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Further generalization of symmetric multiplicity theory to the geometric case over a field

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Abstract: Using the recent geometric Parter-Wiener, etc. theorem and related results, it is shown that much of the multiplicity theory developed for real symmetric matrices associated with paths and generalized stars remains valid for combinatorially symmetric matrices over a field. A characterization of generalized stars in the case of combinatorially symmetric matrices is given.

Keywords: Combinatorially symmetric matrix; Eigenvalue; Generalized star; Geometric multiplicity; Graph of a matrix; Path

MSC: 05C38; 05C50 (primary); 05C05; 15A18 (secondary)

Introduction

The graph of the real symmetric matrix $A = (a_{ij}) \in M_n(\mathbb{R})$ is the graph G on n vertices $1, \dots, n$, with an edge $\{i, j\}$ if and only if $a_{ij} \neq 0$. Denote by $S(G)$ the set of all n-by-n real symmetric matrices whose graph is G. No restriction is placed upon the diagonal entries. The case in which G is a tree T is especially interesting. The recent book [6] develops much of the notation and theory that is relevant.

For a real symmetric (or complex Hermitian) matrix whose graph is a tree, T, on n vertices, much is known about the possible unordered eigenvalue multiplicity lists. Recently, it has been shown that much of this multiplicity theory generalizes to geometric multiplicity for eigenvalues of combinatorially symmetric matrices (i.e., matrices $A = (a_{ij})$ with $a_{ij} \neq 0$ if and only if $a_{ji} \neq 0$) over a field $\mathbb{F}$. Denote by $\mathcal{F}(T)$ the set of combinatorially symmetric matrices over a general field whose graph is T. For $A \in \mathcal{F}(T)$, denote the geometric multiplicity of λ in A as $\mathrm{gm}_A(\lambda)$. Given a set $\alpha \subseteq \{1, \dots, n\}$, denote by $A[\alpha]$ the principal submatrix of A resulting from including only the rows and columns of A indexed by α. Similarly, $A(\alpha)$ is the principal submatrix resulting from excluding those same rows and columns. If v is a vertex of T, then $A[v]$ and $A(v)$ are defined analogously. Similar notation is used for branches T_i of T (at a vertex v), writing $A[T_i]$ and $A(T_i)$, respectively.

Crucial to the study of eignenvalue multiplicity theory is the Parter-Weiner, etc. theorem (and related notions/results). The most general version is given in [2]. For more background, see this source. In [5], it is shown that the theorem is valid for geometric multiplicities of matrices in $\mathcal{F}(T)$ (see also [4]). This is theorem 2,

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below. Several other known results generalize to matrices in $\mathcal{F}(T)$ as well: the downer branch mechanism and that the maximum geometric multiplicity $gM(T)$ (first discussed in [1]) of an eigenvalue of a matrix in $\mathcal{F}(T)$ is equal to the path cover number $P(T)$, [4], [5]. These results, among others, are restated below for convenience. Their proofs, although not provided here, are markedly different from the proofs of the analogous results concerning real symmetric matrices (with the exception of some graph theoretic commonalities).

Unlike real symmetric matrices, however, the interlacing inequalities for eigenvalues do not hold for matrices in $\mathcal{F}(T)$. Consequently, the smallest and largest eigenvalues of $A \in \mathcal{F}(T)$ are not required (as for real symmetric matrices) to have multiplicity 1, and they may occur as eigenvalues of a proper submatrix of A of size one smaller. Despite these differences (see [7] for one prominent example), we show here that using the important results established in [5], much of the remaining specific multiplicity theory for real symmetric matrices [2] may be generalized. This includes both general results and the highly specific results about paths and stars that we give here. These results emphasize the close relationship between symmetric and geometric multiplicity theory.

Throughout this document, we denote by P_n a path on n vertices. A branch of a tree T at v that is a path, with the neighbor of v in this branch a degree 1 vertex of this path, is called a *pendent path* at v. We also refer to a vertex v of a given tree T with $\deg_T(v) \geq 3$ as a high-degree vertex (HDV).

A *generalized star* (or g-star) is a tree T having at most one HDV. We call a vertex v of a g-star a *central vertex* if its neighbors $u_1, \ldots, u_k$ are pendent vertices in their branches (called *arms*) $T_1, \ldots, T_k$, respectively, and each of these branches is a path. We generalize the results from [3] necessary to characterize generalized stars in the case $A \in \mathcal{R}(T)$.

Preliminaries

We now record some important results from [5] (theorem 2 is also proven in [4]).

Lemma 1. *Let G be a graph and $\mathbb{F}$ a field. For $A \in \mathcal{F}(G)$, v a vertex of G, and $\lambda \in \mathbb{F}$,*

$$|\mathrm{gm}_{A(v)}(\lambda) - \mathrm{gm}_A(\lambda)| \leq 1.$$

More precisely, in $A(v)$ there are three possibilities, the third occurring only in case $\mathrm{gm}_A(\lambda) \geq 1$:

1. $\mathrm{gm}_{A(v)}(\lambda) = \mathrm{gm}_A(\lambda) + 1$, *which occurs if and only if*
$$\mathrm{rank}(A(v) - \lambda I) = \mathrm{rank}(A - \lambda I) - 2;$$
2. $\mathrm{gm}_{A(v)}(\lambda) = \mathrm{gm}_A(\lambda)$, *which occurs if and only if*
$$\mathrm{rank}(A(v) - \lambda I) = \mathrm{rank}(A - \lambda I) - 1;$$
3. $\mathrm{gm}_{A(v)}(\lambda) = \mathrm{gm}_A(\lambda) - 1$, *which occurs if and only if*
$$\mathrm{rank}(A(v) - \lambda I) = \mathrm{rank}(A - \lambda I).$$

Thus, removing a vertex v may cause the geometric multiplicity of λ in $A \in \mathcal{F}(G)$ to increase by 1, decrease by 1, or stay the same. We call v a *g-Parter, g-downer* or *g-neutral* vertex, respectively, for λ in $A \in \mathcal{F}(G)$. Below is the geometric Parter-Wiener, etc. theorem.

Theorem 2. *Let $\mathbb{F}$ be a field, T a tree, and $A \in \mathcal{F}(T)$. Suppose that there is a vertex v of T such that $\lambda \in \sigma(A) \cap \sigma(A(v))$. Then*

1. *there is a vertex u of T such that* $\mathrm{gm}_{A(u)}(\lambda) = \mathrm{gm}_A(\lambda) + 1$, *i.e., u is g-Parter for λ (with respect to A and T);*
2. *if* $\mathrm{gm}_A(\lambda) \geq 2$, *then u may be chosen so that* $\deg_T(u) \geq 3$ *and so that there are at least three components T_1, T_2, T_3 of $T - u$ such that* $\mathrm{gm}_{A[T_i]}(\lambda) \geq 1$, $i = 1, 2, 3$*; and*

3. *if* $\mathrm{gm}_A(\lambda) = 1$, *then* u *may be chosen so that* $\deg_T(u) \geq 2$ *and so that there are two components* T_1, T_2 *of* $T - u$ *such that* $\mathrm{gm}_{A[T_i]}(\lambda) = 1, i = 1, 2$.

We also have the following result for paths.

Lemma 3. *If* $\mathbb{F}$ *is a field and* $A \in \mathcal{F}(P_n)$, *then for any* $\lambda \in \sigma(A)$, $\mathrm{gm}_A(\lambda) = n - \mathrm{rank}(A - \lambda I) = 1$.

Lastly, we present what is often referred to as the “downer branch mechanism."

Theorem 4. *Let* T *be a tree,* $A \in \mathcal{F}(T)$, *and* $\lambda \in \sigma(A)$. *Then a vertex* v *of* T *is g-Parter for* λ *(with respect to* A *and* T*) if and only if there is a neighbor* u *of* v *in* T *in whose branch* (T_u) u *is a g-downer vertex for* λ *(with respect to* $A[T_u]$ *and* T_u*).*

We call a branch guaranteed by the above theorem a *g-downer branch* at v.

Results

The following results generalize those from [2] to the case in which $A \in \mathcal{F}(T)$.

Lemma 5. *If* $A \in \mathcal{F}(P_n)$, *then* $\sigma(A) \cap \sigma(A(1)) = \sigma(A) \cap \sigma(A(n)) = \emptyset$.

Proof. We induct on n. By symmetry, we only need to verify the claim for $A(n)$. If $n = 1$, the claim is trivial. If $n = 2$, we suppose for the sake of contradiction that $\lambda \in \sigma(A) \cap \sigma(A(2))$. This implies by Theorem 2 that a g-Parter vertex exists, but clearly neither vertex 1 or 2 can be g-Parter. If $n = 3$, we again suppose for the sake of contradiction that $\lambda \in \sigma(A) \cap \sigma(A(3))$. Since $\mathrm{gm}_A(\lambda) = 1$, by Theorem 2, vertex 2 must be g-Parter. Then, however, $\lambda \in \sigma(A(3)) \cap \sigma(A(\{2, 3\}))$, by part three of Theorem 2, which is contradictory as shown above for $n = 2$.

Now let $n > 3$ and assume the claim holds for $A \in \mathcal{F}(P_i)$, $i < n$. Suppose for the sake of contradiction that $\lambda \in \sigma(A) \cap \sigma(A(n))$. By the induction hypothesis, $\lambda \notin \sigma(A(\{n-1, n\}))$, so that $n - 1$ is a g-downer vertex for λ at n. By Theorem 4, then, vertex n is g-Parter and $\mathrm{gm}_{A(n)}(\lambda) = 1 + 1$, which contradicts Lemma 3. □

This leads to the following corollary, which follows from Theorem 4 and Lemma 5.

Corollary 6. *Let* T *be a tree,* v *a vertex of* T, *and* $A \in \mathcal{F}(T)$. *If there is a direct summand of* $A(v)$ *whose graph is a pendant path at* v *and which has* λ *as an eigenvalue, then* $\mathrm{gm}_{A(v)}(\lambda) = \mathrm{gm}_A(\lambda) + 1$.

We now extend an important result of [2] to the general setting.

Theorem 7. *Let* $i \in \{1, \ldots, n\}$ *and* $A \in \mathcal{F}(P_n)$. *Then* $\lambda \in \sigma(A) \cap \sigma(A(i))$ *if and only if* $1 < i < n$ *and* i *is g-Parter for* λ, *with* $\lambda \in \sigma(A[\{1, \ldots, i-1\}])$ *and* $\lambda \in \sigma(A[\{i+1, \ldots, n\}])$.

Proof. If $\lambda \in \sigma(A) \cap \sigma(A(i))$ then by Lemma 5, $1 < i < n$ and, thus, λ is an eigenvalue of at least one of the two summands of $A(i)$. By Corollary 6, we have $\mathrm{gm}_{A(i)}(\lambda) = \mathrm{gm}_A(\lambda) + 1$. Since P_n is a path $\mathrm{gm}_{A(i)}(\lambda) = 2$ and, because the two branches of P_n at i are paths, $\mathrm{gm}_{A[\{1,\ldots,i-1\}]}(\lambda) = \mathrm{gm}_{A[\{i+1,\ldots,n\}]}(\lambda) = 1$.

For the converse, since λ is an eigenvalue of the two summands of $A(i)$, $1 < i < n$, we can conclude by Lemma 1 that $\lambda \in \sigma(A)$. □

From Lemma 3 and Theorem 7, the following corollary is immediate.

Corollary 8. *If* $A \in \mathcal{F}(P_n)$, *then there are at most* $\min\{i - 1, n - i\}$ *different eigenvalues that are common to both* A *and* $A(i)$.

By Lemma 5 and Theorem 7, when T is a path and $A \in \mathcal{F}(T)$, if $\lambda \in \sigma(A) \cap \sigma(A(v))$, then v must be a degree 2 vertex of T and v is necessarily g-Parter for λ. When $\mathbb{F} = \mathbb{R}$, this property characterizes the path, as in the case of real symmetric matrices. We refer to the set of combinatorially symmetric matrices over the real field whose graph is a given tree T as $\mathcal{R}(T)$.

Theorem 9. *Let T be a tree and v a vertex of T such that, for any $A \in \mathcal{R}(T)$, if $\lambda \in \sigma(A) \cap \sigma(A(v))$ then* $\mathrm{gm}_{A(v)}(\lambda) = \mathrm{gm}_A(\lambda) + 1$. *Then T is a path and v is a degree 2 vertex of T.*

Proof. Suppose that T is a tree but not a path, i.e., T is a tree with at least one HDV. We show that there exists a vertex v of T and a matrix $A \in \mathcal{R}(T)$ with an eigenvalue $\lambda \in \sigma(A) \cap \sigma(A(v))$ satisfying $\mathrm{gm}_{A(v)}(\lambda) = \mathrm{gm}_A(\lambda) - 1$. Since $S(T) \subseteq \mathcal{R}(T)$, it is enough to look in $S(T)$. In order to construct A, consider an HDV u of T whose removal leaves $k \geq 3$ components $T_1, \ldots, T_k$. For each of these components, construct $A_i \in S(T_i)$ whose smallest eigenvalue is λ and let A be any matrix in $S(T)$ with the submatrices A_i in appropriate positions. Since the smallest eigenvalue of a real symmetric matrix whose graph is a tree does not occur as an eigenvalue of any principle submatrix of size one smaller (due to interlacing, see [2]), any T_i is a g-downer branch at u for λ. Since $\mathrm{gm}_{A(u)}(\lambda) = k$, by the real symmetric analogue to Theorem 4, it follows that $\mathrm{gm}_A(\lambda) = k - 1 \geq 2$ and, by interlacing, $\lambda \in \sigma(A) \cap \sigma(A(v))$ for any vertex v of T.

Considering a vertex v of T, with $v \neq u$, let us show that $\mathrm{gm}_{A(v)}(\lambda) = \mathrm{gm}_A(\lambda) - 1$. Observe that λ occurs as an eigenvalue of only one of the direct summands of $A(v)$, corresponding to the component T' of $T - v$ containing the vertex u. Since λ is now an eigenvalue of $k - 1$ components of $A[T' - u]$ (in each one with multiplicity 1), again, by the analogue to Theorem 4, it follows that $\mathrm{gm}_{A[T']}(\lambda) = \mathrm{gm}_{A[T'-u]}(\lambda) - 1 = k - 2$. Since $\mathrm{gm}_{A(v)}(\lambda) = \mathrm{gm}_{A[T']}(\lambda)$, we have $\mathrm{gm}_{A(v)}(\lambda) = \mathrm{gm}_A(\lambda) - 1$. □

We now generalize two results from [3], which characterize generalized stars for the case in which $A \in \mathcal{R}(T)$. We first present a lemma, which holds for any field.

Lemma 10. *Let T be a g-star with central vertex v. If $A \in \mathcal{F}(T)$ and λ is an eigenvalue of $A(v)$, then* $\mathrm{gm}_{A(v)}(\lambda) = \mathrm{gm}_A(\lambda) + 1$

Proof. Observe that if $\deg_T(v) = k$ and $A \in \mathcal{F}(T)$, then $A(v) = A[T_1] \oplus \cdots \oplus A[T_k]$, in which each T_i is a path. By Lemmas 3 and 5, if λ is an eigenvalue of $A(v)$, then at least one arm T_i of T is a g-downer branch for λ, and the result follows from Theorem 4. □

Theorem 11. *Let T be a tree and v a vertex of T such that, for any $A \in \mathcal{R}(T)$ and any eigenvalue λ of $A(v)$,* $\mathrm{gm}_{A(v)}(\lambda) = \mathrm{gm}_A(\lambda) + 1$. *Then T is a g-star and v is a central vertex of T.*

Proof. Suppose that T is a tree but not a g-star. Then T has at least two HDVs. Let v be any vertex of T and choose a vertex u of degree $k \geq 3$ of T, $u \neq v$. We show that there exists $A \in \mathcal{R}(T)$ such that λ is an eigenvalue of $A(v)$ satisfying $\mathrm{gm}_{A(v)}(\lambda) = \mathrm{gm}_A(\lambda) - 1$. Since $S(T) \subseteq \mathcal{R}(T)$, it is enough to look in $S(T)$. In order to construct A, consider the vertex u whose removal leaves $k \geq 3$ components $T_1, \ldots, T_k$. For each of these components, construct $A_i \in S(T_i)$ whose smallest eigenvalue is λ. Let $A \in S(T)$ be any matrix such that $A[T_i] = A_i$, $i = 1, \ldots, k$. Recall that the smallest eigenvalue of a real symmetric matrix whose graph is a tree does not occur as an eigenvalue of any principle submatrix of size one smaller (see [2]). Thus any T_i is a g-downer branch at u for λ. Thus, $\mathrm{gm}_{A(u)}(\lambda) = k$, and, by Theorem 4, it follows that $\mathrm{gm}_A(\lambda) = k - 1$.

Let us see that $\mathrm{gm}_{A(v)}(\lambda) = \mathrm{gm}_A(\lambda) - 1$. Observe that λ occurs as an eigenvalue of only one of the direct summands of $A(v)$, corresponding to the component T' of $T - v$ containing the vertex u. Since now λ is an eigenvalue of $k - 1$ components of $A[T' - u]$ (in each one with multiplicity 1), again, by Theorem 4, it follows $\mathrm{gm}_{A[T'-u]}(\lambda) = \mathrm{gm}_{A[T']}(\lambda) + 1$, i.e., $\mathrm{gm}_{A[T']}(\lambda) = k - 2$. Since $\mathrm{gm}_{A(v)}(\lambda) = \mathrm{gm}_{A[T']}(\lambda)$, we have $\mathrm{gm}_{A(v)}(\lambda) = \mathrm{gm}_A(\lambda) - 1$.

If we assume that T is a g-star and v is not a central vertex, the same argument holds to prove the claimed statement. □

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