes and write $\tilde{\tau}(t)$ and τ. For instance,

$$\dot{x}(t) = x(t - |\sin(t)|), \quad \dot{x}(t) = -Ax(t) + Bx(t - 1),$$

where $A, B \in \mathcal{M}_{2\times 2}(\mathbb{R})$, are DDEs with a single discrete delay and $\tau = 1$. In the first case, the delay is variable, while in the second one it is constant. Moreover, the first example is a scalar non-autonomous DDE, whereas the second one is a multidimensional autonomous DDE.

Example 1.2. If the expression of the function f in (1.1) has a term

$$\int_{t-\tau}^{t} k(t, s) x(s) \, ds, \tag{1.3}$$

for an appropriate nonzero continuous kernel $k(t, s)$, we say that equation (1.3) is a DDE with *distributed delay*. For example, the equation

$$\dot{x}(t) = -a(t)x(t) + \frac{1}{\tau} \int_{t-\tau}^{t} x(s) \, ds,$$

where $\tau > 0$ and $a(t)$ represents a certain scalar function, is a scalar non-autonomous DDE with distributed delay.

After having provided some examples of expression (1.1), it is of interest to recall what solving equation (1.1) means. If $b \in \mathbb{R}^+ \cup \{\infty\}$, a function $x : [\sigma - \tau, \sigma + b) \to \mathbb{R}^n$ is said to be a *solution* of (1.1) if it is continuous, $(t, x_t) \in D$ for any $t \in [\sigma, \sigma + b)$, and it satisfies (1.1).

For a given $(\sigma, \phi) \in D$, we say that a solution of equation (1.1) satisfying the condition $x(\sigma + \theta) = \phi(\theta)$, $\theta \in [-\tau, 0]$, is a *solution* of (1.1) *through* (σ, ϕ) or the solution to the Cauchy problem

$$\begin{cases} \dot{x}(t) = f(t, x_t), \\ x_\sigma = \phi, \end{cases} \tag{1.4}$$

and (σ, ϕ) is its *initial condition*.

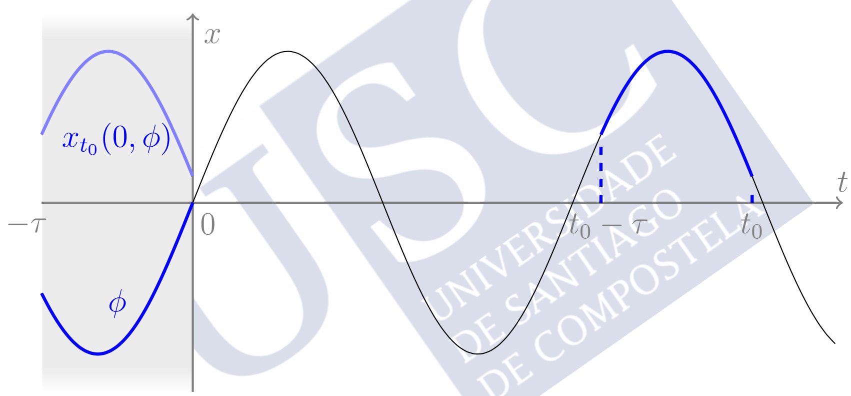

Figure 1.1: A couple of segments related to a certain solution of an equation of the type in (1.4) with $\sigma = 0$. The gray zone aims to represent the points in the plane where the graph of any function in C lies. Notice that this is just a sketch; no differentiability is needed to be assumed for $x_0(0, \phi) = \phi$.

We will say that there exists a *unique solution* of (1.1) through (σ, ϕ) if, given two solutions of (1.1) through (σ, ϕ), they take the same values on the intersection of their domains. If there is a unique solution of (1.1) through (σ, ϕ), it will be denoted by $x(t; (\sigma, \phi))$. In a similar manner, its corresponding segments will be written as $x_t(\sigma, \phi)$.

In case that there exists an open set $U \subset D$ such that there is a unique solution of (1.1) through each $(\sigma, \phi) \in U$, we say that the equation (1.1) shows *continuous dependence on initial data* on U if, for every sequence $((\sigma_k, \phi_k))_{k \in \mathbb{N}}$ in U converging to $(\sigma_0, \phi_0) \in U$, there exist $\delta \in (0, \tau)$ and $b > \sigma$ such that x^k converges uniformly to x^0 as $k \to \infty$ on $[\sigma - \tau + \delta, b]$, where x^k is a solution of (1.1) through (σ_k, ϕ_k), $k \in \mathbb{Z}_+$.

Another natural issue is whether the solutions are defined for every future time or not. At this point, the concept of continuation of solutions comes into play. If $x : [\sigma - \tau, b) \to \mathbb{R}^n$

is a solution of (1.1), then another solution $x^* : [\sigma - \tau, b_*) \to \mathbb{R}^n$ of (1.1) is a *continuation* of x provided that $b_* \geq b$ and $x(t) = x^*(t)$, $t \in [\sigma - \tau, b)$. A solution x of (1.1) will be called *maximal* if it does not admit any proper continuation.

Therefore, under conditions of existence and uniqueness of the solution through an initial condition $(\sigma, \phi) \in D$, there exists $t_{\sigma,\phi} \in (\sigma, \infty]$ such that the maximal solution of (1.4) is defined on $[\sigma - \tau, t_{\sigma,\phi})$ [56, 68].

The following results deal with the previous concepts of existence, uniqueness of solutions of (1.1) and their continuous dependence on initial data. They sum up the main results in [58, Chapter 2], with some support from [68]. They are organised in a hypotheses-dependent way to highlight how further restrictions on the right-hand side of (1.1) actually affect the aforementioned properties.

Just before showing the next results, we remark that it is common to find equation (1.1) in its more regular form

$$x'(t) = f(t, x_t), \tag{1.5}$$

bearing in mind that $x'(\sigma)$ does not necessarily exist. Nevertheless, if the function f in (1.1) is continuous, solutions of (1.1) become differentiable after the initial time[1] and thus, $x'(t)$ makes sense for $t > \sigma$. Since we will always work with a continuous right-hand side in (1.1), we may use its alternative version (1.5) from now on when we refer to the former one. In fact, it is known [58, Chapter 2, Lemma 1.1] that, provided f is continuous, then finding a solution of (1.1) through (σ, ϕ) is equivalent to solving the integral equation

$$\begin{aligned} x(t) &= \phi(0) + \int_\sigma^t f(s, x_s)\, ds, \quad t \in [\sigma, \sigma + b), \\ x_\sigma &= \phi, \end{aligned} \tag{1.6}$$

for a certain $b > 0$. It is worth highlighting [38] that one could have chosen weaker assumptions than the continuity of f. In fact, by relaxing the concept of solution, the study of the integral form (1.6) is still possible by assuming Caratheodory-type conditions [58, Page 58], but, as mentioned above, that will not be the case in this work.

The following result provides further consequences of assuming that the right-hand side of (1.1) is continuous.

Theorem 1.3. *[58, Chapter 2] Let D be an open set of $\mathbb{R} \times C$, $f : D \to \mathbb{R}^n$ be a continuous function and $(\sigma, \phi) \in D$. Then the following statements hold:*

1. *There exists a solution of (1.1) through (σ, ϕ).*

[1] An analogous fact is highlighted, e.g., in [31, Page 453]. For further details on the validity of this fact, see the comments concerning [116, Chapter 7] at the end of the book.

2. *If there is an open neighbourhood* $U \subset D$ *of* (σ, ϕ) *such that there is a unique solution of* (1.1) *through each initial condition in* U*, then* (1.1) *shows continuous dependence on initial data on* U*.*

3. *If* $x : [\sigma - \tau, b) \to \mathbb{R}^n$ *is a maximal solution of* (1.1) *through* (σ, ϕ)*, then, for each compact set* $W \subset D$*, there exists* $t_W \in \mathbb{R}$ *such that* $(t, x_t) \notin W$*, for* $t_W \le t < b$*.*

The proof of the first and second theses of Theorem 1.3 need the construction of an operator, whose expression is based on that one in (1.6) and whose fixed points are associated with the solutions of (1.1) through (σ, ϕ) [58, 68]. The existence of a fixed point is then guaranteed by using Schauder's Fixed Point Theorem [28, Theorem 8.8], since such an operator is completely continuous, a concept that will be stated later.

Remark 1.4. We have seen in Theorem 1.3 that, provided the function f in (1.1) is continuous, the elements (t, x_t), where x is a maximal solution of (1.1) through (σ, ϕ), must leave every compact set in D. As highlighted in [58], this does not mean that such a solution leaves every closed and bounded set of D (for instance, balls), since not all of those sets are compact.

For instance, as it is done in [68, Chapter 3, Theorem 2.1 c)] and its subsequent comments, if we consider the particular domain $D = (a, \infty) \times C$, for some $a \in \mathbb{R} \cup \{-\infty\}$, then any maximal solution $x : [\sigma - \tau, b) \to \mathbb{R}^n$ of (1.1) such that $b < \infty$, does not have a finite limit as $t \to b^-$. In fact, there exist examples (see, e.g., pages 46–48 in [58]) for which a bounded maximal solution of (1.1) oscillates in a way that it is impossible to define a continuous segment x_b. Nevertheless, a result similar to the ordinary case when it comes to leaving every closed and bounded set can be obtained provided we impose further restrictions on f, as we will see later.

We now continue our way regarding the imposition of further constraints on the function f in order to obtain a better behaviour of the solutions of (1.1). In fact, the results that are coming, namely Theorems 1.5 and 1.8, analyse further independent hypotheses.

In the first branch, we recall a way to improve Theorem 1.3 regarding the uniqueness of solutions and, therefore, enhancing and having implications in the first and second theses of such a result. The key idea is to impose Lipschitz-type assumptions on a certain class of subsets of D. For example, we say that $f : D \to \mathbb{R}^n$ is *locally Lipschitzian with respect to the second variable* if, for every $(\sigma, \phi) \in D$, there exists a neighbourhood $W_{(\sigma,\phi)}$ of (σ, ϕ) and a constant $K_{\sigma,\phi} \ge 0$ such that

$$\|f(t, \psi_1) - f(t, \psi_2)\|_{\mathbb{R}^n} \le K_{\sigma,\phi} \|\psi_1 - \psi_2\|, \quad \text{for every } (t, \psi_1), (t, \psi_2) \in W_{(\sigma,\phi)}. \tag{1.7}$$

Theorem 1.5. *[58, Chapter 2, Theorem 2.3] Let* D *be an open set of* $\mathbb{R} \times C$ *and assume that the function* $f : D \to \mathbb{R}^n$ *is continuous and locally Lipschitzian with respect to the second variable. If* $(\sigma, \phi) \in D$*, there exists a unique solution of* (1.1) *through* (σ, ϕ)*.*

The previous result may be proved in terms of well-known arguments on contractivity and can also be stated in a more general sense. We briefly mention two ways to obtain less restrictive hypotheses without affecting the thesis of Theorem 1.5.

Firstly, the Lipschitz-type condition in (1.7) can be considered for compact subsets of D instead of neighbourhoods of points of D. Such condition, together with the continuity of f, is also sufficient for the validity of the thesis of Theorem 1.5 (see the arguments in [58, Section 2.2]). Moreover, it is obvious that a function that is continuous and globally Lipschitz with respect to the second variable will also satisfy the Lipschitz-type hypothesis in Theorem 1.5.

Secondly, the Lipschitz-type assumption does not need to be imposed for every pair $(t, \psi_1), (t, \psi_2) \in W_{(\sigma,\phi)}$, but only to those pairs that also satisfy

$$\psi_1(\theta) = \psi_2(\theta), \quad \theta \in [-\tau, -\varepsilon_{\sigma,\phi}],$$

for a certain $\varepsilon_{\sigma,\phi} \in (0, \tau)$ depending on the initial condition [68, Chapter 3, Subsection 2.2]. Then, in the proof of Theorem 1.5, one can ensure that there is a unique solution on $[\sigma-\tau, \sigma+\alpha_{\sigma,\phi}]$ with $\alpha_{\sigma,\phi} \in (0, \varepsilon_{\sigma,\phi})$ and then recursively repeat the procedure to obtain the existence of a unique solution of (1.1) through (σ, ϕ) in $[\sigma-\tau, \sigma+2\alpha_{\sigma,\phi}]$, $[\sigma-\tau, \sigma+3\alpha_{\sigma,\phi}]$, and so on. This technique can be utilised up to time $t_{\sigma,\phi}$, through which there is no proper continuation.

In fact, the previous comment is related to a method that sometimes leads to obtaining the explicit expression of the solutions on finite time intervals: *the method of steps*, which we explain now before continuing the main path of the current section.

Remark 1.6. [68, Chapter 3, Section 1.3] Assume that f is of the type in (1.2) with $\tilde{\tau}_i(t) > 0$, for every $i \in \{1, \ldots, m\}$ and $t \in \mathbb{R}$. Then, if f^* is continuous and Lipschitzian with respect to the second variable (the one related with $x(t)$), the solutions of the equation (1.2) can be obtained via the solutions of ordinary differential equations on the intervals $[t_i, t_{i+1})$, $i \in \{0, 1, \ldots, p-1\}$, for a certain $p \in \mathbb{N} \cup \{\infty\}$ and

$$\sigma = t_0 < \ldots < t_i < t_{i+1} < \ldots < t_p = t_{\sigma,\phi}.$$

We illustrate this fact in the case of $m = 1$ and a constant positive delay, that is,

$$f(t, x_t) = f^*(t, x(t), x(t-\tau)), \quad \text{with } \tau > 0.$$

Let $(\sigma, \phi) \in D$. Then, for any $t \in [\sigma, \sigma+\tau]$, equation (1.1) takes the form

$$x'(t) = f^*(t, x(t), \phi(t-\sigma-\tau)).$$

Since ϕ is known, the equation can be formally solved as an ODE on $[\sigma, \sigma+\tau]$. Theoretically, we can repeat this process and solve the equation on $[\sigma+\tau, \sigma+2\tau]$ again as an

ODE, bearing in mind that the solution on $[\sigma, \sigma+\tau]$ (which is unique) is actually known from the previous step. This method can be applied until we reach the time $t_{\sigma,\phi}$.

Notice that no Lipschitz-type assumption has been imposed on the third variable of f^*. Thus, one might even have uniqueness of solutions of (1.1) for continuous f that do not satisfy Lipschitz-type assumptions on x_t (see, e.g, [124, Theorem 3.2]).

Example 1.7. [12, Example 3.2] To complement the explanation shown in Remark 1.6, assume, for instance, that $\tau = 1$ and that $f(t, x_t) = x_t(-1) = x(t-1) = f^*(t, x(t), x(t-1))$; then, equation (1.1) takes the form

$$x'(t) = x(t-1). \tag{1.8}$$

If the initial condition satisfies $\phi(\theta) = 1$, for every $\theta \in [-1, 0]$, the solution of (1.8) through $(0, \phi)$ satisfies (see also [124])

$$x(t; (0, \phi)) = \sum_{k=0}^{n+1} \frac{(t-k+1)^k}{k!}, \quad t \in [n, n+1),\ n \in \mathbb{Z}_+.$$

The second branch regarding further hypotheses on Theorem 1.3 deals with a stronger thesis compared to the third one in that result, the one about the behaviour of maximal solutions of (1.1). To do that, we need to recall the following concepts.

If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two real Banach spaces, we say that an operator $g : M \subset X \to Y$ is *bounded* if it takes bounded subsets of M into bounded subsets of Y. Moreover, we say that g is *completely continuous* if it is continuous and takes bounded subsets of M into relatively compact subsets of Y. It is clear that every completely continuous operator is also bounded since every relatively compact set is bounded in a metric space.

In particular, if $(Y, \|\cdot\|_Y)$ is the n-dimensional Euclidean space, then a continuous operator $g : M \to \mathbb{R}^n$ is completely continuous if and only if it is bounded, because of boundedness and relative compactness of a subset being equivalent in $\mathbb{R}^n$ due to the well-known Theorem of Heine-Borel [128, Theorem 5.7.1]. For instance, the function $f : D \to \mathbb{R}^n$ in the right-hand side of (1.1) is completely continuous provided it is bounded and continuous.

Theorem 1.8. *[58, Chapter 2, Theorem 3.1] Let D be an open set of $\mathbb{R} \times C$, $(\sigma, \phi) \in D$, and $f : D \to \mathbb{R}^n$ be a bounded continuous function. If $x : [\sigma - \tau, b) \to \mathbb{R}^n$ is a maximal solution of* (1.1) *through (σ, ϕ), then, for each closed and bounded set $U \subset D$, there exists $t_U \in \mathbb{R}$ such that $(t, x_t) \notin U$, $t_U \leq t < b$.*

Example 1.9. There exist simple sufficient conditions to ensure the applicability of Theorem 1.8. In fact, in the line of [38], the function f is completely continuous if it is continuous and there exist continuous functions $\beta : \mathbb{R} \to \mathbb{R}_+$ and $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|f(t,\phi)\|_{\mathbb{R}^n} \leq \beta(t)\,\mu(\|\phi\|), \quad \forall (t,\phi) \in D.$$

Remark 1.10. Under the hypotheses of Theorem 1.8, if we also assume that the domain of f is $D = (a,\infty)\times C$, for some $a \in \mathbb{R}\cup\{-\infty\}$, then any maximal solution $x : [\sigma-\tau, b) \to \mathbb{R}^n$ of (1.1) such that $b < \infty$, satisfies (see [68])

$$\limsup_{t\to b^-} \|x(t)\|_{\mathbb{R}^n} = \infty,$$

that is, x is unbounded.

We can gather the new assumptions coming from both branches in the following result.

Corollary 1.11. *Let D be an open set of $\mathbb{R} \times C$, $f : D \to \mathbb{R}^n$ be a completely continuous function and locally Lipschitzian with respect to the second variable, and $(\sigma,\phi) \in D$. Then the following statements hold:*

1. *There exists a unique solution of* (1.1) *through (σ,ϕ).*
2. *The equation* (1.1) *shows continuous dependence on initial data.*
3. *If $x : [\sigma-\tau, t_{\sigma,\phi}) \to \mathbb{R}^n$ is the maximal solution of* (1.1) *through (σ,ϕ), then, for each closed and bounded set $U \subset D$, there exists $t_U \in \mathbb{R}$ such that $(t,x_t) \notin U$, for $t_U \leq t < t_{\sigma,\phi}$.*

Example 1.12. In the case of discrete delay, that is, the one illustrated in (1.2), if we further assume that f^* is of class $\mathcal{C}^1$, then the function f fulfils all the related hypotheses in Corollary 1.11.

Example 1.13. If we assume that f is continuous and Lipschitzian with respect to the second variable on every bounded set, that is, if for every bounded subset $V \subset D$, there exists $K_V \geq 0$ such that

$$\|f(t,\phi) - f(t,\psi)\|_{\mathbb{R}^n} \leq K_V\|\phi-\psi\|, \quad \forall (t,\phi),(t,\psi) \in V,$$

then f satisfies the hypotheses of Corollary 1.11 (see [124, Theorem 3.7]). As in Remark 1.10, if we further assume that $D = (a,\infty)\times C$, $a \in \mathbb{R}\cup\{-\infty\}$, and that the Lipschitz constant K_V is independent of V (f is globally Lipschitzian with respect to the second variable), then $t_{\sigma,\phi} = \infty$ for every $(\sigma,\phi) \in D$.

We have presented several basic results for the general non-autonomous equation (1.1). Nevertheless, the case in which f in (1.1) does not depend on the first variable, i.e., the autonomous case of (1.1), is widely used in applications. Hence, we will also consider the equation

$$x'(t) = g(x_t), \tag{1.9}$$

where $g : \tilde{D} \to \mathbb{R}^n$ and $\tilde{D}$ is a subset of C. Note that equation (1.9) is included in the framework of (1.1) by taking $D := \mathbb{R} \times \tilde{D}$ and $f(t, \phi) := g(\phi)$.

Since the behaviour of a solution in the autonomous case does not depend on the initial time σ, but only depends on ϕ, the initial time can be shifted to 0 without loss of generality. Hence, the initial condition will always be considered as $(0, \phi)$ and, therefore, we can refer to solutions of (1.9) through ϕ. Additionally, under uniqueness of solutions of (1.9), whenever it is clear, the solution of (1.9) with condition $x(\theta) = \phi(\theta)$, $\theta \in [-\tau, 0]$, for a given $\phi \in \tilde{D}$, will be denoted by $x(t; \phi)$. Besides, the corresponding segments will be written[2] as $x_t(\phi)$. Finally, in such a case, there exists $t_\phi \in (0, \infty]$ such that the maximal solution of (1.9) through any $\phi \in \tilde{D}$ is defined on $[-\tau, t_\phi)$.

We could rewrite all the results coming from the non-autonomous case for the particular equation (1.9), but they would just be straightforward adaptations. Hence, we only rewrite Corollary 1.11 for such a case.

Corollary 1.14. *Let $g : C \to \mathbb{R}^n$ be a function that is locally Lipschitzian and satisfies*

$$\limsup_{\|\phi\| \to \infty} \frac{\|g(\phi)\|_{\mathbb{R}^n}}{\|\phi\|} \leq K.$$

Then, the following statements hold:

1. *For any $\phi \in C$, there exists a unique solution $x(t; \phi)$ of (1.9), that is defined on $[-\tau, \infty)$ and, thus, it is maximal.*

2. *The equation (1.9) shows continuous dependence on initial data.*

1.2 Asymptotic behaviour and semiflows

If we are interested in the long-term behaviour of the solutions, we obviously need them to be defined for all future times. Hence, the corresponding segments x_t have to be defined for every time $t \in [\sigma, \infty)$ too. This has a straightforward consequence: a natural choice for the domain of the function f in (1.1) is

$$D = (a, \infty) \times \tilde{D},$$

[2]It is also common to find the notation x_t^ϕ (see, e.g., [71, 133]).

where $a \in \mathbb{R} \cup \{-\infty\}$ and $\tilde{D}$ is an open subset of C [58, Page 130].

Notice that, in the autonomous case, we can choose $a = -\infty$ and, therefore, we have $D = \mathbb{R} \times \tilde{D}$. In fact, even for several non-autonomous cases (like the ones considered in Chapter 2), it is also possible to make such choice, as we will see later.

Thus, assume that we also impose that equation (1.1) has a unique maximal solution through each initial condition $(\sigma, \phi) \in D$. Then, one can associate each of those conditions to the long-term behaviour of the corresponding *trajectories* $(t, x_t(\sigma, \phi))$ as $t \to \infty$.

Under such framework, computing the explicit expression of the solutions of a DDE is one way to obtain information with respect to its long-term behaviour. Although there exist particular cases where we can compute an explicit expression for a solution, as it happens with some DDEs with a single discrete delay (see the method of steps in, e.g., Example 1.7), we expect that, in general, such an expression is not known or it is hard to work with. Therefore, we are interested in obtaining information about the solutions without solving the equation and here is where the qualitative theory comes into action. These ideas are presented and extended in the current section.

In particular, most part of this work (Chapters 2–5) is related to results that provide sufficient conditions to ensure that all the solutions of a DDE have a simple asymptotic behaviour. In fact, the following concepts will play a key role in that line.

Definition 1.15. Let $D = (a, \infty) \times \tilde{D}$, where $a \in \mathbb{R} \cup \{-\infty\}$ and $\tilde{D}$ is an open subset of C. If there exist $p \in \mathbb{R}^n$ and $\sigma > a$ such that

$$x(t) = p, \quad t \in [\sigma - \tau, \infty),$$

is a solution of (1.1), then p is an *equilibrium* of (1.1). We will also refer to $\hat{p}$, the element of C that takes the constant value p, as an *equilibrium* of (1.1).

- An equilibrium is *stable* provided that, for every $\sigma > a$ and $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, \sigma) > 0$ such that $\|\phi - \hat{p}\| < \delta$ implies $\|x_t(\sigma, \phi) - \hat{p}\| < \varepsilon$, for every $t \geq \sigma$.

- An equilibrium *attracts points locally* or is *locally attracting* if there exists $k(\sigma) > 0$ such that $\|\phi - \hat{p}\| < k(\sigma)$ implies $x(t; \phi) \to p$ as $t \to \infty$. An equilibrium *attracts points globally* or is *globally attracting* if the latter holds for every positive $k(\sigma)$.

- An equilibrium is *locally asymptotically stable (LAS)* or *globally asymptotically stable (GAS)* if it is stable and, respectively, attracts points locally or globally.

Definition 1.16. Let $D = (a, \infty) \times \tilde{D}$, where $a \in \mathbb{R} \cup \{-\infty\}$ and $\tilde{D}$ is an open subset of C. If there exist $\eta \in \mathbb{R}^+$ and $\sigma > a$ such that a solution of (1.1) satisfies

$$x(t + \eta) = x(t), \quad t \in [\sigma - \tau, \infty),$$

then such a solution is called a *η-periodic solution* of (1.1).

One can also give analogous concepts regarding stability and attractivity of a periodic solution (see, e.g., [58, Chapter 5]).

Following the remarks in [58, Chapter 3], considering the solutions only as functions with values in $\mathbb{R}^n$ does not constitute a fully satisfactory framework regarding the dynamical properties of a solution $x(t)$ of (1.1) as $t \to \infty$, in contrast to what happens in classical ordinary differential equations. For instance, the *phase space* is known to be the set that contains the admissible initial conditions and, in this framework, the initial conditions require the use of segments; in fact, they are a pair formed by initial 'time' σ and an initial 'state', where such state is represented by a segment ϕ, that is, a function of C. Thus, it is more natural to assume that the phase space lies in C and analyse the 'functional' behaviour of $x_t(\sigma, \phi)$ as $t \to \infty$.

For the general non-autonomous equation (1.1), a formal study of this asymptotic behaviour may be done by working with the notion of *process* [58, Chapter 4]. Nonetheless, a great part of the ideas that we will use, e.g., in Chapter 2, are shown in an easier way for the case of autonomous delay differential equations, for which we also recall deeper results. Additionally, the particular expression of the equations that we study in a significant part of this thesis is related to certain autonomous difference equations. Therefore, for the rest of this section, we have restricted our attention to the underlying qualitative features of the autonomous case related to (1.1), i.e., equation (1.9). For the latter equation, the initial time is irrelevant and, hence, we will work with a more restrictive concept than a process: a *semiflow*, which is introduced below. Intuitively, it is a tool that provides a general context to study how a 'state' evolves as 'time' goes by when following a certain rule of evolution, such as a delay differential equation or a difference equation. In order to include both cases under a general framework, we need to define what do 'state' and 'time' mathematically mean.

Firstly, the term 'state' refers to the value that a variable takes in the space of all possible values that it can take. Therefore, this is an idea with a spatial connotation. As mentioned in the introduction, although we can recall several notions and results of the theory of dynamical systems in a very general setting, e.g., when working with dynamics in general sets, metric spaces provide a sufficiently wide framework to tackle many problems [56, 123, 125] and, in particular, the ones considered in this thesis. For a basic reference on metric spaces, we refer the reader to [128]. From now on, unless more explicit conditions are given, we will suppose that (X, d) is a metric space, that is, a set X endowed with a metric d. Whenever the structure of the space is clear, a unique X will refer to both the underlying space and its structure.

Secondly, in order to formally express how the elements of a set change their position over time, we also have to know what the word 'time' actually stands for. Since we will only deal with discrete-time and continuous-time dynamics, the classical choices of $\mathbb{Z}_+$ or

$\mathbb{R}_+$ will be sufficient to state all the subsequent results. Therefore, the set J, that will be used to refer to possible times, will hereafter be either $\mathbb{Z}_+$ or $\mathbb{R}_+$.

The main reason of considering a common context written in terms of J, as it is done in [125], instead of each particular case, is that many results and concepts are completely analogous, so we can avoid repetitive arguments when showing how to include DDEs in this framework and working with an important tool: difference equations. In fact, according to that reference, many results are still valid if J belongs to a broader class of sets, namely the time-sets. In particular, two main and simple time-sets are $\mathbb{Z}_+$ and $\mathbb{R}_+$, i.e., the 'smallest' and the 'biggest' possible ones, respectively.

Once we have stated what 'time' and 'space' mean, we are ready for the following definition.

Definition 1.17. A *semiflow* or *semidynamical system* is a map $\varphi : J \times X \to X$ satisfying the following two properties:

- $\varphi(0, x) = x$, for every $x \in X$,
- $\varphi(t + s, x) = \varphi(t, \varphi(s, x))$, for every $t, s \in J$ and $x \in X$.

The family of mappings $\varphi_t := \varphi(t, \cdot) : X \to X$, $t \in J$, together with the composition of mappings form a semigroup. Each of those mappings represent a picture of the evolution of points in X at a certain time. In fact, the value $\varphi_t(x) = \varphi(t, x)$ represents the position where x is mapped to at time t. Generally speaking, an analogous interpretation can be made for $\varphi_t(B)$ for any subset $B \subset X$. Therefore, the following well-known concept naturally arises: for any subset $B \subset X$, the *forward orbit* [124] is defined as

$$\gamma^+(B) := \{\varphi_t(B) : t \in J\},$$

which can be thought as the 'footprint' of B by following the dynamics given by φ. When we refer to a singleton set, we will simply write $\gamma^+(x)$ instead of $\gamma^+(\{x\})$.

Continuity is a feature that appears in the evolution of many phenomena. Hence, we define a *continuous semiflow* or a *continuous semidynamical system* as a continuous map $\varphi : J \times X \to X$ satisfying the properties in Definition 1.17, where the space $J \times X$ is endowed with the product topology of the one in J (inherited from the usual topology in $\mathbb{R}$) and the one in X (induced by the metric). In the rest of this work, we will only handle continuous semiflows.

In particular, if $J = \mathbb{Z}_+$, i.e., when the *discrete-time* case is considered, a continuous semiflow is generated by the iterations of a certain continuous function $F : X \to X$. In fact, that map must be $F := \varphi(1, \cdot)$. In such a case, it is also common to use the notation F^n instead of $\varphi(n, \cdot)$, where

$$F^n := \overbrace{F \circ \cdots \circ F}^{n}.$$

All along the first part of this dissertation, we will deal with difference equations and the discrete-time semiflows generated by them. By a *difference equation*, we refer to

$$x_{n+1} = F(x_n), \quad n \in \mathbb{Z}_+, \tag{1.10}$$

where $F : X \to X$ is a certain function and we may informally say that it generates the difference equation (1.10). Every solution of (1.10) is a sequence $(x_n)_{n\in\mathbb{Z}_+}$ in X. For instance, if we additionally assume that $X = I$, where I is a real interval, it is common to find in the literature the so-called *cobweb analysis* [69] or *graphical analysis* [29], which is useful to observe the dynamical behaviour of the solutions of (1.10) in a geometrical way (see Figure 1.2).

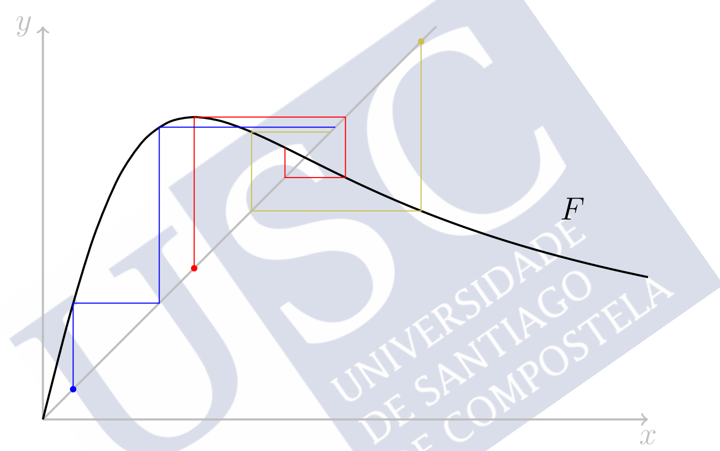

Figure 1.2: Cobweb/Graphical analysis. The behaviour of the forward orbit of different points $x_0 \in X \subset \mathbb{R}$ is represented by the paths (in red, blue or yellow) drawn by the lines connecting the points (x_0, x_0), $(x_0, F(x_0))$, $(F(x_0), F(x_0))$, $(F(x_0), F^2(x_0))$, $(F^2(x_0), F^2(x_0))$,...

When considering $J = \mathbb{R}_+$, i.e., the *continuous-time* case, differential equations may be included within this framework. For instance, an ordinary differential equation, or even a delay differential equation like (1.1) can define a semiflow provided certain suitable conditions are imposed on the phase space and the function f in the right-hand side of the equation. Note that C can be considered as a metric space. In particular, if d is the distance induced by the norm in C, i.e.,

$$d(\phi, \phi_*) = \|\phi - \phi_*\|_\infty = \sup_{\theta\in[-\tau,0]} \|\phi(\theta) - \phi_*(\theta)\|_{\mathbb{R}^n},$$

then (C, d) is a metric space. Thus, any subset of C (e.g., $\tilde{D}$) is also a metric space endowed with the inherited metric and it may be an admissible phase space in the previously-stated theory about dynamical systems. For instance, if $M \subset \mathbb{R}^n$, we define the set

$$C_M := \{\phi \in C : \phi(\theta) \in M,\ \theta \in [-\tau, 0]\},$$

which is, in other words, the set of continuous functions on $[-\tau, 0]$ whose image is contained in M.

In fact, under conditions of existence, uniqueness and continuation of solutions, like those in Corollary 1.14, a semiflow naturally arises as explained below. From now on and for the sake of simplicity, we asume, unless otherwise stated, that $\tilde{D} = C$ in (1.9).

An interesting remark is that a continuous-time semiflow also generates a discrete-time semiflow by, e.g., considering $\varphi_t = \varphi(t, \cdot)$ only for nonnegative integer times.

Remark 1.18. Whenever we refer to a difference equation, we will mainly use capital letters that are common to represent maps in the framework of discrete dynamics, including, for instance, F, G, H, T, Λ,... [29, 30]. Alternatively, we may use certain lower-case letters for every function appearing in the right-hand side of a DDE (1.1): f, g, h, etc. This claim does not constitute an absolute statement for the rest of the thesis, but such difference becomes clear in results or sections where both types of equations are related. This has been chosen to highlight more clearly whether we are referring to a function appearing in the context of a DDE or in the one of a difference equation and thus help avoiding misunderstandings.

Proposition 1.19. *[125, Proposition 5.28] If f is continuous, there exists $V \subset \tilde{D}$ such that there is a unique maximal solution $x(t;\phi)$ of (1.9) through each $\phi \in V$ and $x_t(\phi) \in V$, for every $t \in \mathbb{R}_+$, then the map $\varphi : \mathbb{R}_+ \times V \to V$ given by*

$$\varphi(t, \phi) = x_t(\phi),$$

defines a continuous semiflow on V.

The proof follows from a combination of checking that the main properties of the notion of semiflow hold [124] and considering the theses of Theorem 1.3 for the autonomous case.

Remark 1.20. Actually, the semiflows that can be generated by the DDE in (1.9) are very special among the semiflows on subsets of C, in the sense that they show shift-like properties [68, Page 99]. That particular feature allows working with DDEs via the approach of the infinite-dimensional phase space C and the one of the base space $\mathbb{R}^n$ (the evolution of $x_t(0) = x(t) \in \mathbb{R}^n$ has a major role), which is useful to prove many results.

So, once we have introduced the framework of semiflows, we would like to know the behaviour of $\varphi_t(x)$ as $t \to \infty$ for a certain $x \in X$. In other words, whenever it makes sense, we would like to determine the set of points to which $\varphi_t(x)$ approximates as $t \to \infty$. The following basic concept aims to fix this intuitive idea.

Definition 1.21. Let $\varphi : J \times X \to X$ be a semiflow and $B \subset X$. The *ω-limit set* of B (relative to φ) is defined as

$$\omega(B) := \bigcap_{s \in J} \overline{\bigcup_{t \geq s} \varphi_t(B)}. \tag{1.11}$$

Once more, if $B = \{x\}$, i.e., a singleton subset, the notation $\omega(x)$ will be used instead of $\omega(\{x\})$. Furthermore, if we want to emphasise the dependence of the expression in (1.11) on the semiflow φ, we will write $\omega_\varphi(B)$.

Due to its definition, any ω-limit set is always a closed subset of X and it can be proven that a point $y \in \omega(B)$ if and only if there exists a sequence $(t_n, x_n)_{n \in \mathbb{Z}_+}$ in $J \times B$, with $t_n \to \infty$, such that $\varphi(t_n, x_n) \to y$ [125, Lemma 2.8].

In order to state the main properties of the ω-limit sets, the following concepts become relevant.

Definition 1.22. Let $\varphi : J \times X \to X$ be a semiflow. A set $M \subset X$ is said to be *forward invariant* if $\varphi_t(M) \subset M$ for every $t \in J$. A set $M \subset X$ is said to be *invariant* if $\varphi_t(M) = M$ for every $t \in J$.

Clearly, an invariant set is also a forward invariant set. Additionally, it can be shown that a set is forward invariant under φ if $\gamma^+(x) \subset M$, for every $x \in M$. Furthermore, M is also invariant under φ if there exists what is called a complete orbit $\gamma(x)$ through x satisfying $\gamma(x) \subset M$, for every $x \in M$. As only forward-in-time features of the solutions are considered in this text, we will not focus on backwards states and we refer to [125] for further details on complete orbits. Moreover, it is almost clear that forward invariant sets admit its own inherited semiflow.

Corollary 1.23. *If $\varphi : J \times X \to X$ is a semiflow and M is forward invariant, then $\varphi_{|J \times M} : J \times M \to M$ is also a semiflow.*

Some relevant examples of invariant sets are equilibria and periodic orbits. A point $c \in X$ is said to be an *equilibrium* of the semiflow if $\varphi_t(c) = c$, for all $t \in J$. Moreover, if there are $\eta \in J \setminus \{0\}$ and $x \in X$ such that $\varphi_{t+\eta}(x) = \varphi_t(x)$ for every $t \in J$, we say that the (forward) orbit $\gamma^+(x)$ is *η-periodic*, or simply, *periodic*. Thus, the set

$$P = \{\varphi_t(x) : t \in [0, \eta]\}$$

satisfies $\varphi_t(P) = P$, for every $t \in J$.

In the context of a semiflow regarding the DDE in (1.9), where $J = \mathbb{R}_+$ and X is a subset of C, only the constant functions in C can be equilibria [124, Proposition 5.4]. Therefore, we will identify such type of functions with the value they take, bearing in mind that we are referring to a constant function on $[-\tau, 0]$. To be clear, and following the notation of [124], since any equilibrium of (1.9) is a certain function $\hat{p}(\theta) = p \in \mathbb{R}^n$, for every $\theta \in [-\tau, 0]$, we will make the abuse of notation of saying that $p \in \mathbb{R}^n$ is an equilibrium of (1.9). Additionally, any periodic orbit of the corresponding semiflow comes from a periodic solution of the equation (1.9). Thus, such concepts are coherent with what was said in Definitions 1.15 and 1.16.

Definition 1.24. Let $x \in X$ and $B \subset X$ be non-empty. We define the distance from x to B as

$$\delta(x, B) := \inf_{y \in B} d(x, y).$$

Moreover, if $A \subset X$ is non-empty and bounded, we define the distance from A to B as

$$\delta(A, B) := \sup_{x \in A} d(x, B) = \sup_{x \in A} \inf_{y \in B} d(x, y).$$

Sometimes, the underlying metric space is highlighted as a subscript, that is, by using the notation δ_X [115]. It is known that the previous concept does not define a distance in the usual sense since, for instance, it is not symmetric. Nevertheless, the *Hausdorff distance*, also known as *Pompeiu-Hausdorff distance* [15], fixes that problem.

Since the notion of 'approaching certain values' will be relevant, we need to set a way to measure closeness. As usual, we will say that W is a neighbourhood of a subset M if there exists an open subset U such that $M \subset U \subset W$. In particular, if $\varepsilon > 0$, the ε-neighbourhood of $M \subset X$ will be the set

$$B(M, \varepsilon) := \{x \in X : \delta(x, M) < \varepsilon\} = \left\{ x \in X : \inf_{y \in M} d(x, y) < \varepsilon \right\}.$$

Definition 1.25. Let $\varphi : J \times X \to X$ be a semiflow. If M, B are nonempty subsets of X, then M is said to *attract* B (or B is said to be *attracted by* M) if $\delta(\varphi_t(B), M) \to 0$ as $t \to \infty$.

In other words, a nonempty subset M attracts[3] another subset B if, for any $\varepsilon > 0$, there exists some $t_0 \geq 0$ such that $\varphi_t(B)$ belongs to an ε-neighbourhood of M if $t \geq t_0$.

If there is a nonempty bounded subset B of X that attracts points of X, the semiflow is known to be *point dissipative*. In such a case, every forward orbit is bounded.

Theorem 1.26. *[125, Chapter 2] Let $\varphi : J \times X \to X$ be a continuous semiflow, K be a nonempty subset of X, and assume that $\gamma^+(K)$ is relatively compact. Then, $\omega(K)$ is nonempty, compact, invariant and attracts K. If, additionally, $J = \mathbb{R}_+$ and K is connected, then $\omega(K)$ is also connected.*

[3]See Pages 29-30 in [125] for a brief topological discussion on the idea of 'approaching' a set as time goes by and the role of compact attractors.

Results like Theorem 1.26 might be found, as in [58, Chapter 4, Lemma 2.1], having the hypothesis of precompactness (or total boundedness) of $\gamma^+(K)$. We highlight that precompactness and relative compactness are not equivalent concepts in a general metric space (X, d). Nevertheless, if X is a complete metric space, such as in the framework of Banach spaces, both concepts become equivalent [28]; in fact, this is a common hypothesis and, for instance, it is the case in [58, Chapter 4] and Chapters 2 and 3 in [56]. A proof of such equivalence can be derived via the results in, e.g., [128, Section 10.1].

In fact, a simple application of Arzelà-Ascoli's Theorem concerning compactness in the space of vector continuous functions on a compact subset of a metric space (see [128, Theorem 10.2.7] and the subsequent generalisation) provides us with the following useful result concerning forward orbits of (1.9).

Proposition 1.27. *[124, Proposition 5.5] Assume that V is a closed subset of C, and that, for any $\phi \in V$, there is a unique maximal solution $x(t;\phi)$ of* (1.9) *through ϕ defined on $[-\tau, \infty)$, and whose segments $x_t(\phi) \in V$ generate a continuous semiflow. If f is completely continuous, then any bounded forward orbit $\gamma^+(\phi)$ is also a relatively compact subset.*

Therefore, we can recall Theorem 1.26 for $K = \{\phi\}$ and remark that the hypothesis of relative compactness can be checked via boundedness just by applying Proposition 1.27.

Up to this point, for a certain fixed $x \in X$, we have already argued about the set $\omega(x)$, which gathers the information about the eventual behaviour of $\varphi_t(x)$. Now, a deeper question is whether the ω-limit sets of different points of X are related and, in particular, if there is a set $M \subset X$ that can be the ω-limit set of several points of X. To discuss about these questions, we provide the following concepts.

Definition 1.28. Assume that $\varphi : J \times X \to X$ is a semiflow and that M is a forward invariant subset. Then:

- M is *stable* if, for any neighbourhood V of M, there exists another neighbourhood W of M such that $\varphi_t(V) \subset W$, for any $t \in J$. Otherwise, we say that M is *unstable*.
- M *attracts points locally* or is *locally attracting* if there exists a neighbourhood $\tilde{V}$ of M such that M attracts points of $\tilde{V}$. M *attracts points globally* or is *globally attracting* if it attracts points of X.
- M is *locally asymptotically stable (LAS)* if it is stable and attracts points locally. Analogously, M is *globally asymptotically stable (GAS)* if it is stable and attracts points globally.

In the particular case of a semiflow generated by an autonomous DDE and $M = \{p\}$ where p is a equilibrium, a brief comparison with Definition 1.15 shows its compatibility with the ones in Definition 1.28.

Moreover, also notice that the above-stated concepts strongly depend on the semiflow. In fact, when a DDE or a difference equation generate a semiflow, we say that a set M (in this work, typically, an equilibrium) is stable, attracts points locally or globally, or is LAS or GAS for such equation if they satisfy the properties of the corresponding definition in terms of the semiflow φ generated by the equation.

Furthermore, although the term 'global attractor' can be found having different meanings in the literature [125, Page 37], we will say that a set is a *global attractor* if it is a global attractor of points, i.e., if it attracts points globally. Analogously, the term *local attractor* will be referred to as a set that attracts points locally.

Remark 1.29. As in [56, Page 2] and [118, Page 357], it is worth recalling that a set that attracts points globally does not need to be GAS, which can be seen via simple examples in the Euclidean space $X = \mathbb{R}^2$.

Nonetheless, the case of a continuous semiflow in a real interval $X = I \subset \mathbb{R}$ is special due to the particular features of the real line. In fact, an equilibrium that attracts points globally is always stable and thus, GAS. See [43, 118] for proofs in the discrete-time case, i.e., with $J = \mathbb{Z}_+$. The continuous-time case, $J = \mathbb{R}_+$, is almost obvious since $\varphi_t(x)$ is monotone in t.

Obtaining the asymptotic properties of the solution of a DDE or a difference equation may not be a simple task. Sometimes, it is possible to consider a certain change of variables to transform such an equation into another one that is equivalent in terms of the long-term behaviour of the solutions and yet simpler to study. In that line, and to end this section, we recall the following concept.

Definition 1.30. Let X, Y be metric spaces and $\varphi_X : J \times X \to X$, $\varphi_Y : J \times Y \to Y$ be continuous semiflows. We say that a homeomorphism $h : X \to Y$ is a *topological conjugacy* between φ_X and φ_Y if

$$h(\varphi_X(t, x)) = \varphi_Y(t, h(x)), \tag{1.12}$$

for every $(t, x) \in J \times X$. We say that φ_X and φ_Y are *topologically conjugate* if there exists a topological conjugacy between them.

The previous concept yields a symmetric relation, since h^{-1} would be a topological conjugacy between φ_Y and φ_X. If $J = \mathbb{Z}_+$, then, without loss of generality, expression (1.12) is equivalent to

$$h \circ F = G \circ h,$$

where $F := \varphi_X(1, \cdot)$ and $G := \varphi_Y(1, \cdot)$.

The importance of the previous concept is that, through a topological conjugacy, the dynamical properties are preserved in relation to the asymptotic behaviour and, in particular, to the globally attracting equilibria, as the following result shows. Consequently, one can use this tool to simplify the study of the dynamical properties of an equation.

Proposition 1.31. *Let X, Y be metric spaces and $\varphi_X : J \times X \to X$, $\varphi_Y : J \times Y \to Y$ be continuous semiflows that are topologically conjugate via the homeomorphism h. If $p \in X$ is an equilibrium of φ_X, then $h(p) \in Y$ is an equilibrium of φ_Y. Furthermore, if p attracts points globally (locally) for φ_X, then $h(p)$ attracts points globally (locally, respectively) for φ_Y. Besides, if p is stable for φ_X, then $h(p)$ is stable for φ_Y.*

With all the tools and framework that we have shown, we are able to start studying the asymptotic behaviour of the solutions of certain particular DDEs that are relevant. As announced, we are interested in determining when does a certain $p \in X$ attract points globally. Note that, in the context of a semiflow defined for the autonomous DDE (1.9), that means $\omega(x) = \{p\}$ for every $x \in X \subset C$, or, in other words,

$$\lim_{t\to\infty} \varphi_t(x) = p$$

for every $x \in X$. One of the main issues of the study of DDEs is dealing with a phase space lying in the infinite-dimensional Banach space C. Nonetheless, as we will see in the next section, some information can be derived from another semiflow on a finite-dimensional space, provided (1.1) or (1.9) have some particular form.

1.3 Scalar delay differential equations with a linear decay and delayed feedback

To start with, we consider the following family of DDEs

$$x'(t) = -a(t)x(t) + \hat{f}(t, x_t), \tag{1.13}$$

where $a : \mathbb{R} \to \mathbb{R}_+$ is a continuous bounded function and $\hat{f} : \mathbb{R} \times C \to \mathbb{R}$ is a continuous feedback which is assumed to satisfy

$$\limsup_{\|\phi\|\to\infty} \frac{|\hat{f}(t,\phi)|}{\|\phi\|} \leq a(t). \tag{1.14}$$

Equation (1.13) provides a more particular framework than (1.1), but also serves as a common expression to many equations we will work with, which include DDEs that are found in some models. In particular, it belongs to the family of delay differential equations of production and destruction, introduced in [5]. In equation (1.13), the destruction term is linear and non-delayed, while the production term is a function that incorporates the delay effects. The latter type of constraint was also assumed in a significant part of [5].

Condition (1.14) is key to ensure that the solutions of (1.13) are globally defined. In fact, the proof of the following Proposition 1.32 is based on standard arguments regarding the variation-of-constants formula (see, e.g., [94, Lemma 3.4], and [58, Chapter 6, Theorem 1.1]), condition (1.14) and the classical Gronwall-Bellman's Lemma (see, for instance, [112, Theorem 1.2.2] or [58, Chapter 1, Lemma 3.1]). The latter result will also be a tool of interest in Chapter 5; indeed, we have recalled it in Theorem 5.1.

Proposition 1.32. *Let* $(\sigma, \phi) \in \mathbb{R} \times C$ *and assume that* $x : [\sigma - \tau, b)$ *is a maximal solution of* (1.13) *through* (σ, ϕ)*. Then,* $b = \infty$*.*

Notice that we have not imposed any conditions to ensure the uniqueness of solution through (σ, ϕ). Nevertheless, if we further assume that f is locally Lipschitzian with respect to the second variable (or the generalisations below Theorem 1.5), we could apply Theorem 1.8.

For the rest of the chapter, we will focus on the case of constant coefficient $a(t) = a$, and a single variable delay $\hat{f}(t, \phi) = f(\phi(-\tilde{\tau}(t)))$, for a certain function $f : \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^1$. Notice that the letter f is now used for a real function of real variable in contrast with the notation of Section 1.1. This particular choice is widely used in applications and many results are known, as we will see. Chapters 5 and 6 will be related to the variable coefficient $a(t)$ and a more general function $\hat{f}$. Hence, the remaining part of the current chapter finds its main applications in Chapters 2–4.

1.3.1 Basics and the role of difference equations

As announced, let us focus on equation

$$x'(t) = -ax(t) + f(x(t - \tilde{\tau}(t))), \tag{1.15}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $\tilde{\tau} : \mathbb{R} \to \mathbb{R}_+$ are continuous functions, and $\tilde{\tau}$ is nonzero. Equation (1.15) has received some attention in the last decades (see, e.g., [86] and the references therein).

Hereafter, we will work in a certain interval $I \subset \mathbb{R}$ such that, for any $(\sigma, \phi) \in \mathbb{R} \times C_I$, there exists a globally defined unique solution of (1.15) through $(\sigma, \phi) \in \mathbb{R} \times C_I$. One can use results like the ones in Section 1.1, which provide further assumptions on f for the latter to hold. Hence, the subset $I \subset \mathbb{R}$ will be a non-degenerate interval, which, unless explicitly stated, may be open, closed, or none of them, and may be bounded or unbounded.

A particular case of equation (1.15) is the one of constant delay, i.e., $\tilde{\tau}(t) = \tau > 0$. Thus, the former equation can be rewritten as

$$x'(t) = -ax(t) + f(x(t - \tau)). \tag{1.16}$$

This case has its own interest in modelling (see, e.g., [5]). We continue working with (1.16), but we are going to return to (1.15), since several key dynamical features are still valid for the more general equation (1.15) and equation (1.16) makes explanations easier. In fact, we can use Section 1.2 in a clear way, since equation (1.16) is autonomous while equation (1.15) is not.

Furthermore, equation (1.16) includes two well-known families of delay differential equations, namely the ones of the *Wright-type*, in case $a = 0$ and $I = \mathbb{R}$; and the ones of the *Mackey-Glass-type*, for which $a > 0$ and $I = [0, \infty)$ [88]. Of course, as the title of the current Section 1.3 indicates, the emphasis is going to be put on the case where there is a pure linear decay term, i.e., when $a > 0$ in (1.16). Nevertheless, some results for $a = 0$ are somehow a limit form of the corresponding ones for $a > 0$ as $a \to 0^+$, and tangential remarks with respect to this fact are provided. Analogous comments regarding the cases $\tau \to 0^+$ are done too.

Unless otherwise stated, we assume that $a, \tau > 0$. Thus, for any $\phi \in C$, there is a unique solution of (1.16) through ϕ, which is defined on $[-\tau, \infty)$ without imposing any further restrictions on f. In fact, via the method of steps and the variation-of-constants formula, one can obtain the following expression for the solutions (see [71]):

$$x(t) = x(n\tau)e^{-a(t-n\tau)} + \int_{n\tau}^{t} e^{-a(t-s)} f(x(s-\tau))\, ds, \quad t \in [n\tau, (n+1)\tau],\ n \in \mathbb{Z}_+. \tag{1.17}$$

Indeed, in the latter reference it is assumed that $\tau = 1$, but there is no loss of generality, as we will see later. By bringing together all the above-mentioned information and via Corollary 1.14 and Proposition 1.19, a continuous-time semiflow naturally arises in $X = C$ from (1.16). In fact, such a semiflow $\varphi : \mathbb{R}_+ \times C \to C$ is defined by

$$\varphi(t, \phi) = x_t(\phi).$$

Clearly, in the last paragraph, we have argued in that way for the interval $I = \mathbb{R}$ ($C = C_{\mathbb{R}}$). If I was a proper interval of $\mathbb{R}$, one shall be careful since solutions of (1.16) may not be globally defined. Moreover, there is no big problem in considering a set that is forward invariant by the semiflow φ and the semiflow restricted to it, as we have done in Section 1.2.

In fact, since we are interested in the existence of global attractors, it is natural to choose an interval I where (1.16) has a unique equilibrium, that is, where there is a unique solution of the equation $f(x) = ax$ (recall from Section 1.2 that the equilibrium points are elements of C representing constant functions). This is a natural choice when the function f does not have a unique fixed point on $\mathbb{R}$, which actually happens in many models. In particular, in the framework of population dynamics, it is usual to have, as equilibria, zero and a unique positive value. Therefore, despite the positive equilibrium cannot attract the zero equilibrium, it is quite natural to ask whether it attracts every solution with positive values. In fact, we will see that it is also common that $C_{[0,\infty)}$ and $C_{(0,\infty)}$ are forward invariant subsets.

Regarding equation (1.16), the dynamical features are determined by three items: the rate of linear decay a, the feedback f and the value of the delay τ. The purpose of the remaining part of the current subsection is to show the role of those three objects in relating (1.16) with a certain difference equation whose dynamical properties give us highly relevant information about (1.16).

Reduction of the number of parameters

It is common to find equation (1.16) with less parameters. In fact, as highlighted in [94], we can make a reduction of their number and restrict our attention only to two of them. Both are related to normalising a or τ whenever they are positive. Despite having that option of simplifying equation (1.16), we have decided to maintain the parameters a and τ, due to their meaning in some models, and we just show how the reduction works.

Firstly, since $a > 0$, we consider the change of variables $z(t) = x(t/a)$. Hence,

$$\begin{aligned} z'(t) = \frac{x'(t/a)}{a} &= \frac{-ax(t/a) + f\left(x(t/a - \tau)\right)}{a} \\ &= -x(t/a) + \frac{1}{a} f\left(x\left(\frac{t - a\tau}{a}\right)\right) \\ &= -z(t) + \frac{1}{a} f(z(t - a\tau)). \end{aligned}$$

Thus, provided that $a > 0$, it is clear that (a, f, τ) is reduced to $(1, f/a, a\tau)$.

The strength of this last change is that we could directly study the properties of the map f/a to directly obtain information about the solutions of (1.16), as we will see below. In fact, the equilibria of (1.16) coincide with the ones of $x_{n+1} = \frac{f(x_n)}{a}$.

Secondly, since $\tau > 0$, we can alternatively apply a change of variables to equation (1.16) in order to asssume the delay equals 1 (see, e.g., [124]). Such a change of variables is defined as $z(t) = x(\tau t)$, and it transforms (1.16) into

$$z'(t) = \tau x'(\tau t) = -a\tau x(\tau t) + \tau f(x(\tau t - \tau)) = -a\tau z(t) + \tau f(z(t-1)). \tag{1.18}$$

Therefore, provided that $\tau > 0$, it is clear that (a, g, τ) is identified to $(a\tau, g\tau, 1)$.

The advantage of rewriting (1.16) as (1.18) is that we can work on the same phase space regardless the value of the delay $\tau > 0$, since the right-hand side of (1.18) is equal to

$$g(z_t) := -a\tau z(t) + \tau f(z(t-1)),$$

where $g : \tilde{D} \subset \mathcal{C}([-1, 0], \mathbb{R}) \to \mathbb{R}$.

A singular perturbation and the role of difference equations

The latter change of variables, i.e., the one that involved τ, allows us to rewrite equation (1.18) as

$$\varepsilon \dot{x}(t) = -ax(t) + f(x(t-1)),$$

where $\varepsilon = \frac{1}{\tau}$. Hence, it can be seen as a singular perturbation of

$$0 = -ax(t) + f(x(t-1))$$

or, equivalently,

$$x(t) = \frac{1}{a} f(x(t-1)) =: F(x(t-1)), \tag{1.19}$$

as $\varepsilon \to 0^+$, or equivalently, as $\tau \to \infty$. Consequently, it is quite natural to expect some relation between the asymptotic behaviour of the solutions of (1.16) for sufficiently large τ and that of the solutions of (1.19). Notice that the division by a, related to the alternative change of variables, also appears in (1.19).

Difference equations with continuous argument like (1.19) are similar to DDEs when it comes to the phase space (see, e.g., [65, 120]) since we need to define an initial condition on $[-1, 0]$.

Nonetheless, equation (1.19) may have solutions that are discontinuous at integer times. Such issue can be fixed by imposing the consistency condition that the initial function ϕ belongs to the phase space

$$C_I^* = \{\phi \in \mathcal{C}([-1, 0], I) : \phi(0) = F(\phi(-1))\}.$$

Furthermore, the concepts of solution of (1.19) through an initial function ϕ and segments can also be defined in a similar manner.

Besides, the equation (1.19) represents an iteration by F of every element $\phi(\theta)$, for $\theta \in [-1, 0]$, so its solutions are governed by the difference equation with discrete argument

$$x_{n+1} = F(x_n). \tag{1.20}$$

Additionally, if $F(I) \subset I$, then the solutions of equation (1.19) are unique and globally defined, that is, they exist on $[-1, \infty)$. Moreover, equation (1.20) also generates a continuous discrete-time semiflow on $X = I$, namely $\psi : \mathbb{Z}_+ \times I \to I$, which is defined by $\psi(n, x) = F^n(x)$.

Now we provide some details regarding the comparison between the solutions of (1.16) and (1.19). One could expect some likeness between the solutions of the DDE in (1.16) with large τ and the ones of (1.19), which, formally speaking, corresponds to the limit case $\tau = \infty$. Nevertheless, such limit equation (1.19) shows two major issues that make such comparison tough.

Firstly, no consistent condition is needed to be imposed for the initial functions of the DDE in (1.16) to generate continuous solutions, as it is actually required for (1.19). However, one can flip the reasoning and wonder if equation (1.19) without the consistent condition (thus, having discontinuous solutions) has something to do with the DDE in (1.16) in terms of that similarity of solutions. In such case, there exist results that show the existence of certain continuity features when it comes to the transition between solutions of (1.19) and solutions of (1.16) with small $\varepsilon = \frac{1}{\tau}$ (see, e.g., [65, Section 3]) on bounded time intervals.

Secondly, consider a solution of (1.19) through an initial condition $\phi \in C_I^*$, namely $x(t;\phi)$. Although the solution is continuous due to the consistency condition, the corresponding segments $x_t(\phi)$ might not tend to any element of the phase space as $t \to \infty$: the limit behaviour might be, roughly speaking, a discontinuous segment. In fact, there are examples where (1.20) has a locally attracting 2-periodic orbit and a continuous solution of (1.19) approaches a step function as $t \to \infty$ (known as *convergence to the square wave*) [98].

However, equation (1.20), the discrete-time version of (1.19), is a nice tool to derive several properties about long-term behaviour of the solutions of (1.16). In the following result, we recall that the dynamics of (1.16) are, in some sense, dominated by the dynamics of (1.20).

Theorem 1.33. *[98, Proposition 1.1] Let I be a closed interval and $F : I \to I$ be a continuous function. If $\phi \in C_I$, the solution $x(t;\phi)$ of* (1.16) *satisfies*

$$x(t;\phi) \in I, \quad t \geq 0.$$

Furthermore, the following property holds

$$\omega_\psi(I) = \bigcap_{n=0}^{\infty} \overline{F^n(I)}, \tag{1.21}$$

and, if $\omega_\psi(I) \neq \emptyset$, then

$$\lim_{t\to\infty} d(x(t;\phi), \omega_\psi(I)) = 0,$$

while, if $\omega_\psi(I) = \emptyset$, then $x(t;\phi)$ tends to either $-\infty$ or ∞ as $t \to \infty$.

Remark 1.34. The first thesis of Theorem 1.33 states that any solution of (1.16) such that the image of its initial segment is contained in the interval I (which is assumed to be forward invariant for F or ψ) cannot leave the interval I, and thus the set C_I is forward invariant for the semiflow φ. Therefore, a forward-invariance-type feature is inherited from (1.20) to (1.16). In fact, this has been called the invariance property [65, 120].

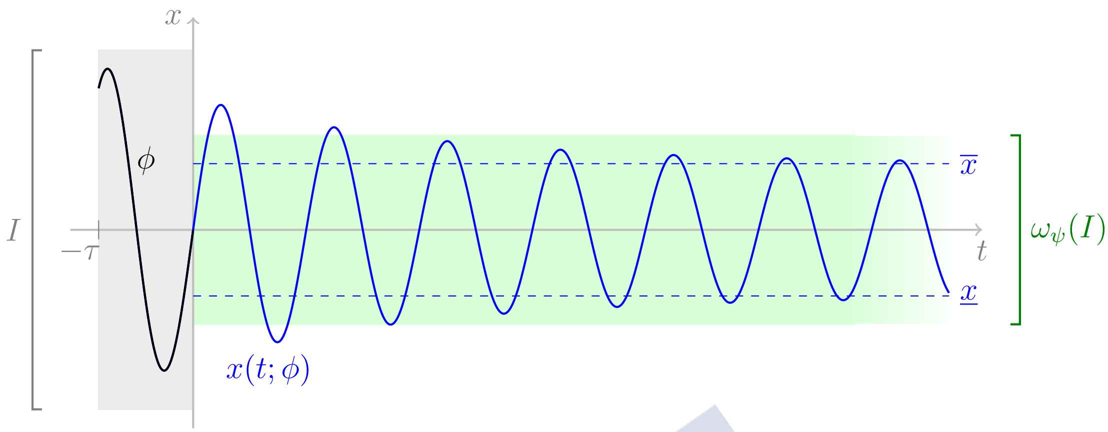

Figure 1.3: The sharp asymptotic bounds for the solution $x(t;\phi)$ of an equation of the form (1.16), $\overline{x} := \limsup_{t\to\infty} x(t;\phi)$ and $\underline{x} := \liminf_{t\to\infty} x(t;\phi)$. They belong to the window $\omega_\psi(I)$, which is the ω-limit set of I with respect to the difference equation (1.20).

The second one gives us more precise information about the long-term behaviour of the solutions of (1.16). Since $F^{n+1}(I) \subset F^n(I)$, for every $n \in \mathbb{Z}_+$, the set $\omega_\psi(I)$ takes the simple expression in the right-hand side of (1.21) (see Definition 1.11). In fact, by the continuity of F and an inductive argument, any set $F^n(I)$ is an interval, so the set $\omega_\psi(I)$ is an intersection of convex sets, and thus, an interval. In addition, the following relation holds:

$$\omega_\varphi(\phi) \subset C_{\omega_\psi(I)}, \quad \forall \phi \in C_I. \tag{1.22}$$

The meaning of the second thesis, which is somehow summed in (1.22), is that the asymptotic behaviour of the solutions of the difference equation (1.20) sets the window where any solution of (1.16) starting at an initial condition with values in I must tend to, as $t \to \infty$. Hence, in some sense, the dynamics of (1.16) are controlled by the ones of (1.20). Notice that the nature of $\omega_\psi(I)$ and $\omega_\varphi(\phi)$ are different: $\omega_\psi(I)$ is a real interval whilst $\omega_\varphi(\phi)$ is a subset of continuous real functions defined on the compact interval $[-\tau, 0]$.

Furthermore, provided that $\omega_\psi(I)$ is non-empty and compact, then we can also rewrite the former relations (see, e.g., [87]) as

$$\left[\liminf_{t\to\infty} x(t;\phi), \limsup_{t\to\infty} x(t;\phi)\right] \subset \omega_\psi(I) = \bigcap_{n=0}^{\infty} \overline{F^n(I)}. \tag{1.23}$$

For instance, the latter happens if I is compact (see Theorem 1.26). Figure 1.3 is a sketch of this situation.

In [98], this kind of result is stated by assuming that I is just closed. Therefore, the authors also admit the choice of a closed unbounded interval I. Nevertheless, this could cause $\omega(I)$ to be empty or unbounded, and one of those cases occurs if and only if $F(I)$ is unbounded. Since the latter will not occur in the forthcoming chapters, we avoid that possibility, having $F(I) \subset K$, for a certain compact subset $K \subset I$, and thus work with $F_{|K} : K \to K$ bearing in mind that every initial condition in I enters K and hence, $\omega(I) = \omega(K)(\subset K)$ is always a nonempty compact set by means of Theorem 1.26. So, in this case, (1.22) is always valid.

When the above-mentioned window $\omega_\psi(I)$ is a singleton subset, we obtain the following result. In particular, if I is compact and p is the global attractor (of points) in I, then $\omega_\psi(I) = \{p\}$ (see, e.g., [98, Proposition 1.2]).

Corollary 1.35. *[98, Corollary 1.2] Let I be a compact interval and $F : I \to I$ be a continuous function. If p is a global attractor for $x_{n+1} = F(x_n)$, then p is a global attractor for*

$$x'(t) = -ax(t) + f(x(t-\tau)).$$

Hereafter, we will say that (1.16) and (1.20) are *corresponding equations* to one another, based on the strong relation regarding the long-term dynamics that we have recalled above.

We will further assume that, in the interval of interest I, the function is of class $\mathcal{C}^1$ on the interior of its domain to facilitate the stability analysis of the unique equilibrium of its interior. In particular, this is trivial if I is open.

Having shown the key properties of (1.16) to simplify its study, we will then focus on the set of triples (a, f, τ) that satisfy the following set of hypotheses, which will be jointly called (T):

(T1) $a, \tau \in \mathbb{R}^+$, $f \in \mathcal{C}^1(I, \mathbb{R})$, $I \subset \mathbb{R}$ is a non-empty open interval,

(T2) $\frac{f}{a} : I \to I$ is well-defined,

(T3) $f(x) = ax$ has a unique solution on I,

(T4) $K \subset I$ is a non-degenerate compact interval that is globally attracting for $\frac{f}{a}$.

Notice that, while the first two parameters are strongly related via the function f/a, there is 'freedom' on τ. Condition (T4) is a technical condition to ensure that the solutions starting close to the boundary of I eventually go away from it, so one avoids solutions 'escaping' from I. Moreover, if (a, f, τ) satisfies the hypotheses (T) then $f'(p) \le a$. Otherwise, if $f'(p) > a$, a simple cobweb analysis of f/a shows that no set K as in (T4) could exist. In fact, the limit case $f'(p) = a$ can only take place under (T) in particular circumstances. For instance, with higher smoothness of f, $f''(p) = 0$ must hold.

Relative position of the graph of the feedback regarding the unique equilibrium

When it comes to the qualitative study of (1.16) with (a, f, τ) satisfying (T), it is common to classify the feedback f in terms of the position of its graph with respect to the unique equilibrium p of (1.10). In particular, this turns out to be relevant, as we will see at the very end of this chapter.

We can make the change $\tilde{x}(t) = x(t) - p$ and $\tilde{f}(y) = f(y + p) - ap$ to assume that 0 is the unique equilibrium of the DDE. In fact,

$$\begin{aligned}\tilde{x}'(t) = x'(t) &= -ax(t) + f(x(t-\tau)) = -ax(t) + ap + f(x(t-\tau)) - ap \\ &= -a\tilde{x}(t) + f(\tilde{x}(t-\tau) + p) - ap = -a\tilde{x}(t) + \tilde{f}(\tilde{x}(t-\tau)).\end{aligned}$$

Therefore, the origin, which is an equilibrium of the equation

$$\tilde{x}'(t) = -a\tilde{x}(t) + \tilde{f}(\tilde{x}(t-\tau)), \tag{1.24}$$

shares the attractivity properties with the equilibrium p of equation (1.16). In particular, we have p being stable or globally attracting for (1.16) if and only if we have 0 being respectively stable or globally attracting for (1.24).

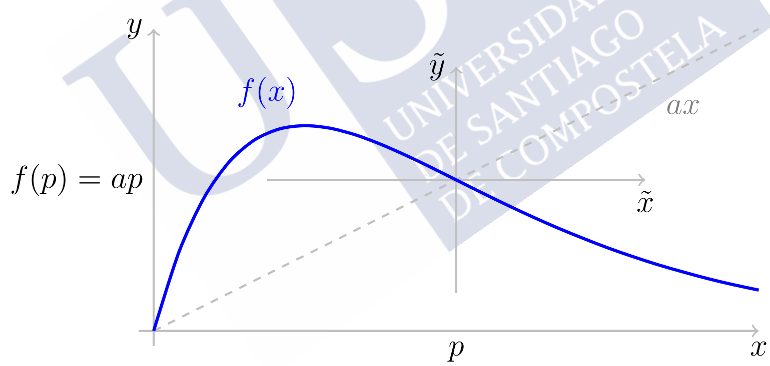

Figure 1.4: Change of variables to the origin in (1.16).

From the previous comments, there is no loss of generality in assuming that $f(0) = 0$. In fact, such condition is actually considered in the literature; see, e.g., the references [71, 94, 97, 113]. We will say that the function $\tilde{f}$ is the *shifted function* of f and its graph is represented in the *shifted variables*.

Regarding the latter, the relative position of the graph with respect to its unique equilibrium p is a key feature to define relevant classes of feedbacks.

Definition 1.36. Let $a > 0$, $f \in \mathcal{C}^1(I, \mathbb{R})$ and assume that $p \in I$ is the unique solution of $f(x) = ax$ on I. If $\tilde{f}$ denotes the shifted function of f, then, f is a *positive feedback* for

(1.16) provided

$$\tilde{x}\tilde{f}(\tilde{x}) = (x-p)(f(x)-ap) > 0, \quad \forall x \in I \setminus \{p\} \ (\forall \tilde{x} \in (I-p)\setminus\{0\}), \tag{1.25}$$

or a *negative feedback* for (1.16) provided

$$\tilde{x}\tilde{f}(\tilde{x}) = (x-p)(f(x)-ap) < 0, \quad \forall x \in I \setminus \{p\} \ (\forall \tilde{x} \in (I-p)\setminus\{0\}). \tag{1.26}$$

Remark 1.37. Notice that, if f belongs to some of the families named above, then so are $\tilde{f}$, f/a and $\tilde{f}/a$; hence, the parameter $a > 0$ does not influence the above-mentioned features. The role of the latter two functions will be clarified later, but we can just highlight that conditions (1.25) and (1.26) can be respectively rewritten as

$$\tilde{x}\,\frac{\tilde{f}(\tilde{x})}{a} = (x-p)\left(\frac{f(x)}{a} - p\right) > 0, \quad \forall x \in I \setminus \{p\} \ (\forall \tilde{x} \in (I-p)\setminus\{0\}),$$
$$\tilde{x}\,\frac{\tilde{f}(\tilde{x})}{a} = (x-p)\left(\frac{f(x)}{a} - p\right) < 0, \quad \forall x \in I \setminus \{p\} \ (\forall \tilde{x} \in (I-p)\setminus\{0\}).$$

Remark 1.38. If (a, f, τ) satisfies (T), then $(a, \tilde{f}, \tau)$ also satisfies (T). In particular, if f is defined on the non-empty open interval $I \subset \mathbb{R}$ and $p \in I$ is the unique root of $f(x) = ax$, then $\tilde{f}$ is defined on the shifted interval $I - p$ and $0 \in I - p$ is the unique root of

$$f(x+p) - ap = \tilde{f}(x) = ax.$$

1.3.2 Some aspects about the local dynamics

For the purposes of this subsection, assume that (a, f, τ) satisfies (T) and that p is the unique root of $f(x) = ax$. At a first glance, we could use Corollary 1.35 to derive a sufficient condition for an equilibrium p to be LAS for (1.16). If we take I_p as the *immediate basin of attraction* of p for F, that is, the connected component of the set

$$\{x \in X : F^n(x) \to p \ \text{as} \ t \to \infty\}$$

that contains p, then we can apply Corollary 1.35 to $F_{|I_p}$ and obtain that, if $x(t;\phi)$ is the solution of (1.16) through $\phi \in C_{I_p}$, then $x(t;\phi) \to p$ as $t \to \infty$.

Corollary 1.39. *If $p \in I$ is LAS for the difference equation* (1.20)*, then p is LAS for the DDE* (1.16)*.*

An easy-to-check well-known sufficient condition for a regular function F to have a locally asymptotically stable equilibrium is the following one (see, e.g., [29, Section 1.4] or [30, Chapter 5]).

Proposition 1.40. *Let $F : I \to I$ be a function of class $\mathcal{C}^1$ and $p \in I$ be a fixed point of F. The following statements hold:*

- *If $|F'(p)| < 1$, then p is LAS for F,*
- *If p is LAS for F, then $|F'(p)| \leq 1$.*

The previous result is 'almost' an equivalence regarding local asymptotic stability when working with a sufficiently regular F. The case $|F'(p)| = 1$, also known as *non-hyperbolic* case, in contrast with the *hyperbolic* case of $|F'(p)| \neq 1$, is special and it highly depends on further properties of F. Furthermore, if $|F'(p)| > 1$ then the equilibrium is called a *repelling point* [29, 30].

Remark 1.41. Bear in mind that, in case we decide not to normalise the parameter a regarding the linear decay of (1.16), we have

$$F'(p) = \frac{f'(p)}{a}.$$

Nevertheless, the condition $|f'(p)| < |a|$ is not the sharpest one, it is only a sufficient condition for an equilibrium to be LAS for (1.16). In order to obtain a refined condition, we have to go deeper than Corollary 1.39 actually does.

It is here where the interesting particular case of (1.16) with $f(x) = bx$, for some $b \in \mathbb{R}$, appears. The dynamics of this linear case are well understood and, additionally, their study provides some useful information for the general equation (1.16). In fact, the local asymptotic stability of the equilibrium p in (1.16) can be studied through its linearised equation at the point p

$$x'(t) = -ax(t) + f'(p)x(t-\tau), \tag{1.27}$$

for which the origin is an equilibrium. Specifically, the following Poincaré-Liapunov-type result, which is a particular case of [12, Theorem 11.2], holds.

Theorem 1.42. *Assume that the origin is a global attractor for*

$$u'(t) = -au(t) + bu(t-\tau), \tag{1.28}$$

and let $\tilde{g} : B_0 \subset \mathbb{R} \to \mathbb{R}$ be a continuous function defined on a neighbourhood B_0 of the origin and such that

$$\lim_{|u|\to 0} \frac{|\tilde{g}(u)|}{|u|} = 0.$$

Then the origin is LAS for

$$u'(t) = -au(t) + bu(t-\tau) + \tilde{g}(u(t-\tau)).$$

We remark that, regarding (1.28), the origin is GAS if and only if it is a global attractor, that is, for the linear equation (1.28), the stability is derived from global attractivity of the origin.

We can take advantage of the statement of Theorem 1.42 for $u(t) = \tilde{x}(t)$, where $\tilde{x}$ is the shifted variable of x, defined in the last subsection. We can write the following computations:

$$\begin{aligned}\tilde{x}'(t) &= -a\tilde{x}(t) + \tilde{f}(\tilde{x}(t-\tau)) \\ &= -a\tilde{x}(t) + \tilde{f}'(p)\tilde{x}(t-\tau) + \tilde{f}(\tilde{x}(t-\tau)) - \tilde{f}'(p)\tilde{x}(t-\tau) \\ &= -a\tilde{x}(t) + \tilde{f}'(p)\tilde{x}(t-\tau) + \tilde{g}(\tilde{x}(t-\tau)),\end{aligned}$$

where $\tilde{g}(\tilde{x}) := \tilde{f}(\tilde{x}) - \tilde{f}'(p)\tilde{x}$. Now, since $\tilde{f}$ is $\mathcal{C}^1$ and $\tilde{f}(0) = 0$, it is a basic fact from Taylor polynomials that

$$\lim_{|\tilde{x}|\to 0} \frac{|\tilde{g}(\tilde{x})|}{|\tilde{x}|} = 0.$$

Consequently, studying the dynamics of the solutions of (1.27) provides us direct information about the local dynamics around the unique equilibrium of the DDE (1.16).

Thus, let us focus on equations that are written as

$$x'(t) = -ax(t) + bx(t-\tau). \tag{1.29}$$

The explicit region of parameters (a, b, τ) for which the origin is globally attracting for (1.29) comes from the study of an equation that, in general, is trascendental. In fact, if we look for solutions of (1.29) of the form $e^{\lambda t}$, with formally allowing λ to take complex values, we would arrive to the equation

$$\lambda + a - be^{-\lambda\tau} = 0. \tag{1.30}$$

The equation (1.30) is called the *characteristic equation* of the DDE in (1.29). Bearing in mind the previously given appetiser regarding the exponential-type solutions, it is natural to expect that the roots of (1.30) induce certain asymptotic behaviour in the solutions of (1.29). In fact, via the use of the Laplace transform, the following property holds (see, e.g., [12, 58, 68]).

Theorem 1.43. *If all the roots of* (1.30) *have negative real parts, then the origin is GAS for* (1.29).

Hence, the study of the attractivity/stability of the origin for (1.16) is reduced to finding the location of the roots of the entire function $w(\lambda) = \lambda + a - be^{-\lambda\tau}$ (also called the characteristic function [127]) for a particular triple of parameters (a, b, τ). Finally, the answer to that issue is known and we can write it in the form of the following criterion.

Theorem 1.44. *[12, Theorem 13.8] Let $a, \tau > 0$. Then, all the roots of* (1.30) *have negative real parts if and only if*

$$-\sqrt{s^2 + (a\tau)^2} < b\tau < a\tau, \tag{1.31}$$

where s is the unique solution of $\frac{-x}{\tan(x)} = a\tau$ in $(\frac{\pi}{2}, \pi)$.

Notice that Theorem 1.44 is sometimes given allowing triples with $a \leq 0$ too [58, Appendix], but we have omitted the triples with that condition since $a > 0$ in all the models that we will study (remember the hypotheses (T)). In particular the limit condition as $a \to 0^+$ is $-\frac{\pi}{2} < b\tau < 0$. Moreover, the condition on the limit case $\tau = 0$ is trivial: $b - a < 0$.

Remark 1.45. If $b \in [-a, a)$, then (1.31) holds. Nevertheless, there are triples (a, b, τ) that also satisfy (1.31) and for which $b < -a$. For the latter ones, there is an upper bound in the delay τ for the origin of (1.29) to be GAS. Thus, for those triples, it is also useful to handle a condition derived from (1.31) that explicitly shows such bound on the delay. In fact, the lower boundary for $b\tau$ for any $a\tau > 0$ is represented by the curve

$$(a\tau, b\tau) = \left(\frac{-s}{\tan(s)}, \frac{-s}{\sin(s)} \right), \quad s \in (\pi/2, \pi). \tag{1.32}$$

From equation (1.32), we can obtain the equalities

$$s = \arccos\left(\frac{a}{b}\right), \quad (b\tau)^2 - (a\tau)^2 = s^2,$$

which can be used to derive that the above-mentioned upper bound for the delay is

$$\bar{\tau} := \frac{\arccos(\frac{a}{b})}{\sqrt{b^2 - a^2}}.$$

More detailed information about the location of the roots of (1.30) is known (see, for instance, [31, 124]). We have only recalled a result excluding roots with nonnegative real part, but one can give analogous conditions to ensure that a certain number of roots lie on the half-right plane, which, for instance, has implications in the dimension of the unstable manifold of the equilibrium of (1.16) [71].

As a consequence, Theorem 1.44 in combination with the sufficient condition of Theorem 1.43 provides a way to compute the stability of the origin regarding the linear equation (1.29) in terms of their parameters.

Corollary 1.46. *If condition* (1.31) *holds, then the origin is GAS for the linear DDE* (1.29).

Finally, due to Theorem 1.42, the previous corollary has a direct implication in the asymptotic behaviour of the solutions of (1.16) that are close to the equilibrium p.

Corollary 1.47. *If condition* (1.31) *holds with* $b = f'(p)$, *then the origin is LAS for the equation* (1.16).

For further references to the previous results, we define L as the region of triples $(a, b, \tau) \in \mathbb{R}^3$ for which the unique equilibrium p of (1.16) is LAS; in particular, those satisfying condition (1.31) with $b = f'(p)$. In Figure 1.5, we can find a couple of two-dimensional slices of the region L. They provide a nice picture of the set L due to the reduction of parameters via the alternative changes of variables previously given.

Besides, notice that the dashed orange lines in Figure 1.5 represent the set of parameters in equation (1.16) for which either $\lambda = 0$ or a couple of conjugated imaginary numbers are roots of the characteristic equation (1.30). In such case, since the origin in equation (1.16) is non-hyperbolic, other techniques need to be considered.

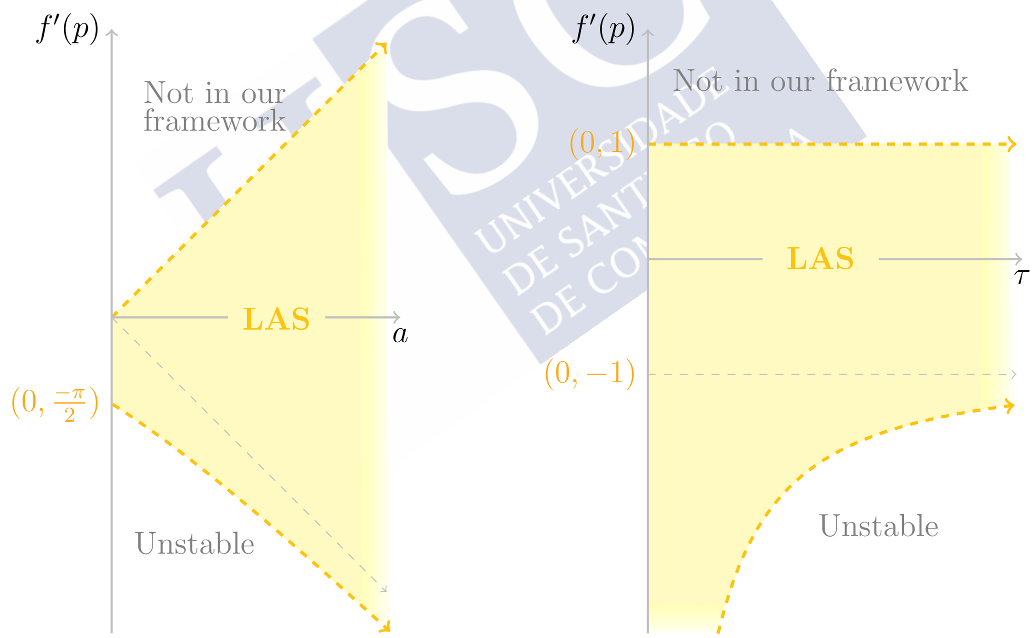

Figure 1.5: Both graphics represent the restriction of the region of parameters L for which the unique equilibrium p of (1.16) is LAS (yellow), for the cases $\tau = 1$ (left) and $a = 1$ (right). The thick dashed orange curves represent the limit cases where either zero (upper case) or a couple of conjugated imaginary numbers (lower case) are roots of the linearised equation and do not fall under the sufficient condition in Corollary 1.47. Gray dashed lines represent the asymptotic behaviour of the nonlinear curves when the first coordinate tends to ∞.

1.3.3 Some aspects about the global dynamics

Let us focus on the study of the global dynamics for (1.16). As a first warning, we highlight that the shape of the feedback f plays a major and subtle role and there are still several open problems regarding the global dynamics of (1.16), including questions related with the global attractivity of the origin [79]. Nonetheless, some partial results can be stated provided f satisfies certain geometrical conditions, as we will see.

A relevant question is whether we can restrict our attention to the cases corresponding to the region L. We remind that, in general, a globally attracting equilibrium does not need to be stable in phase spaces which are subsets of Banach spaces of dimension greater than one (see Remark 1.29). Since we are working in a phase space whose underlying nature is infinite-dimensional, it is quite natural to ask whether the origin can attract every initial condition ϕ in the phase space while being unstable, that is, in cases outside the region L. We derive this question to the end of the subsection and focus on the cases included inside the region L, that is, we will focus on the region of parameters for which the unique equilibrium p of equation (1.16) is LAS.

As at the beginning of Subsection 1.3.2 on local dynamics regarding the DDE in (1.16), we take advantage of the information that directly comes from its corresponding difference equation (1.20). Particularly, we recall the implication of Corollary 1.35 in this framework.

Corollary 1.48. *If $p \in I$ is GAS for the difference equation* (1.20)*, then p is GAS for the DDE* (1.16)*.*

As remarked before, this means that we can deduce deep information about the dynamics of the DDE (1.16), which deals with an infinite-dimensional phase space, via the difference equation (1.20), which is related to a one-dimensional phase space. Therefore, intuition may lead us to think that the difficulty of the problem in the one-dimensional space should be more manageable. In this context, let us analyse when does the hypothesis of Corollary 1.48 hold to see what we may impose to f/a.

We first write a characterisation of globally attracting equilibria in scalar difference equations, the following Theorem 1.49. It can be interpreted as a combination of the criterion to ensure that any initial condition $x_0 \in I$ is attracted by an equilibrium [24, 120] in the particular case where there is, in fact, a unique equilibrium. In order to formulate such result, we introduce, for any continuous function $F : I \to I$ having a unique fixed point $p \in I$, the following auxiliary functions

$$\Delta_n^F(x) := (F^n(x) - x)(x - p), \quad x \in I,\ n \in \mathbb{N}. \tag{1.33}$$

Thus, in the context of the expression (1.33), it is clear that

- $\Delta_1^F(x) = 0$ if and only if $x = p$.

- $\Delta_n^F(p) = 0$, for every $n \in \mathbb{N}$.
- If there existed $n_0 \in \mathbb{N}$ such that $\Delta_{n_0}^F$ has more roots other than p, they would also be equilibria for F^{n_0} and p could not be GAS since a non-constant n_0-periodic orbit would appear.
- If the sign of Δ_n^F is constant on $I \setminus \{p\}$, and we split the plane into two open regions by the graph of the identity function, then the graph of the n-th iterate of F changes only once from one side to the other, exactly at the point (p, p).

Bearing this in mind, we are ready for the announced criterion.

Theorem 1.49. *[42, Lemma 3.4] Let $F : I \to I$ be a continuous function and $p \in I$ be the unique fixed point of F. Then a necessary and sufficient condition for p to be GAS for F is that $\Delta_2^F(x) = (F^2(x) - x)(x - p) < 0$ on $I \setminus \{p\}$.*

One could find more equivalent statements to the global asymptotic stability of the origin in, e.g., [43, Proposition 1]. In particular, an interesting and common alternative equivalent statement to the thesis in Theorem 1.49 would be ‘a necessary and sufficient condition for the fixed point p to be GAS for F is that it is stable and the condition $\Delta_2^F(x) = (F^2(x) - x)(x - p) \neq 0$ holds on $I \setminus \{p\}$’.

Furthermore, if I is compact, Theorem 1.49 recovers the particular case of a unique fixed point in [24, Main Theorem], where the assumptions $p \in I$ being the unique fixed point of F and $\Delta_2^F \neq 0$ on $I \setminus \{p\}$ directly imply that $\Delta_2^F < 0$ on $I \setminus \{p\}$, that is, the stability of the equilibrium is redundant. Nevertheless, if I is not bounded, the latter is no longer true and the stronger negativity condition on Δ_2^F shall be required, as the following example shows.

Example 1.50. Let $I = \mathbb{R}$ and take $F : \mathbb{R} \to \mathbb{R}$, $F(x) = 2x$. It is clear that 0 is the unique fixed point of F and that $\Delta_2^F(x) \neq 0$ on $I \setminus \{0\}$. In fact, $\Delta_2^F > 0$ on $I \setminus \{0\}$ and it is easy to see that $|F^n(x_0)| \to \infty$ as $n \to \infty$, if $x_0 \neq 0$.

The criterion in Theorem 1.49 can be directly used in some applications, yet one shall be aware that challenging computations may appear. Thus, it is recommendable to handle easier sufficient conditions to check global asymptotic stability of the equilibrium for F. Hence, we will show just two of them below. In fact, we anticipate that the second one is somehow based into the first one.

Firstly, we write a result that is useful to compare two maps regarding global attraction of an equilibrium. This technique is known as *enveloping*. There are different meanings for such concept, such as the ones in [25, 33]. We will choose the one coming from the second reference, which will be useful regarding the forthcoming results.

Theorem 1.51. *[33, Theorem B] Assume that* $p \in I$ *is a global attractor for the continuous map* $H : I \to I$*. If* $F : I \to I$ *is another continuous map satisfying the enveloping*

$$\begin{aligned} x < F(x) \leq \max\{H(x), p\}, &\quad x < p, \\ x > F(x) \geq \min\{H(x), p\}, &\quad x > p, \end{aligned}$$

then p *is a global attractor for* F *too.*

In Figure 1.6, we can observe the relative position of the graphs of H and that of an arbitrary function F that is enveloped by H.

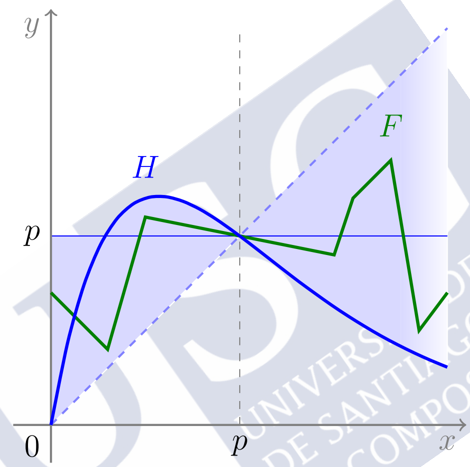

Figure 1.6: For $I = (0, \infty)$, the map H (whose graph is represented by a thick blue curve) envelops in the sense of Theorem 1.51 every continuous map F whose graph (thick green curve) is in the coloured zone. The value of p (narrow blue line) is also relevant in the inequalities of Theorem 1.51.

Theorem 1.51 is useful when we know that the map generating a difference equation we are interested in is enveloped in the sense of Theorem 1.51 by another map whose study turns out to be simpler.

Secondly, some particular easier sufficient conditions for a certain map to have a global attractor, rather than the way of comparing them through enveloping, shall be given. As we highlighted at the beginning of this subsection, the shape of the graph of the feedback f in (1.16) is relevant for studying the global attraction properties of the unique equilibrium p for (1.20).

We restrict our study to two families of continuous maps defined on intervals that have been extensively studied and are widely used in applications. The first family is the one of *(strictly) monotone maps* (from now on, we may call them *M-maps*). This family is split into two subfamilies: the one of *increasing maps*, which are the maps satisfying

$F(x) < F(y)$ if $x < y$, and the one of *decreasing maps*, which are the maps satisfying $F(x) > F(y)$ if $x < y$. The second family is the one of *unimodal maps* (hereafter, *U-maps*). We consider that the latter are maps having a unique critical point, which is not an inflection point (thus associated with a maximum or minimum value of the map).

From a topological point of view, when it comes to M-maps, they are homeomorphisms and the dynamical features are well-known. Unlike the latter maps, unimodal maps are not injective, but their domain is split in two intervals for which the map restricted to each one is a homeomorphism. However, they may actually show complex behaviour, even chaos [30].

Remark 1.52. The aforementioned families of maps are related with the concepts about the relative position of the graph with respect to an equilibrium (see the final part of Subsection 1.3.1). In particular, if $f' > 0$ (f is an increasing M-map), then f is a positive feedback whereas, if $f' < 0$ (f is a decreasing M-map), then f is a monotone negative feedback. If f is unimodal, then it might be a positive/negative feedback or none of them. For instance, the function f depicted in Figure 1.4 is neither a positive nor a negative feedback on the interval where it is defined.

When we impose certain constraints on the graph of a function f that is increasing, decreasing or unimodal, some interesting results about the global dynamics for the difference equation (1.20) and, thus, for the DDE (1.16), arise. In order to show them, we introduce the following relevant concept.

Definition 1.53. Let $F : I \to I$ be a function of class $\mathcal{C}^3$. We define the *Schwarzian derivative* of F as the function SF defined by

$$SF(x) = \frac{F'''(x)}{F'(x)} - \frac{3}{2}\left(\frac{F''(x)}{F'(x)}\right)^2,$$

for every $x \in I$ such that $F'(x) \neq 0$.

In fact, $SF = 0$ if and only if F is a rational map having the form

$$F(x) = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad \alpha\delta - \beta\gamma \neq 0, \tag{1.34}$$

for any $x \in I$ such that $\gamma x + \delta \neq 0$. In the context of complex variables, the map in (1.34) is known as a Möbius transformation. The applications of this concept in difference equations were found by Allwright and Singer [2, 122] and one of the main ingredients for its relevance is the formula of the Schwarzian derivative of the composition of two maps, which can be calculated as follows (see, e.g., [29, Proposition 11.3])

$$S[G \circ H](x) = SG(H(x))(H'(x))^2 + SH(x). \tag{1.35}$$

It is clear that the Schwarzian derivative of a function is not affected by further transformations that have null Schwarzian derivative. In fact, if $SG = 0$, then,

$$S[G \circ H](x) = 0\,(H'(x))^2 + SH(x) = SH(x).$$

For instance, if the Schwarzian derivative of a function F has constant sign, then the Schwarzian derivative of any composition of F with affine transformations

$$A(x) = \alpha x + \beta, \quad \alpha \neq 0,\ \beta \in \mathbb{R},$$

which are rational maps with $\gamma = 0$ and $\delta = 1$, keep the sign of SF. In particular, the latter implies that the changes of variables regarding shifting the equilibrium to 0 (a couple of translations of the type $A(x) = x + k$) or dividing by the parameter a ($A(x) = \frac{x}{a}$), which were considered in Subsection 1.3.1 do not affect the sign of the Schwarzian derivative.

If F is sufficiently regular and such that $SF < 0$, then some geometrical properties of its graph can be derived. The following proposition gathers some of those well-known basic features of maps with negative Schwarzian derivative. Since the Schwarzian derivative of a function F might not be defined on the whole interval I due to the existence of critical points of F, we assume that any condition on SF is assumed to hold on its domain, as natural.

Proposition 1.54. *Let $F : I \to I$ be a function of class $\mathcal{C}^3$ that satisfies $SF < 0$. Then:*

1. *F' does not take values that are either positive minima or negative maxima.*
2. *F has at most one inflection point on each interval of definition of SF.*
3. *$S[F^n] < 0$, for every $n \in \mathbb{N}$.*

The first two parts can be checked, e.g., in [119, Chapter 5, Section 3], and the last one is clear from an inductive reasoning on formula (1.35). Moreover, the last assertion implies that any iterate of F also satisfies the first two theses of the latter result.

The relevance of the geometrical conditions shown in Proposition 1.54 is that we can derive global-type results of the dynamics of (1.16) via the local behaviour of F at the equilibrium if F is a decreasing or unimodal map. In fact, its use provides us with a sufficient condition to check whether p is GAS for (1.20).

Theorem 1.55. *[33, Corollary 2.10] Let $F : I \to I$ be a function of class $\mathcal{C}^3$ such that $\Delta_1^F(x) = (F(x) - x)(x - p) < 0$ on $I \setminus \{p\}$, and having at most one critical point, which would be called c and would also satisfy $F''(c) \neq 0$. Assume that one of the following two conditions holds:*

1. *The function F satisfies $0 \le F'(p) < 1$.*

2. *The function F satisfies $-1 \le F'(p) < 0$ and $SF(x) < 0$, for every $x \in I_*$, where*

   - *$I_* = I$, if there is no extremum,*
   - *$I_* = I \cap (c, \infty)$, if F attains a maximum value at c, and*
   - *$I_* = I \cap (-\infty, c)$, if F attains a minimum value at c.*

Then the equilibrium p is GAS for F.

Originally, the key assumption was $SF < 0$ [2, 122]. In [94, Section 3], the reader can find an alternative proof for this case. The power of the generalisation given by El-Morshedy and Jiménez López [33], which is the version that we have shown in Theorem 1.55, is that the Schwarzian derivative does not need to be negative on the whole interval I in order to obtain that the origin is GAS from being LAS. This turns out to be extremely useful in the analysis of some models, as we will see later. Roughly speaking, if $F'(p) < 0$, then it only assumes negativity on the Schwarzian derivative on the interval where F decreases.

Remark 1.56. Notice that the condition $SF < 0$ and the rest of hypotheses in Theorem 1.55 also allow $F'(p)$ to be equal to -1 in order for the origin to be GAS and, thus, LAS (compare with Proposition 1.40 and the subsequent remark).

The proof of Theorem 1.55 is intimately related with the impossibility of the existence of 2-periodic points different from p for (1.20), the key assumption from Theorem 1.49. While the first part is clear from a reasoning on monotonic sequences, the second one is a subtle combination of the Mean Value Theorem, some parts of Proposition 1.54, and some enveloping-type ideas concerning Theorem 1.51 and the mentioned Theorem 1.49. Even the uniqueness of the critical point (whenever it exists) is relevant in order to prove the non-existence of 2-periodic points different from the equilibrium p (see details in the above-mentioned alternative reference [94]). In particular, we refer to the references [29, 30, 119] to see the role of critical points in maps with negative Schwarzian derivative.

As announced above, there exists some link between the techniques of enveloping and the one of negative Schwarzian derivative. The enveloping-type ideas involved in the proof of Theorem 1.55 are one way to convince ourselves about why that happens. Nevertheless, it becomes much clearer with the following comments.

Remark 1.57. From the work [85], and mainly its Lemma 2.1, it is known that, if F is decreasing or unimodal and 0 is its unique fixed point (otherwise, consider the shifted function $\tilde{F}$) with $F'(0) < 0$, then $SF < 0$ implies that F is enveloped by a rational function $R(x) = \frac{\alpha x}{1+\gamma x}$ such that

$$0 = F(0) = R(0), \quad F'(0) = R'(0) = \alpha \text{ and } F''(0) = R''(0).$$

Some conditions for the map R to have the origin as a globally attracting equilibrium can be easily obtained via Theorem 1.49. The results on enveloping from [25, 33], such as Theorem 1.51, thus yield the global stability result for F.

There exist equivalent conditions to $SF < 0$ that would also make sense with less regularity on F [119, Chapter 5, Section 3]. Nevertheless, such conditions may become quite technical and hard to work with. Thus, since the functions F we are handling are at least $\mathcal{C}^3$, we have assumed such regularity in Theorem 1.55.

Therefore, after having exposed the importance of the Schwarzian derivative for the study of the dynamics of certain maps, we introduce the following concepts.

Definition 1.58. Let $F : I \to I$ be a function of class $\mathcal{C}^3$.

- If F is an M-map and $SF < 0$ on I, then F is an *SM-map*.
- If F is a U-map and $SF < 0$ on I, then F is an *SU-map*.
- Any SM-map or SU-map will be generally called an *S-map*.
- If F is a U-map and $SF < 0$ on the subinterval of I where F decreases, then F is an *SU_*-map*.
- Any SM-map or SU_*-map will be generally called an *S_*-map*.

This type of notation is commonly used in the literature. For instance, see [43, 67, 88]. Obviously, every SU-map is an SU_*-map. Notice that there is no need to define SM_*-maps since the Schwarzian derivative is only used when $F'(p) < 0$; thus, provided F is an M-map, that would mean is decreasing on the whole I.

Recall that, if (a, f, τ) satisfies the hypotheses (T), then $F = f/a$ satisfies $\Delta_1^F < 0$ on $I \setminus \{p\}$. Notice that if Δ_1^F were positive at any side of p, then no K as in (T4) would exist, as a simple graphical analysis shows. Hence, one can combine Theorem 1.55 with Corollary 1.48 to obtain the following result.

Corollary 1.59. *Let (a, f, τ) satisfy (T), p be the unique root of $f(x) = ax$, and assume that f is an S_*-map. If $f'(p) \in [-a, a)$, then p is GAS for the DDE* (1.16).

Let us clarify and sum up the features of the procedure that we have followed. Up to this point, we have already seen that Corollary 1.48 together with the criterion in Theorem 1.49 or Theorem 1.55 provide sufficient conditions in terms of the parameters a and f to deduce the global attraction properties of the equilibrium p in (1.16). Since those conditions come from analysing the difference equation (1.20) and in such an equation the delay τ does not appear, the conditions obtained are then called *delay-independent stability conditions* or *absolute stability conditions*. For instance, the applicability of Corollary 1.59 is independent from the value $\tau > 0$.

Based on these remarks, if we try to go further and obtain sharper conditions, they must be *delay-dependent stability conditions*. In this line, the results in [54], which were extended in [94] in a more general context, are enlightening.

Theorem 1.60. *[54] Let (a, f, τ) satisfy (T) and p be the unique root of $f(x) = ax$. Assume that p is GAS for*

$$x_{n+1} = e^{-a\tau}p + (1 - e^{-a\tau})F(x_n), \tag{1.36}$$

where $F = f/a$. Then, p is GAS for (1.16).

The hypothesis of Theorem 1.60 is related with a difference equation that clearly depends on τ and, thus, a first delay-dependent condition naturally appears, as explained below.

Corollary 1.61. *Let (a, f, τ) satisfy (T), p be the unique root of $f(x) = ax$, and assume that f is an S_*-map. If*

$$f'(p) \in [-a, a) \quad \text{or} \quad \left[f'(p) < -a \ \text{and} \ e^{-a\tau} \geq \frac{f'(p) + a}{f'(p)} \right], \tag{1.37}$$

then the origin is GAS for (1.16).

Theorem 1.51 provides another way to interpret condition (1.37) in terms of enveloping. In fact, if we define the function F_τ as in the right-hand side of (1.36), that is,

$$F_\tau(x) := e^{-a\tau}p + (1 - e^{-a\tau})F(x),$$

and assuming that there exists $\tau^* > 0$ such that the hypotheses in Theorem 1.60 hold, then, since the function F_{τ^*} envelops, in the sense of [33], every F_τ of a lesser τ, the result also holds for every $\tau \in [0, \tau^*]$. In particular, τ^* can be chosen to make the delay-dependent inequality in (1.37) an equation. This situation is depicted in Figure 1.7, where we have to bear in mind that $F'_{\tau^*}(p) = -1$.

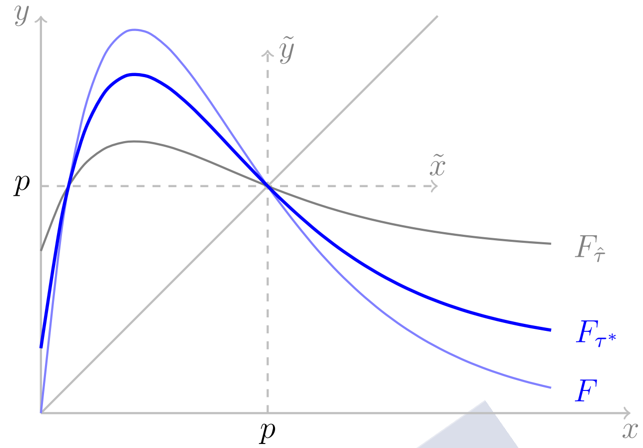

Figure 1.7: For $I = (0, \infty)$, the graph of the map F is drawn in blue. The maps $F_\tau(x)$ are nested in terms of the enveloping in [33]. The thick blue graph is the one of F_{τ^*}, whereas the black one stands for the graph of a certain $F_{\hat{\tau}}$, with $0 < \hat{\tau} < \tau^*$.

Remark 1.62. Despite the geometrical significance of the above-mentioned condition, it is possible to go further [54]. In fact, Theorem 1.60 is sharpened by switching global attraction hypothesis for (1.36) to global attraction for another more complicated difference equation. In particular, in case f were further an S-map, a sharper version of Corollary 1.61 could also be given in case $f'(p) < -a$ by using the relation

$$e^{-a\tau} \geq \frac{f'(p)^2 + af'(p)}{f'(p)^2 + a^2}. \tag{1.38}$$

However, if we directly assume that $Sf < 0$ on the whole I, then the deeper analysis of equation (1.16) in [94], which takes into account the enveloping of funcions with negative Schwarzian derivative by rational functions, in the line of Remark 1.57, actually yields a sharper result that the ones in Corollary 1.61 and Remark 1.62.

Theorem 1.63. *[86, Proposition 1.1] Let (a, f, τ) satisfy (T), p be the unique root of $f(x) = ax$, and assume that f is an S-map. If*

$$f'(p) \in [-a, a) \quad or \quad \left[f'(p) < -a \ and \ e^{-a\tau} > -\frac{f'(p)}{a} \ln\left(\frac{f'(p)^2 - af'(p)}{f'(p)^2 + a^2} \right) \right], \tag{1.39}$$

then the origin is GAS for (1.16).

As remarked in [86], condition (1.39) is sharper than the one in (1.37) due to the linear bound $x > \ln(1+x)$, for every $x > 0$. In fact, provided $f'(p) < -a < 0$, we have

$$e^{-a\tau} \geq \frac{f'(p)+a}{f'(p)} = -\frac{f'(p)}{a}\left(\frac{-af'(p)-a^2}{f'(p)^2}\right) > -\frac{f'(p)}{a}\left(\frac{-af'(p)-a^2}{f'(p)^2+a^2}\right)$$
$$> -\frac{f'(p)}{a}\ln\left(1+\frac{-af'(p)-a^2}{f'(p)^2+a^2}\right) = -\frac{f'(p)}{a}\ln\left(\frac{f'(p)^2-af'(p)}{f'(p)^2+a^2}\right) > 0.$$

Thus, condition (1.37) is more restrictive when f is an S-map and (1.39) opens a larger window of parameters for which the origin is GAS for (1.16). In fact, (1.38) appears between both (see the last expression of the first line in the chain of inequalities above).

The interest of (1.37) is that it is related to easier explicit conditions in applications, while (1.39) might generate implicit conditions and (1.38) involves tougher explicit expressions. Moreover, condition (1.37) is the estimate that we can clearly use when f is only an S_*-map. Moreover, it is important to bear in mind that all these conditions imply that the origin is LAS and thus they must belong to the region L of local asymptotic stability, which was defined at the end of Subsection 1.3.2 and depicted in Figure 1.5.

We are close to the end of this section and the current chapter, so we sum up the information that we have obtained with respect to the qualitative analysis of (1.16) in the remark below. We recall that the negativity condition on the Schwarzian derivative of a map allows us to focus on local dynamics to deduce information regarding global behaviour of solutions and that $F'(p)$ and $f'(p)$ are strongly related (Remark 1.41). In fact, f is an S-map if and only if F is an S-map. Moreover, to write and better understand the following remark, it is useful to define G as the region of triples $(a, f'(p), \tau)$ for which the origin is GAS for (1.16).

The conditions on global and local stability that have appeared in the current chapter, namely (1.37), (1.39) and (1.31), are a motivation to define the following functions. Regarding estimates for the delay τ, we respectively define

$$\rho_1(a, f'(p)) := \frac{1}{a}\ln\left(\frac{f'(p)}{f'(p)+a}\right),$$
$$\rho_2(a, f'(p)) := \frac{-1}{a}\ln\left(-\frac{f'(p)}{a}\ln\left(\frac{f'(p)^2-af'(p)}{f'(p)^2+a^2}\right)\right),$$
$$\rho_3(a, f'(p)) := \frac{\arccos\left(\frac{a}{f'(p)}\right)}{\sqrt{f'(p)^2-a^2}},$$

and, with respect to estimates for $f'(p)$, we respectively define

$$\xi_1(a,\tau) := -\frac{a}{1-e^{-a\tau}},$$
$$\xi_2(a,\tau) := -aM_2^{-1}(e^{-a\tau}),$$
$$\xi_3(a,\tau) := -\sqrt{a^2 + \frac{(M_3^{-1}(a\tau))^2}{\tau^2}},$$

where $M_2 : (1,\infty) \to (0,1)$ and $M_3 : (\pi/2,\pi) \to (0,\infty)$ are the bijective functions defined by

$$M_2(x) := x\ln\left(\frac{x^2+x}{x^2+1}\right), \quad M_3(x) := \frac{-x}{\tan(x)}.$$

The following Remark 1.64 summarises the previous results with respect to the value of the delay and it is graphically supported by Figure 1.8.

Remark 1.64. Let (a, f, τ) satisfy (T), f be an S_*-map, $F = f/a$ and p be the unique fixed point of F. We have the following cases:

- If $\frac{f'(p)}{a} = F'(p) > 1$, then this case does not fall under our framework.
- If $\frac{f'(p)}{a} = F'(p) = 1$, then p is non-hyperbolic for both equations (1.16) and (1.20). However, $f''(p) = 0$ must hold and p is GAS for (1.16) (see [119, Lemma 5.10]).
- If $\frac{f'(p)}{a} = F'(p) \in [-1, 1)$, then p is GAS for F, and, thus, for (1.16) too.
- If $\frac{f'(p)}{a} = F'(p) < -1$, and
$$\tau \leq \rho_1(a, f'(p)),$$
then p is GAS for (1.16).
- If $\frac{f'(p)}{a} = F'(p) < -1$, f is an S-map and
$$\rho_1(a, f'(p)) < \tau < \rho_2(a, f'(p)),$$
then p is GAS for (1.16).
- If $\frac{f'(p)}{a} = F'(p) < -1$ and
$$\rho_2(a, f'(p)) \leq \tau < \rho_3(a, f'(p)),$$
then p is LAS for (1.16) and a general result about the global attractivity of p for such an equation is, as far as we know, not available yet.

- If $\frac{f'(p)}{a} = F'(p) < -1$ and $\tau = \rho_3(a, f'(p))$, then p is non-hyperbolic for (1.16) and other techniques should be considered.
- If $\frac{f'(p)}{a} = F'(p) < -1$ and
$$\rho_3(a, f'(p)) < \tau,$$
then p is unstable for (1.16).

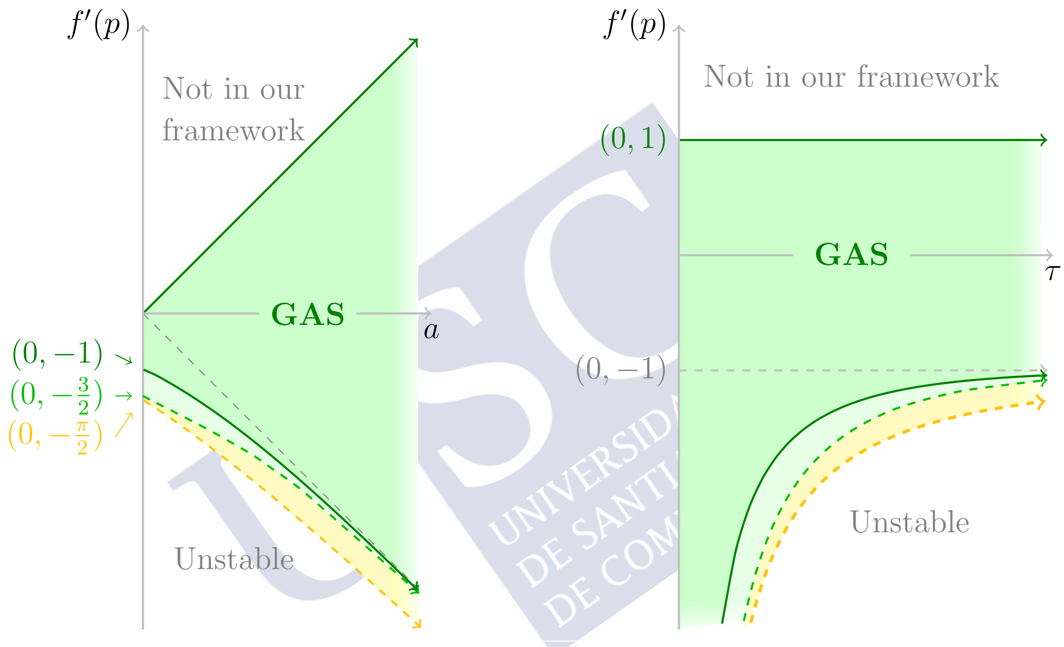

Figure 1.8: These pictures sketch the restriction of the region G that comes from Remark 1.64 for the cases $\tau = 1$ (on the left) and $a = 1$ (on the right). The darker green filling colour represents such region for S_*-maps. It is respectively bounded by the solid green curves $\rho_1(a, f'(p)) = 1$ and $\xi_1(1, \tau) = f'(p)$. The lighter green filling colour represents the extension of G to S-maps and is respectively bounded by the dashed green curves $\rho_2(a, f'(p)) = 1$ and $\xi_2(1, \tau) = f'(p)$. The yellow zone represents the region of L for which the global dynamics of (1.16) are not completely understood and it is respectively bounded by the dashed orange curves $\rho_3(a, f'(p)) = 1$ and $\xi_3(1, \tau) = f'(p)$.

Remark 1.65. Notice that, for each $i \in \{1, 2, 3\}$, any inequality $\tau < \rho_i(a, f'(p))$ (or its non-strict version, alternatively) appearing in Remark 1.64 could have been substituted by $f'(p) > \xi_i(a, \tau)$ (or by its non-strict version, respectively) to produce an analogous version of Remark 1.64.

Remark 1.64 and Figure 1.8 are provided in a general sense. The particular choices of the feedback f may imply the introduction of *feedback inner parameters*, which have a certain meaning depending on what one is attempting to model. For instance, if we take $f(x) = \beta x e^{-\delta x}$, where $\beta, \delta \in \mathbb{R}$, then β and δ would be feedback inner parameters. Specifically, $f'(p)$ may depend on some of those inner parameters. Therefore, in the case $f'(p)$ acquires major relevance when it comes to global asymptotic stability of the equilibrium (e.g., with S-maps), we can provide analogous conditions and figures that highlight the role of any of those parameters compared to, e.g., the delay τ. This will be done afterwards, in Chapters 2 and 3, where we analyse some particular models.

Remark 1.66. We have seen that there is an underlying relation with discrete dynamics provided that $a > 0$. Although we only focus on the latter case, it is worth recalling what happens with $a = 0$ and, furthermore, whether it coincides with the limit version of the given conditions as $a \to 0^+$. The works [63, 85] provide the basis towards a sharp result regarding the global dynamics of equation (1.15) with $a = 0$ and an S-map feedback. Such general result is given in [95, Theorem 2]. The latter work generalises both the well-known work by Yorke [142] and the one by Wright [136] via the use of rational functions (1.34) (recall Remark 1.57). In our framework, such sharp condition from [95] shall be interpreted as

$$0 > \tau f'(p) > -\frac{3}{2},$$

also including the case $\tau f'(p) = -\frac{3}{2}$ if $f''(p) \neq 0$. Through some computations, it can be seen that this condition is consistent with the mentioned limit of (1.39) as $a \to 0^+$. Moreover, the limit form of the LAS-type condition given in (1.31) coincides with the one of the case $a = 0$, which is $0 > \tau f'(p) > -\pi/2$. In fact, by continuity, we can extend the previous functions to $a = 0$ by its limit form as $a \to 0^+$ and define

$$\rho_1(0, f'(p)) := -\frac{1}{f'(p)}, \quad \rho_2(0, f'(p)) := -\frac{3}{2f'(p)}, \quad \rho_3(0, f'(p)) := -\frac{\pi}{2f'(p)}.$$

When the unique equilibrium p of the difference equation (1.20) is unstable and also $f'(p)/a = F'(p) < -1$ (repelling point), the terms of any sequence with x_0 sufficiently close to p go away from the equilibrium. In such case, the instability is inherited by the corresponding DDE (1.16) for large values of the delay τ. In fact, if the function $F = f/a$ is of negative feedback on the set K relative to (T4), then the loss of stability implies the loss of global attractivity [55, 71]. That is one relevant reason for saying at the beginning of this subsection that we were restricting our study to the region L of local asymptotic stability. As a summary of these comments, we write Remark 1.67 and provide Figure 1.9.

Remark 1.67. Consider the assumptions of Remark 1.64 together with $F = f/a$ being of negative feedback on the set K relative to (T4). If $f'(p) < -a$, then the value of the delay affects the stability of the unique equilibrium in the following particular way:

- there exists $\tau^* \in (0, \infty)$ such that the origin is GAS for (1.16) and any $\tau \in [0, \tau^*)$.
- there exists $\bar{\tau} \in (0, \infty)$ such that the origin is unstable for (1.16) and any $\tau \in (\bar{\tau}, \infty)$.

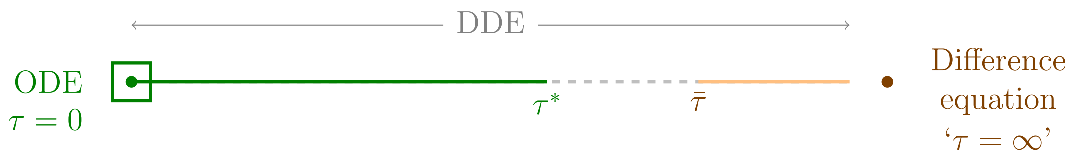

Figure 1.9: Global picture of the role of the delay τ in the dynamics of (1.16) when the equilibrium is unstable for the corresponding difference equation (1.20) in the conditions of Remark 1.67. The unique equilibrium p of (1.16) is stable (green, left zone) for small values of the delay, and unstable (orange, right zone) for large values of it.

Notice that, for general feedbacks, it is not known whether $\tau^* = \bar{\tau}$. In other words, the cases of S-maps that fall into the region between the curves $\rho_2 = 1$ and $\rho_3 = 1$ in the $(a, f'(p))$-plane are not completely studied.

Conjecture 1.68. *[94, Conjecture 2.1] Let (a, f, τ) satisfy (T), p be the unique root of $f(x) = ax$, and assume that f is an S-map. If p is LAS for the DDE* (1.16)*, then it is also GAS.*

Nevertheless, in some particular cases, Conjecture 1.68 is true and the folklore statement 'LAS implies GAS' holds. See [79, Section 6.2] and the references therein to find a case with $a > 0$ where there is an affirmative answer to the above-mentioned conjecture.

Moreover, there is also a very well-known particular case for $a = 0$ that has acquired great attention in the last decades, which is related with the cases in this section. We refer to the famous Wright's equation

$$x'(t) = \beta x(t-1)(1 + x(t)). \tag{1.40}$$

Equation (1.40) is not included in the framework of the current Section 1.3, but, after a change of variables, it can be transformed into

$$x'(t) = \beta(e^{-x(t-1)} - 1) =: f(x(t-1)), \tag{1.41}$$

which is just an example of the equations we have been treating in the current section with $\tau = 1$ and the limit case $a = 0$. Notice that the unique equilibrium of (1.41) on $I = \mathbb{R}$ is $p = 0$ and the linearised equation of (1.41) at $p = 0$ is

$$x'(t) = f'(0)x(t-1) = -\beta x(t-1),$$

so that the unique equilibrium of (1.41) is LAS provided that $\beta \in (0, \pi/2)$. In fact, f is an S-map.

Wright proved [136] that the unique equilibrium of (1.41) is GAS provided that the condition $\beta \in (0, 3/2]$ holds and the problem of extending such a result up to $\beta = \pi/2$ was later named as the Wright's conjecture [71, 79, 132] (compare with Remark 1.64 and Figure 1.8). This conjecture remained unproven until few years ago, when a combination of the works by Bánhelyi et al. [9] and by van der Berg and Jaquette [132] finally achieved this landmark.

Certainly, the issue 'LAS implies GAS' is not only restricted to feedbacks satisfying certain hypotheses regarding the sign of its Schwarzian derivative. Following [79, Section 6.2] and some references therein, there are different kind of maps that satisfy certain a relation between its derivative $f'(x)$ and the quotient $f(x)/x$ for which the property 'LAS implies GAS' is also true. We also refer to the works [43], in the context of difference equations, and [41], regarding its application to DDEs, to consult recent results showing how the negative condition on the Schwarzian derivative can be overcome, and to complement the results exposed above when it comes to the quest of such property.

We conclude this chapter with one important issue that we remarked before. In fact, we have been considering the DDE in (1.16), which has a constant delay. This equation is generalised by the non-autonomous one (of variable delay) in (1.15). It is also natural to wonder whether the unique equilibrium of (1.15) has some attractivity properties. Since the DDE in (1.15) is non-autonomous, it does only fit with the very beginning of Section 1.2; nevertheless, it is still possible to extend the above-stated results.

We refer to the works [64, 86] for further details on this adaptation. While [64] is related to the extension of Corollary 1.48, one can observe that Lemmas 4.5 and 4.7 and Theorem 2.1 in [86] serve as a source to check the validity of the estimates in Theorem 1.60, Remark 1.62 and Theorem 1.63 to the non-autonomous case (1.15). We sum up these cases in the last result of this chapter.

Theorem 1.69. *Let $\tilde{\tau} : \mathbb{R} \to \mathbb{R}_+$ be a bounded continuous function and τ be its supremum. Assume that (a, f, τ) satisfy (T) and let p be the unique equilibrium of $F = f/a$. The following assertions hold:*

- *If p is a global attractor for F, then p is a global attractor for equation* (1.15).
- *If f is an S-map and*

$$f'(p) \in [-a, a) \quad \text{or} \quad [\, f'(p) < -a \text{ and } \tau < \rho_2(a, f'(p))\,], \tag{1.42}$$

  then the unique equilibrium p of (1.15) *is GAS.*

In fact, regarding a proof of the first point in the last theorem, we anticipate that a more general result, in the line of [64], will be needed in Section 2.2.2, namely Theorem 2.32. Therefore, the reader can also check the proof there.

The situation shown in Theorem 1.69 seems to provide some analogy with the constant delay case. Nevertheless, there is a major issue that deserves to be highlighted: as it is recognised in [86], condition (1.42) is sharp: in case τ were at least $\rho_2(a, f'(p))$, it would be possible to find a certain delay function $\tilde{\tau}(t)$ such that τ is its supremum and the equilibrium p is not GAS. This means that the result is optimal as it provides a common estimate for every feedback f, parameter a and bounded continuous delay $\tilde{\tau}(t)$. However, as it has been mentioned before, there exist some particular cases, including some of constant delay for which the 'stability limit' is even larger and the issue 'LAS implies GAS' remains open (see Conjecture 1.68 and the comments below).

Chapter 2

Gamma-models

The main aim of this chapter is to establish, in the line of Section 1.3, sharp global stability conditions for the positive equilibrium of a family of scalar delay differential equations that have been called 'gamma-models' in relation with a certain parameter γ which is present in a particular way in their feedbacks. Firstly, we provide some motivations for their use and study based on some economic and population growth models. Later, in order to deal with a broad scenario, we provide several general results that can be applied to any of these differential equations. Then, we pick up different common choices of the feedback that are relevant in applications and, finally, we provide a brief discussion of the role of such parameter γ in each case, showing to what extent we are able to give information about the asymptotic behaviour of the solutions of the equation modelling the mentioned phenomena. The development of the contents of this chapter is based on the works [17, 20] and short extracts of [83], by the author of this thesis (S. Buedo-Fernández[1]), and, regarding the last two items, also by Eduardo Liz[2]. They are listed below.

S. Buedo-Fernández. On the gamma-logistic map and applications to a delayed neoclassical model of economic growth. *Nonlinear Dynamics* 96(1), 219–227 (2019). Springer, ISSN 1573-269X (Electronic), 0924-090X (Print).

S. Buedo-Fernández, E. Liz. On the stability properties of a delay differential neoclassical model of economic growth. *Electronic Journal of Qualitative Theory of Differential Equations* 2018, No 43, 1–14 (2018). University of Szeged, ISSN 1417-3875.

E. Liz, S. Buedo-Fernández. A new formula to get sharp global stability criteria for one-dimensional discrete-time models. *Qualitative Theory of Dynamical Systems* 18, 813–824 (2019). Springer, ISSN 1662-3592 (Electronic), 1575-5460 (Print).

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2.1 Motivation and background

In this section, we introduce this topic by recalling two applications of scalar delay differential equations whose qualitative study turns out to be strongly related with what was shown in Section 1.3. The first one comes from the field of Economics, while the second is based on models about population growth.

The neoclassical aggregate growth model, also called Solow-Swan model [10], is an economic model that attempts to explain long-run economic growth based on capital accumulation and labour (or population growth). The fundamental equation of this theory was proposed by Solow in 1956 [126], and it is based on the assumption that there is only one commodity and its rate of production is defined by a function $P = P(K, L)$, where K and L are the capital stock and labour's rate of input, respectively. Capital represents the durable physical inputs, such as machines, while labour represents the inputs associated with human body, such as the number of workers and the amount of time they work. It is also supposed that part of the instantaneous output is consumed and the rest is saved and invested. Introducing a new variable $x = K/L$ (the capital-labour ratio, that is, the capital stock per unit of effective labour), and assuming constant returns to scale (which, in mathematical terms, means that the function P is homogeneous of degree one), Solow proposed the ordinary differential equation

$$x'(t) = -ax(t) + s(x(t))P(x(t), 1), \tag{2.1}$$

where a is the depreciation rate of the capital-labour ratio (related with both the depreciation of capital and the population growth rate), and $s(x)$ is the instantaneous rate of saving. See [10, 126] for more details about the derivation of (2.1). Roughly speaking, equation (2.1) states that the rate of change of the capital-labour ratio is the difference between the increment of capital and the increment of labour [126]. The simplest case of equation (2.1) assumes a constant saving rate s, leading to equation

$$x'(t) = -ax(t) + sp_1(x(t)), \tag{2.2}$$

where $p_1(x) = P(x, 1)$. The typical assumptions for the neoclassical production function $P(K, L)$ (see, e.g., [10, pp. 26–28]) ensure that (2.2) has a unique positive equilibrium, which is globally asymptotically stable; this means that, regardless of the initial value of the capital-labour ratio, the system will evolve towards a state of balanced growth at the natural rate [126]. See also [61, 62] for more discussions on the global stability of neoclassical models.

Attempting to show how cyclic and complex behaviour can emerge from a neoclassical model, Day [27] proposed to express the neoclassical growth model (2.1) as a difference equation

$$x_{n+1} = \frac{s(x_n)p_1(x_n)}{1 - \lambda}, \tag{2.3}$$

where λ is the natural rate of population growth. In the same paper, an alternative model is suggested where s is constant but the productivity is reduced by a 'pollution effect' caused by increasing concentrations of capital, thus leading to the equation

$$x_{n+1} = \frac{s\, p_1(x_n) p_2(x_n)}{1-\lambda}, \tag{2.4}$$

where the pollution function $p_2(x)$ is a nonincreasing function of x.

Recently, Matsumoto and Szidarovsky [102, 103] suggested another way to introduce a production lag in Solow's model (2.1) by using a delay differential equation. They consider constant saving rate, constant population growth rate, and a pollution-type factor in the production function in the direction suggested by Day, thus getting the following generalisation of equation (2.2):

$$x'(t) = -ax(t) + sf(x(t-\tau)), \tag{2.5}$$

where $f(x) = p_1(x)p_2(x)$, and $\tau > 0$ represents the time delay inherent to the production process.

The possibility of cycles and complex dynamics both in the discrete models (2.3), (2.4), and in the delay differential equation (2.5) for large values of the delay τ was shown in [27] and [102, 103], respectively. Thus, it is important to find sufficient conditions to ensure the global attractivity of the balanced equilibrium in these models.

A more general form of (2.5), allowing variable and instantaneous saving rate as in (2.1) would lead to equation

$$x'(t) = -ax(t) + s(x(t))f(x(t-\tau)). \tag{2.6}$$

In the line of what we have previously argued in Chapter 1, the issue of global stability for delay differential equations such as (2.6) is, generally speaking, considerably more complicated than for ordinary differential equations, and different approaches have been recently applied to some economic models (see, e.g., [8, 16, 89]). As in [89], in this chapter we apply the approach based on the interplay between delay differential equations and maps shown in Section 1.3, which goes back at least up to the previously discussed works [65, 98], and has been generalised and successfully used to prove global stability results for many different models in the past twenty years (e.g., see [64, 85, 86, 89, 90, 92, 94]).

The delay-differential model (2.6) is written in terms of three factors that need to be fixed: the saving rate $s(x)$ and the two factors of the feedback f, namely the labour-normalised production function $p_1(x)$ and the pollution term $p_2(x)$.

We now recall a common choice for the production function P: the well-known Cobb-Douglas function

$$P(K, L) = BK^{\gamma}L^{1-\gamma},$$

where $B > 0$ refers to the level of labour-augmenting technology (see [10, Pages 23-30] to understand why this constant arises), and $\gamma \in (0,1)$ represents the output elasticity of capital, that is, the part of the output produced by the capital. By using that choice in the equation (2.6), we get

$$x'(t) = -ax(t) + s(x(t))Bx^{\gamma}(t-\tau)p_2(x(t-\tau)). \tag{2.7}$$

In his seminal paper [126], Solow suggested several examples involving the Cobb-Douglas function and some of its generalisations. For example, assuming constant saving rate $s(x) = \bar{s} > 0$ and no pollution effects, that is, $p_2(x) = 1$, the non-delayed equation (2.2) (a particular case of the general DDE (2.7)) becomes

$$x'(t) = -ax(t) + \beta x^{\gamma}(t), \tag{2.8}$$

with $\beta = \bar{s}B$. It is easy to check that the positive balanced equilibrium $p = (\beta/\alpha)^{1/(1-\gamma)}$ is globally asymptotically stable (see, for instance, references [100, 126], or recall the last sentence of Remark 1.29 regarding semiflows generated by scalar ordinary differential equations).

In this chapter, we consider further examples of equation (2.7) apart from the delayed version of (2.8) just by considering different choices of the pollution term and the saving rate. For instance, Matsumoto and Szidarovszky [103] studied the local stability of the equilibrium in (2.7), for constant saving rate s, and the pollution-type function given by $p_2(x) = e^{-\delta x}$, $\delta > 0$, that is, they considered the delay differential equation

$$x'(t) = -ax(t) + \beta x^{\gamma}(t-\tau)e^{-\delta x(t-\tau)}. \tag{2.9}$$

In such direction, Matsumoto and Szidarovszky [102] also proposed a model with a logistic feedback term and fixed delay as follows

$$x'(t) = -ax(t) + \beta x(t-\tau)(1 - x(t-\tau)), \tag{2.10}$$

and they analysed the local stability of its unique positive equilibrium. As recognised in [102], equation (2.10) is a simplification of the more general model

$$x'(t) = -ax(t) + \beta x^{\gamma}(t-\tau)(1 - x(t-\tau)), \tag{2.11}$$

where the term $\beta x^{\gamma}(1-x)$, $\gamma \in (0,1]$, comes from considering the Cobb-Douglas production function $p_1(x) = \beta x^{\gamma}$, as in [27, 126], and $p_2(x) = 1 - x$, which reflects a linear influence of pollution on per-capita output. As highlighted in [102], such production function may lead to solutions that take negative values. They propose to consider the modified DDE

$$x'(t) = -\alpha x(t) + \max\{0, \beta x^{\gamma}(t-\tau)(1 - x(t-\tau))\}, \tag{2.12}$$

which can be used to avoid such problem, as we will see later.

The above-mentioned delay differential models, especially when they are included in the framework of (2.5) (s is constant), are also utilised in the context of population dynamics. Actually, for $\gamma = 1$, (2.9) becomes the celebrated Nicholson's blowflies equation [49]; for $\gamma = 0$, (2.9) is the equation for the red-blood cell system proposed by Ważewska-Czyżewska and Lasota [135]; Lasota himself proposed equation (2.9) with positive values of γ in a later paper [74]; finally, for $\gamma > 1$, (2.9) can be used as a model for populations subject to Allee effects [60, 92].

Moreover, equation (2.10) when considered in the framework of population dynamics is called the blowfly logistic equation [80]. The latter reference provides a discussion about the use of this equation during the last decades.

Additionally, there is another case of equation (2.5) that has acquired much popularity: the Mackey-Glass equation. This latter can be written in the form of

$$x'(t) = -ax(t) + \frac{\beta x^{\gamma}(t-\tau)}{1+\delta x^{m}(t-\tau)},$$

where β, δ, m are also positive constants, and $\gamma \in \{0, 1\}$, which are the options considered in the well-known work by Mackey and Glass [96] to model the concentration of certain blood cells as time goes by.

However, those DDE-based population growth models have their discrete-time version, which in view of the results of Section 1.3 are also of our interest. In fact, we refer to the works by Liz [81, 82] for a discussion of those models and, in particular, for an interpretation about the meaning of γ. Nevertheless, it is worth recalling some brief key ideas. The parameter γ is part of a density-dependent factor that allows to model cooperation and competition. In particular, many of those models fall into the following general expression

$$x_{n+1} = x_n^{\gamma} H(x_n), \tag{2.13}$$

where $\gamma \geq 0$ and H is a nonnegative function defined for nonnegative numbers. The case $\gamma = 1$ has attracted much attention in the framework of population dynamics, and in this case H represents the per capita production function. In compensatory or overcompensatory models, H is decreasing (see, e.g., [69]). Sharp global stability criteria in this particular case can be found in many papers [78].

For $0 \leq \gamma < 1$, and assuming that H is nonincreasing, it is clear that (2.13) has a unique positive equilibrium. For $\gamma = 1$, the existence of an equilibrium requires the extra assumptions $H(0) > 1$, $\lim_{x\to\infty} H(x) < 1$. If $\gamma > 1$, then equation (2.13) can have more than one positive equilibria; in the latter case, it is a suitable model for populations with Allee effects (see, e.g., [81]).

Since we are mostly interested in globally attracting equilibria, we avoid the possibility of having more than one equilibrium by focusing on the case $0 \leq \gamma \leq 1$. For these values of γ, the use of equation (2.13) in population dynamics is intended to gain flexibility to fit population data. For instance, when $H(x) = \beta e^{-\delta x}$, $\beta, \delta > 0$, the equation (2.13) provides the gamma-model included in the list of spawner-recruit models in the book by Quinn and Deriso [114][3]. For example, Zheng and Kruse [148] found that this model fits well the stock-recruitment data for three Alaskan crab stocks; for more references, see [81].

Moreover, the difference equation corresponding to (2.11), generated by the γ-logistic map (in analogy to the so called γ-Ricker map [81]) has been used by Avilés [6] in the framework of cooperative interaction in a group of individuals (for $1 \leq \gamma \leq 2$), and by Eskola and Parvinen [35] in the context of populations with Allee effects (for $\gamma = 2$).

Finally, we refer to the work [5], which deals in a general way with delay differential equations having destruction and production terms like in (2.7). Once more, we recall the reader the existence of [5, Table 1], where many references regarding applications of equation (2.5) to several fields are gathered.

Our main results in this chapter, which can be found in Section 2.2, are organised according to the application on Economics that we have recalled above. Nonetheless, they do not lose their meaning in population dynamics; it is just a matter of the name of parameters. In fact, bearing in mind the meaning of γ on both applications, we focus on the case $\gamma \in (0, 1)$, which allows us to work with a single positive equilibrium and links the limit cases $\gamma \in \{0, 1\}$. Those limit cases can also be studied under our approach, yet we warn the reader that some further special considerations should be given.

2.2 Global dynamics of several gamma-models

To start with, we provide a common framework that includes all the mentioned delay-differential gamma-models. Within the latter framework, we say that the delay differential equation

$$x'(t) = -ax(t) + s(x(t))x^{\gamma}(t - \tilde{\tau}(t))k(x(t - \tilde{\tau}(t))) \tag{2.14}$$

is an example of a *delay differential gamma-model*. In this chapter, we will assume the following set of hypotheses regarding equation (2.14):

(A) $a > 0$, $\gamma \in (0, 1)$, $\tilde{\tau} : \mathbb{R} \to [0, \infty)$ is a bounded continuous function and $\tau > 0$ is its supremum.

(B) $s, k : [0, \infty) \to (0, \infty)$ are nonincreasing functions of class $\mathcal{C}^1$.

[3] This is why we use the name gamma-model for, e.g., equation (2.13).

We could have defined s and k on $(0,\infty)$, since we are only interested in the dynamics of positive solutions, but having a certain regularity at $x = 0$ and thus the bounded character of such functions and their derivatives facilitates the study.

Moreover, equation (2.14) includes the most general model (2.7) that we have mentioned in Section 2.1 by considering k as the product of the constant B and the pollution function p_2. In fact, it is more general than (2.7) since we allow the delay to be variable.

In the following subsections, the reader can find different choices for the functions involved in equation (2.14). We split the section into the following two parts.

Firstly, in Subsection 2.2.1, we assume a constant saving rate term in (2.14). This equation belongs to the well-known family of differential equations with instantaneous linear decay and delayed feedback that has been introduced in Section 1.3, which were partially studied via the corresponding difference equation. Then, we consider different choices of pollution function, yielding equations that have already been considered in the literature for the particular cases $\gamma = 0$ and/or $\gamma = 1$. We provide sharp delay-independent and some delay-dependent conditions for the positive equilibrium to be GAS for each of those choices, which highlights the key role of the pollution function (or the per capita production function in the context of population dynamics) in the stability properties of neoclassical models with delay. Moreover, we analyse how the parameter γ influences both the value and the stability of the equilibrium, sometimes in a subtle way depending on the other parameters of the model, a fact that may lead to stability windows in the bifurcation diagram.

Secondly, in Subsection 2.2.2, we deal with equation (2.14) with variable saving rate term. A suitable framework to look for such case comes from the family of delay differential equations considered by Ivanov et al. in [64], namely,

$$x'(t) = -g_1(x(t))f_2(x(t-\tau)) + f_1(x(t-\tau))g_2(x(t)), \tag{2.15}$$

where f_1, f_2, g_1, g_2 are positive functions. We shall give further details about the analysis of equation (2.15). Nevertheless, we will also relate it with a particular difference equation, using analogous techniques to those in Section 1.3. In fact, we generalise a known global stability result for equation (2.15), allowing variable delays, and apply it to some examples of (2.14).

2.2.1 Constant saving rate

In this subcase, where $s(x) = \bar{s} > 0$, the main equation (2.14) reads as

$$x'(t) = -ax(t) + x^{\gamma}(t-\tilde{\tau}(t))\,h(x(t-\tilde{\tau}(t))), \tag{2.16}$$

with $h(x) := \bar{s}k(x)$. The equation (2.16) is actually included in the setting of equation (1.15) by simply assuming $f(x) = x^\gamma h(x)$. In fact, notice that assumption (B) directly holds for (2.14) since $h : [0, \infty) \to (0, \infty)$ is a nonincreasing function of class $\mathcal{C}^1$.

In order to study the global dynamics of (2.16) as in Section 1.3, we now relate the framework of (2.16) to the hypotheses in (T) (see page 27). Hence, we have to analyse the main features of $F := \frac{f}{a}$ or the difference equation corresponding to (2.16), namely

$$x_{n+1} = F(x_n) = x_n^\gamma H(x_n), \quad H := \frac{h}{a}, \tag{2.17}$$

which may be known as an example of a *discrete gamma-model*. Note that F (or f) has been split into a term x^γ and the remaining part H (respectively, h), which is a nonincreasing function. The shape of the nonlinearity H (or h) will lead to different behaviours.

To start with, we remark that the function $F : I \to I$, with the choice $I = (0, \infty)$, is well-defined and of class $\mathcal{C}^1$ on I. In fact, one has more information, since $F(x) \to 0$ and $F'(x) \to \infty$ as $x \to 0^+$; and $F(x) \to 0$ as $x \to \infty$. Moreover, $a, \tau > 0$ will be provided under condition (A).

The existence of a unique positive root p of $F(x) = x$ is shown in the following result. Besides, Theorem 2.1 also shows the role of the parameter γ on the value of p and reunites some common and underlying analytical features of those gamma-models shown in [17, 20, 81, 82, 83], which will be useful in the subsequent pages. Particularly, it especially resembles the spirit and the ideas of the proofs for the particular cases in [81, Theorem 2], [82, Theorem 5.1] together with the general flavour of [83, Theorem 1].

Theorem 2.1. *Let $a > 0$ and $h : [0, \infty) \to (0, \infty)$ be a nonincreasing function of class $\mathcal{C}^1$. Then, for each $\gamma \in (0, 1)$, there is a unique positive fixed point of the map F in* (2.17), *namely $p := p(\gamma) \in (0, \infty)$, which is the unique root of equation*

$$p^{1-\gamma} = H(p).$$

Moreover, we obtain

$$\Delta_1^F(x) = (F(x) - x)(x - p) < 0, \quad x \in (0, \infty) \setminus \{p\}.$$

Besides, one of the following cases holds:

- *$p(\gamma) < 1$ is decreasing on $(0, 1)$, or, equivalently, $H(1) < 1$;*
- *$p(\gamma) = 1$, for every $\gamma \in (0, 1)$, or, equivalently $H(1) = 1$;*
- *$p(\gamma) > 1$ is increasing on $(0, 1)$, or, equivalently, $H(1) > 1$.*

Finally, we have

$$F'(p) = \gamma + p^{\gamma} H'(p). \tag{2.18}$$

Proof. Let $\gamma \in (0,1)$. If p is a fixed point of F, then

$$p = F(p) = p^{\gamma} H(p) \tag{2.19}$$

which is equivalent to $p^{1-\gamma} = H(p)$, Since $1-\gamma > 0$, the function $p^{1-\gamma}$ is increasing, which combined with the fact of H being nonincreasing, leads to the existence of a unique positive root of (2.19).

From the fact that there is a unique positive fixed point p of $F : (0,\infty) \to (0,\infty)$, we deduce that $(F(x) - x)(x - p) \neq 0$ on $(0,\infty) \setminus \{p\}$. Nevertheless, one can obtain more information from

$$\limsup_{x\to\infty} \frac{F(x)}{x} = \limsup_{x\to\infty} x^{\gamma-1} H(x) \leq H(1) \limsup_{x\to\infty} x^{\gamma-1} = 0,$$
$$\liminf_{x\to 0^+} \frac{F(x)}{x} = \liminf_{x\to 0^+} x^{\gamma-1} H(x) \geq H(1) \liminf_{x\to 0^+} x^{\gamma-1} = \infty,$$

where we have used that H is nonincreasing. Clearly, this can only be possible if $\Delta_1^F < 0$ on $(0,\infty) \setminus \{p\}$.

Consider the auxiliary function $\zeta : (0,\infty) \times (0,1)$ of class $\mathcal{C}^1$ defined by

$$\zeta(p,\gamma) := p^{\gamma-1} - \frac{1}{H(p)}.$$

Clearly, every positive fixed point of F satisfies the equation $\zeta(p,\gamma) = 0$. Moreover, the inequality $H' \leq 0$ yields

$$\partial_1 \zeta(p,\gamma) = (\gamma - 1) p^{\gamma-2} + \frac{H'(p)}{H(p)^2} < 0, \quad \text{for every } (p,\gamma) \in (0,\infty) \times (0,1). \tag{2.20}$$

Therefore, we may apply the Implicit Function Theorem to ζ and derive that there exists a function $p(\gamma)$ of class $\mathcal{C}^1$ on $(0,1)$, which satisfies $\zeta(p(\gamma),\gamma) = 0$ and

$$p'(\gamma) = -\frac{\partial_2 \zeta(p(\gamma),\gamma)}{\partial_1 \zeta(p(\gamma),\gamma)} = \frac{p^{\gamma-1}\ln(p)}{-\partial_1 \zeta(p,\gamma)},$$

where we have finally written $p = p(\gamma)$ for simplicity. Since the denominator in the last expression is positive from (2.20), so as $p^{\gamma-1}$, then the sign of $p'(\gamma)$ coincides with the sign of $\ln(p)$, and thus, on the relative position of p with respect to the value 1.

Besides, $p(\gamma) = 1$ for every $\gamma \in (0,1)$ is equivalent to $H(1) = 1$ via (2.19). Otherwise, if $p(\gamma)$ is not the constant function 1, we can finish this part of the proof by using the condition

$$(F(x) - x)(x - p(\gamma)) < 0, \quad x \in (0,\infty) \setminus \{p(\gamma)\}$$

evaluated at $x = 1$, that is,

$$0 > (F(1) - 1)(1 - p(\gamma)) = (H(1) - 1)(1 - p(\gamma)).$$

In other words, the relative position of p with respect to 1 is known from the relative position of $H(1)$ with respect to 1.

Finally,

$$F'(p) = \gamma p^{\gamma-1} H(p) + p^{\gamma} H'(p) = \gamma + p^{\gamma} H'(p),$$

where expression (2.19) has been used once more. □

Remark 2.2. Notice that, regarding equations (2.16) and (2.17), we consider $F = f/a$. Thus, $f'(p)$ may be computed as

$$f'(p) = a\gamma + \gamma + h'(p).$$

Additionally, it is important to highlight that (2.18) implies

$$F'(p) < 1, \quad f'(p) < a,$$

which will be relevant from the results given in Subsection 1.3.3 (see, in particular, Remark 1.64 and Theorem 1.69).

Regarding (T4) (see page 27), there exists a non-degenerate compact set $K \subset I = (0,\infty)$, which is a globally attracting set for the difference equation (2.17). This can be shown via standard arguments, supported by a cobweb analysis. In fact, let $q = \sup_{x\in(0,p)} F(x)$. If $q = p$, then we can choose $K = \{p\}$. If $q > p$, then our choice could be $K = [w, q]$, where $w = \min_{x\in[p,q]} F(x) > 0$.

All the hypotheses in (T) have been translated into the framework of gamma-models. Thus, we actually know what do we need to impose to equation (2.16) in order to apply the general results of existence and uniqueness of solutions and their continuous dependence on initial data from Section 1.1. Following [86, Theorem 4.1] and allowing an increasing function f too, we adapt the next result from [84, Theorem 3.1].

Theorem 2.3. *Assume that (A) holds and $h : [0,\infty) \to (0,\infty)$ is a nonincreasing function of class $\mathcal{C}^1$. If $\phi \in C_{(0,\infty)}$, then there is a unique solution $x(t;(0,\phi))$ of (2.16) through $(0,\phi)$, which is defined on $[-\tau,\infty)$, positive and*

$$\left[\liminf_{t\to\infty} x(t;(0,\phi)), \limsup_{t\to\infty} x(t;(0,\phi))\right] \subset K, \tag{2.21}$$

for a certain non-degenerate compact set $K \subset (0, \infty)$ that can be chosen independently of ϕ.

Proof. Global existence and uniqueness of solutions come from the results in Chapter 1. The asymptotic behaviour displayed by (2.21) is obtained via [86, Theorem 4.1] and the comments before this result regarding the non-degenerate compact set K, which is globally attracting for the corresponding difference equation (2.17). The case of increasing $f(x) = x^\gamma h(x)$, can be handled in a similar way. □

Remark 2.4. We have restricted our attention to $\gamma \in (0, 1)$, yet many of the results are also directly adaptable to the limit case $\gamma = 0$; in particular, it is the case of Theorem 2.1. The another limit case, $\gamma = 1$, would require the additional assumption

$$\lim_{x\to\infty} h(x) < a < h(0)$$

to ensure that there is a unique equilibrium for (2.23).

As a last general comment, we remark that we will follow a common path for the analysis of the given models, which would include the particular expression of the functions f, F, h, H, stability properties about the unique equilibrium and the role of varying γ with respect those features. In particular, we provide a summary in Table 2.1 at the end of the section to facilitate the comparison between them.

No pollution effects

The first very particular case is the one of no-pollution effects. For instance, if one recalls what was explained in the introduction, we fall into the case $p_2(x) = 1$ and, by virtue of the comments below (2.14), the function k appearing in such general delay-differential gamma-model is constant. Thus, if we rename the product of constants, we can work with (2.16), where the function $h : [0, \infty) \to (0, \infty)$ is defined by $h(x) = \beta > 0$, that is, we deal with equation

$$x'(t) = -ax(t) + \beta x^\gamma(t - \tilde{\tau}(t)). \tag{2.22}$$

In this context, the difference equation corresponding to (2.22) is

$$x_{n+1} = \frac{\beta}{a} x_n^\gamma = x_n^\gamma H(x_n) = F(x_n), \tag{2.23}$$

where $H(x) = \beta/a$, a constant function, so, by applying Theorem 2.1, the positive equilibrium is the unique root of $p^{1-\gamma} = H(p)$ or, in other words,

$$p = p(\gamma) = \left(\frac{\beta}{a}\right)^{\frac{1}{1-\gamma}}. \tag{2.24}$$

Moreover, according to Theorem 2.1, we can view p as a function of γ in (2.24) and analyse the role of γ in the value of the equilibrium.

Corollary 2.5. *Let $a, \beta > 0$. If we denote by $p(\gamma)$ the unique positive equilibrium of* (2.23)*, given by* (2.24) *for $\gamma \in (0,1)$, it satisfies the following properties:*

- *If $\frac{\beta}{a} > 1$, then $p(\gamma) > 1$ for every $\gamma \in (0,1)$ and it is increasing.*
- *If $\frac{\beta}{a} = 1$, then $p(\gamma) = 1$ for every $\gamma \in (0,1)$.*
- *If $\frac{\beta}{a} < 1$, then $p(\gamma) < 1$ for every $\gamma \in (0,1)$ and it is decreasing.*

Under global stability conditions, the dependence of the unique equilibrium p on γ can also be seen as the role of γ in the so-called 'population abundance' [81]. Moreover, the global attractivity of the unique equilibrium of (2.23) is easy to obtain.

Theorem 2.6. *Let $a, \beta > 0$, $\gamma \in (0,1)$. Then the equilibrium p given by* (2.24) *is GAS for the difference equation* (2.23)*.*

Proof. Clearly, the function $F : (0,\infty) \to (0,\infty)$ in (2.23) is increasing. By the first part of Theorem 1.55, we conclude that p is a global attractor for (2.23) for the set $X = I = (0,\infty)$. □

The asymptotic behaviour of the solutions of the DDE (2.22) is also quite simple.

Theorem 2.7. *Assume (A) and $\beta > 0$. Denote by $x(t;(0,\phi))$ the unique solution of* (2.22) *through $(0,\phi)$, with $\phi \in C_{(0,\infty)}$ and by p the unique equilibrium of* (2.23) *given by* (2.24)*. Then, $\lim_{t\to\infty} x(t;(0,\phi)) = p$ for any $\phi \in C_{(0,\infty)}$.*

Proof. It is a straightforward conclusion from Theorem 2.6 and the first part of Theorem 1.69. □

Remark 2.8. While the limit case $\gamma = 0$ of (2.22) is a simple ODE that has a unique equilibrium which is GAS, the one of $\gamma = 1$ cannot be considered under our framework (notice that k would be constant; compare this situation with the contents of Remark 2.4).

Pollution effects

Unlike the previous subcase, where we considered (2.16) with $k(x)$ being constant (so as the saving rate), we now consider a pure pollution effect, that is, a decreasing function k. Therefore, $h = \bar{s}k$ in (2.16) and $H = h/a$ in (2.23) will also be decreasing.

The choice of the pollution function may lead to stability switches, as we will see in the first two choices, or may not, as we consider in the third example. Thus, it might be a critical choice.

One relevant difference with respect to the no-pollution subcase is that the equilibrium may now lie on an interval of decrease of F, which profoundly affects not only the dynamics of the difference equation (2.17), but also the one of the corresponding DDE (2.16), which is the equation we are eventually interested in.

As we have seen at the final part of Subsection 1.3.3, when it comes to stability-related results, we need to work with lower bounds for $f'(p)$ or $F'(p)$ by using the functions ξ_i, $i \in \{1, 2, 3\}$. Since f and F functions will now have particular expressions, we can wonder whether it is possible to find equivalent conditions in terms of their inner parameters to the global asymptotic stability of its unique equilibrium.

Nevertheless, one great issue is that the expression of $F'(p)$ includes the value p (see equation (2.18)), so, at a first glance, it seems that the computation of p is unavoidable. However, assume that we would like to know when does the inequality $\kappa \leq F'(p)$ hold, for a certain real number $\kappa < 1$. Then, from (2.18), such condition is equivalent to

$$W(p) := -p^{\gamma} H'(p) \leq \gamma - \kappa.$$

Hence, whenever W has an inverse (which would be strictly monotone), the condition $\kappa \leq F'(p)$ will be equivalent to a certain inequality concerning p and $\eta := W^{-1}(\gamma - \kappa)$. Then, the condition $(F(x) - x)(x - p) < 0$, $x \neq p$, can be applied to η to yield an equivalent inequality between η and $F(\eta)$ which avoids the computation of the equilibrium p. Instead of fully formalising this fact here in an abstract way, we introduce the reader to the particular cases that we treat, in which we handle an explicit expression of the auxiliary functions.

Lasota equation

Firstly, assume that the pollution function is selected as $p_2(x) = e^{-\delta x}$, with $\delta > 0$. Then, by gathering the constant factors in the feedback under the notation $\beta > 0$, equation (2.16) takes the form

$$x'(t) = -ax(t) + \beta x^{\gamma}(t - \tilde{\tau}(t)) e^{-\delta x(t - \tilde{\tau}(t))}, \tag{2.25}$$

which will be referred to as the *Lasota delay differential equation* or, simply, the *Lasota equation*. Notice that equation (2.25) may also be considered as a *gamma-version of the Nicholson's blowflies equation*. In this case, the function $h : [0, \infty) \to (0, \infty)$ appearing in (2.16) is defined by $h(x) = \beta e^{-\delta x}$ and, thus, it is a decreasing function of class $\mathcal{C}^1$. The corresponding difference equation of (2.25) is

$$x_{n+1} = \frac{\beta}{a} x_n^{\gamma} e^{-\delta x_n}, \tag{2.26}$$

which will be called the *gamma-Ricker difference equation*. It is generated by the *gamma-Ricker map*

$$F(x) = \frac{\beta}{a} x^{\gamma} e^{-\delta x}. \tag{2.27}$$

We list some properties of the γ-Ricker map in the following result. It is based on Propositions 1 and 2, and Theorem 1 in [81]. As usual, we consider the notation $H = h/a$.

Proposition 2.9. *The map $F : (0, \infty) \to (0, \infty)$ defined by*

$$F(x) = \frac{\beta}{a} x^{\gamma} e^{-\delta x}; \quad a, \beta, \delta > 0; \ \gamma \in (0, 1);$$

satisfies the following properties:

(i) *F is of class $\mathcal{C}^{\infty}$ and*

$$\lim_{x \to 0^+} F(x) = 0 = \lim_{x \to \infty} F(x), \quad \lim_{x \to 0^+} F'(x) = \infty.$$

(ii) *F is a U-map, with a unique critical point at $c = \gamma/\delta$, where F attains its global maximum.*

(iii) *There is a unique $p = p(\gamma) \in (0, \infty)$ such that $F(p) = p$, which is given by the unique root of*

$$p^{1-\gamma} e^{\delta p} = \frac{\beta}{a}.$$

Besides, $(F(x) - x)(x - p) < 0$ for every $x \in (0, \infty) \setminus \{p\}$.

(iv) *$SF(x) < 0$, for all $x > c$.*

(v) *$F'(p) = \gamma - \delta p$.*

(vi) *For any $\kappa < 1$, the inequalities $\kappa \le F'(p) < 1$ hold if and only if the following inequality is satisfied:*

$$\frac{\beta}{a} \le e^{\gamma - \kappa} \left(\frac{\gamma - \kappa}{\delta} \right)^{1-\gamma} =: T_{\kappa,\delta}(\gamma). \tag{2.28}$$

Proof. Assertions (i)-(v) are mainly proven in [81], while some of them can also be seen as direct conclusions of the general Theorem 2.1, e.g., Assertion (iii). We also recall the proof of (v) to show how our notation works. In fact, notice that, by virtue of (2.18), we obtain

$$F'(p) = \gamma + p^{\gamma} H'(p) = \gamma - \delta p^{\gamma} \frac{\beta}{a} e^{-\delta p} = \gamma - \delta p^{\gamma} p^{1-\gamma} = \gamma - \delta p,$$

which is always less than 1. Moreover, regarding the proof of (vi), if $\kappa < 1$, then,

$$\kappa \leq F'(p) \iff \delta p \leq \gamma - \kappa \iff p \leq \frac{\gamma - \kappa}{\delta} \iff F\left(\frac{\gamma-\kappa}{\delta}\right) \leq \frac{\gamma-\kappa}{\delta}$$
$$\iff \frac{\beta}{a}\left(\frac{\gamma-\kappa}{\delta}\right)^{\gamma} e^{-\delta\left(\frac{\gamma-\kappa}{\delta}\right)} \leq \frac{\gamma-\kappa}{\delta} \iff \frac{\beta}{a} \leq e^{\gamma-\kappa}\left(\frac{\gamma-\kappa}{\delta}\right)^{1-\gamma},$$

where we have used the condition $(F(x)-x)(x-p) < 0$ on $(0,\infty)\setminus\{p\}$ and $F(p) = p$ in the last equivalence in the first line. This obviously gives us condition (2.28). □

In Assertion (vi) in the last result, we obtained that $F'(p)$ being bounded from below by a certain constant is equivalent to a particular condition that is independent from p, as we announced before starting the current part devoted to Lasota equation. Bear in mind that, in this particular case, the computation of W^{-1} has been easy, since $W(p) = \delta p$, while $\eta = W^{-1}(\gamma - \kappa) = \frac{\gamma-\kappa}{\delta}$.

By an application of Theorem 2.1 to the function F in (2.27), the role of γ in the value of the equilibrium for (2.26) or (2.25) is as follows.

Corollary 2.10. *[81, Theorem 2] Let $a, \beta, \delta > 0$. If we denote by $p(\gamma)$ the unique positive fixed point of the γ-Ricker map* (2.27) *with $\gamma \in (0,1)$, then $p(\gamma)$ satisfies the following:*

- *If $\frac{\beta}{a} > e^{\delta}$, then $p(\gamma) \in (1,\infty)$ is increasing.*
- *If $\frac{\beta}{a} = e^{\delta}$, then $p(\gamma) = 1$, for every $\gamma \in (0,1)$.*
- *If $\frac{\beta}{a} < e^{\delta}$, then $p(\gamma) \in (0,1)$ is decreasing.*

According to the properties of the γ-Ricker map shown in Proposition 2.9, including that F is a S_*-map and that $(F(x)-x)(x-p) < 0$ on $(0,\infty)\setminus\{p\}$, it is possible to apply Theorem 1.55 to the γ-Ricker difference equation.

Theorem 2.11. *[81, Theorem 1(A)] Let $a, \beta, \delta > 0$, $\gamma \in (0,1)$ and denote by p the unique positive fixed point of the γ-Ricker map. Then p is GAS for the γ-Ricker difference equation* (2.26) *if and only if*

$$\frac{\beta}{a} \leq e^{\gamma+1}\left(\frac{\gamma+1}{\delta}\right)^{1-\gamma}. \tag{2.29}$$

After having recalled the main features of the function F and the role of the value of γ on them, we are in a position to analyse the corresponding DDE, namely the Lasota equation in (2.25). The following result provides sufficient conditions for the positive equilibrium p of (2.25) to be GAS.

Theorem 2.12. *Assume that (A) holds together with $\beta, \delta > 0$. Denote by $x(t;(0,\phi))$ the unique solution of* (2.25) *through $(0,\phi)$, with $\phi \in C_{(0,\infty)}$, and by p the unique positive fixed point of the γ-Ricker map* (2.27)*. If*

$$\frac{\beta}{a} \leq e^{\gamma+1} \left(\frac{\gamma+1}{\delta} \right)^{1-\gamma}, \tag{2.30}$$

then $\lim_{t\to\infty} x(t;(0,\phi)) = p$, for every $\phi \in C_{(0,\infty)}$. Moreover, condition (2.30) *is the sharpest absolute stability condition.*

If we further assume that $\tilde{\tau}(t) = \tau > 0$, for all $t \in \mathbb{R}$, condition

$$\frac{\beta}{a} \leq e^{\gamma+\frac{1}{1-e^{-\alpha\tau}}} \left(\frac{\gamma+\frac{1}{1-e^{-\alpha\tau}}}{\delta} \right)^{1-\gamma} \tag{2.31}$$

is also sufficient to ensure that p is GAS for the autonomous DDE (2.25)*.*

Proof. The first part follows from the first statement in Theorem 1.69, whose proof can also be checked by the reader in the forthcoming Theorem 2.32.

Now, assume that $\tilde{\tau}$ takes the constant value $\tau > 0$. If we also fix $a > 0$, then we can apply Corollary 1.61 (see also the summary in Remark 1.64). In particular, if $\tau \leq \rho_1(a, f'(p))$ or its equivalent form $f'(p) \geq \xi_1(a,\tau)$ hold, then the first part of this result will be proven. Such condition is also equivalent to

$$F'(p) = \frac{f'(p)}{a} \geq \frac{\xi_1(a,\tau)}{a} = \frac{-1}{1-e^{-a\tau}} =: \kappa_1.$$

By using Assertion (vi) of Proposition 2.9, we conclude that $\kappa_1 \leq F'(p)$ if and only if condition (2.31) holds. □

Some remarks are given below, providing some explanations that can be analogously adapted to other subsequent models.

Remark 2.13. It is clear from the previous proof that (2.30) implies (2.31). In fact, by using that the value of $T_{\kappa,\delta}(\gamma)$ is decreasing in $\kappa \in (-\infty, 0)$ for any fixed $\delta > 0$ and $\gamma \in (0,1)$, we have that

$$e^{\gamma+1} \left(\frac{\gamma+1}{\delta} \right)^{1-\gamma} < e^{\gamma+\frac{1}{1-e^{-\alpha\tau}}} \left(\frac{\gamma+\frac{1}{1-e^{-\alpha\tau}}}{\delta} \right)^{1-\gamma}.$$

A geometrical way to see that (2.30) constitutes the sharpest absolute stability condition is considering Remark 1.64 and Figure 1.8. For instance, the left picture in the latter figure shows that $F'(p) \in [-1, 1)$ (which in this case is equivalent to condition (2.30)) represents a sector in such graph for which the equilibrium p is globally attracting for every positive value of the delay τ. It is also clear that there is no larger sector for which such a fact holds (see also Remark 1.67).

Furthermore, the delay-dependent global stability condition (2.31) proves that, for a constant delay that is sufficiently small, the positive equilibrium p of (2.25) is GAS, since $(1 - e^{-a\tau})^{-1} \to \infty$ as $\tau \to 0^+$. Compare this fact with the pictures in Figure 1.8 too.

Remark 2.14. The limit form of (2.30) as $\gamma \to 1^-$ provides the well-known absolute global stability condition $\beta \leq ae^2$ for the Nicholson's blowflies equation (see, e.g., [54]). Moreover, the limit of (2.30) as $\gamma \to 0^+$ is coherent with the global stability condition $\beta\delta < ae$ for the Ważewska-Czyżewska and Lasota equation (see, e.g., [50]).

Remark 2.15. In the case of constant delay, that is, if $\tilde{\tau}(t) = \tau > 0$, for all $t \in \mathbb{R}$, one can go even further, at least, in terms of ensuring that p is LAS. In fact, this would happen provided

$$\kappa_3 := -\sqrt{1 + \frac{s^2}{a^2\tau^2}} = \frac{\xi_3(a, \tau)}{a} < \frac{f'(p)}{a} = F'(p)$$

or, equivalently,

$$\frac{\beta}{a} < e^{\gamma + \sqrt{1 + \frac{s^2}{a^2\tau^2}}} \left(\frac{\gamma + \sqrt{1 + \frac{s^2}{a^2\tau^2}}}{\delta} \right)^{1-\gamma}, \tag{2.32}$$

where s is the unique root of $\frac{-x}{\tan(x)} = a\tau$ in $(\frac{\pi}{2}, \pi)$. Indeed, if (2.30) does not hold, then there is $\bar{\tau} > 0$ such that p is LAS for $0 < \tau < \bar{\tau}$ and unstable if $\tau > \bar{\tau}$. Taking into account Assertion (v) in Proposition 2.9 and the equivalent condition regarding ρ_3 in Section 1.3, the value of $\bar{\tau}$ can be calculated as

$$\bar{\tau} = \rho_3(a, f'(p)) = \frac{\arccos\left(\frac{1}{\gamma - \delta p}\right)}{a\sqrt{-1 + (\gamma - \delta p)^2}}. \tag{2.33}$$

From Remark 2.15, it is clear that, for fixed values of the parameters $\alpha, \beta, \gamma, \delta$ for which (2.30) does not hold, an increasing value of τ destabilises the positive equilibrium of (2.25); this property has already been observed in [103]. It is interesting to study the role of the other parameters on the stability properties of (2.25). For example, inequalities (2.30) and (2.31) suggest that the equilibrium is also destabilised by increasing either β or δ, keeping constant the value of the other involved parameters. In [103], δ is assumed to reflect the strength of the negative effect caused by increasing concentration of capital.

The role of parameter γ is subtler. In fact, the role of γ partially depends on the ordering of the elements in $\{\beta/a, e^2, e/\delta\}$. From (2.30), it is easy to prove that p is absolutely stable for all $\gamma \in (0,1)$ if the ratio β/a is small enough; specifically, a sufficient condition is $\beta/a \leq \min\{e/\delta, e^2\}$. However, for other values of $\beta, \gamma, \delta, \tau$, increasing γ can be stabilizing, destabilising or can even give rise to a pair of stability switches (for further details, see, e.g., [81, Proposition 3]).

Now, we provide several stability diagrams for different particular cases of this model. We assume that $a = 1$ and use the corresponding slice of the region $G \subset \mathbb{R}^3$, which was displayed in the right picture of Figure 1.8. The information for other values of a is encoded in this case; the reader is derived to the comments regarding the normalisation of this parameter in Subsection 1.3.1 to see how to obtain analogous diagrams for $a \neq 1$. Bearing in mind that $f'(p) < a$ is always fulfilled due to Remark 2.2, no region of parameters is classified as 'Not in our framework', as it actually happened in Figure 1.8.

Notice that, in the case of the Lasota equation (2.25) with $a = 1$, Proposition 2.9 tells us that the variable $f'(p)$ is rewritten as $\gamma - \delta p$; moreover, since p is the root of $p^{1-\gamma}e^{\delta p} = \beta$, it is also clear that, if we fix the value of the inner parameters β and δ, the value of $f'(p)$ is completely determined by the one of γ. Hence, we can depict diagrams of stability relating the parameter of our interest γ with respect to the delay τ, which are the ones in Figure 2.1. The meaning of each picture is as follows.

- Figure 2.1 (a) corresponds to $\beta = 2$, $\delta = 2$, for which $e^2 > \beta/a > e/\delta$. The positive equilibrium p of (2.25) is globally asymptotically stable for $\gamma \geq \gamma_*$, with $\gamma_* \approx 0.156$, and it is asymptotically stable for $\tau < \overline{\tau}$, with $\overline{\tau} \approx 3.827$. For larger values of τ, increasing γ is stabilizing. In particular, one can ensure that p is GAS provided $\tau < \tau^* \approx 1.783$.

- Figure 2.1 (b) corresponds to $\beta = 7.7$, $\delta = 0.5$, for which $\beta/a > \max\{e/\delta, e^2\}$. The positive equilibrium of (2.25) is globally asymptotically stable for $\gamma \in [\gamma_*, \gamma^*]$, with $\gamma_* \approx 0.469$ and $\gamma^* \approx 0.871$, and it is asymptotically stable for $\tau < \overline{\tau}$ with $\overline{\tau} \approx 4.102$. For values of $\tau > \hat{\tau}$, with $\hat{\tau} \approx 9.858$, increasing γ leads to a pair of stability switches. Moreover, one can affirm that p is GAS if $\tau < \tau^*$, with $\tau^* \approx 1.874$.

- Figure 2.1 (c) corresponds to $\beta = 10$, $\delta = 0.13$, for which $e/\delta > \beta/a > e^2$. The positive equilibrium of (2.25) is globally asymptotically stable for $\gamma \leq \gamma^*$, with $\gamma^* \approx 0.815$, and it is asymptotically stable for $\tau < \overline{\tau}$, with $\overline{\tau} \approx 2.93$. For larger values of τ, increasing γ is destabilising. Furthermore, if $\tau \leq \tau^*$, with $\tau^* \approx 1.46$, then p is GAS.

In Figure 2.2, we plot some numerical simulations for the solutions of (2.25) in the case (b) ($\alpha = 1$, $\beta = 7.7$, $\delta = 0.5$), with $\tau = 15$ and different values of γ, illustrating the stability switches.

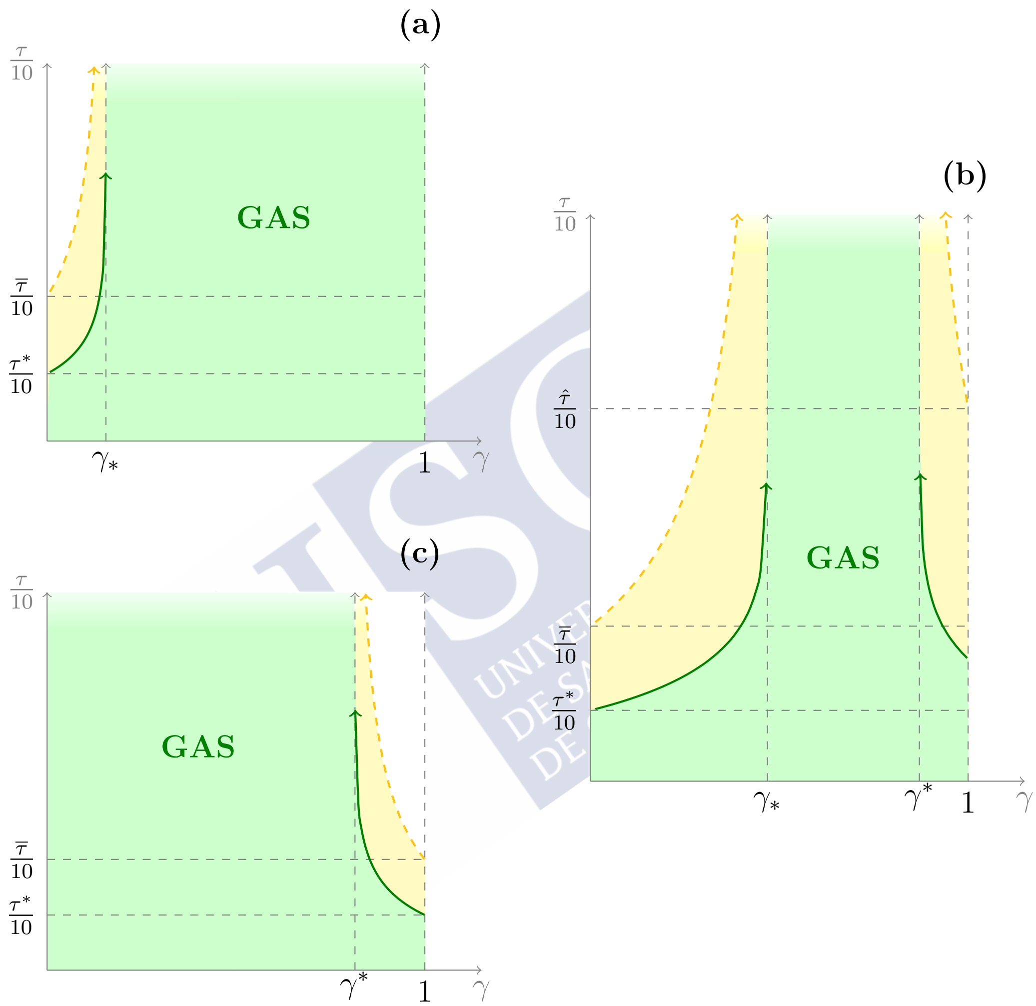

Figure 2.1: Stability diagrams for equation (2.25) with constant delay in the plane $(\gamma, \tau/10)$, with $a = 1$ and different values of β and δ. The green solid lines represent the boundaries of the regions where global asymptotic stability is guaranteed. The orange dashed curves correspond to the boundaries of local asymptotic stability. Dashed gray lines correspond to threshold values of γ and τ (see the text).

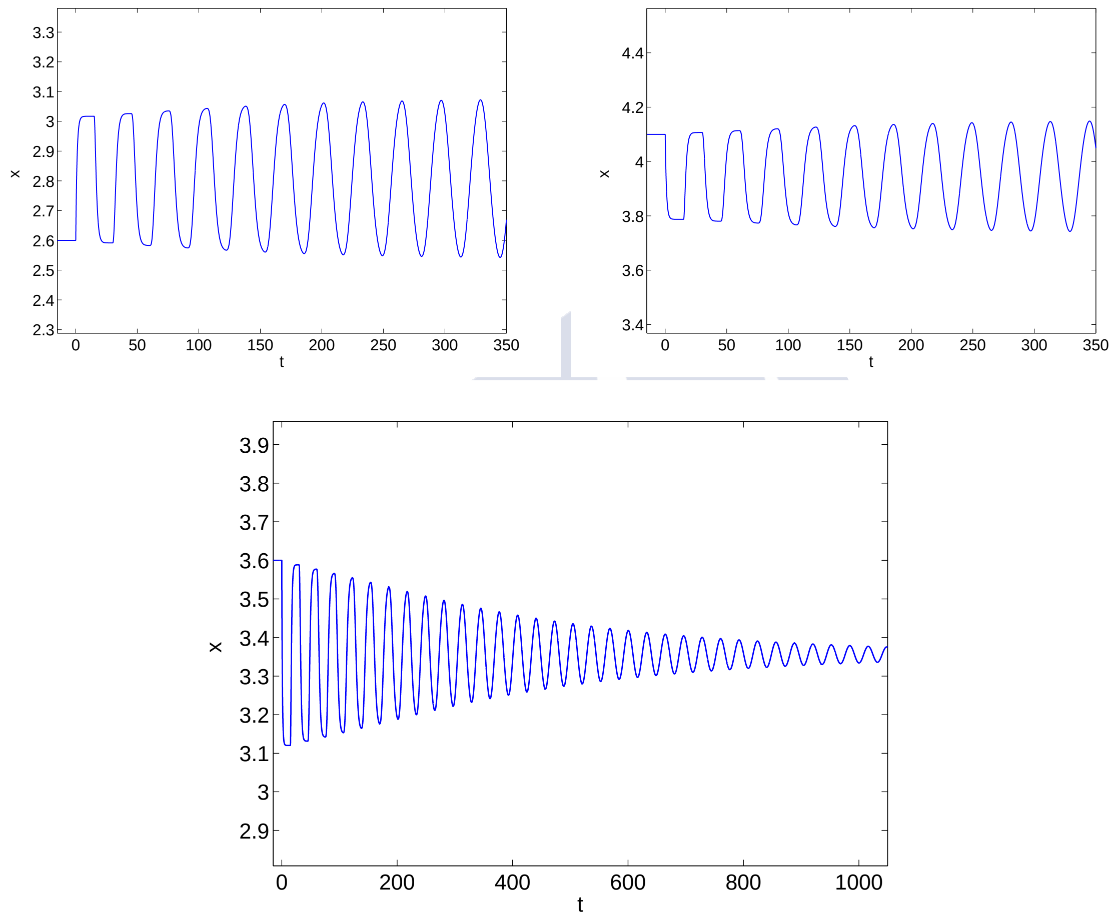

Figure 2.2: Numerical simulations for the solutions of equation (2.25), with $\alpha = 1$, $\beta = 7.7$, $\delta = 0.5$, $\tau = 15$, and different values of γ. For $\gamma = 0.38$ (above, left) and $\gamma = 0.95$ (above, right), the equilibrium p is unstable, and there are sustained oscillations; for $\gamma = 0.7$ (below), p is globally asymptotically stable. These pictures have been obtained via the use of MATLAB and, in particular, its solver *dde23*.

Gamma-logistic DDE

Secondly, we would like to study the following equation

$$x'(t) = -ax(t) + \beta x^{\gamma}(t - \tilde{\tau}(t))(1 - x(t - \tilde{\tau}(t))), \tag{2.34}$$

which would somehow be a gamma-version of the logistic DDE with a linear destruction term. Nevertheless, considering (2.34) would force us to take the pollution function as $p_2(x) = 1 - x$, which is decreasing but takes negative values for $x > 1$. Thus, the function h in (2.16) will no longer be positive (recall the desirable property (B)). The equation (2.34) will be known as the *gamma-logistic delay differential equation*

In fact, if we do not impose an additional constraint on (2.34) solutions might take unrealistic negative values, at least when it comes to the size of a population. For instance, consider the particular case of constant delay $\tilde{\tau} \equiv 1$ and an initial condition $\phi \in C_{(0,\infty)}$ such that $\phi(-1) > 1$ and $\phi(0) = 0$. Then, the solution of (2.34) through ϕ takes negative values on $(0, \varepsilon)$, for a sufficiently small $\varepsilon > 0$.

Hence, we shall fix the above-mentioned problem in order to work with solutions taking values in the interval $I = (0, \infty)$. The key idea is to work with a *modified gamma-logistic DDE*. For example, it could be in the sense of [102], where

$$x'(t) = -ax(t) + \beta x^{\gamma}(t - \tilde{\tau}(t)) \max\{(1 - x(t - \tilde{\tau}(t))), 0\}, \tag{2.35}$$

an equation that truncates the growth of solutions of (2.34) that take a value $x(t) > 1$. Nevertheless, such choice will not properly fit under our framework (once more, recall (B)) since h in (2.35) is neither positive nor differentiable. However, we can consider (2.16) with a general function $h : [0, \infty) \to (0, \infty)$ of class $\mathcal{C}^1$ and a sufficiently small $\varepsilon_* > 0$ such that

$$\begin{cases} h(x) = \beta(1 - x), & x \in [0, 1 - \varepsilon_*], \\ h(x) \text{ is decreasing}, & x \in [1 - \varepsilon_*, \infty). \end{cases} \tag{2.36}$$

A function h with the latter properties is clearly decreasing and the feedback $f(x) = x^{\gamma} h(x)$ coincides with the one in (2.34) on $(0, 1 - \varepsilon_*)$.

The reader may have already guessed our aim: to restrict ourselves to the open set $(0, 1)$ and analyse the corresponding difference equation

$$x_{n+1} = F(x_n) = \frac{\beta}{a} x_n^{\gamma}(1 - x_n), \tag{2.37}$$

whenever $(0, 1)$ is a forward invariant interval. It should not be a surprise that we will refer to equation (2.37) as *gamma-logistic difference equation*, which is generated by the map

$$F(x) = \frac{\beta}{a} x^{\gamma}(1 - x), \tag{2.38}$$

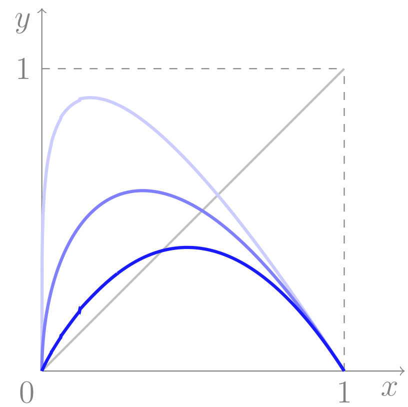

Figure 2.3: The γ-logistic map for $\frac{\beta}{a} = 1.55$ and $\gamma = 0.2, 0.5, 0.925$ with increasing darkness in γ.

analogously called the *gamma-logistic map*.

To ensure that F maps $(0,1)$ into $(0,1)$, and hence to deal with a well-defined difference equation in (2.37), we need to impose a consistency condition. Once we have ensured the mentioned property, then, no matter how we extend h in (2.36), the globally attracting compact set $K \subset (0,\infty)$ for (2.37) would in fact be a subset of $(0,1)$ (recall the introductory part of Subsection 2.2.1). Therefore, all solutions of the modified γ-logistic DDE in the sense of (2.36) through an initial condition $(0,\phi)$, $\phi \in C_{(0,\infty)}$ will eventually enter and remain in $(0,1)$.

Unlike the Lasota equation (2.25), for which we had the work by Liz [81] to support us with many analytical features of its corresponding difference equation (2.26), we now need to study the properties of the γ-logistic map (2.38).

In Proposition 2.16, we show some basic features of the function F such as the existence of a unique fixed point. In Theorem 2.22, a global stability condition for equation (2.34) is provided. Finally, the influence of the parameter γ on the stability and the value of the equilibrium is respectively analysed in Proposition 2.20 and Theorem 2.18. Once more, whenever it is needed, $H = h/a$.

Proposition 2.16. *The map* $F : (0,1) \to (0,\infty)$ *defined by*

$$F(x) = \frac{\beta}{a} x^{\gamma}(1-x); \quad a, \beta > 0;\ \gamma \in (0,1);$$

satisfies the following properties:

(i) F *is of class* $\mathcal{C}^{\infty}$ *and*

$$\lim_{x\to 0^+} F(x) = 0 = \lim_{x\to 1^-} F(x), \quad \lim_{x\to 0^+} F'(x) = \infty.$$

(ii) *F is a concave U-map, with a unique critical point at $c := \frac{\gamma}{\gamma+1}$, where F attains its global maximum.*

(iii) *Condition*

$$\frac{\beta}{a} < (\gamma+1)\left(\frac{\gamma+1}{\gamma}\right)^{\gamma} =: \Gamma(\gamma), \tag{2.39}$$

implies $\operatorname{Im} F \subset (0,1)$.

(iv) *There is a unique $p = p(\gamma) \in (0,1)$ such that $F(p) = p$, which is given by the unique root of*

$$\frac{p^{1-\gamma}}{1-p} = \frac{\beta}{a}.$$

Additionally, $(F(x)-x)(x-p) < 0$ for every $x \in (0,1)\setminus\{p\}$.

(v) *$SF(x) < 0$, for all $x \in (0,1)$, $x > c$.*

(vi) *$F'(p) = \gamma - \frac{\beta}{a}p^{\gamma}$.*

(vii) *For any $\kappa < 1$, the inequalities $\kappa \leq F'(p) < 1$ hold if and only if the following inequality is satisfied*

$$\frac{\beta}{a} \leq (\gamma-\kappa)\left(\frac{\gamma+1-\kappa}{\gamma-\kappa}\right)^{\gamma} =: T_{\kappa}(\gamma). \tag{2.40}$$

Proof. Assertion (i) is trivial from the definition of F (2.38).

The first, second and third derivatives of F are, respectively,

$$\begin{aligned} F'(x) &= \frac{\beta}{a}x^{\gamma-1}[\gamma-(\gamma+1)x], \\ F''(x) &= \frac{\beta}{a}\gamma x^{\gamma-2}[\gamma-1-(\gamma+1)x], \\ F'''(x) &= \frac{\beta}{a}\gamma(\gamma-1)x^{\gamma-3}[\gamma-2-(\gamma+1)x]. \end{aligned}$$

It is easy to check that F'' is negative using that $0 < \gamma < 1$. It is also trivial that $F'(c) = 0$ is equivalent to $c = \frac{\gamma}{\gamma+1}$. By using that $F'(x) > 0$ for $x < c$ and $F'(x) < 0$ for $x > c$, one can affirm that c is a global maximum. This proves Assertion (ii).

As c is a global maximum and F is positive, $F((0,1)) \subset (0,1)$ is equivalent to

$$1 > F(c) = \frac{\beta}{a}\left(\frac{\gamma}{\gamma+1}\right)^{\gamma}\frac{1}{\gamma+1},$$

which proves Assertion (iii).

Let $z(x) := F(x) - x$. Then, $z(0) = 0$, $z'(0^+) = \infty$ and $z(1) = -1$, so there exists at least one fixed point of F in $(0,1)$. Take p as the least positive fixed point of F, which exists because $z'(0^+) = \infty$, which also implies that $F(x) - x > 0$, for $0 < x < p$. Moreover $z'(p) \leq 0$ or, equivalently, $F'(p) \leq 1$. As $F''(x) < 0$, p is the unique fixed point of F in $(0,1)$ and the proof of Assertion (iv) finishes.

By doing some calculations, one can reach

$$SF(x) = \frac{\gamma(\gamma+1)q(x)}{2x^2[\gamma - (\gamma+1)x]^2}, \quad x \in (0,1),\ x \neq c,$$

where q is a polynomial of degree two defined by

$$q(x) := -(\gamma+1)(\gamma+2)x^2 + 2(\gamma+2)(\gamma-1)x - \gamma(\gamma-1).$$

Since $q''(x) = -2(\gamma+1)(\gamma+2) < 0$, for all $x \in (0,1)$, $q(0^+) > 0$ and $q'(0^+) < 0$, then there is at most one root of q in $(0,1)$. By checking $q(c) = -\frac{3\gamma}{\gamma+1} < 0$, we conclude that there is a unique positive root x^* of $q(x)$ on $(0,1)$ and it satisfies $x^* < c < 1$. Then $q(x) < 0$, for every $x > c$. This implies $SF(x) < 0$, for all $x \in (0,1)$, $x > c$, proving Assertion (v).

By applying Theorem 2.1, we have that Assertion (vi) follows from

$$F'(p) = \gamma + p^\gamma H'(p) = \gamma - \frac{\beta}{a}p^\gamma,$$

which is, additionally, less than 1. Finally, if $\kappa < 1$, then

$$\begin{aligned} \kappa \leq F'(p) &\iff \frac{\beta}{a}p^\gamma \leq \gamma - \kappa \iff p \leq \left(\frac{\gamma-\kappa}{\beta/a}\right)^{\frac{1}{\gamma}} \\ &\iff F\left(\left(\frac{\gamma-\kappa}{\beta/a}\right)^{\frac{1}{\gamma}}\right) \leq \left(\frac{\gamma-\kappa}{\beta/a}\right)^{\frac{1}{\gamma}} \\ &\iff \frac{\beta}{a}\frac{\gamma-\kappa}{\beta/a}\left(1 - \left(\frac{\gamma-\kappa}{\beta/a}\right)^{\frac{1}{\gamma}}\right) \leq \left(\frac{\gamma-\kappa}{\beta/a}\right)^{\frac{1}{\gamma}} \\ &\iff \gamma - \kappa \leq (\gamma + 1 - \kappa)\left(\frac{\gamma-\kappa}{\beta/a}\right)^{\frac{1}{\gamma}}, \end{aligned}$$

which yields the desired result regarding Assertion (vii). □

Once more, the computation of W^{-1} and η has not been difficult since $W(p) = \beta p^\gamma / a$.

Remark 2.17. Condition (2.39) becomes relevant to continue the study because it ensures that the solutions of the difference equation (2.37) are well defined for all $x_0 \in (0,1)$ and $n \geq 0$. Hence, in the following, we have to be careful since (2.39) shall hold.

Although we have only supposed that $a, \beta > 0$, condition (2.39) implies that $\frac{\beta}{a} < 4$. In fact, it can be checked that the function Γ in the inequality (2.39) is increasing on $(0,1)$, tends to 1 as $\gamma \to 0^+$ and tends to 4 as $\gamma \to 1^-$. In other words, for any fixed $a, \beta > 0$ such that $\beta/a \in (0,4)$, if $\overline{\gamma} \in (0,1)$ is such that condition (2.39) is fulfilled, then such condition is also satisfied for those $\gamma \in (\overline{\gamma}, 1)$.

Once more, Theorem 2.1 provides us with the role of γ in the value of the equilibrium, which we state in the following result.

Corollary 2.18. *Let $a, \beta > 0$ and $\gamma \in (0,1)$ be such that the consistency condition (2.39) is fulfilled. If we denote by $p(\gamma)$ the unique fixed point of the γ-logistic map (2.37) with $\gamma \in (0,1)$, then the function $p(\gamma) \in (0,1)$ is decreasing.*

Proof. It is trivial from Theorem 2.1 and $\lim\limits_{x \to 1^-} H(x) = 0 < 1$. □

Notice that we have obtained that the value of the equilibrium p decreases as the production function tends to be logistic in the classical sense ($\gamma \to 1^-$).

Theorem 2.19. *Let $a, \beta > 0$ and $\gamma \in (0,1)$ be such that the consistency condition (2.39) is fulfilled, and denote by $p \in (0,1)$ the unique fixed point of the γ-logistic map. Then, p is GAS for the γ-logistic difference equation (2.37) if and only if*

$$\frac{\beta}{a} \leq (\gamma + 1) \left(\frac{\gamma + 2}{\gamma + 1} \right)^{\gamma}. \tag{2.41}$$

Proof. By using Theorem 1.55, we can ensure that p is GAS for (2.37) if $-1 \leq F'(p) < 1$, since $SF(x) < 0$, for any $x > c$ and $\Delta_1^F < 0$ on $(0,1) \setminus \{p\}$ (see Assertions (iv) and (v) in Proposition 2.16). Moreover, $-1 \leq F'(p) < 1$ is equivalent to (2.40) with $\kappa = -1$. □

By using Theorem 2.19 and its key condition (2.41), we can write a result about the influence of γ on the stability of p for (2.37) and each fixed a, β with $\beta/a \in (0,4)$. In fact, we obtain the values of β/a for which γ produces a stability switch in p.

Proposition 2.20. *Let $a, \beta > 0$ such that $\beta/a < 4$. For any $\gamma \in (0,1)$, let $p := p(\gamma)$ be the unique fixed point of F on $(0,1)$. The following assertions are valid:*

- *If $\beta \leq a$, then p is GAS for any $\gamma \in (0,1)$.*
- *If $\beta \in (a, 3a)$, for those $\gamma \in (0,1)$ for which condition (2.39) is fulfilled, the increment of γ switches the type of stability of p: from being unstable to being GAS.*

- *If $\beta \geq 3a$, then p is unstable for any $\gamma \in (0,1)$ for which (2.39) is fulfilled.*

Proof. We analyse the properties of the function $T_{-1} : (0,1) \to \mathbb{R}$

$$T_{-1}(\gamma) := (\gamma+1)\left(\frac{\gamma+2}{\gamma+1}\right)^{\gamma},$$

which is a particular case of (2.40) considered in the proof of Theorem 2.19. First of all, the function T_{-1} is trivially positive. By taking logarithms, one can check that

$$T'_{-1}(\gamma) = T_{-1}(\gamma)\left(\frac{2}{(\gamma+1)(\gamma+2)} + \ln\left(\frac{\gamma+2}{\gamma+1}\right)\right) > 0.$$

Then,

$$\lim_{\gamma\to 0^+} T_{-1}(\gamma) = 1, \quad \lim_{\gamma\to 1^-} T_{-1}(\gamma) = 3.$$

Since $T_{-1}(\gamma)$ is the right-hand side of (2.41), which provides the stability condition for p, the assertions in the statement of the theorem hold (see Figure 2.4). □

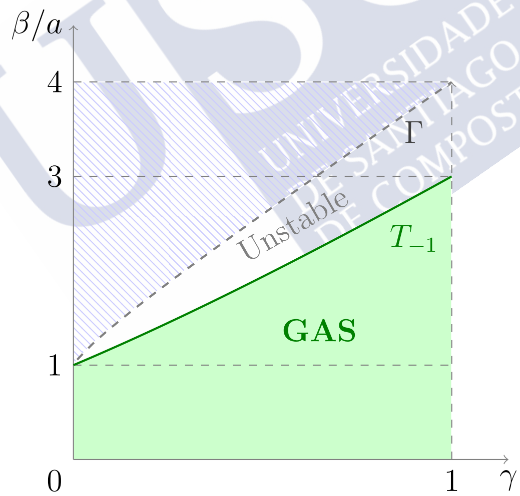

Figure 2.4: The stability diagram of the γ-logistic difference equation (2.37). The graph of the function Γ delimits from above the region of parameters that satisfy the consistency condition (2.39) (solid colours) and from below the ones that do not (filled with blue lines). The graph of the function T_{-1} provides the curve of stability switch for its unique equilibrium p, below which we can find the region of parameters for which p is GAS (in green). Notice that the β/a axis has been rescaled.

Remark 2.21. The sharp estimate for the unique equilibrium to be GAS for $\gamma = 0$ is $\beta < a$, while the limit condition (2.42) as $\gamma \to 0^+$ is $\beta \leq a$. Notice that, while $SF < 0$ if

$\gamma \in (0,1]$, F is a rational map if $\gamma = 0$ and, thus, $SF = 0$. One has to be aware that the limit case $F'(p) = -1$ might cause $F^2 = Id$ [25, Page 995], which is a fact that does not happen with $SF < 0$, so that a tiny difference between conditions is somehow justified.

Moreover, the limit condition of (2.42) as $\gamma \to 1^-$ is $\beta/a \leq 3$ which coincides with the classical estimate for global asymptotic stability of the logistic map.

We have studied the function F in some detail and we are now ready to move into the stability properties of the corresponding delay differential equation (2.34).

Theorem 2.22. *Assume that (A) holds together with $\beta > 0$ and that the consistency condition* (2.39) *is valid. Denote by $x(t;(0,\phi))$ the unique solution of* (2.34) *through $(0,\phi)$, with $\phi \in C_{(0,1)}$, and by p the unique positive fixed point of the γ-logistic map* (2.38)*. If*

$$\frac{\beta}{a} \leq (\gamma+1)\left(\frac{\gamma+2}{\gamma+1}\right)^{\gamma}, \tag{2.42}$$

then $\lim_{t\to\infty} x(t;(0,\phi)) = p$, for every $\phi \in C_{(0,1)}$. Moreover, condition (2.42) *is the sharpest absolute stability condition.*

If we further assume that $\tilde{\tau}(t) = \tau > 0$, for all $t \in \mathbb{R}$, condition

$$\frac{\beta}{a} \leq \left(\gamma + \frac{1}{1-e^{-\alpha\tau}}\right)\left(\frac{\gamma+1+\frac{1}{1-e^{-\alpha\tau}}}{\gamma+\frac{1}{1-e^{-\alpha\tau}}}\right)^{\gamma} \tag{2.43}$$

is also sufficient to ensure that p is GAS for the autonomous DDE (2.34)*.*

Proof. The proof is completely analogous to the one in Theorem 2.12 but with the new definition of the function $T_{-1}(\gamma)$ from the expression in (2.40) instead of $T_{-1,\delta}(\gamma)$ from the one in (2.28). □

The ideas in Remark 2.13 also apply for the previous proof. Moreover, Remark 2.14 is easily adapted to the γ-logistic DDE; in fact, such link between the absolute stability conditions for $\gamma = 0$ and $\gamma = 1$ comes from Remark 2.21. Moreover, we also include the corresponding version of Remark 2.15.

Remark 2.23. If we assume that $\tilde{\tau}(t) = \tau > 0$, for every $t \in \mathbb{R}$, then p is LAS provided

$$\frac{\beta}{a} < \left(\gamma + \sqrt{1+\frac{s^2}{a^2\tau^2}}\right)\left(\frac{\gamma+1+\sqrt{1+\frac{s^2}{a^2\tau^2}}}{\gamma+\sqrt{1+\frac{s^2}{a^2\tau^2}}}\right)^{\gamma}, \tag{2.44}$$

where s is the unique root of $\frac{-x}{\tan(x)} = a\tau$ in $(\frac{\pi}{2}, \pi)$. Condition (2.44) comes from the estimate $F'(p) > \xi_3(a,\tau)/a$; so an analogous condition is

$$\tau < \rho_3(a, f'(p)) = \rho_3(a, a\gamma - \beta p^\gamma) = \frac{\arccos\left(\frac{a}{a\gamma - \beta p^\gamma}\right)}{\sqrt{(a\gamma - \beta p^\gamma)^2 - a^2}}. \tag{2.45}$$

Remark 2.24. The limit version of the local stability condition (2.45) as $\gamma \to 1^-$ coincides with the one in [102, Theorem 3], which is

$$\tau < \frac{\arccos\left(\frac{a}{2a-\beta}\right)}{\sqrt{(2a-\beta)^2 - a^2}}.$$

To see the latter, notice that

$$\lim_{\gamma \to 1^-} f'(p(\gamma)) = \lim_{\gamma \to 1^-} (a\gamma - \beta p(\gamma)^\gamma) = a - \beta p(1) = a - (\beta - a) = 2a - \beta,$$

where we have used that the unique fixed point of the 1-logistic map (2.38) is $p = \frac{\beta - a}{\beta}$.

From the stability issues of the corresponding difference equation, one can give analogous conditions for equation (2.34) provided the consistency condition (2.39) is fulfilled. Since the case of constant delay allows us to deduce more information, we consider such assumption in the remaining part devoted to equation (2.34).

We have seen that, if $\beta/a \le 1$, then the unique equilibrium p of the γ-logistic DDE (2.34) is GAS for any $\gamma \in (0,1)$ and $\tau > 0$. Moreover, if $\beta/a > 1$, then increasing γ forces the equilibrium p to switch from instability to stability. In particular, if $\beta/a \ge 3$, then there exists no absolute stability condition, that is, for any admissible fixed $\gamma \in (0,1)$, there is a sufficient large $\overline{\tau}$ such that, if $\tau > \overline{\tau}$, then p is unstable.

Analogously to the Lasota equation, we give a stability diagram for a particular set of parameters that is of interest. In particular, we depict in Figure 2.5 the stability diagram of the γ-logistic DDE with constant delay, for $\beta = 2.7$ and $a = 1$. First of all, one has to get rid of all the cases that do not satisfy the consistency condition (2.39), that is, the ones with $\Gamma(\gamma) \ge \beta/a = 2.7$. Thus, we only consider the cases of $\gamma > \overline{\gamma}$, where $\overline{\gamma} \approx 0.5358$ is the unique root of $\Gamma(\gamma) = 2.7$. For those cases, we can affirm that p is GAS regardless of the value of $\tau > 0$ if $\gamma \ge \gamma_*$, with $\gamma_* \approx 0.8687$. Moreover, if $\tau \le \overline{\tau}$, with $\overline{\tau} \approx 1.6451$, then p is stable independently of the value of $\gamma \in (0,1)$. In particular, the sufficient conditions that we have used ensure that p is GAS regardless of the value of $\gamma \in (0,1)$ if $\tau \le \tau^*$, with $\tau^* \approx 0.9089$.

In Figure 2.6, we plot a couple of numerical situations corresponding to the case depicted in Figure 2.5 with $\tau = 3$. They correspond to different values of γ (0.6 and 0.9), which have been chosen to show the mentioned stability switch (see also Figure 2.5).

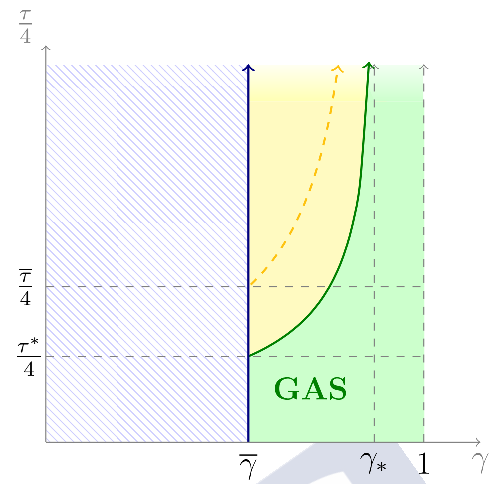

Figure 2.5: Stability diagram for equation (2.34) with constant delay in the plane $(\gamma, \tau/4)$, with $a = 1$ and $\beta = 2.7$. The region with a lined blue pattern represents the values of the parameters that do not satisfy the consistency condition (2.39). The green solid line represents the boundary of the region where global asymptotic stability is guaranteed. The orange dashed curve shows the boundary of local asymptotic stability. Dashed gray lines correspond to threshold values of γ and τ (see the text).

Gamma-Mackey-Glass DDE

Now, assume that the pollution function is $p_2(x) = \frac{1}{1+\delta x^m}$, with $\delta, m > 0$. With this choice, equation (2.16) takes the form

$$x'(t) = -ax(t) + \frac{\beta x^{\gamma}(t - \tilde{\tau}(t))}{1 + \delta x^m (t - \tilde{\tau}(t))}, \tag{2.46}$$

which will be called the *gamma-Mackey-Glass DDE*. This choice means that the function $h : [0, \infty) \to (0, \infty)$, defined by $h(x) = \frac{\beta}{1+\delta x^m}$, is decreasing and of class $\mathcal{C}^1$. Analogously, the map

$$F(x) = \frac{\beta x^{\gamma}}{a(1 + \delta x^m)} \tag{2.47}$$

will be referred to as the *gamma-version of the Maynard Smith and Slatkin map*, which generates the difference equation corresponding to (2.46),

$$x_{n+1} = \frac{\beta x_n^{\gamma}}{a(1 + \delta x_n^m)}, \tag{2.48}$$

from now on called the *gamma-version of the Maynard Smith and Slatkin difference equation* in analogy with the model of Maynard Smith and Slatkin [104, 121], which assumes $\gamma = 1$.

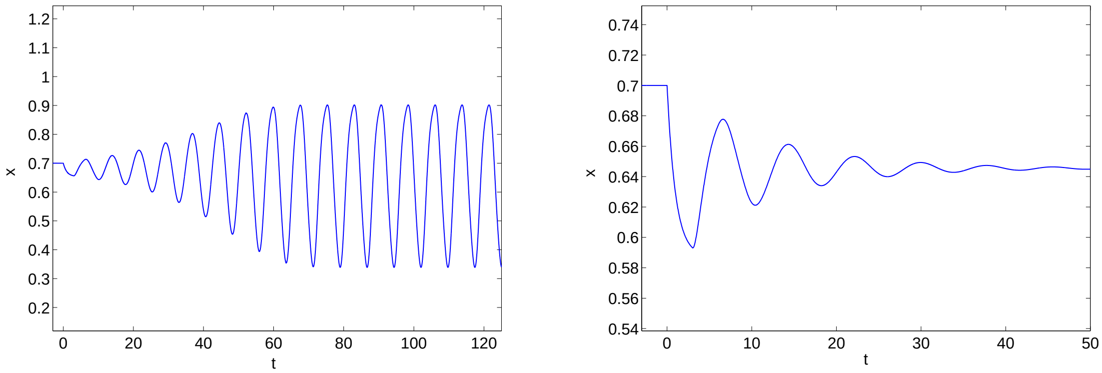

Figure 2.6: Numerical simulations for the solutions of equation (2.34), with $\alpha = 1$, $\beta = 2.7$, $\tau = 3$, and different values of γ. For $\gamma = 0.6$ (left), the equilibrium p is unstable, and there are sustained oscillations; for $\gamma = 0.9$ (right), p is globally asymptotically stable. These pictures have been obtained via the use of MATLAB and, in particular, its solver *dde23*.

Concerning the classical map of Maynard Smith and Slatkin, a well-known particular case is the Beverton-Holt map, for which $m = 1$ (see [82, Section 1] for further information). Hence, the map in (2.47) with $m = 1$ can be called the *gamma-Beverton-Holt map* and, respectively, the difference equation it generates, the *gamma-Beverton-Holt difference equation*. The latter equation and its corresponding DDE (equation (2.46) with $m = 1$) are simpler to study. Thus, let us first state the following two results regarding this choice and then explain the general case.

The following result summarises the main properties of the γ-Beverton-Holt map, and can be derived from Proposition 1 and Theorem 3.1 in [82]. Its third point can also be seen as a corollary of Theorem 2.1.

Proposition 2.25. *The map* $F : (0, \infty) \to (0, \infty)$ *defined by*

$$F(x) = \frac{\beta x^{\gamma}}{a(1 + \delta x)}; \quad a, \beta, \delta > 0;\ 0 < \gamma < 1; \tag{2.49}$$

satisfies the following properties:

(i) F *is of class* $\mathcal{C}^{\infty}$ *and*

$$\lim_{x \to 0^+} F(x) = 0 = \lim_{x \to \infty} F(x), \quad \lim_{x \to 0^+} F'(x) = \infty.$$

(ii) F *is a U-map, with a unique critical point at* $c = \frac{\gamma}{\delta(1-\gamma)}$*, where* F *attains its global maximum.*

(iii) *There is a unique* $p = p(\gamma) \in (0, \infty)$ *such that* $F(p) = p$*, which is given by the unique root of*

$$p^{1-\gamma}(1 + \delta p) = \frac{\beta}{a}.$$

Moreover, $(F(x) - x)(x - p) < 0$ *for every* $x \in (0, \infty) \setminus \{p\}$*.*

(iv) *Equation* $F^2(x) = x$ *has no positive solutions different from* p*. In fact, we have*

$$\Delta_2^F(x) = (F^2(x) - x)(x - p) < 0, \quad x \in (0, \infty) \setminus \{p\}.$$

The role of γ on the value of the unique positive fixed point p of the γ-Beverton-Holt map is as follows.

Corollary 2.26. *Let* $m = 1$ *and* $a, \beta, \delta > 0$*. If we denote by* $p(\gamma)$ *the unique positive fixed point of the* γ*-Beverton-Holt map* (2.49) *with* $\gamma \in (0, 1)$*, then* $p(\gamma)$ *satisfies the following:*

- *If* $\frac{\beta}{a} > 1 + \delta$*, then* $p(\gamma) \in (1, \infty)$ *is increasing.*
- *If* $\frac{\beta}{a} = 1 + \delta$*, then* $p(\gamma) = 1$*, for every* $\gamma \in (0, 1)$*.*
- *If* $\frac{\beta}{a} < 1 + \delta$*, then* $p(\gamma) \in (0, 1)$ *is decreasing.*

Proof. It is a straightforward conclusion from Theorem 2.1, since $H(1) = \frac{\beta}{a(1+\delta)}$. □

Notice that we could have provided Corollary 2.26 without imposing $m = 1$, since $H(1)$ would also be equal to $\frac{\beta}{a(1+\delta)}$ for any $m > 0$. Hence, this fact turns out to be useful for the case $m \neq 1$.

The next result follows as a direct consequence of Proposition 2.25 and the criterion in Theorem 1.49.

Theorem 2.27. *[82, Theorem 3.1] Let* $m = 1$*,* $a, b, \delta > 0$*,* $\gamma \in (0, 1)$*, and denote by* p *the unique positive fixed point of the* γ*-Beverton-Holt map. Then* p *is GAS for the* γ*-Beverton-Holt difference equation.*

Now, Theorem 2.27, combined with Theorem 1.69, yields the natural result concerning the long-term behaviour of the solutions of the DDE (2.46) with $m = 1$.

Theorem 2.28. *Let* $m = 1$*. Assume that (A) holds together with* $\beta, \delta > 0$*. Denote by* $x(t; (0, \phi))$ *the unique solution of* (2.46) *through* $(0, \phi)$*,* $\phi \in C_{(0,\infty)}$*, and by* p *the unique positive fixed point of the* γ*-Beverton-Holt map. Then,* $\lim_{t\to\infty} x(t; (0, \phi)) = p$*, for every* $\phi \in C_{(0,\infty)}$*.*

Remark 2.29. Unlike Theorems 2.12 and 2.22, Theorem 2.28 does only provide a global absolute stability of the equilibrium, yet this is caused by the global asymptotic stability of p for (2.47) with $m = 1$, regardless of the value of other parameters.

One may attempt to provide results analogous to Proposition 2.25 and Theorem 2.28 for the general case of $m > 0$. However, it has not been sufficiently clear how to obtain a condition on the γ-Mackey-Glass map (2.47) to ensure that its unique equilibrium is GAS as it happens with the particular case of the γ-Beverton-Holt map. Neither following the ideas from [82] to prove that (2.47) has no 2-periodic points different from the unique positive equilibrium for $m \neq 1$ (see Theorem 1.49) nor the results that we recalled regarding the Schwarzian derivative (see Theorem 1.55) seem to yield any straightforward conclusion.

Consequently, we shall seek other tools to study the global dynamics of the difference equation (2.48), and, analogously, for (2.46). This will constitute the main motivation for the contents in the next chapter, so we can see that part of the work as a branch arising from and motivated by this issue.

As previously announced, we now provide a summary of the main findings of the current Subsection 2.2.1. It is presented in the form of a table where we give, for each model, its expression, the equation that defines the equilibrium, information about the local dynamics in terms of the auxiliary functions (recall Theorem 2.1 and Remark 2.2), and, finally, the global asymptotic stability conditions in terms of the parameters. Notice that some of them require the use of the function ξ_1 from the final part of Subsection 1.3.3 to obtain a simple explicit delay-dependent global asymptotic stability condition.

2.2.2 Variable saving rate

Next, we consider a variable saving rate $s(x(t))$ in equation (2.14). As we remarked above, the equation (2.14) is not included in the framework of Section 1.3. However, as we will see, the essential tools that we will use are analogous to those in the above-mentioned section. Thus, with that aim of including a variable saving rate factor, we need to consider a slightly more general equation than the scalar one considered in Section 1.3. In fact, let $\tilde{\tau} : \mathbb{R} \to \mathbb{R}_+$ be a bounded continuous function with supremum equal to $\tau > 0$, and consider the equation

$$x'(t) = -g_1(x(t))f_2(x(t - \tilde{\tau}(t))) + f_1(x(t - \tilde{\tau}(t)))g_2(x(t)). \tag{2.50}$$

We will first give some theoretical background. Thus, throughout the first part of this subsection, we will assume that $I = (l, r)$, $0 \leq l < r \leq \infty$, is a real interval and we will consider the following hypotheses on functions f_i, g_i, $i = 1, 2$:

(H1) f_1, f_2, g_1 and g_2 are continuous and positive functions defined on $I = (l, r)$, and $g := g_1/g_2$ is increasing.

$$x'(t) = -ax(t) + f(x(t-\tilde{\tau}(t))) = -ax(t) + x^{\gamma}h(x(t-\tilde{\tau}(t)))$$
$$x_{n+1} = F(x_n) = x_n^{\gamma}H(x_n),$$
$$a > 0,\ \gamma \in (0,1),\ 0 \leq \tilde{\tau}(t) \leq \tau,\ \tau > 0,\ F = f/a,\ H = h/a$$

Model and auxiliary functions	Information about the positive equilibrium	Global asymptotic stability conditions
Delayed Solow Eq. $f(x) = \beta x^{\gamma}$ $h(x) = \beta > 0$	$p = \left(\frac{\beta}{a}\right)^{\frac{1}{1-\gamma}}$ $F'(p) = \gamma$ $H'(p) = 0$	None. p is GAS
Lasota Eq. $f(x) = \beta x^{\gamma}e^{-\delta x}$ $h(x) = \beta e^{-\delta x}$ $\beta, \delta > 0$	Root of $p^{1-\gamma} = \frac{\beta}{a}e^{-\delta p}$ $F'(p) = \gamma - \delta p$ $H'(p) = -\delta p^{1-\gamma}$	*Delay-independent cond.* $\frac{\beta}{a} \leq e^{\gamma+1}\left(\frac{\gamma+1}{\delta}\right)^{1-\gamma}$ *Delay-dependent cond.* *(for constant $\tilde{\tau} \equiv \tau$)* $\frac{\beta}{a} \leq e^{\gamma-\kappa_1}\left(\frac{\gamma-\kappa_1}{\delta}\right)^{1-\gamma}$ $\kappa_1 = \frac{-1}{1-e^{-a\tau}}$
$\gamma -$ logistic Eq. $f(x) = \beta x^{\gamma}(1-x)$ $h(x) = \beta(1-x)$ $\beta > 0$ *Consistency condition* $\frac{\beta}{a} < (\gamma+1)\left(\frac{\gamma+1}{\gamma}\right)^{\gamma}$	Root of $p^{1-\gamma} = \frac{\beta}{a}(1-p)$ $F'(p) = \gamma - \frac{\beta}{a}p^{\gamma}$ $H'(p) = -\frac{\beta}{a}$	*Delay-independent cond.* $\frac{\beta}{a} \leq (\gamma+1)\left(\frac{\gamma+2}{\gamma+1}\right)^{\gamma}$ *Delay-dependent cond.* *(for constant $\tilde{\tau} \equiv \tau$)* $\frac{\beta}{a} \leq (\gamma-\kappa_1)\left(\frac{\gamma+1-\kappa_1}{\gamma-\kappa_1}\right)^{\gamma}$ $\kappa_1 = \frac{-1}{1-e^{-a\tau}}$
$\gamma -$ Mackey-Glass Eq. $f(x) = \frac{\beta x^{\gamma}}{1+\delta x^m}$ $h(x) = \frac{\beta}{1+\delta x^m}$ $\beta, \delta, m > 0$	Root of $p^{1-\gamma} = \frac{\beta}{a}\frac{1}{1+\delta p^m}$ $F'(p) = \gamma - \frac{m\delta p^m}{1+\delta p^m}$ $H'(p) = -\frac{m\delta p^{m-\gamma}}{1+\delta p^m}$	p is GAS if $m = 1$ [See Chapter 3 for $m \neq 1$]

Table 2.1: Summary of the models studied in Subsection 2.2.1. Just above the table, we write the general delay-differential gamma-model studied in this part of the thesis together with its corresponding difference equation. Each row corresponds to a different model for which we provide some relevant details (see the text for additional explanations).

(H2) If $f := f_1/f_2$, then there is a unique solution $p \in I = (l, r)$ of equation $g(x) = f(x)$. Moreover, the function $F(x) := g^{-1}(f(x))$ maps I into I.

Note that, if $g_1(x) = ax$, $a > 0$, $f_1 = f$ and $g_2 = f_2 = 1$, we recover equation (1.15) and, if, for example, we further assume $I = (0, \infty)$, then $F(x) = g^{-1}(f(x)) = \frac{f(x)}{a}$ coincides with the so-used map in Section 1.3.1. Therefore, the equations studied in Subsection 2.2.1 are included in this general framework. We will see that the generalised F defined in (H2) plays an analogous role to the one in the aforementioned particular case, that is, global attractivity of the unique equilibrium of

$$x_{n+1} = F(x_n) = g^{-1}(f(x_n)) \tag{2.51}$$

yields global attractivity of the unique equilibrium of (2.50).

To prove the main result in this subsection, which generalises some previous results in [64, 90], we need to introduce another concept and recall an auxiliary result on scalar discrete dynamics from [90].

Definition 2.30. Let $F : I \to I$ be a continuous map and p be its unique fixed point. We say that p is a *strong attractor* for F on I if, for every compact set $K \subset I$, there exists a family $(I_n)_{n\in\mathbb{N}}$ of non-degenerate compact intervals $I_n = [l_n, r_n]$, $n \in \mathbb{N}$, such that the following properties hold:

(A1) $K \subset Int(I_1) \subset (l, r) = I$.

(A2) $F(I_n) \subset I_{n+1} \subset Int(I_n)$, $\forall n \in \mathbb{N}$.

(A3) $\displaystyle\bigcap_{n=1}^{\infty} I_n = \{p\}$.

Lemma 2.31. *[90] Let $F : I \to I$ be a continuous map and p its unique fixed point. Then p is a global attractor for F on I if and only if p is a strong attractor for F on I.*

Lemma 2.31 can be construed as an additional equivalence for Theorem 1.49; in fact, bear in mind that every equilibrium which is a global attractor is also stable (Remark 1.29).

We are now in a position to prove the main result of this section. We assume that, for any initial condition $(0, \phi)$ with $\phi \in C_{(l,r)}$, the corresponding solution $x(t) = x(t; (0, \phi))$ of (2.50) exists for all $t \geq 0$, and it is unique (recall the results in Chapter 1). For instance, under conditions (H1) and (H2) and $\tau(t) > 0$ for every $t \in \mathbb{R}$, it is enough to assume that g_1 and g_2 are Lipschitz continuous (see [64]). Otherwise, we may need to impose higher regularity on functions f_1 and f_2 (also recall Chapter 1).

Theorem 2.32. *Assume that conditions (H1) and (H2) hold and that p is a global attractor for the difference equation* (2.51)*. Then, p is globally attracting for* (2.50) *on C_I, that is, if $x(t) = x(t;(0,\phi))$ is the solution of* (2.50) *with initial condition $(0,\phi)$, $\phi \in C_I$ then,*

$$\lim_{t\to\infty} x(t) = p.$$

Proof. For a given continuous function $\phi : [-\tau, 0] \to (l, r) = I$, define the compact set

$$K = \left[\min_{t\in[-h,0]} \phi(t), \max_{t\in[-h,0]} \phi(t)\right] \subset I. \tag{2.52}$$

Since p is a global attractor for (2.51), p is also a strong attractor by virtue of Lemma 2.31. Thus, for K defined in (2.52), consider the family $(I_n)_{n\in\mathbb{N}}$, with $I_n = [l_n, r_n]$, which exists by definition of a strong attractor. First we prove that $x(t) \in I_1 = [l_1, r_1]$ for all $t \geq 0$. Assume, by the contrary, that there exists a first instant $t_0 > 0$ such that $x(t_0) = r_1$ (the same argument applies if $x(t_0) = l_1$). Since $F(I_1) \subset Int(I_1)$, it follows that

$$F(x(t_0 - \tilde{\tau}(t_0))) < r_1,$$

which is equivalent to

$$f(x(t_0 - \tilde{\tau}(t_0))) < g(r_1).$$

Then, using (2.50), we have

$$\frac{x'(t_0)}{f_2(x(t_0 - \tilde{\tau}(t_0)))g_2(x(t_0))} = -g(x(t_0)) + f(x(t_0 - \tilde{\tau}(t_0))) = -g(r_1) + f(x(t_0 - \tilde{\tau}(t_0))) < 0,$$

which is a contradiction, because $x'(t_0) \geq 0$.

Next, we show that there is $t_1 > 0$ such that $x(t) \in I_2$ for all $t \geq t_1$. Following the previous argument, and using that $F(I_1) \subset I_2$, it is easy to prove that, if there is t_1 such that $x(t_1) \in I_2$, then $x(t) \in I_2$ for all $t \geq t_1$. Thus, we prove the existence of such a point t_1. Assume, by contradiction, that t_1 does not exist, and, to fix ideas, assume that $x(t) > r_2$ for all $t \geq 0$ (the same arguments apply to the case $x(t) < l_2$ for all $t \geq 0$). Then, using (2.50) and the increasing character of g, we have:

$$\frac{x'(t)}{f_2(x(t - \tilde{\tau}(t))g_2(x(t))} = -g(x(t)) + f(x(t - \tilde{\tau}(t))) < 0, \quad \forall t > 0.$$

Indeed, notice that

$$x(t) > r_2 \Longrightarrow g(x(t)) > g(r_2),$$

$$x(t - \tilde{\tau}(t)) \in I_1 \Longrightarrow F(x(t - \tilde{\tau}(t))) \in I_2 \Longrightarrow f(x(t - \tilde{\tau}(t))) \leq g(r_2).$$

Hence, $x(t)$ is a decreasing function, and there is $\lim_{t\to\infty} x(t) = L \geq r_2$. This is a contradiction because the only possible limit of $x(t)$ is p, and $p < r_2$.

An inductive argument, by using the same reasoning used above, proves that, for each $n > 1$, there exists a point $t_n > 0$ such that $x(t_n) \in I_{n+1}$, for all $t \geq t_n$. Finally, it follows by (A3) that $\lim_{t\to\infty} x(t) = p$. □

We notice that Theorem 2.32 generalises Theorem 3 in [64], and its proof is much simpler. For the special case of constant functions $f_2 = g_2 = 1$, a multidimensional version of Theorem 2.32 has been considered in [90], which was further extended in [18], the core of Chapter 4 of this manuscript. Furthermore, as we announced, Theorem 2.32 also constitutes a generalisation of the first part of Theorem 1.69.

In order to check that p is GAS for $F = g^{-1} \circ f$ in applications, we refer to the results on Section 1.3.3, namely the criterion in Theorem 1.49 or the sufficient conditions on Theorem 1.55.

Remark 2.33. Note that, in order to apply Theorem 1.55, we need to check the condition

$$(F(x) - x)(x - p) < 0, \quad x \neq p. \tag{2.53}$$

By considering both cases $x < p$ and $x > p$, it is easy to see that condition (2.53) is equivalent to

$$(f(x) - g(x))(x - p) < 0, \quad x \neq p.$$

In particular, by assuming that (H1) is satisfied, F is increasing if f is increasing, hence that case is related with the first statement of Theorem 1.55.

We have provided a theoretical background to analyse equation (2.14) with variable saving rate. Now, we show a particular example.

As it was argued by Solow in [126, Page 88], it is natural to assume that the saving rate depends inversely on the capital-labour ratio x, and that, for sufficiently large x, $s(x)$ approaches to zero. Thus, we assume that $s(x)$ is decreasing and tends to 0 as x tends to infinity. We prove the following result for the case of no pollution effects.

Theorem 2.34. *Assume that the function $s : [0,\infty) \to (0,\infty)$ is decreasing and of class $\mathcal{C}^1$, and such that $\lim_{x\to\infty} s(x) = 0$, and $x/s(x)$ is convex. Then, equation*

$$x'(t) = -ax(t) + s(x(t))Bx^{\gamma}(t - \tilde{\tau}(t)) \tag{2.54}$$

has a unique positive equilibrium p, and it is globally attracting on $C_{(0,\infty)}$.

Proof. Equation (2.54) is of the form (2.50), with $g_1(x) = ax$, $f_2(x) = 1$, $f_1(x) = Bx^\gamma$, and $g_2(x) = s(x)$. Since s is decreasing and $\gamma > 0$, it follows that both functions

$$g(x) = \frac{g_1(x)}{g_2(x)} = \frac{ax}{s(x)}, \quad f(x) = \frac{f_1(x)}{f_2(x)} = Bx^\gamma,$$

are increasing; thus $F = g^{-1} \circ f$ is increasing. Next, since f is concave, $\lim_{x \to 0^+} f'(x) = \infty$, and g is convex, it is clear that there exists $p \in (0, \infty)$ such that

$$(f(x) - g(x))(x - p) < 0, \quad x \neq p.$$

From Remark 2.33), it is straightforward that an application of the first part of Theorem 1.55 provides the desired result. □

Remark 2.35. Conditions of Theorem 2.34 are satisfied, for example, if we choose the saving functions $s(x) = s_0 e^{-\delta x}$ or $s(x) = s_0/(1 + \delta x)$, with $s_0 > 0$, $\delta > 0$.

2.3 Discussion about the role of the parameter γ

Solow's paper [126] is one of the most influential works in economic theory [10, 100]. Although the original model is defined by an ordinary differential equation, the role of time delays in the production processes has been discussed both for the one-sector Solow's model by Matsumoto and Szidarovszky [103], and by Gori et al. [45] for the two-sector generalisation proposed by Mankiw et al. [100]. These papers focus on a local stability analysis of the positive equilibrium, and Hopf bifurcations leading to sustained oscillations as time delay increases.

In this chapter, we have carried out a deeper analysis about the role of time delays in the original Solow equation, allowing the possibility of variable saving rate, and a delayed production function with two factors: a positive factor x^γ coming from the Cobb-Douglas production function, and a negative factor $p_2(x)$, which represents a 'pollution effect' due to increasing concentrations of capital [27]. These considerations lead to a general form of Solow's model given by equation (2.7). With this model, we showed that the influence of time delays on the stability properties of the positive equilibrium depends in an essential way on several factors, including the choice of the nonincreasing function $p_2(x)$, and the interplay among the different parameters involved in the model. The role of the parameter γ, which measures the responsiveness of the output to a change in the level of capital used in production, is of particular interest.

Two remarkable consequences of our results are the following: first, under mild assumptions, variable saving rates and variable delays in the production function are not able to destabilise the positive equilibrium if a pollution effect is not considered (Theorem

2.34), so the generalisation of Solow's equation given by (2.54) still predicts convergence to the steady-state capital-labour ratio. Second, even if a negative factor is introduced in the production function, the particular form of this factor may prevent instabilities due to large delays (see, e.g., Theorem 2.28). Roughly speaking, the 'pollution function' $p_2(x)$ needs to have a fast rate of convergence to zero as x tends to infinity.

In contrast with other papers, our stability analysis focuses on global results and allows variable delays. In particular, we establish sharp delay-independent global stability conditions, and also prove that small constant delays cannot destroy the global stability of the equilibrium. The mathematical approach we use to get global stability results for (2.7) is not new. However, the use of the notion of strong attractor introduced in [90] has some advantages: on the one hand, the proofs are much simpler than in previous papers [64, 65]; on the other hand, it is easy to consider variable delays, providing more general results.

Another interesting novelty of our results is that we use a generalisation of Allwright-Singer's theory for maps with negative Schwarzian derivative [122], due to El-Morshedy and Jiménez López [33], that we have recalled in Theorem 1.55. This result is crucial in the analysis of equations (2.9) and (2.11) because, in contrast with other cases ($\gamma = 0$, $\gamma \geq 1$), the respective feedbacks $f(x) = \beta x^\gamma e^{-\delta x}$ and $f(x) = \beta x^\gamma(1-x)$ do not have negative Schwarzian derivative everywhere if $0 < \gamma < 1$: they are what we called S_*-maps.

In this way, our results fill a gap in the stability theory of equation (2.9). As far as we know, equation (2.9) has been introduced for the first time by Lasota in 1977 [74], to model blood cell production (erythropoiesis). Lasota formulated a conjecture concerning the ergodic properties of (2.9), see [107]. An interesting biological interpretation of the parameter γ in the model has been given by Mitkowski in his Ph.D. thesis [106]. It is related to disturbed erythropoiesis (dyserythropoiesis), when the feedback loop that regulates the production of cells in the red bone marrow does not work properly. Roughly speaking, γ represents the degree of disturbance of the normal erythropoietic response. When $\gamma = 0$, the answer is correct, but, when $\gamma > 0$, the response is inhibited and the greater the inhibition is, the greater the value of γ is.

The authors of [102] have dealt with the study of local stability of some particular cases of the logistic delay differential equation with a linear decay term (2.11). In a similar manner to what was shown for Lasota equation (2.25), we have extended the study in [102] by providing global stability results for the delay-differential gamma-model (2.34).

Notwithstanding the advances that we have been able to provide, we have also realised that not all the γ-versions of well-known delay-differential models can be studied in a clear manner via the Allwright-Singer-type of results. For instance, the study of the γ-Mackey-Glass DDE is based on that of the γ-version of the Maynard Smith and Slatkin map (2.48), for which, as far as we know, only results regarding its global stability for $m = 1$

are available [82]; they are based on the Coppel-type criterion recalled in Theorem 1.49. If $m \neq 1$, no simple way to apply the known tools was found, a fact that suggested starting a quest of new sufficient conditions to check when an equilibrium is GAS for a difference equation.

Generally speaking, some further work needs to be done in order to extend the global stability conditions and, in particular, whether the shown models satisfy the 'LAS implies GAS' property. Yet another interesting problem would be to study equation (2.14) with state-dependent delay.

It is remarkable that the same equation with different values of γ has been used for different mathematical models governed by delay differential equations:

- For $\gamma = 0$, it is a model for blood-cell production, proposed by Ważewska-Czyżewska and Lasota [135], and later modified by Lasota [74], allowing positive values of γ.

- For $0 < \gamma < 1$, it is a model in Economics, proposed by Matsumoto and Szidarovszky [103], as a generalisation of the fundamental Solow's equation [126].

- For $\gamma = 1$, it is the famous equation introduced by Gurney, Blythe and Nisbet [49] to explain some qualitative aspects of Nicholson's classical experiments on laboratory cultures of sheep blowflies.

- For $\gamma > 1$, equation (2.9) has been proposed to study the dynamics of single-species populations subject to Allee effects [60, 92].

Finally, it is worth mentioning that the Lasota equation (2.9) can be seen as a continuous version of the γ-Ricker map, which has been used as a flexible discrete model for animal populations, and in the context of cooperative interaction in a group of individuals [81].

Chapter 3

A Schwarzian-type formula for sharp global stability criteria

We present a new formula that makes it possible to get sharp global stability results for one-dimensional discrete-time models in an easy way. In particular, it allows to show that the local asymptotic stability of a positive equilibrium implies its global asymptotic stability for a new family of difference equations that finds many applications in population dynamics, economic models, and also in physiological processes governed by delay differential equations, which have been named gamma-models and which have been studied in Chapter 2. The main ingredients to prove our results are the Schwarzian derivative and some dominance arguments. Finally, we derive some relevant conclusions concerning the dynamics of the so-called gamma-Mackey-Glass models. This chapter explains and extends a great part of the work developed in [83] by Eduardo Liz[1] and the author of this thesis (S. Buedo-Fernández[2]). Further details of the former article may be found in its complete reference below.

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3.1 Introduction

Sufficient conditions to ensure the global attractivity of an equilibrium for a scalar difference equation are facilitated if the function being iterated has negative Schwarzian

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derivative in a certain subinterval of its domain. In such case, as we have seen before, the global attractivity study is moved to the easier task of checking the local asymptotic stability of the equilibrium. Many well-known models and their variants satisfy those types of assumptions concerning the Schwarzian derivative. However, not all maps in population or economic models satisfy such conditions, as shown in Chapter 2. Thus, one may be interested in searching for alternative conditions that preserve the 'LAS implies GAS' property [43]. Another useful tool, enveloping, has been recalled in Theorem 1.51, which, generally speaking, involves a comparison of a map with another one that has the same equilibrium and it is known to be globally attracting. In fact, the approaches in [33] and [78] combine both tools (Schwarzian derivative and dominance).

Our main aim in this chapter consists in proving sharp global stability results for the family of difference gamma-models given in Section 2.2.1. We recall their general expression

$$x_{n+1} = x_n^{\gamma} H(x_n) =: F(x_n), \quad n \in \mathbb{Z}_+, \tag{3.1}$$

where $\gamma \geq 0$, $H : [0, \infty) \to (0, \infty)$ is a smooth function, and any solution of equation (3.1) is a sequence $(x_n)_{n \in \mathbb{Z}_+}$ starting at any initial condition $x_0 > 0$.

The main finding shown in this chapter is a new formula – stated as the forthcoming equation (3.3) – which provides sharp global stability results in an easy way. This formula is applicable in many cases when the criterion of the Schwarzian derivative does not work. In this way, it extends the classical Allwright-Singer result [2, 122]. To illustrate this fact, consider the following equivalent form of (3.3) suggested by Víctor Jiménez López:

$$SH(x) < \frac{H(x) - xH'(x)}{2(xH(x))^2}(xH(x))', \quad \forall x > 0. \tag{3.2}$$

According to our main result, Theorem 3.1, condition (3.2) ensures the global asymptotic stability of the unique equilibrium p of (3.1) if $H : [0, \infty) \to (0, \infty)$ is decreasing, $\gamma \in [0, 1]$, and p is locally asymptotically stable. For instance, if $\gamma = 0$, condition (3.2) is strictly weaker than the classical condition $SH(x) < 0$ when H is decreasing but $xH(x)$ is an increasing function. In fact, in such a case, we provide a positive (functional) upper bound for SF.

For example, consider the map $F(x) = a/(1 + x)^m$, with $a > 0$, $m > 0$. Such map is of the form (3.1), with $\gamma = 0$ (i.e., $H = F$) and satisfies $SF(x) = (1 - m^2)/(2(1 + x)^2)$. If $m > 1$, then $SF(x) < 0$ for all $x > 0$, and therefore the Allwright-Singer result applies (see also Theorem 1.55). However, if $0 < m \leq 1$, then $SF(x) \geq 0$ for all $x > 0$, so this classical result cannot be applied. Nonetheless, as remarked above, since $xF(x)$ is increasing, condition (3.2) is sharper than the classical estimate, and, in fact, this turns out to be useful, since SF satisfies (3.2). Therefore, our results ensure the global stability of the unique positive equilibrium p (notice that $0 > F'(p) = -mp/(1+p) > -1$ if $0 < m \leq 1$).

The principal idea to ensure that (3.3) yields a 'LAS implies GAS' type result regarding (3.1) is to deal with certain consecutive changes of variables that transform such difference equation into one that is easier to handle. Moreover, in Section 1.3, the simple topological conjugacy $\Theta(x) = x - p$ was also proposed to shift the unique equilibrium p to the origin and, thus, to provide a unifying treatment for all the maps considered therein. Bearing in mind such aims, we define the change of variables

$$\Lambda(x) = -\ln(x/p),$$

which shifts the unique equilibrium p to the origin but also becomes essential since it modifies the form of the right-hand side of the discrete gamma-model in (3.1): the product $x^\gamma H(x)$ is transformed into a certain sum $\gamma x + G(x)$, where the function G is defined in terms of H. Thus, by using some known tools as enveloping or functions with negative Schwarzian derivative, any condition on G that is obtained regarding the asymptotic behaviour of the solutions of the difference equation

$$x_{n+1} = \gamma x_n + G(x_n)$$

would translate into conditions on H in the original gamma-model (3.1). Therefore, the term H, which profoundly affected the dynamics, as we have seen in Chapter 2, can be somehow isolated; this was not the case in that chapter, where we have studied the function $F(x) = x^\gamma H(x)$ as a whole.

Two important features of the central result in this chapter are the following. On the one hand, it is quite general, allowing to recover some known results in a unified and simpler way, and to prove some new relevant results for applications. On the other hand, it is easily verifiable, as we show in the applications. Therefore, it may be a useful tool for researchers interested in the global stability of difference equations and delay differential equations.

3.2 Main result and examples

For the main result in this section, we consider equation (3.1) with $0 \leq \gamma \leq 1$. Next, we state the main assumptions for the map H.

(H) $H : [0, \infty) \to (0, \infty)$ is of class $\mathcal{C}^3$ and $H'(x) < 0$ for all $x > 0$. For $\gamma = 1$, we also assume that

$$\lim_{x\to\infty} H(x) < 1 < H(0).$$

Note that, in case $\gamma = 0$, we are assuming that $F = H$, and thus, we assume that F is a decreasing M-map, a constraint that does not appear in the related Theorem 1.55, where unimodal maps were allowed too. We will make further comments regarding this particular case in Subsection 3.3. Besides, most of the reasoning is analogous for the case of a nonincreasing function H (i.e., $H' \leq 0$) if $\gamma \in [0, 1)$.

Furthermore, if (H) holds, then there is a unique $p > 0$ such that $H(p) = p^{1-\gamma}$, that is, a unique equilibrium p of the difference equation (3.1) (see Theorem 2.1 and recall that we usually denote $H = h/a$).

Theorem 3.1. *Assume that $0 \leq \gamma \leq 1$, H satisfies (H), and the following condition holds:*

$$x^2 \left(2\, SH(x) + \left(\frac{H'(x)}{H(x)} \right)^2 \right) < 1, \quad \forall x > 0. \tag{3.3}$$

Then the local asymptotic stability of the unique positive equilibrium p in equation (3.1) *implies its global stability. This stability condition is*

$$W(p) = -p^{\gamma} H'(p) \leq 1 + \gamma. \tag{3.4}$$

Remark 3.2. We notice that an application of Theorem 2.1 yields that condition (3.4) is equivalent to

$$-1 \leq \gamma + p^{\gamma} H'(p) = F'(p).$$

Moreover, in our framework, $F'(p)$ is always less than 1 because $0 \leq \gamma \leq 1$ and $H'(p) < 0$. In this way, the stability condition (3.4) is equivalent to the local asymptotic stability of the equilibrium p.

We give the proof of Theorem 3.1 in Section 3.4. Now we show its applicability by providing two classical examples from the literature about population dynamics. The second one gives absolute stability results for a generalisation of the Mackey-Glass equation [96].

Example 3.3. Consider the gamma-Ricker difference equation [81], previously considered in (2.26),

$$x_{n+1} = \frac{\beta}{a} x_n^{\gamma} e^{-\delta x_n}, \quad n \in \mathbb{Z}_+, \tag{3.5}$$

where $a, \beta, \delta > 0$ and $0 \leq \gamma \leq 1$. In this case, we easily get that

$$H(x) = \frac{\beta}{a} e^{-\delta x}, \quad SH(x) = -\frac{\delta^2}{2}, \quad \frac{H'(x)}{H(x)} = -\delta.$$

Thus,

$$2\,SH(x) + \left(\frac{H'(x)}{H(x)}\right)^2 = 0, \quad \forall x > 0,$$

and (3.3) trivially holds.

Next, condition (3.4) is equivalent to $p\delta \leq 1 + \gamma$, which in turn is equivalent to the global stability condition (see [81] or Assertion (vi) in Theorem 1.3):

$$\frac{\beta}{a} \leq e^{\gamma+1}\left(\frac{\gamma+1}{\delta}\right)^{1-\gamma}.$$

Therefore, likewise Theorem 1.55, an application of Theorem 3.1 also provides the global stability criterion for the γ-Ricker map shown in Theorem 2.11. For $\gamma = 1$, we need the extra condition $\beta > a$, so one gets, once more, the well-known global stability criterion $1 < \beta/a \leq e^2$ for the classical Ricker map.

Example 3.4. Consider

$$x_{n+1} = \frac{\frac{\beta}{a} x_n^{\gamma}}{1 + \delta x_n^m},$$

where $a, \beta, \delta, m > 0$ and $0 \leq \gamma \leq 1$. We recall that the latter equation is a generalisation of the population model proposed by Maynard Smith and Slatkin [104, 121] (they consider the particular case of $\gamma = 1$), which was introduced in Section 2.2.1. We now have

$$H(x) = \frac{\beta/a}{1 + \delta x^m}, \quad SH(x) = \frac{1 - m^2}{2x^2}, \quad \frac{H'(x)}{H(x)} = \frac{-\delta m x^{m-1}}{1 + \delta x^m}.$$

Thus,

$$x^2\left(2SH(x) + \left(\frac{H'(x)}{H(x)}\right)^2\right) = 1 - m^2 + m^2\left(\frac{\delta x^m}{1 + \delta x^m}\right)^2 < 1, \quad \forall x > 0,$$

and condition (3.3) is satisfied. We leave further comments regarding the global stability to Section 3.5, since the difference equation of this example is related with the DDE that remained unstudied in Chapter 2.

3.3 The case $\gamma = 0$

The case $\gamma = 0$ deserves special attention. Firstly, since our assumptions yield $F = H$, Theorem 3.1 with $\gamma = 0$ provides a new formula to prove global stability when the sign of the Schwarzian derivative of F is non-constant. Nevertheless, since $F = H$, it would only be valid for a decreasing function F, letting aside the unimodal case that was actually considered in results like Theorem 1.55. Hence, it is worth formulating a more general result in order to include U-maps. Its proof is also given in Section 3.4.

Theorem 3.5. *Assume that the $\mathcal{C}^3$ map $F : (0,\infty) \to (0,\infty)$ is either decreasing or unimodal (with a unique critical point c, which is a local extremum), and has a unique positive fixed point p, such that $F(x) > x$ if $x < p$ and $F(x) < x$ if $x > p$. If $F'(p) \geq -1$ and* (3.3) *holds for all $x > 0$ such that $F'(x) \neq 0$, then p is a global attractor for*

$$x_{n+1} = F(x_n), \quad n \in \mathbb{Z}_+. \tag{3.6}$$

We give several examples that show the applicability of Theorem 3.5.

Example 3.6. In [89], the study of the absolute global stability of the positive equilibrium of a commodity market model governed by a delay differential equation was reduced to the global stability of the positive equilibrium of the following difference equation:

$$x_{n+1} = \left(-b + \frac{a(d + x_n^m)}{cx_n^m}\right)^{1/k} =: F(x_n), \tag{3.7}$$

where $a, b, c, d, m > 0$, $k \geq 1$, and $bc < a$. The map $F : (0,\infty) \to (0,\infty)$ is decreasing, and [89, Lemma 2.5] ensures that $(SF)(x) < 0$ holds for all $x > 0$ if and only if $m > k$.

However, F satisfies (3.3) also in the case $m \leq k$, so a direct application of Theorem 3.5 proves that the local asymptotic stability of the equilibrium implies its global stability. Indeed, direct computations lead to

$$x^2\left(2(SF)(x) + \left(\frac{F'(x)}{F(x)}\right)^2\right) - 1 = -\frac{(a-bc)m^2x^m(2ad + (a-bc)x^m)}{(ad + (a-bc)x^m)^2} < 0,$$

since, by hypothesis, $a - bc > 0$. In [89], the case $m \leq k$ was solved using the Coppel-type result in Theorem 1.49.

The previous example corresponds to a monotone map F. The following one involves a unimodal map.

Example 3.7. Consider $F(x) = \frac{\beta}{a}\sqrt{x}e^{-x}$, which is a particular case of (3.5) with $\gamma = 1/2$ and $\delta = 1$. We recall that SF does not have constant sign on $(0,\infty)$ (see [81]). However,

$$x^2\left(2(SF)(x) + \left(\frac{F'(x)}{F(x)}\right)^2\right) = \frac{1-8x}{(1-2x)^2} < 1, \quad \forall x > 0,\ x \neq 1/2.$$

Hence, Theorem 3.5 guarantees that the condition $\frac{\beta^2}{a^2} \leq 3e^3/2$ for the local asymptotic stability of the positive equilibrium implies its global asymptotic stability.

An important remark is that, for some difference equations, both Theorem 3.1 and Theorem 3.5 can be applied, as it happens with the γ-Ricker difference equation (compare Examples 3.3 and 3.7). In these cases, Theorem 3.1 is usually easier to use. An additional example showing that Theorem 3.1 works in some situations which do not fall into the scope of Theorem 3.5 is the following one.

Example 3.8. Consider the generalised gamma-Ricker difference equation (see, e.g., [78] for the case $\gamma = 1$):

$$x_{n+1} = x_n^\gamma \left(\alpha + (1-\alpha)e^{r(1-x_n)}\right) =: x_n^\gamma H(x_n), \quad n \in \mathbb{Z}_+, \tag{3.8}$$

where $r > 0$, $0 \leq \gamma \leq 1$, $0 \leq \alpha < 1$. Since the map $F(x) = x^\gamma H(x)$ can have two critical points, Theorem 3.5 does not always apply. However, one can check that

$$2SH(x) + \left(\frac{H'(x)}{H(x)}\right)^2 = -r^2 + r^2 \left(\frac{(1-\alpha)e^{r(1-x)}}{\alpha + (1-\alpha)e^{r(1-x)}}\right)^2 \leq 0, \quad \forall x > 0,$$

and therefore (3.3) holds.

3.4 Proofs

The main ingredient to prove Theorem 3.1 is the following generalisation of Theorem 2.3 in [78], devoted to the particular case $\gamma = 1$.

Theorem 3.9. *Assume that $0 < \gamma \leq 1$ and that there exist constants $\alpha > 0$, $k \geq 0$ such that the continuous function $G : \mathbb{R} \to \mathbb{R}$ satisfies*

$$R_{\alpha,k}(y) < G(y) < 0\,,\ \forall\, y > 0 \quad \textit{and} \quad 0 < G(y) < R_{\alpha,k}(y)\,,\ \forall\, y \in (-1/k, 0)\,, \tag{3.9}$$

where $R_{\alpha,k}(y) = -\alpha y/(1+ky)$, and in case $k = 0$, by $-1/k$ we mean $-\infty$.

If $\gamma - \alpha \geq -1$, then the origin is GAS for equation

$$y_{n+1} = \gamma y_n + G(y_n) =: T(y_n). \tag{3.10}$$

Proof. Assume first that $k > 0$. This case will be proven in several steps that are organised in the following way. We start by highlighting that it is enough to prove the result for $k = 1$. Afterwards, for such case, we will check that the map on the right-hand side of (3.10) is enveloped, in the sense of Theorem (1.51), by another map (involving $R_{\alpha,1}$) which has a globally attracting equilibrium.

Initially, we show why we only need to consider $k = 1$. Take the change of variables $s = ky$. In fact, $H(y) := ky$ stands for a topological conjugacy between (3.10) and

$$s_{n+1} = \gamma s_n + k\, G\left(\frac{s_n}{k}\right) =: T_*(s_n), \tag{3.11}$$

which can be proven by checking $T_* = H \circ T \circ H^{-1}$. Thus, bearing in mind the thesis in the statement and by virtue of Proposition 1.31, we can study equation (3.11). Moreover, $G_*(s) := k\, G\left(\frac{s}{k}\right)$ satisfies (3.9) with $k = 1$:

$$\frac{-\alpha y}{1+ky} = R_{\alpha,k}(y) < G(y) < 0,\ \forall y > 0 \implies \frac{-\alpha s}{1+s} < kG\left(\frac{s}{k}\right) < 0,\ \forall s > 0$$
$$\iff R_{\alpha,1}(s) < G_*(s) < 0,\ \forall s > 0,$$

and

$$\frac{-\alpha y}{1+ky} = R_{\alpha,k}(y) > G(y) > 0,\ \forall y \in (-1/k, 0) \implies \frac{-\alpha s}{1+s} > kG\left(\frac{s}{k}\right) > 0,\ \forall s \in (-1,0)$$
$$\iff R_{\alpha,1}(s) > G_*(s) > 0,\ \forall s \in (-1,0).$$

Hence, having checked that proving the result for $k = 1$ is sufficient to conclude the proof for $k > 0$, we will continue with that special case. Next, we consider the auxiliary equation

$$s_{n+1} = \gamma s_n + R_{\alpha,1}(s_n) := Q(s_n), \tag{3.12}$$

where the map $Q(s) = \gamma s - \alpha s/(1+s)$ is defined for all $s > -1$. It is easy to check that Q has a unique critical point $z = -1 + \sqrt{\alpha/\gamma}$, at which it attains a global minimum. To ensure that (3.12) is well defined, we have to prove that $\operatorname{Im} Q \in (-1, \infty)$. After some computations, we have

$$Q(z) = -\left(\sqrt{\alpha} - \sqrt{\gamma}\right)^2.$$

Since, by hypothesis, $\gamma - \alpha \geq -1$, we get

$$1 + \sqrt{\gamma} > \sqrt{1+\gamma} \geq \sqrt{\alpha} \Longrightarrow \sqrt{\alpha} - \sqrt{\gamma} < 1 \Longrightarrow Q(z) > -1,$$

where we have used that $1 + \sqrt{x} > \sqrt{1+x}$ for all $x \in (0, 1]$.

Next, we have

$$SQ(x) = \frac{-6\alpha\gamma}{(\alpha - \gamma(1+x)^2)^2} < 0, \quad \forall x \neq z.$$

Since $Q'(0) = \gamma - \alpha$, then $-1 \leq Q'(0) < 1$ and, therefore, Theorem 1.55 ensures that the origin is a global attractor for equation (3.12).

Now, relations in (3.9) with $k = 1$ imply that

$$Q(s) = \gamma s + R_{\alpha,1}(s) < \gamma s + G_*(y) < \gamma s \leq s, \quad \forall\, s > 0,$$
$$s \leq \gamma s < \gamma s + G_*(s) < \gamma s + R_{\alpha,1}(s) = Q(s), \quad \forall s \in (-1, 0).$$

Then, as anticipated, the result in case $k = 1$ follows from the enveloping-type result of Theorem 1.51 and, via the previously-stated topological conjugacy, we can also conclude the proof for the case $k > 0$.

Next, we consider the remaining case of $k = 0$. Condition (3.9) implies that

$$\begin{aligned} &(\gamma - \alpha)y < \gamma y + G(y) < \gamma y \leq y, \quad \forall y > 0, \\ &y \leq \gamma y < \gamma y + G(y) < (\gamma - \alpha)y, \quad \forall y < 0. \end{aligned} \tag{3.13}$$

Since $-1 \leq \gamma - \alpha < 1$, it is easy to check that the continuous map $T(y) = \gamma y + G(y)$ cannot have 2-periodic points different from 0. Indeed, the conditions in (3.13) imply that

$$|T^2(y)| \leq |T(y)| \leq |y|, \quad \forall y \in \mathbb{R},$$

so, the unique way for 2-periodic points to exist is that $|T(y)| = |y|$. As $T(y) \neq -y$, for every $y \neq 0$, we arrive to $T(y) = y$. Thus, the unique 2-periodic point is the origin, which is the unique equilibrium. Consequently, we have

$$\Delta_2^T(y) = (T^2(y) - y)y \neq 0, \text{ for every } y \neq 0. \tag{3.14}$$

In fact, notice that $|T^2(y)| < |y|$, for every $y \neq 0$ and thus,

$$T^2(y) > y, \text{ if } y < 0; \quad \text{and} \quad T^2(y) < y, \text{ if } y > 0.$$

Therefore, we obtain that $\Delta_2^T(y) < 0$, for every $y \neq 0$, and the origin is GAS for the difference equation (3.10) by virtue of Theorem 1.49. □

To prove Theorem 3.1, we use the change of variables that transforms (3.1) into (3.10), with $G(y) = -\ln(p^{\gamma-1}H(pe^{-y}))$. Actually, formula (3.3) comes from this change of variables, as the following result shows.

Proposition 3.10. *Assume that $0 \leq \gamma \leq 1$ and H satisfies (H). Let p be the unique positive equilibrium of* (3.1)*, and define*

$$G(y) = -\ln(p^{\gamma-1}H(pe^{-y})). \tag{3.15}$$

Then, condition (3.3) *is equivalent to $SG(y) < 0$ for all $y \in \mathbb{R}$.*

Proof. Firstly, notice that

$$G(y) = -\ln(p^{\gamma-1}) - \ln(H(pe^{-y})).$$

Since the Schwarzian derivative is not affected by affine transformations, SG is equal to the Schwarzian derivative of $L \circ H \circ E$, where $L(y) := \ln(y)$ and $E(y) = pe^{-y}$. We use the formula (1.35) for the Schwarzian derivative of the composition of two maps

$$\begin{aligned} SG(y) &= S[L \circ H](E(y))\,(-pe^{-y})^2 + SE(y) \\ &= [SL(H(E(y)))\,(H'(E(y)))^2 + SH(E(y))](pe^{-y})^2 + SE(y). \end{aligned}$$

Now, taking into account that $SE(y) = -1/2$ and $SL(y) = 1/(2y^2)$, we get

$$SG(y) = \left[\frac{1}{2}\left(\frac{H'(pe^{-y})}{H(pe^{-y})}\right)^2 + SH(pe^{-y})\right](pe^{-y})^2 - \frac{1}{2}.$$

Hence, writing $pe^{-y} = x$, we have:

$$SG(y) < 0,\ \forall y \in \mathbb{R} \Longleftrightarrow \frac{-1}{2} + x^2\left[SH(x) + \frac{1}{2}\left(\frac{H'(x)}{H(x)}\right)^2\right] < 0,\ \forall x > 0,$$

and the last inequality is equivalent to (3.3). □

Now we are in a position to prove Theorems 3.1 and 3.5.

Proof of Theorem 3.1. The change of variables $y = -\ln(x/p)$ transforms (3.1) into (3.10), with $G(y) = -\ln(p^{\gamma-1}H(pe^{-y}))$. In fact, the function $\Lambda(x) := -\ln(x/p)$ represents a topological conjugacy between F and T because $T = \Lambda \circ F \circ \Lambda^{-1}$.

Since H is decreasing, it is clear that G is decreasing from

$$G'(y) = \frac{pe^{-y}H'(pe^{-y})}{H(pe^{-y})}. \tag{3.16}$$

Notice that, in particular, the following relation holds:

$$G'(0) = \frac{pH'(p)}{H(p)} = \frac{pH'(p)}{p^{1-\gamma}} = p^{\gamma}H'(p).$$

Furthermore, by Proposition 3.10, condition (3.3) implies that $SG(y) < 0$, for all $y \in \mathbb{R}$. In order to apply Theorem 3.9, we distinguish three cases:

- Case 1: If $G''(0) = 0$, then $SG(0) = G'''(0)/G'(0) < 0$, and therefore $G'''(0) > 0$. The negativity of the Schwarzian derivative, which is well-defined on the whole $\mathbb{R}$ due to $G' < 0$, ensures that 0 is the unique inflection point of G (see Proposition 1.54); moreover, G' decreases on $(-\infty, 0)$ and increases on $(0, \infty)$. Then, it follows that (3.9) holds with
$$\alpha = -G'(0) > 0 \quad \text{and} \quad k = 0.$$
Hence, this case is proven by bearing in mind that
$$-1 \le \gamma - \alpha = \gamma + G'(0) = \gamma + p^{\gamma} H'(p) = F'(p).$$

- Case 2: If $G''(0) > 0$, then Lemma 2.1 in [85] implies that (3.9) holds with
$$\alpha = -G'(0) > 0 \quad \text{and} \quad k = -\frac{G''(0)}{2\,G'(0)} > 0.$$
Thus, the result when $G''(0) > 0$ follows from Theorem 3.9, again by using the condition $-1 \le \gamma - \alpha = F'(p)$.

- Case 3: If $G''(0) < 0$, then the change of variables $z = -y$ transforms (3.10) into
$$z_{n+1} = \gamma z_n + \hat{G}(z_n), \tag{3.17}$$
with $\hat{G}(z) = -G(-z)$. Then, Lemma 2.1 in [85] can be applied to the map $\hat{G}$ (see [78, Corollary 2.7] for more details). In this way, Case 3 is reduced to Case 2.

All the cases have been checked and the proof is complete. □

Finally, the proof of the case $\gamma = 0$ (and of Theorem 3.5) follows easily and in a similar manner to the case $\gamma > 0$. It is once more worth highlighting that $F = H$ if $\gamma = 0$ and F would be decreasing. The main difference is that we now admit the function F to have at most one critical point.

Proof of Theorem 3.5. The change of variables $y = -\ln(x/p)$ transforms (3.6) into
$$y_{n+1} = G(y_n), \tag{3.18}$$
with $G(y) = -\ln(p^{-1}H(pe^{-y}))$. It is easy to check that G is decreasing or unimodal, which can also be derived from (3.16), a relation independent of $\gamma \in [0, 1]$. In this particular case, we have that $G'(0) = H'(p) = F'(p)$. Additionally, (3.3) ensures that $SG(y) < 0$ for all real y, only excluding $\bar{c} = \ln(p/c)$ in case H had a unique critical value c. Hence, the second statement in Theorem 1.55 ensures that 0 is a global attractor for equation (3.18) if $-1 \le G'(0) < 0$, which is equivalent to $-1 \le H'(p) < 0$.

If $H'(p) \geq 0$, there is no need to use the new change of variables $\Lambda(x) = -\ln(x/p)$, but only apply the first part of Theorem 1.55 to H to finally deduce the global stability of p. Notice that the hypothesis in the above-mentioned theorem regarding the negative feedback condition $\Delta_1^H(x) < 0$ if $x \neq p$ holds due to $H(x) = F(x) > x$ if $x < p$, and $H(x) = F(x) < x$ if $x > p$. □

3.5 The study of the gamma-Mackey-Glass model

In the previous sections, we have provided the machinery to handle the global stability issue for the γ-version of the model by Maynard Smith and Slatkin, which we restate here as

$$x_{n+1} = \frac{\frac{\beta}{a} x_n^\gamma}{1 + \delta x_n^m}, \tag{3.19}$$

where $a, \beta, \delta, m > 0$ and $0 \leq \gamma \leq 1$. Thus, it is an appropriate place to recall the case of Chapter 2 that remained unstudied, i.e., the one that concerns the gamma-Mackey-Glass DDE

$$x'(t) = -ax(t) + \frac{\beta x^\gamma(t - \tilde{\tau}(t))}{1 + \delta x^m(t - \tilde{\tau}(t))}. \tag{3.20}$$

To derive some relevant features of the asymptotic behaviour of the solutions of (3.20) we use the information that comes from the map that generates equation (3.19). With this aim, we write the following result, which gathers the main features of the γ-version of the map given by Maynard Smith and Slatkin, in an analogous way to several propositions in Chapter 2. In particular, it can be interpreted as an extension of Proposition 2.25, which was related to the particular case $m = 1$.

Proposition 3.11. *The map $F : (0, \infty) \to (0, \infty)$ defined by*

$$F(x) = \frac{\beta x^\gamma}{a(1 + \delta x^m)}; \quad a, \beta, \delta, m > 0; \ \gamma \in [0, 1],$$

satisfies the following properties:

(i) *F is of class $\mathcal{C}^\infty$ and*

$$\lim_{x\to\infty} \frac{F(x)}{x} < 1 < \lim_{x\to 0^+} \frac{F(x)}{x}, \quad 0 \leq \gamma < 1,$$

whereas the last condition also holds for $\gamma = 1$ if $\beta > a$.

(ii) *If $m \leq \gamma$, F is an increasing M-map. Alternatively, if $0 < \gamma < m$, then F is a U-map, with a unique critical point at*

$$c = \left(\frac{\gamma/\delta}{m-\gamma}\right)^{\frac{1}{m}},$$

where F attains its global maximum. Otherwise, if $\gamma = 0$, F is a decreasing M-map.

(iii) *There is a unique $p = p(\gamma) \in (0, \infty)$ such that $F(p) = p$, provided $\gamma \in [0, 1)$ or both $\gamma = 1$ and $\beta > a$. In such cases, the root is given by the unique solution of*

$$p^{1-\gamma}(1 + \delta p^m) = \frac{\beta}{a}, \tag{3.21}$$

and $(F(x) - x)(x - p) < 0$ holds for every $x > 0$, $x \neq p$.

(iv) *H satisfies* (3.3)*, where H is such that $F(x) = x^\gamma H(x)$.*

(v) *$F'(p) = \gamma - \frac{m\delta p^m}{1+\delta p^m}$.*

(vi) *For any $\kappa < 1$, the inequalities $\kappa \leq F'(p) < 1$ hold if and only if any of the following two conditions is satisfied:*

$$\begin{aligned} &m \leq \gamma - \kappa; \\ &m > \gamma - \kappa \ \textit{and} \quad \frac{\beta}{a} \leq \frac{m}{m + \kappa - \gamma}\left(\frac{\gamma - \kappa}{\delta(m + \kappa - \gamma)}\right)^{\frac{1-\gamma}{m}} =: T_{\kappa,\delta,m}(\gamma). \end{aligned} \tag{3.22}$$

Proof. The function

$$\frac{F(x)}{x} = \frac{\beta}{ax^{1-\gamma}(1 + \delta x^m)}$$

tends to 0 as $x \to \infty$. Moreover, it tends to ∞ as $x \to 0^+$, provided $\gamma \in [0, 1)$, while assuming $\gamma = 1$ and $\beta > a$ yields that it tends to $\beta/a (> 1)$ as $x \to 0^+$.

The derivative of F takes the form

$$F'(x) = \frac{\beta x^{\gamma-1}}{a(1 + \delta x^m)^2}[\gamma + \delta x^m(\gamma - m)].$$

Consequently, if $\gamma \geq m$, then $F'(x) > 0$ for all $x > 0$. In case $0 < \gamma < m$, then there exists a unique positive root c of $\gamma + \delta c^m(\gamma - m) = 0$, which is a critical point of F; in fact, the function F is increasing on $(0, c)$ and decreasing on (c, ∞). Finally, $F'(x) < 0$ on $(0, \infty)$ if $\gamma = 0$. This proves part (ii).

Equation (3.19) is of the form of (3.1) with $H(x) = \frac{\beta}{a(1+\delta x^m)}$, a decreasing function. From Theorem 2.1 and Remark 2.4, we know that the unique equilibrium p of the difference equation (3.19) is given by the equation $H(p) = p^{1-\gamma}$, an equivalent form of (3.21). Furthermore, from the existence of a unique equilibrium, the graph of F can only intersect the one of the identity function once. From the behaviour of $F(x)/x$ for x close to 0 and for larger values of x (Assertion (i)), one obtains that $\Delta_1^F(x) = (F(x) - x)(x - p) < 0$ on $(0, \infty) \setminus \{p\}$. This ends the proof of (iii).

Regarding (iv), we refer to the comments given in Example 3.4.

Assertion (v) directly follows from Theorem 2.1. Indeed,

$$F'(p) = \gamma + p^\gamma H'(p) = \gamma - p^\gamma \frac{\beta}{a} \frac{m\delta p^{m-1}}{(1+\delta p^m)^2} = \gamma - \left[\frac{\beta p^{\gamma-1}}{a(1+\delta p^m)}\right] \frac{m\delta p^m}{1+\delta p^m}$$

$$= \gamma - p^{\gamma-1} H(p) \frac{m\delta p^m}{1+\delta p^m} = \gamma - \frac{m\delta p^m}{1+\delta p^m}.$$

Now, from (v) and Remark 2.2, condition $\kappa \leq F'(p) < 1$ is equivalent to

$$W(p) := \frac{m\delta p^m}{1+\delta p^m} \leq \gamma - \kappa. \tag{3.23}$$

Condition (3.23) holds if $m \leq \gamma - \kappa$ since, in such case,

$$W(x) < m \leq \gamma - \kappa, \quad \text{for every } x \in (0, \infty).$$

Otherwise, assume that $m > \gamma - \kappa$. We will obtain a condition that avoids the use of p via W^{-1} (notice that W is also increasing on $(0, \infty)$). Condition (3.23) is equivalent to

$$\delta p^m (m + \kappa - \gamma) \leq \gamma - \kappa \tag{3.24}$$

due to $1 + \delta p^m > 0$. Moreover, since in this case $0 < \gamma - \kappa < m$, one can get an equivalent form of (3.24) by writing

$$p \leq \left(\frac{\gamma - \kappa}{\delta(m + \kappa - \gamma)}\right)^{1/m} =: \eta. \tag{3.25}$$

By using (iii), we know that $p \leq \eta$ if and only if $\eta^\gamma H(\eta) = F(\eta) \leq \eta$, so condition (3.25) holds if and only if

$$\frac{\beta(m + \kappa - \gamma)}{am} = H\left(\left(\frac{\gamma - \kappa}{\delta(m + \kappa - \gamma)}\right)^{1/m}\right) \leq \left(\frac{\gamma - \kappa}{\delta(m + \kappa - \gamma)}\right)^{(1-\gamma)/m},$$

which is equivalent to (3.22). $\square$

Theorem 3.1 ensures that p is globally asymptotically stable if the local condition (3.4) holds. Notice that in this case (3.4) is equivalent to (3.22) with $\kappa = -1$.

Theorem 3.12. *Let $a, \beta, \delta, m > 0$, $\gamma \in [0, 1)$ or both $\gamma = 1$ and $\beta > a$. Denote by p the unique positive fixed point of the γ-version of the Maynard Smith and Slatkin map. Then p is GAS for the difference equation* (3.19) *if and only if*

$$m \leq 1 + \gamma \quad \textit{or} \quad \left[m > 1 + \gamma, \quad \frac{\beta}{a} \leq \frac{m}{m - 1 - \gamma} \left(\frac{\gamma + 1}{\delta(m - 1 - \gamma)} \right)^{\frac{1-\gamma}{m}} \right]. \tag{3.26}$$

Thus, Theorem 3.12 gives an absolute stability result for the γ-version of the classical Mackey–Glass DDE from [96].

Theorem 3.13. *Assume that (A) holds together with $\beta, \delta, m > 0$. Denote by $x(t; (0, \phi))$ the unique solution of* (3.20) *through* $(0, \phi)$*, with $\phi \in C_{(0,\infty)}$, and by p the unique positive equilibrium of equation* (3.19)*. If*

$$m \leq 1 + \gamma \quad \textit{or} \quad \left[m > 1 + \gamma, \quad \frac{\beta}{a} \leq \frac{m}{m - 1 - \gamma} \left(\frac{\gamma + 1}{\delta(m - 1 - \gamma)} \right)^{\frac{1-\gamma}{m}} \right], \tag{3.27}$$

then $\lim_{t \to \infty} x(t; (0, \phi)) = p$, for every $\phi \in C_{(0,\infty)}$. Moreover, condition (3.27) *is the sharpest absolute stability condition. Finally, the thesis is also valid by taking $\gamma = 0$ or both $\gamma = 1$ and $\beta > a$.*

Proof. It is just a direct application of Theorem 1.69 (or the more general Theorem 2.32) via the information regarding the long-term behaviour of the solutions of (3.19) that comes from Theorem 3.12. □

Remark 3.14. Notice that in Theorem 2.28, a result related to the particular case of (3.19) with $m = 1$, we have not needed to impose any further constraint on the parameters to ensure that p is GAS for (3.20). In particular, this is consistent with what has been obtained via condition (3.27). In fact, the map generating (3.19) satisfies condition (3.3); hence, by using Theorem 3.1, the global asymptotic stability of p for (3.19) follows from its local asymptotic stability: set $\kappa = -1$ in (3.22) and notice that (3.27) holds since

$$m = 1 \leq \gamma + 1 = \gamma - \kappa,$$

for any $\gamma \in [0, 1]$, without the need of any additional hypotheses on the parameters.

Remark 3.15. Additionally, if $\tilde{\tau}(t) = \tau > 0$, for every $t \in \mathbb{R}$, then p is LAS provided condition (3.22) is satisfied with $\kappa = \kappa_3$, where

$$\kappa_3 = -\sqrt{1 + \frac{s^2}{a^2\tau^2}}, \tag{3.28}$$

with s being the unique root of $\frac{-x}{\tan(x)} = a\tau$ on $(\frac{\pi}{2}, \pi)$.

In [86], the authors gave sharp global asymptotic stability conditions for the gamma-Mackey-Glass DDE with $\gamma = 0$ and $\gamma = 1$ (bounds related with ρ_2 or ξ_2), extending the ones of [44], which deal with delay-independent conditions and bounds related with ρ_1 and Remark 1.62. This comes from the fact that the feedback f is an S-map (recall the ending part of Subsection 1.3.3). Moreover, if the delay is constant, [44] also includes the conditions of local stability that involve ρ_3 or ξ_3 for $\gamma \in \{0, 1\}$. In that line, it would be interesting to sew both cases in terms of the different estimates by letting $\gamma \in (0, 1)$, as we have done in Chapter 2.

Nonetheless, while the following Remarks 3.16 and 3.17, respectively provide a link when it comes to the estimates of absolute stability and local stability, the issue with delay-dependent stability conditions, e.g., the ones that involve ξ_1 or ξ_2, seems not direct since the corresponding results assume some kind of negativity on the Schwarzian derivative.

Notice that in both references [44, 86] the authors consider $\delta = 1$, but such difference is just a matter of straightforward adaptations.

Remark 3.16 (Absolute stability)**.** The limit form of condition (3.27) with $\gamma = 0$,

$$m \leq 1 \quad \text{or} \quad \left[m > 1 \quad \text{and} \quad \frac{\beta}{\alpha} \leq \frac{m}{m-1} \left(\frac{1}{\delta(m-1)} \right)^{\frac{1}{m}} \right],$$

coincides with the one given in [44] for $\delta = 1$. Besides, the limit form of (3.27) as $\gamma \to 1^-$ reads as

$$m \leq 2 \quad \text{or} \quad \left[m > 2 \quad \text{and} \quad \frac{\beta}{\alpha} \leq \frac{m}{m-2} \right],$$

which resembles the one recalled in, e.g., [44, Page 5], for $\delta = 1$. In fact, the latter condition does not depend on δ.

Remark 3.17 (Local asymptotic stability)**.** Notwithstanding the issues with the estimates involving ρ_1 and ρ_2, one can link the cases $\gamma = 0$ and $\gamma = 1$ if the delay is constant. In fact, by taking the unique $s \in (\frac{\pi}{2}, \pi)$ such that $a\tau = \frac{-s}{\tan s}$, we can rewrite (3.28) as

$$\kappa_3 = \frac{-1}{|\cos(s)|} = \frac{1}{\cos(s)}.$$

Therefore, the limit condition of (3.22) with $\kappa = \kappa_3$ and $\gamma = 0$ is

$$m \leq \frac{-1}{\cos(s)} \quad \text{or} \quad \left[m > \frac{-1}{\cos(s)} \quad \text{and} \quad \frac{\beta}{a} \leq \frac{m\cos(s)}{m\cos(s)+1} \left(\frac{-1}{\delta(1+m\cos(s))} \right)^{\frac{1}{m}} \right],$$

which is the one in [44, Remark 3] ($\delta = 1$). Alternatively, the limit condition of (3.22) with $\kappa = \kappa_3$ and $\gamma = 1$ is

$$m \leq 1 - \frac{1}{\cos(s)} \quad \text{or} \quad \left[m > 1 - \frac{1}{\cos(s)} \quad \text{and} \quad \frac{\beta}{a} \leq \frac{m\cos(s)}{1+(m-1)\cos(s)} \right],$$

which also coincides with the corresponding one in [44, Remark 4]. Once more, the limit case for $\gamma = 1$ does not depend on δ.

3.6 Discussion

There are two issues that we would like to emphasise. First, the family (3.1) of discrete gamma-models, as discussed previously, provide great flexibility to fit population data and are also suitable for discrete-time models in Economics; moreover, they can display a very rich dynamics (see, e.g., [27, 81, 82]). Since the model proposed by Maynard Smith and Slatkin, related to $\gamma = 1$ has been already proved to be very flexible in population dynamics [13], we think that (3.19) has a great potential use in applications.

Second, we have been able to prove a global stability result valid for many different choices of the map H, including, for the case $\gamma = 1$, many classical models, as the one by Maynard Smith and Slatkin. In particular, it seems that the issue of global stability for its γ-version, the difference equation (3.19), has been studied for the first time in [83], whose contents have been developed in the current chapter; other properties have been stated in [99] in the framework of delay differential equations. Our examples show that the conditions of Theorem 3.1 are easily verifiable. Thus, although Theorem 3.1 was already known in the case $\gamma = 1$, the new formula (3.3) and its equivalent form (3.2) make the result more friendly. Even in the case $\gamma = 0$, this formula provides a useful tool for proving the folklore statement 'LAS implies GAS' in some cases for which the Schwarzian derivative does not have constant sign and an Allwright-Singer-type result does not apply. Finally, we include the expression of (3.3) without invoking the Schwarzian derivative:

$$x^2 \left(2\frac{F'''(x)}{F'(x)} - 3\left(\frac{F''(x)}{F'(x)}\right)^2 + \left(\frac{F'(x)}{F(x)}\right)^2 \right) < 1,$$

for all $x > 0$ such that $F'(x) \neq 0$.

Chapter 4

The role of multidimensional difference equations

We provide sufficient conditions for a certain type of systems of delay differential equations to have a globally attracting equilibrium. The principal idea is based on a particular type of global attractivity for difference equations in terms of nested, convex and compact sets. We prove that the solutions of the corresponding system of DDEs inherit the convergence to the equilibrium from an associated discrete-time semiflow and thus, we provide a contribution to the extension to the multidimensional case of the ideas recalled in Section 1.3. This chapter reproduces the contents of the work by the author of this thesis (S. Buedo-Fernández[1]) [18] with some updates. The details of such work are also provided below.

S. Buedo-Fernández. Global attraction in a system of delay differential equations via compact and convex sets. *Discrete and Continuous Dynamical Systems Series B* 20, 3171–3181 (2020). American Institute of Mathematical Sciences, ISSN 1531-3492, eISSN 1553-524X.

4.1 Introduction

In previous chapters, we have discussed how to obtain global attractivity conditions for a scalar delay differential equation with both linear destruction and delayed production terms, as in

$$x'(t) = -x(t) + F(x(t-\tau)). \tag{4.1}$$

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Although the study of the long-term behaviour of the solutions of equation (4.1) may become a difficult task, since they are related to dynamical systems with infinite-dimensional phase space, one is able to provide some relevant information. For instance, local stability of its equilibria can be studied through the location of the roots of a certain characteristic equation, as we have seen in Subsection 1.3.2.

However, global dynamics require different and deeper techniques. One of the main ideas shown in Subsection 1.3.3 is to work with the corresponding difference equation

$$x(n+1) = F(x(n)), \tag{4.2}$$

where the existence of a unique equilibrium is assumed. We have adopted the notation in (4.2) just for the purposes of the current chapter[2]. The asymptotic behaviour of the solutions of the difference equation (4.2) influences that of the solutions of the DDE (4.1), as shown in Theorem 1.33. Hence, techniques coming from discrete dynamical systems can be used to study equation (4.1). Equation (4.2) 'shares' the equilibrium with (4.1). Recall from Section 1.2 that, although the usual phase space for (4.1) is the set of continuous functions defined on $[-\tau, 0]$ into $\mathbb{R}$, an equilibrium for (4.1) is a function which is constant on $[-\tau, 0]$ and, therefore, it can be identified with its real value.

In particular, in Corollary 1.48, some key findings from [65, 98] have been used: if F is continuous on a forward invariant interval and the equilibrium is globally attracting for (4.2), then it is globally attracting for (4.1) too. In fact, the convergence to the equilibrium in (4.2) is related to the existence of a family of nested compact intervals containing the equilibrium in its interior and such that the image by F of any set is included in the next one (see Lemma 2.31). A key idea to prove that the solutions of (4.1) inherit the behaviour of the ones of (4.2) is the following: the borders of such nested compact intervals act as 'control points' that the solution of (4.1) must consecutively pass towards the equilibrium. Therefore, the problem of finding out if a delay differential equation has a globally attracting equilibrium is moved to the analogous problem with difference equations, where many results are available (see our Section 1.3.3 or [33, 78] and their references).

It is also natural to wonder if the same relation between discrete-time and continuous-time dynamics holds for the multidimensional case. In particular, we consider a system of delay differential equations of the form

$$x_i'(t) = -x_i(t) + F_i(x_1(t-\tau_{i1}), \ldots, x_s(t-\tau_{is})), \quad i = 1, \ldots, s, \tag{4.3}$$

where s is a positive integer. The latter equation is a possible generalisation of (4.1) while (4.2) is now written with $F \equiv (F_1, \ldots, F_s)$ mapping a subset of $\mathbb{R}^s$ into itself. In [90, 93],

[2] The reader should be aware that, otherwise, three colliding notations would have coexisted: the terms of a sequence x_n, the segments x_t and the coordinate x_i. Regarding the last two items, we include some additional comments in order to avoid misunderstanding.

Liz and Ruiz-Herrera provide some results concerning the present problem by an extension of the ideas from the one-dimensional case to intervals in $\mathbb{R}^s$. They proved that, if for every compact set in a rectangular-type phase space, there exists a family of nested compact intervals of $\mathbb{R}^s$ satisfying certain hypotheses, including that F maps each interval into the next one, then the equilibrium of (4.3) is a global attractor. In such case, the equilibrium of the difference equation is called a 'strong attractor', and as its name evokes, it is a condition strictly stronger than being a global attractor, provided $s > 1$ [90, 91]. Finally, in some cases, the existence of multidimensional strong attractors can be deduced from the study of global attractivity for related scalar difference equations.

Another interesting approach within the multidimensional study is given in [34], in which a 'dominance' condition in the coordinates of F is given to ensure that (4.3) has a globally attracting equilibrium. This technique is utilised to study some well-known models as Nicholson's blowflies model with patch structure.

We would also like to mention that this kind of link between differential equations and difference equations with respect to the long-term behaviour also finds applications in the study of certain partial differential equations [141].

The main goal of our study is to obtain some weaker conditions regarding the global convergence to the equilibrium of the difference equation associated with (4.3) while keeping the inherited global attractivity of the equilibrium of (4.3). The ideas in [90, 93] can be adapted in order to use other 'nested geometries' instead of intervals in $\mathbb{R}^s$, such as usual balls, p-norm balls or even convex and compact sets.

In Section 4.2, we set some notation and definitions and include the discussion about the new concepts, reaching the statement of the main result of this chapter, Theorem 4.5. We also provide an example in which the results of [90] do not apply but Theorem 4.5 does. In Section 4.3, we write several technical lemmas concerning convex and compact sets, which become an important tool to prove the main result (Section 4.4).

4.2 A key concept regarding global attractivity

In this section, we introduce the main concept of this chapter, which is related to attractivity for discrete-time semiflows. Then, we compare such concept with others previously introduced, recovering several ideas in [90, Section 2]. Finally, we state Theorem 4.5, which shows the role that the above-mentioned concept plays in the relation between attractivity of an equilibrium in a DDE and in a difference equation.

In the following, we will assume that $s \in \mathbb{N}$ and $D \subset \mathbb{R}^s$ is an open convex set. We also consider the system of delay differential equations

$$x_i'(t) = -x_i(t) + F_i(x_1(t-\tau_{i1}), \dots, x_s(t-\tau_{is})), \quad i = 1, \dots, s, \tag{4.4}$$

where $\tau_{ik} \geq 0$ and $F_i : D \to \mathbb{R}$ is locally Lipschitzian, for any $i, k \in \{1, \ldots, s\}$. We define $\tau := \max\{\tau_{ik} : i, k \in \{1, \ldots, s\}\}$ and the function $F : D \to \mathbb{R}^s$ by

$$F(x_1, \ldots, x_s) := (F_1(x_1, \ldots, x_s), \ldots, F_s(x_1, \ldots, x_s)). \tag{4.5}$$

Since we are interested in working with the discrete-time semiflow generated by (4.5), we assume that $F(D) \subset D$. Notice that, provided s is fixed, the set C introduced in Section 1.1 is now related with $\mathbb{R}^s$-valued continuous functions defined on a compact interval, that is,

$$C = \mathcal{C}([-\tau, 0], \mathbb{R}^s) = \{\phi : [-\tau, 0] \to \mathbb{R}^s : \phi \text{ is continuous}\}.$$

In particular, C_D denotes the subset of C whose elements ϕ satisfy $\operatorname{Im}\phi \subset D$. In accordance with the notation introduced at the beginning of this work, for each $\phi \in C_D$, we say that the function

$$x(t; \phi) = (x_1(t; \phi), \ldots, x_s(t; \phi))$$

is the unique solution of (4.4) through the initial condition ϕ and its respective segments are referred under the notation x_t^ϕ (see the footnote on page 10), that is, the element of C defined by

$$x_t^\phi(\theta) = x(t + \theta; \phi), \quad \theta \in [-\tau, 0].$$

The latter alternative notation helps us when it comes to distinguishing segments and coordinates. The solution $x(t; \phi)$ is assumed to be defined on its maximal interval $[-\tau, t_\phi)$. We highlight that C_D is an open subset of C and that the right-hand side of (4.4) is such that we can apply the existence and uniqueness results from Section 1.1. Moreover, analogously to what was set in Section 1.3, we will say that equation (4.4) is the DDE corresponding to the difference equation

$$x(n + 1) = F(x(n)), \quad n \in \mathbb{Z}_+, \tag{4.6}$$

and vice versa. The dynamics of (4.4) involve a study in an infinite-dimensional phase space, whilst (4.6) does not. Bearing in mind this fact, the following question naturally arises: as in the particular case of a one-dimensional equation, do the dynamics of the general system of delay differential equations (4.4) show any particular feature of the dynamics of its corresponding difference equation (4.6)? Under some circumstances, the answer is affirmative.

Since we are looking for sufficient conditions to have a globally attracting equilibrium for (4.4), we will assume that (4.4) has a unique equilibrium, which will be called p. Since p will be the unique fixed point of F, equation (4.6) has only one equilibrium too.

In the scalar case ($s = 1$), Corollary 1.48 tells us that, if p is globally attracting for (4.6), then p is also globally attracting for (4.4). Therefore, our work can be focused on techniques to ensure global convergence to p in (4.6). For example, Theorem 1.49 uses the information of $F \circ F$, while Theorem 1.55 utilises some negativity hypotheses on the Schwarzian derivative of F.

The multidimensional case is more challenging. In [91], an example for $s = 2$ is provided to show that global convergence to p in (4.6) is not sufficient to guarantee the analogous property for (4.4). Indeed, Liz and Ruiz-Herrera [90] proposed a concept which recovers the principal needs of the proof in the scalar case. The key idea is that global attraction to p in the scalar case of (4.6) is equivalent to the existence of a family of compact intervals $I_n = [c_n, d_n]$, $n \in \mathbb{N}$, which are nested ($F(I_n) \subset I_{n+1} \subset Int(I_n)$) and shrink around the equilibrium ($\cap_{n=1}^{\infty} I_n = \{p\}$). Such concept is the following one.

Definition 4.1. [90, Definition 2.1] Assume that $D = (a_1, b_1) \times \cdots \times (a_s, b_s)$. An equilibrium $p \in D$ of the system (4.6) is a *strong attractor* in D if, for every compact set $K \subset D$, there exists a family of sets $\{I_n\}_{n\in\mathbb{N}}$, where every I_n is the product of s non-empty compact real intervals, satisfying that

(A1) $K \subset Int(I_1) \subset D$,

(A2) $F(I_n) \subset I_{n+1} \subset Int(I_n)$, $\forall n \in \mathbb{N}$,

(A3) $\displaystyle\bigcap_{n=1}^{\infty} I_n = \{p\}$.

Notice that we have previously recalled this concept in Definition 2.30 for the case $s = 1$ in order to study certain gamma-models. In fact, as commented above, the notions of strong attractor and global attractor coincide in the scalar case, a fact that we have highlighted in Lemma 2.31 (see the Appendix in [90] for a proof). Finally, under this concept, one can link the long-term dynamics in the multidimensional case for both equations (4.4) and (4.6).

Theorem 4.2. *[90, Theorem 2.5] Assume that* $D = (a_1, b_1) \times \cdots \times (a_s, b_s)$ *and that* $F : D \to D$ *is a locally Lipschitzian map. Let* p *be a strong attractor for* (4.6) *in* D*. Then, for each* $\phi \in C_D$*, the unique solution* $x(t;\phi)$ *of* (4.4) *through* ϕ *is defined for any* $t \geq 0$ *and* $\lim_{t\to\infty} x(t;\phi) = p$*.*

Sometimes, we refer to a strong attractor in the sense of [90] as an *interval-strong attractor* (I-strong attractor) to highlight the geometry involved in the convergence to p.

A careful analysis of the proof of Theorem 2.5 in [90] shows that more general sets can play the role of the intervals in $\mathbb{R}^s$. We introduce the following weaker concept, which becomes the key tool for the current part of the thesis.

Definition 4.3. Assume that $D \subset \mathbb{R}^s$ is an open convex set. An equilibrium $p \in D$ of (4.6) is a *CC-strong attractor* if, for every compact set $K \subset D$, there exists a family $\{K_n\}_{n\in\mathbb{N}}$ of compact and convex sets with non-empty interior, satisfying that

(C1) $K \subset Int(K_1) \subset D$,

(C2) $F(K_n) \subset K_{n+1} \subset Int(K_n)$, $\forall n \in \mathbb{N}$,

(C3) $\bigcap_{n=1}^{\infty} K_n = \{p\}$.

The name has been chosen to emphasise the important features of the family of subsets $\{K_n\}_{n\in\mathbb{N}}$ involved in the latter concept: compactness and convexity. Notice that convexity is a property that extends the case of intervals in $\mathbb{R}$ to sets in $\mathbb{R}^s$, $s > 1$. Moreover, it also generalises the extension via products of real intervals in [90]. In fact, the following example, somehow close to the one in [90, Appendix], shows that the new concept includes further cases of globally attracting equilibria.

Example 4.4. Let $D = \mathbb{R}^2$ and $H : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mapping associated to the rotation of $\frac{\pi}{4}$ radians, i.e.,

$$H(x, y) = \left(\frac{1}{\sqrt{2}}(x - y), \frac{1}{\sqrt{2}}(x + y)\right), \quad \forall (x, y) \in \mathbb{R}^2.$$

Fix $m \in (\frac{1}{\sqrt{2}}, 1)$ and let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mapping defined by

$$G(x, y) = (mx, my), \quad \forall (x, y) \in \mathbb{R}^2.$$

If we define $F := G \circ H$, then

$$F(x, y) = \left(\frac{m}{\sqrt{2}}(x - y), \frac{m}{\sqrt{2}}(x + y)\right), \quad \forall (x, y) \in \mathbb{R}^2.$$

First, we will prove that the origin is not a strong attractor for the system

$$(x(n + 1), y(n + 1)) = F(x(n), y(n)), \quad n \in \mathbb{Z}_+, \tag{4.7}$$

in $\mathbb{R}^2$. Take any compact set $K \subset \mathbb{R}^2$ and choose an arbitrary compact interval I_1 in $\mathbb{R}^2$ satisfying $K \subset Int(I_1)$, $0 \in Int(I_1)$. We will write $I_1 := [c_1, c_2] \times [d_1, d_2]$ for certain $c_1, d_1 < 0$ and $c_2, d_2 > 0$.

Now, consider the value $l = \min\{|c_1|, |c_2|, |d_1|, |d_2|\}$. We sketch the proof for $l = |d_1|$, yet the remaining cases would be similarly treated. Take the point $q \in \partial I_1$ related to an angle of $\frac{5\pi}{4} = \frac{3\pi}{2} - \frac{\pi}{4}$ with $(1,0)$, which is mapped by H into the negative y-semiaxis. It is clear that $\|H(q)\| = \|q\| = l\sqrt{2}$ (see Figure 4.1).

By using that $m \in (\frac{1}{\sqrt{2}}, 1)$, it is easy to see that $q \in I_1$, but $F(q) \notin I_1$ due to

$$\|F(q)\| = \|mH(q)\| = m\|H(q)\| = ml\sqrt{2} > l = |d_1|.$$

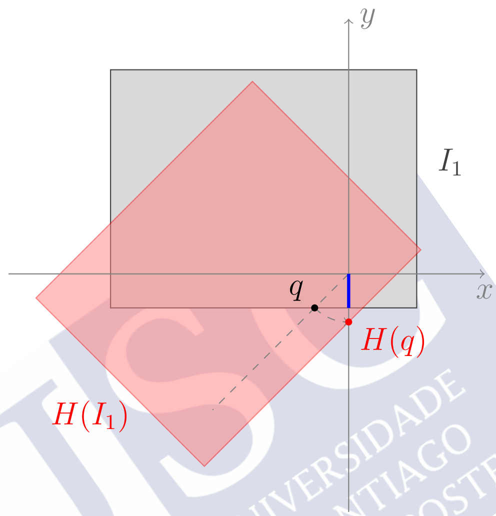


Figure 4.1: The crucial step to see why the origin is an equilibrium that is not a strong attractor. We represent the set $I_1 = [c_1, c_2] \times [d_1, d_2]$ (gray) and its image $H(I_1)$ (red). In case $l = |d_1|$ (see the blue thick segment), we choose q as the point in ∂I_1 that makes an angle of $\frac{5\pi}{4}$ with the positive x-semiaxis.

However, it is simple to show that $(0,0)$ is a CC-strong attractor for (4.7) in $\mathbb{R}^2$. In fact, for any compact set $K \subset \mathbb{R}^2$, we can choose a $r_K > 0$ and a sequence of closed (2-norm) balls $B_n = B[(0,0), r_K m^{n-1}]$, $n \in \mathbb{N}$, such that $K \subset Int(B_1)$. Finally, we remark that the latter family $(B_n)_{n\in\mathbb{N}}$ satisfies conditions (C1)–(C3).

Now, we are ready to formulate the main result of this part of the thesis.

Theorem 4.5. *Assume that $D \subset \mathbb{R}^s$ is an open convex set and that $F : D \to D$ is a locally Lipschitzian map. Let $p \in D$ be a CC-strong attractor for* (4.6) *in D. Then, for each $\phi \in C_D$, the unique solution $x(t;\phi)$ of* (4.4) *through ϕ is defined for any $t \geq 0$ and $\lim_{t\to\infty} x(t;\phi) = p$.*

In Section 4.4, we provide its proof. It requires some technical lemmas, which can be found in the following section.

Remark 4.6. Assuming that F is just a continuous map would somehow be harmless when it comes to uniqueness of solutions, provided $\tau_{ik} > 0$, for every $i, k \in \{1, \dots, s\}$ (see the comments below Theorem 1.5 and the method of steps in Remark 1.6). Otherwise, we can still keep such general hypothesis by doing a simple adaptation in the statement of Theorem 4.5 to consider a context of non-uniqueness. For instance, we can do that by choosing a more general notation than the one of $x(t;\phi)$, in the line of replacing *'the unique solution $x(t;\phi)$ of* (4.4) *through ϕ'* by *'any solution $x(t)$ of* (4.4) *through ϕ'*.

Remark 4.7. It is clear that, regardless of the dimension s of the system, every I-strong attractor is a CC-strong attractor, since the Cartesian products of real intervals are convex sets, and every CC-strong attractor is a global attractor (see Definition 4.3 with K being a singleton set).

Indeed, those three concepts are equivalent for the scalar case. Notice that the unique convex and compact sets with nonempty interior in $\mathbb{R}$ are the non-degenerated compact intervals of $\mathbb{R}$. Hence, if $s = 1$, then

$$\text{I-strong attractivity} \iff \text{CC-strong attractivity} \iff \text{Global attractivity.}$$

Nevertheless, this is no longer true in the context of a higher dimensional DDE (4.4). In fact, a 2-dimensional example of a global attractor which is not an I-strong attractor is shown in [90, Appendix]. For instance, this can also be seen via [91], where the authors provide an example of a global attractor for a difference equation in $\mathbb{R}^2$ which is not a global attractor for the corresponding system of delay differential equations in $\mathbb{R}^2$ and, hence, it cannot be an I-strong attractor by virtue of Theorem 4.2. Therefore, being an I-strong attractor in a rectangular phase space in $\mathbb{R}^2$ is a condition strictly stronger than being a global attractor. Nonetheless, one can adapt the difference equation in, e.g, [91] by adding independent variables driven to 0 in one iteration to see that the former condition holds for any $\mathbb{R}^s$, $s > 1$.

Now, the issue is to relate the concept in Definition 4.3 with the other two. In fact, by using our main result, Theorem 4.5, one can see that the example in [91] also serves as an example of a global attractor which is not a CC-strong attractor.

In addition to the former explanations, Example 4.4 shows that (interval) strong attractivity is a condition strictly stronger than CC-strong attractivity in a phase space contained in $\mathbb{R}^2$, even when the phase space is rectangular. Again, if we consider independent variables driven to 0 in one iteration then the same reasoning is also valid for $\mathbb{R}^s$, $s > 1$. Thus, if $s > 1$, then

$$\text{I-strong attractivity} \underset{\not\Longleftarrow}{\Longrightarrow} \text{CC-strong attractivity} \underset{\not\Longleftarrow}{\Longrightarrow} \text{Global attractivity.}$$

Of course, finding a valid family of subsets of D to check CC-strong attractivity may become a difficult task. The next corollary, based on the ideas of Example 4.4, provides sufficient conditions to ensure the existence of such a family of sets. Hereafter, the notation $\|\cdot\|_*$ will refer to a norm in $\mathbb{R}^s$.

Corollary 4.8. *Assume that $D \subset \mathbb{R}^s$ is an open convex subset. If there exists $k \in [0,1)$ and a norm $\|\cdot\|_*$ in $\mathbb{R}^s$ such that*

$$\|F(x) - p\|_* \le k\|x - p\|_*,$$

for any $x \in D$, then, for each $\phi \in C_D$, the solution $x(t;\phi)$ of (4.4) through ϕ is defined for any $t \ge 0$ and $\lim_{t\to\infty} x(t;\phi) = p$.

Proof. We only have to check that p is a CC-strong attractor for (4.6) in D. However, this directly comes from the following fact: the sets $A_r := \{x \in \mathbb{R}^s : \|x - p\|_* \le r\}$, $r \ge 0$, are compact and convex. Therefore, by choosing $r_0 > 0$ such that $K \subset A_{r_0}$, the family $\{A_{r_0 k^{n-1}}\}_{n\in\mathbb{N}}$ satisfies the conditions (C1)–(C3). $\square$

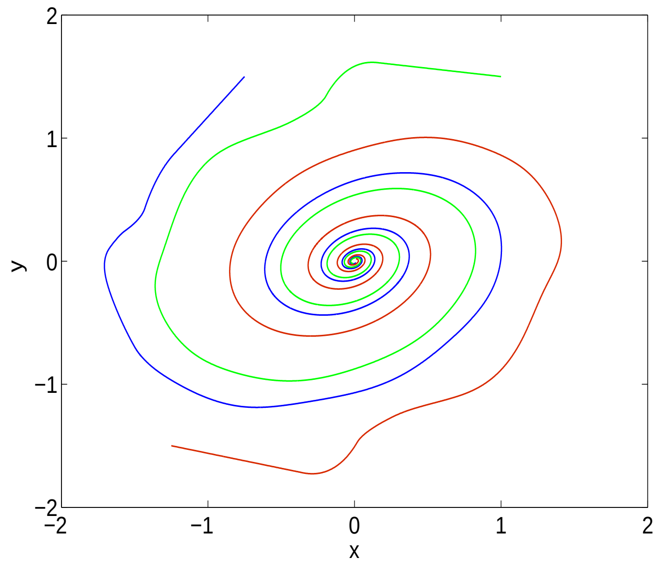
Based on Example 4.4, we provide an example of a system of delay differential equations for which we cannot use Theorem 2.5 of [90] but the general case of Theorem 4.5 applies.

Example 4.9. Let $\tau_{ik} \ge 0$, with $i, k \in \{1,2\}$, $m \in (\frac{1}{\sqrt{2}}, 1)$, and consider the system

$$\begin{aligned} x'(t) &= -x(t) + \tfrac{m}{\sqrt{2}}(x(t-\tau_{11}) - y(t-\tau_{12})),\\ y'(t) &= -y(t) + \tfrac{m}{\sqrt{2}}(x(t-\tau_{21}) + y(t-\tau_{22})). \end{aligned} \tag{4.8}$$

Its corresponding difference equation is (4.7). We have checked in Example 4.4 that the origin is a CC-strong attractor but not an I-strong attractor for (4.6) in $\mathbb{R}^2$. Therefore, by applying Theorem 4.5, we conclude that the origin is a global attractor for the system (4.8).

In Figure 4.2, we plot numerical simulations for several solutions of (4.8) projected into $\mathbb{R}^2$, with the particular choices $m = 0.95$, $\tau_{11} = 2.25$, $\tau_{12} = 3.5$, $\tau_{21} = 1$ and $\tau_{22} = 3$. We take as initial segments the constant functions $\phi_1, \phi_2, \phi_3 \in C = \mathcal{C}([-3.5,0],\mathbb{R}^2)$ that are given, respectively, by $\phi_1 \equiv (-0.75, 1.5)$, $\phi_2 \equiv (-1.25, -1.5)$ and $\phi_3 \equiv (1, 1.5)$. The solutions corresponding to such segments approach the origin as t increases. In particular, we also numerically compute the two coordinates of the solution $(x(t;\phi_1), y(t;\phi_1))$.



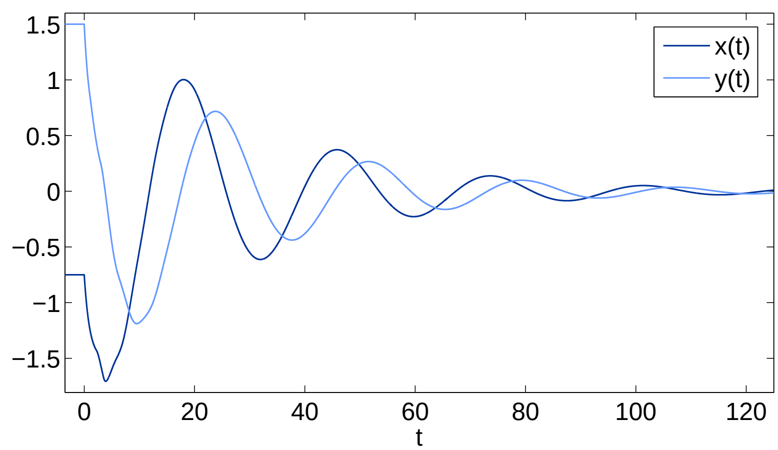


Figure 4.2: Above, we plot numerical simulations for some of the solutions of the system (4.8) projected into $\mathbb{R}^2$, with $m = 0.95$, $\tau_{11} = 2.25$, $\tau_{12} = 3.5$, $\tau_{21} = 1$ and $\tau_{22} = 3$ and different initial conditions (see the text). Below, the two coordinates of a solution depicted in the image above are numerically computed. This picture has been obtained via the use of MATLAB and, in particular, its solver *dde23*.

4.3 Some results concerning convex sets

In this section, we provide some lemmas that will be used in the proof of Theorem 4.5, in Section 4.4.

We make use of the usual norm in $\mathbb{R}^s$, i.e., the one defined by

$$\|z\|_2 = \left(\sum_{j=1}^{s} {z_j}^2 \right)^{\frac{1}{2}}, \quad z \equiv (z_1, \ldots, z_s).$$

Finally, if $v \in \mathbb{R}^s$ and $\lambda \in \mathbb{R}^+$, we will also consider the sets

$$M + v := \{x + v : x \in M\}, \quad \lambda M := \{\lambda x : x \in M\},$$
$$B_2(v, \lambda) := \{z \in \mathbb{R}^s : \|z - v\|_2 < \lambda\}.$$

Lemma 4.10. *Let $K \subsetneq \mathbb{R}^s$ be a convex subset such that $Int(K)$ is non-empty. If $x \in \partial K$ and $v \in Int(K)$, then*

$$(1 - \lambda)x + \lambda v \in Int(K), \quad \forall \lambda \in (0, 1].$$

Moreover, there exist an $\varepsilon > 0$ such that

$$Q_{x,v,\varepsilon} := \{(1 - \lambda)x + \lambda z : \ \lambda \in (0, 1], \ z \in B_2(v, \varepsilon)\} \subset Int(K)$$

and $\beta_\varepsilon \in [0, 1)$ such that, if v_* satisfies

$$\|v_* - x\|_2 \leq \|v - x\|_2 - \varepsilon,$$
$$\cos(\angle(v_* - x, v - x)) := \frac{\langle v_* - x, v - x\rangle}{\|v_* - x\|_2 \|v - x\|_2} > \beta_\varepsilon,$$

then $v_* \in Q_{x,v,\varepsilon}$.

Proof. The first part of the lemma can be derived from, e.g., [131, Theorem 1.11].

The second part is just a direct application of the first part: if $v \in Int(K)$, then, there exists an $\varepsilon > 0$ such that $B_2(v, \varepsilon) \subset Int(K)$. Therefore, we can apply the first part of the lemma with a general $z \in B_2(v, \varepsilon)$ instead of v.

The last assertion is trivial in case $s = 1$. Otherwise, if $s \geq 2$, we pick a plane containing x and spanned by $v - x$ and one of its orthogonal vectors. Then, a simple trigonometric reasoning (see Figure 4.3) will lead to see that

$$1 > \beta_\varepsilon := \frac{\sqrt{\|v - x\|_2^2 - \varepsilon^2}}{\|v - x\|_2} \geq 0.$$

□

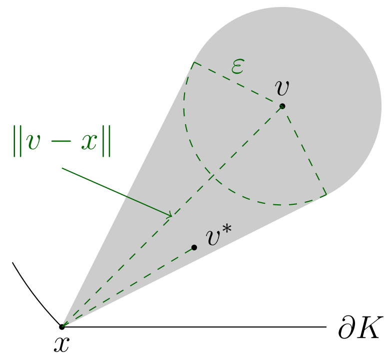


Figure 4.3: A sketch of the set $Q_{x,v,\varepsilon}$ is represented in gray. Distances are highlighted with dashed green lines. A particular element v^* satisfying the hypotheses of the last assertion of Lemma 4.10 is also depicted.

In the following lemma, a similar assertion to the first one in Lemma 4.10 is proved: in some sense, the result is robust under small perturbations.

Lemma 4.11. *Let $K \subsetneq \mathbb{R}^s$ be a convex subset such that $Int(K)$ is non-empty. If $x \in \partial K$, $v \in Int(K)$ and $g : \mathbb{R} \to \mathbb{R}^s$ is such that*

$$\lim_{\lambda \to 0} \frac{\|g(\lambda)\|_2}{|\lambda|} = 0,$$

then there exists $\lambda_0 > 0$ such that

$$(1-\lambda)x + \lambda v + g(\lambda) \in Int(K),$$

for any $\lambda \in (0, \lambda_0)$.

Proof. Let v and g satisfy the given hypotheses. We will prove this result by reaching a contradiction. In fact, assume that there exists a sequence of positive real numbers $(\lambda_k)_{k\in\mathbb{Z}_+}$ such that $\lambda_k \to 0$ as $k \to \infty$ and

$$(1-\lambda_k)x + \lambda_k \left[v + \frac{g(\lambda_k)}{\lambda_k} \right] = (1-\lambda_k)x + \lambda_k v + g(\lambda_k) \notin Int(K),$$

for any $k \in \mathbb{Z}_+$. On the one hand, by using the second assertion of Lemma 4.10, we conclude that, for a sufficiently small $\varepsilon > 0$, we obtain

$$(1-\lambda_k)x + \lambda_k \left[v + \frac{g(\lambda_k)}{\lambda_k} \right] \notin Q_{x,v,\varepsilon}, \quad k \in \mathbb{Z}_+.$$

Moreover, we can pick a certain $k_0 \in \mathbb{Z}_+$ such that

$$\|x + \lambda_k(v - x) + g(\lambda_k) - x\|_2 = \|\lambda_k(v - x) + g(\lambda_k)\|_2 < \|v - x\|_2 - \varepsilon, \quad k \geq k_0,$$

and, by the last assertion of the same lemma, there exists a $\beta_\varepsilon \in [0, 1)$ such that

$$\cos(\angle(\lambda_k(v - x) + g(\lambda_k), v - x)) \leq \beta_\varepsilon < 1, \quad k \geq k_0. \tag{4.9}$$

On the other hand,

$$\begin{aligned}\lim_{k\to\infty} \cos(\angle(\lambda_k(v - x) + g(\lambda_k), v - x)) &= \lim_{k\to\infty} \frac{\langle \lambda_k(v - x) + g(\lambda_k), v - x\rangle}{\|\lambda_k(v - x) + g(\lambda_k)\|_2 \|v - x\|_2} \\ &= \lim_{k\to\infty} \frac{\left\langle v - x + \frac{g(\lambda_k)}{\lambda_k}, v - x\right\rangle}{\|v - x + \frac{g(\lambda_k)}{\lambda_k}\|_2 \|v - x\|_2} = 1. \end{aligned}\tag{4.10}$$

By (4.9) and (4.10), we obtain a contradiction. □

We will also need the following result, which can be easily proved.

Lemma 4.12. *If $v \in \mathbb{R}^s$, $\lambda \geq 0$ and $M, W \subset \mathbb{R}^s$ are convex sets, then the sets $M + v$, $M \cap W$ and λM are also convex. Moreover, if $0 \in M$ and $\lambda < \mu$, then $\lambda M \subset \mu Int(M)$.*

4.4 Proof of Theorem 4.5

Before giving the proof, we provide an alternative way to write the system (4.4) that will be helpful. In fact, define the function $\hat{f} : C_D \to \mathbb{R}^s$ by

$$\hat{f}(\psi) = (F_1(\psi_1(-\tau_{11}), \dots, \psi_s(-\tau_{1s})), \dots, F_s(\psi_1(-\tau_{s1}), \dots, \psi_s(-\tau_{ss}))), \tag{4.11}$$

for any $\psi \equiv (\psi_1, \dots, \psi_s) \in C_D$, then, we can rewrite the DDE in (4.4) as

$$x'(t) = -x(t) + \hat{f}(x_t), \tag{4.12}$$

Notice that, in the expressions (4.11) and (4.12), ψ and x_t obviously represent segments, while the items ψ_i, $i \in \{1, \dots, s\}$, represent the composition of the segment ψ with the i-th projection.

The proof of Theorem 4.5 is based on the one in [90, Theorem 2.5].

Proof. Fix an arbitrary function ϕ as in the statement of the theorem, that is, belonging to C_D. Since p is a CC-strong attractor for (4.6) in D, if we choose the compact set

$$K = \{\phi(t) : t \in [-\tau, 0]\},$$

then there exist a family $\{K_n\}_{n\in\mathbb{N}}$ of compact and convex sets with non-empty interior satisfying conditions (C1)–(C3). Firstly, we prove that

$$x(t;\phi) \in K_1, \quad \forall t \geq 0. \tag{4.13}$$

We remark that the right-hand side of (4.12) is completely continuous and Theorem 1.8 asserts that if $x(t;\phi)$ is not defined for all $t \geq 0$, then it must leave each closed and bounded set of D, in particular, K_1, as $t \to t_\phi^-$ (notice the different meaning of the notation D here in comparison with that in the statement of Theorem 1.8).

Now, we verify that $x(t;\phi)$ cannot leave K_1, so (4.13) would be true. First, notice that $x(t;\phi) = \phi(t) \in K \subset Int(K_1)$, for any $t \in [-\tau, 0]$. We proceed by reaching a contradiction. Indeed, assume that there exists a first time $\tilde{t} > 0$ such that $x(\tilde{t};\phi) \in \partial K_1$ and $x(\tilde{t}+\lambda;\phi) \notin K_1$, $\lambda \in (0, \lambda_*)$, for some $\lambda_* > 0$. The solution $x(t;\phi)$ is of class $\mathcal{C}^1$ for $t > 0$ and, therefore, there exists a continuous function g such that

$$x(\tilde{t}+\lambda;\phi) = x(\tilde{t};\phi) + \lambda x'(\tilde{t};\phi) + g(\lambda) \tag{4.14}$$

and

$$\lim_{\lambda\to 0} \frac{\|g(\lambda)\|_2}{|\lambda|} = 0.$$

We can use (4.12) in (4.14) and obtain

$$x(\tilde{t}+\lambda;\phi) = x(\tilde{t};\phi) + \lambda\left(\hat{f}(x_{\tilde{t}}^{\phi}) - x(\tilde{t};\phi)\right) + g(\lambda). \tag{4.15}$$

Since $x(\tilde{t};\phi) \in \partial K_1$ and

$$\hat{f}(x_{\tilde{t}}^{\phi}) \subset F(K_1) \in Int(K_1),$$

a direct application of Lemma 4.11 to (4.15) gives us that $x(\tilde{t}+\lambda;\phi) \in Int(K_1)$ for sufficiently small $\lambda > 0$. Therefore, condition (4.13) is true. In fact, since ϕ has been arbitrarily chosen inside C_D, we obtain that, for any $\phi \in C_D$, there is a unique and globally defined solution of (4.4) through ϕ. Hence, by virtue of Theorem 1.19, a continuous semiflow on C_D is defined via the corresponding segments.

Now, we prove that there is a time $t_1 > 0$ such that

$$x(t;\phi) \in K_2, \quad t \geq t_1. \tag{4.16}$$

First, we prove that, if $x(t^*;\phi) \in K_2$, for some $t^* \geq 0$, then $x(t;\phi) \in K_2$, for $t \geq t^*$. In other words, once the solution has entered K_2, it cannot go outside the set. So, assume that there exist $t^* \geq 0$ such that $x(t^*;\phi) \in K_2$ and $t' > t^*$ satisfying

$$x(t';\phi) \notin K_2.$$

Then, since the solution is continuous, $x(t;\phi)$ must intersect ∂K_2 at least, at some t'' such that $t^* \leq t'' < t'$. Take the maximum t when this happens and rename it as t''. Again, as an application of Lemma 4.11, we get that $x(t''+\lambda,\phi) \in Int(K_2)$, for a sufficiently small $\lambda > 0$. Then, $x(t;\phi)$ must belong to K_2 at $t = t'$ because it cannot intersect the boundary of K once more between t'' and t'.

Finally, we prove that there exists $t_1 \geq 0$ so that the assertion related to (4.16) holds. Assume that $x(t;\phi) \notin K_2$, $\forall t \geq 0$. Define the sets (see Figure 4.4)

$$K_{2,\mu} := \{p + \mu(x-p) :\ x \in K_2\}, \quad \mu \geq 1.$$

By consecutive applications of Lemma 4.12, it is clear that, $K_2 - p$, $\mu(K_2 - p)$ and, finally, $K_{2,\mu}$ are convex sets. Moreover, by applying the same lemma, we use that $0 \in \mu(K_2 - p)$ to write $K_{2,\mu_1} \subset Int(K_{2,\mu})$ if $\mu_1 < \mu$. Then if we set $G_\mu = K_1 \cap K_{2,\mu}$, which is a convex set (Lemma 4.12), we obtain the following chain of inclusions:

$$F(G_\mu) \subset F(K_1) \subset K_2 \subset Int(K_1) \cap Int(K_{2,\mu}) = Int(G_\mu) \subset G_\mu,$$

which holds for every $\mu > 1$.

The condition $x(t;\phi) \notin K_2$, $\forall t \geq 0$, is equivalent to the assumption that $x(t;\phi) \in \partial G_\mu$, with $\mu(=\mu(t)) > 1$, for any $t \geq 0$. Pick an arbitrary $t_* \geq 0$, then

$$x(t_*+\lambda;\phi) = x(t_*;\phi) + \lambda x'(t_*;\phi) + g^*(\lambda) = x(t_*;\phi) + \lambda\left(\hat{f}(x^\phi_{t_*}) - x(t_*;\phi)\right) + g^*(\lambda),$$

where

$$\lim_{\lambda \to 0} \frac{\|g^*(\lambda)\|_2}{|\lambda|} = 0.$$

Reasoning in a similar way to what we have done before and using Lemma 4.11, there exists $\lambda_{t_*} > 0$ such that $x(t_*+\lambda;\phi) \in Int(G_{\mu(t_*)})$, for $\lambda \in (0, \lambda_{t_*})$. This situation happens again if $x(t;\phi)$ eventually returns to $\partial G_{\mu(t_*)}$ at another $t_{**} > t_*$. Hence, $\mu(t_*) \geq \mu(t_*+\lambda)$, for every $\lambda \in \mathbb{R}^+$, and since the reasoning is valid for any $t_* \geq 0$, the function μ is nonincreasing on $[0,\infty)$. Moreover, by our assumption ($x(t;\phi)$ stays outside K_2), μ is bounded from below by 1 ($G_1 = K_2$), which implies $\mu(t) \to r \in [1, \mu(0)]$ and $x(t;\phi) \to \partial G_r$ as $t \to \infty$. Notice that the latter is just an intuitive way to say that

$$\delta_{\mathbb{R}^s}(x(t;\phi), \partial G_r) \to 0, \quad \text{as } t \to \infty, \tag{4.17}$$

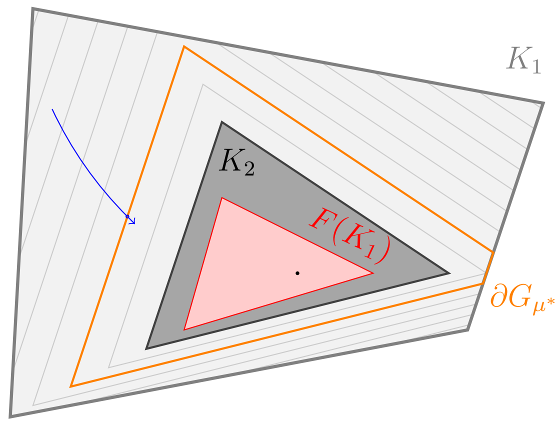


Figure 4.4: A sketch of the behaviour of $x(t;\phi)$ (blue) inside the set K_1 (light gray) for a particular case of dimension $s=2$. The set K_2 is represented with dark grey. The boundaries of $G_\mu = K_1 \cap K_{2,\mu}$, for some values $\mu > 1$, are represented surrounding K_2 (gray lines). We highlight the whole boundary of a particular $G_{\mu *}$ (orange) and the blue arrow represents $x'(t;\phi)$, which means that $x(t;\phi)$ enters such subset. The set $F(K_1)$ is represented in red. The equilibrium p is depicted as a point inside $F(K_1)$.

where δ represents the distance from a point to a subset of $\mathbb{R}^s$ (see Definition 1.24). Now, by applying Proposition 1.27 for $V = C_{K_1} \subset C_D$, which is a closed subset of C, we obtain that the the forward orbit of ϕ, the set $\gamma^+(\phi)$, is a relatively compact set. Thus, Theorem 1.26 tells us that $\omega(\phi)$ is nonempty, invariant and

$$\delta_{C_D}(x_t^\phi, \omega(\phi)) \to 0, \quad \text{as } t \to \infty.$$

By recalling the asymptotic behaviour of $x(t;\phi)$ shown in (4.17), we must have the relation $\omega(\phi) \subset C_{\partial G_r}$. Nevertheless, a nonempty subset of $C_{\partial G_r}$, $r \geq 1$, cannot be invariant. In fact, if $x \equiv x(\hat{t};\phi) \in \partial G_r$ for a certain time $\hat{t}$, then there exists a sufficiently small λ_0 such that $x(\hat{t}+\lambda;\phi)$ belongs to $Int(G_r)$ for any $\lambda \in (0,\lambda_0)$, as we have seen several paragraphs above. Obviously, this is a contradiction and the assertion related to (4.16) is true.

The previous procedure can be repeated with K_1, K_2 being replaced by K_2, K_3, and so on. We have an inductive argument, obtaining an increasing sequence of nonnegative real numbers $(t_n)_{n\in\mathbb{Z}_+}$ (it can be chosen with $t_n \to \infty$ as $n \to \infty$) such that $x(t;\phi) \in K_{n+1}$, $\forall t \geq t_n$. In fact, if, for every $n \in \mathbb{Z}_+$, we take $t_{n+1} \geq t_n + \tau$, we ensure that the solution $x(t;\phi)$ remains at least a time τ in the set K_{n-1}, which is key for some parts of the reasoning made with K_1 and K_2 to be valid for K_{n-1} and K_n. Finally, since, by (C3), $\cap_{n=1}^{\infty} K_n = \{p\}$, it clearly follows that $\lim_{t\to\infty} x(t;\phi) = p$. □

Remark 4.13. If the elements of the family $\{K_n\}_{n\in\mathbb{N}}$ satisfying the conditions of CC-strong attractivity were balls with respect to the p-norm, $p > 1$, in $\mathbb{R}^s$ (not necessarily with the same center, but 'shrinking' around the strong attractor) then the proof of Theorem 4.5 would be obtained in an easier way. The idea is that the sets K_n have 'regular boundary' and we can use a discrete Hölder inequality [116, Chapter 3] to prove that the solution enters each K_n. Thus, in that case we would avoid working with general convex sets.

Remark 4.14. A result of independent interest concerning forward invariance in equation (4.4) is hidden inside the reasoning that we used in the proof of Theorem 4.5. In fact, assume that $M \subset \mathbb{R}^s$ is a compact and convex set with $F(M) \subset Int(M)$; if $\phi \in C_M$, the solution $x(t;\phi)$ of (4.4) through ϕ remains in M, for $t \geq 0$. Even if we replace the compactness of M by its closed character, an analogous statement can be given, provided all the discrete delays in (4.4) are positive. That would somehow constitute an extension of the first part of Theorem 1.33.

4.5 A short discussion

Theorem 4.5 constitutes an extension to the multidimensional case of the ideas in the works by Mallet-Paret and Nussbaum [98] and by Ivanov and Sharkovsky [65] in the line highlighted by Liz and Ruiz-Herrera [90], that is, by working with a stronger concept than global attractivity for its corresponding difference equation.

The concept proposed in [90], shown in Definition 4.1, recovers the essential properties one needs to handle in order to give an inheritance-type result for the corresponding DDE. In fact, it is proven there that, while every globally attracting equilibrium shows strong attractivity features in the scalar case, that is not true for higher dimensions, so the introduction of such concept is clearly justified to handle the multidimensional case. However, we have realised that their reasoning can be sharpenned and we proposed a concept, CC-strong attractivity (Definition 4.3), that aims to achieve such goal. This concept is strictly weaker than the strong attractivity in the sense of [90] and strictly stronger than just the global attractivity (Remark 4.7), provided the dimension of the system is $s \geq 2$. Nevertheless, it constitutes a sufficient condition in order to obtain the global attractivity of the equilibrium for the corresponding DDE.

In [90, Section 2.2], the reader can find several sufficient conditions to ensure that an equilibrium is an I-strong attractor. Some of them reduce the multidimensional difference equation to a scalar difference equation, for which many results are available (see the previous chapters). Nonetheless, as recognised in [34, Section 6], checking that an equilibrium is an I-strong attractor is not a simple task. In fact, we also expect that checking that an equilibrium is a CC-strong attractor may generally be a puzzling goal. In this line, we have also provided Corollary 4.8, where we have CC-strong attractivity if the convergence towards the equilibrium is 'controlled' by a norm. Definitely, some further efforts deserve to be devoted to study this line.

Besides, we have only been concerned with providing a theoretical extension of the known results in contrast with [90], where, e.g., an application to neural networks is studied. Therefore, it would also be interesting to show some relevant applications.

Chapter 5

Gronwall-Bellman estimates for delayed inequalities of Volterra-type

We obtain some estimates of Gronwall-Bellman-type for Volterra's inequalities, which have applications to the study of the stability properties of the solutions to some nonautonomous linear functional differential equations. This chapter is related with the contents of [21], a work by the author of this thesis (S. Buedo-Fernández[1]) and Rosana Rodríguez-López[1]. We also recall its reference below.

S. Buedo-Fernández, R. Rodríguez-López. Gronwall-Bellman estimates for linear Volterra-type inequalities with delay. Electronic Journal of Qualitative Theory of Differential Equations 2018, No 83, 1–16 (2018). University of Szeged, ISSN 1417-3875.

5.1 Introduction

The Gronwall-Bellman estimates are a very useful tool in order to study the stability properties of the solutions to differential equations. In particular, they provide explicit estimates of the functions satisfying certain implicit inequalities, which are frequently related to solutions of linear differential equations. The well-known classical Gronwall-Bellman Lemma [11, 47] on a real interval $[t_0, T)$ is stated as follows.

Theorem 5.1. *[112, Theorem 1.2.2] Let $c \in \mathbb{R}_+$ and $b : [t_0, T) \to \mathbb{R}_+$ be a continuous function. If $x : [t_0, T) \to \mathbb{R}_+$ is a continuous function satisfying*

$$x(t) \leq c + \int_{t_0}^{t} b(s)x(s)\, ds, \quad t \in [t_0, T), \tag{5.1}$$

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then the following estimate holds:

$$x(t) \leq c \exp\left(\int_{t_0}^{t} b(s)\, ds\right), \quad t \in [t_0, T).$$

Any function playing the role of x in the framework of Theorem 5.1 is called a solution to (5.1). There have been several generalisations of this type of results since the works [11, 47]. For further details, we refer the reader to references [32, 112]. Nevertheless, for the aim of this chapter, we will only highlight two of them.

The first one deals with the addition of a new variable in the definition of b, so that b becomes a kernel k defined on the set $\mathcal{T} := \{(t,s) \in \mathbb{R}^2 : t_0 \leq s \leq t < T\}$. Norbury and Stuart [110] obtained an explicit estimate involving the exponential of the integral of the kernel by considering that it is nondecreasing in the first variable, as we recall below.

Theorem 5.2. *[112, Theorem 1.4.2(i)] Let $k : \mathcal{T} \to \mathbb{R}_+$ be a continuous function such that $k(\cdot, s)$ is nondecreasing for each fixed $s \in [t_0, T)$. If $x : [t_0, T) \to \mathbb{R}_+$ is a continuous function satisfying*

$$x(t) \leq c + \int_{t_0}^{t} k(t,s)x(s)ds, \quad t \in [t_0, T),$$

then the following estimate holds:

$$x(t) \leq c \exp\left(\int_{t_0}^{t} k(t,s)ds\right), \quad t \in [t_0, T).$$

The other generalisation deals with the concept of delay by considering in (5.1) an integral between t_0 and $m(t)$, for a certain function $m : [t_0, T) \to [t_0, T)$ such that $m(t) \leq t$, for every $t \in [t_0, T)$. Although one can find some results in this direction (see [32, Chapter 1], which includes a result by Ahmedov et al. [1]), we will focus on the one proposed by Győri and Horváth [52]. In this work, the authors use the characteristic equation and the so-called *characteristic inequality* coming from the former equation in order to study some nonautonomous linear delay differential equations.

Theorem 5.3. *[52, Theorem 2.2] Let $b : [t_0, T) \to \mathbb{R}_+$ be a locally integrable function. Assume that $r \geq 0$ and $\tau : [t_0, T) \to \mathbb{R}_+$ is a measurable function such that*

$$t_0 - r \leq t - \tau(t), \quad t_0 \leq t < T.$$

If x is a nonnegative Borel measurable and locally bounded function defined on $[t_0 - r, T)$ such that

$$x(t) \leq c + \int_{t_0}^{t} b(u)x(u - \tau(u))\, du, \quad t_0 \leq t < T,$$

then

$$x(t) \le K \exp\left(\int_{t_0}^{t} \gamma(s)ds\right), \quad t_0 \le t < T,$$

where the function $\gamma : [t_0 - r, T) \to \mathbb{R}_+$ *is locally integrable and satisfies the characteristic inequality*

$$\gamma(t) \ge b(t) \exp\left(-\int_{t-\tau(t)}^{t} \gamma(s)ds\right), \quad t_0 \le t < T,$$

and

$$K := \max\left\{c \exp\left(\int_{t_0-r}^{t_0} \gamma(s)ds\right), \sup_{t_0-r\le s\le t_0} x(s)\ \exp\left(\int_{s}^{t_0} \gamma(w)\,dw\right)\right\}.$$

The previous result is also interesting because it does not require the continuity of the functions involved.

The main objective of this chapter is to provide an extension of Theorems 5.2 and 5.3 at the same time. We will provide some results that follow the main lines in [52] by introducing a new variable, as it is done in Theorem 5.2 with respect to Theorem 5.1, and, thus, by working with the integral inequality with delay

$$x(t) \le c + \int_{t_0}^{t} k(t,u)x(u-\tau(u))\,du, \quad t_0 \le t < T, \tag{5.2}$$

where τ, k are functions endowed with some suitable measurability and integrability properties. In order to obtain estimates for the solutions to the previous integral inequality, we make a revision of some of the procedures and proofs in [52], and derive a generalised characteristic inequality involving the kernel k of the inequality, just by using the identity

$$\int_{t_0}^{t} k(t,u)\,du = \int_{t_0}^{t}\left[k(u,u) + \int_{t_0}^{u} \partial_1 k(u,s)\,ds\right]du, \quad t_0 \le t < T,$$

where, as natural, $\partial_1 k$ means the partial derivative of k with respect to the first variable.

Our main result, Theorem 5.6, extends Theorem 5.3, since, if we take the particular case of $k(t,u) = b(u)$, the imposed hypotheses on k are consistent with the hypotheses in [52] concerning the function b. Theorem 5.2 is also generalised, not only by avoiding the continuity of the involved functions, but also by introducing the delay. We also discuss the sharpness of the estimates obtained for the solutions to (5.2) that our results give.

Finally, we discuss the utility of the results given in this chapter and provide an example of a functional differential equation whose stability properties can be derived from Theorem 5.6. For more applications of generalised characteristic inequalities arising from delay differential equations, we refer to [53].

5.2 Preliminaries

In this section, we summarise the notation that we are going to use, set the definitions, and recall some needed results.

Let $t_0 \in \mathbb{R}$ and $T \in \mathbb{R} \cup \{\infty\}$ be such that $t_0 < T$ and assume that $c, r \in \mathbb{R}_+$. We define the sets (see Figure 5.1)

$$\mathcal{T} := \{(t,u) \in \mathbb{R}^2 : t_0 \le u \le t < T\},$$
$$\mathcal{T}_\eta := \{(t,u) \in \mathcal{T} : t \le \eta\}, \quad \eta \in [t_0, T).$$

Moreover, for any given function $\zeta : \mathcal{T} \to \mathbb{R}$, we define the function $\zeta_* : [t_0, T) \to \mathbb{R}$ as

$$\zeta_*(u) = \zeta(u,u), \quad u \in [t_0, T).$$

It is obvious that, if $\operatorname{Im} \zeta \subset \mathbb{R}_+$, then $\operatorname{Im} \zeta_* \subset \mathbb{R}_+$.

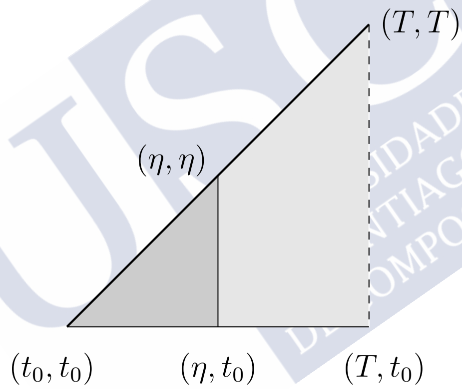

Figure 5.1: A sketch of the set $\mathcal{T}$. The subset $\mathcal{T}_\eta$, for some $\eta \in [t_0, T)$, is represented with darker colour. Given a function ζ defined on $\mathcal{T}$, the restriction to the diagonal (thick line) is ζ_*.

Following the definitions in [52], measurability will refer to Lebesgue measurability, and integrable will mean here Lebesgue integrable [116]. Furthermore, we say that a function $f : [t_0, T) \to \mathbb{R}$ is *locally integrable* if it is integrable over $[t_0, t^*]$, for every $t^* \in [t_0, T)$. In fact, for each locally integrable function $f : [t_0, T) \to \mathbb{R}_+$, we define the function $I_f : [t_0, T) \to \mathbb{R}_+$ as

$$I_f(t) = \int_{t_0}^{t} f(s)\, ds, \quad t_0 - r \le t < T.$$

Analogously, we say that the function f is *locally bounded* if it is bounded on $[t_0, t^*]$, for every $t^* \in [t_0, T)$.

We recall from [116] that the function $g : K \to \mathbb{R}$, with K being a real compact interval, is called *absolutely continuous* if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{i=1}^{n} |g(\beta_i) - g(\alpha_i)| < \varepsilon$$

for any $n \in \mathbb{N}$, and any disjoint set of intervals $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) \subset K$, whose lengths satisfy that

$$\sum_{i=1}^{n} (\beta_i - \alpha_i) < \delta.$$

Thus, we will say that a function $f : [t_0, T) \to \mathbb{R}$ is *locally absolutely continuous* if it is absolutely continuous on each $[t_0, t^*]$, for every $t^* \in [t_0, T)$.

In the next section, we will need the following couple of results. The first one is the Fundamental Theorem of Calculus for Lebesgue integration, which can be found in a large list of references such as [116], while the second one is a technical lemma.

Theorem 5.4. *[116, Theorem 7.20] Let I be a real interval and $g : I \to \mathbb{R}$ be an absolutely continuous function. Then g' exists almost everywhere (a.e.), it is integrable and*

$$g(b) = g(a) + \int_a^b g'(s)\, ds, \quad a, b \in I.$$

In the cases for which the Fundamental Theorem of Calculus applies to a function g, the existence of derivatives of g on the whole interval is not generally guaranteed, but we will still handle a function g' as some function defined on the same interval as g (it could be extended anyway to the remaining set of measure zero).

Lemma 5.5. *Let $\zeta : \mathcal{T} \to \mathbb{R}$ be such that $\zeta(\cdot, u) : [u, T) \to \mathbb{R}$ is locally absolutely continuous for each fixed $u \in [t_0, T)$. Moreover, assume that $\zeta_* : [t_0, T) \to \mathbb{R}$ is locally integrable and that $\partial_1 \zeta : \mathcal{T} \to \mathbb{R}$ is integrable on each $\mathcal{T}_\eta$, for all $\eta \in [t_0, T)$. Then, the following identity holds:*

$$\int_{t_0}^{t} \zeta(t, u)\, du = \int_{t_0}^{t} \left[\zeta(u, u) + \int_{t_0}^{u} \partial_1 \zeta(u, s)\, ds \right] du, \quad t_0 \le t < T.$$

Proof. For each $u \in [t_0, T)$, the function $\zeta(\cdot, u)$ is locally absolutely continuous and, by virtue of Theorem 5.4, the function $\partial_1 \zeta$ is defined a.e. on $\mathcal{T}$. As previously said, we extend it in an arbitrary way to $\mathcal{T}$. Hence, for a given $(t, u) \in \mathcal{T}$, we have

$$\zeta(t, u) = \zeta(u, u) + \int_u^t \partial_1 \zeta(s, u)\, ds. \tag{5.3}$$

Note that the integrability of $\partial_1\zeta$ on each $\mathcal{T}_\eta$, $\eta \in [t_0, T)$, implies that

$$y(t,u) := \int_u^t \partial_1\zeta(s,u)\,ds$$

is integrable in u for each t. Then, if we integrate (5.3) with respect to the second variable between t_0 and t, we reach to

$$\int_{t_0}^t \zeta(t,u)\,du = \int_{t_0}^t \left[\zeta(u,u) + \int_u^t \partial_1\zeta(s,u)\,ds\right] du. \tag{5.4}$$

Once more, by using the integrability of $\partial_1\zeta$ on each $\mathcal{T}_\eta$, $\eta \in [t_0, T)$, we can change the order of variables of integration in (5.4) (see Fubini's Theorem in, e.g., [116, Theorem 8.8]) and obtain

$$\int_{t_0}^t \zeta(t,u)\,du = \int_{t_0}^t \zeta(u,u)\,du + \int_{t_0}^t \int_{t_0}^s \partial_1\zeta(s,u)\,du\,ds. \tag{5.5}$$

□

5.3 Volterra-type inequalities with delay dependence

In this section, we present the principal results of this chapter. Theorems 5.6 and 5.9 provide its core, while Theorems 5.11 and 5.14 are a couple of extensions. One of the key tools is a characteristic inequality that generalises the one in Theorem 5.3.

Theorem 5.6. *Let $k : \mathcal{T} \to \mathbb{R}_+$ be such that $k(\cdot, u) : [u, T) \to \mathbb{R}$ is locally absolutely continuous for each fixed $u \in [t_0, T)$. Moreover, assume that $k_* : [t_0, T) \to \mathbb{R}_+$ is locally integrable and that $\partial_1 k : \mathcal{T} \to \mathbb{R}_+$ is integrable on each $\mathcal{T}_\eta$, for all $\eta \in [t_0, T)$. Assume that $\tau : [t_0, T) \to \mathbb{R}_+$ is a measurable function such that*

$$t_0 - r \le t - \tau(t), \quad t_0 \le t < T.$$

If $x : [t_0 - r, T) \to \mathbb{R}_+$ is Borel measurable and locally bounded such that

$$x(t) \le c + \int_{t_0}^t k(t,u)x(u - \tau(u))\,du, \quad t_0 \le t < T, \tag{5.6}$$

then

$$x(t) \le K \exp\left(\int_{t_0}^t \gamma(s)\,ds\right), \quad t_0 \le t < T, \tag{5.7}$$

where the function $\gamma : [t_0 - r, T) \to \mathbb{R}_+$ is locally integrable and satisfies the characteristic inequality

$$\gamma(t) \geq k(t,t)\exp\left(-\int_{t-\tau(t)}^{t} \gamma(s)\,ds\right) + \int_{t_0}^{t} \partial_1 k(t,s)\exp\left(-\int_{s-\tau(s)}^{t} \gamma(w)\,dw\right) ds, \quad (5.8)$$

for $t_0 \leq t < T$ a.e., and

$$K := \max\left\{ c\exp\left(\int_{t_0-r}^{t_0} \gamma(s)\,ds\right), \sup_{t_0-r\leq s\leq t_0} x(s)\ \exp\left(\int_{s}^{t_0} \gamma(w)\,dw\right)\right\}.$$

Proof. First, we consider that the function x is continuous on $[t_0, T)$. Now, assume that the function $y : [t_0 - r, T) \to \mathbb{R}_+$ is defined by

$$y(t) = x(t)\exp\left(-\int_{t_0-r}^{t} \gamma(s)\,ds\right), \quad t_0 - r \leq t < T.$$

Then, the hypothesis (5.6) implies that

$$y(t) \leq \exp\left(-\int_{t_0-r}^{t} \gamma(s)\,ds\right)\left[c + \int_{t_0}^{t} k(t,u)y(u-\tau(u))\exp\left(\int_{t_0-r}^{u-\tau(u)} \gamma(s)\,ds\right) du\right],$$

for each $t \in [t_0, T)$. In order to handle a less tedious notation in the following steps, we also consider the function $R : [t_0, T) \to \mathbb{R}_+$, which is defined by

$$R(t) = y(t-\tau(t))\exp\left(\int_{t_0-r}^{t-\tau(t)} \gamma(s)\,ds\right),$$

so we can continue our computations with the inequality

$$y(t) \leq \exp\left(-\int_{t_0-r}^{t} \gamma(s)ds\right)\left(c + \int_{t_0}^{t} k(t,u)R(u)\,du\right), \quad t_0 \leq t < T.$$

Now, we are going to apply Lemma 5.5 to the function $\zeta(t,u) := k(t,u)R(u)$. First, $\zeta(\cdot,u)$ is locally absolutely continuous for each fixed $u \in [t_0, T)$ due to the hypotheses on k. The function R is measurable and locally bounded, which combined with the local integrability of k_*, yields that ζ_* is locally integrable. By considering an analogous reasoning, and using both the previous properties of R and the hypotheses concerning k, one can conclude that $\partial_1\zeta$ is integrable on each $\mathcal{T}_\eta$, $\eta \in [t_0, T)$. Then,

$$y(t) \leq \exp\left(-\int_{t_0-r}^{t} \gamma(s)\,ds\right)\left(c + \int_{t_0}^{t}\left[k(u,u)R(u) + \int_{t_0}^{u} \partial_1 k(u,s)R(s)\,ds\right] du\right),$$

for any $t \in [t_0, T)$. If we take

$$L := \max \left\{ c, \sup_{u \in [t_0 - r, t_0]} x(u) \exp \left(- \int_{t_0 - r}^{u} \gamma(s) \, ds \right) \right\},$$

we arrive, for $L_1 > L$ arbitrarily chosen, to the following inequality:

$$y(t) \leq L < L_1, \quad t_0 - r \leq t \leq t_0.$$

Due to the continuity of y, there exists $\varepsilon > 0$ such that $y(t) < L_1$ for $t \in [t_0 - r, t_0 + \varepsilon]$. Assume that there exists $t_1 > t_0 + \varepsilon$ such that $y(t_1) = L_1$, which can be chosen as the least real number satisfying such condition. Then,

$$\begin{aligned} L_1 \leq c \exp \left(- \int_{t_0 - r}^{t_1} \gamma(s) \, ds \right) &+ L_1 \exp \left(- \int_{t_0}^{t_1} \gamma(s) \, ds \right) \\ &\times \int_{t_0}^{t_1} \left(k(u,u) e^{I_\gamma(u - \tau(u))} + \int_{t_0}^{u} \partial_1 k(u,s) e^{I_\gamma(s - \tau(s))} ds \right) du, \end{aligned}$$

where we have used that $y(u) \leq L_1$, for $u \leq t_1$. By using the characteristic inequality (5.8) in the latter expression, we write

$$L_1 \leq c \exp \left(- \int_{t_0 - r}^{t_1} \gamma(s) \, ds \right) + L_1 \exp \left(- \int_{t_0}^{t_1} \gamma(s) \, ds \right) \int_{t_0}^{t_1} \gamma(u) e^{I_\gamma(u)} du. \tag{5.9}$$

By computing the last integral in expression (5.9), we get

$$L_1 \leq c \exp \left(- \int_{t_0 - r}^{t_1} \gamma(s) \, ds \right) + L_1 \left(1 - \exp \left(- \int_{t_0}^{t_1} \gamma(s) \, ds \right) \right).$$

Then, we rearrange the terms and obtain, from the choices of L and L_1, that

$$L_1 \leq L_1 + \exp \left(- \int_{t_0}^{t_1} \gamma(s) \, ds \right) \left(c \exp \left(- \int_{t_0 - r}^{t_0} \gamma(s) \, ds \right) - L_1 \right) < L_1,$$

which is a contradiction. We can thus ensure that $y(t) \leq L$, for $t \in [t_0 - r, T)$. Therefore, it only remains to undo the change of variables considered at the beginning of the proof and obtain the estimate

$$\begin{aligned} x(t) &= y(t) \exp \left(\int_{t_0 - r}^{t} \gamma(s) \, ds \right) \\ &\leq L \exp \left(\int_{t_0 - r}^{t} \gamma(s) \, ds \right) = K \exp \left(\int_{t_0}^{t} \gamma(s) \, ds \right), \quad t_0 \leq t < T. \end{aligned}$$

To prove the result for the general case of x, we proceed in an analogous way to [52, Theorem 2.2], that is, by using the function z defined as x on $[t_0 - r, t_0)$ and as the right-hand side of (5.6) on $[t_0, T)$. The function z is continuous on $[t_0, T)$ and is an upper bound of x. Therefore, any upper bound of z is also an upper bound of x. □

In accordance with [52], any function x satisfying the integral inequality (5.6) will hereafter be called a *solution* of (5.6).

Remark 5.7. The assumptions of Theorem 5.3 are generalised by the ones in the previous theorem. In fact, if $k(t, u) = b(u)$, for some locally integrable and nonnegative function b, then this kernel satisfies all the hypotheses written in Theorem 5.6. Firstly, as k does not depend on the first variable, $k(\cdot, u)$ is, trivially, locally absolutely continuous, for all $u \in [t_0, T)$, and $\partial_1 k \equiv 0$ on $\mathcal{T}$. Secondly, the function k_* is locally integrable because $k_*(u) = k(u, u) = b(u)$.

As a final comment regarding Theorem 5.6, notice that the characteristic inequality (5.8) does not need to be satisfied at every point of $[t_0, T)$.

As a consequence of Theorem 5.6, we derive the next result, which provides a simple estimate for the solutions of (5.6) through the use of k.

Corollary 5.8. *Let all the hypotheses of Theorem 5.6 hold. Then,*

$$x(t) \leq K \exp\left(\int_{t_0}^{t} k(t, s)\, ds\right), \quad t_0 \leq t < T,$$

where K is the same as in Theorem 5.6.

Proof. By applying Lemma 5.5 to the function $\zeta = k$, we get

$$\int_{t_0}^{t} k(t, u)\, du = \int_{t_0}^{t} \left[k(u, u) + \int_{t_0}^{u} \partial_1 k(u, s)\, ds\right] du, \quad t_0 \leq t < T.$$

Now, one can conclude the proof by choosing

$$\gamma(t) = k(t, t) + \int_{t_0}^{t} \partial_1 k(t, s)\, ds, \quad t_0 \leq t < T, \tag{5.10}$$

and

$$\gamma(t) = 0, \quad t_0 - r \leq t < t_0. \tag{5.11}$$

Indeed, it is obvious that the function γ defined by (5.10) and (5.11) satisfies the characteristic inequality (5.8). □

The previous result yields that the classical estimate is also valid, but it is not necessarily the best upper bound for the solutions of (5.6). In fact, the next result states which is the sharpest one and how it is characterised.

Theorem 5.9. *Let all the hypotheses of Theorem 5.6 hold. Then, the following assertions are valid:*

1. *There exists a unique locally integrable function $\hat{\gamma} : [t_0 - r, T) \to \mathbb{R}_+$ satisfying*

$$\hat{\gamma}(t) = k(t,t) \exp\left(-\int_{t-\tau(t)}^{t} \hat{\gamma}(s)\, ds\right) + \int_{t_0}^{t} \partial_1 k(t,s) \exp\left(-\int_{s-\tau(s)}^{t} \hat{\gamma}(w)\, dw\right) ds,$$

for every $t \in [t_0, T)$, and

$$\hat{\gamma}(t) = 0, \quad t \in [t_0 - r, t_0).$$

2. *Any function $\gamma : [t_0 - r, T) \to \mathbb{R}_+$ satisfying* (5.8) *also satisfies*

$$\hat{\gamma}(t) \le \gamma(t), \quad t_0 \le t < T.$$

3. *The function $\hat{x} : [t_0 - r, T) \to \mathbb{R}_+$ defined as*

$$\hat{x}(t) = c \exp\left(\int_{t_0}^{t} \hat{\gamma}(s)\, ds\right),$$

is the unique solution to the integral equation

$$x(t) = c + \int_{t_0}^{t} k(t,u) x(u - \tau(u))\, du, \quad t_0 \le t < T,$$

with

$$x(t) = c, \quad t_0 - r \le t \le t_0.$$

Proof. The proof is analogous to that of [52, Theorem 2.7]. We will only remark some key points.

In order to prove Assertion 1, we first extend the function k to

$$\tilde{\mathcal{T}} := \{(t,u) \in \mathbb{R}^2 : t_0 - r \le u \le t < T\},$$

by defining

$$k(t,u) = 0, \quad (t,u) \in \tilde{\mathcal{T}}, \quad u \in [t_0 - r, t_0).$$

Then, we choose a sequence $(\gamma_n)_{n\in\mathbb{Z}_+}$ of functions with domain $[t_0-r,T)$ defined by

$$\gamma_0(t) = k(t,t) + \int_{t_0}^{t} \partial_1 k(t,s)\,ds, \quad t_0 - r \le t < T,$$

and then, by recurrence, for $n \in \mathbb{Z}_+$,

$$\gamma_{n+1}(t) = k(t,t)\exp\left(-\int_{t-\tau(t)}^{t}\gamma_n(s)\,ds\right) + \int_{t_0}^{t}\partial_1 k(t,u)\exp\left(-\int_{u-\tau(u)}^{t}\gamma_n(s)\,ds\right)du,$$

for each $t_0 \le t < T$ and

$$\gamma_{n+1}(t) = 0, \quad t_0 - r \le t < t_0.$$

Following the same reasoning of the proof of [52, Theorem 2.7], one can ensure that

$$0 \le \gamma_{2k+1}(t) \le \gamma_{2k+3}(t) \le \cdots \le \gamma_{2k+4}(t) \le \gamma_{2k+2}(t) \le \gamma_0(t),$$

for any $k \in \mathbb{Z}_+$ and $t \in [t_0 - r, T)$. Then, by virtue of the Theorem of Dominated Convergence (see, e.g., [116, Theorem 1.34]), there exist

$$\gamma_{up} := \lim_{k\to\infty}\gamma_{2k}, \quad \gamma_{low} := \lim_{k\to\infty}\gamma_{2k+1},$$

and these limits satisfy the system

$$\gamma_{up}(t) = k(t,t)\exp\left(-\int_{t-\tau(t)}^{t}\gamma_{low}(s)\,ds\right) + \int_{t_0}^{t}\partial_1 k(t,u)\exp\left(-\int_{u-\tau(u)}^{t}\gamma_{low}(s)\,ds\right)du,$$
$$\gamma_{low}(t) = k(t,t)\exp\left(-\int_{t-\tau(t)}^{t}\gamma_{up}(s)\,ds\right) + \int_{t_0}^{t}\partial_1 k(t,u)\exp\left(-\int_{u-\tau(u)}^{t}\gamma_{up}(s)\,ds\right)du,$$

for $t_0 \le t < T$ and

$$\gamma_{up}(t) = \gamma_{low}(t) = 0, \quad t_0 - r \le t < t_0.$$

By using that $|e^{-x} - e^{-y}| \le |x-y|$, for $x, y \ge 0$, we get to

$$\begin{aligned} 0 \le \gamma_{up}(t) - \gamma_{low}(t) = k(t,t)&\int_{t-\tau(t)}^{t}[\gamma_{up}(s) - \gamma_{low}(s)]\,ds \\ &+ \int_{t_0}^{t}\partial_1 k(t,u)\int_{u-\tau(u)}^{t}[\gamma_{up}(s) - \gamma_{low}(s)]\,ds\,du, \end{aligned}$$

for any $t \in [t_0, T)$. By using the nonnegativity of $\gamma_{up}(t) - \gamma_{low}(t)$, we obtain

$$0 \le \gamma_{up}(t) - \gamma_{low}(t) \le \left(k(t,t) + \int_{t_0}^{t}\partial_1 k(t,u)\,du\right)\int_{t_0-r}^{t}[\gamma_{up}(s) - \gamma_{low}(s)]\,ds.$$

By applying a simple generalisation of the classical Gronwall-Bellman result in Theorem 5.1 (see, e.g., [112, Theorem 1.3.2]), we conclude that $\gamma_{up} - \gamma_{low} = 0$ and, then, we define $\hat{\gamma} := \gamma_{up} = \gamma_{low}$ on $[t_0 - r, T)$. It is clear that $\hat{\gamma}$ satisfies the thesis of Assertion 1 because of the system written some steps before. The uniqueness of $\hat{\gamma}$ follows analogously to the procedure in [52], by using the same Gronwall-Bellman-type result.

The second assertion can be proved in the same way as in [52]. In fact, one can take the sequence $(\gamma_n)_{n\in\mathbb{Z}_+}$ that starts with $\gamma_0 = \gamma$, where γ is as in the statement of this assertion, and it is defined in the same way for $n \geq 1$. Then,

$$\hat{\gamma} = \lim_{n\to\infty} \gamma_n \leq \gamma_0 = \gamma.$$

It only remains to complete the proof of Assertion 3. Indeed, we have that

$$c + \int_{t_0}^{t} k(t,u)\hat{x}(u-\tau(u))\,du = c + \int_{t_0}^{t} k(t,u)\,c\exp\left(\int_{t_0}^{u-\tau(u)} \hat{\gamma}(s)\,ds\right)du, \tag{5.12}$$

with the right-hand side of equation (5.12) being equal to

$$c + c\int_{t_0}^{t}\left[k(u,u)e^{I_{\hat{\gamma}}(u-\tau(u))} + \int_{t_0}^{u} \partial_1 k(u,s)e^{I_{\hat{\gamma}}(s-\tau(s))}ds\right]du.$$

By using the characteristic inequality in its equation version (see also the equation related to Assertion 1) in this case, the right-hand expression of (5.12) is also equal to

$$c\left(1 + \int_{t_0}^{t} \gamma(u)e^{I_{\hat{\gamma}}(u)}du\right) = ce^{I_{\hat{\gamma}}(t)} = c\exp\left(\int_{t_0}^{t} \hat{\gamma}(s)\,ds\right) = \hat{x}(t).$$

In order to prove the uniqueness, one can reproduce similar arguments as those used in the proof of the first assertion, reaching an inequality of the type of (5.6), to which we can apply Theorem 5.6. □

In the following example we provide a family of functions k for which we can find the sharpest estimate.

Example 5.10. Let us consider

$$x(t) \leq c + b(t)\int_{1}^{\sqrt{t}} \frac{1}{\sqrt{u}}x(u)\,du, \quad t \geq 1. \tag{5.13}$$

Then, after making the substitution $u = \sqrt{z}$ and renaming the variable, we obtain

$$x(t) \leq c + b(t)\int_{1}^{t} \frac{1}{2u^{\frac{3}{4}}}x(u-(u-\sqrt{u}))\,du, \quad t \geq 1.$$

In the Example 3.5(a) of [52], the function $\gamma(t) = \frac{1}{2t}$ provides the best estimate one could give for the solutions of (5.13) with $b(t) = 1$. This remains true when

$$b(t) = 1 - K \exp\left(-\int_1^t \frac{1}{2(s-\sqrt{s})}\, ds\right),$$

for any $K \in [0,1]$. This can be proved by checking that (5.8) holds with equality a.e. on $[1,\infty)$ and, then, Theorem 5.9 can be applied.

Now, we show some generalisations of Theorem 5.6. The first one is related to the solutions to another inequality of the type of (5.6), which has another integral term (without 'delay') on the right-hand side.

Theorem 5.11. *Let $k, g : \mathcal{T} \to \mathbb{R}_+$ be such that $k(\cdot,u), g(\cdot,u) : [u,T) \to \mathbb{R}$ are locally absolutely continuous for each fixed $u \in [t_0,T)$. Moreover, assume that $k_*, g_* : [t_0,T) \to \mathbb{R}_+$ are locally integrable and $\partial_1 k, \partial_1 g : \mathcal{T} \to \mathbb{R}_+$ are integrable on each $\mathcal{T}_\eta$, for all $\eta \in [t_0,T)$. Assume that $\tau : [t_0,T) \to \mathbb{R}_+$ is a measurable function such that*

$$t_0 - r \le t - \tau(t), \quad t_0 \le t < T.$$

If $x : [t_0 - r, T) \to \mathbb{R}_+$ is Borel measurable and locally bounded such that

$$x(t) \le c + \int_{t_0}^t g(t,u)x(u)\, du + \int_{t_0}^t k(t,u)x(u-\tau(u))\, du, \quad t_0 \le t < T, \tag{5.14}$$

then

$$x(t) \le K \exp\left(\int_{t_0}^t [\gamma(s) + g(t,s)]\, ds\right), \quad t_0 \le t < T, \tag{5.15}$$

where the function $\gamma : [t_0 - r, T) \to \mathbb{R}_+$ is locally integrable and satisfies the characteristic inequality

$$\gamma(t) \ge \psi(t,t)\exp\left(-\int_{t-\tau(t)}^t \gamma(w)\, dw\right) + \int_{t_0}^t \partial_1\psi(t,s)\exp\left(-\int_{s-\tau(s)}^t \gamma(w)\, dw\right) ds, \tag{5.16}$$

$$\psi(t,u) := k(t,u)\exp\left(-\int_{u-\tau(u)}^u g(t,s)\, ds\right)$$

for $t_0 \le t < T$ a.e., and K is equal to

$$\max\left\{c\exp\left(\int_{t_0-r}^{t_0} \gamma(s)\, ds\right),\ \sup_{t_0-r\le s\le t_0} x(s)\ \exp\left(\int_s^{t_0} \gamma(w)\, dw\right)\right\}. \tag{5.17}$$

Proof. The key point of the proof is to manipulate the inequality (5.14) in order to reach an expression that satisfies Theorem 5.6.

It can be proven, with a reasoning based on Lemma 5.12 below, that

$$\begin{aligned} x(t) \leq \exp\left(\int_{t_0}^{t} g(t,u)\,du\right) \int_{t_0}^{t} k(t,u)x(u-\tau(u)) \exp\left(-\int_{t_0}^{u} g(u,s)\,ds\right) du \\ + c\exp\left(\int_{t_0}^{t} g(t,u)\,du\right), \end{aligned} \tag{5.18}$$

for each $t \in [t_0, T)$. Then, by defining

$$G(t) := g(t,t) + \int_{t_0}^{t} \partial_1 g(t,s)\,ds, \tag{5.19}$$

and

$$y(t) := x(t)\exp\left(-\int_{t_0}^{t} G(s)ds\right),$$

it is possible to rewrite (5.18) as

$$\begin{aligned} y(t) &\leq c + \int_{t_0}^{t} k(t,u)y(u-\tau(u)) \exp\left(\int_{t_0}^{u-\tau(u)} G(s)\,ds - \int_{t_0}^{u} G(s)\,ds\right) du, \\ &= c + \int_{t_0}^{t} k(t,u)y(u-\tau(u)) \exp\left(-\int_{u-\tau(u)}^{u} G(s)\,ds\right) du. \end{aligned}$$

Now, by using arguments analogous to the ones in the beginning of the proof of Theorem 5.6, the function

$$\psi(t,u) := k(t,u)\exp\left(-\int_{u-\tau(u)}^{u} G(s)\,ds\right) = k(t,u)\exp\left(-\int_{u-\tau(u)}^{u} g(t,s)\,ds\right)$$

can play the role of k in Theorem 5.6 and

$$y(t) \leq K\exp\left(\int_{t_0}^{t} \gamma(s)\,ds\right), \quad t_0 \leq t < T,$$

where $\gamma : [t_0 - r, T) \to \mathbb{R}_+$ satisfies

$$\gamma(t) \geq \psi(t,t)\exp\left(-\int_{t-\tau(t)}^{t} \gamma(w)\,dw\right) + \int_{t_0}^{t} \partial_1\psi(t,s)\exp\left(-\int_{s-\tau(s)}^{t} \gamma(w)\,dw\right) ds$$

and K is as in (5.17). $\square$

Next, we prove the remaining step in the proof of Theorem 5.11.

Lemma 5.12. *Let the hypotheses of Theorem 5.11 hold. Then,*

$$x(t) \le c\exp\left(\int_{t_0}^{t} g(t,u)du\right) + \exp\left(\int_{t_0}^{t} g(t,u)du\right) \times \int_{t_0}^{t} k(t,u)x(u-\tau(u))\exp\left(-\int_{t_0}^{u} g(u,w)dw\right)du, \tag{5.20}$$

for each $t_0 \le t < T$.

Proof. Define z as the right-hand side of (5.14). This function is absolutely continuous on each $[t_0,t^*]$. Then, z is differentiable almost everywhere, i.e., on some set Ω_{t^*}, with $[t_0,t^*]\setminus\Omega_{t^*}$ having measure zero. Besides,

$$z'(t) = g(t,t)x(t) + \int_{t_0}^{t} \partial_1 g(t,u)x(u)du + k(t,t)x(t-\tau(t)) + \int_{t_0}^{t} \partial_1 k(t,u)x(u-\tau(u))du, \quad t\in\Omega_{t^*},$$

and using $x(t)\le z(t)$ for $t\in[t_0,t^*]$, then

$$z'(t) \le g(t,t)z(t) + \int_{t_0}^{t} \partial_1 g(t,u)z(u)du + k(t,t)x(t-\tau(t)) + \int_{t_0}^{t} \partial_1 k(t,u)x(u-\tau(u))du, \quad t\in\Omega_{t^*}. \tag{5.21}$$

As $z(u)\le z(t)$, for $u\in[t_0,t]$ a.e., we can deduce that

$$\left[z(t)\exp\left(-\int_{t_0}^{t} g(t,u)du\right)\right]' \le \left[z'(t) - g(t,t)z(t) - \int_{t_0}^{t} \partial_1 g(t,u)z(u)du\right] \times \exp\left(-\int_{t_0}^{t} g(t,u)du\right), \quad t\in\Omega_{t^*}. \tag{5.22}$$

Then, by using (5.21) and (5.22), we get

$$\left[z(t)\exp\left(-\int_{t_0}^{t} g(t,u)du\right)\right]' \le \left[k(t,t)x(t-\tau(t)) + \int_{t_0}^{t} \partial_1 k(t,u)x(u-\tau(u))du\right] \times \exp\left(-\int_{t_0}^{t} g(t,u)du\right), \quad t\in\Omega_{t^*}. \tag{5.23}$$

By integrating (5.23) between t_0 and t^*, we obtain

$$z(t)\exp\left(-\int_{t_0}^{t} g(t,u)du\right) - z(t_0) \leq \int_{t_0}^{t}\left[k(s,s)x(s-\tau(s)) + \int_{t_0}^{s}\partial_1 k(s,u)x(u-\tau(u))du\right] \times \exp\left(-\int_{t_0}^{s} g(s,w)dw\right)ds,\ \ t\in[t_0,t^*]. \quad (5.24)$$

Now, if we use the nonnegativity of g and $\partial_1 g$, then the right-hand side of the inequality (5.24) is less than or equal to

$$\int_{t_0}^{t} k(s,s)x(s-\tau(s))\exp\left(-\int_{t_0}^{s} g(s,w)dw\right)ds + \int_{t_0}^{t}\int_{t_0}^{s}\partial_1 k(s,u)x(u-\tau(u))\exp\left(-\int_{t_0}^{u} g(u,w)\,dw\right)du\,ds. \quad (5.25)$$

Then, by using Lemma 5.5 with ζ being defined as

$$\zeta(t,u) = k(t,u)x(u-\tau(u))\exp\left(-\int_{t_0}^{u} g(u,w)\,dw\right),$$

the expression (5.25) is equal to

$$\int_{t_0}^{t} k(t,u)x(u-\tau(u))\exp\left(-\int_{t_0}^{u} g(u,w)\,dw\right)du.$$

From the previous reasoning, one can rewrite (5.24) as

$$z(t) \leq c\exp\left(\int_{t_0}^{t} g(t,u)\,du\right) + \exp\left(\int_{t_0}^{t} g(t,u)\,du\right)\int_{t_0}^{t} k(t,u)x(u-\tau(u))\exp\left(-\int_{t_0}^{u} g(u,w)\,dw\right)du,$$

where we have used $z(t_0) = c$. Now the result follows from the definition of z, since

$$x(t) \leq z(t), \quad \text{for any } t\in[t_0,T),$$

and from the fact that t^* was arbitrarily chosen in $[t_0,T)$. □

Remark 5.13. Notice that the proof of the previous theorem is supported in the thesis of Theorem 5.6. Besides, the assertions of Theorem 5.9 can be adapted to this case and the sharpest estimate is produced by the unique function $\hat{\gamma}$ that satisfies (5.16) with equality.

Finally, as done in [52], it is possible to substitute c by a positive, measurable and nondecreasing function with the goal of obtaining a result analogous to the previous ones.

Theorem 5.14. *Let $p : [t_0, T) \to \mathbb{R}_+$ be a measurable, positive and nondecreasing function. Let $k, g : \mathcal{T} \to \mathbb{R}_+$ be such that $k(\cdot, u), g(\cdot, u) : [u, T) \to \mathbb{R}$ are locally absolutely continuous for each fixed $u \in [t_0, T)$. Moreover, assume that $k_*, g_* : [t_0, T) \to \mathbb{R}_+$ are locally integrable and $\partial_1 k, \partial_1 g : \mathcal{T} \to \mathbb{R}_+$ are integrable on each $\mathcal{T}_\eta$, for all $\eta \in [t_0, T)$. Assume that $\tau : [t_0, T) \to \mathbb{R}_+$ is a measurable function such that*

$$t_0 - r \le t - \tau(t), \quad t_0 \le t < T.$$

If $x : [t_0 - r, T) \to \mathbb{R}_+$ is Borel measurable and locally bounded such that

$$x(t) \le p(t) + \int_{t_0}^{t} g(t,u)x(u)\,du + \int_{t_0}^{t} k(t,u)x(u - \tau(u))\,du, \quad t_0 \le t < T, \tag{5.26}$$

then

$$x(t) \le Kp(t)\exp\left(\int_{t_0}^{t} [\gamma(s) + g(t,s)]\,ds\right), \quad t_0 \le t < T, \tag{5.27}$$

where the function $\gamma : [t_0 - r, T) \to \mathbb{R}_+$ is locally integrable and satisfies the characteristic inequality

$$\gamma(t) \ge \psi(t,t)\exp\left(-\int_{t-\tau(t)}^{t} \gamma(w)\,dw\right) + \int_{t_0}^{t} \partial_1\psi(t,s)\exp\left(-\int_{s-\tau(s)}^{t} \gamma(w)\,dw\right)\,ds, \tag{5.28}$$

$$\psi(t,u) := k(t,u)\exp\left(-\int_{u-\tau(u)}^{u} g(t,s)\,ds\right)$$

for $t_0 \le t < T$ a.e., and

$$K := \max\left\{\exp\left(\int_{t_0 - r}^{t_0} \gamma(s)\,ds\right), \sup_{t_0 - r \le s \le t_0} \frac{x(s)}{p(s)}\,\exp\left(\int_{s}^{t_0} \gamma(w)\,dw\right)\right\}.$$

Proof. The proof follows by dividing by $p(t)$ in (5.26) and applying Theorem 5.11 with $c = 1$ and the function

$$z(t) := \frac{x(t)}{p(t)}$$

playing the role of x. □

5.4 An application to functional differential equations

The results of the previous sections can be used to study some types of functional differential equations. In particular, some delay differential equations similar to those included in the expression (1.13) can be treated via the techniques shown in this chapter. Nevertheless, notice that, in Section 1.3, we assumed that the delay was finite and bounded. Now we get rid of the long-term boundedness of the delay, that is, we allow situations in which the larger t is, the larger the delay may be. For instance, consider the equation

$$x'(t) = -a(t)x(t) + \int_{\sigma}^{t} \eta(t,s)x(\nu(s))\,ds, \quad t \geq \sigma, \tag{5.29}$$

for $\sigma \in \mathbb{R}$, the continuous functions $\eta : \mathcal{T} \to \mathbb{R}_+$ and $a : [\sigma, \infty) \to \mathbb{R}_+$ and the increasing function $\nu : [\sigma, \infty) \to \mathbb{R}$ of class $\mathcal{C}^1$ such that

$$\sigma - r \leq \nu(t) \leq t,$$

for a certain $r \in \mathbb{R}_+$. One can identify in (5.29) both a linear destruction term and a production term, which is given by the delayed feedback

$$\hat{f}(t,\phi) = \int_{\nu(\sigma)-t}^{\nu(t)-t} \hat{\eta}(t,u)\,\phi(u)\,du,$$

for some $\hat{\eta} : \mathcal{T} \to \mathbb{R}_+$ and a certain domain $\operatorname{Dom} \hat{f}$. Notice that, at each instant t, the evolution of $x(t)$ depends on the values of x on $[\nu(\sigma), \nu(t)]$. Therefore, the length of the time interval for which we need to know the solution is $t - \nu(\sigma)$, which is obviously unbounded on $[\sigma, \infty)$. In other words, we deal with a finite delay which is unbounded as $t \to \infty$.

The peculiarity of equation (5.29) deserves some remarks when it comes to global existence of solutions. Although we only need the values of x on $[\sigma - r, \sigma]$, the equation instantly shows a delay higher than r. Therefore, in order to show that any solution of (5.29) through (σ, ϕ), $\phi \in \mathcal{C}([-r,0],\mathbb{R})$, is globally defined, we need to use a reasoning on finite time intervals, that is, we need to apply the results of Section 1.1 with an arbitrary $\tau > r$ and, then, compute the solutions on $[\sigma, \sigma + \tau)$. In fact, we can choose an increasing sequence of real numbers $(\tau_k)_{k \in \mathbb{Z}_+}$ such that $\tau_0 > r$ and $\tau_k \to \infty$ as $k \to \infty$, and obtain a solution defined globally, that is, a solution defined on $[\sigma, \infty)$. Moreover, equation (5.29) also shows uniqueness of solutions on finite time intervals. Hence, we obtain that there exists a unique globally defined solution of (5.29) through (σ, ϕ), where the function $\phi : [-r, 0] \to \mathbb{R}$ is continuous. For further details, see, e.g., [68, Chapter 3, Subsection 2.2].

Having shown that equation (5.29) has desirable properties, it is time to study the long-time behaviour of its solutions. Indeed, one can integrate (5.29) and obtain

$$x(t)\exp\left(\int_\sigma^t a(w)\,dw\right) = x(\sigma) + \int_\sigma^t \exp\left(\int_\sigma^s a(w)\,dw\right)\int_\sigma^s \eta(s,u)x(\nu(u))\,du\,ds. \quad (5.30)$$

By the change of variables

$$y(t) := x(t)\exp\left(\int_\sigma^t a(w)\,dw\right),$$

we rewrite (5.30) as

$$y(t) = y(\sigma) + \int_\sigma^t \int_\sigma^s \eta(s,u)\exp\left(\int_{\nu(u)}^s a(w)\,dw\right) y(\nu(u))\,du\,ds.$$

Now, if we consider

$$\tau(t) := t - \nu(t), \quad \tilde{\eta}(t,u) := \eta(t,u)\exp\left(\int_{\nu(u)}^t a(w)\,dw\right),$$

then, we obtain

$$y(t) = y(\sigma) + \int_\sigma^t \int_\sigma^s \tilde{\eta}(s,u)y(u-\tau(u))\,du\,ds$$

so, by virtue of Lemma 5.5, one can write

$$|y(t)| \le |y(\sigma)| + \int_\sigma^t \chi(t,s)|y(s-\tau(s))|\,ds, \quad (5.31)$$

provided that the function χ is nonnegative and such that $\tilde{\eta} = \partial_1\chi$.

It should be clear that (5.31) fits into the expressions handled in this chapter. In particular, one could have repeated the same kind of arguments for some more general equations like

$$x'(t) = -a(t)x(t) + \beta(t)x(\nu(t)) + \int_\sigma^t \hat{\eta}(t,s)x(\nu(s))\,ds, \quad (5.32)$$

as we show in the example below.

Example 5.15. Consider the functional differential equation

$$x'(t) = -\frac{2}{t}x(t) - \frac{1}{4t^2}x\left(\frac{\sqrt{t}}{2}\right) + \int_1^t \frac{1}{4t^2 s}x\left(\frac{\sqrt{s}}{2}\right)ds, \quad t \ge 1, \quad (5.33)$$

which is (5.32) with

$$\sigma = 1, \quad a(t) = \frac{2}{t}, \quad \beta(t) = -\frac{1}{4t^2}, \quad \hat{\eta}(t,s) = \frac{1}{4t^2 s}, \quad \nu(t) = \frac{\sqrt{t}}{2}, \quad r = \frac{1}{2}.$$

We will see that all solutions to (5.33) tend to 0 as $t \to \infty$. By integrating and applying analogous arguments to those previously seen, we get

$$\begin{aligned} x(t)\, t^2 - x(1) &= x(t) \exp\left(\int_1^t \frac{2}{w}\, dw \right) - x(1) \\ &= \int_1^t \left[-\frac{1}{4s^2} x\left(\frac{\sqrt{s}}{2}\right) + \int_1^s \frac{1}{4s^2 u} x\left(\frac{\sqrt{u}}{2}\right) du \right] \exp\left(\int_1^s \frac{2}{w}\, dw \right) ds \\ &= \int_1^t \left[-\frac{1}{4s^2} x\left(\frac{\sqrt{s}}{2}\right) + \int_1^s \frac{1}{4s^2 u} x\left(\frac{\sqrt{u}}{2}\right) du \right] s^2\, ds. \end{aligned}$$

The former computations lead to

$$t^2 x(t) = x(1) + \int_1^t \left[-\frac{1}{4} x\left(\frac{\sqrt{s}}{2}\right) + \int_1^s \frac{1}{4u} x\left(\frac{\sqrt{u}}{2}\right) du \right] ds. \tag{5.34}$$

If we define

$$y(t) := t^2 x(t) = x(t) \exp\left(\int_1^t \frac{2}{w}\, dw \right),$$

we can write (5.34) in terms of y and obtain

$$y(t) = y(1) + \int_1^t \left[-\frac{1}{s} y\left(\frac{\sqrt{s}}{2}\right) + \int_1^s \frac{1}{u^2} y\left(\frac{\sqrt{u}}{2}\right) du \right] ds.$$

Then,

$$|y(t)| \leq |y(1)| + \int_1^t \left[\frac{1}{s} \left| y\left(\frac{\sqrt{s}}{2}\right) \right| + \int_1^s \frac{1}{u^2} \left| y\left(\frac{\sqrt{u}}{2}\right) \right| du \right] ds. \tag{5.35}$$

By applying Lemma 5.5 for $\zeta(t,s) = k(t,s)|y(\sqrt{s}/2)|$, with

$$k(t,s) = \frac{t}{s^2},$$

we can write (5.35) as

$$|y(t)| \leq |y(1)| + \int_1^t \frac{t}{s^2} \left| y\left(\frac{\sqrt{s}}{2}\right) \right| ds, \tag{5.36}$$

which is included in the framework of Theorem 5.6. Now, we can choose any locally integrable function $\gamma : [\frac{1}{2}, \infty) \to \mathbb{R}_+$ such that

$$\gamma(t) := \frac{1}{t}, \quad t \geq 1.$$

It can be checked, by doing some calculations, that the generalised characteristic inequality (5.8) holds for this case, i.e.,

$$\frac{1}{t} \geq \frac{1}{t} \exp\left(- \int_{\frac{\sqrt{t}}{2}}^{t} \frac{1}{s}\, ds \right) + \int_1^t \frac{1}{u^2} \exp\left(- \int_{\frac{\sqrt{u}}{2}}^{t} \frac{1}{s}\, ds \right) du.$$

Hence, we can ensure that there exists some $K \in \mathbb{R}_+$ such that

$$|y(t)| \leq K \exp\left(\int_1^t \gamma(s)\, ds \right) = K \exp\left(\int_1^t \frac{1}{s}\, ds \right) = Kt, \quad t \geq 1.$$

Then,

$$|x(t)| \leq \frac{K}{t}, \quad t \geq 1,$$

which obviously leads to

$$\lim_{t\to\infty} x(t) = 0.$$

Remark 5.16. The classical estimate that Corollary 5.8 provides for the solutions of the inequality (5.36) is

$$|y(t)| \leq K' \exp\left(\int_1^t k(t,s)\, ds \right) = K' \exp\left(\int_1^t \frac{t}{s^2}\, ds \right) = K' e^{t-1}, \quad t \geq 1,$$

for some $K' \in \mathbb{R}_+$. This estimate is not good enough to deduce global attractivity of the origin. In fact, we would have

$$|x(t)| \leq K' \frac{e^{t-1}}{t^2} \to \infty, \quad \text{as } t \to \infty.$$

Therefore, Example 5.15 shows the importance of results like [52, Theorem 2.2] or Theorem 5.6. The reason is that the characteristic inequality opens a window for more functions than the classical estimates with $b(u)$ and $k(t,u)$, respectively, that serve to 'build' estimates for the solutions to certain integral inequalities.

5.5 Conclusion

In this chapter, we have presented some generalisations of the results of [52] and some classical results as Theorems 5.1 and 5.2. Theorems 5.6 and 5.9 represent the central part of the exposed contents and provide, respectively, estimates for the solutions to the integral inequality (5.6) and their sharpness properties. Finally, we have proposed an example of a non-autonomous functional differential equation (with an integral term) whose stability properties are derived from the aforementioned results.

In fact, another look into the hypotheses of our main results may be sufficient to realise that not too much regularity was imposed on the solutions of the integral inequalities, so it would be interesting to relate these results with more general situations when it comes to the concept of solution of a DDE (recall the comments on page 5).

Chapter 6

Periodic solutions for scalar delay differential equations with impulses and infinite delay

Sufficient conditions for the existence of at least one positive periodic solution are established for a family of scalar periodic differential equations with infinite delay and nonlinear impulses. Our results, obtained by applying a fixed point argument to an original operator constructed here, allow to deal with equations incorporating a rather general nonlinearity and impulses whose signs may vary. Applications to some classes of Volterra integro-differential equations with unbounded or periodic delay and nonlinear impulses are given, extending and improving results in the literature. The current chapter is based on the work published in [19], an article by the author of this thesis (S. Buedo-Fernández[1]) and Teresa Faria[2], which arose from the internship that the first one did in CMAF-CIO (Universidade de Lisboa). We also write below the complete reference of the above-mentioned article.

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6.1 Introduction

In recent years, much attention has been dedicated to the study of differential equations with delays and impulses, since they often produce realistic models for evolutionary systems which go through abrupt changes, caused by random or predictable external factors. In the case of autonomous equations, it is particularly relevant to study the existence and attractivity of equilibria. However, periodic phenomena contribute significantly in population dynamical systems, artificial neural networks and many other biological and physical processes as well, in which case the models are better portrayed by periodic differential equations. For periodic differential equations, without and with impulses, a key question is whether there exists any periodic solution, which, in many models, due to their real world interpretation, is required to be positive.

In this chapter, we consider a family of periodic scalar differential equations with infinite delay and impulses, written in the general abstract form

$$\begin{cases} x'(t) = -a(t)x(t) + g(t, x_t), & t \geq 0,\ t \neq t_k,\ k \in \mathbb{N}, \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), & k \in \mathbb{N}, \end{cases} \tag{6.1}$$

where the scalar functions $a(t)$ and $g(t, \varphi)$ are continuous, nonnegative and periodic in t with a common period $\omega > 0$, the impulse functions $I_k(u)$ are continuous, $k \in \mathbb{N}$, and the impulses $\Delta x(t_k) := x(t_k^+) - x(t_k)$ occur at times $t_k \to \infty$ with a periodicity ω to be specified in the next section. Here, $g : [0, \infty) \times D \to \mathbb{R}$, where $D \subset \mathcal{B}$ and $\mathcal{B}$ is an adequate Banach space of piecewise continuous functions defined on $(-\infty, 0]$ with values in $\mathbb{R}$, which are left-continuous. As usual, x_t denotes the entire past history of the system up to time t, or, in other words, $x_t(s) = x(t + s)$ for $s \leq 0$. Hence, in this context, the segments are functions $x_t : (-\infty, 0] \to \mathbb{R}$ (compare with the case of finite delay in Chapters 1–5). Indeed, equation (6.1) sets a general framework for many relevant models from mathematical biology and other sciences, for which only positive (or nonnegative) solutions of (6.1) are meaningful.

The main aim of the current part of the thesis is to establish sufficient conditions for the existence of positive periodic solutions for (6.1). Our technique is based on a version of the Krasnoselskii Fixed Point Theorem in cones [48, 70], which is applied to a convenient operator whose fixed points are precisely the ω-periodic solutions of (6.1) we are looking for.

In fact, there is an extensive literature on the application of fixed point theorems, such as Banach contraction principle, Schauder or Krasnoselskii theorems, the continuation theorem, as well as other methods, which include lower and upper solutions, monotone iterative schemes or combinations with Lyapunov functionals, to the quest of positive periodic solutions to delay differential equations. Related to our research, we refer to the early works by Nieto [108], Chen [23], Jiang and Wei [66], and to [4, 46, 76, 77, 105, 129, 134, 137, 140, 145, 146], most of them considering DDEs with a single discrete delay. See also [36, 147], for results about permanence implying existence of positive periodic solutions for DDEs. In spite of this wide array of techniques from nonlinear analysis (see [28] for a theoretical background), our method is essentially new, since it relies on a fixed point argument applied to an original operator constructed here. Our method, applicable under very mild restrictions, improves previous results in the literature, and has straightforward extensions to other classes of impulsive DDEs.

The work reported here was highly motivated by the papers [40, 66, 76], where further references can be found. In [76], extending the ideas in [66, 108], Li et al. considered a family of impulsive DDEs, with a single discrete delay and positive impulses:

$$\begin{cases} x'(t) = -a(t)x(t) + f(t, x(t - \tilde{\tau}(t))), & t \geq 0,\ t \neq t_k,\ k \in \mathbb{N}, \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), & k \in \mathbb{N}, \end{cases} \tag{6.2}$$

with the functions $a(t)$, $f(t,u)$, $\tilde{\tau}(t)$ being continuous, nonnegative and ω-periodic in t, with the sequences $(t_k)_{k\in\mathbb{N}}$, $(I_k)_{k\in\mathbb{N}}$ as in (6.1), and, in addition, with the impulses satisfying $I_k(u) > 0$ for $u > 0$, $k \in \mathbb{N}$. As in [105, 139, 145, 146] and many other papers, in [76] the authors found positive periodic solutions as fixed points for a certain operator via a Krasnoselskii-type fixed point theorem; since the definition of such operator requires summing all the impulses (multiplied by a Green function) up to time t, the impulses must be positive in order to be sure that the operator maps a cone of positive functions into itself. This creates some difficulties in applications, and it turns out that some results in [76] cannot be used in the simple case of positive linear impulses (i.e., when $I_k(u) = b_k u$ with $b_k > 0, k \in \mathbb{N}$). Furthermore, from the point of view of applications, it is very restrictive to consider only positive impulses. For a discussion on the role of impulses, see e.g. [77, 117].

Recently, by introducing a different operator, Faria and Oliveira [40] studied more general equations with either finite multiple discrete delays or finite distributed delay. However, due to technical difficulties, only linear impulses were considered, as follows:

$$\begin{cases} x'(t) + a(t)x(t) = g(t, x_t), & t_0 \leq t \neq t_k, \\ x(t_k^+) - x(t_k) = b_k x(t_k), & k \in \mathbb{N}, \end{cases} \tag{6.3}$$

with the functions $a(t)$, $g(t,\varphi)$ and the sequence $(t_k)_{k\in\mathbb{N}}$ as in (6.1) and with constants $b_k \in (-1, \infty)$. The existence of positive periodic solutions for periodic DDEs (6.1) with impulses given by general continuous functions $I_k(u)$ which may change sign was the principal motivation for this chapter, which can be seen as a continuation of the work in [40]. Nonetheless, we emphasise that, even for impulsive DDEs (6.3) with finite delay, the results in this chapter are more general than the ones in [40].

One should stress that DDEs with infinite delay require a rigorous abstract formulation in an *admissible* phase space [57, 59], in order to guarantee that the initial value problems are well-posed and that the standard qualitative properties of solutions are valid.

The contents and organisation of the next sections are now described. Section 6.2 contains some preliminaries, and begins with the choice of an appropriate Banach phase space to study general equations with infinite delays and impulses. Afterwards, we set the main hypotheses for (6.1), and define a suitable cone K and an adequate operator Φ on $K \setminus \{0\}$, whose fixed points are precisely the positive ω-periodic solutions to (6.1).

Section 6.3 contains the main results of the paper, which establish sufficient conditions for the existence of fixed points for Φ. Such conditions arise by analysing the interplay between the behaviour of the impulse functions and that of the nonlinear terms in the equation, for solutions next to zero or to infinity, and are easily verified in practice. A version of the Krasnoselskii Fixed Point Theorem [48, 70] is used, both in its compressive and expansive forms. Several criteria, based on either a pointwise or an average comparison of $a(t)$ and $g(t, \varphi)$ for $t \in [0, \omega]$ and $\|\varphi\|$ in the vicinity of zero or infinity, are derived. Our results are illustrated and analysed within the context of some related literature.

In Section 6.4, applications to impulsive Volterra integro-differential equations with delay are given, with emphasis on the case of infinite distributed delay. We remark that the situation with infinite delay has not often been addressed, the works of Zhao [147] and Jiang and Wei [66] for DDEs without impulses being exceptions. Generalisations of results in [66] are given in Subsection 6.4.1, namely, we highlight the treatment of a mixed monotonicity model. The case of impulsive DDEs with a bounded periodic distributed delay is analysed in Subsection 6.4.2, extending and enhancing some previous criteria in, e.g., [4]. The models in these subsections are broad enough to include generalised Nicholson's blowflies and Mackey-Glass equations as particular examples. Some attention to impulses satisfying the requirements set in Section 6.2 is paid in Subsection 6.4.3. We only consider a few applications and some selected examples in order to avoid making this part excessively large; many other examples can be given, e.g., straightforward generalisations of the models studied in [4, 40, 76, 145]. A short section of final comments ends the chapter.

6.2 An abstract framework and preliminary results

We now set an appropriate phase space to deal with impulsive differential equations with infinite delay. Although this abstract formulation does not seem to be relevant for the strict purpose of finding periodic solutions of (6.1), it is essential to consider equations with abstract nonlinear terms $g(t, x_t)$, instead of the form $g(t, x(t - \tilde{\tau}(t)))$, or even

$$g\left(t, \int_{-\tilde{\tau}(t)}^{0} k(s)x(t+s)\, ds\right)$$

for some $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. Moreover, for other reasons, e.g., to pursue with the study of their attractivity, it is important to realise that we shall look for periodic solutions inside a Banach space X contained naturally in the phase space for (6.1). For more details and some results of existence, uniqueness, global continuation of solutions for DDEs with impulses and infinite delay, we refer to [37, 111] and the references therein. On the other hand, a reader mostly motivated by particular applications may simply focus on the definition of the space X in (6.4).

For a real interval I, let $\mathcal{B}(I,\mathbb{R})$ be the Banach space of bounded functions $\varphi : I \to \mathbb{R}$, endowed with the supremum norm $\|\cdot\|_\infty$. For a compact interval $I = [\alpha,\beta]$ ($\alpha < \beta$), consider the subspace $PC(I,\mathbb{R})$ of $\mathcal{B}(I,\mathbb{R})$ of the piecewise continuous functions defined on I which are left-continuous on $(\alpha,\beta]$, i.e.,

$$PC(I,\mathbb{R}) := \left\{ \varphi : I \to \mathbb{R} \;\middle|\; \begin{array}{c} \varphi \text{ is continuous except for a finite number of elements} \\ \text{of } I \text{ for which } \varphi(s^-) \text{ and } \varphi(s^+) \text{ exist and } \varphi(s) = \varphi(s^-) \end{array} \right\}.$$

In fact, the closure of $PC(I,\mathbb{R})$ in $\mathcal{B}(I,\mathbb{R})$ is the space $\mathcal{R}(I,\mathbb{R})$ of normalised regulated functions on I.

Next, we shall consider systems with 'infinite memory' (also known as *infinite delay*) and, thus, we take the interval $I = (-\infty, 0]$. Define

$$PC := PC((-\infty,0],\mathbb{R}) = \left\{ \varphi : (-\infty,0] \to \mathbb{R} \;\middle|\; \begin{array}{c} \text{the restriction of } \varphi \text{ to any} \\ \text{compact interval } [\alpha,\beta] \subset (-\infty,0] \\ \text{is in } \mathcal{R}([\alpha,\beta],\mathbb{R}) \end{array} \right\}.$$

The elements of PC are left-continuous and may have a countable number of discontinuities of the first kind. However, note that PC is not contained in $\mathcal{B}((-\infty,0],\mathbb{R})$. We need to choose a subset of PC and an appropriate norm, in such a way that the new space is a Banach space providing a suitable framework to handle DDEs with infinite delays and impulses [57, 59, 111]. We use a norm with a weight function f satisfying the properties below:

(F1) $f : (-\infty,0] \to [1,\infty)$ is a nonincreasing function with $f(0) = 1$,

(F2) $\displaystyle\lim_{u\to 0^-} \frac{f(t+u)}{f(t)} = 1$ uniformly on $(-\infty,0]$,

(F3) $f(s) \to \infty$ as $s \to -\infty$.

The space

$$PC_f := \left\{ \varphi \in PC((-\infty,0],\mathbb{R}) \,:\, \sup_{s\le 0} \frac{|\varphi(s)|}{f(s)} < \infty \right\}$$

is a Banach space (see [37, Lemma 3.1]) with the norm

$$\|\varphi\|_f := \sup_{s\le 0} \frac{|\varphi(s)|}{f(s)}.$$

In the space PC_f, we consider a DDE with impulses in the abstract form (6.1), where $a : \mathbb{R}_+ \to \mathbb{R}_+$, $g : \mathbb{R}_+ \times PC_f \to \mathbb{R}_+$ (or $g : \mathbb{R}_+ \times D \to \mathbb{R}_+$ with $D \subset PC_f$), $I_k : \mathbb{R}_+ \to \mathbb{R}$ (for any $k \in \mathbb{N}$) are continuous functions, and $(t_k)_{k\in\mathbb{N}}$ is an increasing sequence of positive real numbers, with $t_k \to \infty$. As mentioned in the introduction of the current chapter, for any $b \in \mathbb{R}$ and a piecewise continuous function $x : (-\infty, b) \to \mathbb{R}$, the *segment* $x_t : (-\infty, 0] \to \mathbb{R}$ for $t < b$ is defined by

$$x_t(s) = x(t + s), \quad s \in (-\infty, 0].$$

For non-impulsive versions of (6.1), we take as phase space the subset of PC_f of continuous functions, that is,

$$C_f := \left\{ \varphi \in \mathcal{C}((-\infty, 0], \mathbb{R}) \,:\, \sup_{s\le 0} \frac{|\varphi(s)|}{f(s)} < \infty \right\},$$

endowed with the above-mentioned norm $\|\cdot\|_f$. This space has often been considered in the literature of (non-impulsive) DDEs with infinite delay [59].

As usual, by a *solution* x of equation (6.1) on $[\sigma, b)$, with $0 \le \sigma < b \le \infty$, we mean a function $x : (-\infty, b) \to \mathbb{R}$ such that $x_t \in PC_f$ for $t \in [\sigma, b)$, x, x' are continuous on $[\sigma, b) \setminus \{t_k : k \in \mathbb{N}\}$, and satisfying (6.1). For (6.1), the subset of bounded functions

$$BPC := PC((-\infty, 0], \mathbb{R}) \cap \mathcal{B}((-\infty, 0], \mathbb{R}) \subset PC_f$$

is usually taken as the space of admissible initial conditions [37, 111].

We recall that a function $y : I \to \mathbb{R}$, with $I = \mathbb{R}, I = \mathbb{R}_+$ or $I = \mathbb{R}_-$, is called ω-*periodic* if $y(t + \omega) = y(t)$, for any $t, t + \omega \in I$. Since we are only interested in obtaining periodic solutions, the spaces PC_f and BPC are still too large for our purposes. Hence, we choose a subset of PC_f that fits well with the nature of the solutions we are looking for.

Consider $\omega > 0$ and points $t_1, \dots, t_p$ (for some $p \in \mathbb{N}$) such that $0 \le t_1 < \cdots < t_p < \omega$, and define the sequence $(t_k)_{k\in\mathbb{N}}$ by

$$t_{k+np} = t_k + n\omega, \quad \text{for all } n \in \mathbb{Z},\ k = 1, \dots, p.$$

Take X as the space

$$X := \left\{ y : \mathbb{R} \to \mathbb{R} \;\middle|\; \begin{array}{c} y \text{ is } \omega\text{-periodic, continuous for all } t \ne t_k, \\ \text{and } y(t_k^-) = y(t_k),\ y(t_k^+) \in \mathbb{R}, \text{ for } k \in \mathbb{Z} \end{array} \right\}, \tag{6.4}$$

and let

$$\tilde{X} := \{y_t : y \in X,\, t \in \mathbb{R}\}.$$

For any fixed $y \in X$, its corresponding segments satisfy $y_t \in PC_f$, for all t. Thus, $\tilde{X} \subset PC_f$. In this way, with the identification $y \equiv y_0 = y_{|(-\infty,0]}$, the space X can also be identified with a subspace of ω-periodic functions in PC_f, and, therefore, seen as a (closed) subset of PC_f. We now take the supremum norm in X and $\tilde{X}$, that is, $\|y\|_\infty = \sup_{t\in[0,\omega]} |y(t)|$.

Lemma 6.1. *There exists $L > 0$ such that, for any $y \in X$ and $t \in \mathbb{R}$,*

$$\|y_t\|_f \le \|y\|_\infty = \|y_t\|_\infty \le L\|y_t\|_f.$$

In particular, the norms $\|\cdot\|_f$ and $\|\cdot\|_\infty$ are equivalent in both $\tilde{X}$ and X.

Proof. Let $y \in X, t \in \mathbb{R}$. Notice that $\|y\|_\infty = \|y_t\|_\infty$. On the one hand, we have

$$\|y_t\|_f = \sup_{s\le 0} \frac{|y(t+s)|}{f(s)} \le \sup_{s\le 0} |y(t+s)| = \sup_{s\in[0,\omega]} |y(s)| = \|y\|_\infty.$$

On the other hand,

$$\|y\|_\infty = \sup_{s\in[-\omega,0]} |y(t+s)| \le L \sup_{s\le 0} \frac{|y(t+s)|}{f(s)} = L\|y_t\|_f,$$

where $L := \sup\{f(s) : s \in [-\omega, 0]\} = f(-\omega)$. □

In view of the previous lemma, from now on we shall work in the Banach space X with the norm $\|\cdot\|_\infty$, simply denoted by $\|\cdot\|$. Thus, $\mathbb{R}_+ \times \tilde{X}$ is taken as the domain of $g(t,\varphi)$.

The following hypotheses on (6.1) will be assumed:

(A1) The functions $a : \mathbb{R}_+ \to \mathbb{R}$, $g : \mathbb{R}_+ \times \tilde{X} \to \mathbb{R}$ are nonnegative, continuous, nonzero, ω-periodic in $t \in \mathbb{R}_+$, for some $\omega > 0$, and, additionally, g is bounded on bounded sets of $\mathbb{R}_+ \times \tilde{X}$.

(A2) The functions $I_k : [0,\infty) \to \mathbb{R}$, $k \in \mathbb{N}$, are continuous and there is a positive integer p such that $0 \le t_1 < \cdots < t_p < \omega$ and $t_{k+p} = t_k + \omega,\ I_{k+p} = I_k$, $k \in \mathbb{N}$.

(A3) There exist constants $a_k > -1$ and b_k such that $a_k y \le I_k(y) \le b_k y$, for $y \ge 0$ and every $k \in \{1, \dots, p\}$.

(A4) $\displaystyle\prod_{i=1}^{p}(1+b_k) < \exp\left(\int_0^\omega a(t)\,dt\right)$.

We give a few comments about the choice of the above hypotheses. Firstly, under (A2)–(A3), the constants a_k and b_k can be extended for $k > p$ by ω-periodicity, i.e., $a_{k+p} = a_k$, $b_{k+p} = b_k$, $k \in \mathbb{N}$, and chosen as the sharpest ones, that is,

$$a_k = \inf_{u>0} \frac{I_k(u)}{u}, \quad b_k = \sup_{u>0} \frac{I_k(u)}{u}, \quad k \in \mathbb{N}.$$

Secondly, we highlight that, in [40], Faria and Oliveira studied the impulsive periodic scalar equations (with finite delay) (6.3) subject to linear impulses

$$\Delta x(t_k) := x(t_k^+) - x(t_k) = b_k x(t_k),$$

with t_k and $I_k(u) = b_k u$ satisfying (A2). For such systems, it was imposed in [40] that $b_k > -1$, in order to guarantee that, after suffering an impulse at the instant t_k, a positive solution remains positive. Hypothesis (A3) above is a generalisation of such assumption. It also implies that $I_k(0) = 0$ for all k, a constraint considered by many authors, see, e.g., [138]. The situation $I_k(0) \neq 0$ could also be analysed. Related to this aspect, see [109] and the references therein, for possible extensions of our method to impulsive DDEs (6.1) for which $g(t, \varphi)$ has a singularity at $\varphi = 0$. Thirdly, condition (A4), also imposed in [40, 77], expresses that the impulses are not too large when compared with the average of $a(t)$ over an interval of length ω. We remark that, even in the case of linear impulses $I_k(u) = b_k u$, the stronger restriction $\prod_{i=1}^{p}(1 + b_k) = 1$ has often been imposed [137, 140, 144, 146] (see further comments in [77]).

Let X^+ be the subset of X of nonnegative functions, i.e., of functions $y \in X$ such that $y(t) \geq 0, t \in [0, \omega]$. In order to simplify the writing, we define the following auxiliary functions:

$$A(t) := \int_0^t a(u)\, du, \quad \text{for } t \geq 0;$$

$$J_k(u) := \frac{u}{u + I_k(u)}, \quad \text{for } u > 0, \ k \in \{1, \dots, p\};$$

$$B(t; y) := \prod_{k: t_k \in [0,t)} J_k(y(t_k)), \quad \text{for } y \in X^+ \setminus \{0\}; \text{ and}$$

$$\tilde{B}(s, t; y) := \frac{B(s; y)}{B(t; y)} = \prod_{k: t_k \in [t,s)} J_k(y(t_k)), \quad \text{for } 0 \leq t \leq s \leq t + \omega, \ y \in X^+ \setminus \{0\}.$$

Bearing in mind the previous definitions, we adopt, throughout the chapter, the standard convention that a product is equal to one when the number of factors is zero.

From the definitions above and hypotheses (A2)–(A4), it is clear that

$$\frac{1}{1 + b_k} \leq J_k(u) \leq \frac{1}{1 + a_k}, \quad u > 0, \ k \in \{1, \dots, p\}. \tag{6.5}$$

Furthermore, the function $\tilde{B}(s,t;y)$ has the property

$$\tilde{B}(s+\omega,t+\omega;y)=\tilde{B}(s,t;y),\quad \text{for } 0\le t\le s\le t+\omega,\ y\in X^+\setminus\{0\}.$$

Indeed, since there is a finite number of impulses on each interval of length less than or equal to ω, the function $\tilde{B}(s,t;y)$ is uniformly bounded from above and from below by constants $\overline{B},\underline{B}\in(0,\infty)$,

$$\underline{B}\le\tilde{B}(s,t;y)\le\overline{B}\quad\text{for}\quad 0\le t\le s\le t+\omega,\ \ y\in X^+\setminus\{0\},$$

with

$$\overline{B}\le\max\left\{\prod_{k=j}^{j+l-1}\frac{1}{1+a_k}:j=1,\dots,p;\ l=0,\dots p\right\},$$
$$\underline{B}\ge\min\left\{\prod_{k=j}^{j+l-1}\frac{1}{1+b_k}:j=1,\dots,p;\ l=0,\dots,p\right\}.$$

We stress that there are significant differences between our setting and the situation in [40], where only linear impulses given by functions $I_k(u)=b_ku$ are allowed: in contrast with [40], where $J_k(u)\equiv(1+b_k)^{-1}$ is constant, in the present setting $J_k(u)$ depends on u, need not be defined at $u=0$ ($k\in\mathbb{N}$) and the functions $B(t;y)$ and $\tilde{B}(s,t;y)$ now depend on $y\in X^+\setminus\{0\}$.

Consider the partial order in X induced by the cone X^+, i.e., for $x,y\in X$, $x\le y$ means that $y-x\in X^+$. For any $\sigma\in(0,1)$, we consider a new cone in X, which has been utilised in many papers (see, e.g., [108]), and takes the form

$$K_\sigma:=\{y\in X^+:y(t)\ge\sigma\|y\|\}.$$

For a fixed $\sigma\in(0,1)$, we shall refer to K_σ simply as K. Next, we define the operator Φ on $K\setminus\{0\}$ by

$$(\Phi y)(t):=(B(\omega;y)e^{A(\omega)}-1)^{-1}\int_t^{t+\omega}\tilde{B}(s,t;y)g(s,y_s)e^{\int_t^s a(u)\,du}ds,\quad t\ge 0,\tag{6.6}$$

for $y\in K\setminus\{0\}$, where, according to the above notation,

$$A(\omega)=\int_0^\omega a(t)\,dt,\quad B(\omega;y)=\prod_{k=1}^p J_k(y(t_k)).$$

Remark 6.2. It is worth mentioning that, at each impulse instant t_k, a solution y of (6.1) satisfies $J_k(y(t_k)) = y(t_k)/y(t_k^+)$, i.e., $J_k(y(t_k))$ gives the ratio between the two one-sided limits of $y(t)$ at t_k. As a consequence, the function $x(t) := B(t;y)y(t)$ is continuous (as already observed in [138]), and this fact plays an important role, as the fixed points of the operator Φ turn out to be periodic solutions of (6.1), cf. Lemma 6.4 below.

The construction in the remainder of this section, as well as the proofs of the next lemmas, follow along the main ideas in [40, Section 2], which however have to be carefully adapted, in order to tackle the problems caused by the dependence of the functions $B(t;y)$ and $\tilde{B}(s,t;y)$ on $y \in K \setminus \{0\}$, and by the fact that Φ may not be defined at $y = 0$. Most of the arguments are included here, nevertheless, the reader can check [40] for some omitted details.

Lemma 6.3. *Assume (A1)–(A4) and take $\sigma \leq (\underline{B}/\overline{B})e^{-A(\omega)}$. Then, $\Phi(K \setminus \{0\}) \subset K$.*

Proof. Let $y \in K \setminus \{0\}$ be fixed. As seen above, $y_t \in \tilde{X} \subset PC_f$ for $t \geq 0$. The definition of Φ, the assumptions (A1)–(A4) and a simple change of variables show that $\Phi y \geq 0$ and that Φy is an ω-periodic function. Moreover, it is clear that $t \mapsto (\Phi y)(t)$ is continuous for every $0 \leq t \neq t_k$ and left-continuous at t_k, for every $k \in \{1, \dots, p\}$.

Take $\varepsilon > 0$ with $\varepsilon < \min_{1 \leq k \leq p}(t_{k+1} - t_k)$, $0 \leq t \leq \omega$, and $k \in \{1, \dots, p\}$. We have

$$J_k(y(t_k))\tilde{B}(s, t_k + \varepsilon; y) = \tilde{B}(s, t_k; y),$$

for any $t_k + \varepsilon \leq s \leq t_k + \omega$, while $\tilde{B}(s, t_k + \varepsilon; y) = \tilde{B}(\omega; y)$ if $t_k + \omega < s \leq t_k + \omega + \varepsilon$. Hence,

$$\begin{aligned}(\Phi y)(t_k + \varepsilon) = &(B(\omega; y)e^{A(\omega)} - 1)^{-1} e^{-\int_{t_k}^{t_k+\varepsilon} a(u)du} \\ &\times \left[\begin{array}{l} J_k(y(t_k))^{-1} \displaystyle\int_{t_k+\varepsilon}^{t_k+\omega} \tilde{B}(s, t_k; y) g(s, y_s) e^{\int_{t_k}^{s} a(u)du} \, ds \\ +\tilde{B}(\omega; y) \displaystyle\int_{t_k+\omega}^{t_k+\omega+\varepsilon} g(s, y_s) e^{\int_{t_k}^{s} a(u)du} \, ds \end{array} \right].\end{aligned}$$

Since $\tilde{B}(\omega; y) \leq \overline{B}$ and g is bounded on bounded sets of $\mathbb{R}_+ \times \tilde{X}$, it follows

$$0 \leq \tilde{B}(\omega; y) \int_{t_k+\omega}^{t_k+\omega+\varepsilon} g(s, y_s) e^{\int_{t_k}^{s} a(u)du} \, ds \leq e^{2A(\omega)} \overline{B} \int_{t_k+\omega}^{t_k+\omega+\varepsilon} g(s, y_s) \, ds \to 0 \quad \text{as} \quad \varepsilon \to 0^+.$$

Therefore, by letting $\varepsilon \to 0^+$, we obtain

$$\Phi y(t_k^+) = J_k(y(t_k))^{-1} \Phi y(t_k) \in \mathbb{R}. \tag{6.7}$$

Thus, we conclude that $\Phi(K \setminus \{0\}) \subset X^+$.

Now, we check that $\Phi(K \setminus \{0\}) \subset K$. Take $y \in K \setminus \{0\}$ and $t \geq 0$. Then,

$$\|\Phi y\| \leq (B(\omega; y)e^{A(\omega)} - 1)^{-1}\overline{B}e^{A(\omega)} \int_0^\omega g(s, y_s)\, ds,$$
$$(\Phi y)(t) \geq (B(\omega; y)e^{A(\omega)} - 1)^{-1}\underline{B} \int_0^\omega g(s, y_s)\, ds.$$

These two inequalities imply that

$$(\Phi y)(t) \geq (\underline{B}/\overline{B})e^{-A(\omega)}\|\Phi y\| \geq \sigma\|\Phi y\|,$$

which leads to $\Phi(K \setminus \{0\}) \subset K$. □

From the former lemma, we can suppose hereafter that σ is chosen so that $K = K_\sigma$ satisfies $\Phi(K \setminus \{0\}) \subset K$.

Lemma 6.4. *Assume (A1)–(A4). Then $y \in K \setminus \{0\}$ is a positive ω-periodic solution of* (6.1) *if and only if y is a fixed point of Φ.*

Proof. Let y be a positive ω-periodic solution to (6.1). As observed in Remark 6.2, the function $x(t) := B(t; y)y(t)$ is continuous, and it satisfies

$$x'(t) = B(t; y)y'(t) = -a(t)x(t) + B(t; y)g(t, y_t),$$

for any $t \geq 0$, $t \neq t_k$, $k \in \{1, \ldots, p\}$. Integration over $[t, t + \omega]$ leads to

$$\begin{aligned} x(t+\omega)e^{A(t+\omega)} - x(t)e^{A(t)} &= \left[B(t+\omega; y)e^{A(\omega)} - B(t; y)\right] y(t)e^{A(t)} \\ &= \int_t^{t+\omega} B(s; y)g(s, y_s)e^{A(s)}\, ds. \end{aligned}$$

Since $B(t + \omega; y) = B(\omega; y)B(t; y)$ and $\tilde{B}(s, t; y) = B(s; y)B(t; y)^{-1}$, one obtains

$$\left(B(\omega; y)e^{A(\omega)} - 1\right) y(t) = \int_t^{t+\omega} \tilde{B}(s, t; y)g(s, y_s)e^{\int_t^s a(u)\, du}\, ds,$$

and, thus, $y = \Phi y$. Finally, the converse follows along similar lines. □

In view of Lemma 6.4, finding positive periodic solutions of (6.1) in a cone K is equivalent to finding fixed points of the operator Φ. To provide some criteria for the existence of a fixed point of Φ in $K \setminus \{0\}$, the Krasnoselskii Fixed Point Theorem in the version given below will be used.

Theorem 6.5. *[28] Let X be a Banach space, K be a cone in X and consider the conical shell $A_{r,R} := \{y \in K : r \leq \|y\| \leq R\}$, for some $0 < r < R$. Let $T : A_{r,R} \to K$ be a completely continuous operator. Suppose that there exist r, R with $0 < r < R$ such that one of the following forms is satisfied:*

(a) compressive form, given by the conditions

(a1) $\|Ty\| \leq R$, if $y \in K$, $\|y\| = R$;

(a2) there exists $\psi \in K \setminus \{0\}$ such that $y \neq Ty + \lambda\psi$ for all $y \in K$ with $\|y\| = r$ and $\lambda > 0$;

(b) expansive form, given by the conditions

(b1) $\|Ty\| \leq r$, if $y \in K$, $\|y\| = r$;

(b2) there exists $\psi \in K \setminus \{0\}$ such that $y \neq Ty + \lambda\psi$ for all $y \in K$ with $\|y\| = R$ and $\lambda > 0$.

Then, there exists a fixed point y^* of T in $A_{r,R}$.

In order to apply Theorem 6.5 to our setting, we need to check that Φ is completely continuous on a conical shell $A_{r,R}$. In particular, the proof of the continuity of Φ requires an extra technical hypothesis, as follows:

(A5) The family of operators $g(t, \cdot)$, with $t \in [0, \omega]$, is *uniformly equicontinuous* on bounded sets of $\tilde{X} \setminus \{0\}$, in the sense that, for any bounded set $A \subset \tilde{X} \setminus \{0\}$ and $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $|g(t, \varphi_1) - g(t, \varphi_2)| < \varepsilon$, for all $t \in [0, \omega]$ and $\varphi_1, \varphi_2 \in A$ with $\|\varphi_1 - \varphi_2\| < \delta$.

Note that in (A5) one can replace $\tilde{X}$ by $\tilde{K} := \{y_t : y \in K, t \geq 0\}$. It is clear that (A5) is satisfied if g is uniformly continuous on bounded sets of $[0, \omega] \times \tilde{X}$. However, this latter requirement is too strong for our purposes.

Lemma 6.6. *If (A1)–(A5) hold, Φ is completely continuous on any conical sector $A_{r,R}$, $0 < r < R$.*

Proof. First, we prove that Φ is continuous on $K \setminus \{0\}$. In view of (A2), the functions J_k, $k \in \mathbb{N}$, are continuous on $(0, \infty)$. Thus, $B(\omega; y)$ and $\tilde{B}(s, t; y)$ are continuous at $y \in K \setminus \{0\}$. Recall also that $B(\omega; y)$, $\tilde{B}(s, t; y)$ are uniformly bounded from above and below by positive constants $\underline{B}, \overline{B}$ on $D_\omega \times (X^+ \setminus \{0\})$, where $D_\omega = \{(s, t) : t \in [0, \omega], s \in [t, t + \omega]\}$, and g is bounded on bounded sets of $[0, \omega] \times \tilde{X}$.

Fix $y_* \in K$, $y_* \neq 0$. For any $\varepsilon \in (0, \|y_*\|)$, $t \in [0, \omega]$ and $y \in K \cap B_\varepsilon(y_*)$, write

$$\begin{aligned} |\Phi y(t) - \Phi y_*(t)| \leq & \left|(B(\omega; y)e^{A(\omega)} - 1)^{-1} - (B(\omega; y_*)e^{A(\omega)} - 1)^{-1}\right| \overline{B} e^{A(\omega)} \int_0^\omega g(s, y_s)\, ds \\ & + (B(\omega; y_*)e^{A(\omega)} - 1)^{-1} e^{A(\omega)} \overline{B} \int_0^\omega |g(s, y_s) - g(s, y_{*,s})|\, ds \\ & + (B(\omega; y_*)e^{A(\omega)} - 1)^{-1} e^{A(\omega)} \int_t^{t+\omega} \left|\tilde{B}(s, t; y) - \tilde{B}(s, t; y_*)\right| g(s, y_{*,s})\, ds. \end{aligned}$$

Clearly,

$$(B(\omega; y)e^{A(\omega)} - 1)^{-1} - (B(\omega; y_*)e^{A(\omega)} - 1)^{-1} \to 0 \quad \text{as} \quad y \to y_*.$$

By induction on p, it is easy to prove that

$$\left|\tilde{B}(s, t; y) - \tilde{B}(s, t; y_*)\right| \leq \overline{B} \sum_{k=1}^{p} |J_k(y(t_k)) - J_k(y_*(t_k))|$$

for $0 \leq t \leq s \leq t + \omega$. Consequently, there exists $\delta_1 \in (0, \varepsilon)$ such that, if $\|y - y_*\| < \delta_1$, then

$$\left|\tilde{B}(s, t; y) - \tilde{B}(s, t; y_*)\right| \leq \varepsilon \quad \text{for all} \quad (s, t) \in D_\omega,$$

and we derive that

$$\max_{t \in [0,\omega]} \int_t^{t+\omega} \left|\tilde{B}(s, t; y) - \tilde{B}(s, t; y_*)\right| g(s, y_{*,s})\, ds \to 0 \quad \text{as} \quad y \to y_*.$$

Finally, for $R = \varepsilon + \|y_*\|$, we obtain from (A5) the existence of $\delta_2 \in (0, \varepsilon)$ such that

$$|g(s, \varphi_1) - g(s, \varphi_2)| < \frac{\varepsilon}{\omega},$$

for $s \in [0, \omega]$ and $\varphi_1, \varphi_2 \in \tilde{X} \setminus \{0\}$ with $\|\varphi_1\|, \|\varphi_2\| \leq R$ and $\|\varphi_1 - \varphi_2\| < \delta_2$.

Note also that $\|y_s - y_{*,s}\| = \|y - y_*\|$ for all $s \geq 0$, implying that, for $y \in K \cap B_{\delta_2}(y_*)$, we have

$$\int_0^\omega |g(s, y_s) - g(s, y_{*,s})|\, ds < \varepsilon.$$

From the three-term upper bound of $|\Phi y(t) - \Phi y_*(t)|$ given at the beginning of the current proof and the above computations, we conclude that

$$\|\Phi y - \Phi y_*\| \to 0 \quad \text{as} \quad \|y - y_*\| \to 0.$$

Next, fix $A_{r,R}$ with $0 < r < R$. To show that $\Phi : A_{r,R} \to K$ takes bounded subsets of $A_{r,R}$ into relatively compact sets of K, we define the operator

$$(\mathcal{F}y)(t) = B(t;y)(\Phi y)(t), \quad y \in K \setminus \{0\}.$$

From formula (6.7), $(\mathcal{F}y)(t)$ is continuous on $[0,\infty)$. By reasoning as in [40, Lemma 2.3], one proves that $\mathcal{F}_0 := \{(\mathcal{F}y)|_{[0,\omega]} : y \in A_{r,R}\} \subset \mathcal{C}([0,\omega],\mathbb{R})$ is bounded and equicontinuous, and this procedure allows us to conclude that $\Phi(A_{r,R})$ is relatively compact in K (some remaining details regarding such fact are omitted). □

Remark 6.7. For the case of DDEs with finite delay (6.3), in [40], a technical condition was also imposed, in order to prove that Φ is a continuous operator: hypothesis (h5) in [40] requires that the function $(t,y) \mapsto g(t,y_t)$ is uniformly continuous on bounded sets of $[0,\omega] \times K$. However, this condition is not fulfilled by most of the functions g, since the map $t \mapsto y_t$ is not continuous in the impulsive case; in fact, the additional assumption (h5) in [40] should be replaced by the above requirement (A5).

6.3 Existence of positive periodic solutions

We are now ready to state the main result of this chapter. Beforehand, we introduce some further notation, which allows us to simplify the presentation.

For any $k \in \{1,\dots,p\}$, the function $J_k(u)$ is bounded from above and below by positive constants and, thus, the items

$$J_k(0)^s := \limsup_{u\to 0^+} J_k(u), \quad J_k(0)^i := \liminf_{u\to 0^+} J_k(u),$$
$$J_k(\infty)^s := \limsup_{u\to \infty} J_k(u), \quad J_k(\infty)^i := \liminf_{u\to \infty} J_k(u)$$

lie in $(0,\infty)$. We further denote

$$B^0 := \prod_{k=1}^{p} J_k(0)^s, \quad B_0 := \prod_{k=1}^{p} J_k(0)^i, \tag{6.8}$$
$$B^\infty := \prod_{k=1}^{p} J_k(\infty)^s, \quad B_\infty := \prod_{k=1}^{p} J_k(\infty)^i.$$

For a continuous function $h : [0,\infty) \to [0,\infty)$, let $L_i(h), L^i(h) \in [0,\infty]$, $i \in \{0,\infty\}$, be defined by

$$L_0(h) := \liminf_{u\to 0^+} \frac{h(u)}{u}, \quad L_\infty(h) := \liminf_{u\to \infty} \frac{h(u)}{u},$$
$$L^0(h) := \limsup_{u\to 0^+} \frac{h(u)}{u}, \quad L^\infty(h) := \limsup_{u\to \infty} \frac{h(u)}{u}. \tag{6.9}$$

If it is clear which function h we are referring to, we may simply write L_i, L^i instead of $L_i(h), L^i(h)$, for $i \in \{0, \infty\}$.

In the sequel, we shall impose one of the following assumptions:

(A6) There are $r_0, R_0 \in \mathbb{R}$ with $0 < r_0 < R_0$ and continuous functions $b, h : \mathbb{R}_+ \to \mathbb{R}_+$, with $b \not\equiv 0$ and ω-periodic, such that, for $r > 0$, $y \in K$ and $t \geq 0$, the following estimates hold:

$$\begin{aligned} g(t, y_t) &\leq b(t)h(r), \quad \text{if } R_0 \leq y \leq r, \\ g(t, y_t) &\geq b(t)h(r), \quad \text{if } r \leq y \leq r_0; \end{aligned} \tag{6.10}$$

(A7) There are $r_0, R_0 \in \mathbb{R}$ with $0 < r_0 < R_0$ and continuous functions $b, h : \mathbb{R}_+ \to \mathbb{R}_+$, with $b \not\equiv 0$ and ω-periodic, such that, for $r > 0$, $y \in K$ and $t \geq 0$, the following estimates hold:

$$\begin{aligned} g(t, y_t) &\geq b(t)h(r), \quad \text{if } y \geq r \geq R_0, \\ g(t, y_t) &\leq b(t)h(r), \quad \text{if } 0 < y \leq r \leq r_0. \end{aligned} \tag{6.11}$$

Theorem 6.8. *Consider* (6.1) *and assume (A1)–(A5). In addition, assume one of the following set of hypotheses:*

(i) (sublinear case) condition (A6) holds with b, h satisfying

$$C_1 L_0 > 1 \quad \text{and} \quad C_2 L^\infty < 1, \tag{6.12}$$

where $L_0 = L_0(h), L^\infty = L^\infty(h)$ are as in (6.9) *and $C_i = C_i(b)$, $i \in \{1, 2\}$, are given by*

$$\begin{aligned} C_1(b) &:= \left(e^{A(\omega)} B^0 - 1\right)^{-1} \min_{t \in [0,\omega]} \int_t^{t+\omega} b(s) e^{\int_t^s a(u)\, du} \prod_{k: t_k \in [t,s)} J_k(0)^i \, ds, \\ C_2(b) &:= \left(e^{A(\omega)} B_\infty - 1\right)^{-1} \max_{t \in [0,\omega]} \int_t^{t+\omega} b(s) e^{\int_t^s a(u)\, du} \prod_{k: t_k \in [t,s)} J_k(\infty)^s \, ds; \end{aligned} \tag{6.13}$$

(ii) (superlinear case) condition (A7) holds with b, h satisfying

$$C_4 L^0 < 1 \quad \text{and} \quad C_3 L_\infty > 1, \tag{6.14}$$

where $L^0 = L^0(h), L_\infty = L_\infty(h)$ are as in (6.9) *and $C_i = C_i(b)$, $i \in \{3, 4\}$, are given by*

$$\begin{aligned} C_3(b) &:= \left(e^{A(\omega)} B^\infty - 1\right)^{-1} \min_{t \in [0,\omega]} \int_t^{t+\omega} b(s) e^{\int_t^s a(u)\, du} \prod_{k: t_k \in [t,s)} J_k(\infty)^i \, ds, \\ C_4(b) &:= \left(e^{A(\omega)} B_0 - 1\right)^{-1} \max_{t \in [0,\omega]} \int_t^{t+\omega} b(s) e^{\int_t^s a(u)\, du} \prod_{k: t_k \in [t,s)} J_k(0)^s \, ds. \end{aligned} \tag{6.15}$$

Then, there exists at least one positive ω-periodic solution of (6.1).

Proof. We shall use Theorem 6.5, to conclude that $\Phi : A_{r,R} \to K$ has a fixed point in a conical sector $A_{r,R} := \{y \in K : r \leq \|y\| \leq R\}$, for some $0 < r < R$.

First, suppose the sublinear case (i) is satisfied. From (A4), (6.5), (6.8) and $C_2 L^{\infty} < 1$, where C_2 is as in (6.13), one can choose $\varepsilon > 0$ sufficiently small so that $e^{A(\omega)}(B_{\infty} - \varepsilon) - 1 > 0$ and $C_2(\varepsilon) L^{\infty} < 1$, where

$$C_2(\varepsilon) = \left(e^{A(\omega)}(B_{\infty} - \varepsilon) - 1\right)^{-1} \max_{t \in [0,\omega]} \int_t^{t+\omega} \left[\prod_{k: t_k \in [t,s)} J_k(\infty)^s + \varepsilon\right] b(s) e^{\int_t^s a(u)\, du}\, ds.$$

Take $R_1 \geq R_0$ such that

$$\prod_{k=j}^{j+l-1} J_k(u_k) \leq \prod_{k=j}^{j+l-1} J_k(\infty)^s + \varepsilon \quad \text{for } u_k \geq R_1,\ k \in \{j, \dots, j+l-1\},\ j, l \in \{1, \dots, p\},$$

and

$$\prod_{k=1}^{p} J_k(u_k) \geq B_{\infty} - \varepsilon \quad \text{for} \quad u_1, \dots, u_p \geq R_1.$$

For any $R \geq \sigma^{-1} R_1$ and $y \in K$ such that $\|y\| = R$, from the definition of K we obtain that $R_0 \leq R_1 \leq y(t) \leq R$ and $\tilde{B}(s,t;y) \leq \prod_{k: t_k \in [t,s)} J_k(\infty)^s + \varepsilon$. Therefore, the first inequality in (6.10) implies

$$\begin{aligned}(\Phi y)(t) &\leq \left(e^{A(\omega)}(B_{\infty} - \varepsilon) - 1\right)^{-1} h(R) \int_t^{t+\omega} \left[\prod_{k: t_k \in [t,s)} J_k(\infty)^s + \varepsilon\right] b(s) e^{\int_t^s a(u) du}\, ds \\ &\leq h(R) C_2(\varepsilon), \quad \text{for } t \in [0,\omega].\end{aligned} \tag{6.16}$$

Since $C_2(\varepsilon) L^{\infty} < 1$, there exists $M \in \mathbb{R}$ such that $C_2(\varepsilon)\frac{h(u)}{u} < 1$ for $u \geq M$, hence we can choose any $R \geq \max\{M, \sigma^{-1} R_1\}$, to obtain

$$\|\Phi y\| < R \quad \text{for } y \in K,\ \|y\| = R.$$

Now, from $C_1 L_0 > 1$, where C_1 is as in (6.13), one can choose $\varepsilon > 0$ such that

$$\prod_{k=j}^{j+l-1} J_k(0)^i - \varepsilon > 0, \quad j, l \in \{1, \dots, p\},$$

and $C_1(\varepsilon)L_0 > 1$, where

$$C_1(\varepsilon) = \left(e^{A(\omega)}(B^0+\varepsilon)-1\right)^{-1} \min_{t\in[0,\omega]} \int_t^{t+\omega} \left[\prod_{k:t_k\in[t,s)} J_k(0)^i - \varepsilon\right] b(s) e^{\int_t^s a(u)\,du}\,ds.$$

If one considers $r \in (0, r_0)$ small enough so that $C_1(\varepsilon)\frac{h(u)}{u} > 1$, if $0 < u \le r$,

$$\prod_{k=j}^{j+l-1} J_k(u_k) \ge \prod_{k=j}^{j+l-1} J_k(0)^i - \varepsilon \quad \text{for} \quad u_k \le r,\ k\in\{j,\dots,j+l-1\},\ j,l=1,\dots,p,$$

and

$$\prod_{k=1}^{p} J_k(u_k) \le B^0 + \varepsilon \quad \text{for} \quad 0 < u_1,\dots,u_p \le r.$$

If $\psi \equiv 1$, we claim that for any $y \in K$ with $\|y\| = r$ and $\lambda > 0$, one obtains $y \neq \Phi y + \lambda\psi$ (compare with (a2) in Theorem 6.5). Indeed, assume that there exist $y_* \in K$ with $\|y_*\| = r$ and $\lambda_* > 0$ such that $y_* = \Phi y_* + \lambda_*\psi$. Then, from the second condition in (6.10), if we define $\mu := \min_{t\in[0,\omega]} y_*(t) \in [\sigma r, r]$, it holds

$$\begin{aligned}(\Phi y_*)(t) &\ge \left(e^{A(\omega)}(B^0+\varepsilon)-1\right)^{-1} h(\mu) \int_t^{t+\omega} \left[\prod_{k:t_k\in[t,s)} J_k(0)^i - \varepsilon\right] b(s) e^{\int_t^s a(u)\,du}\,ds \\ &\ge h(\mu)C_1(\varepsilon) > \mu, \quad \text{for } t\in[0,\omega].\end{aligned} \tag{6.17}$$

Hence, if we take $t_* \in [0,\omega]$ such that $y_*(t_*) = \mu$, we obtain a contradiction, since

$$(\Phi y_*)(t_*) > \mu > \mu - \lambda_* = y_*(t_*) - \lambda_*.$$

Thus, (a2) in Theorem 6.5 holds for Φ and $\psi \equiv 1$ and, finally, from Theorem 6.5(a), we conclude that there exists at least one ω-periodic solution of (6.1) in the sector $A_{r,R}$.

For the superlinear case (ii), the proof follows from Theorem 6.5(b) by arguing in a similar way, as the reader can easily check. To avoid repetitions, we do not include it here. □

Remark 6.9. As in, e.g., [3, 46, 76, 105, 146], under suitable hypotheses, a combination of both the compressive and expansive forms of Krasnoselskii's cone theorem (see [48, 70]) can lead to the existence of more than one positive periodic solution to (6.1).

Remark 6.10. It is apparent that the above method can be extended to other families of impulsive scalar DDEs. For instance, under the same general assumptions (A1)–(A3), (A5), and with (A4) being replaced by $\prod_{i=1}^{p}(1+a_k) > \exp\left(-\int_0^\omega a(t)\,dt\right)$, Theorem 6.8 applies to impulsive DDEs where the equation $x'(t) = -a(t)x(t) + g(t, x_t)$ is replaced by $x'(t) = a(t)x(t) - g(t, x_t)$. By straightforward adjustments, the above technique can also be used to study DDEs in the more general form $x'(t) = (-1)^i[-a(t)x(t)g_0(t, x(t)) + g(t, x_t)]$, with $i \in \{1, 2\}$, g_0 continuous, nonnegative and bounded from above and below by positive constants, and subject to impulses as in (6.1). For results along these lines, see [4, 105, 139, 145]. Extensions to DDEs where $g(t, \varphi)$ has a singularity at $\varphi = 0$ are also feasible, since the operator Φ in (6.6) is not required to be defined at $y = 0$.

Whenever either (A6) or (A7) is satisfied by some functions b, h, now we would like to establish some alternative criteria for the existence of a positive periodic solution based on either a pointwise or an average comparison between $a(t)$ and $b(t)$. Of course, the contribution of the impulses has to be taken into account. Some results using a pointwise comparison between $a(t)$ and $g(t, \cdot)$ can be found in [4, 40, 75, 76, 134], where a restriction $b(t) > a(t)$ is typically imposed. The second approach, relating the integral averages of $a(t), b(t)$ over $[0, \omega]$, has rarely been used in the literature, even for the case of DDEs without impulses; see [23, 40, 105]. We explore these ideas separately for the sublinear and the superlinear cases in the next corollaries, which turn out to be very usual in applications.

Corollary 6.11 (Sublinear case)**.** *Consider* (6.1) *and suppose that (A1)–(A6) hold. For functions b, h as in (A6), let $L_0 = L_0(h), L^\infty = L^\infty(h)$ be as in* (6.9) *and define*

$$\underline{B}(0) := \min\left\{\prod_{k=j}^{j+l-1} J_k(0)^i : 1 \le j \le p, 0 \le l \le p\right\},$$
$$\overline{B}(\infty) := \max\left\{\prod_{k=j}^{j+l-1} J_k(\infty)^s : 1 \le j \le p, 0 \le l \le p\right\}.$$

Assume, in addition, that either

$$m_1 C_1^* L_0 > 1, \quad m_2 C_2^* L^\infty < 1, \tag{6.18}$$

where $m_1, m_2 > 0$ are such that $m_1 a(t) \le b(t) \le m_2 a(t)$, for $t \in [0, \omega]$, and

$$\begin{aligned} C_1^* &:= \left(e^{A(\omega)}B^0 - 1\right)^{-1}\underline{B}(0)(e^{A(\omega)} - 1),\\ C_2^* &:= \left(e^{A(\omega)}B_\infty - 1\right)^{-1}\overline{B}(\infty)(e^{A(\omega)} - 1); \end{aligned} \tag{6.19}$$

or

$$C_1^{**} L_0 \ge 1, \quad C_2^{**} e^{A(\omega)} L^\infty \le 1, \tag{6.20}$$

where

$$\begin{aligned} C_1^{**} &:= \left(e^{A(\omega)}B^0 - 1\right)^{-1} \underline{B}(0) \int_0^\omega b(t)\,dt, \\ C_2^{**} &:= \left(e^{A(\omega)}B_\infty - 1\right)^{-1} \overline{B}(\infty) \int_0^\omega b(t)\,dt. \end{aligned} \tag{6.21}$$

Then, there exists at least one positive ω-periodic solution of (6.1).

Proof. If $m_1 a(t) \le b(t) \le m_2 a(t)$ for $t \in [0,\omega]$, we have

$$m_1 \int_t^{t+\omega} a(s) e^{\int_t^s a(u)\,du}\,ds \le \int_t^{t+\omega} b(s) e^{\int_t^s a(u)\,du}\,ds \le m_2 \int_t^{t+\omega} a(s) e^{\int_t^s a(u)\,du}\,ds$$

and $\int_t^{t+\omega} a(s) e^{\int_t^s a(u)\,du}\,ds = e^{A(\omega)} - 1$. For C_i and C_i^*, $i = 1, 2$, given by (6.13) and (6.19), we get the estimates

$$C_1 \ge m_1 C_1^* \quad \text{and} \quad C_2 \le m_2 C_2^*, \tag{6.22}$$

and the result follows from Theorem 6.8(a).

On the other hand, since the function a is nonzero,

$$\int_0^\omega b(s)\,ds < \int_t^{t+\omega} b(s) e^{\int_t^s a(u)\,du}\,ds < e^{A(\omega)} \int_0^\omega b(s)\,ds, \quad \text{for } t \in [0,\omega].$$

Thus, from (b) and the definition of C_i, C_i^{**}, $i = 1, 2$, it is clear that

$$C_1 > C_1^{**} \quad \text{and} \quad C_2 < C_2^{**} e^{A(\omega)}.$$

Once more, we obtain the result from Theorem 6.8(a). □

Remark 6.12. In (a) of the above corollary, if the expression $b(t) - m_1 a(t) \ge 0$ (respectively, $m_2 a(t) - b(t) \ge 0$) is nonzero, one can replace the strict inequality $m_1 C_1^* L_0 > 1$ by $m_1 C_1^* L_0 \ge 1$ (respectively, $m_2 C_2^* L^\infty < 1$ by $m_2 C_2^* L^\infty \le 1$) in (6.18). Also, if (6.10) is fulfilled with $b(t) = a(t)$, condition (6.18) reduces to $C_2^* L^\infty < 1 < C_1^* L_0$. Moreover, in the context of (6.19) and (6.21), one can obviously replace $\underline{B}(0)$ by $\underline{B}$ and $\overline{B}(\infty)$ by $\overline{B}$.

Similar arguments to those concerning Corollary 6.11, including an application of Theorem 6.8(b), lead to the following result.

Corollary 6.13. *(superlinear case) Consider* (6.1) *and suppose that (A1)–(A5) and (A7) hold. For functions b, h as in (A7), let $L^0 = L^0(h), L_\infty = L_\infty(h)$ be as in* (6.9),

$$\overline{B}(0) := \max\left\{\prod_{k=j}^{j+l-1} J_k(0)^s : 1 \le j \le p, 0 \le l \le p\right\},$$

$$\underline{B}(\infty) := \min\left\{\prod_{k=j}^{j+l-1} J_k(\infty)^i : 1 \le j \le p, 0 \le l \le p\right\}$$

and further define C_i^* and $C_i^{**} = C_i^{**}(b)$, $i \in \{3, 4\}$, by

$$C_3^* := \left(e^{A(\omega)}B^\infty - 1\right)^{-1}\underline{B}(\infty)(e^{A(\omega)} - 1),\ C_4^* := \left(e^{A(\omega)}B_0 - 1\right)^{-1}\overline{B}(0)(e^{A(\omega)} - 1), \quad (6.23)$$

$$C_3^{**} := \left(e^{A(\omega)}B^\infty - 1\right)^{-1}\underline{B}(\infty)\int_0^\omega b(t)dt,\ C_4^{**} := \left(e^{A(\omega)}B_0 - 1\right)^{-1}\overline{B}(0)\int_0^\omega b(t)dt. \quad (6.24)$$

Then, (6.1) *has at least one positive ω-periodic solution if one of the following conditions is satisfied:*

(a) $m_2 C_4^* L^0 < 1 < m_1 C_3^* L_\infty$, *where m_1, m_2 are such that $m_1 a(t) \le b(t) \le m_2 a(t)$ for $t \in [0, \omega]$;*

(b) $C_4^{**} e^{A(\omega)} L^0 \le 1 \le C_3^{**} L_\infty$.

Remark 6.14. Note that the constants in (6.19) and (6.23) do not depend explicitly on the function $b(t)$, but this dependence appears in the choice of m_1, m_2. In fact, if $a(t) > 0$ on $[0, \omega]$, in (a) of Corollaries 6.11 and 6.13 one may take

$$m_1 = \min_{t\in[0,\omega]} \frac{b(t)}{a(t)}, \quad m_2 = \max_{t\in[0,\omega]} \frac{b(t)}{a(t)}.$$

Remark 6.15. If $\lim_{u\to\infty} h(u)/u = 0$, then condition (6.12) reduces to $C_1 L_0 > 1$. Indeed, by replacing $h(u)$ by $\bar{h}(u) = C_1 h(u)$ and $b(t)$ by $\bar{b}(t) = C_1^{-1} b(t)$, this latter condition reads as $L_0 = L_0(\bar{h}) > 1$. In an analogous way, if $\lim_{u\to 0} h(u)/u = \infty$, then (6.12) reduces to $C_2 L^\infty < 1$, and, by rescaling the functions b, h, this latter requirement is given by condition $L^\infty = L^\infty(\bar{h}) < 1$ for $\bar{h}(u) = C_2 h(u)$, $\bar{b}(t) = C_2^{-1} b(t)$. Similar considerations can be given for the superlinear case when either $\lim_{u\to\infty} h(u)/u = \infty$ or $\lim_{u\to 0} h(u)/u = 0$. Hence, when one (or more) of the limits L_i or L^i is 0 or ∞, the criteria above can be simplified.

From the viewpoint of applications, the sublinear case is more useful. Note also that, for many models from biomathematics, the nonlinearity g has a strictly 'sublinear' growth at ∞, thus $L^\infty = 0$ (e.g., this always happens if g is bounded). We portray this situation below.

Corollary 6.16. *Consider* (6.1), *assume that (A1)–(A6) hold, with functions b, h in (A6) such that* $\lim_{u\to\infty} \frac{h(u)}{u} = 0$ *and, for* C_1^*, C_1^{**} *as in* (6.19), (6.21), *one of the following conditions is satisfied:*

(a) $b(t) \geq a(t)$, $t \in [0, \omega]$, *and* $C_1^* L_0 > 1$;

(b) $C_1^{**} L_0 \geq 1$.

Then, (6.1) *has a positive* ω*-periodic solution.*

Next, we use the framework above to derive results for the non-impulsive version of equation (6.1), that is, the delay differential equation

$$x'(t) = -a(t)x(t) + g(t, x_t), \quad t \geq 0, \tag{6.25}$$

which is the particular case of (6.1) with $I_k \equiv 0$, for every $k \in \mathbb{N}$. In this situation, X is simply the space of continuous and ω-periodic functions $y : \mathbb{R} \to \mathbb{R}$, endowed with the supremum norm. For (6.25), Theorem 6.8 reads as follows.

Theorem 6.17. *Consider equation* (6.25) *and assume (A1), (A5) and one of the following sets of requirements:*

(a) condition (A6) holds with b, h *satisfying* $C_{\min} L_0 > 1, C_{\max} L^\infty < 1$;

(b) condition (A7) holds with b, h *satisfying* $C_{\max} L^0 < 1, C_{\min} L_\infty > 1$;

where $L_i = L_i(h)$, $L^i = L^i(h)$, $i \in \{0, \infty\}$, *and* $C_{\min} = C_{\min}(b), C_{\max} = C_{\max}(b)$ *are defined by*

$$\begin{aligned} C_{\min} &= \left(e^{A(\omega)} - 1\right)^{-1} \min_{t\in[0,\omega]} \int_t^{t+\omega} b(s) e^{\int_t^s a(u)\,du}\, ds, \\ C_{\max} &= \left(e^{A(\omega)} - 1\right)^{-1} \max_{t\in[0,\omega]} \int_t^{t+\omega} b(s) e^{\int_t^s a(u)\,du}\, ds. \end{aligned} \tag{6.26}$$

Then, (6.25) *has at least one positive* ω*-periodic solution.*

The non-impulsive versions of the corollaries above are easily derived. For the sake of illustration, here we only write Corollary 6.11 for the situation without impulses, for which $C_1^* = C_2^* = 1$ and $C_1^{**} = C_2^{**} = (e^{A(\omega)} - 1)^{-1} \int_0^\omega b(t)\, dt$.

Corollary 6.18. *Consider* (6.25) *and assume (A1), (A5), (A6), where* b, h *in (A6) are such that either*

$$m_1 a(t) \leq b(t) \leq m_2 a(t), \ \textit{for } t \in [0,\omega], \quad \textit{and} \quad m_2 L^\infty < 1 < m_1 L_0;$$

or

$$L_0 \int_0^\omega b(t)\,dt \geq e^{A(\omega)} - 1, \quad L^\infty \int_0^\omega b(t)\,dt \leq 1 - e^{-A(\omega)}.$$

Then, (6.25) *admits at least one positive* ω*-periodic solution.*

We end this section with a few examples, to which our results are applied and analysed within the context of the related literature.

Example 6.19. In [134], Wan et al. considered the following family of non-impulsive scalar DDEs with a single discrete delay:

$$x'(t) = -a(t)x(t) + f(t, x(t - \tilde{\tau}(t))), \quad t \geq 0, \tag{6.27}$$

where the functions $a(t) > 0$, $\tilde{\tau}(t) \geq 0$, $f(t,u) \geq 0$ are continuous and ω-periodic in t. This equation has the form (6.25), for $g(t,\varphi) = f(t, \varphi(-\tilde{\tau}(t)))$, and also constitutes a periodic version of (1.15), which is one of the main equations in the first part of the thesis and has been previously studied in Section 1.3. In order to conclude the existence of an ω-periodic solution, in [134, Theorem 2.1] the authors prescribed the following sufficient conditions on f:

$$f_0 := \liminf_{u \to 0^+} \min_{t \in [0,\omega]} \frac{f(t,u)}{a(t)u} > 1, \quad f^\infty := \limsup_{u \to \infty} \max_{t \in [0,\omega]} \frac{f(t,u)}{a(t)u} < 1. \tag{6.28}$$

Note that condition (6.28) means that there exist constants $\varepsilon, M, \alpha, \beta$ with $M > \varepsilon > 0$ and $f^\infty \leq \beta < 1$, $f_0 \geq \alpha > 1$ such that

$$\frac{f(t,u)}{a(t)u} \geq \min_{t \in [0,\omega]} \frac{f(t,u)}{a(t)u} \geq \alpha, \quad \text{if } 0 < u < \varepsilon,$$
$$\frac{f(t,u)}{a(t)u} \leq \max_{t \in [0,\omega]} \frac{f(t,u)}{a(t)u} \leq \beta, \quad \text{if } u > M,$$

which implies

$$f(t,u) \geq a(t)\,\alpha u, \ \text{if } 0 < u < \varepsilon, \quad \text{and} \quad f(t,u) \leq a(t)\beta u, \ \text{if } u > M.$$

Consequently, Corollary 6.18(a) applies with $b(t) = a(t)$, $m_1 = m_2 = 1$ and a continuous function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$h(u) = \begin{cases} \alpha u, & 0 < u < \varepsilon, \\ \beta u, & u > M. \end{cases} \tag{6.29}$$

With our notations, $L_0 = \alpha > 1 > \beta = L^\infty$. Therefore, the result in [134] is generalised here via Corollary 6.18(a).

On the other hand, Amster and Idels [4] considered (6.27) with a possible state dependent delay $\sigma(t, x(t))$ instead of $\tilde{\tau}(t)$. It was shown in [4, Theorem 2.6] that (6.27) has a positive ω-periodic solution if

$$\gamma_\infty := \limsup_{u\to\infty} \max_{t\in[0,\omega]} \frac{f(t,u)}{u} < a(t) < \liminf_{u\to 0^+} \min_{t\in[0,\omega]} \frac{f(t,u)}{u} =: \gamma_0 \quad \text{for} \quad t \in [0,\omega].$$

For f_0, f^∞ as in (6.28), we have $f_0 \geq \gamma_0/a(t)$ and $f^\infty \leq \gamma_\infty/a(t)$ for $t \in [0,\omega]$. Clearly, $\gamma_\infty < a(t) < \gamma_0$ implies $f^\infty < 1 < f_0$. Hence, this result is again a particular case of that in Corollary 6.18(a).

Example 6.20. In [76], the impulsive version of (6.27) was considered (see also [139, 145]); we refer to

$$\begin{cases} x'(t) = -a(t)x(t) + f(t, x(t-\tilde{\tau}(t))), & t \geq 0,\ t \neq t_k,\ k \in \mathbb{N}, \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), & k \in \mathbb{N}, \end{cases} \tag{6.30}$$

where the functions $a(t)$, $f(t,u)$, $\tilde{\tau}(t)$ are as in (6.27) and all the impulses are given by nonnegative continuous functions $I_k(u)$ satisfying (A2). For $b(t) = a(t)$ and $h(u)$ as in (6.29), from the arguments above and Corollary 6.11(a), we get the existence of at least one positive periodic solution if $f^\infty C_2^* < 1 < f_0 C_1^*$.

As verified in many applications, we also suppose that $f^\infty = 0$.

Proposition 6.21. *For the impulsive DDE* (6.30)*, with the functions* $a(t) > 0$, $\tilde{\tau}(t) \geq 0$, $f(t,u) \geq 0$, $I_k(u) \geq 0$ *being continuous and* ω*-periodic in* t*, for* $t, u \in \mathbb{R}_+$, $k \in \mathbb{N}$*, assume (A2)–(A4) and* $f^\infty = 0$*. If*

$$f_0(e^{A(\omega)} - 1)B_0 > e^{A(\omega)}B^0 - 1, \tag{6.31}$$

then there exists at least one positive ω*-periodic solution. In particular, the latter holds if* $f_0 B_0 > 1$.

Proof. From (6.31), choose α such that $f_0 > \alpha$ and $\alpha(e^{A(\omega)} - 1)B_0 > e^{A(\omega)}B^0 - 1$, define $b(t) = a(t)$ and $h(u)$ as in (6.29). As observed above, we only need to verify that $\alpha C_1^* > 1$ for C_1^* as in (6.19). Since $I_k(u) \geq 0$ for $u > 0$, then $J_k(u) \leq 1$ for all k, $\underline{B}(0) = B_0$, and, therefore,

$$C_1^* = (e^{A(\omega)} - 1)\left(e^{A(\omega)}B^0 - 1\right)^{-1} B_0. \tag{6.32}$$

The above choice of α implies $\alpha C_1^* > 1$. Note also that $B^0 \leq 1$, and, thus, $C_1^* \geq B_0$. □

For the impulsive DDE (6.30), with $a(t), f(t,u), \tilde{\tau}(t)$ as above and $I_k(u)$ nonnegative functions satisfying (A2), Li et al. [76] studied both the sublinear and superlinear cases. For the sublinear situation, Theorem 2.3(i) in [76] asserts that (6.30) has a positive ω-periodic solution provided that

$$f_0 + (e^{A(\omega)} - 1)^{-1} I_0 > 1 \quad \text{and} \quad f^\infty + e^{A(\omega)}(e^{A(\omega)} - 1)^{-1} I^\infty < 1, \tag{6.33}$$

where f_0 and f^∞ are already defined in (6.28) and

$$I_0 = \liminf_{u\to 0^+} \sum_{k=1}^{p} \frac{I_k(u)}{u}, \quad I^\infty = \limsup_{u\to\infty} \sum_{k=1}^{p} \frac{I_k(u)}{u}.$$

Similar results can be found in [139]. In addition, suppose, e.g., that $f^\infty = 0$ and also $J_k(0)^i = J_k(0)^s = \lim_{u\to 0^+} J_k(u)$ for every $k \in \{1, \dots, p\}$. In this setting, we observe that only the first condition in (6.33) is more restrictive than (6.31). In fact, since $I_k(u) \geq 0$ and $B_0 = B^0$, we have

$$1 + \sum_{k=1}^{p} \frac{I_k(u)}{u} \leq \prod_{k=1}^{p} \Big(1 + \frac{I_k(u)}{u}\Big),$$

which implies $1 + I_0 \leq (B^0)^{-1}$. Thus, for C_1^* in (6.32) we get

$$\begin{aligned} \frac{1}{C_1^*} - \left[1 - (e^{A(\omega)} - 1)^{-1} I_0\right] &= (e^{A(\omega)} - 1)^{-1} \left[\left(e^{A(\omega)} B^0 - 1\right) B_0^{-1} - (e^{A(\omega)} - 1 - I_0)\right] \\ &= (e^{A(\omega)} - 1)^{-1} [-(B^0)^{-1} + 1 + I_0] \leq 0, \end{aligned}$$

which shows that $f_0 + (e^{A(\omega)} - 1)^{-1} I_0 > 1$ implies $f_0 C_1^* > 1$. In conclusion, the criterion in Proposition 6.21 is proven under less restrictions than in [76].

Remark 6.22. Due to the nature of our operator Φ, where the impulses intervene in a multiplicative mode (rather than additive as in [76, 139, 145] and many other papers) by means of the products of the auxiliary functions $J_k(u)$, the major results given here and in [76] are not always comparable. In [76], a key idea is that, e.g., a decay at 0 faster than linear can be tackled with an appropriate choice of positive impulses. For example, if one chooses

$$g(t, \varphi) = a(t)\varphi(-\tilde{\tau}(t))^2 e^{-\varphi(-\tilde{\tau}(t))},$$

so that $g(t,\varphi) = f(t, \varphi(-\tilde{\tau}(t)))$ for $f(t,u) := a(t)u^2 e^{-u}$ and functions $a(t)$, $\tilde{\tau}(t)$ as in (6.30), we have $f_0 = 0$, $f^\infty = 0$; from (6.33), in [76] the authors established the existence of ω-periodic orbits for (6.30) provided that

$$I_0 > e^{A(\omega)} - 1 \quad \text{and} \quad I^\infty < 1 - e^{-A(\omega)},$$

whereas in this situation we are not able to apply Theorem 6.8. On the other hand, as pointed out in Remark 6.9, when $L_0 = 0, L^\infty = 0$, one could introduce impulses in such a way that Krasnoselskii's results yield the existence of two fixed points of Φ in a suitable conical sector $A_{r,R} \subset K$. In conclusion, our Theorem 6.8 is not a proper extension of the main results in [76]. Nevertheless, our criteria apply to equations which are much more general than (6.30), and with impulses which may change sign.

Finally, it is worth adding another remark with respect to the contents of [19] in order to relate this context with the one of previous chapters.

Remark 6.23. The function f appearing in Remark 6.22 resembles the feedback map of the Lasota equation (2.25) from Chapter 2, but with $\gamma = 2 > 1$. Such example allows us to wonder whether we can provide some information in the line of our results if the feedback f in equation (6.27) (or in its impulsive version (6.30)) is similar to those chosen in Section 2.2.1. The main difference from equation (2.16) is that a, $\tilde{\tau}$ and the coefficients in the corresponding function h change from constant to periodic functions. For instance, let $\gamma \in (0,1)$, $b, d, \tilde{\tau} : \mathbb{R}_+ \to \mathbb{R}_+$ be ω-periodic functions, $\underline{d} = \min_{t\in[0,\omega]} d(t)$, $\overline{d} = \max_{t\in[0,\omega]} d(t)$ and consider the following cases:

- The periodic Lasota equation, given by

$$x'(t) = -a(t)x(t) + b(t)x^\gamma(t - \tilde{\tau}(t))e^{-d(t)x(t-\tilde{\tau}(t))}. \tag{6.34}$$

  The equation (6.34) is of the form (6.27) by considering $f(t,u) = b(t)u^\gamma e^{-d(t)u}$, and

$$f(t,u) \le b(t)u^\gamma e^{-\underline{d}u}, \quad f(t,u) \ge b(t)u^\gamma e^{-\overline{d}u}.$$

  In fact, according to [134, Theorem 2.1], or our generalised Corollary 6.18, there exists a positive ω-periodic solution of (6.34), since

$$f_0 \ge \liminf_{u\to 0^+} \min_{t\in[0,\omega]} \frac{b(t)u^\gamma e^{-\overline{d}u}}{a(t)u} = \lim_{u\to 0^+} u^{\gamma-1}e^{-\overline{d}u} \min_{t\in[0,\omega]} \frac{b(t)}{a(t)} = \infty,$$
$$f^\infty \le \limsup_{u\to\infty} \max_{t\in[0,\omega]} \frac{b(t)u^\gamma e^{-\underline{d}u}}{a(t)u} = \lim_{u\to\infty} u^{\gamma-1}e^{-\underline{d}u} \max_{t\in[0,\omega]} \frac{b(t)}{a(t)} = 0.$$

- The periodic γ-Mackey-Glass DDE, given by

$$x'(t) = -a(t)x(t) + \frac{b(t)x^\gamma(t-\tilde{\tau}(t))}{1 + d(t)x^m(t-\tilde{\tau}(t))}. \tag{6.35}$$

In this case, (6.35) is of the form (6.27) with $f(t,u) = \frac{b(t)u^\gamma}{1+d(t)u^m}$, $m > 0$, and

$$f(t,u) \le b(t)\frac{u^\gamma}{1+\underline{d}u^m}, \quad f(t,u) \ge b(t)\frac{u^\gamma}{1+\overline{d}u^m}.$$

Once more, we can apply the above-mentioned results and obtain that there exists a positive ω-periodic solution of (6.35), since

$$f_0 \ge \liminf_{u\to 0^+} \min_{t\in[0,\omega]} \frac{b(t)u^\gamma/(1+\overline{d}u^m)}{a(t)u} = \lim_{u\to 0^+} \frac{1}{u^{1-\gamma}(1+\overline{d}u^m)} \min_{t\in[0,\omega]} \frac{b(t)}{a(t)} = \infty,$$
$$f^\infty \le \limsup_{u\to\infty} \max_{t\in[0,\omega]} \frac{b(t)u^\gamma/(1+\underline{d}u^m)}{a(t)u} = \lim_{u\to\infty} \frac{1}{u^{1-\gamma}(1+\underline{d}u^m)} \max_{t\in[0,\omega]} \frac{b(t)}{a(t)} = 0.$$

6.4 Applications to delayed integro-differential equations of Volterra-type

In this section, we apply the results of Section 6.3 to some Volterra integro-differential equations which are based on well-known models from mathematical biology. In view of the applications, we will focus on some consequences of the sublinear case of Theorem 6.8, and on equations with distributed, possibly unbounded delay, since the situation of (multiple) discrete delays was largely illustrated in [40]. We begin with a general setting of Volterra integro-differential equations.

Consider a DDE with distributed delay and nonlinear impulses of the form

$$\begin{cases} x'(t) = -a(t)x(t) + \displaystyle\int_{-\infty}^{0} k(t,s)F(t,s,x(t+s))\,ds, \quad t\ge 0,\ t\ne t_k,\ k\in\mathbb{N}, \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), \quad k\in\mathbb{N}, \end{cases} \tag{6.36}$$

where:

(h1) the function $a:[0,\infty)\to[0,\infty)$ is continuous, ω-periodic and $a\not\equiv 0$;

(h2) the function $k:[0,\infty)\times(-\infty,0]\to[0,\infty)$ is measurable, ω-periodic and continuous in t, and $k(t_*,\cdot)\not\equiv 0$ a.e. on $(-\infty,0]$, for some t_*; furthermore, for any $t_0\in[0,\omega]$, there exist a constant $\varepsilon=\varepsilon(t_0)>0$ and a function $K_{t_0}(s)$ that is Lebesgue integrable on $(-\infty,0]$, such that

$$k(t,s) \le K_{t_0}(s) \quad \text{for} \quad t\in I_\varepsilon(t_0) := [t_0-\varepsilon, t_0+\varepsilon]; \tag{6.37}$$

(h3) the function $F : [0,\infty) \times (-\infty, 0] \times [0,\infty) \to [0,\infty)$ is continuous, ω-periodic in the first variable and $F \not\equiv 0$; furthermore, $F(t,s,x)$ is bounded and uniformly equicontinuous with respect to the variable x, on sets $[0,\omega] \times (-\infty, 0] \times A$ for any bounded set $A \subset [0,\infty)$, i.e., for any $L > 0$ and $\varepsilon > 0$, there exist $M > 0$ and $\delta > 0$ such that, for all $(t,s) \in [0,\omega] \times (-\infty, 0]$ and $x_1, x_2 \in [0, L]$ with $|x_1 - x_2| < \delta$,

$$|F(t,s,x_1)| \leq M \quad \text{and} \quad |F(t,s,x_1) - F(t,s,x_2)| < \varepsilon.$$

For (6.36), we shall also assume that the impulsive assumptions (A2)–(A4) are satisfied.

Under (h2), the functions $s \mapsto k(t,s)$ are summable on $(-\infty, 0]$, and

$$b(t) := \int_{-\infty}^{0} k(t,s)\, ds, \quad t \geq 0, \tag{6.38}$$

is a continuous function. Of course, $b(t)$ is also nonnegative, ω-periodic and $b(t_*) > 0$ for some t_*. Together with (6.36), we shall also consider its non-impulsive version

$$x'(t) = -a(t)x(t) + \int_{-\infty}^{0} k(t,s) F(t,s,x(t+s))\, ds, \quad t \geq 0. \tag{6.39}$$

System (6.36) has the form of our general model (6.1), with the nonlinearity given by

$$g(t,\varphi) = \int_{-\infty}^{0} k(t,s) F(t,s,\varphi(s))\, ds, \quad t \in \mathbb{R}_+, \;\; \varphi \in \tilde{X}, \tag{6.40}$$

and is sufficiently general to encompass models with both discrete and distributed delays. Without loss of generality, suppose that $g \not\equiv 0$.

Lemma 6.24. *Under (h1)–(h3), $a(t)$ and the nonlinearity $g(t,\varphi)$ in* (6.40) *satisfy (A1) and (A5).*

Proof. Fix $L > 0$. From (h3), there is $M > 0$ such that $0 \leq F(t,s,\varphi(s)) \leq M$ for $t \in [0,\omega]$, $s \in \mathbb{R}_-$ and $\varphi \in \tilde{X}$ with $\|\varphi\| \leq L$. Thus, $0 \leq g(t,\varphi) \leq M \max_{t\in[0,\omega]} b(t)$, which proves that g is well-defined on $\mathbb{R} \times \tilde{X}$ and bounded on $\mathbb{R} \times (\tilde{X} \cap \overline{B(0,L)})$.

We now observe that, for any fixed $\varphi \in \tilde{X}$ with $\|\varphi\| \leq L$, the map $t \mapsto g(t,\varphi)$ is continuous on $[0,\omega]$. In fact, for $t, t_0 \in [0,\omega]$ and $s \in \mathbb{R}_-$,

$$k(t,s)F(t,s,\varphi(s)) \to k(t_0,s)F(t_0,s,\varphi(s)) \quad \text{as } t \to t_0,$$

with $0 \leq k(t,s)F(t,s,\varphi(s)) \leq MK_{t_0}(s)$ for $t \in I_\varepsilon(t_0)$, where K_{t_0} is as in (h2). By the Lesbesgue Theorem of Dominated Convergence, we have $g(t,\varphi) \to g(t_0,\varphi)$ as $t \to t_0$.

Next, fix $\varepsilon > 0$. Since the family $\{F(t,s,\cdot) : (t,s) \in [0,\omega] \times \mathbb{R}_-\}$ is uniformly equicontinuous on the interval $[0,L]$, there exists $\delta > 0$ such that, for any $t \in [0,\omega]$, $s \in \mathbb{R}_-$ and $\varphi_1, \varphi_2 \in \tilde{X} \cap \overline{B(0,L)}$ with $\|\varphi_1 - \varphi_2\| < \delta$, we have

$$|F(t,s,\varphi_1(s)) - F(t,s,\varphi_2(s))| < \varepsilon,$$

and therefore obtain

$$|g(t,\varphi_1) - g(t,\varphi_2)| \le \varepsilon \int_{-\infty}^{0} k(t,s)\,ds \le \varepsilon \max_{t\in[0,\omega]} b(t).$$

This proves that g satisfies (A5). Together with the continuity of $t \mapsto g(t,\varphi)$ (for any $\varphi \in \tilde{X}$) it also shows that $g(t,\varphi)$ is continuous on $[0,\omega] \times \tilde{X}$. □

Fix r_0, R_0 with $0 < r_0 < R_0$. Under (h2)–(h3), it is clear that (A6) is satisfied with $b(t)$ defined by (6.38) and any continuous function h such that

$$\begin{aligned} h(u) &= \sup\{F(t,s,x) : t \in [0,\omega], s \in \mathbb{R}_-, x \in [R_0,u]\} \quad \text{for} \quad u \ge R_0, \\ h(u) &= \inf\{F(t,s,x) : t \in [0,\omega], s \in \mathbb{R}_-, x \in [u,r_0]\} \quad \text{for} \quad u \le r_0. \end{aligned} \tag{6.41}$$

From Theorem 6.8(a), we derive the following result.

Theorem 6.25. *Consider* (6.36)*, assume (h1)–(h3) and (A2)–(A4), let* $b(t)$ *be as in* (6.38) *and* $h(u)$ *as in* (6.41)*. Then, there exists a positive periodic solution of* (6.36) *if* $L_0C_1 > 1$ *and* $L^\infty C_2 < 1$*.*

Similarly, we can construct $h(u)$ such that (6.11) holds. This shows that, not only the results in Section 6.3 apply to broad classes of impulsive and non-impulsive models included in the form (6.36), but also that the sufficient conditions in Theorem 6.8 and its corollaries are easy to check.

Also recall that, as observed previously, in many applications the nonlinearity $g(t,\varphi)$ is strictly sublinear at ∞, i.e., $L^\infty(h) = 0$. Consequently, condition (6.12) reduces to $L_0(h)C_1(b) > 1$. Besides, after rescaling b and h as described in Remark 6.15, we may assume that $L_0(h) = 1$.

To illustrate our results, we further analyse two selected families of impulsive DDEs which fall within the scope of (6.36). They are most relevant in mathematical biology and other sciences, as they include many important models. A few concrete classical examples will also be considered.

6.4.1 Integro-differential equations with infinite distributed delay

The examples studied in [66] can be inserted in the following class of integro-differential equations, written as

$$x'(t) = -a(t)x(t) + b(t)\int_{-\infty}^{0} k(s)F(t, x(t+s))\,ds, \quad t \geq 0, \tag{6.42}$$

with the functions $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ being continuous and ω-periodic, $k : \mathbb{R}_- \to \mathbb{R}_+$ being integrable and normalised so that $\int_{-\infty}^{0} k(s)\,ds = 1$, $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ being continuous and ω-periodic in the first variable, and a, b, F also assumed to be nonzero. This equation has the form (6.39), it satisfies (h1)–(h3), and the nonlinearity in (6.40) reads as

$$g(t, \varphi) = b(t)\int_{-\infty}^{0} k(s)F(t, \varphi(s))\,ds.$$

Next, we consider the impulsive version of (6.42).

Theorem 6.26. *Consider the system*

$$\begin{cases} x'(t) = -a(t)x(t) + b(t)\displaystyle\int_{-\infty}^{0} k(s)F(t, x(t+s))\,ds, & t \geq 0,\ t \neq t_k,\ k \in \mathbb{N}, \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), & k \in \mathbb{N}, \end{cases} \tag{6.43}$$

assume that the functions a, b, k, F satisfy the above-mentioned properties, as in (6.42), *and that conditions (A2)–(A4) are satisfied. For $C_i = C_i(b)$, $i \in \{1, 2, 3, 4\}$, as in* (6.13), (6.15) *and $F_0, F^0, F_\infty, F^\infty \in \mathbb{R}_+ \cup \{\infty\}$ defined by*

$$\begin{aligned} F_0 &:= \liminf_{u \to 0^+} \left(\min_{t \in [0,\omega]} \frac{F(t,u)}{u} \right), \quad & F^0 &:= \limsup_{u \to 0^+} \left(\max_{t \in [0,\omega]} \frac{F(t,u)}{u} \right), \\ F_\infty &:= \liminf_{u \to \infty} \left(\min_{t \in [0,\omega]} \frac{F(t,u)}{u} \right), \quad & F^\infty &:= \limsup_{u \to \infty} \left(\max_{t \in [0,\omega]} \frac{F(t,u)}{u} \right), \end{aligned} \tag{6.44}$$

assume also that either

$$C_2 F^\infty < 1 < C_1 F_0,$$

or

$$C_3 F_\infty > 1 > C_4 F^0.$$

Then, (6.43) *has at least one positive ω-periodic solution.*

Proof. Assume that $C_2F^\infty < 1 < C_1F_0$. For any r_0, R_0 with, e.g., $0 < r_0 < 1 < R_0$, condition (A6) holds with $b(t)$ being the coefficient in (6.43) and $h_{r_0,R_0} : \mathbb{R}_+ \to \mathbb{R}_+$ being a continuous function defined on $[0, r_0] \cup [R_0, \infty)$ by (6.41), that is,

$$h_{r_0,R_0}(u) := \begin{cases} \min\limits_{t\in[0,\omega],x\in[u,r_0]} F(t,x) & \text{if} \quad 0 \le u \le r_0, \\ \max\limits_{t\in[0,\omega],x\in[R_0,u]} F(t,x) & \text{if} \quad u \ge R_0. \end{cases} \tag{6.45}$$

For $L_0(h_{r_0,R_0})$ as in (6.9), we have $L_0(h_{r_0,R_0}) \le F_0$ for any $r_0 \in (0,1)$. If $F_0 = \infty$, it is clear that $L_0(h_{r_0,R_0}) \to \infty$ as $r_0 \to 0^+$. Otherwise, simple calculations show that, for any $\varepsilon > 0$, we may choose r_0 small enough such that $L_0(h_{r_0,R_0}) \ge F_0 - \varepsilon$. In this way, if $1 < C_1(F_0 - \varepsilon)$, the assertion $C_1L_0(h_{r_0,R_0}) > 1$ holds true for r_0 small. An analogous procedure allows us to conclude that we can choose $R_0 > 1$ sufficiently large such that $C_2L^\infty(h_{r_0,R_0}) < 1$. The superlinear case is treated in a similar way. Theorem 6.8 implies the result. □

For the non-impulsive version (6.42), Corollary 6.16 and Remark 6.12 yield the following result.

Corollary 6.27. *For* (6.42)*, with the functions a, b, k, F being as above, assume $F^\infty = 0$ and either $b(t) \ge a(t)$ with $b \not\equiv a$ on $[0,\omega]$ and $F_0 \ge 1$; or $F_0 \int_0^\omega b(t)\,dt \ge e^{A(\omega)} - 1$. Then,* (6.42) *has at least one positive ω-periodic solution.*

Remark 6.28. Theorem 6.17 and Corollary 6.27 generalise the work by Jiang and Wei [66], where the authors obtained two results about the existence of a positive ω-periodic solution for (6.42): in [66, Theorem 2.1] under the assumptions $F_0 = \infty, F^\infty = 0$, and in [66, Theorem 2.2] assuming that $F_0 \ge 1, F^\infty = 0$, and $b(t) > a(t)$ for $t \in [0,\omega]$. In fact, these latter hypotheses are slightly stronger than the requisites in the first part of Corollary 6.27.

Some straightforward generalisations of these results are derived if (6.42) is replaced by

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{m} b_i(t) \int_{-\infty}^{0} k_i(s)F_i(t, x(t+s))\,ds,$$

where the functions b_i, k_i, F_i, $i \in \{1, \dots, m\}$, are as b, k, F in (6.42). We now give some particular examples.

Example 6.29. Consider an impulsive equation with

$$\begin{cases} x'(t) = -a(t)x(t) + b(t) \displaystyle\int_{-\infty}^{0} k(s) \frac{x(t+s)}{1 + c(t)x(t+s)^n} ds, & t \ge 0,\ t \ne t_k,\ k \in \mathbb{N}, \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), \quad k \in \mathbb{N}, \end{cases} \tag{6.46}$$

where $a, b, c : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and ω-periodic, with $a \not\equiv 0$, $b \not\equiv 0$ and $c(t) > 0$ for $t \in [0, \omega]$, $k : \mathbb{R}_- \to \mathbb{R}_+$ is integrable with $\int_{-\infty}^0 k(s)\,ds = 1$ and $n > 0$. Suppose also that the instants t_k and the functions I_k ($k \in \mathbb{N}$) satisfy (A2)–(A4). Equation (6.46) is a particular case of (6.43) with

$$F(t, x) = \frac{x}{1 + c(t)x^n}.$$

It deserves to be highlighted that (6.46) is written in terms of the generalised version of the feedback appearing in the Mackey-Glass DDE (recall equation (2.46) with $\gamma = 1$). With the previous notations, we obtain $F_0 = 1, F^\infty = 0$ in (6.44). For the constants defined in (6.19) and (6.21), and by applying Corollary 6.11, we have that (6.46) admits at least one positive ω-periodic solution if one of the following conditions is satisfied:

(a) $b(t) \geq m_1 a(t)$ for $t \in [0, \omega]$ with $\max_{t\in[0,\omega]}(b(t) - m_1 a(t)) > 0$, and $m_1 C_1^* \geq 1$;

(b) $C_1^{**} \geq 1$.

Thus, for (6.46) with no impulses, Corollary 6.27 asserts that a positive ω-periodic solution must exist if either

$$\left[\max_{t\in[0,\omega]}(b(t) - a(t)) > 0 \text{ and } \min_{t\in[0,\omega]}(b(t) - a(t)) \geq 0\right] \quad \text{or} \quad \int_0^\omega b(t)\,dt \geq e^{A(\omega)} - 1.$$

For the sake of illustration, assume that $a, b : \mathbb{R}_+ \to (0, \infty)$ are functions satisfying

$$m_1 = \min_{t\in[0,\omega]} \frac{b(t)}{a(t)} \quad \text{and} \quad b(t_0) > m_1 a(t_0), \text{ for some } t_0,$$

and suppose, in addition, that all the impulses are nonpositive, i.e., $I_k(u) \leq 0$ for every $u \geq 0$ and $k \in \{1, \ldots, p\}$. Then, we have $J_k(0)^s \geq J_k(0)^i \geq 1$ for all $k \in \mathbb{N}$, $B_0 \geq 1 = \underline{B}(0)$ and

$$C_1^* = \left(e^{A(\omega)}B^0 - 1\right)^{-1}(e^{A(\omega)} - 1), \quad C_1^{**} = \left(e^{A(\omega)}B^0 - 1\right)^{-1}\int_0^\omega b(s)\,ds.$$

Therefore, from Corollary 6.11, the existence of a positive ω-periodic solution of (6.46) follows if

$$\max\left\{m_1(e^{A(\omega)} - 1), \int_0^\omega b(s)\,ds\right\} \geq e^{A(\omega)}B^0 - 1.$$

On the contrary, if all the impulses are nonnegative, i.e., $I_k(u) \geq 0$ for $u \geq 0$ and all $k \in \{1, \ldots, p\}$, then $J_k(0)^i \leq J_k(0)^s \leq 1$ for all $k \in \mathbb{N}$, $B_0 = \underline{B}(0)$, and

$$C_1^* = \left(e^{A(\omega)}B^0 - 1\right)^{-1}B_0(e^{A(\omega)} - 1), \quad C_1^{**} = \left(e^{A(\omega)}B^0 - 1\right)^{-1}B_0\int_0^\omega b(s)\,ds.$$

Thus, since $B^0 \le 1$, Corollary 6.11 guarantees that (6.46) admits a positive ω-periodic solution if

$$\max\left\{m_1, (e^{A(\omega)}-1)^{-1}\int_0^\omega b(s)\,ds\right\} \ge B_0^{-1}.$$

Recently, there has been an increasing interest in DDEs with mixed monotonicity, where the delayed feedback (the production function, in many of our models) involves one or more functions with different delays, e.g., of the form $f(t, x(t-\tilde{\tau}(t)), x(t-\sigma(t)))$, with $f(t,x,y)$ increasing in the variable x and decreasing in y. Although small delays are generally harmless, in the sense that the delayed model has the same global properties as the equation without delays, the presence of two or more delays in the same nonlinear function may change this situation drastically, as illustrated in [14]. However, DDEs with different delays in the same nonlinear term appear naturally in real-world models; for instance, see [73, 143] and the references therein. The global dynamics of such equations have been the subject of a few recent studies, where questions of stability, persistence, permanence, and/or existence of periodic solutions were addressed [3, 14, 23, 40, 51, 143]. Nevertheless, as far as we know, only the case of discrete delays has been considered.

The next example is a model with mixed monotonicity, where the different delays in the nonlinear terms have a different nature: discrete and distributed. Two further comments are in order. Firstly, this example shows that the choice of σ for the cone $K = K_\sigma$ may have an important role. Secondly, to work on K is very useful to treat DDEs with a mixed monotonicity, since the definition of K provides natural uniform upper and lower bounds for functions $y \in K$: $\sigma\|y\| \le y(t) \le \|y\|$ on $[0,\omega]$.

Example 6.30. Consider an impulsive Nicholson-type equation given by

$$\begin{cases} x'(t) = -a(t)x(t) + \displaystyle\sum_{i=1}^m b_i(t)x(t-\sigma_i(t))\int_{-\infty}^0 k_i(s)e^{-c_i(t)x(t+s)}\,ds, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \ge 0,\ t \ne t_k, \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), \quad k \in \mathbb{N}, \end{cases} \tag{6.47}$$

where $a, b_i, c_i, \sigma_i : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and ω-periodic functions satisfying $a \not\equiv 0$, $b(t) := \sum_{i=1}^m b_i(t)$, $b \not\equiv 0$, and $c_i(t) > 0$ for $t \in [0,\omega]$, $k_i : \mathbb{R}_- \to \mathbb{R}_+$ are integrable functions with $\int_{-\infty}^0 k_i(s)\,ds = 1$, $i \in \{1,\dots,m\}$, and $t_k, I_k(u)$, $k \in \mathbb{N}$, satisfy (A2)–(A4). Here, g in (6.40) reads as

$$g(t,\varphi) = \sum_{i=1}^m f_i(t, \varphi(-\sigma_i(t)), \varphi), \quad t \ge 0,\ \varphi \in \tilde{X},$$

where

$$f_i(t,x,\varphi) := b_i(t)x\int_{-\infty}^0 k_i(s)e^{-c_i(t)\varphi(s)}ds$$

increases in the argument $x \in \mathbb{R}_+$ and decreases in the argument $\varphi \in \tilde{X}^+$, for every $i \in \{1, \dots, m\}$. In the next result, we explore the criteria in Corollary 6.11.

Proposition 6.31. *Fix $K = K_\sigma$, where $\sigma = (\underline{B}/\overline{B})e^{-A(\omega)}$, as in Lemma 6.3. Under the above conditions and notations, (6.47) admits at least one positive ω-periodic solution if, in addition, one of the following statements holds:*

(a) $C_1^* \sum_{i=1}^m b_i(t) \geq \frac{\overline{B}}{\underline{B}} a(t)e^{A(\omega)}$ on $[0,\omega]$ and $C_1^* \sum_{i=1}^m b_i(t_0) > \frac{\overline{B}}{\underline{B}} a(t_0)e^{A(\omega)}$ for some t_0;

(b) $C_1^{**} \geq \frac{\overline{B}}{\underline{B}} e^{A(\omega)}$.

In particular, the non-impulsive version of (6.47) has a positive ω-periodic solution if either

$$\gamma(t) := \sum_{i=1}^m b_i(t) - a(t)e^{A(\omega)} \geq 0 \ \text{ on } [0,\omega] \quad \text{and} \quad \gamma(t) \not\equiv 0;$$

or

$$\sum_{i=1}^m \int_0^\omega b_i(t)\,dt \geq e^{A(\omega)}(e^{A(\omega)} - 1).$$

Proof. We adapt the reasoning in the former example. In fact, we write $b(t) = \sum_{i=1}^m b_i(t)$ and define

$$\underline{c_i} := \min_{[0,\omega]} c_i(t), \quad \overline{c_i} := \max_{[0,\omega]} c_i(t),$$
$$\underline{c} := \min_{1\leq i\leq m} \underline{c_i}, \quad \overline{c} := \max_{1\leq i\leq m} \overline{c_i}.$$

Recall that the function $\theta(u) = ue^{-cu}$, where $c > 0$, attains its maximum $(ce)^{-1}$ at $u = c^{-1}$ and is increasing on $[0, c^{-1}]$. Take $r_0, R_0 > 0$, with $r_0 < \overline{c}^{-1}$ and $\sigma R_0 > \underline{c}^{-1}$.

Let $t \geq 0$, and $y \in K$, and note that $y(t) \geq \sigma\|y\|$. If $R_0 \leq y \leq R$, it holds

$$g(t, y_t) \leq \sum_{i=1}^m b_i(t)y(t - \sigma_i(t))e^{-\underline{c}\sigma\|y\|} \leq b(t)\|y\|e^{-\underline{c}\sigma\|y\|} \leq b(t)(\underline{c}\sigma e)^{-1}.$$

Moreover, if $r \leq y \leq r_0$, we also have

$$g(t, y_t) \geq \sum_{i=1}^m b_i(t)y(t - \sigma_i(t))e^{-\overline{c}\|y\|} \geq b(t)\sigma\|y\|e^{-\overline{c}\|y\|} \geq b(t)\sigma re^{-\overline{c}r}.$$

Thus, assumption (6.10) is satisfied with $h : \mathbb{R}_+ \to \mathbb{R}_+$ being a continuous function such that $h(R) = (\underline{c}\sigma e)^{-1}$ for $R \geq R_0$ and $h(r) = \sigma re^{-\overline{c}r}$ for $r \leq r_0$. Hence, $L_0 = \sigma$ and $L^\infty = 0$. If (a) is satisfied, then take $m_1 = (C_1^*\sigma)^{-1}$; the result follows from the first part of Corollary 6.11 (see (6.18) and (6.19)) and Remark 6.12. If (b) holds, we apply the second part of Corollary 6.11 (see (6.20) and (6.21)). □

We remark that Chen [23] considered the DDE

$$y'(t) + a(t)y(t) = b(t)y(t - \sigma(t))e^{-c(t)y(t-\tilde{\tau}(t)))},$$

with $a(t), b(t), \sigma(t), \tilde{\tau}(t), c(t)$ being positive, ω-periodic continuous functions, and used the Continuation Theorem of Degree Theory to prove the existence of a positive periodic solution provided that $\int_0^\omega b(s)\,ds > A(\omega)e^{2A(\omega)}$; this condition is more restrictive than the one of $\int_0^\omega b(s)\,ds \geq e^{A(\omega)}(e^{A(\omega)} - 1)$ in Proposition 6.31.

6.4.2 Integro-differential equations with periodic distributed delay

Consider the family of impulsive DDEs

$$\begin{cases} x'(t) = -a(t)x(t) + \beta(t)\displaystyle\int_{t-\tilde{\tau}(t)}^{t} c(s)H(s, x(s))\,ds, \quad t \geq 0,\ t \neq t_k,\ k \in \mathbb{N}, \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), \quad k \in \mathbb{N}, \end{cases} \tag{6.48}$$

with the functions $a, \beta, \tilde{\tau} : \mathbb{R}_+ \to \mathbb{R}_+$, $c : \mathbb{R} \to \mathbb{R}_+, H : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ being continuous, ω-periodic in t and nonzero. Assume also that assumptions (A2)–(A4) are satisfied. See, e.g., [4, 36] for non-impulsive versions of (6.48). This system has the form (6.36), with

$$k(t, s) = \beta(t)c(s + t)\chi_{[-\tilde{\tau}(t),0]}(s), \quad F(t, s, x) = H(s + t, x),$$

so that the nonlinearity is given by

$$g(t, \varphi) = \beta(t) \int_{-\tilde{\tau}(t)}^{0} c(s + t)H(s + t, \varphi(s))\,ds.$$

Of course, (h1)–(h3) are fulfilled. Set

$$b(t) = \beta(t) \int_{-\tilde{\tau}(t)}^{0} c(s + t)\,ds.$$

Condition (A6) is satisfied with a continuous function $h : \mathbb{R}_+ \to \mathbb{R}_+$ if we define $h(u)$ as in (6.41) on $[0, r_0] \cup [R_0, \infty)$, for $0 < r_0 < R_0$. Thus, the results for the sublinear case in Section 6.3 can be applied.

Example 6.32. System (6.48) represents a periodic Nicholson's blowfly model with a periodic distributed finite delay $\tilde{\tau}(t)$ if

$$F(t, s, x) = H(s + t, x) = xe^{-d(t+s)x}, \tag{6.49}$$

where $d : \mathbb{R} \to (0,\infty)$ is a continuous ω-periodic function, in which case, we obtain the system

$$\begin{cases} x'(t) = -a(t)x(t) + \beta(t)\displaystyle\int_{t-\tilde{\tau}(t)}^{t} c(s)x(s)e^{-d(s)x(s)}\, ds, \quad t \geq 0,\ t \neq t_k,\ k \in \mathbb{N}, \\ x(t_k^+) - x(t_k) = I_k(x(t_k)), \quad k \in \mathbb{N}. \end{cases} \tag{6.50}$$

Following along the lines in Example 6.30, for

$$\overline{d} := \max_{s\in[0,\omega]} d(s), \quad \underline{d} := \min_{s\in[0,\omega]} d(s),$$

choose $r_0 = \overline{d}^{-1}$ and $R_0 = \underline{d}^{-1}$, so that, in (6.41), we obtain

$$h(u) = ue^{-\overline{d}u}, \text{ if } 0 \leq u < r_0, \quad h(u) = R_0 e^{-\underline{d}R_0}, \text{ if } u \geq R_0,$$

for which $L_0(h) = 1, L^\infty(h) = 0$. Under the stated conditions on the functions $a, \beta, \tilde{\tau}, c, d$ and (A2)–(A4), Theorem 6.25 asserts that (6.50) has at least one positive ω-periodic solution if $C_1(b) > 1$. In particular, the non-impulsive Nicholson equation

$$x'(t) = -a(t)x(t) + \beta(t)\int_{t-\tilde{\tau}(t)}^{t} c(s)x(s)e^{-d(s)x(s)}\, ds, \quad t \geq 0, \tag{6.51}$$

admits at least one positive ω-periodic solution if either

$$\max_{t\in[0,\omega]} \left(\beta(t)\int_{t-\tilde{\tau}(t)}^{t} c(s)\, ds - a(t)\right) > 0, \quad \min_{t\in[0,\omega]} \left(\beta(t)\int_{t-\tilde{\tau}(t)}^{t} c(s)\, ds - a(t)\right) \geq 0, \tag{6.52}$$

or

$$\int_0^\omega \beta(t)\int_{t-\tilde{\tau}(t)}^{t} c(s)\, ds\, dt \geq e^{\int_0^\omega a(t)\, dt} - 1. \tag{6.53}$$

In [4], Amster and Idels proved this result for (6.51) under the sufficient condition

$$\min_{t\in[0,\omega]} c(t) > \max_{t\in[0,\omega]} \frac{a(t)}{\beta(t)\tilde{\tau}(t)}, \tag{6.54}$$

which is much more restrictive than (6.52). In summary, we improve significantly the criterion in [4].

If, instead of $H(t,x) = xe^{-d(t)x}$ in (6.48), we choose

$$H(t,x) = x^\gamma e^{-d(t)x}, \quad \text{where } \gamma \in (0,1),$$

we obtain a modified version of the Nicholson model, which can be seen as a gamma-model with distributed delay and for which there are $b(t), h(u)$ such that (6.10) is true and $L_0(h) = \infty, L^\infty(h) = 0$. Theorem 6.8 guarantees that there exists a positive ω-periodic solution, regardless of the sign or profile of the impulses, provided that they satisfy (A2)–(A4). If one chooses

$$H(t,x) = \frac{x}{1+d(t)x^m}, \quad \text{where } m > 0,$$

in (6.48), a version of the Mackey-Glass DDE with distributed periodic delay is obtained; conclusions as the ones for (6.50) can be stated. Indeed, one can go further and give analogous conclusions for the distributed gamma-version of such DDE, that is, the one with

$$H(t,x) = \frac{x^\gamma}{1+d(t)x^m}, \quad \text{where } m > 0,\ \gamma \in (0,1).$$

6.4.3 Effect of the impulses

It turns out that the introduction of impulses in periodic scalar DDEs can create positive periodic solutions, which do not exist otherwise. For instance, in [26, 40], particular examples of a periodic DDE of the form (6.3) exhibiting a positive periodic solution were given, for which the zero solution of the associated equation without impulses is a global attractor of all its positive solutions. Consequently, it is worth illustrating some circumstances under which the impulses satisfy our set of assumptions, and moreover either generate or at least do not destroy periodic orbits.

Thus, we go back to impulsive equations of the form of (6.1) and suppose that $a(t)$ and the term $g(t,\varphi)$ in (6.40) satisfy (A1), (A5) and (A6) with, as often occurs in applications (see the above examples), functions $b(t), h(u)$ such that $L_0 = 1, L^\infty = 0$. Within this scenario, we now give examples of impulse functions $I_k(u)$ whose sign may vary, for which (A2)–(A4) and condition $C_1 > 1$ are fulfilled, ensuring that a positive ω-periodic solution must exist.

Example 6.33. Take $I_k(u) = I(u) := \sin(u)$ for all $k \in \mathbb{N}$, and suppose that $2^p < e^{A(\omega)}$. Then, (A3), (A4) are satisfied with $a_k = -1/\pi$, $b_k = 1$ and $J(0) := J_k(0)^s = J_k(0)^i = 2^{-1}$, for all $k \in \mathbb{N}$. Moreover, $C_1 = C_1(b) > 1$ holds if

$$\min_{t\in[0,\omega]} \int_t^{t+\omega} b(s) e^{\int_t^s a(u)du}\, ds > e^{A(\omega)} - 2^p.$$

In particular, $C_1 > 1$ if

$$\int_0^\omega b(s)\, ds \geq e^{A(\omega)} - 2^p.$$

Thus, impulses can create a periodic solution.

To illustrate this situation, consider a periodic DDE with multiple discrete delays and subject to the above impulses:

$$\begin{cases} x'(t) = -a(t)x(t) + \displaystyle\sum_{i=1}^{m} f_i(t, x(t - \tilde{\tau}_i(t))), \ 0 \le t \ne t_k, \\ x(t_k^+) - x(t_k) = \sin(x(t_k)), \qquad k \in \mathbb{N}, \end{cases} \tag{6.55}$$

with the functions $a(t) > 0$, $\tilde{\tau}_i(t) \ge 0$, $f_i(t, u) \ge 0$ being continuous, ω-periodic in t, $f_i(t, u)$ being bounded, for every $i \in \{1, \dots, m\}$, and such that

$$b(t)h^-(u) \le \sum_{i=1}^{m} f_i(t, u) \le b(t)h^+(u) \quad \text{for} \quad t \in [0, \omega], \ u \ge 0, \tag{6.56}$$

for some continuous functions $b, h^-, h^+ : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\begin{aligned} &h^+(u) < u \quad \text{for} \quad u > 0, \\ &h^\pm(0) = 0, \quad (h^\pm)'(0) = 1. \end{aligned} \tag{6.57}$$

For the non-impulsive equation $x'(t) = -a(t)x(t) + \sum_{i=1}^{m} f_i(t, x(t - \tilde{\tau}(t)))$, [39, Theorem 3.2] implies that its zero solution is globally asymptotically stable (in the set of all nonnegative solutions) if $b(t) \le a(t)$, $t \in [0, \omega]$. On the other hand, condition (A6) holds. Thus, if there exists $p \in \mathbb{N}$ such that

$$e^{A(\omega)} - \int_0^\omega b(s)\, ds \le 2^p < e^{A(\omega)},$$

the introduction of p impulses on each interval of length ω generates at least one positive ω-periodic solution for (6.55); e.g., with

$$\int_0^\omega a(s)\, ds = \int_0^\omega b(s)\, ds = 1,$$

it suffices to implement one impulse $\Delta(x(t_1)) = \sin(x(t_1))$, for any $t_1 \in [0, \omega)$, an repeat it at times $t_1 + k\omega$, $k \in \mathbb{N}$. Note also that the requirements in (6.56), (6.57) are satisfied by a broad class of equations, which include cases of the family of Nicholson and Mackey-Glass delay differential equations.

Example 6.34. Let $p = 2m$ with $m \in \mathbb{N}$, $\beta \in (0, 1)$ and $\delta > 0$. Define the impulse functions as $I_k(u) = (-1)^k \beta u e^{-\delta u}$ for $k \in \mathbb{N}$ and $u \ge 0$. Hence, $|I_k(u)|/u < \beta$, and there exist

$$J_k(0) := \lim_{u \to 0^+} J_k(u) = (1 + (-1)^k \beta)^{-1}, \quad \text{for all } u > 0 \text{ and } k \in \mathbb{N}.$$

Here, $a_k = -\beta$, $b_k = 0$ for $k \in \mathbb{N}$ odd and $a_k = 0$, $b_k = \beta$ for $k \in \mathbb{N}$ even, and conditions (A2)–(A4) are satisfied if $(1+\beta)^m < e^{A(\omega)}$. Moreover, we have $C_1 > 1$ if

$$\min_{t\in[0,\omega]} \int_t^{t+\omega} b(s) e^{\int_t^s a(u)du} \prod_{k:t_k\in[t,s)} J_k(0)\, ds > \frac{e^{A(\omega)}}{(1-\beta^2)^m} - 1.$$

In particular, this condition is fulfilled if

$$\int_0^\omega b(s)\, ds \geq (1+\beta)\left(\frac{e^{A(\omega)}}{(1-\beta^2)^m} - 1\right).$$

Example 6.35. Define the impulse functions as $I_k(u) = I(u) := \beta u e^{-\delta u} - \alpha u$, for any $k \in \mathbb{N}$, where $\beta > 0 > \alpha - 1$ and $\delta > 0$. Note that, for $u > 0$, the impulses are always negative if $\beta \leq \alpha$, always positive if $\alpha < 0$, whereas the sign of the impulses changes if $\beta > \alpha > 0$, with $I'(0) = \beta - \alpha > 0$ and $\lim_{u\to\infty} I(u) = -\infty$. With the previous notation, $a_k = -\alpha,\ b_k = \beta - \alpha,\ J_k(u) = (1 + \beta e^{-\delta u} - \alpha)^{-1}$, so, we obtain

$$\begin{aligned} J(0) &:= \lim_{u\to 0^+} J_k(u) = (1+\beta-\alpha)^{-1}, \quad k \in \mathbb{N}, \\ J(\infty) &:= \lim_{u\to\infty} J_k(u) = (1-\alpha)^{-1}, \quad k \in \mathbb{N}, \\ B_0 &= B^0 = (1+\beta-\alpha)^{-p}. \end{aligned}$$

Hypotheses (A2)–(A4) are fulfilled if $(1+\beta-\alpha)^p < e^{A(\omega)}$. Here, $C_1 > 1$ holds if

$$\min_{t\in[0,\omega]} \int_t^{t+\omega} b(s) e^{\int_t^s a(u)du} \prod_{k:t_k\in[t,s)} (1+\beta-\alpha)^{-1}\, ds > \left(e^{A(\omega)}(1+\beta-\alpha)^{-p} - 1\right).$$

In particular, $J(0) \geq 1$ if $\beta \leq \alpha$, and we get $C_1 > 1$ provided that

$$\int_0^\omega b(s)\, ds \geq e^{A(\omega)}(1+\beta-\alpha)^{-p} - 1.$$

For $\beta > \alpha$, $J(0) < 1$; thus, $C_1 > 1$ if

$$\text{either } b(t) \geq a(t),\ t \in [0,\omega], \quad \text{or} \quad \int_0^\omega b(s)\, ds \geq e^{A(\omega)} - (1+\beta-\alpha)^p.$$

6.5 Final comments

In this chapter, we have applied a version of the Krasnoselskii Fixed Point Theorem to establish sufficient conditions under which the impulsive periodic DDE (6.1) admits at least one positive periodic solution. The novelty of our approach derives from the operator Φ constructed in (6.6), whose fixed points are the periodic solutions we are looking for. Our method allows the treatment of equations with a very general feedback map (production term in the sense of [5]) and infinite delay, subject to impulses whose signs may vary. Applications to Volterra integro-differential equations (6.36) are given. In particular, the study of a Nicholson-type equation with mixed monotonicity is included as an illustration of our results. Equation (6.1) is broad enough to incorporate as special cases a large number of problems studied by many authors, nevertheless our criteria strongly improve several known results.

The present chapter pursues the study in the former work by Faria and Oliveira [40], who dealt with equation (6.3). The main result here, Theorem 6.8, generalises [40, Theorem 2.3] in several ways: (i) as mentioned above, infinite delays as well as nonlinear impulses are allowed in (6.1); (ii) rather than being bounded (as a map), the feedback nonlinearity is now allowed to have sublinear growth at infinity; the superlinear case is also addressed; (iii) finally, following the suggestion in [40, Remark 2.1], here superior and inferior limits are used in (6.12) and (6.14), instead of limits.

Although several open problems posed in [40] were solved here, relevant lines of future investigation were not addressed. Namely, the global attractivity of a positive periodic solution to (6.1) was not studied: it depends heavily on the particular shape of the feedback nonlinearity, and more conditions on the impulses should be imposed. Also, it would be desirable to adapt the present approach to quasi-periodic DDEs, due to their relevance in real world phenomena. On the other hand, our method can be extended to other classes of impulsive scalar DDEs, as outlined briefly in Remark 6.10.

Chapter 7

General discussion, conclusions and future work

This project has become a great opportunity for the author to learn many facts within the study of dynamical systems, differential equations and other related areas. In particular, discrete-time dynamics, integral inequalities and fixed point theorems have been the principal tools used.

Many of those basic facts, already known in the above-mentioned fields, were reviewed in Chapter 1, where several explanations, pictures and results have been recalled in order to organise the underlying ideas appearing throughout the work, and to provide the reader with the tools that are needed afterwards.

Firstly, the particular features of delay differential equations lead us to work with a higher level of abstraction than the one required for classical ordinary differential equations (Sections 1.1 and 1.2). Such parts set a general framework for which a great part of the equations studied in this thesis fit properly. In fact, we have put a particular emphasis on scalar delay differential equations that correspond to the type of dynamics of production and destruction, in the sense of [5], where the destruction term is linear and the production term is a general 'delayed' feedback, as in

$$x'(t) = -a(t)x(t) + \hat{f}(t, x_t).$$

Indeed, if $a(t)$ is constant and the feedback is written in terms of a single bounded delay, i.e., $\hat{f}(t, x_t) = f(x(t - \tilde{\tau}(t)))$, the behaviour of the iterations of the map f/a influences the asymptotic behaviour of the solutions of the DDE (Section 1.3). In fact, it is possible to obtain that an equilibrium of the former DDE is globally attracting by finding conditions to ensure a similar situation for the corresponding difference equation. In particular, if the discrete delay $\tilde{\tau}(t)$ is constant, that is, if

$$x'(t) = -ax(t) + f(x(t - \tau)), \tag{7.1}$$

some refined results can be given. Finally, in order to study the iterates of the map f/a, we were interested in cases where we can extract information about the global dynamics via the behaviour of the map near its unique equilibrium. For instance, the maps with negative Schwarzian derivative fall into this context.

In Chapters 2 and 3, we have developed the contents of the works [17, 20, 83]. Those works present a class of delay differential equations that belong to the family of gamma-models. The value γ appears as a feedback inner parameter and has a particular meaning in many ecological and economical models. Indeed, the assumption $\gamma \in [0, 1]$ allows us to work with a general framework under which we have provided some results regarding global attractivity, complementing the studies on the global dynamics of the limit cases $\gamma \in \{0, 1\}$ and on the local attractivity for the intermediate cases $\gamma \in (0, 1)$ already available in the literature.

In Chapter 4, we have made a contribution [18] to the extension of the ideas shown in Section 1.3 to equations in $\mathbb{R}^s$, $s \geq 2$. It is known that global attractivity in a system of DDEs is not guaranteed by global attractivity of its corresponding difference equation, in fact, some more restrictive conditions are needed. While the emergence of the concept of strong attractor has become one relevant step towards the understanding of how such inherited properties work, we have seen that weaker global attractivity conditions are still sufficient to keep such link valid. Such new weaker condition (CC-strong attractivity), which is written in terms of families of nested, compact and convex sets, opens a window that would potentially serve to study the global dynamics of more systems of delay differential equations of the type of production and destruction.

When it comes to scalar equations where the feedback is linear but the delay is not bounded as $t \to \infty$, the situation seems tricky. The work [52] provides a tool to deal with this kind of equations if they have a single discrete increasing delay: a type of Gronwall-Bellman estimates. We have extended their results in [21], as illustrated in Chapter 5, and showed that they also work for linear feedbacks with distributed increasing delay. Indeed, we have also obtained estimates that are sharp.

Furthermore, when all the coefficients in the scalar DDE are periodic for a common period $\omega > 0$, the question of the existence of a periodic solution with period ω arises naturally. In fact, such solution can be interpreted as a fixed point of the map that takes initial conditions into their state after a time ω and, thus, it resembles the notion of an equilibrium. Many researchers have studied this problem in the context of a DDE with a linear destruction term and a feedback with finite delay. We have gone further: we have obtained conditions to ensure the existence of periodic solutions, even with infinite delay and the introduction of impulses (Chapter 6). The latter features require handling even a more complicated phase space when it comes to dynamics, yet merely showing the existence of such solution can be somehow simplified. Our results are also valid for the non-impulsive case, improving many well-known conditions in the literature.

Regarding the lines that have been started within this project, the author is willing to continue studying them and, in particular, working in and learning about the following topics that are commented below.

For instance, the author is interested in continuing learning about the problem 'LAS implies GAS', both in difference equations and DDEs.

In the case of the DDE (7.1), it would be thrilling to know whether $\tau^* = \overline{\tau}$ in the context of Remark 1.67, in other words, if there exists a unique global stability switch, that is, whether increasing τ yields to a loss of global attractivity that is not recovered for any larger delay. For S-maps, such aim involves the study of the global stability in the region of parameters between conditions related to ρ_2 and ρ_3, which seems to constitute a challenging problem due to the amount of computations [85]. Recently, some techniques involving computer-assisted proofs (see, e.g., [9]) have been developed to deal with it. This is a tool that is also worth learning about.

Nevertheless, some work can be done before approaching the boundary of local stability of (7.1). For instance, it would be interesting to write the stability conditions of the delay-differential gamma-models shown in Chapter 2 by using the curves ρ_2 and/or ξ_2 and, thus, extend the corresponding given conditions. Some comments need to be done since the feedback f does not satisfy $Sf < 0$ wherever the Schwarzian derivative is defined; a sufficient condition should be provided in order to envelope f by a rational function, which is the key assumption in [86]. However, those changes seem to be more dramatic in the case of the γ-Mackey-Glass DDE (2.46) since the Schwarzian-type condition shown in Chapter 3 is different in such an extent that the results in [86] cannot be straightforwardly adapted. Thus, even the relative globally attracting condition with ρ_1 or ξ_1 for (2.46) deserves a careful study.

The more general case of variable delay in (7.1) requires some further study too. We have only recalled Theorem 1.69, for which a delay-dependent condition is stated just for S-maps [86]. Hence, it is also relevant to ask whether it can be extended to S_*-maps, in order to provide an appropriate framework for all the γ-models.

When it comes to the absolute stability of the DDE in (7.1) (or even its variable-delay version), obtaining sharp global attractivity conditions for their corresponding difference equation becomes the natural goal. One should be aware that the classical Schwarzian negativity condition is not the only one that ensures the property 'LAS implies GAS' in the context of a difference equation. We have seen in Chapter 3 that the values of the Schwarzian derivative might be somehow controlled in another way to produce the same output. In this line, the condition in [43, Corollary 1] might be considered as a third example of a Schwarzian-type condition. In fact, when it comes to the property 'LAS implies GAS', the approach shown in [43] is a nice supporting point to continue studying this type of conditions.

Another line of work shall be to seek easy necessary or sufficient conditions for an equilibrium of the system (4.4) to be a CC-strong attractor. In this line, multidimensional discrete-time dynamics become relevant and the author is involved in an ongoing work with Daniel Cao Labora. Moreover, the author is also interested in the models that can be studied through this concept.

Furthermore, there exist results that extend Gronwall-Bellman-type estimates to non-linear integral inequalities [112]. It is interesting to check to what extent they are useful for DDEs and whether one can improve some known estimates by utilizing ideas analogous to the ones in [52, 53], which have been recalled in our generalisations in Chapter 5.

When it comes to the impulsive models shown in Chapter 6, both with and without impulses, the existence of a periodic orbit was established, yet its stability requires special attention too. Therefore, obtaining conditions for such an orbit to be globally attracting becomes a possible future task. In particular, if the γ-models considered in Chapter 2 are written in terms of periodic coefficients, it is interesting to study whether the periodic orbit is globally attracting.

Finally, the author has started a joint work with Gergely Röst on a family of scalar DDEs which find applications in avascular tumor growth. For further details on how this situation can be modelled, we refer to [22].

As a conclusion, we may say that this thesis provides several improvements to the knowledge of the dynamics of some models of delay differential equations which are written in the line of [5] and have appeared in several recent articles. For instance, we have provided a unified framework to study the so-called delay-differential gamma-models and some results concerning their global dynamics which, as far as we know, fill several gaps in the literature. Moreover, we have contributed to the the study of the above-mentioned type of DDEs when their feedback is linear and allows the existence of a long-term unboundedness of the delay. Such improvements have been accomplished by extending some tools providing sharp estimates in order to permit cases of distributed delay. Additionally, we have generalised some results on the existence of periodic solutions when the coefficients in the equations are periodic and impulses are allowed: we have got rid of the finiteness of the delay, the linearity of the impulse functions, and several constraints on the production function.

Moreover, the author would also like to highlight that writing this thesis as a monograph, and thus aiming to provide a common framework for all the contents of the current project, has become an exceptional way not only to better present the work already done, but also to let many questions arise. This has yielded several lines of work which will be hopefully developed in the near future.

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Notation

$\mathbb{Z}$	integer numbers
$\mathbb{Z}_+$	nonnegative integer numbers
$\mathbb{N}$	positive integer numbers
$\mathbb{R}$	real numbers
$\mathbb{R}_+$	nonnegative real numbers
$\mathbb{R}^+$	positive real numbers
$\mathbb{R}_-$	nonpositive real numbers
$P(X)$	parts of a set X
$\mathrm{Dom}\, f$	domain of a function f
$\mathrm{Im}\, f$	image of a function f
$f \circ g$	composition of functions f and g
$\mathcal{C}(X, Y)$	set of continuous functions from X to Y
$\mathcal{C}^n(X, Y)$	set of functions from X to Y with continuous n-th derivative
$\partial_j f$	j-th partial derivative of a function with multivariable domain
$\overline{B}$	closure of a set B
$Int(B)$	interior of a set B
∂B	boundary of a set B
$\mathcal{M}_{n\times n}(\mathbb{R})$	space of real $n \times n$ matrices

Index

Further information

In these pages, we give some information concerning this document that is mandatory and/or of interest.

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Article [17]

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Article [20] (co-author: Eduardo Liz)

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Article [21] (co-author: Rosana Rodríguez-López)

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